Extraordinary exam solution

Problema 1. Using either **the inclusion-exclusion principle** or **generating functions**, find the number of solutions to the following distribution:

$$\begin{cases} n_1 + n_2 + n_3 + \ldots + n_7 = 32, \\ 2 \le n_i \le 8. \end{cases}$$

Solución. (Using **the inclusion-exclusion principle**) We apply the change of variable $n_i \mapsto 2 + y_i$. The distribution is now

$$\begin{cases} y_1 + y_2 + y_3 + \ldots + y_7 = 18, \\ 0 \le y_i \le 6. \end{cases}$$

Let S be the set of solutions to $y_1 + y_2 + \ldots + y_7 = 18$ with $y_i \ge 0$ (i.e. no upper constraint). Let A_k, with $k \in \{1, 2, \ldots, 7\}$, the subset of S for which $y_k \ge 7$ (i.e. for which *at least* y_k has way too many elementos).

The number of solutions to the original distribution can be found by computing $|S \setminus (A_1 \cup A_2 \cup \ldots \cup A_7)| = |S| - |A_1 \cup A_2 \cup \ldots \cup A_7|$. Using the inclusion-exclusion principle, we have

$$\begin{aligned} \#\text{Solutions} = |S| - (&|A_1| + \ldots + |A_7| - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_6 \cap A_7| + \\ &+ |A_1 \cap A_2 \cap A_3| + \ldots + |A_5 \cap A_6 \cap A_7| - \ldots) \,. \end{aligned}$$

The intersections of three or more subsets A_i will be empty as they require three or more y_i to be at least 7 and that will make the equation to be unfeasible. Moreover, due to symmetry, we have that $|A_1| = |A_i|$ for $i \in \{1, 2, \ldots, 7\}$ and $|A_1 \cap A_2| = |A_i \cap A_h|$ for $i \ne j \in \{1, 2, \ldots, 7\}$. Thus, taking into account the number of possible intersections,

$$\#\text{Solutions} = |S| - \binom{7}{1}|A_1| + \binom{7}{2}|A_1 \cap A_2|.$$

The cardinals we are left to compute are "easy" distributions, and they are

$$|S| = \binom{18+(7-1)}{18}, \ |A_1| = \binom{11+(7-1)}{11}, \ |A_1 \cap A_2| = \binom{4+(7-1)}{4}.$$

Hence the answer is

$$\boxed{\#\text{Solutions} = \binom{24}{18} - \binom{7}{1}\binom{17}{11} + \binom{7}{2}\binom{10}{4}}.$$

Solución. (Using **generating functions**) Let $f_i(x) = x^2 + x^3 + \ldots + x^8$ be the functions that represent the value to put into the position n_i. Observe that $f_i(x) = f_1(x)$ for every $i \in \{1,2,\ldots,7\}$. Moreso, $f_1(x) = x^2 \frac{1-x^7}{1-x}$.

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the function that encodes the solution to the distributions for every number of objects. In order to include te particular constraints, we define $F(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_7(x) = \left[f_1(x)\right]^7 = x^{14}\left(1-x^7\right)^7 \frac{1}{(1-x)^7}$.

First using the generalized binomial theorem, and then the Newton binomial theorem, we get

$$\begin{aligned}
F(x) &= x^{14}\left(1-x^7\right)^7 \frac{1}{(1-x)^7} = \\
&= x^{14}\left(1-x^7\right)^7 \sum_{n=0}^{\infty} \binom{n+(7-1)}{n} x^n = \\
&= x^{14} \left[\sum_{n=0}^{7} \binom{7}{n} (-1)^n x^{7n}\right] \left[\sum_{n=0}^{\infty} \binom{n+6}{n} x^n\right]
\end{aligned}$$

The number of solutions to the distribution will be encoded in the coefficient a_{32}. Since $32 = 14 + 0 + 18 = 14 + 7 + 11 = 14 + 14 + 4$, the answer will then be

$$\boxed{\#\text{Solutions} = a_{32} = \binom{7}{0}\binom{24}{18} - \binom{7}{1}\binom{17}{11} + \binom{7}{2}\binom{10}{4}}.$$

Problema 2. Solve this recurrence relation:

$$\begin{cases} a_n = 4a_{n-2} + (-2)^{2n}, \\ a_0 = 7/3; \ a_1 = 16/3. \end{cases}$$

Solución. This is a non-homogeneous linear recurrence relation of second order. The solution is $a_n = a_n^h + a_n^p$, where a_n^h solves the related homogeneous recurrence relation and a_n^p is a particular solution.

The characteristic equation is $x^2 = 4$. Hence $a_n^h = A2^n + B(-2)^n$, with $A, B \in \mathbb{R}$.

The non-homogeneous term is $t_n = (-2)^{2n} = \left((-2)^2\right)^n = 4^n$. Thus, the particular solution has the shape $a_n^p = n^0 4^n \left[p_0\right]$. As this is a particular solution, it checks $a_n^p = 4a_{n-2}^p + t_n$. I.e. $p_0 4^n = 4p_0 4^{n-2} + 4^n$, hence $p_0 = 4/3$.

