Remember - Signal Energy :

- The Energy of a signal is defined by:

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt \;\; *$$

If $g(t)$ is complex, then $|g(t)|^2$ is equivalent to $\rightarrow |g(t)|^2 = g(t) \cdot g^*(t)$ ← complex conjugate

- If we know our Fourier Transform, we can also relate Energy to the signal Spectrum

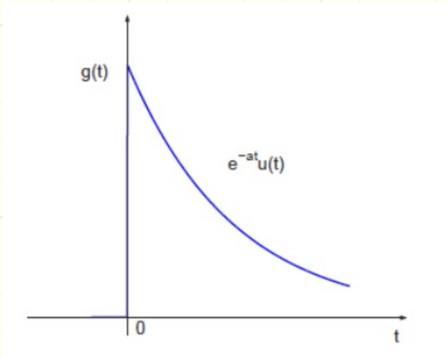
$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega \qquad \text{Parseval's Theorem}$$

Parseval's Theorem - The total energy of a signal calculated in the time domain is equal to the total energy calculated in the frequency domain.

Example ①: Parseval's Theorem

Consider the signal $g(t) = e^{-\alpha t} u(t)$ for $(\alpha > 0)$, calculate its signal energy in both time and frequency domains.

$|u(t)|^2 \rightarrow 1$ for $t > 0$
0 for $t < 0$



Step ①: Find E_g in the time-domain

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |e^{-\alpha t} u(t)|^2 dt = \int_{0}^{\infty} e^{-2\alpha t} dt$$

$$E_g = -\frac{1}{2\alpha} e^{-2\alpha t} \Big|_0^{\infty} = -\frac{1}{2\alpha}\left[e^{-\infty} - e^0\right] = -\frac{1}{2\alpha}[0 - 1]$$

$$\boxed{E_g = \frac{1}{2\alpha} \quad \text{Time - Domain Energy}}$$

Example 1: Parseval's Theorem - Continued

Step 2: Find E_g in the frequency-domain

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

$$g(t) = e^{-\alpha t}u(t) \longrightarrow G(\omega) = \frac{1}{\alpha + j\omega}. \text{ (when } \alpha > 0)$$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{\alpha + j\omega}\right|^2 d\omega$$

* $|\alpha + j\omega|^2 = (\alpha + j\omega)(\alpha - j\omega)$

* $|\alpha + j\omega|^2 = \alpha^2 + \omega^2$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} d\omega$$

* $\int \frac{1}{\alpha^2 + \omega^2} d\omega = \frac{1}{\alpha}\arctan\left(\frac{\omega}{\alpha}\right)$

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} d\omega = \frac{1}{2\pi}\cdot\frac{1}{\alpha}\arctan\left(\frac{\omega}{\alpha}\right)\Big|_{-\infty}^{+\infty}$$

$$E_g = \frac{1}{2\pi\alpha}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{1}{2\cancel{\pi}\alpha}\left[\cancel{\pi}\right] = \frac{1}{2\alpha}$$

$$\boxed{E_g = \frac{1}{2\alpha} \quad \text{Frequency Domain}}$$

$$\boxed{E_g = \frac{1}{2\alpha} \quad \text{Time - Domain Energy}}$$

Energy Spectral Density - How Is Energy Distributed Across Frequency?

- We can think of the signal Energy (E_g) as the result of energies contributed by all spectral components

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

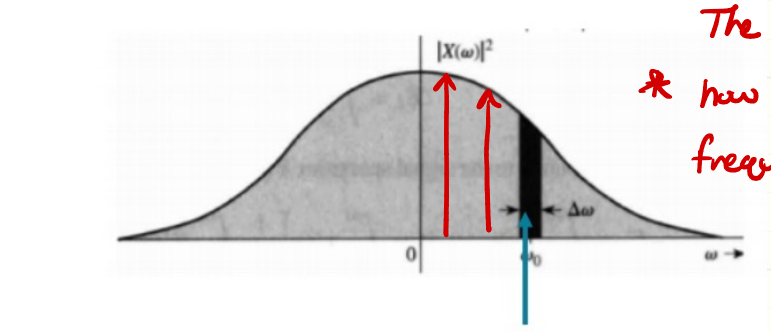
Resulting Total Energy

Energy of each individual Frequency Component "Spectrum"

- We say that ESD is the energy per unit bandwidth of the spectral components of $g(t)$ centered at frequency (ω)

$$\Psi(\omega) = |G(\omega)|^2 \quad \text{and} \quad E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)\, d\omega$$

- We can also call ESD the Spectral Contribution



The "Height" tells * how much each frequency contributes!

