Lecture 3.5 → Practice Problems

Example 1: Calculating Power

Calculate the power of the following signals

$x_1(t) = 10\cos(100t + \frac{\pi}{3})$ $\quad$ $x_2(t) = 10\cos(5t)\cos(10t)$

Step 1: Find Power P_{x_1}

$$P_g = \frac{1}{T}\int_0^T |g(t)|^2 dt$$

$$P_{x_1} = \frac{1}{T}\int_0^T |10\cos(100t + \tfrac{\pi}{3})|^2 dt$$

$$P_{x_1} = \frac{1}{T}\int_0^T 100\cos^2(100t + \tfrac{\pi}{3})dt$$

* $\cos^2(a) = \frac{1}{2}[1 + \cos(2a)]$

$$P_{x_1} = \frac{100}{2T}\int_0^T [1 + \cos(200t + \tfrac{2\pi}{3})]dt$$

$$P_{x_1} = \frac{50}{T}\left[\int_0^T 1\, dt + \int_0^T \cos(200t + \tfrac{\pi}{3})dt\right]$$

* Integral of cos(x) → zero *

$$P_{x_1} = \frac{50}{T}(t)\Big|_0^T + \frac{50}{T}[0]$$

$$P_{x_1} = \frac{50}{\cancel{T}}\cdot \cancel{T} + 0 \rightarrow \boxed{P_{x_1} = 50 \text{ watts}}$$

* Can check by saying $P_{\cos(x)} \rightarrow \frac{A^2}{2} \rightarrow \frac{10^2}{2} = \frac{100}{2} = 50$ *

Step 2: Find Power of P_{x_2}

* $\cos(a)\cos(b) = \frac{1}{2}\cos(a-b) + \cos(a+b)$ *

$$P_{x_2} = \frac{1}{T}\int_0^T |10\cos(5t)\cos(10t)|^2 dt =$$

$$P_{x_2} = \frac{1}{T}\int_0^T \left|10\cdot\frac{1}{2}[\cos(-5t) + \cos(15t)]\right|^2 dt$$

$$P_{x_2} = \frac{1}{T}\int_0^T |5[\cos(-5t) + \cos(15t)]|^2 dt$$

* We know that the power of cos(x) is $\frac{A^2}{2}$, use this! $A = 5$

$$P_{x_2} = \frac{A^2}{2} + \frac{A^2}{2} = \frac{25}{2} + \frac{25}{2} \rightarrow \boxed{P_{x_2} = 25 \text{ watts}}$$

Example 2: Fourier Transform

If $f(t) \longleftrightarrow F(\omega)$ and $\omega = 2\pi f$, find the Fourier Transform of the following functions: ① $f(t-2)$ ② $f(t)\cos(\pi t)$

Step 1: Find $\mathcal{F}[x_1(t)]$

$$x_1(t) = f(t-2)$$

* In this function we see a time shift. If we identify the base form, we can use our Fourier Transform Pair *

- The Fourier Transform Time-Shift property tells us that

$$g(t \pm t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{\pm j\omega t_0} G(\omega)$$

$$\boxed{\mathcal{F}[x_1(t)] = e^{-j\omega 2} F(\omega)}$$

Step 2: Find $\mathcal{F}[x_2(t)]$

$$x_2(t) = f(t)\cos(\pi t)$$

* From $x_2(t)$ we see that $\omega_0 = \pi$ *

$$x_2(t) = f(t) \cdot \frac{1}{2}\left[e^{j\pi t} + e^{-j\pi t}\right]$$

* $\cos(\omega t)$ can be rewritten as $\frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$

$$x_2(t) = \frac{1}{2} f(t) e^{j\pi t} + \frac{1}{2} f(t) e^{-j\pi t}$$

* The Fourier Transform of this frequency shift is:

$$g(t)e^{j\omega_0 t} \longleftrightarrow G(\omega - \omega_0)$$

$$\boxed{\mathcal{F}[x_2(t)] = \frac{1}{2} F(\omega - \pi) + \frac{1}{2} F(\omega + \pi)}$$

Example 3: Energy Spectral Density (ESD)

