Remember - Signal Energy :

- The Energy of a signal is defined by:

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt \; *$$

If $g(t)$ is complex, then $|g(t)|^2$ is equivalent to $\rightarrow |g(t)|^2 = g(t) \cdot g^*(t)$ ← complex conjugate

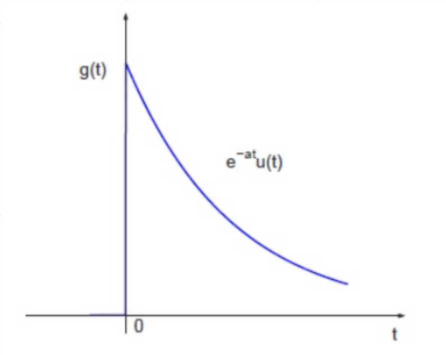
- If we know our Fourier Transform, we can also relate Energy to the signal Spectrum

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega \quad \text{Parseval's Theorem}$$

Parseval's Theorem - The total energy of a signal calculated in the time domain is equal to the total energy calculated in the frequency domain.

Example 1 : Parseval's Theorem

Consider the signal $g(t) = e^{-\alpha t}u(t)$ for $(\alpha > 0)$, calculate its signal energy in both time and frequency domains.



$|u(t)|^2 \rightarrow 1$ for $t > 0$
0 for $t < 0$

Step 1: Find E_g in the time-domain

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-\alpha t} u(t)|^2 dt = \int_{0}^{\infty} e^{-2\alpha t} dt$$

$$E_g = -\frac{1}{2\alpha} e^{-2\alpha t}\Big|_0^{\infty} = -\frac{1}{2\alpha}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{2\alpha}[0-1]$$

$$\boxed{E_g = \frac{1}{2\alpha} \quad \text{Time-Domain Energy}}$$

Example ①: Parseval's Theorem - Continued

Step ②: Find E_g in the frequency-domain

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

$$g(t) = e^{-\alpha t}u(t) \longrightarrow G(\omega) = \frac{1}{\alpha + j\omega}. \text{ (when } \alpha > 0)$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{\alpha + j\omega}\right|^2 d\omega$$

* $|\alpha + j\omega|^2 = (\alpha + j\omega)(\alpha - j\omega)$
* $|\alpha + j\omega|^2 = \alpha^2 + \omega^2$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} d\omega$$

* $\int \frac{1}{\alpha^2+\omega^2} d\omega = \frac{1}{\alpha}\arctan\left(\frac{\omega}{\alpha}\right)$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} d\omega = \frac{1}{2\pi}\cdot\frac{1}{\alpha}\arctan\left(\frac{\omega}{\alpha}\right)\Big|_{-\infty}^{+\infty}$$

$$E_g = \frac{1}{2\pi\alpha}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{2\cancel{\pi}\alpha}\left[\cancel{\pi}\right] = \frac{1}{2\alpha}$$

$$\boxed{E_g = \frac{1}{2\alpha} \quad \text{Frequency Domain}}$$

$$\boxed{E_g = \frac{1}{2\alpha} \quad \text{Time-Domain Energy}}$$

Energy Spectral Density - How Is Energy Distributed Across Frequency?

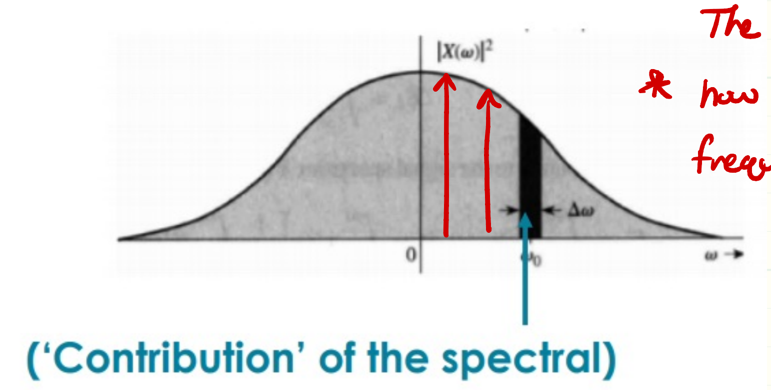
- We can think of the signal Energy (E_g) as the result of energies contributed by all spectral components

$$\underbrace{E_g}_{\text{Resulting Total Energy}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|G(\omega)|^2}_{\text{Energy of each individual Frequency Component "Spectrum"}} d\omega$$

- We say that ESD is the energy per unit bandwidth of the spectral components of $g(t)$ centered at frequency (ω)

$$\Psi(\omega) = |G(\omega)|^2 \quad \text{and} \quad E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)\, d\omega$$

- We can also call ESD the Spectral Contribution



The "Height" tells * how much each frequency contributes!

