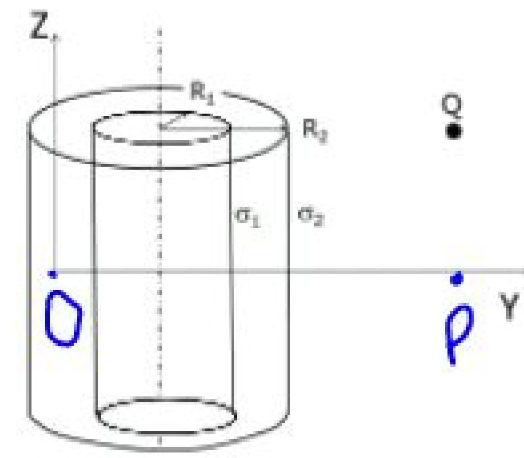
P2. ***(3 p)*** You are given the following charge distributions:

- A uniformly charged cylindrical surface of infinite length and radius R_1, whose axis is parallel to the Z-axis and passes through the point (0,2,0), with surface charge density σ_1.
- A uniformly charged cylindrical surface of infinite length and radius R_2, coaxial to the previous distribution and with surface charge density σ_2.
- A point charge Q >0 located at (0, a, b).



a) If a point charge q is placed at (0,a,0), it experiences an electric force with magnitude F= 4.85×10^{-4} N. Calculate the value of the charge Q.

b) Calculate the electric force (expressed in Cartesian coordinates) that the charge q would experience if it were placed at the origin of coordinates.

DATA: R_1= 0.5 m; R_2= 2.5 m; σ_1= -2.3×10^{-6} C/m^2; σ_2= 5.4×10^{-6} C/m^2; a= 8 m; b=1.5 m; q= 1.3×10^{-9} C

a) If point P (0, a, 0) then the electric force that will act on a charge q placed at P will be $\vec{F}_{Pq} = \vec{E}_P \cdot q$

Where $\vec{E}_P$ is the net electric field at point P: $\vec{E}_P = \vec{E}_1(P) + \vec{E}_2(P) + \vec{E}_Q(P)$

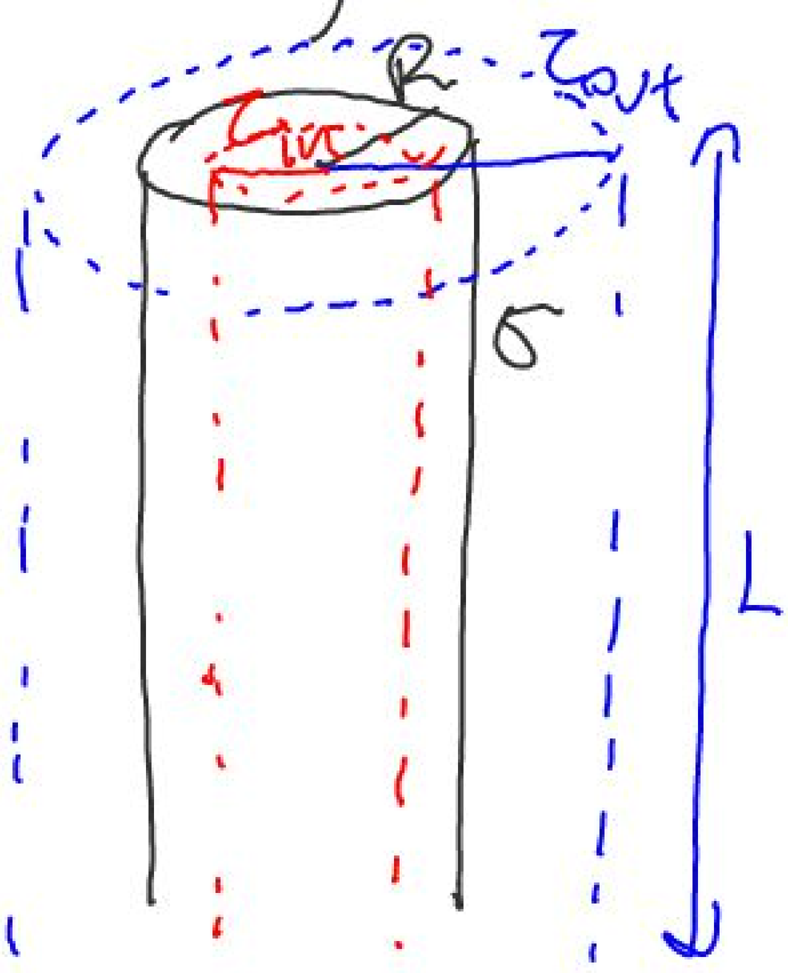
with $\vec{E}_1(P)$ the electric field at P created by the charged cylindrical surface with radius R_1, $\vec{E}_2(P)$ the electric field at P created by the charged cylindrical surface with radius R_2 and $\vec{E}_Q(P)$ the electric field at P created by the point charge Q.

