CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Functions: properties and continuity

Problem 4.2.

1) The domain is $\mathbb{R}$, the image is $\mathbb{Z}$, and f is continuous in $\mathbb{R} \setminus \mathbb{Z}$.
2) The domain is $\mathbb{R}$, the image is $[0, 1)$, and f is continuous in $\mathbb{R} \setminus \mathbb{Z}$.
3) The domain is $\mathbb{R}$, the image is $[0, 1)$, and f is continuous in $\mathbb{R} \setminus \mathbb{Z}$.
4) The domain is $\mathbb{R}$, the image is $\mathbb{R}$, and f is continuous in $\mathbb{R}$.
5) The domain is $\mathbb{R}\setminus\{0\}$, the image is $\mathbb{Z}$, and f is continuous in $\mathbb{R}\setminus\left\{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots\right\}$.

Problem 4.3.

1) The function f is continuous in $\mathbb{R}$.
2) The function f is continuous in $\mathbb{R}$.
3) The function f is continuous in $\mathbb{R} \setminus \{0\}$.
4) The function f is continuous in $[-1, 1]$.
5) The function f is continuous in $(4/9, +\infty)$.
6) The function f is continuous in $(4/9, 1]$.

Problem 4.4.

- The function f is continuous in $\mathbb{R} \setminus \{0\}$ as product of continuous functions ($\cos(1/x)$ is the continuous composition of continuous functions). At $x = 0$, the function f is also continuous because $\lim_{x\to 0} f(x) = f(0) = 0$ (the limit can be calculated taking into account that $\cos(1/x)$ is bounded).

- The function g is continuous in $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as composition of continuous functions. At $x = 1$, the function g is also continuous because $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x) = g(1) = 0$ (here, we need to use lateral limits as $g(x)$ is defined by two different expressions on the left and the right of $x = 1$). At $x = 0$, the function g is not continuous since $\lim_{x\to 0^+} g(x) = 0$ and $\lim_{x\to 0^-} g(x) = 1$, hence the limit of g when $x \to 0$ does not exist (again, we need to use lateral limits).

- The function h is continuous in $\mathbb{R}\backslash\{0\}$ being a sum of continuous functions. At $x = 0$, the function h is not continuous since we have that $\lim_{x\to 0^+} h(x) = 1$ and $\lim_{x\to 0^-} h(x) = 5$, hence the limit of h when $x \to 0$ does not exist (we need to use lateral limits).

Problem 4.5. In order to prove that f is bounded in the interval $[-7, 5]$ (closed and bounded), we can show that it is continuous. Then, f is continuous in $[-7, 0) \cup (0, 5]$ as composition of continuous functions. At $x = 0$, f is also continuous because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 0$ (we need to use lateral limits). Thus, we can conclude that f is continuous, hence bounded, in $[-7, 5]$.

Problem 4.6. Apply Bolzano's theorem to the function $f(x) = \cos x - x$ on, for instance, $[0, 1]$.

Problem 4.7. (A) Apply Bolzano's theorem to the function $F(x) = f(x) - x$ on $[0, 1]$. (B) Apply Bolzano's theorem to the function $F(x) = f(x) - g(x)$ on $[x_1, x_2]$.

Functions: derivative

Problem 5.1. The function f is continuous at $x = 0$ because $\lim_{x\to 0} f(x) = f(0) = 0$ (the limit can be calculated taking into account that $\sin(1/x)$ is bounded). Also, f is differentiable at $x = 0$ because $f'(0) = \lim_{h\to 0} [f(0 + h) - f(0)]/h = 0$.

Problem 5.2. The function f is continuous in $[-2, -1]$ and differentiable in $(-2, -1)$.

Problem 5.3.

1. $f'(x) = \dfrac{6x - 7}{2\sqrt{3x^2 - 7x - 2}}$.

2. $f'(x) = x\sin(x)\left(2\tan(x) + x + \dfrac{x}{\cos^2(x)}\right).$

3. $f'(x) = \dfrac{2}{3(x-1)^{2/3}(x+1)^{4/3}}.$

4. $f'(x) = \dfrac{-\sin(x)\,\cos\left(\sqrt{1+\cos(x)}\right)}{2\sqrt{1+\cos(x)}}.$

5. $f'(x) = \dfrac{2}{x} + \dfrac{1}{\tan(x)} - \dfrac{1}{2x+2}.$

Problem 5.4. An equation for the desired tangent line is $y = -2x + 7$.

Problem 5.5.

1. For $x \neq 0$, f is differentiable and we have $f'(x) = 1/(3x^{2/3})$. At $x = 0$, f is not differentiable.

2. For $x \neq 0$, f is differentiable and we have $f'(x) = 1/x$. At $x = 0$, f is not differentiable.

Problem 5.6. The function is differentiable in $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as defined in terms of differentiable functions (an expression for $f'(x)$ in the three intervals is obtained by differentiating the corresponding elementary functions). At $x = 0$, we have that $\lim_{h\to 0^+}[f(0+h) - f(0)]/h = \lim_{h\to 0^-}[f(0+h) - f(0)]/h = 0$, thus $f(x)$ is differentiable and $f'(0) = 0$ (we need to use lateral derivatives). At $x = 1$, the function is not continuous (thus not differentiable) as $\lim_{x\to 1^+} f(x) = \pi/4$ and $\lim_{x\to 1^-} f(x) = 0$, hence the limit of $f(x)$ when $x \to 1$ does not exist.

Problem 5.7. By applying the chain rule we get the following expressions.

