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ESCUELA POLITÈCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

Discrete-Mathematics Problem Set

Grado en Ingeniería en Informática

Doble Grado en Ingeniería en Informática y Administración de Empresas

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Contents

1 Set theory and functions

Question 1.1 Let $A = \{x \in \mathbb{Z} : \ x^2 < 16\}$. For each of the following statements, determine whether it is true or not:

1. $\{0, 1, 2, 3\} \subset A$
2. $\{3, 1\} \in A$
3. $\{x \in \mathbb{Z} : \ |x| < 4\} \subset A$
4. $\varnothing \subset A$
5. $3 \in A$
6. $\{3\} \in A$
7. $A \subset \{-3, -2, -1, 0, 1, 2, 3\}$

Question 1.2 Prove the following identities:

1. $A \cup (A \cap B) = A \cap (A \cup B) = A$
2. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$
3. $A \setminus (B \cup C) = (A \setminus B) \setminus C$
4. $(A \triangle B) \triangle C = A \triangle (B \triangle C)$
5. $A \setminus B = A \triangle (A \cap B)$
6. $\overline{(A \triangle B)} = \overline{A} \triangle B = A \triangle \overline{B}$

Question 1.3 Simplify the following expressions:

1. $[\overline{B} \cap (\overline{A} \cup C) \cap D] \cup \overline{[(A \cup B) \cap B]}$
2. $\overline{(\overline{[(A \cup B) \cap C]} \cup \overline{B})}$

Question 1.4 Determine if the functions $f, g \colon \mathbb{R} \to \mathbb{R}$ below are injective or not:

$$f(x) \ = \ \begin{cases} 2x & \text{if } x \geq 0, \\ -2x - 1 & \text{if } x < 0. \end{cases}, \qquad g(x) \ = \ \begin{cases} 3x + 1 & \text{if } x \geq 0, \\ 2x & \text{if } x < 0. \end{cases}$$

Question 1.5 Given the function $F \colon \mathbb{R} \setminus \{-1/2\} \to \mathbb{R} \setminus \{1/2\}$, defined as follows:

$$F(x) = \frac{x + 3}{1 + 2x},$$

determine if its inverse exists or not. In case it exists, find F^{-1}.

Question 1.6 Let us consider the floor and ceiling functions. Then,

1. Compute the following quantities:

$$\lfloor 1/2 \rfloor,\ \lceil 1/2 \rceil,\ \lfloor -1/2 \rfloor,\ \lceil -1/2 \rceil,\ \lfloor \pi \rfloor,\ \lceil \pi \rceil,\ \lfloor 1/2 + \lceil 1/2 \rceil \rfloor,\ \lceil \lfloor 1/2 \rfloor + \lceil 1/2 \rceil + 1/2 \rceil$$

2. Draw the graph of both functions $y = \lfloor x \rfloor$ and $y = \lceil x \rceil$.

3. In a certain communication protocol, the data is transmitted in groups of 53 bytes. How many groups can be transmitted in one minute through a connection working at a rate of 500 Kilobits per second?
   NOTE: Each byte contains 8 bits.

Question 1.7 Determine if each one of the following functions $f : A \to B$ is injective (one–to–one), surjective (onto), bijective, or none of these:

1. $A \neq \varnothing$, $B = \mathcal{P}(A)$, $f(a) = \{a\}$.

2. $A = B = \mathcal{P}(\{a,b,c,d\})$, $f(X) = \overline{X}$.

3. $A = B = \mathcal{P}(\{a,b,c,d\})$, $f(X) = X \cup \{a,b\}$.

4. $A = B = \mathcal{P}(\{a,b,c,d\})$, $f(X) = X \cap \{a,b\}$.

Question 1.8 Compute the quotient and the remainder when

1. 44 is divided by 8?

2. 19 is divided by 7?

3. -1 is divided by 3?

4. -123 is divided by 19?

5. -100 is divided by -101?

A note on the greatest common divisor

Once the decomposition of two integers a and b into prime factors is known, it is very easy to compute their greatest common divisor and their least common multiple:

Proposition 1 *If the integers $a, b > 1$ can be factorized in the form*

$$\begin{aligned} a &= p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}, \\ b &= p_1^{m_1} \cdot p_2^{m_2} \cdots p_k^{m_k}, \end{aligned}$$

with $n_i, m_i \geq 0$ and all prime factors of a and b appear in both decompositions, then

$$\begin{aligned} \gcd(a,b) &= p_1^{\min(n_1,m_1)} \cdot p_2^{\min(n_2,m_2)} \cdots p_k^{\min(n_k,m_k)}, \\ \operatorname{lcm}(a,b) &= p_1^{\max(n_1,m_1)} \cdot p_2^{\max(n_2,m_2)} \cdots p_k^{\max(n_k,m_k)}. \end{aligned}$$

Question 1.9 Find the greatest common divisor gcd and the least common multiple lcm of 500 and 120 by first computing the representation of both numbers as products of prime numbers.

2 Elementary combinatorics I

Question 2.1 Computers represent information using **bits**. A bit has two possible values: 0 or 1. A **bit string** of length n is a sequence of n bits $b_1b_2b_3 \dots b_n$.

1. How many different bit strings are there of length n?
2. How many bit strings of length $n \geq 2$ start and end with 1?
3. How many bit strings have length smaller or equal to n?
4. How many bit strings of length smaller or equal to $n \in \mathbb{N}$ contain only 1's?
5. How many bit strings of length $n \geq 6$ contain at least three 0's and three 1's?
6. How many bit strings of length $n \geq 5$ either start with 00 or end with three 1's?
7. A **palindrome** is a bit string such its inverse is identical to itself (e.g., 0010110100). How many bit strings of length n are palindromes?

Question 2.2 A wedding photographer likes to make pictures of the 10 people present in the banquet (including the groom and the bride) in groups of 6 persons. How many distinct pictures can the photographer take if

1. the bride is in the picture?
2. both the bride and the groom are in the picture?
3. exactly one of them (bride or groom) is in the picture?
4. both the bride and the groom are next to each other in the picture?
5. both the bride and the groom are in the picture but not next to each other?
6. both the groom and the bride are next to each other in the picture, and the bride is on the left of the groom?

Question 2.3 Find how many five-digit numbers n can be formed with the set $A = \{1, 2, 3\}$, and such that each digit in A must appear in the number n at least once.

Question 2.4 Find how many three–letter words can be formed with the 10–element set $\{A, B, \dots, J\}$, and such that the letters are all distinct and ordered in the standard lexicographic way.

Question 2.5 The USS Arizona had 12 distinct flags and the sailors could put up to 3 flags in the corresponding flagpole, each flag configuration describing a distinct circumstance in the ship.

- How many distinct circumstances using at least one flag could be described with this set-up?
- How many distinct circumstances using at least one flag could be described if there were three equal sets of the above-mentioned 12 flags?

Question 2.6 Find in how many ways we can put three letters A and seven letters B such that there are no two consecutive A's.

Question 2.7 We have a three-dimensional space and we are allowed to move with jumps of length 1 unit in the direction of three coordinate axis. In other words, each trajectory is composed by moves of the following three types:

(H) $(x, y, z) \to (x+1, y, z)$,

(V) $(x, y, z) \to (x, y+1, z)$,

(L) $(x, y, z) \to (x, y, z+1)$.

How many trajectories are there joining the points $(-1, 2, 0)$ and $(1, 3, 7)$? How many of them go through the point $(0, 3, 4)$?

The pigeonhole principle

Proposition 2 (The pigeonhole principle) *If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.*

Remark: Given a function $f\colon A \to B$ with $|A| > |B|$, then f cannot be injective. Therefore, there exist at least two distinct elements $a, b \in A$ such that $f(a) = f(b) \in B$.

