CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Applications of the derivative

Problem 6.1. Consider $k \in \mathbb{R}$ and the functions

$$f_1(x) = |x|^k , \quad f_2(x) = x\,|x|^{k-1} .$$

- For $x \neq 0$, calculate $f_1'(x)$ and $f_2'(x)$.
- For $k > 1$, prove that both functions are differentiable at $x = 0$ and calculate $f_1'(0)$, $f_2'(0)$.
- Prove that, if $f(x)$ is a function verifying $|f(x)| \leq |x|^k$ for $k > 1$ and all x in a neighborhood of $x_0 = 0$, then $f(x)$ is differentiable at $x_0 = 0$. Finally, calculate $f'(0)$.

Problem 6.2. Analyze the continuity and differentiability of the function

$$f(x) = \begin{cases} (3 - x^2)/2 & \text{if } x < 1 , \\ 1/x & \text{if } x \geq 1 . \end{cases}$$

Can you apply the Lagrange's mean-value theorem in the interval $[0, 2]$? If you can, find the point(s) of the theorem statement.

Problem 6.3. The function $f(x) = 1 - x^{2/3}$ vanishes at $x = -1$ and $x = 1$. However, $f'(x) \neq 0$ for all $x \in (-1, 1)$. Explain this apparent contradiction of Rolle's theorem.

Problem 6.4. Let $h(x)$ be a continuous function in $\mathbb{R}$, with $h'(x)$ and $h''(x)$ also continuous in $\mathbb{R}$. Then, consider

$$f(x) = \begin{cases} h(x)/x^2 & \text{if } x \neq 0 , \\ 1 & \text{if } x = 0 . \end{cases}$$

Supposing that $f(x)$ is continuous in $\mathbb{R}$, calculate $h(0)$, $h'(0)$, and $h''(0)$.

Problem 6.5. Let $f(x)$ be a continuous function in $\mathbb{R}$, with $f'(x)$ also continuous in $\mathbb{R}$, such that

$$\lim_{x\to 0} \frac{f(2x^3)}{5x^3} = 1\,.$$

- Prove that $f(0) = 0$ and $f'(0) = 5/2$.
- Calculate $\lim\limits_{x\to 0} \dfrac{f(f(2x))}{3f^{-1}(x)}$ (supposing that f^{-1} exists).

Problem 6.6. Prove the following two theorems.

THEOREM 1. Let $f(x)$ be a differentiable function in $[x_1, x_2]$. If $f(x)$ has $k \geq 2$ roots in $[x_1, x_2]$, then $f'(x)$ has at least $k-1$ roots in the same interval.

THEOREM 2. Let $f(x)$ be k-times differentiable in $[x_1, x_2]$. If $f(x)$ has $k+1 \geq 2$ roots in $[x_1, x_2]$, then $f^{(k)}(x)$ has at least one root in the same interval.

Problem 6.7. Find the exact number of real solutions of the given equations.

$$\begin{aligned}
&a)\ \ x^7 + 4x = 3\,, \quad x \in \mathbb{R}\,.\\
&b)\ \ x^5 = 5x - 6\,, \quad x \in \mathbb{R}\,.\\
&c)\ \ x^4 - 4x^3 = 1\,, \quad x \in \mathbb{R}\,.\\
&d)\ \ \sin(x) = 2x - 1\,, \quad x \in \mathbb{R}\,.\\
&e)\ \ x^2 = \ln(1/x)\,, \quad x \in (1, +\infty)\,.
\end{aligned}$$

Problem 6.8. Calculate the following limits.

- $\lim\limits_{x\to 0} \dfrac{e^x - \sin(x) - 1}{x^2}$.
- $\lim\limits_{x\to 0^+} \dfrac{\ln(\sin(7x))}{\ln(\sin(x))}$.

Extra problem. Use the Lagrange's mean-value theorem to calculate the limit

$$\lim_{x\to +\infty} \left[(1+x)^{1+\frac{1}{1+x}} - x^{1+\frac{1}{x}} \right].$$