Extraordinary exam (English)

Problem 1. We have built a calculator with infinite memory that can properly do sums, differences, and multiplications with integer numbers. Sadly, the display was damaged and we can only see the last digit from the number being displayed (i.e. if $n = 1234567$, we can only see the 7).

1. Using induction, prove that, if $n \geq 2$ is a natural number, and we ask the calculator to compute 2^{2^n+1}, the damaged display will always show a 2. REMARK: the input for the calculator is `2^(2^(n) + 1)`.

2. What will the calculator show if we ask it to compute 2^{1032}?

Solution. As we are only interested in the last digit of the number, we are working on $\mathbb{Z}_{10}$. Thus, we want to justify that, for $n \geq 2$, we have $2^{2^n+1} \equiv 2 \pmod{10}$.

1. For $n = 2$, we have $2^{2^n+1} = 2^{4+1} = 2^5 = 32 \equiv 2 \pmod{10}$.

   We assume $2^{2^k+1} \equiv 2 \pmod{10}$. Then

   $$2^{2^{k+1}+1} \equiv 2^{2\cdot 2^k+1} \equiv 2^{2^k+2^k+1} \equiv {\color{red}2^{2^k+1}} \cdot 2^{2^k} \equiv {\color{red}2} \cdot 2^{2^k} \equiv {\color{blue}2^{2^k+1}} \equiv {\color{blue}2} \pmod{10},$$

   where we have used the induction hypothesis two times (marked in color, each time).

   By the induction principle, we can say that $2^{2^n+1} \equiv 2 \pmod{10}$ for every $n \geq 2$.

2. We can observe that $1032 = (1024+1)+7 = (2^{10}+1)+7$. Then $2^{1032} \equiv 2^{(2^{10}+1)+7} \equiv 2^{2^{10}+1} \cdot 2^7 \equiv 2 \cdot 2^7 \equiv 2^8 \pmod{10}$, for which we have used the results from above.

   As $2^8 = 256$, then $2^{1032} \equiv 6 \pmod{10}$.

Problem 2. Let $G_n = (V_n, E_n)$ be a simple graph, where $V_n = \{v_0, v_1, \ldots, v_n\}$ is the vertex set, and $E_n = \{\{v_0, v_k\}, 1 \le k \le n\} \cup \{\{v_k, v_{k+1}\}, 1 \le k \le n-1\}$ is the edge set.

1. For $n \ge 2$, compute $P_{G_n}(q)$ and find $\chi(G_n)$.

2. For $n \ge 3$, if we add the edge $\{v_1, v_n\}$ to E_n we get a graph isomorphic to W_n. Compute $P_{W_n}(q)$.

Solution. 1. Once we have represented the graph in an adequate way, we can conclude that

$$P_{G_n}(q) = \frac{P_{K_3}(q) P_{G_{n-1}}(q)}{P_{K_2}(q)} = \ldots = \frac{P_{K_3}(q)^{n-2} P_{G_2}(q)}{P_{K_2}(q)^{n-2}} \ ,$$

and, as $G_2 \simeq K_3$,

$$\boxed{P_{G_n}(q) =} \frac{P_{K_3}(q)^{n-1}}{P_{K_2}(q)^{n-2}} = \boxed{q(q-1)(q-2)^{n-1}} \, .$$

Moreover, we can say $\chi(G_n) = 3$ for every value of $n \ge 2$.

2. Using the contraction-deletion theorem on the added edge, we get that $P_{W_n}(q) = P_{G_n}(q) - P_{W_{n-1}}(q)$, which is a non-homogeneous linear recurrence relation of order 1. Since $W_3 \simeq K_4$, we have the initial condition $P_{W_3}(q) = q(q-1)(q-2)(q-3)$.

   The characteristic equation to the related homogeneous recurrence relation has -1 as a single root. Thus the homogeneous contribution to the solution is $A(-1)^n$, with $A \in \mathbb{R}$.