- If $x(t)$ and $y(t)$ are the input and output of LTI system

$$Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$

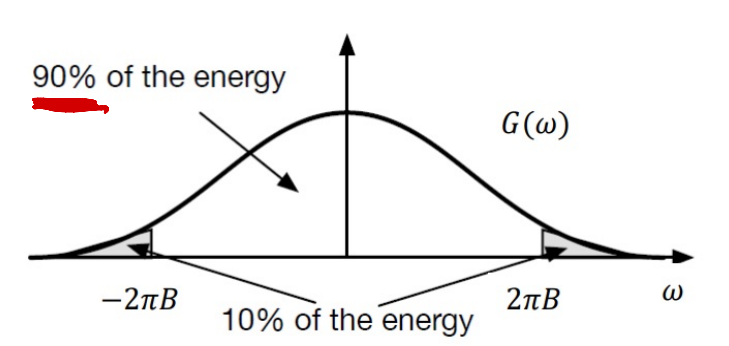
$$\Psi_y(\omega) = |H(\omega)|^2 \Psi_x(\omega)$$

$\Psi(\omega)$ - ESD

The ESD of an output signal is the multiplication of $|H(\omega)|^2$ and the ESD of the system input signal

Essential Bandwidth

- The essential bandwidth is the range of frequencies where most of the signal's energy is held.
  - ↳ "most" depends on the application (could be 90%, 95%, etc)
  - ↳ The percentage is with respect to your overall energy



- Using the picture above, if $G(\omega)$ is a low pass signal and E_B is the Energy from $-2\pi B$ to $2\pi B$, then

$$E_B = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |G(\omega)|^2 \, d\omega$$

$$\frac{E_B}{E_g} = \underline{0.9}$$

- These two equations help you find B if you know all other information

Correlation Function

- Correlation is the measurement of similarity between two signals

$$R_{12} = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$$

- This function has two use cases, to find the Time Delay between two signals or to examine the Similarity of two functions
  - ↳ If τ is zero → Peak Correlation

- In a RADAR example, τ can be thought of as a range variable :

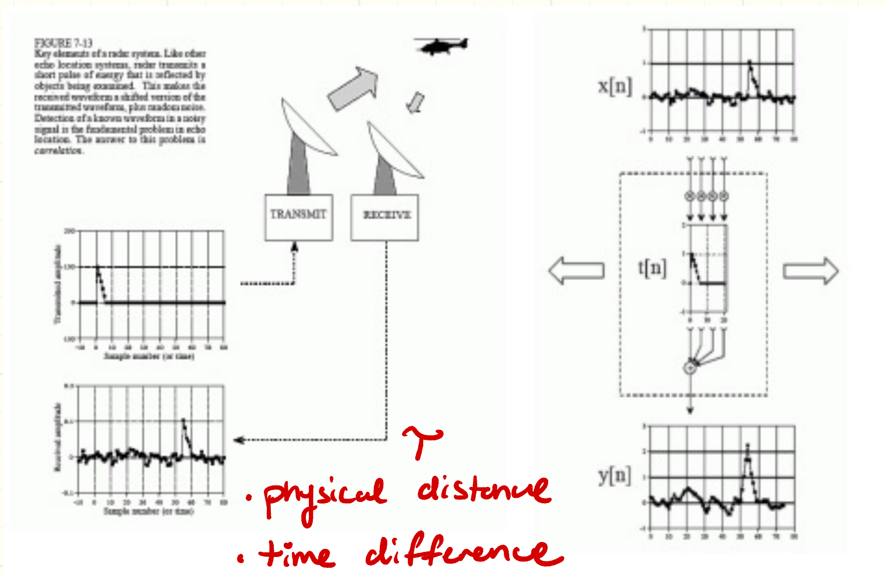


FIGURE 7-13
Key elements of a radar system. Like other echo location systems, radar transmits a short pulse of energy that is reflected by objects being examined. This makes the received waveform a shifted version of the transmitted waveform, plus random noise. Detection of a known waveform in a noisy signal is the fundamental problem in echo location. The answer to this problem is correlation.

