SYSTEM IDENTIFICATION AND PID TUNING

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1. Introduction

In the Tau Labs software, the autotuning algorithm is actually performing system identification using an EKF that estimates the latency of the motor responses (approximated as a first order low pass filter) and the gain of each axis. Based on these parameters, we can calculate the optimal PID coefficients to achieve snappy responses without overshoot. We also want to optimize parameters such as the low pass filtering that is applied to the gyro signal before the derivative stage, which allows quick responses while not being overly sensitive to noise.

Typically, optimizing a control loop with this many parameters would be done by specifying certain properties of the desired step response and then performing many simulation to achieve these constraints. However, the goal of this work is to find analytic approximations that can ultimately be implemented onboard. This could ultimately lead to systems that adapt constantly based on throttle setting (accounting for both the non-linear gain response and change in latency) or things like sagging battery voltage.

2. Rate loop PD controller

From the parameters measured above, we want to calculate the PID settings for the rate loop that will achieve a pre specified performance. Specifically, we will be look at the positions of the poles of the closed loop response and the sensitivity of the controller output to the measured gyro noise.

2.1. **Rate loop.** We model each axis as a latency τ for the rotors to reach speed and then an effectiveness in terms of generating rotation ω.

$$G(s) = \frac{\beta}{s(1+\tau s)} \tag{1}$$

and use a PD controller

$$C(s) = K_p + K_d s \tag{2}$$

which can be manipulated to give a closed loop response of

$$\begin{align}
CL(s) &= \frac{G(s)C(s)}{1+G(s)C(s)} \tag{3}\\
&= \frac{\beta(K_d s + K_p)}{s(s\tau+1)+\beta K_p + \beta K_d s} \tag{4}\\
&= \frac{\beta K_d s + \beta K_p}{\tau s^2 + (1+\beta K_d)s + \beta K_p)} \tag{5}\\
&= \frac{\beta/\tau K_d s + \beta/\tau K_p}{s^2 + (1+\beta K_d)/\tau s + \beta/\tau K_p} \tag{6}
\end{align}$$

From which we can calculate the characteristic polynomial

$$\begin{align}
&s^2 + 2\zeta\omega_n + \omega_n^2 \tag{7}\\
\omega_n &= \sqrt{\frac{\beta K_p}{\tau}} \tag{8}\\
\zeta &= \frac{1+\beta K_d}{\sqrt{\beta K_p \tau}} \tag{9}
\end{align}$$

Then we can set the desired response frequency, ω_0 and damping ζ_0 and solve for K_p and K_d.

$$\begin{align}
K_p &= \frac{\tau}{\beta} w_0^2 \tag{10}\\
K_d &= 2\zeta_0 \sqrt{\tau \beta K_p}/\beta \tag{11}
\end{align}$$

2.2. **Rate loop PD controller with filtering.** Unfortunately using a derivative component without a low pass filter adds too much high frequency noise. Thus, we must replace the controller with one of the following form:

$$C(s) = K_p + \frac{K_d s}{1 + \tau_d s} \tag{12}$$

(13)

Performing the same manipulations as above, we can get it in the form:

$$CL(s) = \frac{\beta\left(K_p\tau_d + K_d\right)s + \beta K_p}{\tau\tau_d s^3 + (\tau + \tau_d)s^2 + \left(1 + \beta K_p\tau_d + \beta K_d\right)s + \beta K_p} \tag{14}$$

In this form, it is not as easy to interpret the terms of the equation as for a second order system.

