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ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

Discrete Mathematics Solved Exams

Grado en Ingeniería en Informática

Doble Grado en Ingeniería en Informática y Administración de Empresas

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Grupo de Modelización, Simulación Numérica y Matemática Industrial
Universidad Carlos III de Madrid
Avda. de la Universidad, 30
28911 Leganés

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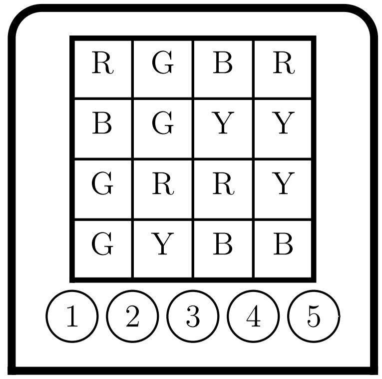
1 Final exam June 2015

Question 1.1 An electronic toy displays a 4×4 grid of colored squares. At all times, four are red (R), four are green (G), four are blue (B), and four are yellow (Y). For example, here is one possible configuration:



1. How many such configurations are possible?

   Solution.

   We have to permute 16 elements such that they contain 4 groups of 4 identical elements each (4 B's, 4 R's, 4 G's, and 4 Y's). Therefore, the number of distinct screens is:

$$\frac{16!}{(4!)^4} .$$

Below the display, there are five buttons numbered 1, 2, 3, 4, and 5. The player may press a sequence of buttons; however, the same button can not be pressed twice in a row.

2. How many different sequences of n button-presses are possible?

   Solution.

   This is an application of the product principle: the first press can be done in 5 different ways, and for the other presses there are only 4 available options (as we cannot press the same bottom two consecutive times). The number of distinct sequences of button presses is:

$$5 \times 4^{n-1} .$$

3. Prove **by induction** that $n^n > n!$ for any integer $n \geq 2$.

   Solution.

   The base step is easy: $2^2 = 4 > 2 = 2!$. Let us assume that for any arbitrary fixed integer k, $k^k > k!$. Let us see what happens for the next integer $k+1$:

$$(k+1)^{k+1} \;=\; (k+1)\,(k+1)^k \;>\; (k+1)\,k^k \;>\; (k+1)\,k! \;=\; (k+1)!$$

   because $k+1 > k$ for any $k \geq 2$. Then, the induction principle ensures that $n^n > n!$ for any integer $n \geq 2$.

Each button press scrambles the colored squares in a complicated, but nonrandom way.

4. Using result (3), prove that there exist two different sequences of 32 button presses that both produce the same configuration, if the puzzle is initially in the state shown above.

   SOLUTION.

   In this case we have sequences of 32 button presses, so the total number of sequences of button presses is 5×4^{31}. Then

   $$5 \times 4^{31} \; > \; 4^{32} \; = \; 16^{16} \; > \; 16! \; > \; \frac{16!}{(4!)^4}\,.$$

   Therefore, there are more distinct press sequences than distinct screens. The pigeon principle ensures that there are at least two sequences of button presses corresponding to the same screen. `Remark:` The role of pigeon holes is played by the screens, and the role of the pigeons, by the sequences of button presses.

Question 1.2

1. Compute the multiplicative inverse (if any) of 2^{68} in $\mathbb{Z}_{19}$.

   SOLUTION.

   This multiplicative inverse exists and it is unique because 2^{68} and 19 are relatively prime. Question 12.9 of the Problem Set shows that $2^{68} \equiv 6 \pmod{19}$. Therefore, we look for an x such that
   $$2^{68} \cdot x \equiv 6 \cdot x \equiv 1 \pmod{19}\,.$$
   As $6 \cdot 3 = 18 \equiv -1 \pmod{19}$, then
   $$6 \cdot (-3) = -18 \equiv 1 \pmod{19}\,.$$
   Therefore the sought multiplicative inverse is $x \equiv -3 \pmod{19} \equiv 16 \pmod{19}$.

2. Find the zero divisors (if any) of $\mathbb{Z}_{16}$.

   SOLUTION.

   A zero divisor of $\mathbb{Z}_{16}$ is an element $x \not\equiv 0 \pmod{16}$ such that there exists an element $y \not\equiv 0 \pmod{16}$ such that $xy \equiv 0 \pmod{16}$. As $16 = 4^2$, we should look for pairs of elements of $\mathbb{Z}_{16}$ such that their multiplication contains the factor 16:

   - $2 \times 8 = 16 \equiv 0 \pmod{16}$.
   - $4 \times 4 = 16 \equiv 0 \pmod{16}$.
   - $4 \times 12 = 48 \equiv 0 \pmod{16}$.
   - $6 \times 8 = 48 \equiv 0 \pmod{16}$.
   - $8 \times 10 = 80 \equiv 0 \pmod{16}$.

- $8 \times 14 = 112 \equiv 0 \pmod{16}$.

The zero divisor of $\mathbb{Z}_{16}$ are its even non-zero elements $\{2n \in \mathbb{Z}_{16} \colon 1 \leq n \leq 7\}$.

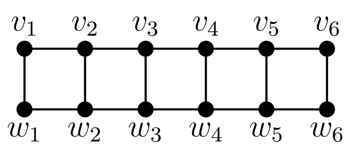
3. Let A be the set $A = \{0, 1, 2\} \times \{2, 4, 7, 9\}$. On A, we define the following relation $(a, b)\, \mathcal{R}\, (c, d) \;\Leftrightarrow\; (a + b) \mid (c + d)$. Is $(A, \mathcal{R})$ a lattice?

   SOLUTION.

   The set $(A, \mathcal{R})$ is not a partially ordered set because $\mathcal{R}$ on A is not antisymmetric. On one side, we have $\{(0, 4), (2, 2)\} \subset A$, $(0, 4)\mathcal{R}(2, 2)$, and $(2, 2)\mathcal{R}(0, 4)$ (because $4 \mid 4$), but on the other hand, $(0, 4) \neq (2, 2)$; therefore, $\mathcal{R}$ is not antisymmetric. If $(A, \mathcal{R})$ is not a partially ordered set, it cannot be a lattice.

Question 1.3 Let $\{G_n\}_{n \in \mathbb{N}}$ be a graph family such that each member $G_n = (V_n, E_n)$ is defined as follows:

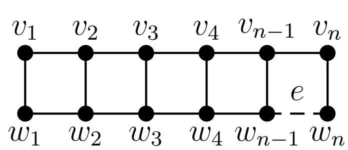
- $V_n = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$
- $E_n = \{\{v_i, v_{i+1}\} \colon 1 \leq i \leq n-1\} \cup \{\{w_i, w_{i+1}\} \colon 1 \leq i \leq n-1\} \cup \{\{v_i, w_i\} \colon 1 \leq i \leq n\}$.



1. Find, using the contraction-deletion theorem, a recurrence for the chromatic polynomial P_{G_n} of G_n. Find the necessary initial conditions.

   SOLUTION.

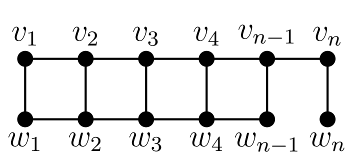
   We apply the contraction-deletion theorem on G_n. We illustrate this argument with the case $n = 6$, but the discussion is valid for any n.



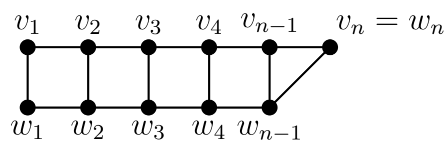
   The contraction-deletion theorem (Theorem 132) ensures that

$$P_{G_n} \;=\; P_{G_n - e} - P_{G_n / e}\,,$$

   where e is the edge depicted as a dashed line on the above figure. The graph $G_n - e$ (the edge e is deleted) is given by



The graph G_n/e (with the edge e contracted) is given by



If we apply Theorem 134 of the Lecture notes, we get

$$P_{G_n} = P_{G_n-e} - P_{G_n/e} = \frac{P_{G_{n-1}}\, P_{P_3}}{P_{K_1}} - \frac{P_{G_{n-1}}\, P_{K_3}}{P_{K_2}} = (q^2-3q+3)\, P_{G_{n-1}}\,,$$

which is valid for any integer $n \geq 2$. This is a linear recurrence of order 1 for P_{G_n}, therefore, we need a single initial condition:

$$P_{G_1}(q) = P_{K_2}(q) = q(q-1)\,.$$

2. Solve such recurrence using generating functions.

<u>Solution.</u>

We define the generating function in the usual way (but starting the sum at $n = 1$):

$$F(x) = \sum_{n=1}^{\infty} P_{G_n}\, x^n = q(q-1)\, x + \sum_{n=2}^{\infty} P_{G_n}\, x^n\,.$$

From the recurrence, we obtain

$$\sum_{n=2}^{\infty} P_{G_n}\, x^n = (q^2-3q+3) \sum_{n=2}^{\infty} P_{G_{n-1}}\, x^n\,.$$

Therefore,

$$F - q(q-1)x = (q^2-3q+3)xF\,.$$

The solution for F is

$$F(x) = \frac{q(q-1)x}{1-(q^2-3q+3)x} = \sum_{n=1}^{\infty} q(q-1)(q^2-3q+3)^{n-1}\, x^n\,,$$

then

$$P_{G_n}(q) \;=\; q\,(q-1)\,(q^2-3q+3)^{n-1}\,, \quad n \geq 1\,.$$

3. Compute $\chi(G_n)$ using the expression for $P_{G_n}(q)$.

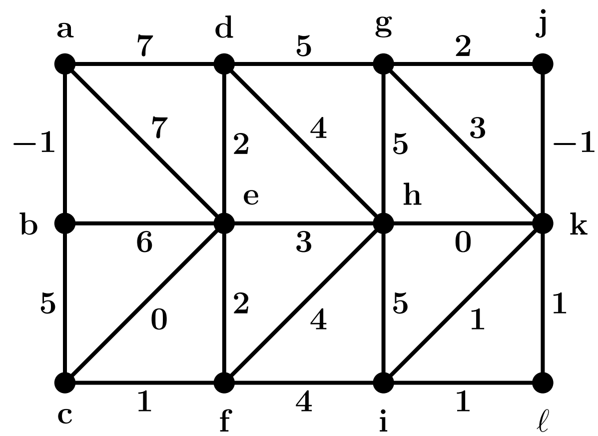
<u>Solution.</u>

$P_{G_n}(q)$ vanishes at $q = 0, 1$. However, the roots of $q^2 - 3q + 3$ are not real; therefore, $\chi(G_n) = 2$. As a matter of fact, $P_{G_n}(2) = 2$, as G_n is bipartite.

2 Mid-term exam April 2016

Question 2.1

1. For the following weighted graph



compute the minimum weight spanning tree T:

(a) Write the edges of T in the order they are chosen according to Prim's algorithm.

(b) Compute the weight of this tree T.

<u>Solution.</u>

Let $T = (V, F)$ be the sought spanning tree using Prim's algorithm.

(a) The edges are $F = \{\{a, b\}, \{b, c\}, \{c, e\}, \{c, f\}, \{e, d\}, \{e, h\}, \{h, k\}, \{k, j\}, \{k, \ell\}, \{\ell, i\}, \{j, g\}\}$.