Proposition 3 (The generalized pigeonhole principle) *If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.*

Question 2.8 Show that, given any five distinct integers, there are at least two of them with the same remainder when divided by four.

3 Graph theory I

Question 3.1 Let V be the set of those two–letter words formed with the alphabet $\{w, x, y, z\}$ and starting with either y or z. We define the graph $G = (V, A)$ in such a way that two words in V determine an edge in A if they differ in exactly one letter:

1. How many vertices does G have?
2. Draw the graph G.
3. Prove that G is regular and compute its degree.
4. Determine whether G is bipartite or not.

Question 3.2 Let G be a graph with 20 edges, and such that its edge-complement $\overline{G}$ has 25 edges. How many vertices does G have?

Question 3.3 Compute the number of vertices of simple connected graphs G if:

1. G is a 2–regular graph with 9 edges.

2. G is a regular graph with 6 edges.
3. G has 10 edges, 2 vertices of degree 4, and all other vertices have degree 3.

Question 3.4 Let K_n be the complete graph of n vertices.

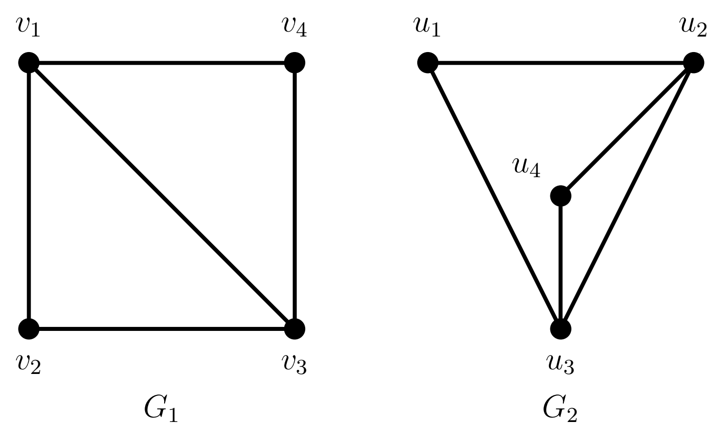
- Draw K_1, K_2, K_3, K_4, and K_5.
- Which is the degree of the vertices of K_n?
- How many edges does K_n have?
- Prove that K_n is a sub-graph of K_m for all $n < m$.

Question 3.5 For which values of $n \geq 3$ are the graphs K_n, P_n, Q_n, and C_n bipartite?

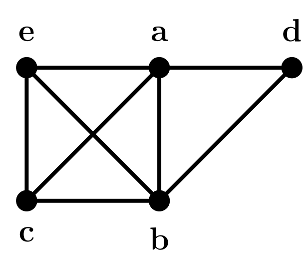
Question 3.6 Prove that in any simple graph with no isolated vertices, there are at least two vertices with the same degree.

Question 3.7 Find the minimum number of vertices of a graph with 7 edges if every vertex has a degree at most 3.

Question 3.8 Write the adjacency matrices A_1 and A_2 corresponding to the graphs shown in the figure, and prove that these two graphs are isomorphic by finding a change–of–basis matrix P satisfying $A_2 = P^{-1} \cdot A_1 \cdot P$.



Question 3.9 Let G be the graph shown in the figure below.



We now consider the following walks on G:

1. $C_1 = (a, e, b, c, b)$
2. $C_2 = (e, b, a, d, b, e)$
3. $C_3 = (a, e, a, d, b, c, a)$
4. $C_4 = (c, b, d, a, e, c)$

Determine which ones are trails, paths, cycles, or circuits, and compute their corresponding lengths.

4 Graph theory II

Question 4.1 Let K_n be the complete graph of n vertices.

1. How many length–3 cycles does it contain?
2. How many triangles does each edge of K_n belong to?

Question 4.2 Show that the paraffins $C_n H_{2n+2}$ have tree–like molecules [Arthur Caley, 1857].

Question 4.3 Prove that in a rooted tree such that all vertices that are not leaves have degree 3, then the tree has an even number of vertices.

Question 4.4 Prove that there is no planar and connected graph satisfying that each vertex has at least degree 8, and each face is bounded by at least 8 edges.

Question 4.5 Let G and G' be two distinct connected graphs. G is a plane graph with 10 vertices, such that it splits the plane into 3 regions. G' is a 10-vertex graph with all vertices of degree at least 3. Are G and G' isomorphic?

Question 4.6 Give an example (if any) of

1. a regular and bipartite graph,
2. a 3–regular graph with 9 vertices,
3. a graph with n vertices and $(n-1)(n-2)/2$ edges,
4. a 4–regular connected multi-graph,
5. a graph isomorphic to its edge-complement,
6. a graph isomorphic to its dual.

Question 4.7 How many trees does a forest of 62 vertices and 51 edges contain?

Question 4.8 Let X be the set $X = \{A, B, C\}$. We define the simple graph $G = (V, E)$ in the following way: the set of vertices is given by the power set of X ($V = \mathcal{P}(X)$), and two vertices $R, S \in V$ are adjacent if and only if $R \subset S$ or $S \subset R$.

- How many vertices and edges does G contain?
- Determine the degree of each vertex in V. Is G regular?
- Is G planar?
- Is G bipartite?

Question 4.9 Show that if $G = (V, E)$ is a simple graph and

$$|E| > \binom{|V|-1}{2},$$

then G is connected.

Question 4.10 If the mean degree of a connected graph is greater than 2, then show that there exist at least two independent cycles.

Question 4.11 Let $G = (V, E)$ be a graph whose adjacency matrix A_G is given by

$$A_G = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right).$$

Answer the following questions using arguments based solely on the matrix A_G (<u>without using</u> any graphical representation of G that can be obtained from A_G):

a) Tell whether G is a simple graph, a multi-graph, or a pseudo-graph.

b) How many vertices and edges does G contain?

c) Is G a regular graph? If yes, tell the common degree, and if not, give the degree sequence of G.

d) Let $i \neq j$ be two distinct vertices of G ($i, j \in V$). Let n_{ij} be the number of walks from i to j of <u>length 3</u>. Find all the possible values of n_{ij} for G.

e) Which is the length of a minimum-length <u>cycle</u> of G?

5 Graph theory III

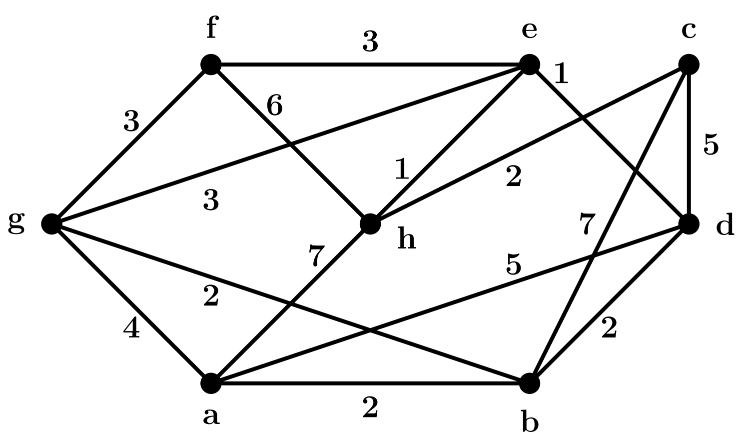
Question 5.1 Let $G = (V, E)$ be the graph defined by the following adjacency matrix:

$$A_G = \left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right).$$

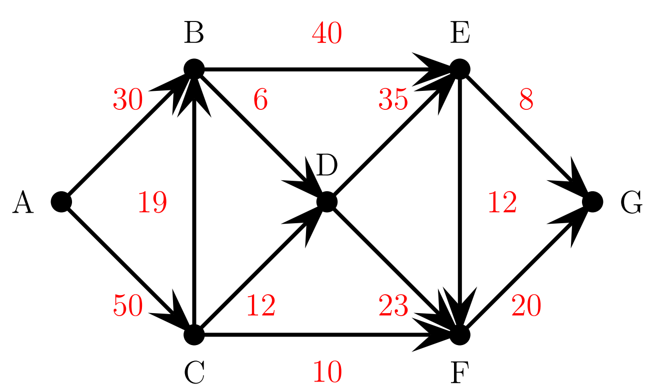
1. Is it bipartite? Is it planar?