2.2.1. *Better factorization.* In order to analytically analyze map from this equation to the behavior of the system, we want to factor the equation into the product of a single pole and a second order system: $(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

$$(s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2) = s^3 + \frac{\tau + \tau_d}{\tau\tau_d}s^2 + \frac{1 + \beta K_p\tau_d + \beta K_d}{\tau\tau_d}s + \frac{\beta K_p}{\tau\tau_d} \tag{15}$$

(16)

$$s^3 + (a + 2\zeta\omega_n)s^2 + (\omega_n^2 + 2\zeta a\omega_n)s + a\omega_n^2 = s^3 + \frac{\tau + \tau_d}{\tau\tau_d}s^2 + \frac{1 + \beta K_p\tau_d + \beta K_d}{\tau\tau_d}s + \frac{\beta K_p}{\tau\tau_d}$$

Which gives a set of constraints:

$$a + 2\zeta\omega_n = \frac{\tau + \tau_d}{\tau\tau_d} \tag{17}$$

$$\omega_n^2 + 2\zeta a\omega_n = \frac{1 + \beta K_p\tau_d + \beta K_d}{\tau\tau_d} \tag{18}$$

$$a\omega_n^2 = \frac{\beta K_p}{\tau\tau_d} \tag{19}$$

So by specifying the values of ζ, ω_n, and a we can solve for an appropriate combination of K_p, K_d and τ_d. We know to have a nice stable response without much overshoot we can use a higher value of ζ on the order of $1 - 1.5$. However, specifying both ω_n and a separately is more parameters than we would like to control the response.

2.3. **Sensitivity Analysis.** A practical limitation is that the gyro measurement has a substantial amount of noise, specifically at the rotor RPM. We want to limit the gain of the controller to the measurement noise which is not captured by the above analysis. We currently use a default low pass filter of the gyro at 20 Hz for the derivative component, but this is determined by the value of τ_d.

One constraint on the system is to limit the range of the output used by noise. In order to do this we want to calculate the sensitivity of the controller output on the gyros. The full equation for this is

$$\frac{-C(s)}{1 + C(s)G(s)} \tag{20}$$

But this is very close to $C(s)$ for high frequencies because $G(s)$ goes to zero while $C(s)$ plateaus.

$$C(s) = K_p + \frac{K_d s}{1 + \tau_d s} + K_i \frac{1}{s} \tag{21}$$

$$= \frac{K_p s(1 + \tau_d s) + K_d s^2 + K_i(1 + \tau_d s)}{s(1 + \tau_d s)} \tag{22}$$

$$= \frac{(K_p \tau_d + K_d)s^2 + (K_p + \tau_d K i)s + K_i}{\tau_d s^2 + s} \tag{23}$$

Which shows the high frequency response as approaching

$$\frac{K_p \tau_d + K_d}{\tau_d} = K_p + \frac{K_d}{\tau_d} \tag{24}$$

Giving us the constraint:

$$\left(K_p + \frac{K_d}{\tau_d} \right) \sigma_{gyro} < 0.2 \tag{25}$$

Commonly this noise is less than 20 deg/s if unmeasured so can also be the same as keeping the gain at less than 1% for higher frequencies.

2.3.1. *Solving for the fastest τ_d that fits the noise.* We can manipulate the equation

$$\left(K_p + \frac{K_d}{\tau_d} \right) = g_{hf} \tag{26}$$

, above to give

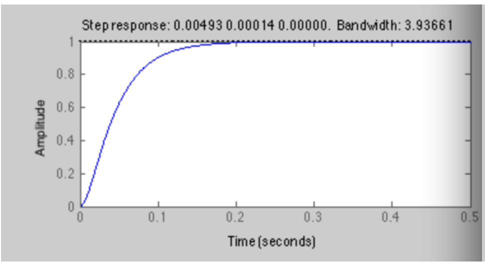
$$\frac{(-2\zeta\tau\omega_n^3 + \omega_n^2)\tau_d^2 + (\tau\omega_n^2 + \omega_n^2 - 2\zeta\tau\omega_n^3)\tau_d + \tau\omega_n^2}{\beta\tau_d} = g_{hf} \tag{27}$$