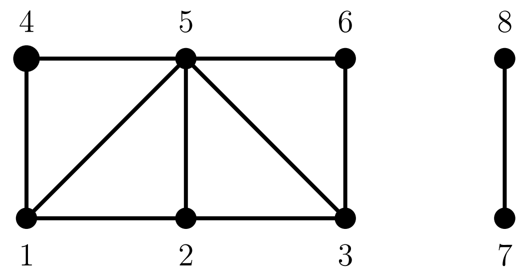
(b) The minimum weight is $\omega_{\min} = 13$.

2. Let N be a fixed natural number. Find a simple graph with N vertices that is both Hamiltonian and Eulerian, and such that its Hamiltonian cycle does not coincide with its Eulerian circuit.

<u>Solution.</u>

It suffices to consider the graph $C_N = (V, E)$ and add it three edges $F = \{\{1, 3\},\{3, 5\},\{5, 1\}\}$. Then, the sought graph is $G = (V, E \cup F)$. This procedure makes sense only if $N \geq 6$.

3. Consider the graph G in the picture:



Compute the number of proper colorings $P_G(q)$ of such graph G with q colors, and its chromatic number.

SOLUTION.

The graph is not connected $G = (V_1 \cup V_2, E_1 \cup E_2)$ with two connected components. The chromatic polynomial of the component $G_2 = (V_2, E_2) \simeq K_2$ with $V_2 = \{7, 8\}$ is $P_{G_2}(q) = q(q-1)$.

For the connected component on the left hand-side $G_1 = (V_1, E_1)$ with $V_1 = \{1, 2, 3, 4, 5, 6\}$, we can use the product principle:

- Vertex 1 can be colored in q distinct ways.
- Once vertex 1 is colored, vertex 4 can be colored in $q-1$ distinct ways.
- Once vertices 1 and 4 are colored, vertex 5 can be colored in $q-2$ distinct ways.
- Once vertices 1,4,5 are colored, vertex 2 can be colored in $q-2$ distinct ways.
- Once vertices 1,2,4,5 are colored, vertex 3 can be colored in $q-2$ distinct ways.
- Once vertices 1–5 are colored, vertex 6 can be colored in $q-2$ distinct ways.

Then, the product principle tells us that $P_{G_1}(q) = q(q-1)(q-2)^4$. Finally, the same principle ensures that

$$P_G(q) \;=\; P_{G_1}(q) P_{G_2}(q) \;=\; q^2(q-1)^2(q-2)^4 \,.$$

The real roots of P_G are $q = 0, 1, 2$; therefore, the first natural such that $P_G(q) > 0$ is $q = 3$. Then, $\chi(G) = 3$ ($P_G(3) = 12$).

Question 2.2 Note: results may contain numbers, factorials, or binomial coefficients.

1. A commercial vehicles company makes white trucks and blue vans. The company wants to ship P trucks and Q vans, in such a way that only vans have license plates on (all of them different). How many different convoys can be formed with all the vehicles such that there are no consecutive vans?

Solution.

The trucks are identical; therefore they can be lined up in a unique way. If I now assume that the vans are identical, then I can insert one of them in front of all trucks, in between two trucks, or at the end of the convoy. In each of the possible $P+1$ places, I can either insert a single van or none of them. Therefore, there are

$$\binom{P+1}{Q}$$

ways to achieve this. As the vans are all distinct, for each one of the above configurations I have $Q!$ permutations of the vans. Finally, the product principle ensures that the answer is

$$Q!\binom{P+1}{Q}.$$

2. Solve the following recurrence relation

$$a_n \;=\; 2a_{n-1} + 2a_{n-2}\,, \quad n \geq 3\,, \quad a_1 = 1\,, \quad a_2 = 4\,.$$

Solution.

The roots of the characteristic polynomial are $x = 1 \pm \sqrt{3}$, then the general solution is $a_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$. If $a_1 = 1$ and $a_2 = 4$, we have that $a_2 = 2a_1 + 2a_0 = 4 = 2 + 2a_0$. Therefore, $a_0 = 1$.

Using the initial conditions $a_0 = a_1 = 1$, we obtain $A = B = 1/2$. Then,

$$a_n = \frac{1}{2}\left[(1+\sqrt{3})^n + (1-\sqrt{3})^n\right]\,.$$

3. A company needs to inspect a series of cities and wants to group them into 3 groups of 3, 4 groups of 4, 5 groups of 5, and 6 groups of 6. In how many different ways can they do so?

Solution.

We have $9 + 16 + 25 + 36 = 86$ cities. The number of partitions is given by

$$\frac{86!}{(3!)^3(4!)^4(5!)^5(6!)^6}\,\frac{1}{3!4!5!6!}\,,$$

because the order of the cities within a group is unimportant, as well as the order of the groups with the same cardinal.

3 Final exam May 2016

Question 3.1 Let $\mathbb{Z}_+ = \{x \in \mathbb{Z} \colon x \geq 0\}$ be the set of non-negative integers. We define the following relation $\mathcal{T}$ on $\mathbb{Z}_+$:

$$a \,\mathcal{T}\, b \quad \Leftrightarrow \quad \lfloor \sqrt{a} \rfloor \; = \; \lfloor \sqrt{b} \rfloor \, .$$

1. Prove that $\mathcal{T}$ is an equivalence relation.
2. Find the equivalence classes.
3. Compute the cardinal of each equivalence class.
4. Find the quotient set $\mathbb{Z}_+/\mathcal{T}$.

<u>SOLUTION.</u>

1. It is an equivalence relation because $a \,\mathcal{T}\, b \;\Leftrightarrow\; f(a) = f(b)$ where $f \colon \mathbb{Z}_+ \to Z_+$ is given by $f(x) = \lfloor \sqrt{x} \rfloor$. The quotient set then satisfies $\mathbb{Z}_+/\mathcal{T} \simeq \mathrm{Im}(f) = Z_+$.
2. The equivalence classes are: $[n^2]_{\mathcal{T}} = \{k \in \mathbb{Z}_+ \colon n^2 \leq k \leq n^2 + 2n\}$ for each $n \in \mathbb{Z}_+$.
3. $|[n^2]_{\mathcal{T}}| = 2n + 1$ for each $n \in \mathbb{Z}_+$.
4. $\mathbb{Z}_+/\mathcal{T} = \{[n^2]_{\mathcal{T}} \colon n \in \mathbb{Z}_+\}$.

Question 3.2

1. Solve the linear congruence $6x \equiv 9 \pmod{15}$.
2. Find the remainder of dividing $20^{234123456702702}$ by 101.

<u>SOLUTION.</u>

1. This linear congruence is equivalent to $2x \equiv 3 \pmod{5}$. As $\gcd(2, 5) = 1$ and $1 \mid 3$, there is a unique solution modulo 5. This solution can be seen easily: $x \equiv 4 \pmod{5}$ $(2 \cdot 4 = 8 \equiv 3 \pmod{5})$. The three solutions modulo 15 can be obtained by writing the above solution as $x = 4 + 5k$ and considering the 3 congruence classes modulo 3 of k:

   - $k = 3p$, then $x = 4 + 15p$ ó $x \equiv 4 \pmod{15}$.
   - $k = 3p + 1$, then $x = 9 + 15p$ ó $x \equiv 9 \pmod{15}$.
   - $k = 3p + 2$, then $x = 14 + 15p$ ó $x \equiv 14 \pmod{15}$.

   These are the three solutions modulo 15 of the congruence $6x \equiv 9 \pmod{15}$.

2. As 101 is a prime number, $\phi(101) = 100$, and Fermat's little theorem implies that $20^{100} \equiv 1 \pmod{101}$. Then

$$\begin{aligned} 20^{2341234567 02702} \pmod{101} &\equiv 20^{234123456702700} \cdot 20^2 \pmod{101} \\ &\equiv \left(20^{100}\right)^{2341234567027} 400 \pmod{101} \\ &\equiv 400 \pmod{101} \equiv 97 \pmod{101}. \end{aligned}$$

As $0 \leq 97 \leq 100$, the sought remainder is $20^{234123456702702} \bmod 101 = 97$.

Question 3.3 Let $W_2, W_3, W_4, \ldots$ be the family of wheel graphs.
Reminder: the wheel graph W_n has n vertices forming a cycle, and an extra vertex inside this cycle, and such that it is neighbour of the n vertices on that cycle.

1. If $q \geq 2$ is an arbitrary positive integer, prove that the chromatic polynomial $P_{W_n}(q) = p_n$ of W_n satisfies the following recurrence relation

$$p_n = -p_{n-1} + q(q-1)(q-2)^{n-1}, \quad n \geq 4.$$

2. Find the initial condition $p_3 = P_{W_3}(q)$.

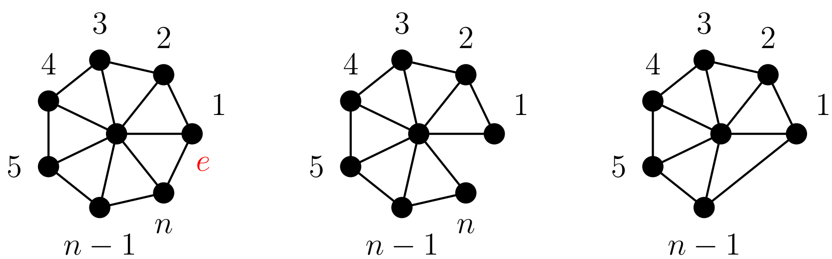
3. Solve the recurrence relation given in point 1. with the initial condition found in point 2.

4. Is the expression for $p_n = P_{W_n}(q)$ found in point 3. above (and valid for any $n \geq 3$) also valid when applied to the case $n = 2$?

5. Compute $\chi(W_n)$ for any $n \geq 2$.

SOLUTION.

1. On the right-hand side of the following figure you can find the graph W_7, with the internal vertex without its label $n+1$. Even though we are using W_7 as an example, the argument will be general for all $n \geq 4$. The recurrence relation is obtained by using the deletion-contraction theorem on any edge belonging to the outer cycle; in particular, to the edge $e = \{n, 1\}$. The graphs $W_n - e$ and $W_n \setminus e$ are represented in the next figure.



$\boldsymbol{W_n}$ $\quad$ $\boldsymbol{W_n - e}$ $\quad$ $\boldsymbol{W_n \setminus e}$

The graph $W_n \setminus e$ is isomorphic to W_{n-1}, and the chromatic polynomial of the graph $W_n - e$ can be computed easily by using the product principle: the central vertex can be colored in q distinct ways; the vertex n cam be then colored in $q-1$ distinct ways; and each one of the $n-1$ other vertices ($1 \leq k \leq n-1$) can then be sequentially colored in $q-2$ distinct ways. Finally, the sought chromatic polynomial is $P_{W_n-e}(q) = q(q-1)(q-2)^{n-1}$, and

$$P_{W_n}(q) \;=\; P_{W_n-e}(q) - P_{W_n \setminus e} \;=\; P_{W_n-e}(q) - P_{W_{n-1}}\,.$$

In terms of $p_n = P_{W_n}(q)$, we find the recurrence

$$p_n \;=\; -p_{n-1} + q(q-1)(q-2)^{n-1}\,, \quad \text{for all } n \geq 4.$$

2. The graph W_3 is simple, with 4 vertices, and 3-regular. Then, it is isomorphic to K_4 and

$$p_3 \;=\; P_{W_3}(q) \;=\; P_{K_4}(q) \;=\; q(q-1)(q-2)(q-3)\,.$$

3. The recurrence is linear, of degree one, non-homogeneous, and with constant coefficients (they do not depend on n). Then, its general solution is the sum of the general solution of the homogeneous part and a particular solution of the full non-homogeneous recurrence.

