1 Binary Child given one binary parent

Let X and Y be two binary variables, where the conditional probability of Y given X is expressed as follows:

$$p(Y|X) = p_{y|x}^{I(X=0)I(Y=0)} p_{\bar{y}|x}^{I(X=0)I(Y=1)} p_{y|\bar{x}}^{I(X=1)I(Y=0)} p_{\bar{y}|\bar{x}}^{I(X=1)I(Y=1)}$$

This conditional probability can be expressed in different exponential forms as follows:

$$\begin{aligned}
\ln p(Y|X) &= \theta^T s(Y,X) - A(\theta) \\
&= \begin{pmatrix} \ln p_{y|x} \\ \ln p_{\bar{y}|x} \\ \ln p_{y|\bar{x}} \\ \ln p_{\bar{y}|\bar{x}} \end{pmatrix}^T \begin{pmatrix} I(Y=0)I(X=0) \\ I(Y=1)I(X=0) \\ I(Y=0)I(X=1) \\ I(Y=1)I(X=1) \end{pmatrix} - 0
\end{aligned}$$

$$\begin{aligned}
\ln p(Y|X) &= \theta(X)^T s(Y) - A(X) \\
&= \begin{pmatrix} I(X=0)\ln p_{y|x} + I(X=1)\ln p_{y|\bar{x}} \\ I(X=0)\ln p_{\bar{y}|x} + I(X=1)\ln p_{\bar{y}|\bar{x}} \end{pmatrix}^T \begin{pmatrix} I(Y=0) \\ I(Y=1) \end{pmatrix} - 0 \\
&= \begin{pmatrix} m_0^X \cdot \theta_0 + m_1^X \cdot \theta_2 \\ m_0^X \cdot \theta_1 + m_1^X \cdot \theta_3 \end{pmatrix}^T \begin{pmatrix} I(Y=0) \\ I(Y=1) \end{pmatrix} - 0
\end{aligned}$$

$$\begin{aligned}
\ln p(Y|X) &= \theta(Y)^T s(X) - A(Y) \\
&= \begin{pmatrix} I(Y=0)\ln p_{y|x} + I(Y=1)\ln p_{\bar{y}|x} \\ I(Y=0)\ln p_{y|\bar{x}} + I(Y=1)\ln p_{\bar{y}|\bar{x}} \end{pmatrix}^T \begin{pmatrix} I(X=0) \\ I(X=1) \end{pmatrix} - 0 \\
&= \begin{pmatrix} m_0^Y \theta_0 + m_1^Y \theta_1 \\ m_0^Y \theta_2 + m_1^Y \theta_3 \end{pmatrix}^T \begin{pmatrix} I(X=0) \\ I(X=1) \end{pmatrix} - 0
\end{aligned}$$

2 Multinomial Child given a set of multinomial parents

Let us Y be a multinomial variable whose state space is $\{y_1, ..., y_k\}$. Let us also be $X_1, \ldots, X_p$ the set of parents of variable Y, all them also multinomial. The log-conditional probability of Y given X can be expressed as follows:

$$\ln p(Y|X_1, \ldots, X_p) =$$

Similarly the above conditional probability can be expressed in the following exponential forms:

$$\ln p(Y|X_1,\ldots,X_p) \quad = \quad \theta(X_1,\ldots,X_p)^T s(Y) - A(X_1,\ldots,X_p)$$

$$\ln p(Y|X_1,\ldots,X_p) \quad = \quad \theta(Y,X_1,\ldots,X_{L-1},X_L,\ldots,X_p)^T s(X_L) - A(Y,X_1,\ldots,X_{L-1},X_L,\ldots,X_p)$$

$$\ln p(Y|X_1,\ldots,X_p) \quad = \quad \theta^T s(Y,X_1,\ldots,X_p) - A(\theta)$$

3 Normal Child given a set of normal parents

Let us Y be a normal variable and $X_1,\ldots,X_p$ the set of parents of variable Y, all them also normal. The log-conditional probability of Y given X can be expressed as follows:

$$\ln p(Y|X_1,\ldots,X_p) = \ln\left(\frac{1}{\sigma\sqrt{2(\beta_0+\sum_i \beta_i X_i)}}\mathrm{e}^{-\frac{(x-(\beta_0+\sum_i \beta_i X_i))^2}{2\sigma^2}}\right)$$

