Last Class

- Derivation of Energy and Power → How to calculate both
- How to classify signals as Energy, Power, or Neither
- Logarithmic vs. Linear Power Ratios
  - ↳ Rule of 10's and 3's → 10x and 2x

Example ③:

If a signal has a logarithmic gain of 29 dBm, what's the linear gain of the signal?

29 dBm (Logarithmic) → X (Linear)?

* Rule of 10's and 3's *

29 dBm = 10 dBm + 10 dBm + 3 dBm + 3 dBm + 3 dBm

29 dBm → (10) times (10) times (2) times (2) times (2)

$$\boxed{29\text{ dBm} \rightarrow (10)(10)(2)(2)(2) = 800\text{X Gain (in mWatts)}}$$

Example ④:

A system has input voltage $V_1(t)$ and output voltage $V_2(t)$. Find gain (dB)

$$V_1(t) = \cos(\omega t + \theta) \qquad V_2(t) = 2\cos(\omega t + \theta)$$

Power → $10\log_{10}\left(\frac{P_2}{P_1}\right)$ Voltage → $20\log_{10}\left(\frac{V_2}{V_1}\right)$

$$\text{Gain (dB)} = 10\log_{10}\left(\frac{P_2}{P_1}\right)$$

$$\text{Gain (dB)} = 10\log_{10}\left(\frac{(V_2)^2/\cancel{R}}{(V_1)^2/\cancel{R}}\right) = 10\log_{10}\left(\frac{V_2^{\,2}}{V_1^{\,2}}\right) = 10(2)\log_{10}\left(\frac{V_2}{V_1}\right)$$

$$\boxed{\text{Gain (dB)} = 20\log_{10}\left(\frac{V_2}{V_1}\right) = 20\log_{10}\left(\frac{2}{1}\right) \approx 6\text{dB Gain}}$$

* Since voltage is squared when calculating power, we must distribute that square as a multiple of $10\log_{10}(x)$ *

. Let's now look at some very important signals for Telecom

The Complex Exponential (Complex Sinusoid)

If we have the following complex exponential:

$x(t) = Ce^{j(\omega t + \theta)}$, we can use Euler's Identity to convert it to a complex sinusoid

Euler's Identity: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

Therefore, we can say that: Remember → $\omega = 2\pi f$

$$x(t) = Ce^{j(\omega t + \theta)} = \underbrace{C\cos(\omega t + \theta)}_{\text{Real}} + \underbrace{jC\sin(\omega t + \theta)}_{\text{Imaginary}}$$

$$= C\cos(2\pi f t + \theta) + jC\sin(2\pi f t + \theta)$$

The Unit Impulse Function:

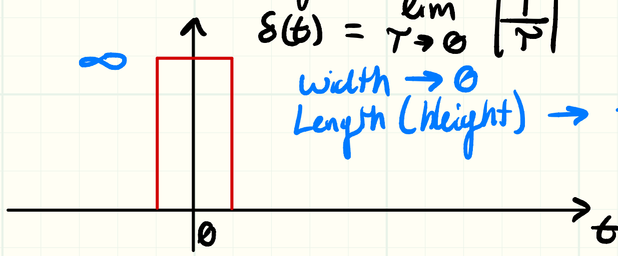
. The Unit Impulse (Dirac Delta) Function is given as:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$\delta(t) = \lim_{T \to 0} \left| \frac{1}{T} \right|$

width → 0

Length (Height) → $+\infty$



. Very sharp Impulse @ $t = 0$

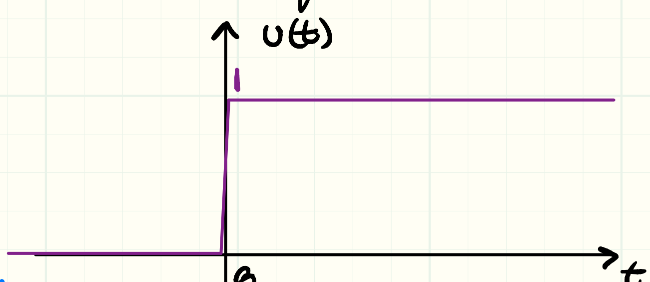
. Commonly use $\delta(t)$ to sample * Very, very, very, zoomed in perspective! *

