Last Class

- Derivation of Fourier Series - Trigonometric and Complex Exponential
- How to visualize Euler's → Trigonometric → Complex Exponential

This Class → Fourier Transform

- Remember, the Fourier Series is represented by

$$x(t) = \sum_{-\infty}^{+\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad D_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt$$

- In Time Domain → This is an expansion of the periodic signal in terms of an infinite set of sines and cosines
- In Frequency Domain → The spectral representation of periodic signals

Fourier Transform

- Very important tool to allow us to analyze frequencies in systems
  - ↳ we can use too "see" our fundamental frequencies
- We say that $G(\omega)$ is the Direct Fourier Transform of $g(t)$ and we say that $g(t)$ is the Reverse Fourier Transform of $G(\omega)$

$G(\omega)$ → Frequency Domain

$g(t)$ → Time Domain

* $\omega = 2\pi f$ *

$$G(\omega) = \mathcal{F}[g(t)] \quad \text{and} \quad g(t) = \mathcal{F}^{-1}[G(\omega)]$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \qquad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

- might see $G(\omega)$ as $G(f)$
- $|G(\omega)|$ is the continuous amplitude spectrum of $G(\omega)$
- $\varphi(\omega)$ is the continuous phase spectrum of $G(\omega)$
  - ↳ we often ignore the phase spectrum

$$G(\omega) = |G(\omega)| e^{j\varphi(\omega)}$$

Complex → Real and Imaginary

Example 6 :

a) Find the Fourier Transform of the Unit Impulse Function $\delta(t)$

$$G(\omega) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \rightarrow G(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

* Remember → for $\delta(t)$ → Impulse @ time $t = 0$ → Area = 1
* Also → $\delta(t)$ times any function will just give that functions value @ time $t = 0$ → $\delta(t)$ 0 everywhere else

$$G(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega 0} dt \rightarrow \int_{-\infty}^{\infty} \delta(t) e^{0} dt = \int_{-\infty}^{\infty} \delta(t)(1)\, dt$$

$$G(\omega) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

* Remember → Integral of $\delta(t)$ is finding the area → ~~T~~ $\frac{1}{T}$ → Area = 1 *

$G(\omega) = 1$ Fourier Transform → $G(\omega)$ * $\delta(t) \leftrightarrow 1$ *

b) Find the Inverse Fourier Transform of $\delta(\omega)$:

$$g(t) = \mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j\omega t} d\omega$$

* Remember, integrating with respect to ω *

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j0t} d\omega$$ * using similar thought, only when $\omega = 0$ *

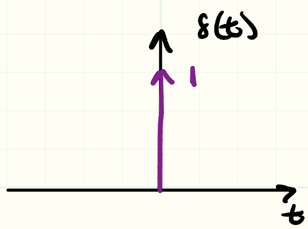
$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot e^{0} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)\, d\omega = \frac{1}{2\pi}$$

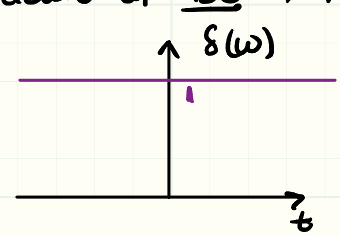
$g(t) = \frac{1}{2\pi}$ Inverse Fourier Transform → $g(t)$ * $1 \leftrightarrow 2\pi\delta(\omega)$ *

- What does $G(\omega) = 1$ and $g(t) = \frac{1}{2\pi}$ really mean?
- $G(\omega) = 1$ → the frequency content is spread across all frequencies

  $\delta(t)$ → All the area packed at instant in time

  $\delta(\omega)$ → All the area packed at DC → flat in freq





Example ⑦: You Try This One!

- Find the Inverse Fourier Transform of $\delta(\omega-\omega_0)$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

* step ①: our function given as $G(\omega) = \delta(\omega-\omega_0)$ *

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega t} d\omega$$

* step ②: Realize that delta function shifted to right by ω_0 *

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega_0 t} d\omega$$

* step ③: Because integral → $d\omega$ → complex exponential is a constant (ω_0) *

$$g(t) = \frac{1}{2\pi}\cdot e^{j\omega_0 t}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) d\omega = \frac{e^{j\omega_0 t}}{2\pi} * (1) = \frac{e^{j\omega_0 t}}{2\pi}$$

$$g(t) = \frac{e^{j\omega_0 t}}{2\pi}$$

* $e^{-j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega+\omega_0)$

$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$ *

- Another important relationship is :

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

- This gives us the following Fourier / Inverse Fourier Pair

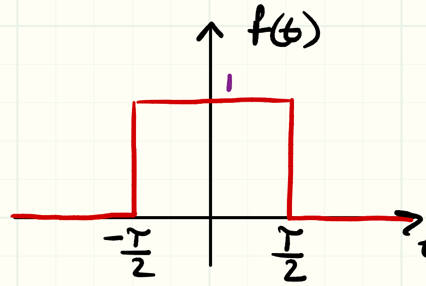
* $\cos(\omega_0 t) \longleftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$ *

* This is one of the golden telecom equations *

Example 8:

Find the Fourier Transform of the Unit Gate Function $f(t)$

$$f(t) = \text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| < \frac{T}{2} \\ 0 & |t| \geq \frac{T}{2} \end{cases}$$



$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$G(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot e^{-j\omega t} dt = \int_{-T/2}^{\frac{T}{2}} e^{-j\omega t} dt$$