   The general solution of the homogeneous recurrence $p_n = -p_{n-1}$ is trivial: $x = -1$ and $p_n = A(-1)^n$.

   The particular solution of the full recurrence is of the form $p_n = B(q-2)^n$, as $q-2$ can never be equal to -1 if $q \geq 2$. If we substitute $p_n = B(q-2)^n$ in the recurrence, we obtain that $B = q$. Then, the particular solution is $p_n = q(q-2)^n$.

   The general solution of the recurrence equation is $p_n = A(-1)^n + q(q-2)^n$. As $p_3 = q(q-1)(q-2)(q-3)$, we have that

$$q(q-1)(q-2)(q-3) \;=\; -A + q(q-2)^3\,, \quad \Rightarrow \quad A \;=\; q(q-2)\,.$$

   Therefore, the sought solution is:

$$p_n \;=\; P_{W_n}(q) \;=\; q(q-2)(-1)^n + q(q-2)^n\,, \quad n \;\geq\; 3\,.$$

4. The graph W_2 has 3 vertices. The central vertex has degree 2 and the two external vertices are joined together by two edges. But its chromatic polynomial is equal to that of K_3, as multi-edges are irrelevant to compute this polynomial. Then $P_{W_2}(q) = P_{K_3}(q) = q(q-1)(q-2)$.

   The solution of point 3. evaluated at $n = 2$ gives:

$$P_{W_2}(q) \;=\; q(q-2) + q(q-2)^2 \;=\; q(q-1)(q-2)\,,$$

   which is equal to the above result. Therefore, the general solution is also valid for $n = 2$ and

$$P_{W_n}(q) \;=\; q(q-2)(-1)^n + q(q-2)^n\,, \quad n \;\geq\; 2\,.$$

5. For each arbitrary $n \geq 2$, we can take $q(q-2)$ as a common factor in $P_{W_n}(q)$. Then, $P_{W_n}(0) = P_{W_n}(2) = 0$. This should also imply that $P_{W_n}(1) = 0$, and this is so: $P_{W_n}(1) = (-1)^{n+1} + (-1)^n = 0$. All these facts imply that $\chi(W_n) \geq 3$.

   Let us see what is $P_{W_n}(3)$:

$$P_{W_n}(3) \;=\; 3(-1)^n + 3 \;=\; \begin{cases} 6 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

   Therefore, $\chi(W_n) = 3$ for all even integer $n \geq 2$ par. If $n \geq 3$ is an odd integer, $P_{W_n}(4) = 8(2^{n-1} - 1) > 0$. In conclusion,

$$\chi(W_n) \;=\; \begin{cases} 3 & \text{if } n \geq 2 \text{ is even,} \\ 4 & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

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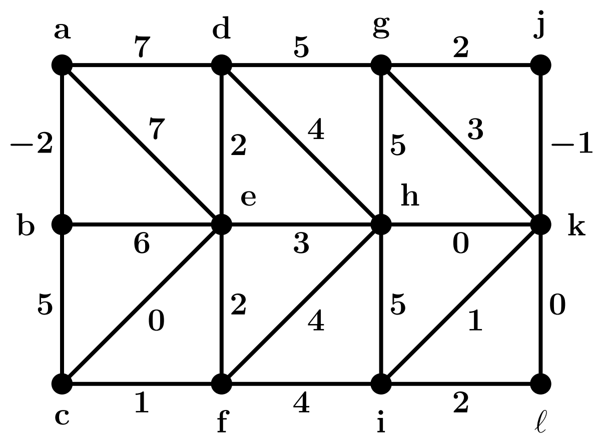
4 Final exam June 2016

Question 4.1

1. For the following weighted graph



   compute a minimum-weight spanning tree T.

   (a) Write the edges of T in the order they are chosen according to Prim's algorithm.

   (b) Compute the weight of this tree T.

2. Let N be a fixed natural number. Find a simple graph with N vertices that is both Hamiltonian and Eulerian, and such that its Hamiltonian cycle does not coincide with its Eulerian circuit. **Remark:** it is not valid to give an example for a numerical value of N; for instance, $N = 6$.

SOLUTION.

1. Let $T = (V, F)$ be the sought spanning tree using Prim's algorithm.
   (a) The edges are $F = \{\{a,b\},\{b,c\},\{c,e\},\{c,f\},\{e,d\},\{e,h\},\{h,k\},\{k,j\},\{k,\ell\},$ $\{k,i\},\{j,g\}\}$.
   (b) The minimum weight is $\omega_{\min} = 11$.
2. It suffices to consider the graph $C_N = (V, E)$ and add it three edges $F = \{\{1,3\},\{3,5\},$ $\{5,1\}\}$. Then, the sought graph is $G = (V, E \cup F)$. This procedure makes sense only if $N \geq 6$.

Question 4.2 **Note:** results may contain numbers, factorials, or binomial coefficients.

1. A company manufactures golf balls. These can be standard (all of them are identical) or customized by printing the initials of the customer (all of them are distinct). If we have N balls in total, in how many ways is it possible to line up these balls if there are exactly M standard balls and they cannot go in consecutive positions?
2. Solve the following recurrence relation

$$a_n = 2a_{n-1} + 6a_{n-2}\,, \quad n \geq 2\,, \quad a_0 = 1\,, \quad a_1 = 1\,.$$

SOLUTION.

1. There are M standard balls (all of them identical) and $N - M$ customized balls (all of them distinct). The first task is to line up the customized ones, and there are $(N - M)!$ distinct ways to do this. The next task is to place the standard balls. There are $N - M + 1$ available positions: in front of all customized balls, in-between two balls, or at the end of the customized balls. In each of these positions only one standard ball can be placed. Therefore, there are

$$\binom{N-M+1}{M}$$

   distinct ways to do this task. The product principle ensures that the result is

$$(N-M)!\binom{N-M+1}{M}\,.$$

2. The roots of the characteristic polynomial are $x = 1 \pm \sqrt{7}$, then the general solution of this recurrence is $a_n = A(1+\sqrt{7})^n + B(1-\sqrt{7})^n$. Using the initial conditions $a_0 = a_1 = 1$, we obtain $A = B = 1/2$. The final solution is:

$$a_n = \frac{1}{2}\left[(1+\sqrt{7})^n + (1-\sqrt{7})^n\right]\,, \quad n \geq 0\,.$$

Question 4.3

1. Let $\mathbb{Z}_+ = \{x \in \mathbb{Z} \colon x \geq 0\}$ be the set of non-negative integers. We define the following equivalence relation $\mathcal{T}$ on $\mathbb{Z}_+$:

$$a \,\mathcal{T}\, b \quad \Leftrightarrow \quad \lfloor \sqrt{a} \rfloor \;=\; \lfloor \sqrt{b} \rfloor \,.$$

   (a) Find the equivalence classes and the quotient set.

   (b) Compute the cardinal of each equivalence class.

2. Solve the linear congruence $6x \equiv 9 \pmod{15}$.

<u>SOLUTION.</u>

1. See Question 1 of the May exam.

2. See Question 2(a) of the May exam.

Question 4.4 Let $W_2, W_3, W_4, \ldots$ be the family of wheel graphs.
<u>Reminder</u>: the wheel graph W_n has n vertices forming a cycle, and an extra vertex inside this cycle, and such that it is adjacent to all of the n vertices on that cycle.

1. If $q \geq 2$ is an arbitrary positive integer, prove that the chromatic polynomial $P_{W_n}(q) = p_n$ of W_n satisfies the following recurrence relation

$$p_n \;=\; -p_{n-1} + q(q-1)(q-2)^{n-1}\,, \quad n \;\geq\; 4\,.$$

2. Solve the above recurrence relation by finding the necessary initial conditions.

<u>SOLUTION.</u>

1. See Question 3(a) of May exam.

2. See Questions 3(b) and 3(c) of May exam.

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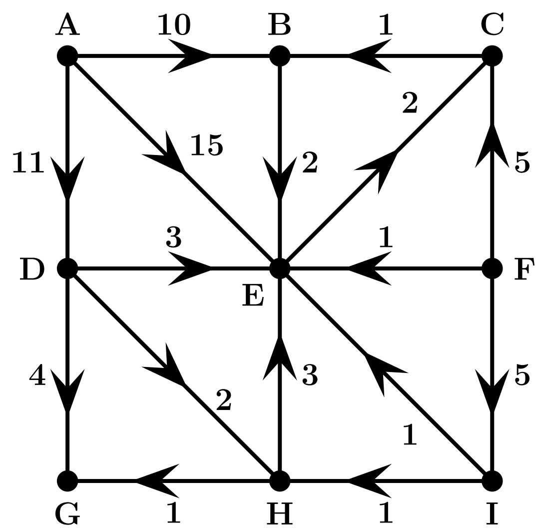
5 Mid-term exam April 2017

Question 5.1

1. For the following weighted directed graph



compute the minimum-length paths from vertex **A** to the rest of the vertices of the graph, as well as the corresponding minimum lengths.

<u>Solution.</u>

Running Dijkstra's algorithm on this graph (which is simple, connected and with all weights positive) we have the following table:

A	$\boxed{(0,A)}$	*	*	*	*	*	*
B	(10,A)	$\boxed{(10,A)}$	*	*	*	*	*
C	∞	∞	∞	(14,E)	(14,E)	$\boxed{(14,E)}$	*
D	(11,A)	(11,A)	$\boxed{(11,A)}$	*	*	*	*
E	(15,A)	(12,B)	(12,b)	$\boxed{(12,B)}$	*	*	*
F	∞	∞	∞	∞	∞	∞	∞
G	∞	∞	(15,D)	(15,D)	(14,H)	(14,H)	$\boxed{(14,H)}$
H	∞	∞	(13,D)	(13,D)	$\boxed{(13,D)}$	*	*
I	∞	∞	∞	∞	∞	∞	∞

Therefore, the solution is:

- $d(A,B) = 10$ along the path (A,B).
- $d(A,C) = 14$ along the path (A,B,E,C).

- $d(A, D) = 11$ along the path (A,D).
- $d(A, E) = 12$ along the path (A,B,E).
- $d(A, F) = \infty$. It is impossible to reach F from A.
- $d(A, G) = 14$ along the path (A,D,H,G).
- $d(A, H) = 13$ along the path (A,D,H).
- $d(A, I) = \infty$. It is impossible to reach I from A.

2. Let $G = (V, E)$ be a plane connected graph such that the degree of each vertex is at least 2, and each face is bounded by at least 6 edges. Tell whether such a graph exists or not. If it does, find the minimum number of vertices it should contain; and if it does not exist, find the minimum maximum-degree it should have in order to exist.

SOLUTION.

