CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Functions: properties and continuity

Problem 4.1. Sketch the graph of the following functions.

1) $f(x) = \sqrt{x-3}\,.$
2) $f(x) = x^2 - 4x + 5\,.$
3) $f(x) = \left|\, (x-2)^3 + 1 \,\right|\,.$

Problem 4.2. Identify domain and image of the following functions. Then, for each case, study the continuity of $f(x)$ and sketch its graph.

1) $f(x) = \lfloor x \rfloor$ ($\lfloor x \rfloor$ equals the largest integer $\leq x$).
2) $f(x) = x - \lfloor x \rfloor\,.$
3) $f(x) = \sqrt{x - \lfloor x \rfloor}\,.$
4) $f(x) = \lfloor x \rfloor + \sqrt{x - \lfloor x \rfloor}\,.$
5) $f(x) = \left\lfloor \dfrac{1}{x} \right\rfloor\,.$

Problem 4.3. Study the continuity of the following functions.

1) $f(x) = \dfrac{e^x + 2\cos(x) - 8x + 5}{e^x + \sin^2(x) + 5}\,.$
2) $f(x) = \sqrt{x^4+3} + e^{-x^2+\cos(x)} \sin(4x^5 + 3x^2 + 2x - 5 + \cos(x)) + 2\arctan(3^x - 5)\,.$
3) $f(x) = e^{4/x} + x^4 - 7\,.$
4) $f(x) = \arccos^5(x)\,.$
5) $f(x) = (x-3)\ln(9x-4)\,.$
6) $f(x) = (4x^6 + 3x^3 - 2x + 6)\ln(x) + \ln(9x-4)\arccos(x)\,.$

Problem 4.4. Analyze the continuity of the given functions.

$$f(x) = \begin{cases} x\cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$g(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ x^2 - x & \text{if } 0 < x < 1 \\ \cos\left(\pi|2-x^2|\right) + 1 & \text{if } x \geq 1 \end{cases}$$

$$h(x) = \begin{cases} x^3 - x + 5 & \text{if } x < 0 \\ e^x + \sin(x) & \text{if } x \geq 0 \end{cases}$$

Problem 4.5. Prove that the following function is bounded.

$$f(x) = \begin{cases} e^{1/x} & \text{if } -7 \leq x < 0 \\ 0 & \text{if } 0 \leq x \leq 5 \end{cases}$$

Problem 4.6. Prove that the equation $\cos(x) = x$ has *some* solution.

Problem 4.7. Prove the following theorems.

(A) Let $f : [0,1] \to [0,1]$ be a continuous function. Then, there exists $x_0 \in (0,1)$ such that $f(x_0) = x_0$.

(B) Let $f,\ g : [x_1, x_2] \to \mathbb{R}$ be continuous functions such that $f(x_1) > g(x_1)$ and $f(x_2) < g(x_2)$. Then, there exists $x_0 \in (x_1, x_2)$ verifying $f(x_0) = g(x_0)$.