* We integrate this complex exponential *

$$\int e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_a^b$$

"constant" pull out

$$G(\omega) = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{1}{j\omega}\left[e^{-j\omega \frac{T}{2}} - e^{-j\omega \cdot -\frac{T}{2}}\right]$$

$$G(\omega) = -\frac{1}{j\omega}\left[e^{-j\omega \frac{T}{2}} - e^{j\omega \frac{T}{2}}\right]$$

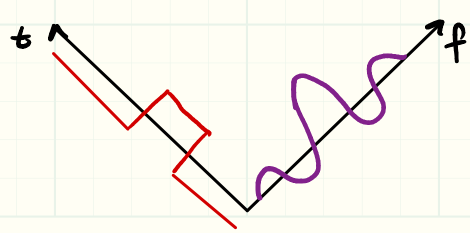
* This is the result without any further processing ... 'Simplify' *

- Because this function is "messy" → Created the "Sinc" function
- If you have $\sin\left(\frac{\omega T}{2}\right) = \frac{1}{2j}\left[e^{j\omega \frac{T}{2}} - e^{-j\omega \frac{T}{2}}\right]$

$$G(\omega) \rightarrow T\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right] = T \,\text{sinc}\left(\frac{\omega T}{2}\right)$$

* $\text{sinc}(x) = \frac{\sin(x)}{x}$ *

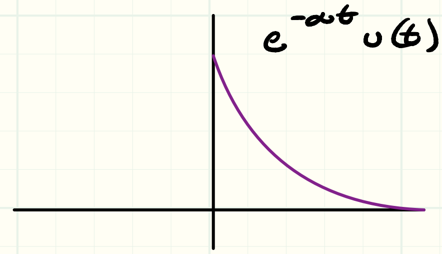
$$\underline{G(\omega) = T\,\text{sinc}\left(\frac{\omega T}{2}\right)}$$



Example 9:

Find the Fourier Transform of $g(t) = e^{-\alpha t} u(t)$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \rightarrow$$



$$G(\omega) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) \cdot e^{-j\omega t} \, dt = \int_{0}^{+\infty} e^{-\alpha t} \cdot e^{-j\omega t} \, dt$$

$$G(\omega) = \int_{0}^{+\infty} e^{-(\alpha t + j\omega t)} \, dt = -\frac{1}{\alpha + j\omega} e^{-(\alpha + j\omega)t} \Big|_{0}^{\infty}$$

$$G(\omega) = -\frac{1}{\alpha + j\omega} \left[e^{-(\alpha + j\omega)\infty} - e^{-(\alpha + j\omega)0} \right] \quad \text{when } \alpha > 0$$

$$G(\omega) = -\frac{1}{\alpha + j\omega} \left[e^{-\infty} - e^{0} \right] = -\frac{1}{\alpha + j\omega} \left[0 - 1 \right]$$

$$\underline{G(\omega) = \frac{1}{\alpha + j\omega}}$$

what about when $\alpha < 0$?

↳ not All Signals are Fourier Transformable

ECE 3710 - Fourier Transform Pairs

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a>0$
2	$e^{-at}u(-t)$	$\frac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_o t}$	$2\pi\delta(\omega-\omega_o)$	

No.	$x(t)$	$X(\omega)$	
9	$\cos(\omega_o t)$	$\pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$	
10	$\sin(\omega_o t)$	$j\pi[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$	
13	$cos(\omega_o t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]+\frac{j\omega}{\omega_o^2-\omega^2}$	
14	$sin(\omega_o t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]+\frac{\omega_o}{\omega_o^2-\omega^2}$	
15	$e^{-at}sin(\omega_o t)u(t)$	$\frac{\omega_o}{(a+j\omega)^2+\omega_o^2}$	$a>0$
16	$e^{-at}cos(\omega_o t)u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_o^2}$	$a>0$

No.	$x(t)$	$X(\omega)$	
17	$rect(\frac{t}{\tau})$	$\tau sinc(\frac{\omega\tau}{2})$	
18	$\frac{B}{2\pi}sinc(\frac{Bt}{2})$	$rect(\frac{\omega}{B})$	
19	$\Delta(\frac{t}{\tau})$	$\tau sinc^2(\frac{\omega\tau}{4})$	
20	$\frac{B}{2\pi}sinc^2(\frac{Bt}{2})$	$\Delta(\frac{\omega}{2B})$	
21	$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\omega_o\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_o)$	$\omega_o=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Relationship of (ω) and (f) In Fourier Transforms

- The variables f (Hz) and ω ($\frac{rad}{sec}$) are interchangeable
- By substituting $f = \frac{\omega}{2\pi}$ into $G(\omega)$ → we get f-domain

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \qquad G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df \qquad g(t) = \frac{1}{\boxed{2\pi}} \int_{-\infty}^{\infty} g(t) e^{j\omega t} d\omega$$

* we mostly will use $G(\omega)$ *

But are interchangeable

Fourier Transform of Periodic Signals

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0}$$

Fourier Series

* Remember *

Constant "weight"

- The Fourier Transform is denoted by:

$$\mathcal{F}[x(t)] = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}\right] = \sum_{n=-\infty}^{\infty} D_n \mathcal{F}[e^{jn\omega_0 t}]$$

$$\mathcal{F}[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

- The Fourier Transform of a Periodic Signal consists of a set of Impulses @ multiples of the fundamental frequency

$f = \frac{1}{T}$ → T = Signal's Period

Properties of Fourier Transform - Please Review, We will Use These!

1. Linearity
2. Duality
3. Complex Conjugate
4. Time - Scaling
5. Time - Shifting
6. Frequency - Shifting
7. Convolution
8. Time Differentiation
9. Time Integration