2. Find a spanning tree (if any).

Question 5.2 Tell whether the following weighted graph has a spanning tree of weight equal or smaller than 12:



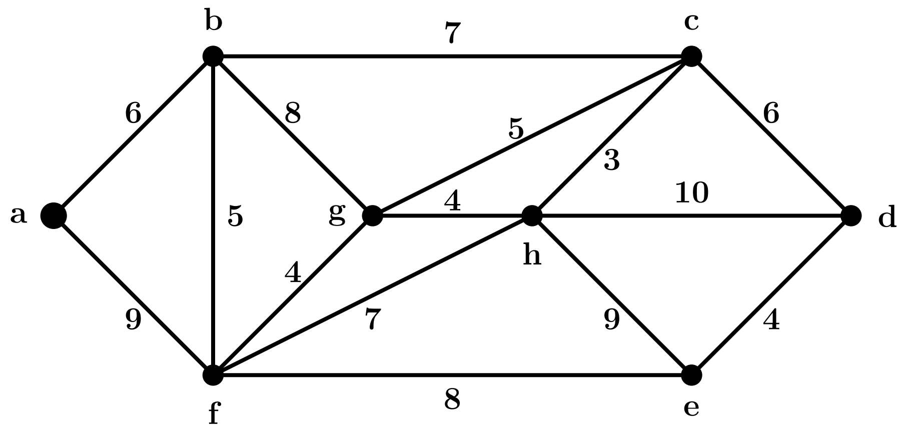
Question 5.3 Let us consider the following directed graph:



1. Find the shortest path to go from A to G.

2. Find the shortest path to go from A to G, assuming that the edges CB and EF are not directed (i.e., you can use both directions).

Question 5.4 In the weighted graph shown in the figure below, compute the distances $d(a,h)$, $d(a,e)$, $d(d,a)$, $d(d,g)$, and $d(b,e)$.



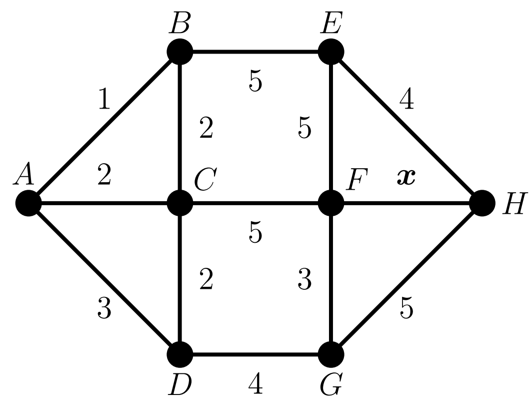
Question 5.5 Let $V = \{A, B, C, D, E, F, G, H, I, J, K, L, M\}$ be the set of vertices of the weighted graph $G = (V, E)$. Its weight matrix is given below: e.g., for $(C, D) \in E$, its weight is $w(C, D) = 10$:

V	A	B	C	D	E	F	G	H	I	J	K	L	M
A	·	5	·	·	7	14	·	·	·	·	·	·	·
B	5	·	7	·	1	·	2	·	·	·	·	·	·
C	·	7	·	10	·	·	6	6	·	·	·	·	·
D	·	·	10	·	·	·	·	4	·	·	·	·	·
E	7	1	·	·	·	4	·	·	·	·	·	·	·
F	14	·	·	·	4	·	8	·	6	4	·	·	·
G	·	2	6	·	·	8	·	7	6	4	·	·	·
H	·	·	6	4	·	·	7	·	·	·	1	2	·
I	·	·	·	·	·	6	6	·	·	·	·	·	·
J	·	·	·	·	·	4	4	·	·	·	11	13	·
K	·	·	·	·	·	·	·	1	·	11	·	·	2
L	·	·	·	·	·	·	·	2	·	13	·	·	5
M	·	·	·	·	·	·	·	·	·	·	2	5	·

1. Compute the minimum weight (or shortest) path from vertex A to vertex M. Compute the total weight of such path.

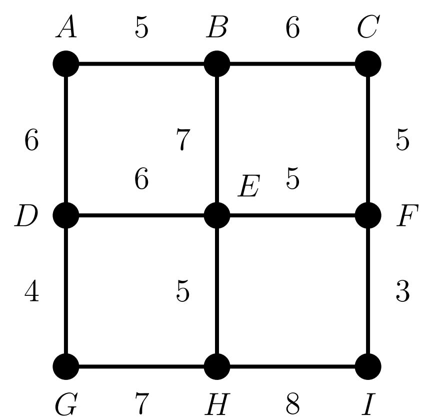
2. Find a minimum-weight spanning tree and give its weight.

Question 5.6 Let $G = (V, E, \omega)$ be the following weighted graph (with the weight of the edge $\{F, H\}$ equal to $\boldsymbol{x} \in \mathbb{R}$):



Compute the range of values of the weight $\boldsymbol{x} \in \mathbb{R}$, so that the minimum-length path from A to H goes through the edge $\{F, H\}$.

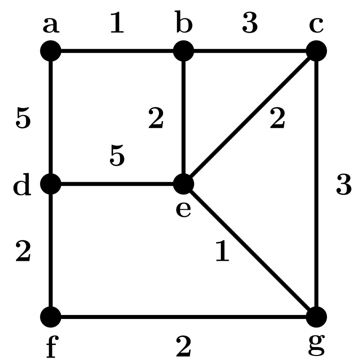
Question 5.7 A constructor is planning a new urban development formed by 9 family houses, and now he is designing the water supply. As he knows some graph theory, he defines a weighted graph $G = (V, E, \omega)$, where the vertices $V = \{A, B, C, D, E, F, G, H, I\}$ correspond to the houses, two vertices are adjacent if the houses can be connected by a water pipe, and the weight of each edge is the cost (in thousands of euros) of placing the corresponding water pipe. The graph G is given by



- If the constructor places the water supply on house A, compute using Dijsktra's algorithm the minimum-cost path to reach house I (where he will live). As a side result, compute the total cost of the **rooted** (at A) spanning tree that connects A to all the other vertices.
- When they heard this idea, the other neighbors complained about the price. They preferred to place the pipes using a minimum-cost spanning tree. Find one of these subgraphs using Prim's algorithm, and compute the total cost of the pipes.

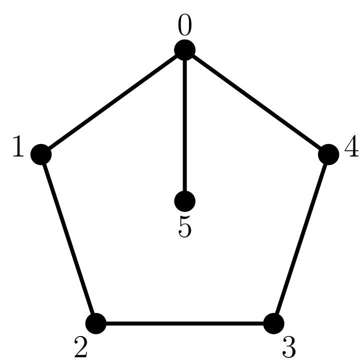
6 Graph theory IV

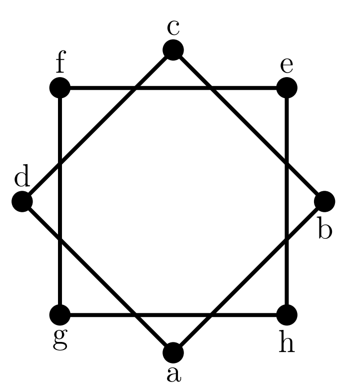
Question 6.1 Let G be the following weighted graph, let H be the simple graph obtained by erasing the weights of G.