We then have $a_n = A2^n + B(-2)^n + 4^{n+1}/3$. The initial conditions will determine A and B:

$$\begin{cases} a_0 = A + B + \dfrac{4}{3} = \dfrac{7}{3} \\ a_1 = 2A - 2B + \dfrac{16}{3} = \dfrac{16}{3} \end{cases}$$

That is,

$$\begin{cases} A + B = 1 \\ A - B = 0 \end{cases}$$

From which we can gather that $A = B = 1/2$.

In conclusion,

$$\boxed{a_n = 2^{n-1} + \frac{(-2)^n}{2} + \frac{4^{n+1}}{3}}. \tag{1}$$

Problema 3. Let $V = \{20, 28, 30, 70, 75, 99\}$. Let $G = (V, E)$ be the simple graph where two vertices $a, b \in V$ are adjacent IF, AND ONLY IF, $a \neq b$ and $\gcd(a, b) \neq 1$. Let $\overline{G} = (V, \overline{E})$ be its complementary graph.

1. Find $|E|$ and compute, in a justified way, $|\overline{E}|$. Find a spanning tree for $\overline{G}$.
2. Check whether G has an Euler tour, Euler trail, Hamiltonian cycle, and/or a Hamiltonian path. In case it does, write the walk that proves it.
3. Choose an edge from G and compute the number of perfect matchings that contain that edge.

Solución. Factorize the numbers: $20 = 2^2 \cdot 5$, $28 = 2^2 \cdot 7$, $30 = 2 \cdot 3 \cdot 5$, $70 = 2 \cdot 5 \cdot 7$, $75 = 3 \cdot 5^2$, y $99 = 3^2 \cdot 11$.

The vertices' degrees are: $d(20) = 4$, $d(28) = 3$, $d(30) = 5$, $d(70) = 4$, $d(75) = 4$, y $d(99) = 2$.

By the handshake theorem, we get that $\boxed{|E| = 11}$. Moreso, as $|E| + \left|\overline{E}\right| = 6 \cdot 5/2 = 15$ (both together will define a K_6), we arrive to $\boxed{\left|\overline{E}\right| = 4}$.

Since a tree has one edge less than vertices, and that $\overline{G}$ has 6 vertices but 4 edges, $\boxed{\text{There's no spanning tree}}$.

As there are two odd-degree vertices, the graph has an Euler trail. A possible trail is: $28 \to 30 \to 70 \to 28 \to 20 \to 30 \to 75 \to 70 \to 20 \to 75 \to 99 \to 30$.

Even more, it is Hamiltonian. A possible cycle is $99 \to 30 \to 70 \to 28 \to 20 \to 75 \to 99$.

Lastly, there's some freedom for choosing the edge, and the answer will depend on that. For instance, (and not limited to these examples):

- If we choose the edge that connects 30 with 75, since 99 is only connected to these two, there will be no perfect matching containing the edge 30—75.

- If we choose the edge 75—70, there is a single perfect matching containing it, since 99 must connect to 30, which leads to 28 being connected with 20.

Problema 4. Answer the following:

1. Prove, using **induction**, that $3^{2^n+1} \equiv 3 \pmod{10}$ for every natural $n \geq 2$. IMPORTANT: 3^{2^n+1} is not the same as 3^{2n+1} nor $3^{2^{n+1}}$.

2. Compute 3^{1030} in $\mathbb{Z}_{10}$.

Solución. If $n = 2$, we have that $3^{2^2+1} = 3^{4+1} = 3^5 = 3^4 \cdot 3 = 81 \cdot 3 \equiv 1 \cdot 3 \pmod{10}$. This is the base step.

Assume that $3^{2^n+1} \equiv 3 \pmod{10}$. Let's see what happens with $3^{2^{n+1}+1}$:

$$3^{2^{n+1}+1} = 3^{2^n \cdot 2+1} = 3^{2^n+2^n+1} = 3^{2^n} \cdot {\color{red}3^{2^n+1}} \equiv 3^{2^n} \cdot 3 = {\color{red}3^{2^n+1}} \equiv 3 \pmod{10},$$

where we have used the Induction Hypothesis twice (see the terms in red).

By inductions (weak version), we conclude $3^{2^n+1} \equiv 3 \pmod{10}$ for every natural $n \geq 2$

For the second half of the exercise, we might use Euler's theorem (which will require us to compute $\Phi(10) = \Phi(2) \cdot \Phi(5) = 1 \cdot 4 = 4$) or, using the previous result. Notice that $1030 = 1024 + 6 = 2^{10} + 6 = \left(2^{10} + 1\right) + 5$. Hence

$$3^{1030} = 3^{2^{10}+1} \cdot 3^5 \equiv 3 \cdot 3^5 = 3 \cdot 3^{2^2+1} \equiv 3 \cdot 3 \equiv \boxed{9 \pmod{10}}.$$