Let us suppose that such a graph G does exist. From the hand-shake lemma applied to G, we obtain $|E| \geq |V|$.

Because G is plane and connected, its dual graph $G^* = (V^*, E^*)$ exists. Using the hand-shake lemma on G^* we obtain $|E| \geq 3R$, where R is the number of regions determined by G on the plane.

Euler's theorem ensures that $|V| - |E| + R = 2$, hence

$$|E| \ = \ |V| + R - 2 \leq \frac{|E|}{3} + |V| - 2 \ \Rightarrow \ |E| \leq \frac{3|V|}{2} - 3\,.$$

Therefore, the number of vertices should satisfy

$$|V| \leq |E| \leq \frac{3|V|}{2} - 3\,.$$

This equation has a solution for $|V| \geq 6$. Therefore, G exists whenever this condition on $|V|$ holds. Then,

$$|V|_{\min} \ = \ 6$$

and C_6 is a simple example of such a graph.

Question 5.2 Note: results may contain numbers, factorials, or binomial coefficients.

1. A farmer wants to distribute his fruit harvest among 30 distinct people. The harvest consists in 100 identical oranges and 300 identical apples. The farmer also wants to distribute all the fruit pieces in such a way that each person gets at least one orange and 3 apples. In how many distinct ways can he achieve this task?

   SOLUTION.

   The problem can be split into two consecutive and independent tasks: first, we distribute the oranges, and then we distribute the apples.

   In the former case, we have to distribute 100 identical objects into 30 distinct boxes in such a way that no box can stay empty. To achieve this, we line up the 100 oranges, so that there are 99 spaces in-between two consecutive oranges. In those 99 places we can insert one of the 29 mobile bars that define the 30 boxes. Therefore, the number of ways of performing this task is $\binom{99}{29}$.

Once the oranges are distributed, we distribute the apples. Now we have to distribute 300 identical objects into 30 distinct boxes in such a way that in each box we have at least 3 objects. This is equivalent to distribute $300 - 30 \times 3 = 210$ objects into 30 boxes without any constraint on the minimum number of objects in a given box. This is equivalent to permute 210 identical oranges $+29$ identical mobile bars $= 239$ objects. Therefore, the number of ways of performing this task is $\binom{239}{29}$.

The product principle ensures us that the solution of this problem is:

$$\binom{99}{29} \times \binom{239}{29}.$$

2. Solve the following recurrence relation

$$a_n \;=\; 6a_{n-1} - 9a_{n-2} + 2 \times 3^n, \quad n \geq 2\,, \quad a_0 = a_1 = 3\,.$$

<u>Solution.</u>

The general solution of this non-homogeneous recursion is the sum of the general solution of the homogeneous one plus a particular solution of the full recursion.

The homogeneous recursion has the characteristic polynomial $x^2-6x+9 = (x-3)^2 = 0$. Then it has the root $x = 3$ with multiplicity 2. Therefore, the general solution of this recursion is $a_n^{(h)} = 3^n(A + Bn)$.

The form of the particular solution is $a_n^{(p)} = Cn^2 3^n$, as the non-homogeneous term is proportional to 3^n, and 3 is a root of the above characteristic polynomial with multiplicity 2. The value of C can be computed by substituting the expression for $a_n^{(p)}$ into the full recursion

$$C3^n n^2 \;=\; 6C3^{n-1}(n-1)^2 - 9C3^{n-2}(n-2)^2 + 2 \times 3^n\,.$$

This expression can be simplified a lot: $-2C + 2 = 0$, so that $C = 1$.

The general solution for the full recurrence

$$a_n \;=\; 3^n(A + Bn + n^2)\,.$$

The parameters A and B are computed by using the initial conditions: $a_0 = 3 = A$ and $a_1 = 3 = 3(A + B + 1)$. Then $A = -B = 3$, and the sought solution is

$$a_n \;=\; 3^n(3 - 3n + n^2)\,, \quad n \geq 0.$$

6 Final exam May 2017

Question 6.1

1. Find the solutions in $\mathbb{Z}_{33}$ (if any) of the linear congruence $18x \equiv 30 \pmod{33}$.

2. Let us consider the set $A = \{2,5,8\} \times \{0,1,2,3\}$. We define on A the binary relation $\mathcal{R}$ as follows: $(a,b)\mathcal{R}(c,d) \Leftrightarrow (a+b) \mid (c+d)$. Justify whether $\mathcal{R}$ is an order relation or not. If yes, find its Hasse diagram.

SOLUTION.

- As $\gcd(2,33) = 1$, the congruence is equivalent to $9x \equiv 15 \pmod{33}$. Furthermore, as $\gcd(3,33) = 3$, the last congruence can be written as $3x \equiv 5 \pmod{11}$.

  As $\gcd(3,11) = 1$ and $1 \mid 5$, the previous congruence has a unique solution mod11. Because $3(-2) = -6 \equiv 5 \pmod{11}$, then $x \equiv -2 \pmod{11} \equiv 9 \pmod{11}$ is precisely such (unique) solution mod11.

  To write the solutions in $\mathbb{Z}_{33}$, it suffices to consider the previous solution as the equality $x = 9 + 11k$ with $k \in \mathbb{Z}$. If we now take $k = 3p$, $k = 3p+1$ and $k = 3p+2$ with $p \in \mathbb{Z}$, we will obtain the three solutions in $\mathbb{Z}_{33}$ of the original congruence: $x \equiv 9 \pmod{33}$, $x \equiv 20 \pmod{33}$, and $x \equiv 31 \pmod{33}$.

- $\mathcal{R}$ is not an order relation because it fails to be anti-symmetric: $(5,0),(2,3) \in A$, $(5,0)\mathcal{R}(2,3)$, $(2,3)\mathcal{R}(5,0)$, but $(5,0) \neq (2,3)$.

Question 6.2 Let us consider the following graph families:

- The hypercube graph Q_n, whose vertices are the bit strings of length n.
- The path graph P_n on n vertices.
- The cycle graph C_n on n vertices.
- The complete graph K_n on n vertices.
- The trivial graph t_n on n vertices: $t_n = (\{1,2,\dots,n\}, \varnothing)$

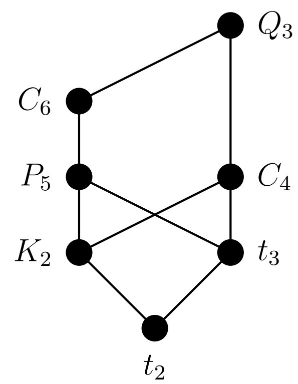
Let us now define the set of graphs $A = \{Q_3, P_5, C_6, C_4, K_2, t_3, t_2\}$, and the order relation $\preceq$ on A: if $G_1, G_2 \in A$,

$$G_1 \preceq G_2 \quad \Leftrightarrow \quad G_1 \text{ is a subgraph of some graph that is isomorphic to } G_2.$$

Justify whether the set $(A, \preceq)$ has the structure of a Boolean algebra or not.

SOLUTION.

The corresponding Hasse diagram is



Therefore, $(A, \preceq)$ is not a lattice, as $\sup(K_2, t_3)$ does not exist. This implies that it cannot be either a Boolean algebra.

Question 6.3 Find the solution of the following recursion

$$a_n \;=\; 4a_{n-1} - 4a_{n-2} + (e-2)^2\, e^n\,, \quad n \geq 2\,,$$

with the initial conditions $a_0 = e^2$, $a_1 = e^3 + 1$.

Solution.

The general solution of this recurrence is equal to the general solution of the homogeneous part plus a particular solution of the full non-homogeneous recurrence.

The general solution of the homogeneous recurrence is obtained from the roots of their characteristic polynomial $x^2 - 4x + 4 = (x-2)^2 = 0$. Then, it only has the root $x = 2$ with multiplicity 2. Therefore, the general solution of the homogeneous recurrence is

$$a_n^{(h)} \;=\; (A + B\,n)\,2^n\,.$$

Because e is not a root of the above characteristic polynomial, the form of the particular solution of the non-homogeneous recurrence is $C\,e^n$. The value of the constant C if found by substituting this expression into the full recurrence:

$$C\,e^n \;=\; 4\,C\,e^{n-1} - 4\,C\,e^{n-2} + (e-2)^2\,e^n\,.$$

If we divide by e^{n-2}, we get

$$C\,e^2 \;=\; 4\,C\,e - 4\,C + (e-2)^2\,e^2\,.$$

Then,

$$C \;=\; \frac{(e-2)^2\,e^2}{e^2 - 4\,e + 4} \;=\; \frac{(e-2)^2\,e^2}{(e-2)^2} \;=\; e^2\,.$$

and the sought particular solution is $a_n^{(p)} = e^{n+2}$.

The general solution of the recurrence is

$$a_n \;=\; (A + B\,n)\,2^n + e^{n+2}$$

and we compute the constants A y B by using the initial conditions:

$$a_0 \;=\; e^2 \;=\; A + e^2 \;\Rightarrow\; A \;=\; 0\,, \quad a_1 \;=\; e^3 + 1 \;=\; 2\,B + e^3 \;\Rightarrow\; B \;=\; \frac{1}{2}\,.$$

Finally, the solution is given by

$$a_n \;=\; e^{2+n} + 2^{n-1}\,n\,, \quad n \geq 0\,.$$

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7 Mid-term exam April 2018

Question 7.1 **Note:** results may contain numbers, factorials, or binomial coefficients.

1. Compute the number of distinct words of length n that can be formed by using the alphabet $\{A, B, C, D\}$ and such that they do not contain 2 consecutive B nor 2 consecutive C nor 2 consecutive D.

   Solution.

   Let X be the condition that a given string (word) does not contain the substrings BB, CC, or DD (so it cannot contain two consecutive B, nor two consecutive C, nor two consecutive D). We then define:

   - x_n is the number of words of length n that satisfy X.
   - a_n is the number of words of length n that satisfy X and start with an A.
   - b_n is the number of words of length n that satisfy X and start with a B.
   - c_n is the number of words of length n that satisfy X and start with a C.
   - d_n is the number of words of length n that satisfy X and start with a D.

   As the alphabet is $\{A, B, C, D\}$, the following identity holds (by the sum principle)

   $$x_n \;=\; a_n + b_n + c_n + d_n\,, \quad \forall\, n \geq 1\,.$$

   The initial conditions can be easily computed

   - $x_1 = 4$, as the strings are $\{A, B, C, D\}$.
   - $x_2 = 13$, as the strings are $\{AA, AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC\}$.