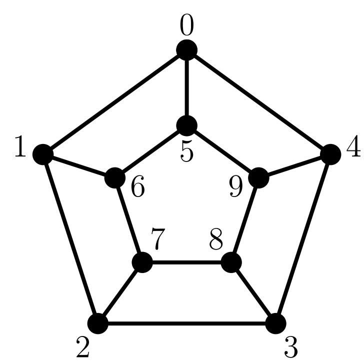


1. Find a minimum-weight spanning tree for G.
2. Is H bipartite? If yes, give the partition of V into two disjoint sets that provides a proof for bipartiteness.
3. Tell whether there are Euler tours/trails, and Hamilton cycles/paths.
4. Find a graph with exactly the same degree sequence as H, but not isomorphic to H. Explain why it is not isomorphic to H.

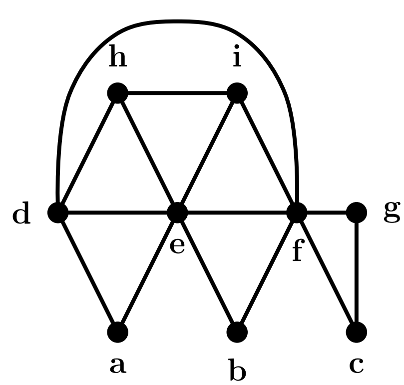
Question 6.2 Tell whether there are Euler tours/trails, and Hamilton cycles/paths in the following graphs:







Question 6.3 Tell whether the following graph admits an Euler tour/trail, and find it (if any).



Question 6.4 In a film festival there are 6 films on the first day. Films 1, 3, and 5 are dramas; films 2, 4, and 6 are comedies; films 3 and 4 are indies, and films 5 and 6 are blockbusters. Each film lasts two hours. What is the minimum number of hours needed to show all films, in such a way that films of the same type do not overlap?

Question 6.5 Given a set of intervals in $\mathbb{R}$, we can construct a graph named **interval graph** in the following way: each interval is a vertex of the graph, and two vertices are adjacent if and only if the corresponding intervals have a non-empty intersection.

1. Compute the interval graph G associated to the intervals:

$$\{(1,9),(7,8),(0,3),(4,10),(2,6),(5,11)\}$$

2. Tell whether this graph is bipartite, admits an Euler tour or a Hamilton cycle. In case there is an Euler tour/trail and/or a Hamilton cycle/path, find one example of each.

Question 6.6 Find examples of **simple** graphs G that satisfy the following conditions:

- G has 7 vertices, is Hamiltonian, and is Eulerian.
- G has 8 vertices, is Hamiltonian and Eulerian, but there is at least one Hamilton cycle that does not coincide with any Euler tour.
- G has 7 vertices, is Hamiltonian, is not Eulerian, and has no cut-edges.
- G has 7 vertices, is not Hamiltonian, and is Eulerian.
- G has 7 vertices, is Hamiltonian, is not Eulerian, and is bipartite.

Question 6.7 Let $G = (V, E)$ be the graph defined by the following adjacency matrix:

$$A_G = \left(\begin{array}{cccc|cccc}
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}\right).$$

1. Find its chromatic number using a graph-theoretic algorithm.
2. Find a spanning tree of G (if any).
3. Does G admit an Euler trail? Which is the minimum number of edges we should add to G to make it an Eulerian graph?

Question 6.8 Let $G_n = (V_n, E_n)$ with $n \in \mathbb{N}$ be a graph family defined as follows:

- Each vertex $v \in V_n$ corresponds to a bit string of length n with an **even** number of ones.

- Two vertices $x, y \in V_n$ are adjacent ($\{x, y\} \in E_n$) if and only if they differ exactly in two bits.

If n is a fixed natural number, compute $|V_n|$ and $|E_n|$. Is G_n a regular graph? If the answer is positive, then give the common degree; and if the answer is negative, then give the degree sequence. For which values of n the graph G_n is Eulerian?

7 Elementary combinatorics II

Question 7.1 We have 4 golf balls and 10 distinct boxes. Compute the number of distinct ways of distributing the golf balls in the boxes if:

1. All golf balls are distinct, and each box may contain at most one ball.
2. All golf balls are indistinguishable, and each box may contain at most one ball.
3. All golf balls are indistinguishable, and we can put as many balls as we wish in each box.
4. All golf balls are distinct, and we can put as many balls as we wish in each box.

Question 7.2 If we simultaneously toss 6 identical dice, how many distinct results are possible?

Question 7.3 We want to line up a white balls and b black balls. In how many distinct ways can we arrange these balls such that there are exactly $k+1$ groups of black balls?
NOTE: A group of equal balls may consist in a single ball.

Question 7.4 Find the number of 4–element subsets taken from the set $\{1, 2, 3, \ldots, 15\}$, and such that they do not contain consecutive integers.

Question 7.5 Find the number of p–element subsets taken from the n–element set $\{a_1, a_2, ..., a_n\}$, and such that they do not contain consecutive elements.

Question 7.6 Find the number of distinct solutions of the equation $x_1 + x_2 + x_3 = 17$ when the x_i all belong to the set of the non-negative integers $\mathbb{Z}_+$.

Question 7.7 Find the number of distinct solutions of the equation $x_1 + x_2 + x_3 = 17$ when the x_i all belong to the set of the natural numbers $\mathbb{N}$.

Question 7.8 Find the number of distinct solutions of the equation $x_1 + x_2 + x_3 = 17$ when the x_i all belong to the set $\{0, 1, 2, 3, 4, 5, 6\}$.

Question 7.9 We have 7 distinct object types, and we want to choose 25 objects in such a way that there are always at least 2 objects and at most six objects of each type. In how many distinct ways can we achieve this task?

Question 7.10

1. Eight people go out for dinner and on the dessert menu there are four distinct desserts. How many distinct orders can the waiter have?

2. How many distinct solutions exist of the equation

$$x_1 + x_2 + x_3 + \ldots + x_n \ = \ r\,,$$

with $x_i \in \mathbb{N}$?

3. How many distinct positive integer solutions exist of the equation

$$x_1 + x_2 + x_3 + \ldots + x_n \ = \ 21\,,$$

with the constraint $x_1 > 1$?

Question 7.11 A sales company wants to audit the results in 20 cities, and there are 5 employees willing to do that job; each of them will be in charge of 4 cities.

1. In how many distinct ways can the company manager split the 20 cities in 5 groups of 4 cities each?

2. In how many distinct ways can the company manager assign the cities to the 5 employees?

8 Recurrences

Question 8.1 For each integer $n \in \mathbb{N}$, let us consider the set of n lines in the plane with the following properties:

P1. There are no parallel lines.

P2. At each intersection point, only two lines meet.

Let S_n be the number of regions of the plane defined by n lines with the above properties:

- Find a recurrence equation for S_n.

- Solve this equation.

Question 8.2 Let us consider numerical strings of 10 digits formed with elements in the set $\{0, 1, 2\}$. How many of these strings are such that the sum of the 10 digits is an even number?

