CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Sequences of real numbers

Problem 2.1. Consider the following sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers and study whether they are bounded, monotone, and convergent.

$$a)\quad a_n = \frac{1 + (-1)^n}{2}.$$

$$b)\quad a_n = \frac{(-1)^{n+1}}{n}.$$

$$c)\quad a_n = \frac{n}{n+2}.$$

$$d)\quad a_n = \frac{\lfloor n/2 \rfloor}{n}.$$

$$e)\quad a_n = \frac{\lfloor nx \rfloor}{n}, \quad x \in \mathbb{R}.$$

$$f)\quad a_n = \frac{n + \sin(\pi n/2)}{2n + 1}.$$

$$g)\quad a_n = \sqrt[n]{\pi^n + (\sqrt{7})^n}.$$

$$h)\quad a_n = \frac{1}{n^2 + 1}\sum_{k=1}^{n} k.$$

$$i)\quad a_n = \sqrt[n]{x^n + y^n}, \quad 0 < y \le x.$$

Problem 2.2. Calculate the limit as $n \to \infty$ of the following sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers.

$$a)\quad a_n = \frac{n^2}{(n-7)!} \quad (n \ge 7).$$

$$b)\quad a_n = \frac{n!}{n^n}.$$

$$c)\quad a_n = \sqrt{n^3 - 1} - n.$$

$$d)\quad a_n = \frac{\sqrt{n^3 - 1} - n}{5n^2 - 7\sqrt{n}}.$$

$$e)\quad a_n = \frac{3^n + 2^{n+1}}{3^{n+1} + 2^n}.$$

Problem 2.3. Prove that the given *recursive* sequences of real numbers are bounded and monotone. Then, calculate their limit as $n \to \infty$.

a) $a_1 = \sqrt{3}, \; a_2 = \sqrt{3\sqrt{3}}, \; a_3 = \sqrt{3\sqrt{3\sqrt{3}}}, \; \ldots .$

b) $a_n = 5 + \dfrac{a_{n-1}}{4} \quad \forall n \geq 2, \quad a_1 = 0.$

c) $a_n = \dfrac{1 + 3a_{n-1}^2}{4} \quad \forall n \geq 2, \quad 1/3 \leq a_1 < 1.$

d) $a_{n+1} = \sqrt{2a_n + 3} \quad \forall n \geq 1, \quad a_1 = 1.$

Problem 2.4. Calculate the following limits.

a) $\displaystyle\lim_{n \to \infty} n^{1/(n-1)}.$

b) $\displaystyle\lim_{n \to \infty} (7n^3 - 1)^{1/n}.$

c) $\displaystyle\lim_{n \to \infty} \left(\dfrac{3n^2 + 1}{3n^2 + 2}\right)^{-n^2}.$