1. $h'(x) = f'(g(x))\,g'(x)\,e^{f(x)} + f(g(x))\,f'(x)\,e^{f(x)}.$

2. $h'(x) = \dfrac{-f'(x) - 2g(x)g'(x)}{(f(x) + g^2(x))\ln^2(f(x) + g^2(x))}.$

3. $h'(x) = \dfrac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}.$

4. $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{f^2(x) + g^2(x)}.$

5. $h'(x) = \dfrac{g'(x)}{g(x)} - f'(x)\tan(f(x))\,.$

Problem 5.8. For each case, calculate the necessary derivatives, substitute their expressions into the left-hand side of the equation, and verify that an equality is obtained.

Problem 5.9. For each case, calculate the first derivative of the function appearing in the left-hand side of the equality and realize that such derivative is equal to zero for all given values of x. As a consequence, the value of the involved function must be the same for all indicated values of x. Thus, in order to prove the equality, we can evaluate it at some convenient x.

Problem 5.10. The slope of the desired tangent line from the right is given by $\text{lím}_{h\to 0^+}\,[f(0+h)-f(0)]\,/h = 0$, thus such line is parallel to the x-axis. The slope of the desired tangent line from the left is given by $\text{lím}_{h\to 0^-}\,[f(0+h)-f(0)]\,/h = 1$, thus such line is parallel to the line $y = x$. As a consequence, the two tangent lines form an angle equal to $\pi/4$.

Applications of the derivative

Problem 6.1.

- For $x \neq 0$, we have $f_1'(x) = k\,x\,|x|^{k-2}$ and $f_2'(x) = k\,|x|^{k-1}$.
- Using the definition of differentiability, we get $f_1'(0) = f_2'(0) = 0$.
- From $0 \leq |f(x)| \leq |x|^k$ for $x = 0$, we deduce that $f(0) = 0$. In addition, we have $0 \leq |f(x)\,/\,x| \leq |x|^{k-1}$ for each $x \neq 0$ in a neighborhood of $x_0 = 0$, which implies that $f'(0) = \text{lím}_{x\to 0} f(x)\,/\,x = 0$ (by the sandwich theorem, noting that $k > 1$).

Problem 6.2. The function $f(x)$ is continuous in $\mathbb{R}$ with $f(1) = 1$ and differentiable in $\mathbb{R}$ with $f'(1) = -1$. In the interval $[0, 2]$, all assumptions of Lagrange's mean-value theorem are satisfied and the points of the theorem statement (namely, points $c \in (0, 2)$ where $f'(c) = -1/2$) are $c = 1/2,\ \sqrt{2}$.

Problem 6.3. The function $f(x)$ satisfies all assumptions of Rolle's theorem except for the differentiability in the whole interval $(-1, 1)$. Indeed, $f'(0)$ does not exist, hence the mentioned theorem cannot be applied.

Problem 6.4. The desired values are $h(0) = 0$, $h'(0) = 0$, and $h''(0) = 2$. All of them are obtained by first noting that $\text{lím}_{x\to 0} h(x)/x^2 = f(0) = 1$ (since $f(x)$ is continuous at $x = 0$) and then using this limit in the definition of continuity and (twice) differentiability of $h(x)$ at $x = 0$ (in the calculation of $h''(0)$ it may be convenient to apply l'Hôpital's rule).

Problem 6.5. As $f(x)$ is continuous at $x = 0$, we have $\text{lím}_{x\to 0} f(2x^3) = f(0)$. Then, since the given limit is finite, it must be $f(0) = 0$. The value $f'(0) = 5/2$ is obtained by using the given limit in the definition of differentiability of $f(x)$ at $x = 0$. Finally, we have

$$\operatorname*{lím}_{x\to 0} \frac{f(f(2x))}{3f^{-1}(x)} = \operatorname*{lím}_{x\to 0} \frac{f(f(2x))}{f(2x)} \operatorname*{lím}_{x\to 0} \frac{f(2x)}{x} \operatorname*{lím}_{x\to 0} \frac{x}{3f^{-1}(x)},$$

where each one of the limits on the right can be calculated after transformation (by a suitable change of variable) into $\text{lím}_{t\to 0} f(t)/t = f'(0) = 5/2$. The value of the above limit is then $125/12$.

Problem 6.6.

THEOREM 1. Let $f(x)$ vanish at $k \geq 2$ points in $[x_1, x_2]$, say $\bar{x}_1, \ldots, \bar{x}_k$. Thus, we get $k - 1$ intervals $[\bar{x}_1, \bar{x}_2], [\bar{x}_2, \bar{x}_3], \ldots, [\bar{x}_{k-1}, \bar{x}_k]$ where $f(x)$ satisfies the assumptions of Rolle's theorem. Hence, there are $k - 1$ points in $[x_1, x_2]$ (one per interval) where $f'(x)$ must be zero.

THEOREM 2. The statement is proved by repeatedly applying THEOREM 1 above to $f(x), f'(x), f''(x), \ldots, f^{(k-1)}(x)$.

Problem 6.7. Write the given equations as $f(x) = 0$, where $f(x)$ is a function to be conveniently defined in each case. Then, analyze the sign of $f'(x)$ in the indicated intervals.

a) 1 real solution.

b) 1 real solution.

c) 2 real solutions.

d) 1 real solution.

e) No real solutions.

Problem 6.8.

- The value of the limit is 1/2 (use l'Hôpital's rule two times).
- The value of the limit is 1 (use l'Hôpital's rule two times).

Extra problem. Consider $f(x) = x^{1+\frac{1}{x}}$. Then, by using the Lagrange's mean-value theorem, the given limit can be calculated as

$$\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} x^{\frac{1}{x}} \left(1 + \frac{1}{x} - \frac{\ln(x)}{x}\right) = \lim_{x\to+\infty} x^{\frac{1}{x}} \lim_{x\to+\infty} \left(1 + \frac{1}{x} - \frac{\ln(x)}{x}\right) = 1\,.$$