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CALCULUS FINAL EXAM

January 2020 **Bachelor in Informatics Engineering**

SURNAME			
NAME		GROUP	

Problem 1. *[1 point]* Consider the *alternating* series of real numbers

$$\sum_{n=1}^{\infty} (-1)^n a_n , \quad \text{with } a_n = n \int_0^{1/n} e^{-x^2} \, dx - 1 .$$

Apply Leibniz test to prove its convergence.

SOLUTION

According to Leibniz test, the given alternating series is convergent if $\lim_{n\to\infty} a_n = 0$. In order to calculate this limit, let us note that

$$e^{-x^2} = 1 - x^2 + o(x^2)$$

thanks to Taylor theory. Hence, after integrating the previous expression, we can write that

$$a_n = n \int_0^{1/n} \left[1 - x^2 + o(x^2)\right] dx - 1 = n \left[\frac{1}{n} - \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right] - 1 = -\frac{1}{3}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) .$$

Thus, thanks to Leibniz test, we can conclude that the series is convergent.

Problem 2. Let

$$f(x) = \begin{cases} \alpha \ln(1-2x) + \cos(x) & \text{if} \quad x \leq 0\,, \\ \alpha\, x^2 + \dfrac{\beta}{\sqrt{1+x}} & \text{if} \quad x > 0\,. \end{cases}$$

(a) *[1 point]* Study the continuity and differentiability of $f(x)$ in terms of $\alpha, \beta \in \mathbb{R}$.

(b) *[0.5 points]* Prove whether the tangent line to $f(x)$ at $x = 1$ is parallel to the line $y = 2x$.

SOLUTION

(a) For $x \neq 0$, the given function is continuous as composition of continuous elementary functions. In addition, $f(x)$ is continuous at $x = 0$ if $\lim_{x\to 0} f(x) = f(0) = 1$. Since

$$\lim_{x\to 0^-} f(x) \;=\; \lim_{x\to 0^-} \big[\,\alpha \ln(1-2x) + \cos(x)\,\big] = 1\,,$$

$$\lim_{x\to 0^+} f(x) \;=\; \lim_{x\to 0^+} \left[\,\alpha\, x^2 + \frac{\beta}{\sqrt{1+x}}\,\right] = \beta\,,$$

we need that $\beta = 1$ to ensure the continuity of $f(x)$ at $x = 0$, hence in $\mathbb{R}$. On the other hand, for $x \neq 0$, the given function is differentiable as composition of differentiable elementary functions. In addition, taking $\beta = 1$, $f(x)$ is differentiable at $x = 0$ if the following lateral limits

$$f'_-(0) \;=\; \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0^-} \frac{\alpha \ln(1-2x) + \cos(x) - 1}{x} \;=\; -2\alpha\,,$$

$$f'_+(0) \;=\; \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0^+} \frac{\alpha\, x^2 + \frac{1}{\sqrt{1+x}} - 1}{x} \;=\; -\frac{1}{2}\,,$$

provide the same finite result. Thus, $\alpha = 1/4$ ensures the differentiability of $f(x)$ at $x = 0$ (hence in $\mathbb{R}$), being $f'(0) = -1/2$.

(b) For the values of α and β found in (a), the tangent line to $f(x)$ at $x = 1$ has slope equal to $1/2 - 1/(2\sqrt{8})$. Hence it's not parallel to the given line.

Problem 3. Let $F(x) = \int_0^{\frac{x^2}{2}} \cos(t^5)\, dt\,.$

(a) *[1 point]* Find the *local* behavior of $F(x)$ close to $x = 0\,.$

(b) *[1 point]* Calculate $\lim\limits_{x\to 0} \dfrac{2F(x) - x^2}{x^{22}}\,.$

SOLUTION

(a) According to Taylor theory, we have that

$$\cos(t^5) = 1 - \frac{t^{10}}{2} + o(t^{10})\,,$$

hence

$$F(x) = \int_0^{\frac{x^2}{2}} \left[1 - \frac{t^{10}}{2} + o(t^{10})\right] dt = \frac{x^2}{2} - \frac{x^{22}}{22\cdot 2^{11}} + o(x^{22})\,.$$

Being $x^2/2$ the first nonzero term of the Taylor polynomial for $F(x)$ around $a = 0$, we can conclude that $x = 0$ is a local minimum for $F(x)$ and $F(x)$ is locally concave up.

(b) The given limit provides an indeterminate form of the type $0/0$. Thus, using the Taylor polynomial for $F(x)$ around $a = 0$ found in (a), we can write

$$\lim_{x\to 0} \frac{2F(x) - x^2}{x^{22}} = \lim_{x\to 0} \frac{-x^{22}/(11\cdot 2^{11}) + o(x^{22})}{x^{22}} = -\frac{1}{11\cdot 2^{11}}\,.$$

Problem 4. *[1.5 points]* Calculate

$$\int \frac{1}{e^{12x} - 1}\, dx$$

using the change of variable $u = e^{6x}$.

SOLUTION

Using the change of variable

$$u = e^{6x}, \quad du = 6\,u\,dx,$$

the given integral becomes

$$\int \frac{1}{e^{12x} - 1}\, dx = \frac{1}{6} \int \frac{1}{u(u^2 - 1)}\, du.$$

Now, using the partial fractions method, we can write

$$\frac{1}{u(u^2-1)} = \frac{1}{u(u-1)(u+1)} = -\frac{1}{u} + \frac{1/2}{u-1} + \frac{1/2}{u+1},$$

which yields

$$\int \frac{1}{u(u^2-1)}\, du = -\ln|u| + \frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1| + c,$$

where c is an arbitrary constant. Thus, we can finally write

$$\int \frac{1}{e^{12x} - 1}\, dx = -x + \frac{1}{12}\ln|e^{12x} - 1| + c.$$