   The sought recurrence is

   $$x_n \;=\; \underbrace{x_{n-1}}_{A} + \underbrace{a_{n-1} + c_{n-1} + d_{n-1}}_{B} + \underbrace{a_{n-1} + b_{n-1} + d_{n-1}}_{C} + \underbrace{a_{n-1} + b_{n-1} + c_{n-1}}_{D}\,.$$

   In the previous equation, the character below each brace shows the letter each type of word starts with. Then,

   $$x_n \;=\; x_{n-1} + a_{n-1} + 2(a_{n-1} + b_{n-1} + c_{n-1} + d_{n-1}) \;=\; 3x_{n-1} + a_{n-1}$$

As $a_{n-1} = x_{n-2}$, we obtain the recurrence

$$x_n \;=\; 3x_{n-1} + x_{n-2}\,, \quad n \geq 3\,, \quad x_1 = 4\,, \quad x_2 = 13\,.$$

This is a linear, homogeneous recurrence with constant coefficients and order 2. The characteristic polynomial associated to this recurrence and its roots are

$$x^2 - 3x - 1 \;=\; 0 \quad \Rightarrow \quad x \;=\; \frac{3 \pm \sqrt{13}}{2}\,.$$

Then, the general solution is given by

$$x_n \;=\; \alpha \left(\frac{3+\sqrt{13}}{2} \right)^n + \beta \left(\frac{3-\sqrt{13}}{2} \right)^n\,.$$

To compute α and β, we use the initial conditions. As the second one $x_2 = 13$ implies doing very cumbersome algebra, it is more convenient to obtain the value of x_0 predicted by the recurrence

$$x_2 \;=\; 3x_1 + x_0 \quad \Rightarrow \quad x_0 \;=\; x_2 - 3x_1 = 1\,,$$

and use as initial conditions the values $x_0 = 1$ and $x_1 = 4$. The system of two linear equations obtained in this way is

$$\begin{cases} 1 &= \alpha + \beta\,, \\ 4 &= \alpha\,\dfrac{3+\sqrt{13}}{2} + \beta\,\dfrac{3-\sqrt{13}}{2}\,. \end{cases}$$

The solution is easy to find:

$$\alpha \;=\; \frac{\sqrt{13}+5}{2\sqrt{13}}\,, \quad \beta \;=\; \frac{\sqrt{13}-5}{2\sqrt{13}}\,.$$

Therefore, the final solution is

$$x_n \;=\; \frac{1}{26}\left[(13+5\sqrt{13}) \left(\frac{3+\sqrt{13}}{2} \right)^n + (13-5\sqrt{13}) \left(\frac{3-\sqrt{13}}{2} \right)^n \right], \quad n \geq 1\,.$$

Indeed, $x_0 = 1, x_1 = 4$ y $x_2 = 13$.

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2. Find the number of distinct integer solutions of the equation

$$x_1 + x_2 + \ldots + x_{n-1} + x_n \;=\; M\,,$$

if $x_i \geq 2$ for all $1 \leq i \leq n$. What conditions should n and M satisfy so that the number of solutions is non-zero?

Solution.

We first re-write the equation in such a way that the variables can take any non-negative integer value. To achieve that, we perform the following change of variables $x_i = 2 + u_i$ for any $1 \leq i \leq n$. In this way, we get that $u_i \geq 0$ for any $1 \leq i \leq n$. Therefore,

$$u_1 + u_2 + \ldots + u_{n-1} + u_n \; = \; M - 2n \,, \quad u_i \geq 0 \,.$$

This is a standard distribution of $M - 2n$ identical objects in n (distinct) boxes represented by $n - 1$ identical bars. Therefore, the number of solutions is

$$\binom{M - 2n + n - 1}{n - 1} \; = \; \binom{M - n - 1}{n - 1} .$$

In order to ensure that $\binom{p}{q} \neq 0$, we need that $p \geq 0$ and $0 \leq q \leq p$. The first condition implies that $M \geq n + 1$. The condition $n - 1 \geq 0$ implies that $n \geq 1$. Finally, the condition $M - n - 1 \geq n - 1$ implies that $M \geq 2n$. As $n \geq 1$, $M \geq 2n$ is stronger than $M \geq n + 1 \geq 2$. Therefore, the sought conditions on M and n are:

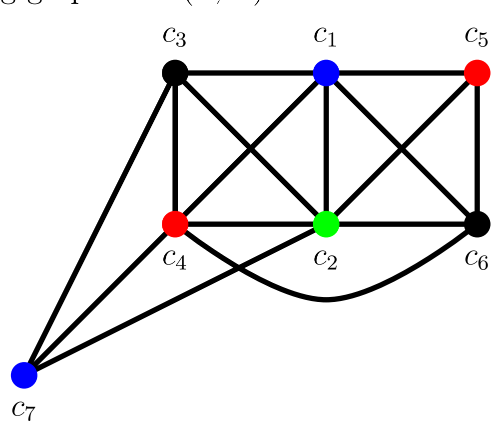
$$n \geq 1 \,, \quad M \geq 2n \,.$$

Question 7.2

1. The first day in a workshop there are 7 talks related to 4 distinct topics. Talks $\{c_1, c_2, c_5, c_6\}$ are about Topic 1; talks $\{c_1, c_2, c_3, c_4\}$, about Topic 2; talks $\{c_2, c_3, c_4, c_7\}$, about Topic 3, and talks $\{c_4, c_6\}$, about Topic 4. If each talk lasts one hour, find (using graph-theoretic techniques) the minimum number of hours needed to organize the talks in such a way that any person interested in a given topic cannot miss any of corresponding talks. What is the minimum number of lecture halls needed to arrange this inaugural session?

Solution.

We model this problem using a graph in such a way that each talk is represented by a vertex. Two vertices are adjacent if and only if the corresponding talks belong to the same topic. The resulting graph $G = (V, E)$ is:



The final result is just the chromatic number $\chi(G)$, as the time slots play the role of colors. Because there are several subgraphs of G that are K_4 (e.g., those corresponding to the first three topics), then $\chi(G) \geq \chi(K_4) = 4$.

In addition, it is fairly easy to find an ordering of the vertices of G such that the greedy algorithm for proper q-colorings finds a proper 4-coloring for G. Therefore, $\chi(G) \leq 4$. This ordering is e.g., $(c_1, c_5, c_2, c_6, c_4, c_3, c_7)$. The colors assigned to each vertex in the previous picture correspond to the proper 4-coloring obtained by using this algorithm.

Therefore the unique solution of both inequalities ($\chi(G) \geq 4$ and $\chi(G) \leq 4$) is $\chi(G) = 4$. We now arrange these results in this table:

Time slots	Color	vertices
1	blue	c_1, c_7
2	red	c_4, c_5
3	green	c_2
4	black	c_3, c_6

It is clear that the minimum number of time slots is 4 and, as each time slot corresponds to one hour, the minimum number of needed hours is also 4. In addition, by inspection from the above table, the maximum number of vertices colored alike is 2. Therefore, the minimum number of lecture halls needed to organize the talks is 2.

---

2. Given a plane graph G with 68 vertices, such that it splits the plane into 52 regions and the degree of each vertex is at most 3, what is the minimum number of connected components G has?

Solution.

Let us assume that $G = (V, E)$ is a graph that satisfies all these conditions. The hand-shacking lemma tells us that

$$2|E| \;=\; \sum_{x\in V} d_x \;\leq\; 3|V| \quad \Rightarrow \quad |E| \;\leq\; \frac{3}{2}\,|V|\,.$$

On the other hand, Euler's theorems ensures that

$$|V| - |E| + R \;=\; 1 + n\,,$$

where n is the number of connected components of G. Therefore,

$$n \;=\; |V| - |E| + R - 1 \geq |V| - \frac{3}{2}\,|V| + R - 1 \;=\; R - 1 - \frac{1}{2}\,|V| \;=\; 17\,.$$

Then, $n \geq 17$ and hence, $n_{\min} = 17$. As a matter of fact, this is the result if G is a graph with 17 connected components and each of them is a K_4 ($d = 3$).

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8 Final exam May 2018

Question 8.1

1. Find the solutions in $\mathbb{Z}_{69}$ (if any) of the linear congruence $12x \equiv 30 \pmod{69}$.

2. Let us define the sets $A = \{a, b, c, d\}$ and $B = \mathcal{P}(A) \setminus \{A, \varnothing\}$, where $\mathcal{P}(A)$ is the power set of A. Is the partially ordered set $(B, \subseteq)$ a distributive lattice?

SOLUTION.

- The congruence $12x \equiv 30 \pmod{69}$ is equivalent to $6x \equiv 15 \pmod{69}$ because $\gcd(2, 69) = 1$. As $\gcd(3, 69) = 3$, this latter congruence can be written as $2x \equiv 5 \pmod{23}$.

  The previous congruence has a unique solution in $\mathbb{Z}_{23}$ because $\gcd(2, 23) = 1$ and $1 \mid 5$. It is now clear that $2 \times 12 = 24 \equiv 1 \pmod{23}$. Therefore, $2^{-1} \equiv 12 \pmod{23}$ and $x \equiv 5 \times 12 \pmod{23} \equiv 60 \pmod{23} \equiv 14 \pmod{23}$ is the claimed (unique) solution in $\mathbb{Z}_{23}$.

  To get the solutions in $\mathbb{Z}_{69}$, it suffices to rewrite the latter solution as the identity $x = 14 + 23\,k$ for any $k \in \mathbb{Z}$. If we now take $k = 3p$, $k = 3p + 1$, and $k = 3p + 2$ with $p \in \mathbb{Z}$, we obtain the three solutions of the original linear congruence in $\mathbb{Z}_{69}$: $x \equiv 14 \pmod{69}$, $x \equiv 37 \pmod{69}$, and $x \equiv 60 \pmod{69}$.

- No, because $(B, \subseteq)$ is not a lattice. For instance, given $\{a\}, \{b\} \in B = \mathcal{P}(A) \setminus \{A, \varnothing\}$, there is no $\inf(\{a\}, \{b\})$ because $\text{minor}(\{a\}, \{b\}) = \varnothing$.

Question 8.2 Let us consider the complete graph $K_3 = (V_3, E_3)$ with $V_3 = \{a, b, c\}$ and $E_3 = \{\{a, b\}, \{b, c\}, \{c, a\}\}$. Let us now consider the set A of all spanning subgraphs of K_3:

$$A = \{G = (V_3, E) \colon E \subseteq E_3\} .$$

What is the cardinality $|A|$ of the set A? We now consider the poset $(A, \preceq)$, where the order relation $\preceq$ is given by:

$$\text{For any } G_1, G_2 \in A, \qquad G_1 \preceq G_2 \quad \Leftrightarrow \quad G_1 \text{ is a subgraph of } G_2.$$

Justify whether the poset $(A, \preceq)$ has the structure of a Boolean algebra or not.

SOLUTION.