- If $x(t)$ and $y(t)$ are the input and output of LTI system

$$Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$

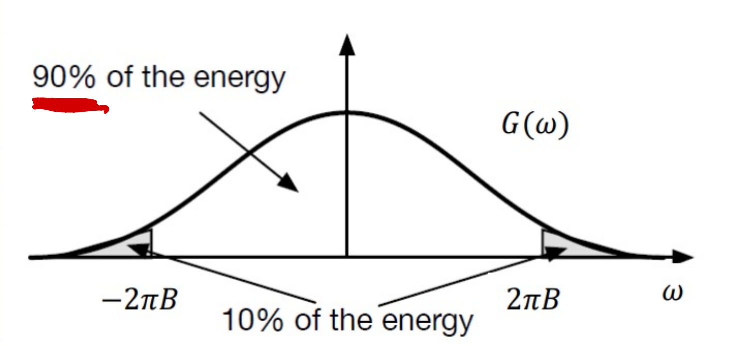
$$\Psi_y(\omega) = |H(\omega)|^2 \Psi_x(\omega)$$

$\Psi(\omega)$ - ESD

The ESD of an output signal is the multiplication of $|H(\omega)|^2$ and the ESD of the system input signal

Essential Bandwidth

- The essential bandwidth is the range of frequencies where most of the signal's energy is held.
  - ↳ "most" depends on the application (could be 90%, 95%, etc)
  - ↳ The percentage is with respect to your overall energy



- Using the picture above, if $G(\omega)$ is a low pass signal and E_B is the Energy from $-2\pi B$ to $2\pi B$, then

$$E_B = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |G(\omega)|^2 \, d\omega$$

$$\frac{E_B}{E_g} = 0.9$$

- These two equations help you find B if you know all other information

Correlation Function

- Correlation is the measurement of similarity between two signals

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t - \tau)\, dt$$

- This function has two use cases, to find the Time Delay between two signals or to examine the Similarity of two functions

  ↳ If τ is zero → Peak Correlation

- In a RADAR example, τ can be thought of as a range variable :

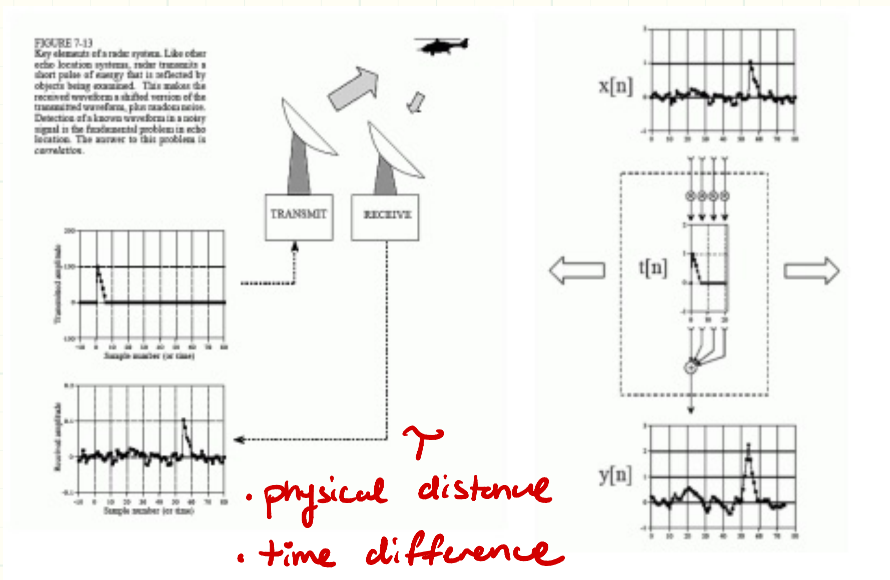


FIGURE 7-13
Key elements of a radar system. Like other echo location systems, radar transmits a short pulse of energy that is reflected by objects being examined. This makes the received waveform a shifted version of the transmitted waveform, plus random noise. Detection of a known waveform in a noisy signal is the fundamental problem in echo location. The answer to this problem is correlation.

