CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Real numbers: inequalities, subsets; methods of proof

Problem 1.1.

1. $x \in (-\infty, -2) \cup (0, 1) \cup (1, +\infty)$
2. $x \in [0, 25]$
3. $\forall\, x \in \mathbb{R}$
4. $x \in [-5, 11]$
5. $x \in \left(\dfrac{3}{2}, 2\right) \cup \left(2, \dfrac{5}{2}\right)$
6. $x \in (-\infty, 2] \cup [3, +\infty)$
7. $x \in (-3, 0) \cup (5, +\infty)$
8. $x \in (-7, -4) \cup (-1, +\infty)$
9. $x \in (-\infty, 1) \cup (2, +\infty)$
10. $x = \dfrac{-1 \pm \sqrt{21}}{2}$
11. $x \in (-\sqrt{2} + 1, 1) \cup (1, 1 + \sqrt{2})$

Problem 1.2.

1. $\sup(A_1) = 1$, $\text{ínf}(A_1) = 0$, $\text{máx}(A_1) = 1$, no $\text{mín}(A_1)$
2. $\sup(A_2) = 1$, $\text{ínf}(A_2) = -1$, $\text{máx}(A_2) = 1$, $\text{mín}(A_2) = -1$
3. $\sup(A_3) = \sqrt{2}$, $\text{ínf}(A_3) = 0$, no $\text{máx}(A_3)$, $\text{mín}(A_3) = 0$
4. no $\sup(A_4)$, no $\text{ínf}(A_4)$, no $\text{máx}(A_4)$, no $\text{mín}(A_4)$
5. $\sup(A_5) = \dfrac{\sqrt{5} - 1}{2}$, $\text{ínf}(A_5) = \dfrac{-\sqrt{5} - 1}{2}$, no $\text{máx}(A_5)$, no $\text{mín}(A_5)$

6. $\sup(A_6) = 0$, $\inf(A_6) = \dfrac{-\sqrt{5}-1}{2}$, no $\max(A_6)$, no $\min(A_6)$

7. $\sup(A_7) = 1 + \dfrac{1}{2}$, $\inf(A_7) = -1$, $\max(A_7) = 1 + \dfrac{1}{2}$, no $\min(A_7)$

8. $\sup(A_8) = 3$, $\inf(A_8) = \dfrac{1}{3}$, no $\max(A_8)$, no $\min(A_8)$

9. $\sup(A_9) = d$, $\inf(A_9) = a$, no $\max(A_9)$, no $\min(A_9)$

10. $\sup(A_{10}) = \dfrac{7}{10}$, $\inf(A_{10}) = 0$, $\max(A_{10}) = \dfrac{7}{10}$, no $\min(A_{10})$

Problem 1.3.

1. Proof by contradiction.

2. For instance, by the method of induction.

3. For instance, by the method of induction.

4. *Hint*: prove the three inequalities on the right separately, using properties of the square root if necessary.

5. *Hint*: find for which values of x and y the inequality on the right is satisfied.

6. *Hint*: in order to prove $\Longrightarrow$, take the square of both sides of the equality on the left; in order to prove $\Longleftarrow$, distinguish between three cases, namely $x = 0$ or $y = 0$, $x > 0$ and $y > 0$, $x < 0$ and $y < 0$.

Sequences of real numbers

Problem 2.1.

a) Bounded; not monotone; not convergent.

b) Bounded; not monotone; convergent to 0 (for instance, use either the limit properties or the sandwich theorem).

c) Bounded; monotone; convergent to 1.

d) Bounded; not monotone; convergent to $1/2$.

e) Bounded; not monotone; convergent to x (for instance, use the sandwich theorem).

f) Bounded; not monotone; convergent to $1/2$ (for instance, use the sandwich theorem).

g) Bounded; monotone; convergent to π (it may be useful to check the behavior of $\ln(a_n)$).

h) Bounded; monotone for $n \geq 2$; convergent to $1/2$ (for instance, consider the formula for the sum of the first n natural numbers).

i) Bounded; monotone; convergent to x (distinguish cases $x = y$ and $x \neq y$).

Problem 2.2.

a) The sequence converges to 0.

b) The sequence converges to 0 (for instance, use the sandwich theorem).

c) The sequence diverges.

d) The sequence converges to 0.

e) The sequence converges to $1/3$.

Problem 2.3.

a) The sequence can be written as $a_n = \sqrt{3\,a_{n-1}}\ \ \forall n \geq 2,\ \ a_1 = \sqrt{3}$. Use the method of induction to prove that a_n is bounded. In addition, the sequence is increasing and converges to 3.

b) Use the method of induction to prove that a_n is bounded. In addition, the sequence is increasing and converges to $20/3$.

c) Use the method of induction to prove that a_n is bounded. In addition, the sequence is decreasing and converges to $1/3$.

d) Use the method of induction to prove that a_n is bounded. In addition, the sequence is increasing and converges to 3.

Problem 2.4.

a) The limit is 1 (use that $\sqrt[n]{n} \to 1$ as $n \to \infty$).

b) The limit is 1 (use that $\sqrt[n]{n} \to 1$ as $n \to \infty$).

c) The limit is $e^{1/3}$ (use that $(1 + a_n)^{1/a_n} \to e$ as $n \to \infty$, if $a_n \to 0$).

Series of real numbers

Problem 3.1.

a) Convergent telescoping series (as indicated, the sum of the series is 1).

b) Convergent (for instance, use the comparison test with $b_k = 1/k^2$).

c) Divergent (for instance, use the comparison test with $b_k = 1/k$).

d) Convergent (for instance, use the limit comparison test with $b_k = 1/k^{3/2}$).

e) Convergent (for instance, use the limit comparison test with $b_k = 1/k^2$).

f) Convergent (for instance, use the limit comparison test with $b_k = (2/3)^k$).

g) Convergent (for instance, use the limit comparison test with $b_k = 1/k^3$).

h) Divergent (for instance, use the comparison test with $a_k = 1/k$).

i) Convergent (for instance, use the limit comparison test with $b_k = 1/k^{3/2}$).

j) Divergent (for instance, use the comparison test with $a_k = 1/k$).

Problem 3.2.

a) Alternating series: convergent by Leibniz test.

b) Convergent. For instance, consider the series of $|a_k|$ and use the comparison test with $b_k = (1/5)^k$; then, convergence of $\sum_{k=1}^{\infty} |a_k|$ implies convergence of $\sum_{k=1}^{\infty} a_k$.

c) Alternating series: convergent by Leibniz test.

d) Convergent by ratio test.

e) Divergent by root test.

f) Convergent by root test.

g) Convergent by ratio test.

h) Divergent telescoping series.

Problem 3.3.

1) Convergent for $|b| > 1$ and $a > 0$, divergent for $|b| < 1$ ($b \neq 0$) and $a > 0$ (using the ratio test). Divergent for $b = \pm 1$ ($a > 0$) as the general term of the series does not tend to zero.

2) Convergent for all values of $b \in \mathbb{R}$ by ratio test.

3) Convergent for $|\alpha| < \sqrt[3]{7}/2$ and divergent for $|\alpha| > \sqrt[3]{7}/2$ (using the ratio test). For $\alpha = \sqrt[3]{7}/2$ the series is convergent by Leibniz test, while for $\alpha = -\sqrt[3]{7}/2$ it is divergent (for instance, use the limit comparison test with $b_k = 1/k^{2/3}$).