   On the other hand, if $q > 1$, a particular solution to the main recurrence relation will have the shape $B(q-2)^n$, with $B \in \mathbb{R}$. Replacing this expression in the recurrence relation and simplifying we get $B = q$.

   Using the initial conditions we have $q(q-1)(q-2)(q-3) = P_{W_3}(q) = A(-1)^3 + q(q-2)^3$. From which we can say: $A = q(q-2)[(q-2)^2 - (q-1)(q-3)] = q(q-2)$.

   We conclude that $P_{W_n}(q) = (-1)^n q(q-2) + q(q-2)^n$.

Problem 3. In the fictional Kingdom of Anheria they use the Erky as legal tender, and due to historical reasons, they only print bills of 6 and 70 Erkys.

1. The owner of an antiquities shop need to price-tag the different item that are being sold. What condition do these prices need to check in order to be purchasable using Erkys?

2. If we want to buy an item tagged at 400 Erkys, what kind of exchange must be done with the owner in order to pay this using Erkys?

3. If the owner has no money on him (to pay back whatever we might have overpaid), justify, **using the previous results**, whether it is possible to acquire the item or not. In case it is, in how many different ways can we pay for it?

Solution. Let x and y be the number of bills of 6 and 70 Erkys being used in a transaction, respectively. If k is the price of an item, the equation $6x + 70y = k$ must have a solution $x, y \in \mathbb{Z}$, and we consider that a positive value of x or y mean that we give the corresponding number of bills to the shop owner, whereas a negative value means that the owner gives them to us.

1. For the equation $6x + 70y = k$ to have an integer solution, we need the greatest common divisor of 6 and 70 to divide k. As this one is 2, the condition is that the prices must be even values.

2. One way to approach the problem is to obtain the value x as solution to the congruence equation $6x \equiv 400 \pmod{70}$. Solving the related Bézout Identity using Euclid's Lemma, we obtain one possible solution: $x \equiv 300 \equiv 20 \pmod{70}$.

   As for y, it can be worked out from the equation itself: $70y = 400 - 6x = 400 - 6 \cdot 20 = 280$, thus $y = 4$.

   All in all, $(x, y) = (20, 4)$ is solution. This means, it is enough to give the owner 20 bills of 6 Erkys and 4 bills of 70 Erkys.

3. From the answer above we can tell that que can pay for this item even if owner has no money at hand.

   If we had obtained another solution, like $(x, y) = (90, -2)$, it would be enough to find another solution knowing that $(x_k, y_k) = (90, -2) + k(35, -3)$, with $k \in \mathbb{Z}$, is also solution.

   In any case, there are only two valid solutions in which both x and y are non-negative, and those are $(x, y) = (20, 4)$ and $(x, y) = (55, 1)$. Thus, there would be two ways to make the payment.

Problem 4. Let X be a finite set with n elements (i.e. $|X| = n$). We know that its power set, $P(X)$, checks $|P(X)| = 2^n$.

1. Prove that $|P(X)| = 2^n$ using only elementary counting techniques (e.g. the product principle).

2. Prove that $|P(X)| = 2^n$ using Newton's binomial theorem.

Solution.

1. If we make an ordered list of the elements of X, like $x_1, x_2, \ldots, x_n$, we can encode each subset $A \subseteq X$ by means of a bit-string of length n, where the 1's and their positions represent which elements of X belong to the subset A. This encoding is bijective. In other words, there's a function $S : \{0,1\}^n \to P(X)$ that is bijective. Then, by the product principle, $|P(X)| = 2 \cdot 2 \overset{n)}{\cdots} 2 = 2^n$, which is the number of bit-strings of length n.

2. The number of subsets with k elements is $\binom{n}{k}$. As the number of elements contained by a subset is an exclusive property, we can apply the sum principle and get that

$$|P(X)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}.$$

   Hence,

$$\boxed{|P(X)| =} \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k \overset{(*)}{=} (1+1)^n = \boxed{2^n},$$

   where, in $(*)$, we have used Newton's binomial theorem.