τ
- physical distance
- time difference

- This Correlation Function is extremely similar to Energy and ESD

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g^*(t)\, g(t + \tau)\, dt$$

Auto Correlation
$x_1(t) = x_2(t)$

Cross Correlation
$x_1(t) \neq x_2(t)$

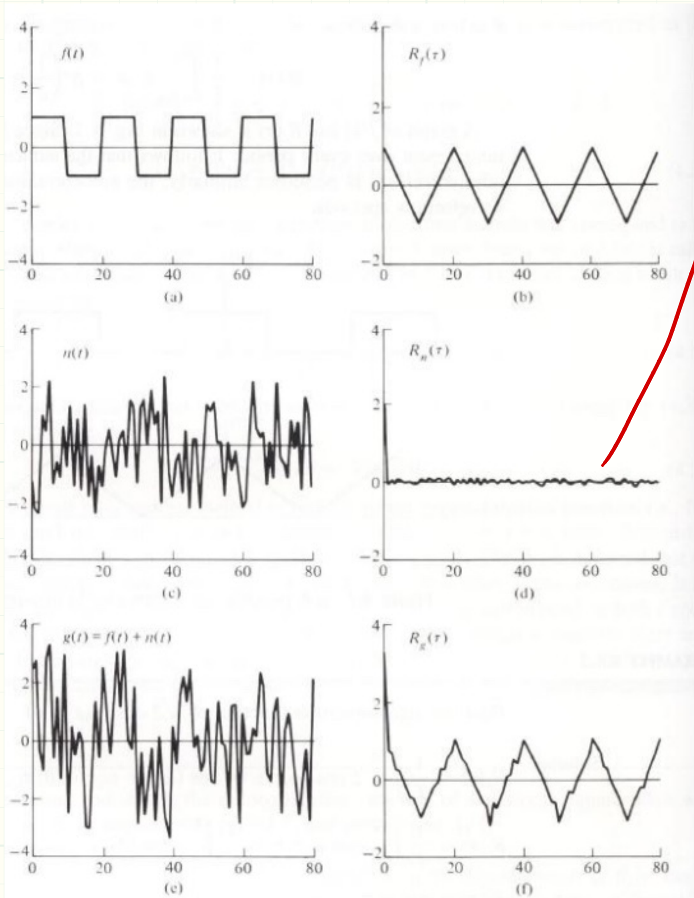
Time Delay

* more complex *

- In this class, we will mostly deal with Auto Correlation

Correlation Example

- Original Signal
- Channel Noise
- Resulting Signal
  - ↳ Very hard to extract any good info!



Random noise low average distribution

Auto Correlation

Allows us to examine meaningful similarities

- Remember, Auto - Correlation and ESD → very similar

$\Psi_g(\tau)$ Auto - Correlation

$\Psi(\omega)$ ESD

Transform Pair

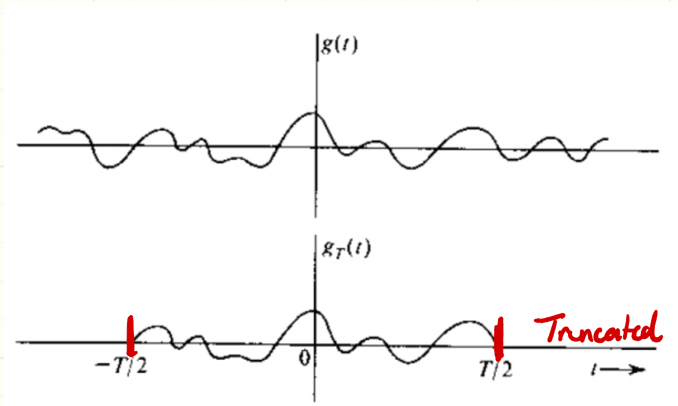
- Both of these are wanting to examine a $|G(\omega)|^2$ signal!

Power Spectral Density - How Is My Power Distributed Across Frequency

Recall that:

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

- Let's say I have some truncated signal $g_T(t)$, shown below:



$$E_{g_T} = \int_{-\infty}^{\infty} |g_T(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_T(\omega)|^2 d\omega$$

* Remember → Spectral Density means the contributions @ certain frequency

- We say that Power Spectral Density (PSD) is given as:

$$S_g(\omega) = \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T} \qquad \text{(PSD)}$$

* As T increases, $|G_T(\omega)|^2$ also increases

* Since P_g is finite (power signal), the increase rates of T and $|G_T(\omega)|^2$ must be the same

$$P_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T}\, d\omega$$

Auto - Correlation of Power signal

$$R_g(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} g^*(t)\, g(t+\tau)\, dt$$

$$R_g(\tau) = \frac{1}{T_0} \int_{0}^{T_0} g^*(t)\, g(t+\tau)\, dt$$

* Periodic Function and what we will use the most in our course

- As with ESD previously, the auto-correlation function and PSD are a Transform Pair

$$R_g(\tau) \longleftrightarrow S_g(\omega)$$

- Remember, we use transform pairs to help simplify !

Example ②: PSD vs. Power

Find the power spectral density (PSD) and the power of the function $x(t) = A\cos(\omega_0 t + \theta)$ Theta

* Try and work this problem for next class *

* Next class we will work examples and I will assign Homework ① *