. If we multiply a signal by impulse → can look at that signals value @ certain times

The Unit Step Function

. The Unit Step (Heaviside) Function is given as:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

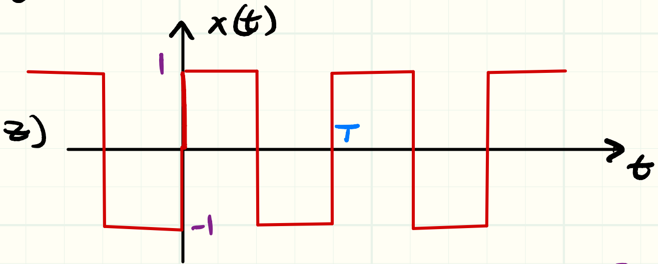


* Unit Step → Integral of Unit Impulse! *

- Now we will start examining the Fourier Series and Fourier Transform
- Let's say we have a square wave function

Periodic → $T = 2\pi$

Fundamental Frequency → $f = \frac{1}{2\pi}$ (Hz)



- Fourier Series tries to show that when we have a Periodic function, we can represent it as an Infinite Summation of sine and cosine functions of different frequencies.

- To do this, we have the basic format:

This a_0 Represents DC (offset)

$$x(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \dots$$
$$+ b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \dots$$

- The different frequencies are "multiples" of original period (t)

- The a_n and b_n coefficients show how much different frequencies contribute to the final equation → Generating Waveforms

a_0 → DC component

- Remember, the Fourier Series has two forms, first we look @ :

Trigonometric Form

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]; \quad \omega_0 \rightarrow \text{Angular velocity of our original signal}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt \rightarrow \text{average value of the signal}$$

$$a_n = \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)\,dt \quad \text{where } n \neq 0$$

- If $a_1 \rightarrow \cos(\omega_0 t)$ .....

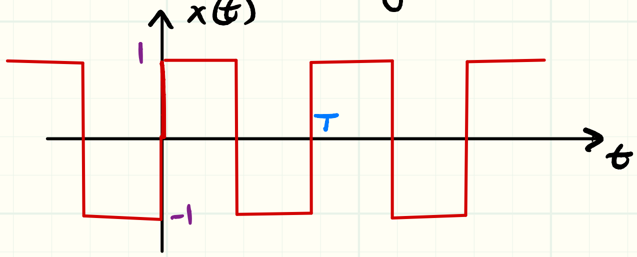
$$b_n = \frac{2}{T}\int_T x(t)\sin(n\omega_0 t)\,dt$$

- If $b_1 \rightarrow \sin(\omega_0 t)$ ....

- If even function → $x(-t) = x(t)$ → $b_n = 0$!
- If odd function → $x(-t) = -x(t)$ → $a_n = 0$!

Example 5

What is the Trigonometric Fourier Series of the following signal?



$$x(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$

- Remember → for this type of problem, want to find a_0, a_n, and b_n → then assemble into overall equation

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

- Looking here, T is your period (2π)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} 1\,dt + \int_{\pi}^{2\pi} -1\,dt\right]$$

$$a_0 = \frac{1}{2\pi}\left[t\Big|_0^{\pi} + (-t)\Big|_{\pi}^{2\pi}\right] = \frac{1}{2\pi}\left[\pi + (-2\pi) + \pi\right]$$

$$a_0 = \frac{1}{2\pi}[0] \rightarrow \underline{a_0 = 0}$$

* Does this make sense? * Yes, no DC offset!

a.k.a. the signal average (area) is zero! Pos. + neg. cancels out!