Question 8.3 A software company is trying to solve a very complicated problem. They say they are in the n-th phase of the project when there are n steps left to finish it. Let us suppose that in each phase there are five options. Two of them, lead to the $(n-1)$-th phase; but the other three options are better in the sense that they lead directly to the $(n-2)$-th phase. Let us denote by a_n the number of ways of achieving the solution of the project from the n-th phase. If $a_1 = 2$, check that $a_2 = 7$, and obtain a recurrence equation for a_n. Prove that

$$a_n = \frac{1}{4}\left[3^{n+1} + (-1)^n\right] .$$

Question 8.4 Compute the number of bit strings of length $n \geq 1$ such that they do not have two consecutive zeros.

Question 8.5 There are 3^n strings of length n formed by $\{0, 1, 2\}$. Compute the number of those strings with an odd number of zeros a_n, by first proving that the recurrence equation for a_n is

$$a_n \;=\; a_{n-1} + 3^{n-1}\,, \quad n \geq 2\,, \quad a_1 = 1\,,$$

and then, by solving this equation.

Question 8.6 Solve the equation:

$$a_n \;=\; 4a_{n-1} - 4a_{n-2}\,, \quad n \geq 3\,, \quad a_1 = a_2 = 1\,.$$

Question 8.7 Let a_n be the number of strings of length n which can be formed with $\{0, 1, 2\}$ in such a way that there are no consecutive 1's nor two consecutive 2's.

1. Show that $a_n = 2a_{n-1} + a_{n-2}\,, \quad n \geq 3$.
2. Find an explicit solution for a_n.

Question 8.8 Solve the equation:

$$a_n \;=\; -a_{n-1} + 3 \cdot 2^{n-1}\,, \quad n \geq 2\,, \quad a_1 = 0\,.$$

Question 8.9 Let us consider the following **recursive** algorithm to compute the exponential a^n with $n \in \mathbb{N}$:

```
procedure exp1(a,n)
if (n = 1)
  return a
else
  m = floor(n/2)
  return exp1(a,m) ⋆ exp1(a,n-m)
```

Let b_n be the number of multiplications needed to compute a^n:

- Compute b_1, b_2, b_3, and b_4.
- Find a recurrence equation for $\{b_n\}$.
- Solve this recurrence when n is a power of 2.
- Prove that $b_n = n - 1$ for any $n \in \mathbb{N}$.

9 Generating functions

Question 9.1 Find the number of distinct integer solutions of the linear equation

$$x_1 + x_2 + x_3 \;=\; 17\,,$$

if the variables are constrained as follows:

1. $x_i \in \{0, 1, 2, \ldots, 6\}$.

2. $x_1, x_2 \in 2\mathbb{N}$ are even integers, and $x_3 \geq 0$ is an odd integer.

3. x_i are non-negative odd integers.

Question 9.2 Solve the following recurrences using generating functions:

1. $a_{n+1} - a_n = 3^n$, $n \geq 0$, $a_0 = 1$.

2. $a_{n+1} - a_n = n^2$, $n \geq 0$, $a_0 = 1$.

3. $a_n - a_{n-1} = 5^{n-1}$, $n \geq 1$, $a_0 = 1$.

4. $a_{n+2} - 3a_{n+1} + 2a_n = 0$, $n \geq 0$, $a_0 = 1$, $a_1 = 6$.

5. $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $n \geq 0$, $a_0 = 1$, $a_1 = 2$.

6. $a_{n+2} - 2a_{n+1} - a_n = 2^n$, $n \geq 0$, $a_0 = 1$, $a_1 = 2$.

Question 9.3 Prove that given a positive integer N, the number of *partitions* of N into *distinct* positive integers is equal to the number of partitions of N into *odd* positive integers. For instance, the integer $N = 4$ has two partitions into distinct positive integers ($3 + 1$ y 4), and two partitions into odd positive integers ($1 + 1 + 1 + 1$ y $3 + 1$). The integer $N = 6$ has four partitions into distinct positive integers ($1 + 2 + 3$, $2 + 4$, $1 + 5$, and 6), and four partitions into odd positive integers ($1 + 1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 3$, $3 + 3$, and $1 + 5$).

Question 9.4 Let F be the generating function that solves the following recurrence with n-dependent coefficients:

$$n\, a_n \;=\; 2(a_{n-1} + a_{n-2})\,, \quad n \geq 2\,, \quad a_0 = e\,, \quad a_1 = 2e\,.$$

Find the equation satisfied by F.
HINT: The equation may involve F'.

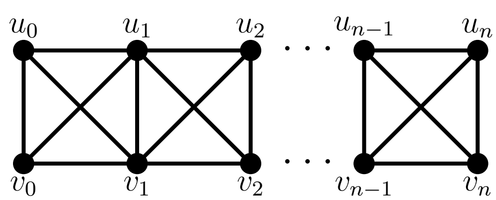
Question 9.5 Find, using generating functions, the number of distinct integer solutions of the equation

$$x_1 + x_2 + x_3 \;=\; N\,,$$

with $x_i \geq 0$.

10 Graph theory V

Question 10.1 Let G_n be the graph with $2(n + 1)$ vertices shown below:

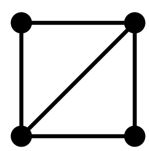


- Is G_n bipartite? Is it planar?
- Compute the number of perfect matchings a_n of G_n:
  - Show that a_n satisfies the recurrence $a_n = a_{n-1} + 2a_{n-2}$ for all $n \geq 3$, with $a_1 = 3$, and $a_2 = 5$.
  - Solve the recurrence equation and prove that $a_n = \frac{1}{3}\left[2^{n+2} + (-1)^{n+1}\right]$.

Question 10.2 Find the chromatic polynomial P_{C_n} of a cycle of n vertices:

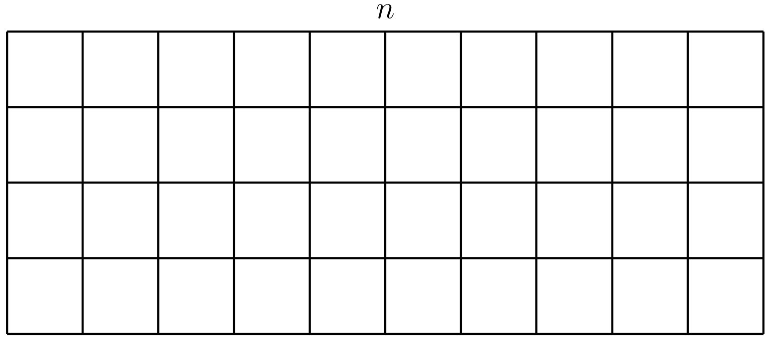
- Find the recurrence equation for P_{C_n} by using the deletion-contraction theorem.
- Solve the above recurrence.

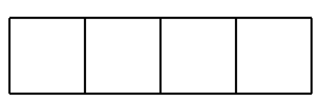
Question 10.3 Find the chromatic polynomial P_G of the graph G defined as follows:



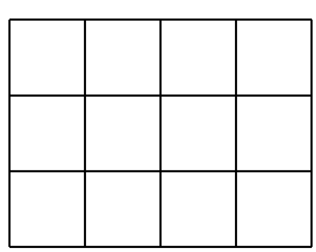
Question 10.4 We have a corridor of size $n \times 4$, and we want to tile it with tiles of sizes 4×1 and 4×3 in such a way that the corridor is completely tiled without overlapping tiles. Notice that each type of tile has two positions: horizontal and vertical.

n

4  Corridor of size $n \times 4$

1  4×1 tile

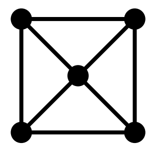
4

3  4×3 tile

4

Let a_n be the number of ways one can tile a corridor of length $n \geq 1$. Find a recurrence equation for a_n, and the necessary initial conditions needed to solve it. **Note**: you don't have to solve the recurrence equation!

