

CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Integrals: Fundamental Theorem of Calculus

Problem 9.1. The desired area is equal to 1 .

Problem 9.2. The desired area is equal to $2 - \ln(3)$.

Problem 9.3. We get

$$F(x) = \begin{cases} \sin(x) & \text{for } 0 \leq x \leq \pi/2, \\ 1 + \pi/2 - x & \text{for } \pi/2 < x \leq \pi. \end{cases}$$

In addition, by the Fundamental Theorem of Calculus we have that $F'(x) = f(x)$ for each $x \in (0, \pi)$ with $x \neq \pi/2$ (indeed $F(x)$ is not differentiable at $x = \pi/2$).

Problem 9.4. The equation of the tangent line is

$$y = F(1) + F'(1)(x - 1) = -\frac{1}{3}x + \frac{1}{3}.$$

Problem 9.5. The function $F(x)$ is strictly increasing in \mathbb{R} (as $F'(x) > 0$), hence $F(x)$ is one-to-one for any value of $x \in \mathbb{R}$.

Problem 9.6.

- (a) The value of the limit is 1 (for instance, by l'Hôpital's rule).
- (b) The value of the limit is 0 (for instance, by the sandwich theorem).

Problem 9.7. We have

$$H'(x) = \int_{2x}^{3x} e^{-t^2} dt + x \left\{ 3e^{-9x^2} - 2e^{-4x^2} \right\},$$
$$H''(x) = 2 \left\{ 3e^{-9x^2} - 2e^{-4x^2} \right\} + 2x^2 \left\{ 8e^{-4x^2} - 27e^{-9x^2} \right\}.$$

Problem 9.8. The function $H(x)$ is decreasing in $[0, 1/2]$ as $H'(x) = \ln(1 - x^2) < 0$ in that interval.

Problem 9.9. The global maximum is at $x = 3$ and the global minimum is at $x = 1$. In addition

$$H(3) = 2 \int_0^1 e^{-t^4} dt > 2 \int_0^1 e^{-1} dt = \frac{2}{e} > \frac{2}{3}.$$

Problem 9.10. (a) 0. (b) $1/3$.

Problem 9.11.

- We have that $F'(x) = 1 + \sin(\sin(x)) > 0$ for each $x \in \mathbb{R}$. Thus, the function $F(x)$ is strictly increasing and one-to-one in \mathbb{R} . In addition, it's easy to see that $F(0) = 0$, which implies that $F^{-1}(0) = 0$. As a consequence, we get

$$(F^{-1})'(0) = \frac{1}{F'(0)} = \frac{1}{1 + \sin(\sin(0))} = 1.$$

- Observe that

$$G(x) = \int_1^0 \sin(\sin(t)) dt + \int_0^x \sin(\sin(t)) dt,$$

where the first integral is independent of x (say, equal to $G_0 \in \mathbb{R}$) and the second integral is a function $H(x)$ that is invariant under replacing x with $-x$ (this can be shown by means of the change of variable $u = -t$). As a consequence, we get $G(-x) = G_0 + H(-x) = G_0 + H(x) = G(x)$, which means that $G(x)$ is even and not one-to-one, namely G^{-1} does not exist.

Problem 9.12. The desired Taylor polynomial is given by $P_3(x) = x^3/3$ and the value of the limit is $1/3$.

Problem 9.13. We have

$$(a) \quad H'(x) = \sin^3(x) \left\{ 1 + \left(\int_1^x \sin^3(t) dt \right)^2 + \sin^6 \left(\int_1^x \sin^3(t) dt \right) \right\}^{-1}.$$

$$(b) \quad K'(x) = \cos \left(\int_0^x \sin \left(\int_0^t \sin^3(s) ds \right) dt \right) \sin \left(\int_0^x \sin^3(s) ds \right).$$