We have to derive the general expressions for the electric field created by an infinite, uniformly charged cylindrical surface with radius R and surface charge density σ.

For points with $r_{in} < R$, where r_{in} is the radial distance from the axis of the cylinder $\boxed{\vec{E}_{r_{in}} = 0}$

because if we consider any Gaussian surface within the cylinder it contains no charge.

For $r_{out} > R$ we will consider a cylindrical Gaussian surface co-axial to the cylinder and depicted in blue.



From Gauss' law

$$\Phi_S = \frac{Q_{ins}}{\varepsilon_0} = \frac{\sigma \cdot 2\pi R \cdot L}{\varepsilon_0} \quad (1)$$

The electric field flux through the Gaussian surface $\Phi_s = \oint \vec{E}\, d\vec{s} \overset{\vec{E} \parallel d\vec{s}}{=}$

$$= \oint E\, ds \overset{E = \text{constant along } S}{=} E \oint ds = E \cdot 2\pi r_{out} L \quad (2)$$

$$(1),(2) \rightarrow \boxed{\vec{E}_{out} = \frac{\sigma R}{\varepsilon_0 r_{out}} \hat{u}}$$ with $\hat{u}$ the unit vector

Point P is located outside both cylinders, therefore

$$\vec{E}_1(P) = \frac{\sigma_1 R_1}{\varepsilon_0 \cdot (a - y_0)} \vec{j}$$ and

$$\vec{E}_2(P) = \frac{\sigma_2 R_2}{\varepsilon_0 (a - y_0)} \vec{j}$$, where $y_0 = 2$ m

is the point on the y-axis from where the axis of the cylindrical surfaces passes.

And $$\vec{E}_q(P) = -\frac{1}{4\pi\varepsilon_0} \frac{q}{b^2} \vec{k}$$

Therefore

$$\vec{E}_P = \left(\frac{\sigma_1 R_1}{\varepsilon_0 (a - y_0)} + \frac{\sigma_2 R_2}{\varepsilon_0 (a - y_0)}\right)\vec{j} - \frac{Q}{4\pi\varepsilon_0 b^2}\vec{k}$$

The magnitude of the force acting on q

$$F_q = |\vec{E}_P| \cdot q = \sqrt{\left(\frac{\sigma_1 R_1 + \sigma_2 R_2}{\varepsilon_0 (a - y_0)}\right)^2 + \left(\frac{Q}{4\pi\varepsilon_0 b^2}\right)^2}$$

from where we obtain:

$$\boxed{Q = 4\pi\varepsilon_0 b^2 \sqrt{\frac{F_q^2}{q^2} - \left(\frac{\sigma_1 R_1 + \sigma_2 R_2}{\varepsilon_0 (a - y_0)}\right)^2} = 0.73\mu C}$$

b) If now the charge q is placed at poin O (0, 0, 0) it will act upon force

$$\vec{F}_q(0) = q\,\vec{E}(0) = q\left(\vec{E}_1(0) + \cancel{\vec{E}_2(0)}^{\,0} + \vec{E}_a(0)\right)$$

with $\vec{E}_2(0) = 0$ because point O is inside the cylinder of radius R_2.

$$\vec{E}_1(0) = -\frac{\sigma_1 R_1}{\varepsilon_0 y_0}\vec{j} \quad \text{and}$$

$$\vec{E}_Q(0) = \frac{Q}{4\pi\varepsilon_0(a^2+b^2)}\frac{(-a\vec{j}-b\vec{k})}{\sqrt{a^2+b^2}}$$

So that

$$\boxed{\begin{aligned}\vec{F}_q(0) &= -q\left[\frac{\sigma_1 R_1}{\varepsilon_0 y_0}\vec{j} + \frac{Q}{4\pi\varepsilon_0(a^2+b^2)^{3/2}}\cdot(a\vec{j}+b\vec{k})\right] \\ &= 7.18\cdot 10^{-5}\,\vec{j} - 2.37\cdot 10^{-6}\,\vec{k} \quad N\end{aligned}}$$