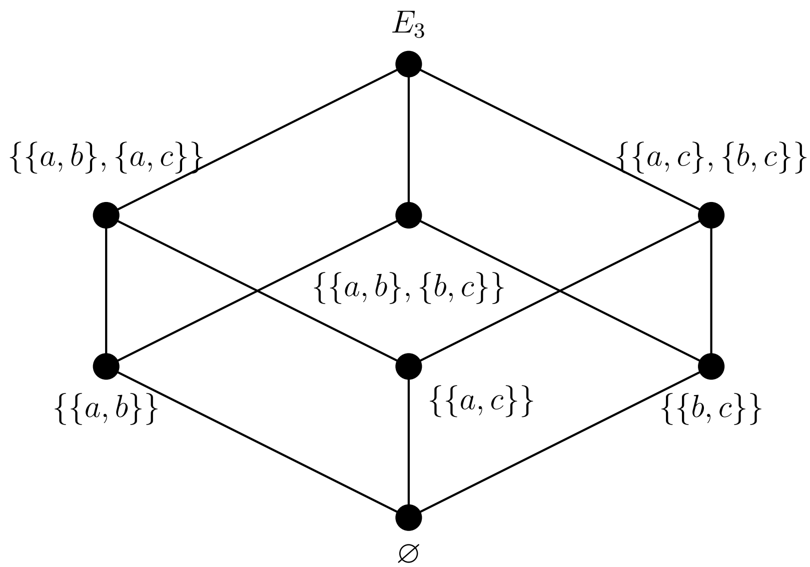
Because that each spanning subgraph $H = (V_3, E) \in A$ de $K_3 = (V_3, E_3)$ corresponds to a unique subset $E \subseteq E_3$ and *vice versa*, the cardinality of A should be equal to that of the power set $\mathcal{P}(E_3)$:

$$|A| \;=\; |\mathcal{P}(E_3)| \;=\; 2^{|E_3|} \;=\; 2^3 \;=\; 8\,.$$

By using this bijection, the order relation can be rewritten as follows: for every $G_1 = (V_3, F_1)$ and $G_2 = (V_3, F_2)$ belonging to A,

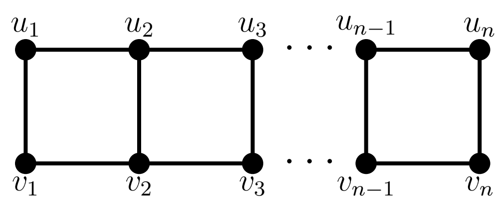
$$G_1 \preceq G_2 \quad \Leftrightarrow \quad F_1 \subseteq F_2\,.$$

The Hasse diagram of this poset is



The poset $(\mathcal{P}(E_3), \subseteq)$ is a lattice with $\sup(A, B) = A \cup B$ and $\inf(A, B) = A \cap B$. Moreover, it is also a distributive, bounded and complemented lattice (the complementary of any element A is given by $\overline{A} = E_3 \setminus A$). All these results are included in the course notes. Therefore, $(\mathcal{P}(E_3), \cup, \cap, \setminus, \varnothing, E_3)$ is a Boolean algebra.

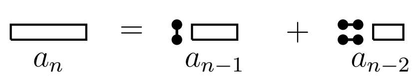
Question 8.3 Let G_n be the graph with $2n$ vertices shown below:



A perfect matching of a graph with $2p$ vertices is a spanning subgraph formed by p disjoint edges. Compute the number of perfect matchings a_n of G_n by using recurrence relations.

Solution.

The initial conditions are easily computed: $a_1 = 1$ () y $a_2 = 2$ ( and ).
The recurrence relation can be obtained as follows:



In other words,

$$a_n = a_{n-1} + a_{n-2}\,, \quad n \geq 3\,, \quad a_1 = 1\,, \quad a_2 = 2\,.$$

This recursion is identical to Fibonacci's one $f_n = f_{n-1} + f_{n-2}$, but with distinct initial conditions. Fibonacci's sequence is $(f_n)_{n \geq 1} = (1, 1, 2, 3, 5, \ldots)$, therefore the initial conditions of our problem correspond to f_2 and f_3, respectively. These arguments show that the final solution is given by $a_n = f_{n+1}$:

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right], \quad n \geq 1\,.$$

9 Final exam June 2018

Question 9.1 **Note:** results may contain numbers, factorials, or binomial coefficients.
In a meeting there are n men and n women. How many distinct n couples can be formed with all the attendees if each couple consists in a man and a woman? How many distinct n couples can be formed with all the attendees if each couple can be of any possible type?

SOLUTION.

The first part can be solved as follows. We first order the n men on a line and keep their positions fixed. Then, we sequentially place the n women in front of the men. There are $n!$ ways of placing the women, and therefore, there are $n!$ man-woman couples.

The second part is nothing but computing the number of partitions of type $\underbrace{(2, 2, \ldots, 2)}_{n \text{ couples}}$ we can obtain from a set with $2n$ distinct objects. The solution is

$$\frac{(2n)!}{2^n \, n!}$$

because, even though there are $(2n)!$ distinct permutations of the $2n$ objects, the ordering within each couple is unimportant (hence, we have to divide by a factor 2 for each one of the n couples), and, as the order of the couples is irrelevant, we have to divide by the number of permutations of such couples ($n!$).

Question 9.2 Prove that there is no plane and connected graph satisfying that each vertex has at least degree 8, and each face is bounded by at least 8 edges.

SOLUTION.

Let us assume that such graph $G = (V, E)$ exists. The hand-shacking lemma implies that

$$\sum_{x \in V} d(x) \;=\; 2|E| \;\geq\; 8|V| \quad \Rightarrow \quad |E| \;\geq\; 4|V|\,.$$

As G is plane and connected, its dual graph G^* exists. If we apply the hand-shacking lemma to G^*, we obtain

$$\sum_{r \in R} d_r \;=\; 2|E| \;\geq\; 8R \quad \Rightarrow \quad |E| \;\geq\; 4R\,.$$

Finally, as G is plane and connected, we can use Euler's formula

$$|V| - |E| + R \;=\; 2\,, \quad 2 \;\leq\; |V| - |E| + \frac{1}{4}|E| \;=\; |V| - \frac{3}{4}|E|\,.$$

Therefore,

$$|E| \;\leq\; \frac{4}{3}|V| - \frac{8}{3} \;<\; 2|V|\,.$$

The set of simultaneous solutions for these two inequalities $|E| \geq 4|V|$ and $|E| < 2|V|$ is empty. Then, the initial hypothesis should be false, so such a graph G cannot exist.

Question 9.3 **Note:** results may contain numbers, factorials, or binomial coefficients. How many distinct integer solutions does the equation $x_1 + x_2 + x_3 + x_4 = 21$ have if each variable x_i is constrained to take values in the set $\{0, 1, 2, 3, 4, 5, 6\}$? `Hint:` generating functions could be useful.

<u>Solution.</u>

The generating function $f_i(x)$ associated to each variable x_i is the same for all $i = 1, 2, 3, 4$, and it is given by

$$f_i(x) \;=\; 1 + x + x^2 + \cdots + x^6 \;=\; \frac{1 - x^7}{1 - x}\,.$$

The generating function encoding the full problem is

$$F(x) \;=\; \prod_{i=1}^{4} f_i(x) \;=\; \frac{(1 - x^7)^4}{(1 - x)^4} \;=\; \left[\sum_{k=0}^{4} \binom{4}{k} (-x^7)^k\right] \left[\sum_{k=0}^{\infty} \binom{-4}{k} (-x)^k\right]$$

where we have used Newton's binomial formula to expand $(1 - x^7)^4$, and the generalized version of Newton's formula to express $(1 - x)^{-4}$. Then,

$$F(x) \;=\; (1 - 4x^7 + 6x^{14} - 4x^{21} + x^{28}) \sum_{k=0}^{\infty} \binom{k+3}{3} x^k\,.$$

The sought solution is

$$[x^{21}]\, F(x) \;=\; \underbrace{\binom{24}{3}}_{k=21} - 4 \underbrace{\binom{17}{3}}_{k=14} + 6 \underbrace{\binom{10}{3}}_{k=7} - 4 \underbrace{\binom{3}{3}}_{k=0} \;=\; 20\,.$$

Question 9.4 Compute the remainder of dividing 3^{1492} by 20.

<u>Solution.</u>

As $\gcd(3, 20) = 1$, we can use Euler's theorem to simplify this expression

$$3^{\Phi(20)} \;\equiv\; 1 \pmod{20}\,.$$

Because $20 = 2^2 \times 5$, the value of $\Phi(20)$ is given by

$$\Phi(20) \;=\; 20\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) \;=\; 8\,.$$

In addition, we have that $1492 = 186 \times 8 + 4$, so

$$3^{1492} \;=\; \left(3^8\right)^{186} 3^4 \;\equiv\; 3^4 \pmod{20} \;\equiv\; 81 \pmod{20} \;\equiv\; 1 \pmod{20}\,.$$

Therefore, the sought remainder is $3^{1492} \mod 20 = 1$.

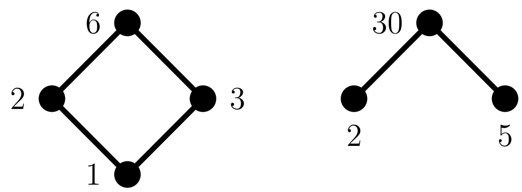
Question 9.5 Let X be the set $X = \{1, 2, 3, 6\} \times \{2, 5, 30\}$. We define on X the lexicographic order relation $\mathcal{R}$

$$(a, b)\,\mathcal{R}\,(c, d) \quad \Leftrightarrow \quad ((a \neq c) \wedge (a \mid c)) \vee ((a = c) \wedge (b \mid d))$$

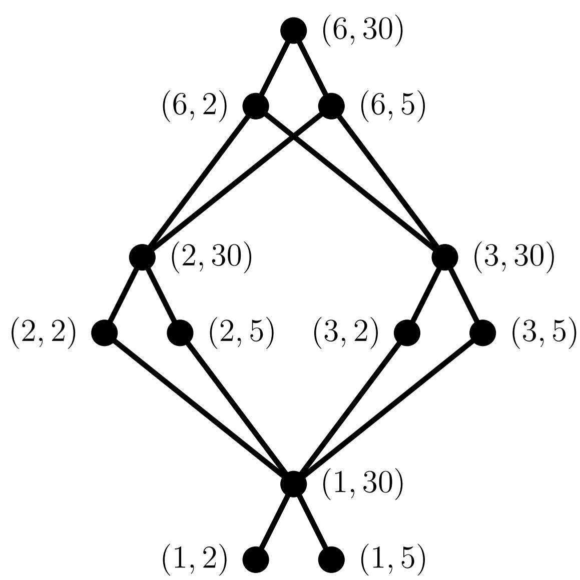
where $\wedge$ corresponds to the logical AND, and $\vee$ to the logical OR. Find the Hasse diagram of the partially ordered set $(X, \mathcal{R})$.

<u>SOLUTION.</u>

This is the lexicographic product of two posets: $(\{1, 2, 3, 6\}, |)$, and $(\{2, 5, 30\}, |)$. The corresponding Hasse diagrams are, respectively



The Hasse diagram of $(X, \mathcal{R})$ is obtained by substituting each vertex of the Hasse diagram of $(\{1, 2, 3, 6\}, |)$ by the Hasse diagram of $(\{2, 5, 30\}, |)$. The result is:



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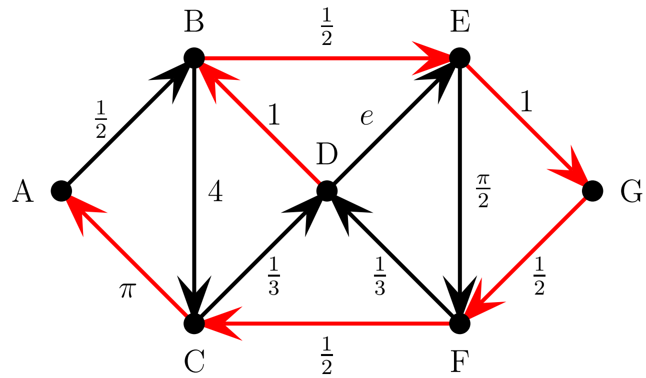
10 Mid-term exam April 2019

Question 10.1 Let us consider the following directed and weighted graph



Using an algorithm from graph theory, compute the shortest path and its corresponding distance from D to A. **Remark:** some of the weights contain the standard numerical constants π and e.