τ
- physical distance
- time difference

- This Correlation Function is extremely similar to Energy and ESD

$$\psi_g(\tau) = \int_{-\infty}^{\infty} g^*(t)\, g(t+\tau)\, dt$$

Auto Correlation

$x_1(t) = x_2(t)$

Cross Correlation

$x_1(t) \neq x_2(t)$

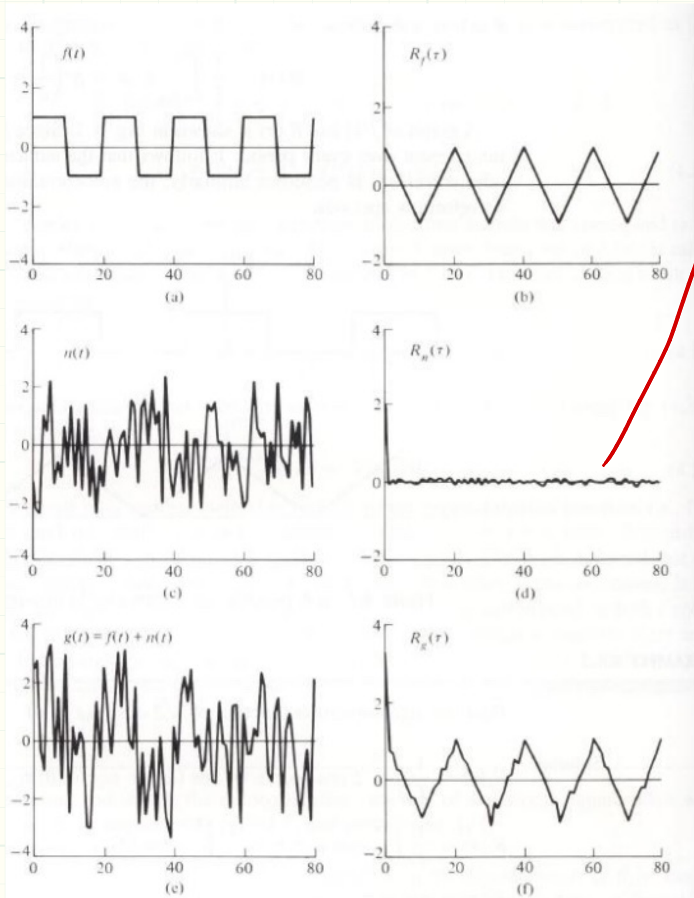
↗ Time Delay

* more complex *

- In this class, we will mostly deal with Auto Correlation

Correlation Example

- Original Signal
- Channel Noise
- Resulting Signal
  - ↳ Very hard to extract any good info!



★ Random noise ★ low average distribution

Auto Correlation

Allows us to examine meaningful similarities

- Remember, Auto - Correlation and ESD → Very Similar

$\psi_g(\tau)$ Auto - Correlation

$\Psi(\omega)$ ESD

] Transform Pair

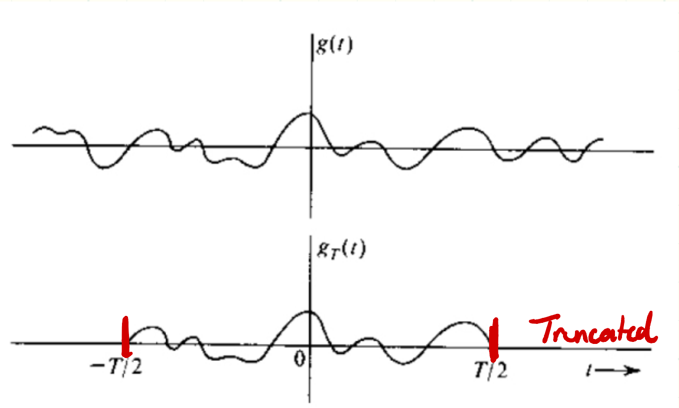
- Both of these are wanting to examine a $|G(\omega)|^2$ signal!

Power Spectral Density - How Is My Power Distributed Across Frequency

Recall that:

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

- Let's say I have some truncated signal $g_T(t)$, shown below:



$$E_{g_T} = \int_{-\infty}^{\infty} |g_T(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_T(\omega)|^2 d\omega$$

* Remember → Spectral Density means the contributions @ certain frequency

- We say that Power Spectral Density (PSD) is given as:

$$S_g(\omega) = \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T} \qquad \text{(PSD)}$$

* As T increases, $|G_T(\omega)|^2$ also increases

* Since P_g is finite (power signal), the increase rates of T and $|G_T(\omega)|^2$ must be the same

$$P_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T}\, d\omega$$

Auto - Correlation of Power signal

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} g^*(t)\, g(t+\tau)\, dt$$

* Having a plus τ shift in the Auto - Correlation function is okay due to Hermition Symmetry $R_x(\tau) = R_x(-\tau)$

$$R_g(\tau) = \frac{1}{T_0} \int_{0}^{T_0} g^*(t)\, g(t+\tau)\, dt$$

Periodic Function and
* what we will use the most in our course

- As with ESD previously, the auto-correlation function and PSD are a Transform Pair

$$R_g(\tau) \longleftrightarrow S_g(\omega)$$

- Remember, we use transform pairs to help simplify ! :)