Question 10.5 Find the chromatic polynomial P_G of the graph G defined as follows:



Question 10.6 Find the number of perfect matchings for the wheel graph W_n (please, recall that W_n has n vertices on the "wheel" and an extra vertex at the "axis").

Question 10.7 Find the chromatic polynomial of a forest with n vertices and k connected components.

Question 10.8 Let $G_n = (V_n, E_n)$ be the graph family defined as follows: the vertex set is $V_n = \{1, \ldots, n+1\}$, and the edge set is given by

$$E_n \;=\; \{\{i, i+1\} \colon 1 \leq i \leq n-1\} \cup \{\{n, 1\}\} \cup \{\{n+1, k\} \colon 1 \leq k \leq 3\}\,.$$

Compute its chromatic polynomial. `Hint:` draw G_n as a cycle of n vertices with the vertex $n+1$ placed inside that cycle.

11 Equivalence relations

Question 11.1 Let A and B be two sets, and let $f \colon A \to B$ be a certain function. Show that any binary relation defined on A of the form

$$a\mathcal{R}b \;\Leftrightarrow\; f(a) \;=\; f(b)\,, \quad a, b \in A\,,$$

is an equivalence relation for any f. Find the quotient set $A/\mathcal{R}$.

Question 11.2 Let A be the set $A = \{6, 10, 12, 18, 21, 40, 441, 1323\}$. We define the following binary relation $\mathcal{R}$ on A:

$$x\mathcal{R}y \;\Leftrightarrow\; x \text{ and } y \text{ have the same prime divisors}\,.$$

If $\mathcal{R}$ is an equivalence relation, find its classes of equivalence, or if it is not an equivalence relation, say which properties do not hold.

Question 11.3 Let A be a set, and $B \subset A$ a **fixed** subset of A. We now consider the power set $\mathcal{P}(A)$, and define the following relation on $\mathcal{P}(A)$: for any subsets $X, Y \subseteq A$:

$$X\mathcal{R}Y \quad \Leftrightarrow \quad X \cap B \;=\; Y \cap B\,.$$

1. Prove that $\mathcal{R}$ is an equivalence relation.

2. Compute its quotient set $\mathcal{P}(A)/\mathcal{R}$, and prove that there is a bijection between this quotient set and $\mathcal{P}(B)$.

Question 11.4 Let $\mathcal{R}$ be a relation defined on $\mathbb{N} \times \mathbb{N}$, such that $(a,b)\mathcal{R}(c,d)$ if and only if $a+b = c+d$. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$, and that there exists a bijection between the quotient set $(\mathbb{N} \times \mathbb{N})/\mathcal{R}$ and $\mathbb{N}$.

Question 11.5 Let A be the set $A = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$, and let $\mathcal{R}$ be a relation on A, such that $a\,\mathcal{R}\,b$ if and only if $\lfloor\sqrt{a}\rfloor = \lfloor\sqrt{b}\rfloor$. Show that $\mathcal{R}$ is an equivalence relation, find its classes of equivalence, and its quotient set.

Question 11.6 We define the relation $\mathcal{R}$ on $\mathbb{R}_2 = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ such that

$$(a,b)\mathcal{R}(c,d) \quad \Leftrightarrow \quad ad = bc\,.$$

Show that this is an equivalence relation, and obtain the quotient set $\mathbb{R}_2/\mathcal{R}$.

Question 11.7 A relation $\mathcal{R}$ defined on a set A is a **circular** relation if it verifies the following property:

$$a\mathcal{R}b \quad \text{and} \quad b\mathcal{R}c \quad \Rightarrow \quad c\mathcal{R}a\,.$$

Prove that a relation is an equivalence relation if and only if it is circular and reflexive.

Question 11.8 A relation $\mathcal{R}$ on a set A is **weakly transitive** if, for all elements $a,b,c,d \in A$, the relations $a\mathcal{R}b$, $b\mathcal{R}c$, and $c\mathcal{R}d$ imply that $a\mathcal{R}d$. Determine which one of the following two statements is true and which one is false (by proving the former, and giving a counter-example of the latter):

1. Every symmetric and weakly transitive relation is transitive.

2. Every reflexive, symmetric, and weakly transitive relation is an equivalence relation.

Question 11.9 The adjacency matrix of a binary relation $\mathcal{R}$ is given by

$$A_{\mathcal{R}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & b \\ 1 & a & c \end{pmatrix},$$

where $a,b,c = 0,1$. Which conditions should a, b, and c satisfy so that $\mathcal{R}$ becomes an equivalence relation?

Question 11.10 Show that the following relations are equivalence relations. Find the corresponding classes of equivalence and the quotient set $V/\mathcal{R}$:

1. $V = \mathbb{Z}$ and $v\mathcal{R}w$ if $|v-w|$ is a multiple of 2.

2. $V = \mathbb{Z}$ and $v\mathcal{R}w$ if $v^2 - w^2 = v - w$. Describe the equivalence class containing the element 2005.

3. $V = \mathbb{R}^2$ and $(x,y)\mathcal{R}(u,w)$ if $xy = uw$.

4. $V = \mathbb{R}^2$ and $(x,y)\mathcal{R}(u,w)$ if $(x-y)(x+y) = (u-w)(u+w)$.

5. $V = \mathbb{R}^2$ and $(x,y)\mathcal{R}(u,w)$ if $x^2+y^2 = u^2+w^2$.

12 Modular arithmetic

Question 12.1 Given $a = 92$ and $b = 84$, use Euclid's algorithm to compute $d = \gcd(a, b)$, Find integers $x, y \in \mathbb{Z}$ such that $ax + by = d$.

Question 12.2 The product of two natural numbers is 1260, and their lcm is 630. Find those numbers.

Question 12.3 How many positive divisors does the number $29338848000 = 2^8 \cdot 3^5 \cdot 5^3 \cdot 7^3 \cdot 11$ have? How many of them are multiple of 99? And how many of them are multiple of 39?

Question 12.4 Prove that $\log_2 3$ is an irrational number.

Question 12.5 Prove that 101 is a prime number.

Question 12.6 Prove that $6 \mid a(a + 1)(2a + 1)$ for any $a \in \mathbb{Z}$.

Question 12.7 Find the integer solutions of the Diophantine equations:

1. $28x + 36y = 44.$
2. $66x + 550y = 88.$

Question 12.8 Solve the following congruence equations:

1. $3x \equiv 5 \pmod{13}$.
2. $8x \equiv 2 \pmod{10}$.
3. $5x \equiv 7 \pmod{15}$.
4. $3x \equiv 9 \pmod{15}$.

Question 12.9 Find the remainder of the integer 2^{68} when divided by 19.

Question 12.10 Prove that $30 \mid (a^{25} - a)$ for any $a \in \mathbb{Z}$.

Question 12.11 Compute the last two digits of the integer 3^{1492}.

Question 12.12 Find the remainder of the hexadecimal number A1F05FFA01AFA0F when divided by 5.

13 Order relations

Question 13.1 Given the matrix representing the relation $\mathcal{R}$ on a set A

$$\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

1. Compute $\mathrm{Dom}(\mathcal{R})$ and $\mathrm{Im}(\mathcal{R})$.

2. Draw its Hasse diagram.

3. Find a total order compatible with $\mathcal{R}$.

Question 13.2 Let A the set $A = \{0, 1, 2\} \times \{2, 5, 8\}$, and let us define the order relation $\mathcal{R}$ on A such that $(a, b)\mathcal{R}(c, d) \Leftrightarrow (a + b) \mid (c + d)$. Find the maximal, minimal, maximum, and minimum elements of the poset $(A, \mathcal{R})$.

