

CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Functions: derivative

Problem 5.1. Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- Is $f(x)$ continuous at $x = 0$?
- Is $f(x)$ differentiable at $x = 0$?

Problem 5.2. Study the continuity and differentiability of the function

$$f(x) = \sqrt{x+2} \arccos(x+2).$$

Problem 5.3. Find the first derivative of the following functions.

1. $f(x) = \sqrt{3x^2 - 7x - 2}.$
2. $f(x) = x^2 \sin(x) \tan(x).$
3. $f(x) = \sqrt[3]{\frac{x-1}{x+1}}.$
4. $f(x) = \sin\left(\sqrt{1 + \cos(x)}\right).$
5. $f(x) = \ln\left(\frac{x^2 \sin(x)}{\sqrt{1+x}}\right).$

Problem 5.4. Find an equation for the *tangent line* to the graph of the function

$$g(x) = \frac{x+1}{x-1}$$

at $x = 2$.

Problem 5.5. Study the differentiability of the following functions and calculate the corresponding derivatives.

1. $f(x) = x^{1/3}$.
2. $f(x) = \ln|x|$.

Problem 5.6. Consider the function

$$f(x) = \begin{cases} \cos(x) & \text{if } x \leq 0, \\ 1 - x^2 & \text{if } 0 < x < 1, \\ \arctan(x) & \text{if } x \geq 1, \end{cases}$$

and say whether it is differentiable in \mathbb{R} . In addition, find an expression for $f'(x)$ where it exists.

Problem 5.7. Let f, g be differentiable functions in \mathbb{R} . Then, write an expression for the first derivative of the functions below.

1. $h(x) = f(g(x)) e^{f(x)}$.
2. $h(x) = \frac{1}{\ln(f(x) + g^2(x))}$.
3. $h(x) = \sqrt{f^2(x) + g^2(x)}$.
4. $h(x) = \arctan\left(\frac{f(x)}{g(x)}\right)$.
5. $h(x) = \ln(g(x) \cos(f(x)))$.

Problem 5.8. Let c , c_1 , and c_2 be constants. For each case, prove that the function $f(x)$ is solution of the corresponding *differential* equation.

- | | |
|---|-------------------------|
| 1. $f(x) = c/x$; | $xf' + f = 0$. |
| 2. $f(x) = x \tan(x)$; | $xf' - f - f^2 = x^2$. |
| 3. $f(x) = c_1 \sin(3x) + c_2 \cos(3x)$; | $f'' + 9f = 0$. |
| 4. $f(x) = c_1 e^{3x} + c_2 e^{-3x}$; | $f'' - 9f = 0$. |
| 5. $f(x) = c_1 e^{2x} + c_2 e^{5x}$; | $f'' - 7f' + 10f = 0$. |
| 6. $f(x) = \ln(c_1 e^x + c_2 e^{-x}) + c_2$; | $f'' + (f')^2 = 1$. |

Problem 5.9. Prove the following equalities.

1. $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ (for $x > 0$).
2. $\arctan\left(\frac{1+x}{1-x}\right) - \arctan(x) = \frac{\pi}{4}$ (for $x < 1$).
3. $2 \arctan(x) + \arcsin\left(\frac{2x}{1+x^2}\right) = \pi$ (for $x > 1$).

HINT: calculate the first derivative of the function on the left of each equality.

Problem 5.10. Calculate the angle formed by the tangent lines from the right and from the left, at $x = 0$, to the graph of the function

$$f(x) = \begin{cases} \frac{x}{1 + e^{1/x}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$