CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Real numbers: inequalities, subsets; methods of proof

Problem 1.1. Find all values of $x \in \mathbb{R}$ satisfying the following conditions.

$$
\begin{aligned}
&1) \quad x^2 + \frac{2}{x} > 3\,. \\
&2) \quad |\sqrt{x} - 2| \le 3\,. \\
&3) \quad -8 \le |x-5| - |x-3| \le 8\,. \\
&4) \quad |x-3| \le 8\,. \\
&5) \quad 0 < |x-2| < \frac{1}{2}\,. \\
&6) \quad x^2 - 5x + 6 \ge 0\,. \\
&7) \quad x^3(x+3)(x-5) > 0\,. \\
&8) \quad \frac{2x+8}{x^2+8x+7} > 0\,. \\
&9) \quad |x-1| + |x-2| > 1\,. \\
&10) \quad |x-1||x+2| = 3\,. \\
&11) \quad |x^2 - 2x| < 1\,.
\end{aligned}
$$

Problem 1.2. Find supremum, infimum, maximum, and minimum (if they exist) of the following subsets of $\mathbb{R}$.

1) $A_1 = \{1/n \,:\, n \in \mathbb{N}\}$.
2) $A_2 = \{1/n \,:\, n \in \mathbb{Z},\ n \neq 0\}$.
3) $A_3 = \{x \in \mathbb{Q} \,:\, 0 \le x \le \sqrt{2}\}$.
4) $A_4 = \{x \in \mathbb{R} \,:\, x^2 + x + 1 \ge 0\}$.
5) $A_5 = \{x \in \mathbb{R} \,:\, x^2 + x - 1 < 0\}$.
6) $A_6 = \{x \in \mathbb{R} \,:\, x < 0,\ x^2 + x - 1 < 0\}$.
7) $A_7 = \{1/n + (-1)^n \,:\, n \in \mathbb{N}\}$.
8) $A_8 = \{x \in \mathbb{R} \,:\, 3x^2 - 10x + 3 < 0\}$.
9) $A_9 = \{x \in \mathbb{R} \,:\, (x-a)(x-b)(x-c)(x-d) < 0\,,\ a, b, c, d \in \mathbb{R}\,,\ a < b < c < d\}$.
10) $A_{10} = \{2^{-p} + 5^{-q} \,:\, p, q \in \mathbb{N}\}$.

Problem 1.3. Prove the following properties by the most appropriate method.

1) $\sqrt{2}$ is an irrational number.

2) $\displaystyle\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \quad r \in \mathbb{R}, \quad r \neq 1, \quad N \in \mathbb{N}.$

3) $\displaystyle\sum_{n=1}^{N} n = \frac{N(N+1)}{2}, \quad N \in \mathbb{N}.$

4) $0 < x < y \implies x < \sqrt{xy} < \dfrac{x+y}{2} < y \quad (x, y \in \mathbb{R}).$

5) $0 < x < y \implies \dfrac{x}{y} < \dfrac{x+k}{y+k}, \quad \forall k > 0 \quad (x, y \in \mathbb{R}).$

6) $|x+y| = |x| + |y| \iff xy \geq 0 \quad (x, y \in \mathbb{R}).$