Similarly the above conditional probability can be expressed in the following exponential forms:

$$
\begin{aligned}
\ln p(Y|X) &= \theta^T s(Y,X) - A(\theta) + h(X) \\
&= \begin{pmatrix}
\frac{-1}{2\sigma^2} & = & \theta_{\text{-}1} \\
\frac{-\beta_1^2}{2\sigma^2} & = & \theta_{1^2} \\
\cdots & & \\
\frac{-\beta_p^2}{2\sigma^2} & = & \theta_{p^2} \\
\frac{\beta_0}{\sigma^2} & = & \theta_0 \\
\frac{\beta_1}{\sigma^2} & = & \theta_1 \\
\cdots & & \\
\frac{\beta_p}{\sigma^2} & = & \theta_p \\
\frac{-\beta_0\beta_1}{\sigma^2} & = & \theta_{01} \\
\cdots & & \\
\frac{-\beta_0\beta_p}{\sigma^2} & = & \theta_{0p} \\
\frac{-\beta_1\beta_2}{\sigma^2} & = & \theta_{12} \\
\cdots & & \\
\frac{-\beta_1\beta_p}{\sigma^2} & = & \theta_{1p} \\
\cdots & & \\
\frac{-\beta_{p-1}\beta_p}{\sigma^2} & = & \theta_{p\text{-}1p}
\end{pmatrix}^T
\begin{pmatrix}
Y^2 \\ X^2 \\ \cdots \\ X_p^2 \\ Y \\ YX_1 \\ \cdots \\ YX_p \\ X_1 \\ \cdots \\ X_p \\ X_1X_2 \\ \cdots \\ X_1X_p \\ \cdots \\ X_{p-1}X_p
\end{pmatrix}
- \left(\frac{\beta_0^2}{2\sigma^2} + \ln\sigma\right) + \frac{1}{\ln\sqrt{2\mu_{Y|X}}}
\end{aligned}
$$

$$
\begin{aligned}
\ln p(Y|X) &= \theta(X)^T s(Y) - A(\theta(X)) + h(X) \\
&= \begin{pmatrix} \frac{\mu_{Y|X}}{\sigma^2} \\ \frac{-1}{2\sigma^2} \end{pmatrix}^T \begin{pmatrix} Y \\ Y^2 \end{pmatrix} - \left(\frac{\mu_{Y|X}^2}{2\sigma^2} + \ln\sigma\right) + \ln\frac{1}{\sqrt{2\mu_{Y|X}}} \\
&= \begin{pmatrix} \theta_0 + \theta_i m_0^{X_i} \\ \theta_{\text{-}1} \end{pmatrix}^T \begin{pmatrix} Y \\ Y^2 \end{pmatrix} - \left(\frac{\mu_{Y|X}^2}{2\sigma^2} + \ln\sigma\right) + \ln\frac{1}{\sqrt{2\mu_{Y|X}}}
\end{aligned}
$$

where $\mu_{Y|X} = \beta_0 + \sum_i \beta_i X_i$

$$
\begin{aligned}
\ln p(Y|X) &= \theta(Y)^T s(X) - A(\theta(Y)) + h(X) \\
&= \begin{pmatrix} -\frac{\beta_1^2}{2\sigma^2} \\ \cdots \\ -\frac{\beta_p^2}{2\sigma^2} \\ \frac{\beta_1(Y-\beta_0)}{\sigma^2} \\ \cdots \\ \frac{\beta_p(Y-\beta_0)}{\sigma^2} \\ -\frac{\beta_1\beta_2}{\sigma^2} \\ \cdots \\ -\frac{\beta_1\beta_p}{\sigma^2} \\ \cdots \\ -\frac{\beta_{p-1}\beta_p}{\sigma^2} \end{pmatrix}^T \begin{pmatrix} X_1^2 \\ \cdots \\ X_p^2 \\ X_1 \\ \cdots \\ X_p \\ X_1X_2 \\ \cdots \\ X_1X_p \\ \cdots \\ X_{p-1}X_p \end{pmatrix} - \left(\frac{(Y-\beta_0)^2}{\sigma^2} + \ln\sigma\right) + \frac{1}{\ln\sqrt{2\mu_{Y|X}}} \\
&= \begin{pmatrix} \theta_{1^2} \\ \cdots \\ \theta_{p^2} \\ \theta_1 m_0^Y + \theta_{01} \\ \cdots \\ \theta_p m_0^Y + \theta_{0p} \\ \theta_{12} \\ \cdots \\ \theta_{1p} \\ \cdots \\ \theta_{p\text{-}1p} \end{pmatrix}^T \begin{pmatrix} X_1^2 \\ \cdots \\ X_p^2 \\ X_1 \\ \cdots \\ X_p \\ X_1X_2 \\ \cdots \\ X_1X_p \\ \cdots \\ X_{p-1}X_p \end{pmatrix} - \left(\frac{(Y-\beta_0)^2}{\sigma^2} + \ln\sigma\right) + \frac{1}{\ln\sqrt{2\mu_{Y|X}}}
\end{aligned}
$$