Question 13.3 Let us consider the relation $\mathcal{R}$ on $\mathbb{R}^2$ given by

$$(a, b)\mathcal{R}(c, d) \quad \Leftrightarrow \quad a \leq c \quad \text{and} \quad b \leq d\,.$$

Find the maximal and minimal elements of the set $C \subseteq \mathbb{R}^2$:

$$C \;=\; \{(x, y) \in \mathbb{R}^2 :\; x^2 + y^2 = 1\}\,.$$

Find $\sup(C)$ and $\inf(C)$ by considering C as a subset of $\mathbb{R}^2$.

Question 13.4 Let A be the set $A = \{n \in \mathbb{Z} \;:\; 2 \leq n \leq 12\}$, and let us define on A the order relation $\mathcal{R}$ given by

$$n\,\mathcal{R}\,m \Leftrightarrow n \mid m, \text{ or } n \text{ is prime and } n \leq m\,.$$

Tell the maximal, minimal, maximum, and minimum elements of the poset $(A, \mathcal{R})$

Question 13.5 Let us consider two binary relations on the set $\mathbb{N}$.

$$\begin{aligned} a\mathcal{R}_1 b \;&\Leftrightarrow\; \exists\, n \in \mathbb{N} \quad \text{such that } a = b^n\,,\\ a\mathcal{R}_2 b \;&\Leftrightarrow\; \exists\, n \in \mathbb{N} \cup \{0\} \quad \text{such that } a = b^n\,. \end{aligned}$$

1. Show that $\mathcal{R}_1$ is an order relation. Is $\mathcal{R}_2$ also an order relation? Is $\mathcal{R}_1$ a total order?

2. Find the Hasse diagram of both relations on the set

$$A = \{n \in \mathbb{N} :\; 1 \leq n \leq 9\}\,.$$

3. Find for $\mathcal{R}_1$ and $\mathcal{R}_2$ the maximal, minimal, maximum, and minimum elements on A. Find also the supremum and infimum of A as a subset of $\mathbb{N}$.

Question 13.6 Let us consider the cycle $C_4 = (V_4, E_4)$ with labelled vertices $V_4 = \{a, b, c, d\}$.

1. If A is the set of the spanning subgraphs of C_4:

$$A \;=\; \{G = (V_4, E) \mid E \subseteq E_4\}\,,$$

compute the cardinal of A.

2. We define on A the following equivalence relation $\mathcal{R}$: if $G_1, G_2 \in A$,

$$G_1\,\mathcal{R}\,G_2 \quad \Leftrightarrow \quad G_1 \text{ is isomorphic to } G_2.$$

Find the equivalence classes $[G]_{\mathcal{R}}$, and the quotient set $C = A/\mathcal{R}$.

3. We now define the order relation $\preceq$ on the quotient set C as follows: $[A]_{\mathcal{R}} \preceq [B]_{\mathcal{R}}$ if and only if there exist graphs $G_1 = (V_4, E_1) \in [A]_{\mathcal{R}}$ and $G_2 = (V_4, E_2) \in [B]_{\mathcal{R}}$ such that $E_1 \subseteq E_2$. Find the Hasse diagram associated to the set $(C, \preceq)$. Is $(C, \preceq)$ a totally ordered set?

4. Let $Z \subset C$ be the subset of C containing the classes of equivalence that contain at least one representative with two edges. Compute $\sup(Z)$ and $\inf(Z)$.

Question 13.7 Prove that for all $n \in \mathbb{N}$ the following equation holds:

$$3 \mid \left(4^n - 1\right).$$

Question 13.8 A polygon P is **convex** if, for any two points $a, b \in P$, the segment $\overline{ab}$ joining both points is totally contained inside the polygon. Prove that the sum of the interior angles of a convex polygon of $n \geq 3$ sides is $(n-2)\pi$.

Question 13.9 Prove that $1 + 2^n < 3^n$ for each $n \geq 2$.

Question 13.10 Prove using induction that the number of odd–degree vertices in any graph G is an even number.

Question 13.11 Prove by induction that the Fibonacci numbers F_n and F_{n+1} are relatively prime for all integer $n \geq 0$:

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2, \qquad F_0 = 0, \qquad F_1 = 1.$$

`Hint:` You do not need to solve the recurrence equation to perform the proof.

14 Lattices and Boolean algebras

Question 14.1 Let us consider the following relation $\mathcal{R}$ on $\mathbb{R}^2$:

$$(x, y)\ \mathcal{R}\ (z, t) \Leftrightarrow x^2 + y^2 = z^2 + t^2.$$

1. Prove that $\mathcal{R}$ is an equivalence relation, find its classes of equivalence, and the quotient set $\mathbb{R}^2/\mathcal{R}$.

2. We define a new relation $\preceq$ on the quotient set $\mathbb{R}^2/\mathcal{R}$ in the following way:

$$[(x, y)]_{\mathcal{R}} \preceq [(z, w)]_{\mathcal{R}} \Leftrightarrow x^2 + y^2 \leq z^2 + w^2.$$

   Show that this relation is well defined; i.e., that the result is independent of the representative elements chosen for each class of equivalence.

3. Prove that $\preceq$ is an order relation.

4. Prove that the set $(\mathbb{R}^2/\mathcal{R}, \preceq)$ is a lattice.

Question 14.2 Which ones of the following subsets of $\mathbb{N}$ are lattices with respect to the order $\preceq$ given by $x \preceq y \Leftrightarrow x \mid y$?

1. $\{5, 10, 15, 30\}$.

2. $\{2, 3, 5, 6, 10, 30\}$.

3. $\{1, 2, 3, 4, 6, 9\}$.

4. $\{1, 3, 7, 15, 21, 105\}$.

Question 14.3 Prove that the poset $(\mathbb{N}, |)$ with the binary operations $\sup(a, b) = \mathrm{lcm}(a, b)$ and $\inf(a, b) = \gcd(a, b)$ is a distributive lattice. Is it a Boolean algebra?

Question 14.4 Let us consider the set $A = \{a, b, c\}$. Find the Hasse diagram of the poset $(\mathcal{P}(A), \subseteq)$, and prove that it is also a lattice.

1. Find a sub-lattice of that lattice.

The set $(\mathcal{P}(A), \cup, \cap, \setminus, \varnothing, A)$ is a Boolean algebra. Tell whether the following subsets $S_i \subseteq \mathcal{P}(A)$ are Boolean algebras, Boolean sub-algebras of $(\mathcal{P}(A), \cup, \cap, \setminus, \varnothing, A)$, or none of those two options:

2. $S_1 = \{\varnothing, \{a, c\}, \{b\}, A\}$.

3. $S_2 = \{\{a, c\}, \{c\}, \{a\}, A\}$.

4. $S_3 = \{\varnothing, \{b, c\}, \{c\}, \{b\}\}$.

5. $S_4 = \{\varnothing, \{a, c\}, \{b, c\}, A\}$.

Question 14.5 Let A be the set $A = \{1, 2\} \times \{1, 2, 4, 3, 12\}$. We define on A the following binary relation

$$(x, y)\ \mathcal{R}\ (u, w) \quad \Leftrightarrow \quad x < u, \text{ or } (x = u \text{ and } y \mid w)\,.$$

- Prove that $\mathcal{R}$ is an order relation.
- Find its Hasse diagram.
- Prove that $(A, \mathcal{R})$ is a lattice.
- Is $(A, \mathcal{R})$ a bounded lattice? If so, which are the elements 1 and 0 of such lattice?
- Is $(A, \mathcal{R})$ a Boolean algebra?

