CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Taylor polynomial

Problem 7.1. Approximate the following values within an error bound given by the indicated ε, using suitable Taylor polynomials.

- $\sin(1)\,, \quad \varepsilon = 10^{-5}\,.$
- $\sqrt[5]{\dfrac{3}{2}}\,, \quad \varepsilon = 10^{-2}\,.$

Problem 7.2. Write the Maclaurin polynomial of the indicated degree n for the following functions.

1. $f(x) \;=\; \sqrt{1+x}\,, \quad n = 3\,.$
2. $f(x) \;=\; \sin(3x^2)\,, \quad n \in \mathbb{N}$ (generic).
3. $f(x) \;=\; \tan(x)\,, \quad n = 5\,.$
4. $f(x) \;=\; e^{-x^2}\cos(x)\,, \quad n = 3\,.$
5. $f(x) \;=\; (1+e^{x})^2\,, \quad n \in \mathbb{N}$ (generic).

Problem 7.3. Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in terms of powers of $x - 4$.

Problem 7.4. Write the Taylor formula of degree $n \in \mathbb{N}$ about point $a = -1$ for the function $f(x) = 1/x$.

Problem 7.5. Find the Maclaurin polynomial of degree 5 for $f(x) = e^{x}\sin(x)$.

Problem 7.6. Compute the coefficient of x^4 in the Maclaurin polynomial for the function $f(x) = \ln(\cos(x))$.

Problem 7.7. Calculate the Taylor polynomial of degree 3 about point $a = 0$ for the following functions.

$$\begin{aligned} f(x) &= \sin(2x)\,. \\ f(x) &= e^{3x}\,. \\ f(x) &= x\,e^{-x}\,. \\ f(x) &= e^{x}\ln(1-x)\,. \\ f(x) &= \sin^2(x)\,. \\ f(x) &= \frac{\sqrt{1+x^2}\,\sin(x)}{1+\ln(1+x)}\,. \end{aligned}$$

Problem 7.8. Write the Maclaurin polynomial of degree $n \in \mathbb{N}$ for the following functions ($a \in \mathbb{R}$).

$$\begin{aligned} f(x) &= \cos(ax)\,. \\ f(x) &= \frac{e^{ax} - e^{-ax}}{2}\,. \\ f(x) &= e^{ax^2}\,. \\ f(x) &= \frac{1+x}{1-x}\,. \end{aligned}$$

Problem 7.9. The Taylor polynomial of degree 4 about point $a = 1$ for a function $f(x)$ is given by $P_{4,1}(x) = 2(x-1)^3 - 3(x-1)^4$.

- Find an equation for the tangent line to the graph of $f(x)$ at $x = 1$.
- Calculate $\displaystyle \lim_{x\to 1} \frac{f(x)}{(x-1)^3}$.
- Compute $f^{(4)}(1)$.

Problem 7.10. Prove what follows.

$$\begin{aligned} &\forall a < 1: \quad \sin(x) = o(x^a)\,, \quad \text{as } x \to 0\,. \\ &\ln(1+x^2) = o(x)\,, \quad \text{as } x \to 0\,. \\ &\tan(x) - \sin(x) = o(x^2)\,, \quad \text{as } x \to 0\,. \\ &\ln(x) = o(x)\,, \quad \text{as } x \to +\infty\,. \end{aligned}$$

Problem 7.11. Find a polynomial $P(x)$ such that

$$\lim_{x\to 0} \frac{\sqrt{1-x^4} - P(x)}{x^7} = 0\,.$$

Problem 7.12. Approximate $f(x) = \cos(x) + e^x$ by means of a polynomial of degree 3 about $a = 0$ and estimate the involved error when such approximation is used for $x \in [-1/4, 1/4]$.

Problem 7.13. How many terms should you consider in the Maclaurin polynomial for $f(x) = e^x$, with $x \in [-1, 1]$, in order to get an approximation with three exact decimal places?

Problem 7.14. Using a Taylor polynomial of degree 3, approximate the value

$$\frac{1}{\sqrt{1.1}}$$

and find an upper bound of the involved error.

Problem 7.15. Using suitable Taylor polynomials, find an approximation within an error smaller than 10^{-3} for the following values.

- $\sin(2)$.
- $\ln(4/5)$.
- $\cos(1)$.
- e^{-2}.
- $\ln(2)$.

Problem 7.16. Find how many terms of the Taylor series for $f(x) = \sin(x)$, about $a = 0$, you need to consider to approximate $\sin(1/2)$ within an error smaller than 10^{-12}.

Problem 7.17. Calculate the given limits by using appropriate Taylor polynomials.

$$\text{(a)} \quad \lim_{x\to 0} \frac{e^x - \sin(x) - 1}{x^2}.$$

$$\text{(b)} \quad \lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{x^5}.$$

$$\text{(c)} \quad \lim_{x\to 0} \frac{\cos(x) - \sqrt{1-x}}{\sin(x)}.$$

$$\text{(d)} \quad \lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x^3}.$$

(e) $\displaystyle \lim_{x \to 0} \frac{x - \sin(x)}{x\,[1 - \cos(3x)]}$.

(f) $\displaystyle \lim_{x \to 0} \frac{\cos(x) + e^{x} - x - 2}{x^3}$.

(g) $\displaystyle \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right)$.

(h) $\displaystyle \lim_{x \to 0} \frac{1}{x} \left(\frac{1}{x} - \frac{\cos(x)}{\sin(x)} \right)$.

(i) $\displaystyle \lim_{x \to +\infty} x^{3/2} \left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x} \right)$.

(j) $\displaystyle \lim_{x \to +\infty} \left[x - x^2 \ln\left(1 + \frac{1}{x}\right) \right]$.

Problem 7.18. Calculate the given limits by using either Taylor polynomials or l'Hôpital's rule.

(a) $\displaystyle \lim_{x \to 0^+} x \ln(e^{x} - 1)$.

(b) $\displaystyle \lim_{x \to +\infty} \frac{e^{x} - \arctan(x)}{\ln(1 + x)}$.

(c) $\displaystyle \lim_{x \to 0^+} x^{x}$.

(d) $\displaystyle \lim_{x \to +\infty} \frac{\ln(x)}{x^{a}}, \quad a > 0$.

(e) $\displaystyle \lim_{x \to +\infty} \frac{x}{e^{ax}}, \quad a > 0$.

(f) $\displaystyle \lim_{x \to +\infty} x^{1/x}$ (use the change of variable $t = 1/x$) .

(g) $\displaystyle \lim_{x \to 0} (1 + x)^{1/x}$.