SOLUTION.

Using Dijkstra's algorithm, we obtain the table

D	$\boxed{(0,\text{D})}$	*	*	*	*	*	*
A	∞	∞	∞	∞	∞	$(\frac{7}{2}+\pi,\text{C})$	$\boxed{(\frac{7}{2}+\pi,\text{C})}$
B	(1,D)	$\boxed{(1,\text{D})}$	*	*	*	*	*
C	∞	(5,B)	(5,B)	(5,B)	$(\frac{7}{2},\text{F})$	$\boxed{(\frac{7}{2},\text{F})}$	*
E	(e,D)	$(\frac{3}{2},\text{B})$	$\boxed{(\frac{3}{2},\text{B})}$	*	*	*	*
F	∞	∞	$(\frac{3+\pi}{2},\text{E})$	(3,G)	$\boxed{(3,\text{G})}$	*	*
G	∞	∞	$(\frac{5}{2},\text{E})$	$\boxed{(\frac{5}{2},\text{E})}$	*	*	*

Therefore the shortest distance is $d(D,A) = \dfrac{7}{2}+\pi$ and the shortest path is (D,B,E,G,F,C,A), which is colored in red in the figure. We have used that $e \approx 2.71828$ y $\pi \approx 3.14159$, as it is well-known from calculus.

Question 10.2 Compute the number of distinct subsets containing 10 elements that can be taken from the set $X = \{x \in \mathbb{N}\colon 1 \leq x \leq p\}$ with $p > 10$, and such that each subset contains at least two consecutive elements of X. **Remark:** the results may contain numbers, factorials, or binomial coefficients; but not expressions like C_r, V_r, $V_{r,k}$, $\mathrm{CR}_{m,n}$, et cetera, which have not been defined in this course.

SOLUTION.

The number of sought solutions is equal to the number of distinct subsets of cardinality 10 that can be taken from the set X minus the number of distinct subsets of 10 elements that can be taken from X, and such that each subset does not contain any pair of consecutive elements.

The solution for the first case is $\binom{p}{10}$. The second case is equivalent to place 10 black balls and $p-10$ white balls in such a way that there are consecutive black balls. The black balls correspond to the chosen elements of X, and the white balls to elements not chosen in the subset. If we place the $p-10$ white balls on a line, there are $p-10+1=p-9$ places where we can place the black balls. In each one of these possible places, we can either not place any ball or just place a single black ball (but we cannot place more than one black balls). Therefore, out of the possible $p-9$ places, we have to choose 10. The number of distinct ways of performing such operation is $\binom{p-9}{10}$. The final solution is

$$\binom{p}{10} - \binom{p-9}{10}.$$

Question 10.3 Solve using combinatorial techniques the following recursion

$$a_n \;=\; 2\,a_{n-1} \;+\; 6\,n\,2^n\,, \quad n \geq 2\,, \qquad a_1 \;=\; 18\,.$$

Solution.

This is linear recurrence of order 1, with constant coefficients and not homogeneous. Its general solution is the sum of the general solution for the homogeneous recursion plus a particular solution of the non-homogeneous one.

- Solution of the homogeneous recursion: $a_n = 2a_{n-1}$. Its characteristic polynomial is $x = 2$; therefore, its general solution is

$$a_n \;=\; A\,2^n\,.$$

- Particular solution of then non-homogeneous recursion. As the non-homogeneous term is of the form $2^n\,P_1(n)$ where $P_1(n) = 6n$ is a polynomial of degree 1, there is a theorem that ensures that the form of this particular solution is $a_n = n(C+Dn)\,2^n = (Cn+Dn^2)\,2^n$. The values of the constants C and D are obtained by substituting this general expression into the non-homogeneous recursion:

$$(Cn+Dn^2)2^n \;=\; 2(Cn-C+Dn^2-2Dn+D)\,2^{n-1}+6n2^n\,.$$

  If we divide by 2^n, we get the polynomial

$$(6-2D)n+D-C=0 \quad \Rightarrow \quad D \;=\; C \;=\; 3\,.$$

  Therefore, the particular solution is

$$a_n \;=\; 3n(1+n)\,2^n\,.$$

- The general solution of the non-homogeneous recursion is

$$a_n \;=\; A\,2^n+3n(1+n)\,2^n\,.$$

  The constant A is determined by the initial condition $a_1 = 18$. However, it is easier to compute a_0 from the original recurrence $a_1 = 18 = 2a_0 + 12$; hence $a_0 = 3$. So $a_0 = 3 = A$, and the final solution is

$$a_n \;=\; 3\,(n^2+n+1)\,2^n\,, \quad n \geq 1\,.$$

Question 10.4 Compute the number of perfect matchings of the graph K_{2n} with $n \in \mathbb{N}$.

SOLUTION.

One way of solving this problem is to use the principle of the product. The first task consists in choosing the first edge of the perfect matching. There are $2n(2n-1)$ ways of doing that task. The second task is to choose the second edge, and there are $(2n-2)(2n-3)$ possible ways of doing this. If we follow this procedure until we run out of edges (there are n consecutive tasks in total), the result is

$$2n(2n-1)(2n-2)(2n-3)\cdots 2\cdot 1 \;=\; (2n)!$$

However, this result is not right: this number is larger than the true number of perfect matchings. On one side, the edges are not directed, so any permutation of its two elements produce the same perfect matching. Hence, we should divide the previous result by $(2!)^n = 2^n$. On the other side, the order for choosing the edges is also not relevant; so we should also divide by $n!$ to take into account this effect. The final result is

$$\frac{(2n)!}{2^n\, n!}\,.$$

This problem is equivalent to find the number of partitions of a set of $2n$ distinct elements into n subsets, each of them with 2 elements. The argument is the same as above, and also the result

$$\frac{(2n)!}{2^n\, n!}\,.$$

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11 Final exam May 2019

Question 11.1

- Is 127 a prime number?
- Solve the linear congruence $32\,x \equiv 39 \pmod{127}$.

SOLUTION.

- Let us assume that 127 is not a prime number, therefore, there should be a prime divisor p of 127 such that $p \le \sqrt{127}$. As $12^2 = 144$, the possible candidates belong to the set $A = \{2, 3, 5, 7, 11\}$. It is clear that $2 \nmid 127$, $3 \nmid 127$, and $5 \nmid 127$. In addition, $127 \equiv 57 \pmod 7 \equiv 1 \pmod 7$, so $7 \nmid 127$. Finally, $127 \equiv 17 \pmod{11} \equiv 6 \pmod{11}$, so $11 \nmid 127$. Therefore, as none of the candidates is a divisor of 127, the initial hypothesis is false and 127 is a prime number.

- Obviously, $\gcd(32, 127) = 1$ and $1 \mid 39$, hence there exists a unique solution $\pmod{127}$ to this congruence. One possible method to solve it is to obtain the multiplicative inverse of 32 (mod 127) (this inverse exists because 127 and 32 are relatively prime). As $32 \cdot 4 = 128 \equiv 1 \pmod{127}$, then the multiplicative inverse of 32 (mod 127) is $32^{-1} \equiv 4 \pmod{127}$. Therefore,

$$32 \cdot 32^{-1} \cdot x \equiv 4 \cdot 39 \pmod{127} \equiv 156 \pmod{127},$$

which implies that the sought solution is

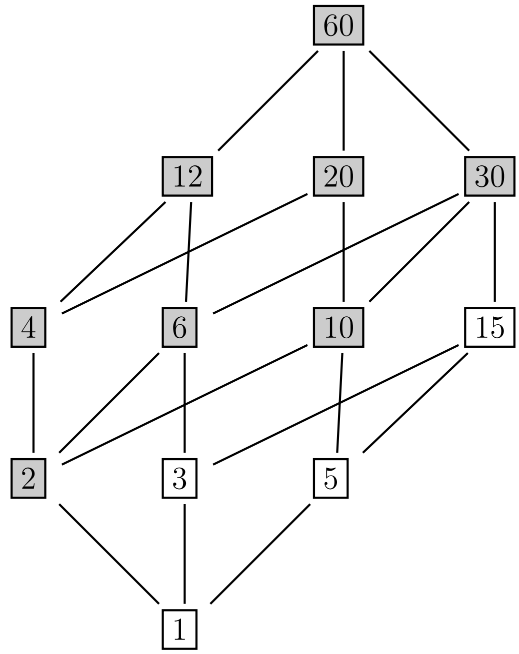
$$x \equiv 29 \pmod{127}.$$

Question 11.2 Let D_n be the set of positive integer divisors of $n \in \mathbb{N}$. Let us consider the partially ordered set $(D_{60}, |\,)$.

- Compute $|D_{60}|$.
- Find the Hasse diagram associated to $(D_{60}, |\,)$.
- If $C = D_{60} \setminus D_{15}$, compute $\sup(C)$ and $\inf(C)$.

<u>SOLUTION.</u>

- As $60 = 2^2\, 3\, 5$, each divisor d of 60 has the form $d = 2^a\, 3^b\, 5^c$ with $0 \leq a \leq 2$, $0 \leq b, c \leq 1$. If each task is to choose the power a, b, c in a sequential way, the principle of the product guarantees that $|D_{60}| = 3 \cdot 2 \cdot 2 = 12$. The sought set is $D_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$.
- The most efficient method to build the Hasse diagram is to do it layer by layer, in such a way that each layer contains the elements of D_{60} whose decomposition in prime factors has a fixed number $0 \leq k \leq 4$ of terms. The result is given by the following figure. The white nodes correspond to the subset D_{15}, while the gray nodes, to the subset C.



- If we focus on the gray nodes (that correspond to the subset C), we have that $\text{major}(C) = \{60\}$ and $\text{minor}(C) = \{2, 1\}$. Hence, $\sup(C) = \min(\text{major}(C)) = 60$ and $\inf(C) = \max(\text{minor}(C)) = 2$.

Question 11.3 Compute the number of strings of length n that can be formed with the elements of the set $\{0, 1, 2\}$, and such that they contain an odd number of zeros.

- Prove that, if a_n is the number of the above-mentioned strings, then it satisfies the recursion $a_n = a_{n-1} + 3^{n-1}$, for any $n \geq 2$.
- Solve this recursion.

<u>Solution.</u>

Let A_n be the set that contains the strings of length n that can be formed with the elements of $\{0, 1, 2\}$, and such that they contain an odd number of zeros. Then, $a_n = |A_n|$. Let B_n be the set that contains the strings of length n that can be formed with the elements of $\{0, 1, 2\}$, and such that they contain an even number of zeros. Let us define $b_n = |B_n|$. Obviously, if we use the principle of the product, we obtain the relation

$$a_n + b_n = 3^n, \quad n \geq 1.$$

The recurrence is obtained as follows: As every string in A_n starts with 0, 1, or 2, then

$$\underbrace{\boxed{}}_{a_n} = 0\,\underbrace{\boxed{}}_{b_{n-1}} + 1\,\underbrace{\boxed{}}_{a_{n-1}} + 2\,\underbrace{\boxed{}}_{a_{n-1}}$$

When the string starts with 0, to fill in the empty part of the string we can take any element of B_{n-1}; in the same way, if the string starts with 1 or 2, we can take any element of A_{n-1}. Therefore,

$$a_n \;=\; 2\,a_{n-1} + b_{n-1} \;=\; a_{n-1} + 3^{n-1}\,, \quad n \geq 2\,.$$

As the recursion is of order 1, we need one initial condition: $a_1 = 1$, as $A_1 = \{0\}$.