$$a_n = \frac{2}{T}\int_T x(t)\cos(n\omega t)$$

* Note! $T = 2\pi \rightarrow \omega = \frac{2\pi}{T} = 1$

* $n \neq 0$

$$a_n = \frac{2}{T}\int_T x(t)\cos(nt) = \frac{2}{2\pi}\left[\int_0^{\pi}(1)\cos(nt)\,dt + \int_{\pi}^{2\pi}(-1)\cos(nt)\,dt\right]$$

* Remember $\int \cos(\omega t) = \frac{1}{\omega}\sin(\omega t)\Big|_a^b$

$$a_n = \frac{1}{\pi}\left[\left(\frac{1}{n}\sin(nt)\Big|_0^{\pi}\right) + \left(-\frac{1}{n}\sin(nt)\Big|_{\pi}^{2\pi}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\left(\cancelto{0}{\sin(n\pi)} - \cancelto{0}{\sin(0)}\right) - \frac{1}{n}\left(\cancelto{0}{\sin(n2\pi)} - \cancelto{0}{\sin(n\pi)}\right)\right]$$

$$a_n = \frac{1}{\pi}(0) \rightarrow \underline{a_n = 0}$$

* Remember, before we said that for odd functions → $a_n = 0$! *

Example 5 continued :

$* \ \omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$$b_n = \frac{2}{T}\int_T x(t)\sin(n\omega t)dt = \frac{2}{T}\int_T x(t)\sin(nt)dt$$

$$b_n = \frac{2}{2\pi}\left[\int_0^{\pi}(1)\sin(nt)dt + \int_{\pi}^{2\pi}(-1)\sin(nt)dt\right]$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi}\sin(nt)dt - \int_{\pi}^{2\pi}\sin(nt)dt\right]$$

$* \int \sin(\omega t) = -\frac{1}{\omega}\cos(\omega t)$

$$b_n = \frac{1}{\pi}\left[-\frac{1}{n}\left(\cos(nt)\Big|_0^{\pi}\right) + \frac{1}{n}\left(\cos(nt)\Big|_{\pi}^{2\pi}\right)\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{1}{n}\left(\cos(n\pi) - \cos(0)\right) + \frac{1}{n}\left(\cos(n2\pi) - \cos(n\pi)\right)\right]$$

$$b_n = \frac{1}{n\pi}\left[-\cos(n\pi) + \overset{1}{\cancel{\cos(0)}} + \overset{1}{\cancel{\cos(n2\pi)}} - \cos(n\pi)\right]$$

$$b_n = \frac{1}{n\pi}\left[2 - 2\cos(n\pi)\right]$$

$* \ \cos(n\pi) \rightarrow -1$ when $n =$ odd
$+1$ when $n =$ even

$$b_n = \begin{cases} \frac{1}{n\pi}\left[2 - 2(-1)\right] & \text{when } n \text{ is odd} \\ \frac{1}{n\pi}\left[2 - 2(1)\right] & \text{when } n \text{ is even} \end{cases}$$

$\downarrow$

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

* Now, this is very drawn out, we can start using these simplifications as we go forward! *

- Remember → Fourier Series has two forms!

Exponential Form

- The more compact and commonly used form uses complex exponential

$$x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi f_0$

f_0 = Fundamental Frequency (original)

- Even when real-valued, D_n coefficients are complex numbers

$$D_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

→ This is basically allowing for correlation to act as an inner product and only extract n^{th} term

- This form is more unified because its more similar to the Fourier Transform

- Let's go a bit more indepth for complex Exponential
  - If you have a complex exponential

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

  - Magnitude $|e^{j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$
  - Phase $\angle e^{j\theta} = \tan^{-1}\left(\frac{\sin(\theta)}{\cos\theta}\right) = \theta$

- We analyze in terms of Magnitude + Phase
  - cosine → Real
  - Sine → Imaginary
- Rotates around center-clockwise based on angle (θ)

- What does this mean for Fourier Series?

