2nd Midterm – 87

Problem 1. Solve the following recurrence relation:

$$\begin{cases} a_{n+1} = (e+\pi)a_n - e\pi a_{n-1}, \\ a_0 = e+\pi; \quad a_1 = 2e\pi. \end{cases}$$

Solution. This is a second order linear homogeneous recurrence relation, and can be written as

$$a_n = (e+\pi)a_{n-1} - e\pi a_{n-2}.$$

The characteristic equation is $x^2 = (e+\pi)x - e\pi$. That is, $0 = x^2 - (e+\pi)x - e\pi = (x-e)(x-\pi)$. Hence we have two different roots of algebraic multiplicity one. This makes

$$a_n = Ae^n + B\pi^n.$$

Applying the initial conditions, we get $e+\pi = a_0 = A+B$ and $2e\pi = a_1 = Ae+B\pi$, which make a system of two linear equations for two unknowns. Some little algebra leads to $A = \pi$ and $B = e$.

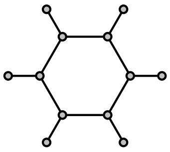
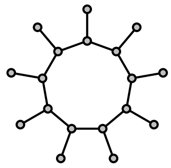
In conclusion,

$$\boxed{a_n = \pi e^n + e\pi^n}.$$

Problem 2. Let $G_n = (V_n, E_n)$, $n \geq 3$, where

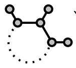
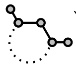
- $V_n = \{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_n\}$, hence $|V_n| = 2n$, and
- $E_n = \{\{a_i, a_{i+1}\}, 1 \leq i \leq n-1\} \cup \{a_n, a_1\} \cup \{\{a_i, b_i\}, 1 \leq i \leq n\}$, hence $|E_n| = 2n$.

For instance,

$G_6 \equiv$  or $G_9 \equiv$ 

Find, using suitable properties of the chromatic polynomial, $P_{G_n}(q)$ and $\chi(G_n)$, for $n \geq 3$.

Solution. G_n is a C_n with n extra edges sticking out. Those sticks are a K_2, and overlap with the graph forming a K_1. Using the factorization theorem, we know:

$P_{G_n}(q) = P\Bigg($  $\Bigg) = \dfrac{P_{K_2}(q)}{P_{K_1}(q)} P\Bigg($  $\Bigg).$

Doing the same for the remaining sticks, we end up with a stripped down graph that is a C_n. Since G_n has a total of n sticks, we end up with:

$$P_{G_n}(q) = \left(\frac{P_{K_2}(q)}{P_{K_1}(q)}\right)^n P_{C_n}(q) = (q-1)^n P_{C_n}(q).$$

Now our problem is to find $P_{C_n}(q)$. This has been done in class, and if $a_n = P_{C_n}(q)$, then we have to solve the recurrence

$$\begin{cases} a_n = q(q-1)^{n-1} - a_{n-1}, \\ a_3 = q(q-1)(q-2). \end{cases}$$

Once solved, we know $P_{C_n}(q) = (q-1)^n + (-1)^n(q-1)$.

In conclusion,

$$\boxed{P_{G_n}(q) = (q-1)^n \left[(q-1)^n + (-1)^n(q-1)\right]}.$$

This also means that $P_{G_n}(q)$ has the same roots as $P_{C_n}(q)$. Hence $\chi(G_n) = \chi(C_n)$. That is,

$$\boxed{\chi(G_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}}$$

Problem 3. Solve, using generating functions techniques, the following distribution:

$$\begin{cases} n_1 + n_2 + n_3 = 35, \\ n_i \in \{0, 1, 2, \ldots, 15\}. \end{cases}$$

Solution. Encode the problem as $x^{35} = x^{n_1} \cdot x^{n_2} \cdot x^{n_3}$. By means of the sum principle, encode the values of each group into functions $f_1(x) = f_2(x) = f_3(x) = 1 + x + x^2 + \ldots + x^{15}$, and by the product principle, encode the solutions into $F(x) = f_1(x) \cdot f_2(x) \cdot f_3(x)$, which turns out to be $F(x) = \big(f_1(x)\big)^3$.

Notice that

$$f_1(x) = 1 + x + x^2 + \ldots + x^{15} = \sum_{n=0}^{15} x^n = \frac{1 - x^{16}}{1 - x}.$$

Hence,

$$F(x) = \left(\frac{1 - x^{16}}{1 - x}\right)^3 = \big(1 - x^{16}\big)^3 \cdot \frac{1}{(1 - x)^3}.$$

Then apply the binomial theorem to the first power, and the generalized binomial theorem to the power in the denominator:

$$F(x) = \big(1 - 3x^{16} + 3x^{32} - x^{48}\big) \cdot \left[\sum_{n=0}^{\infty} \binom{n+2}{n} x^n\right]$$

As $F(x)$ is a product of polynomials, then it will have the shape $F(x) = a_0 + a_1 x + a_2 x^2 + \ldots$, and we only care for the value of a_{35} as it encodes the **number** of solutions to the distribution we are dealing with. By comparison to the previous expression of $F(x)$ as a product of a polynomial and a series, we find out that $a_{35} = \binom{35+2}{35} - 3\binom{19+2}{19} + 3\binom{3+2}{3}$. That is,

$$\boxed{\#\text{solutions } = a_{35} = \binom{37}{35} - 3\binom{21}{19} + 3\binom{5}{3}}$$