This is a non-homogeneous recurrence, so its general solution is the sum of the general solution of homogeneous part and a particular solution of the full non-homogeneous recursion.

The homogeneous part of the recurrence is $a_n = a_{n-1}$; its characteristic polynomial is $x = 1$. Then, its general solution is $a_n = A$.

The particular solution of the full non-homogeneous recursion has the form $a_n = B\,3^n$. The value of B is computed by substituting this form into the recursion

$$B\,3^n \;=\; B\,3^{n-1} + 3^{n-1}\,.$$

If we divide by 3^{n-1}, we get

$$3B \;=\; B + 1 \quad \Rightarrow \quad B \;=\; \frac{1}{2}\,.$$

Then, the particular solution is $a_n = \dfrac{1}{2}\,3^n$.

The general solution of this recurrence is

$$a_n \;=\; A + \frac{1}{2}\,3^n\,.$$

If we use the initial condition $a_1 = 1 = A + \dfrac{3}{2}$, we obtain $A = -\dfrac{1}{2}$. Therefore, the sought solution is

$$a_n \;=\; \frac{1}{2}\,(3^n - 1)\,, \quad n \geq 1\,.$$

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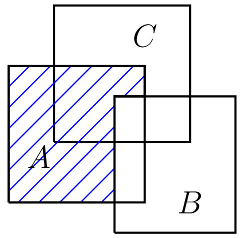
12 Final exam June 2019

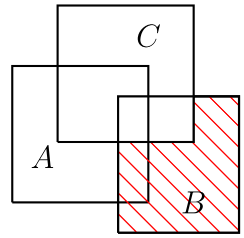
Question 12.1 Justify whether the set difference operation $A \setminus B$ is associative or not.

<u>Solution.</u>

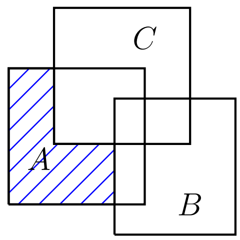
We want to decide whether, given 3 arbitrary sets A, B and C, the expression $(A \setminus B) \setminus C$ is equal or not to the expression $A \setminus (B \setminus C)$. We are going to use Venn diagrams to solve this question:



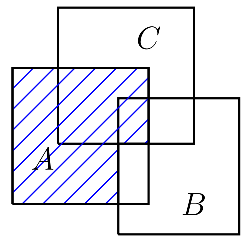
$(A \setminus B)$



$(B \setminus C)$



$(A \setminus B) \setminus C$



$A \setminus (B \setminus C)$

It is clear that the set difference is not an associative operation, as both sets differ in every element in $A \cap C$.

Question 12.2 Let $\mathcal{M}$ be the set of matrices M of dimension 10×10, whose entries are integers satisfying $m_{ij} \in \{0, 1, 2, 3, 4, 5, 6\}$ and such that its trace $\mathrm{tr}(M) = \sum_{i=1}^{10} m_{ii} = 30$. Compute the cardinal $|\mathcal{M}|$ using combinatorial methods. **Warning:** The results might only contain numbers, factorials or binomial coefficients.

<u>Solution.</u>

This question can be solved sequentially. The first task consists in placing the non-diagonal elements of the matrix. As there are $10^2 - 10 = 90$ non-diagonal entries in each matrix, and each of them can take 7 distinct values, the principle of the product ensures that

$$N_{\text{non-diagonal}} \;=\; 7^{90}\,.$$

The second task consists in placing the diagonal elements with the condition that their sum should be equal to 30. In other words, if we name $x_i = m_{ii}$, we look for the integer solutions of the linear equation

$$x_1 + x_2 + \cdots + x_{10} \;=\; 30\,, \qquad \text{with } x_i \in \{0, 1, \ldots, 6\}.$$

The best way to solve this part is by using generating functions. If $f(x)$ is the generating function that encodes each one of the variables (notice that all of them have the same range), then:

$$f(x) \;=\; 1 + x + x^2 + \cdots + x^6 \;=\; \frac{1 - x^7}{1 - x}\,.$$

The generating function that encodes the full problem is

$$F(x) \;=\; f(x)^{10} \;=\; \frac{(1 - x^7)^{10}}{(1 - x)^{10}}\,.$$

The numerator can be expanded via Newton's theorem:

$$(1-x^7)^{10} \;=\; \sum_{k=0}^{10} \binom{10}{k} (-x^7)^k \,.$$

The denominator can be expanded using the generalized Newton's theorem:

$$(1-x)^{-10} \;=\; \sum_{k=0}^{\infty} \binom{k+9}{9} x^k \,.$$

The final solution to this pat is $N_{\text{diagonal}} = [x^{30}]F(x)$:

$$\begin{aligned} N_{\text{diagonal}} &= \sum_{k=0}^{\infty} (-1)^k \binom{10}{k} \binom{39-7k}{9} \\ &= \binom{10}{0}\binom{39}{9} - \binom{10}{1}\binom{32}{9} + \binom{10}{2}\binom{25}{9} - \binom{10}{3}\binom{18}{9} + \binom{10}{4}\binom{11}{9} \\ &= 17538157\,. \end{aligned}$$

Notice that the terms with $k \leq 4$ actually count, as if $k \geq 5$, $7k \geq 35 > 30$.

The product principle ensures that the final result is:

$$|\mathcal{M}| \;=\; N_{\text{non-diagonal}} \times N_{\text{diagonal}} \;=\; 7^{90} \times 17\,538\,157 \;\approx\; 2.0082 \times 10^{83}\,.$$

Question 12.3 We define on the set $A = \mathbb{N} \cup \{0\}$ the following equivalence relation:

$$a \,\mathcal{R}\, c \quad \Leftrightarrow \quad \lceil \sqrt{a}\, \rceil \;=\; \lceil \sqrt{c}\, \rceil \,.$$

- Compute the equivalence classes.
- Compute the cardinal of each equivalence class.
- Find the quotient set, and show that this set is isomorphic to A.

<u>Solution.</u>

- First, it is obvious that
$$a \,\mathcal{R}\, b \quad \Leftrightarrow \quad f(a) \;=\; f(b)$$
where $f\colon A \to A$ is a function given by $f(x) = \lceil \sqrt{x}\, \rceil$ any whose image is A (because f only returns integer values, $f(0) = 0$, and f is monotonously non-decreasing). This implies that $A/\mathcal{R} \simeq A$.

- The first equivalence classes are
  1. $[0]_{\mathcal{R}} = \{0\}$.
  2. $[1]_{\mathcal{R}} = \{1\}$.

3. $[4]_{\mathcal{R}} = \{2, 3, 4\}$.
4. $[9]_{\mathcal{R}} = \{5, 6, 7, 8, 9\}$.

The class $[0]_{\mathcal{R}}$ is trivial; for any other class, if we choose as its representative a perfect square $n^2 > 0$, such class will contain all the integers between the previous perfect square (excluded) and n^2. In other words,

$$[n^2]_{\mathcal{R}} = \begin{cases} \{0\} & \text{if } n = 0, \\ \{k \in \mathbb{N} \colon (n-1)^2 + 1 \le k \le n^2\} & \text{if } n \ge 1. \end{cases}$$

The last expression can be simplified:

$$[n^2]_{\mathcal{R}} = \begin{cases} \{0\} & \text{if } n = 0, \\ \{k \in \mathbb{N} \colon n^2 - 2n + 2 \le k \le n^2\} & \text{if } n \ge 1. \end{cases}$$

- The cardinal of each equivalence class can be computed from the previous equation:

$$|[n^2]_{\mathcal{R}}| = \begin{cases} 1 & \text{if } n = 0, \\ 2n - 1 & \text{if } n \ge 1. \end{cases}$$

This result can be double-checked easily with the results obtained at the beginning of this question.

- The quotient set is

$$A/\mathcal{R} = \{[n^2]_{\mathcal{R}} \colon n \in A\} \simeq A\,.$$

because each equivalence class is associated to a unique element of A, and *vice versa*.

Question 12.4 Let us consider the n-cube graphs Q_n with $n \ge 2$. Justify your answers to the following questions

- For which values of n are these graphs bipartite?
- For which values of n are these graphs Eulerian?
- Assuming that none of these graphs contains a cycle of length 3, for which values of n are these graphs planar?

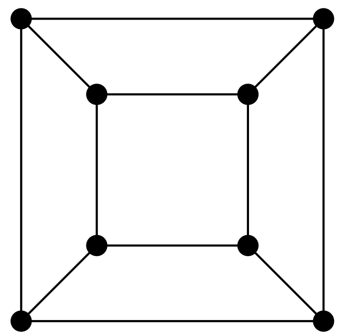
SOLUTION.

- Every n-cube $Q_n = (V_n, E_n)$ is bipartite for $n \ge 2$ because we can split the vertex set into two disjoint subsets $V_n = V_e \cup V_o$ (with $V_o \cap V_e = \varnothing$), where V_e contains all the vertices associated to bit strings with an even number of zeros. Analogously, V_o contains all the vertices associated to bit strings with an odd number of zeros. As two vertices are neighbors if and only if their corresponding bit strings differ in exactly one bit, then it is impossible that ant edge $e \in E_n$ can be adjacent to two vertices a, b that belong to the same subset, either V_e or V_o.

- As the Q_n are connected and regular graphs, and the degree of every vertex is n, only those graphs with even n can be Eulerian:

$$Q_n \begin{cases} \text{is Eulerian} & \text{if } n \geq 2 \text{ is even,} \\ \text{is not Eulerian} & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

- The cases Q_2 and Q_3 are obviously planar. $Q_2 \simeq C_4$, which is planar. Q_3 can also be drawn in such a way that its edges do not cross each other:



Let us now consider the graph $Q_n = (V_n, E_n)$ with $n \geq 4$. If this graph Q_n were planar, as it is a simple and connected graph that contains more than 3 vertices and has no cycle of length 3, then the following inequality should be satisfied:

$$|E_n| \; < \; 2|V_n| - 4\,.$$

But $|V_n| = 2^n$ and, by using the hand-shaking lemma, $|E_n| = n\,2^{n-1}$. Therefore, if Q_n were planar, it would satisfy $n\,2^{n-1} < 2^{n+1} - 4$. If we divide by 2^{n-1}, we get

$$n \; < \; 4 - 4 \times 2^{1-n} \; = \; 4 - 2^{3-n}\,.$$

As $n \geq 4$, the previous result implies that

$$n \; < \; 4$$

which contradicts $n \geq 4$. Therefore, the hypothesis about the planarity of Q_n is false: Q_n is not planar for any $n \geq 4$, The solution is

$$Q_n \text{ is } \begin{cases} \text{planar} & \text{if } n = 2, 3, \\ \text{non-planar} & \text{if } n \geq 4. \end{cases}$$