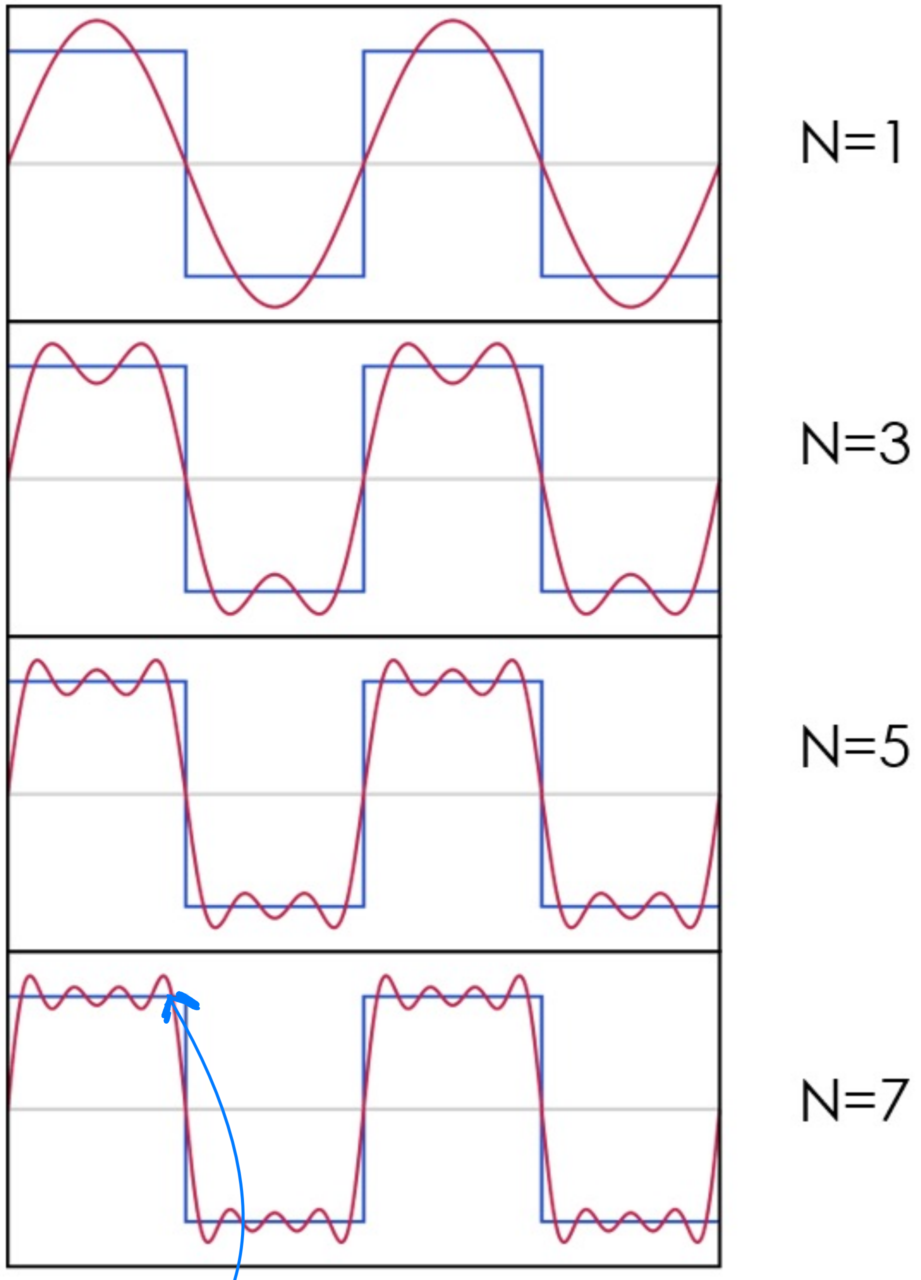
$$x(t) = \sum_{-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

magnitude = 1

Phase = $n\omega_0 t$

- when n large
  ↳ "rotate speed fast"
- when n small
  ↳ "rotate speed slow"

An Example of Square Wave Generation - Fourier Series



Gibb's Phenomenon

- An overshoot of Fourier Series occuring at simple discontinuities (≈ 9% overshoot)