Properties of the Fourier Transform

- Linearity
- Duality
- Complex Conjugate
- Time-scaling
- Time-shifting
- Frequency-shifting
- Convolution
- Time differentiation*
- Time integration*

Linearity Property

- **Fourier transform is a linear transformation**

$$Given\ f(t) \leftrightarrow F(\omega)\ and\ g(t) \leftrightarrow G(\omega)$$

$$Then\ af(t) + bg(t) \leftrightarrow aF(\omega) + bG(\omega)$$

$$where\ a, b\ are\ arbitrary\ constants$$

The results can be extended to any finite number of terms:

$$\sum_k a_k g_k(t) \leftrightarrow \sum_k a_k G_k(\omega)$$

Complex Conjugate Property

- **If $g(t)$ is a real function of t, then $G(\omega)$ and $G(-\omega)$ are complex conjugate**

$$G(-\omega) = G^*(\omega) \quad \begin{cases} |G(-\omega)| = |G(\omega)| \text{ (even function)} \\ \theta_g(-\omega) = -\theta_g(\omega) \text{ (odd function)} \end{cases}$$

Duality Property

$$
\begin{tikzcd}
 & \displaystyle\int_{-\infty}^{+\infty} g(t)e^{-j\omega t} \arrow[dr, bend left] & \\
g(t) \arrow[ur, bend left] & & G(\omega) \arrow[dl, bend left] \\
 & \displaystyle\frac{1}{2\pi}\int_{-\infty}^{+\infty} G(\omega)e^{j\omega t}d\omega \arrow[ul, bend left] &
\end{tikzcd}
$$

$$
g(t) \leftrightarrow G(\omega) \quad \overset{\text{Duality}}{\Longleftrightarrow} \quad \begin{array}{l} \mathcal{F}\{G(t)\} = 2\pi g(-\omega) \\ \mathcal{F}\{G(-t)\} = 2\pi g(\omega) \end{array}
$$

Time-scaling Property

If $g(t) \leftrightarrow G(\omega)$

Then, for any real constant a $$g(at) \leftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$$

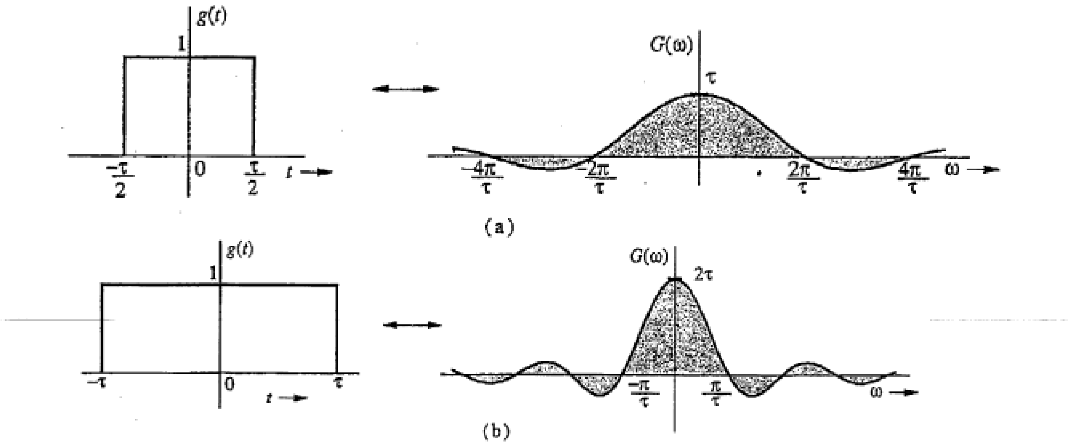

Fig (a): $g(t) = rect(t)$

Fig (b): $g(t) = rect\left(\frac{t}{2}\right)$

- 0<a<1:
  - ✓ g(at): expanded version of g(t)
  - ✓ $\mathcal{F}\{g(at)\}$: compressed version of $\mathcal{F}\{g(t)\}$
- a>1:
  - ✓ g(at): compressed version of g(t)
  - ✓ $\mathcal{F}\{g(at)\}$: expanded version of $\mathcal{F}\{g(t)\}$

Time compression of a signal results in spectral expansion and time expansion of a signal results in spectral compression

Time-shifting Property

If $g(t) \leftrightarrow G(\omega)$

$g(t - t_o) \leftrightarrow e^{-j\omega t_o} G(\omega)$ and $g(t + t_o) \leftrightarrow e^{j\omega t_o} G(\omega)$

What is the Fourier Transform of $rect(t-5)$ (using the time-shifting property)?

Frequency-shifting Property

If $g(t) \leftrightarrow G(\omega)$

$$g(t)e^{j\omega_o t} \leftrightarrow G(\omega - \omega_o) \quad \text{and} \quad g(t)e^{-j\omega_o t} \leftrightarrow G(\omega + \omega_o)$$

- **Important applications**

$$g(t)\cos(\omega_o t) = \frac{1}{2}\left(g(t)e^{j\omega_o t} + g(t)e^{-j\omega_o t}\right)$$

$$g(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}[G(\omega - \omega_o) + G(\omega - \omega_o)]$$

Multiplication of a signal $g(t)$ by the factor $[cos(\omega_o t)]$ places $G(\omega)$ centered at $\omega = \pm\omega_o$

Convolution Property

- **The convolution of two functions *g(t)* and *w(t)*, denoted by *g(t)*w(t)*, is defined as:**

$$g(t) * w(t) = \int_{-\infty}^{+\infty} g(\tau)w(t-\tau)\,d\tau$$

- **Time/Frequency convolution property**

If $g_1(t) \leftrightarrow G_1(\omega)$ and $g_2(t) \leftrightarrow G_2(\omega)$

Time convolution:

$$g_1(t) * g_2(t) \leftrightarrow G_1(\omega)G_2(\omega)$$

Frequency convolution:

$$g_1(t)g_2(t) \leftrightarrow \frac{1}{2\pi}G_1(\omega) * G_2(\omega)$$

Time differentiation Property

If $g(t) \leftrightarrow G(\omega)$

$$\frac{dg(t)}{dt} \leftrightarrow j\omega G(\omega)$$

Time integration Property

If $g(t) \leftrightarrow G(\omega)$

$$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega)$$

Summary

▪ Linearity	$af(t) + bg(t) \leftrightarrow aF(\omega) + bG(\omega)$
▪ Duality	$G(t) \leftrightarrow 2\pi g(-\omega)$
▪ Complex conjugate	$G(-\omega) = G^*(\omega)$
▪ Time-scaling	$g(at) \leftrightarrow \frac{1}{\lvert a\rvert} G\left(\frac{\omega}{a}\right), a \neq 0$
▪ Time-shifting	$g(t \pm t_o) \leftrightarrow e^{\pm j\omega t_o} G(\omega)$
▪ Frequency-shifting	$g(t)e^{\pm j\omega_o t} \leftrightarrow G(\omega \mp \omega_o)$
▪ Convolution	$g_1(t) * g_2(t) \leftrightarrow G_1(\omega)G_2(\omega)$ (time)
▪ Time differentiation*	$\frac{dg(t)}{dt} \leftrightarrow j\omega G(\omega)$
▪ Time integration*	$\int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j\omega} G(\omega) + \pi G(0)\delta(\omega)$