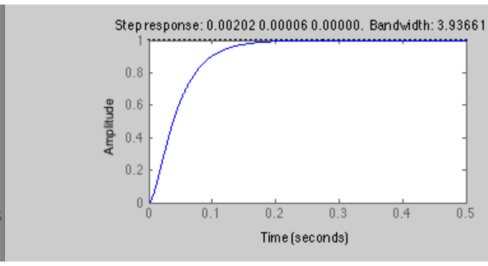
$$(-2\zeta\tau\omega_n^3 + \omega_n^2)\tau_d^2 + (\tau\omega_n^2 + \omega_n^2 - 2\zeta\tau\omega_n^3)\tau_d + \tau\omega_n^2 = g_{hf}\beta\tau_d \tag{28}$$

$$(-2\zeta\tau\omega_n^3 + \omega_n^2)\tau_d^2 + (\tau\omega_n^2 + \omega_n^2 - 2\zeta\tau\omega_n^3 - g_{hf}\beta)\tau_d + \tau\omega_n^2 = 0 \tag{29}$$

Which can then be dropped into the quadratic equation

$$\tau_d = \frac{2\zeta\tau\omega_n - 1}{4\tau\zeta^2\omega_n^2 - 2\zeta\omega_n - \tau\omega_n^2 + \beta g_{hf}} \tag{30}$$





We check this for the beta for all axis and select the largest value to ensure the noise sensitivity is limited.

2.4. **Combined constraints.** With this value of τ_d we can then solve for the other parameters, given a value of ω_n and ζ.

$$a = \frac{\tau + \tau_d}{\tau \tau_d} - 2\zeta\omega_n \tag{31}$$

$$Kp = a\omega_n^2 \tau \tau_d / \beta \tag{32}$$

$$Kd = \frac{\tau \tau_d(\omega_n^2 + 2\zeta a \omega_n) - 1 - \beta K_p \tau_d}{\beta} \tag{33}$$

Typically using $\omega_n = 1/\tau_d$ and $\zeta = 1$ gives empirically nice results.

2.5. **Results.** TODO: collect a number of results for N trials with various copters.

On an asymmetrical frame (QAV500) I got a τ of -3.30 and a β of 9.2 and 8.3 for roll and pitch, respectively. This resulted in these tunings and clean step responses:

3. Rate loop PID controller

$$C(s) = K_p + \frac{K_d s}{1 + \tau_d s} + \frac{K_i}{s} \tag{34}$$

$$= \frac{(K_p \tau_d + K_d)\, s^2 + (K_p + K_i \tau_d)\, s + K_i}{s\,(1 + \tau_d s)} \tag{35}$$

(36)
$$CL(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$
(37)
$$CL(s) = \frac{\beta\left(K_p\tau_d + K_d\right)s^2 + \beta\left(K_p + K_i\tau_d\right)s + \beta K_i}{s^2\left(1+\tau s\right)\left(1+\tau_d s\right) + \beta\left(K_p\tau_d + K_d\right)s^2 + \beta\left(K_p + K_i\tau_d\right)s + \beta K_i}$$
(38)
$$CL(s) = \frac{\beta\left(K_p\tau_d + K_d\right)s^2 + \beta\left(K_p + K_i\tau_d\right)s + \beta K_i}{\tau\tau_d s^4 + \left(\tau+\tau_d\right)s^3 + \left[1 + \beta\left(K_p\tau_d + K_d\right)\right]s^2 + \beta\left(K_p + K_i\tau_d\right)s + \beta K_i}$$
(39)

We want to factor the characteristic equation to something like this:

$$(s+a)\left(s+b\right)\left(s^2 + 2\zeta\omega s + \omega^2\right) \tag{40}$$
(41)
$$s^4 + \left(2\zeta\omega + a + b\right)s^3 + \left[ab + \omega^2 + (a+b)2\zeta\omega\right]s^2 + \left[(a+b)\,\omega^2 + ab2\zeta\omega\right]s + ab\omega^2$$