Techniques of integration

Problem 10.1. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

- $I(x) = x \arctan(3x) - \frac{1}{6} \ln(1 + 9x^2) + k$ (integration by parts)
- $I(x) = \frac{1}{2} e^x (\sin(x) - \cos(x)) + k$ (integration by parts)
- $I(x) = \frac{1}{2} x (\cos(\ln(x)) + \sin(\ln(x))) + k$ (change of variable $t = \ln(x)$ and integration by parts)
- $I(x) = \frac{x}{2} + \frac{x}{10} \cos(2 \ln(x)) + \frac{x}{5} \sin(2 \ln(x)) + k$ (change of variable $t = \ln(x)$, identity $\cos(2\alpha) = 2 \cos^2(\alpha) - 1$ and integration by parts)
- $I(x) = \arctan\left(\frac{1}{2} \sqrt{e^x - 4}\right) + k$ (change of variable $t = \sqrt{e^x - 4}$)
- $I(x) = \arctan\left(\sqrt{x^2 - 1}\right) + k$ (change of variable $t = \sqrt{x^2 - 1}$)

Problem 10.2. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

- $I = \sqrt{3} - \frac{\pi}{3}$ (change of variable $u = \sqrt{t^2 - 1}$)
- $I = 2 - \frac{\pi}{2}$ (change of variable $u = \sqrt{e^t - 1}$)
- $I(x) = 2 \arctan\left(\sqrt{1+x}\right) + k$ (change of variable $t = \sqrt{1+x}$)
- $I(x) = -\frac{3}{2}(1-x)^{2/3} + 3(1-x)^{1/3} - 3 \ln|(1-x)^{1/3} + 1| + k$ (change of variable $t = (1-x)^{1/3}$)

Problem 10.3. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

- $I(x) = \frac{1}{\sqrt{2}} \arctan\left(\frac{3}{\sqrt{2}}x + \sqrt{2}\right) + k$
- $I(x) = \frac{1}{4} \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+1} + \frac{1}{2} x^2 + k$

- $I(x) = \frac{1}{x} + \ln|x-1| - \ln|x+1| + k$
- $I(x) = \frac{3}{2} \ln(x^2 + 4x + 13) + \frac{47}{3} \arctan\left(\frac{x+2}{3}\right) + \frac{1}{2}x^2 - 4x + k$
- $I(x) = \frac{3}{2} \ln|x-1| - \frac{1}{2} \ln|x-3| - \frac{13}{x-3} + k$

Problem 10.4. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

1. $I(x) = \sin(x) - \frac{1}{3} \sin^3(x) + k$
2. $I(x) = \frac{3}{8}x - \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + k$
3. $I(x) = \frac{1}{2}e^{2x} - 2e^x + \ln(e^{2x} + 2e^x + 2) + 2 \arctan(e^x + 1) + k$
4. $I(x) = \cos(x) - 2 \arctan(\cos(x)) + k$
5. $I(x) = -\frac{1}{2} \ln|1 - \sin(x)| + \frac{1}{2} \ln|1 + \sin(x)| + k$
6. $I(x) = \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + \frac{x}{2} \sqrt{a^2 - x^2} + k$

Improper integrals

Problem 11.1.

- Divergent (for instance, by the definition of improper integral).
- Divergent (for instance, by limit comparison test with $\int_1^{+\infty} dx/x$).
- Convergent (for instance, by comparison test with $\int_1^{+\infty} dx/x^3$ after using the absolute value of the integrand).
- Convergent (for instance, by limit comparison test with $\int_1^{+\infty} dx/x^{\alpha+1}$).
- Convergent (for instance, by limit comparison test with $\int_1^{+\infty} dx/x^{3/2}$; note that the change of variable $t = 1/\sqrt{x}$ may be useful).

- Convergent. Apply the method of induction together with the definition of improper integral.
- Convergent. First note that e^{-x^2} is even, thus the integral can be written as $2 \int_0^{+\infty} e^{-x^2} dx$. Then, the latter converges by limit comparison test with $\int_0^{+\infty} e^{-x} dx$.