Example 2 : PSD vs. Power

Find the power spectral density (PSD) and the power of the function $x(t) = A\cos(\omega_0 t + \theta)$ Theta

* Try and work this problem for next class *

* Next Class we will work examples and I will assign Homework 1 *

Example 2: PSD vs. Power

Find the power spectral density (PSD) and the power of the function $x(t) = A\cos(\omega_0 t + \theta)$ Theta

Step 1: Remember the PSD ⟷ Autocorrelation Fourier Transform Pair

PSD: $S_x(\omega) = \lim_{T\to\infty} \frac{|G_x(\omega)|^2}{T}$

Auto Correlation: $R_x(\tau) = \frac{1}{T}\int_0^{T_0} x^*(t)\, x(t+\tau)\, dt$

Step 2: Find the Power Spectral Density (PSD)

$$S_x(\omega) = \lim_{T\to\infty} \frac{|G_x(\omega)|^2}{T}$$

* We can use $R_x(\tau)$ to find the Power Spectral Density! * Fourier Transform Pair

$$R_x(\tau) = \frac{1}{T}\int_0^{T_0} \underline{[A\cos(\omega_0 t + \theta)]}[A\cos(\omega_0(t+\tau)+\theta)]\, dt$$

* Remember, $x^*(t)$ of a real function is itself! $(\text{Real})^* = \text{Real}$

* We can use the identity → $\cos(A)\cos(B) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$$R_x(\tau) = \frac{1}{T}\int_0^{T_0} A^2 \cdot \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right] dt \quad \Bigg|\; \begin{aligned} a &= [\omega_0 t + \theta] \\ b &= [\omega_0(t+\tau)+\theta] \end{aligned}$$

$$a - b = [\omega_0 t + \theta] - [\omega_0(t+\tau) + \theta]$$

$$= \cancel{\omega_0 t} + \cancel{\theta} - \cancel{\omega_0 t} - \omega_0\tau \cancel{- \theta} = -\omega_0\tau$$

$$a + b = [\omega_0 t + \theta] + [\omega_0(t+\tau) + \theta]$$

$$= \omega_0 t + \theta + \omega_0 t + \omega_0\tau + \theta = 2\omega_0 t + \omega_0\tau + 2\theta$$

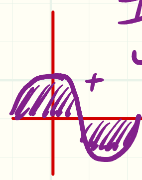
$$R_x(\tau) = \frac{1}{T}\int_0^{T_0} \frac{A^2}{2}\left[\cos(-\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau + 2\theta)\right] dt$$

Example 2: PSD vs. Power - Continued

$$R_x(\tau) = \frac{1}{T}\int_0^{T_0} \frac{A^2}{2}\left[\underbrace{\cos(-\omega_0\tau)}_{\substack{\text{Even Function} \\ \cos(\omega_0\tau)}} + \cos(2\omega_0 t + \omega_0\tau + 2\theta)\right] dt$$

$$R_x(\tau) = \frac{1}{T}\int_0^{T_0} \frac{A^2}{2}\cos(\omega_0\tau)\,dt + \frac{1}{T}\int_0^{T_0} \frac{A^2}{2}\cos(2\omega_0 t + \omega_0\tau + 2\theta)\,dt$$

"constants" so integral is (t)



Integral is area under the curve will be zero when looking over one period

$$R_x(\tau) = \frac{1}{T}\cdot\frac{A^2}{2}\cos(\omega_0\tau)\,t\Big|_0^{T_0} + \frac{1}{T}\cdot 0$$

$$R_x(\tau) = \frac{A^2}{2T}\cos(\omega_0\tau)T_0 - \cancel{\frac{A^2}{2T}\cos(\omega_0\tau)0} + 0$$

$$R_x(\tau) = \frac{A^2}{2\cancel{T}}\cos(\omega_0\tau)\cancel{T_0} = \frac{A^2}{2}\cos(\omega_0\tau)$$

PSD $S_x(\omega) \overset{\mathcal{F}}{\longleftrightarrow} R_x(\tau)$

$$S_x(\omega) = \mathcal{F}\left[\frac{A^2}{2}\cos(\omega_0\tau)\right] = \frac{A^2}{2}\cdot\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\boxed{S_x(\omega) = \frac{A^2}{2}\cdot\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$

Step 3: Find the Power (P_x)

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{A^2\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]d\omega$$

$$P_x = \frac{A^2\cancel{\pi}}{4\cancel{\pi}}\int_{-\infty}^{\infty}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]d\omega = \frac{A^2}{4}(1+1)$$

* Area of $\delta(\omega)$ is 1

$$P_x = \frac{A^2}{2}$$

$$\boxed{\text{Power } (P_x) = \frac{A^2}{2}}$$