From which we get:

$$2\zeta\omega + a + b = \frac{\tau + \tau_d}{\tau\tau_d} \tag{42}$$
$$ab + \omega^2 + (a+b)2\zeta\omega = \frac{1 + \beta\left(K_p\tau_d + K_d\right)}{\tau\tau_d} \tag{43}$$
$$(a+b)\,\omega^2 + ab2\zeta\omega = \frac{\beta\left(K_p + K_i\tau_d\right)}{\tau\tau_d} \tag{44}$$
$$ab\omega^2 = \frac{\beta K_i}{\tau\tau_d} \tag{45}$$

We want to determine the high frequency noise, $g_{hf} = K_p + K_d/\tau_d$, in terms of the poles.

$$K_i = ab\omega^2\tau\tau_d/\beta \tag{46}$$
$$K_p = \left[(a+b)\,\omega^2 + ab2\zeta\omega\right]\tau\tau_d/\beta - K_i\tau_d \tag{47}$$
$$K_d = \left(\left[ab + \omega^2 + (a+b)2\zeta\omega\right]\tau\tau_d - 1\right)/\beta - K_p\tau_d \tag{48}$$

Which we can substitute in to get:

$$g_{hf} = K_p + K_d/\tau_d \tag{49}$$

$$= K_p + \left(\left(\left[ab + \omega^2 + (a+b)2\zeta\omega\right] \tau\tau_d - 1\right)/\beta - K_p\tau_d\right)/\tau_d \tag{50}$$

$$= \frac{\left[ab + \omega^2 + (a+b)2\zeta\omega\right] \tau\tau_d - 1}{\beta\tau_d} \tag{51}$$

And solving for τ_d for a given g_{hf}.

$$\beta g_{hf}\tau_d = \left[ab + \omega^2 + (a+b)2\zeta\omega\right] \tau\tau_d - 1 \tag{52}$$

$$1 = \left[ab + \omega^2 + (a+b)2\zeta\omega\right] \tau\tau_d - \beta g_{hf}\tau_d \tag{53}$$

$$1 = \tau_d\left[\tau\left(ab + \omega^2 + (a+b)2\zeta\omega\right) - \beta g_{hf}\right] \tag{54}$$

$$\tau_d = 1/\left[\tau\left(ab + \omega^2 + (a+b)2\zeta\omega\right) - \beta g_{hf}\right] \tag{55}$$

To keep all the components at the same speed, $a = b = \omega = z$. Need to expand derivation to include the K_i term. This makes the system fourth order. At this point we will also need another constraint - the most likely being either the maximum allowable overshoot, the stability margins, or the time for load rejection.

Regardless, the stability margins should probably be addressed, although typically I'm seeing things ¿ 60 even with the integral component.

4. Attitude loop P controller

If we approximate the inner loop as a unity gain with a lag (like in the beginning), followed by the integration of rate to attitude:

$$I(s) = \frac{1}{s(1+\tau s)} \tag{56}$$

And then wrap it in a proportional controller:

$$CO_o(s) = \frac{K_p/(s+\tau s^2)}{1 + K_p/(s+\tau s^2)} \tag{57}$$

$$CO_o(s) = \frac{K_p}{\tau s^2 + s + K_p} \tag{58}$$

$$CO_o(s) = \frac{K_p/\tau}{s^s + 1/\tau s + K_p/\tau} \tag{59}$$

and again rewrite it in terms of a standard second order equation, $s^2 + 2\zeta\omega + \omega^2$.

$$\omega = \sqrt{K_p/\tau} \tag{60}$$

$$2\zeta\omega = 1/\tau \tag{61}$$

and finally if we solve for the K_p that makes it critically damped:

$$2\zeta\sqrt{K_p/\tau} = 1/\tau \tag{62}$$

$$K_p = \frac{1}{4\zeta^2\tau} \tag{63}$$

5. Implementation

These optimizations have been implemented in the Tau Labs software and are being widely used to both estimate the properties of various sizes of multirotor, as well as optimize the control loops.