Problem 5. Using **the inclusion-exclusion principle** or **generating functions**, find the number of solutions to the distribution below:

$$\begin{cases} n_1 + n_2 + n_3 + \ldots + n_7 = 32, \\ 2 \le n_i \le 8. \end{cases}$$

Solution. (Using **the inclusion-exclusion principle**) We apply the change of variable $n_i \mapsto 2 + y_i$. The distribution is now

$$\begin{cases} y_1 + y_2 + y_3 + \ldots + y_7 = 18, \\ 0 \le y_i \le 6. \end{cases}$$

Let S be the set of solutions to $y_1 + y_2 + \ldots + y_7 = 18$ with $y_i \ge 0$ (i.e. no upper constraint). Let A_k, with $k \in \{1, 2, \ldots, 7\}$, the subset of S for which $y_k \ge 7$ (i.e. for which *at least* y_k has way too many elementos).

The number of solutions to the original distribution can be found by computing $|S \setminus (A_1 \cup A_2 \cup \ldots \cup A_7)| = |S| - |A_1 \cup A_2 \cup \ldots \cup A_7|$. Using the inclusion-exclusion principle, we have

$$\begin{aligned} \#\text{Solutions} = |S| - (|A_1| + \ldots + |A_7| - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_6 \cap A_7| + \\ + |A_1 \cap A_2 \cap A_3| + \ldots + |A_5 \cap A_6 \cap A_7| - \ldots). \end{aligned}$$

The intersections of three or more subsets A_i will be empty as they require three or more y_i to be at least 7 and that will make the equation to be unfeasible. Moreover, due to symmetry, we have that $|A_1| = |A_i|$ for $i \in \{1, 2, \ldots, 7\}$ and $|A_1 \cap A_2| = |A_i \cap A_h|$ for $i \neq j \in \{1, 2, \ldots, 7\}$. Thus, taking into account the number of possible intersections,

$$\#\text{Solutions} = |S| - \binom{7}{1}|A_1| + \binom{7}{2}|A_1 \cap A_2|.$$

The cardinals we are left to compute are "easy" distributions, and they are

$$|S| = \binom{18 + (7-1)}{18}, \quad |A_1| = \binom{11 + (7-1)}{11}, \quad |A_1 \cap A_2| = \binom{4 + (7-1)}{4}.$$

Hence the answer is

$$\boxed{\#\text{Solutions} = \binom{24}{18} - \binom{7}{1}\binom{17}{11} + \binom{7}{2}\binom{10}{4}}.$$

Solution. (Using **generating functions**) Let $f_i(x) = x^2 + x^3 + \ldots + x^8$ be the functions that represent the value to put into the position n_i. Observe that $f_i(x) = f_1(x)$ for every $i \in \{1, 2, \ldots, 7\}$. Moreso, $f_1(x) = x^2 \frac{1-x^7}{1-x}$.

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the function that encodes the solution to the distributions for every number of objects. In order to include te particular constraints, we define $F(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_7(x) = \left[f_1(x)\right]^7 = x^{14}\left(1 - x^7\right)^7 \frac{1}{(1-x)^7}$.

First using the generalized binomial theorem, and then the Newton binomial theorem, we get

$$\begin{aligned} F(x) &= x^{14}\left(1-x^7\right)^7 \frac{1}{(1-x)^7} = \\ &= x^{14}\left(1-x^7\right)^7 \sum_{n=0}^{\infty} \binom{n+(7-1)}{n} x^n = \\ &= x^{14} \left[\sum_{n=0}^{7} \binom{7}{n} (-1)^n x^{7n}\right] \left[\sum_{n=0}^{\infty} \binom{n+6}{n} x^n\right] \end{aligned}$$

The number of solutions to the distribution will be encoded in the coefficient a_{32}. Since $32 = 14 + 0 + 18 = 14 + 7 + 11 = 14 + 14 + 4$, the answer will then be

$$\boxed{\#\text{Solutions} = a_{32} = \binom{7}{0}\binom{24}{18} - \binom{7}{1}\binom{17}{11} + \binom{7}{2}\binom{10}{4}}.$$