Find the Energy Spectral Density (ESD) of the output $y(t)$ given that $x(t) = e^{-100t} u(t)$ and $h(t) = e^{-10t} u(t)$

Step 1: Map out Input, System, and output

Input $\rightarrow$ $x(t) = e^{-100t} u(t)$

System Impulse Response $\rightarrow$ $h(t) = e^{-10t} u(t)$

Output $\rightarrow$ $y(t) = x(t) * h(t)$

Step 2: Finding ESD of $y(t)$

- Remember $\rightarrow$ ESD refers to $\Psi_y(\omega) = |Y(\omega)|^2$
- If $y(t) = x(t) * h(t)$ and if $Y(\omega) = X(\omega) H(\omega)$

  then $\Psi_y(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \, |X(\omega)|^2$

* The Fourier Transform Pair of $e^{-\alpha t} u(t)$ when $\alpha > 0$ is: $\frac{1}{\alpha + j\omega}$

$X(\omega) = \frac{1}{100 + j\omega}$ and $H(\omega) = \frac{1}{10 + j\omega}$

$$|Y(\omega)|^2 = \left|\frac{1}{100 + j\omega}\right|^2 \left|\frac{1}{10 + j\omega}\right|^2 = \left(\frac{1}{100^2 + \omega^2}\right)\left(\frac{1}{10^2 + \omega^2}\right)$$

$$\boxed{\Psi_y(\omega) = |Y(\omega)|^2 = \frac{1}{100^2 + \omega^2} \cdot \frac{1}{10^2 + \omega^2}}$$

Energy Spectral Density (ESD)

Example 4: Power Spectral Density (PSD)

Given a signal $g(t) = 4\cos(2\pi 200 t)$, calculate its autocorrelation function and corresponding Power Spectral Density (PSD)

Step 1: Autocorrelation

$$R_g(\tau) = \frac{1}{T}\int_0^T g^*(t)\, g(t+\tau)\,dt$$

* Remember $\underset{A.C.}{R(\tau)} \overset{\mathcal{F}}{\longleftrightarrow} \underset{PSD}{S_g(\omega)}$ *

$$R_g(\tau) = \frac{1}{T}\int_0^T [4\cos(2\pi 200 t)\cdot 4\cos(2\pi 200(t+\tau))]\,dt$$

* Remember, $\omega_0 = 2\pi\cdot 200 \rightarrow \omega_0 = 400\pi \rightarrow T = \frac{2\pi}{\omega_0} = \frac{1}{200}$

$$R_g(\tau) = \frac{1}{\frac{1}{200}}\int_0^{1/200} 16\cos(400\pi t)\cos(400\pi(t+\tau))\,dt$$

* Remember, $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b)+\cos(a+b)]$

$$R_g(\tau) = 200\int_0^{1/200} 16\cdot\frac{1}{2}\Big[\cos\big(400\pi t - 400\pi(t+\tau)\big) + \cos\big(400\pi t + 400\pi(t+\tau)\big)\Big]dt$$

$$R_g(\tau) = 1{,}600\int_0^{1/200} [\underbrace{\cos(-400\pi\tau)}_{\text{Integrating this "constant" gives } t} + \underbrace{\cos(800\pi t + 400\pi\tau)}_{\text{Integrating this cos function tends to zero (area cancels)}}]\,dt$$

0

$$R_g(\tau) = 1600\cos(-400\pi\tau)\cdot t\,\Big|_0^{1/200} = 1600\cdot\cos(-400\pi\tau)\cdot\frac{1}{200}$$

$$R_g(\tau) = 8\cos(-400\pi\tau) = 8\cos(400\pi\tau)$$

* cosine is even * $g(t) = g(-t)$

$$\boxed{R_g(\tau) = 8\cos(400\pi\tau)}$$

Step 2: Find PSD

PSD $S_g(\omega)$ is the Fourier Transform Pair of Auto Correlation

$$S_g(\omega) = \mathcal{F}[8\cos(400\pi\tau)] = 8[\pi\,\delta(\omega-400\pi) + \pi\,\delta(\omega+400\pi)]$$

$$\boxed{S_g(\omega) = 8\pi\,\delta(\omega-400\pi) + 8\pi\,\delta(\omega+400\pi)}$$