A Answers to exercises

Set 1. Set theory and functions

Question 1.1 1. True. 2. False. 3. False. 4. True. 5. True. 6. False. 7. False.

Question 1.2 `Hint`: use Venn diagrams o truth tables.

Question 1.3 1) $\overline{B}$. 2) $B \cap C$.

Question 1.4

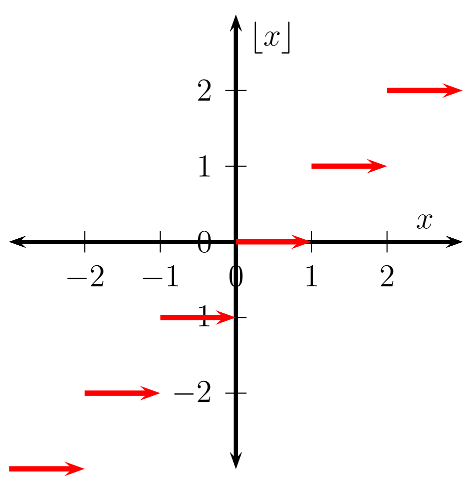
- f is not injective nor surjective.
- g is inyective; but it is not surjective.

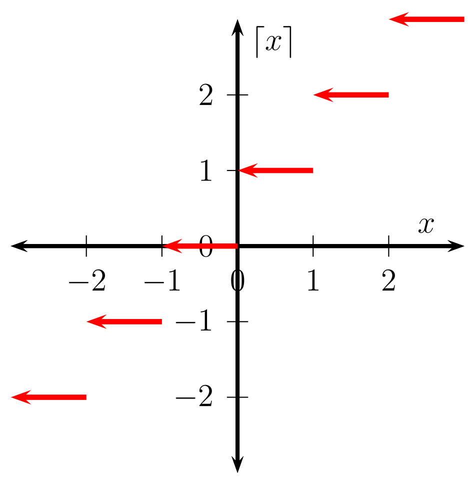
Question 1.5 F is bijective, and there exists the inverse function $F^{-1} \colon \mathbb{R} \setminus \{1/2\} \to \mathbb{R} \setminus \{-1/2\}$ defined as follows:

$$F^{-1}(x) = \frac{x-3}{1-2x}.$$

Question 1.6

1. $\lfloor 1/2 \rfloor = 0$. $\lceil 1/2 \rceil = 1$. $\lfloor -1/2 \rfloor = -1$. $\lceil -1/2 \rceil = 0$. $\lfloor \pi \rfloor = 3$. $\lceil \pi \rceil = 4$. $\lfloor 1/2 + \lceil 1/2 \rceil \rfloor = 1$. $\lceil \lfloor 1/2 \rfloor + \lceil 1/2 \rceil + 1/2 \rceil = 2$.
2. The graphics are





3. 72 452 groups.

Question 1.7

1. Injective, not surjective, and not bijective.
2. Injective, surjective, and bijective.
3. It is not injective, not surjective, and not bijective.
4. It is not injective, not surjective, and not bijective.

Question 1.8

1. $q = 5$ and $r = 4$.
2. $q = 2$ and $r = 5$.
3. $q = -1$ and $r = 2$.
4. $q = -7$ and $r = 10$.
5. $q = 1$ and $r = 1$.

Question 1.9 $\gcd(500, 120) = 2^2 \times 5 = 20$ and $\operatorname{lcm}(500, 120) = 2^3 \times 3 \times 3^3 = 3000$.

Set 2. Basic combinatorics I

Question 2.1

1. 2^n.
2. 2^{n-2} with $n \geq 2$.
3. $\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$.
4. n.
5. $\sum_{k=3}^{n-3} \binom{n}{k}$ with $n \geq 6$.
6. $2^{n-2} + 2^{n-3} - 2^{n-5}$ with $n \geq 5$.
7. $2^{\lceil n/2 \rceil}$.

Question 2.2

1. 90 720.
2. 50 400.
3. 80 640.
4. 16 800.
5. 33 600.
6. 8 400.

Question 2.3 $3^5 - 3 \times 2^5 + 3 = 150$.

Question 2.4 120.

Question 2.5 If we assume that the spaces between two consecutive flags are immaterial, the results are: 1) 1 464. 2) 1 884.

Question 2.6 $\binom{8}{3}$.

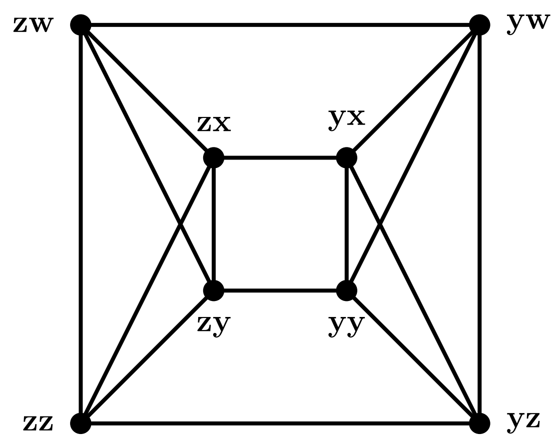
Question 2.7 1) 360. 2) 120.

Question 2.8 `Hint`: Use the pigeon-hole principle.

Set 3. Graph theory I

Question 3.1

1. 8.
2. A graphical representation of the graph G is the following:



3. The common degree is 4.
4. It is not bipartite.

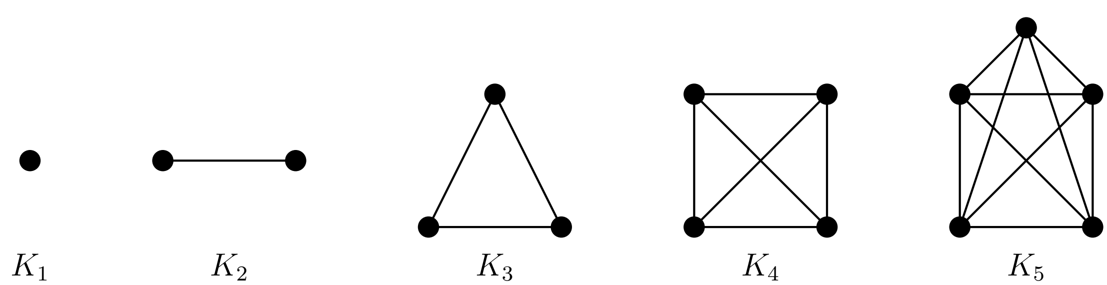
Question 3.2 10.

Question 3.3

1. $|V| = 9$.
2. There are only two possible cases: if the degree is$d = 2$, then $|V| = 6$; and if the degree is $d = 3$, then $|V| = 4$.
3. $|V| = 6$.

Question 3.4

1. A graphical representation of the graphs K_n with $1 \leq k \leq 5$ is:



2. The degree of K_n is $n-1$.

3. $|E_n| = \binom{n}{2}$.

4. `Hint`: you have to prove that $V_n \subset V_m$ and $E_n \subset E_m$ whenever $n < m$.

Question 3.5

1. False for any $n \geq 3$.

2. True for any $n \geq 3$.

3. True for any $n \geq 3$.

4. True for any even $n \geq 4$.

Question 3.6 `Hint`: Use the pigeon-hole principle.

Question 3.7 $|V|_{\min} = 5$.

Question 3.8 The adjacency matrices A_1 (with the vertex ordering (v_1, v_2, v_3, v_4)) and A_2 (with the vertex ordering (u_1, u_2, u_3, u_4)), and the permutation matrix P are:

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Question 3.9

1. Walk of length 4.

2. Closed walk of length 5.

3. Closed walk of length 6.

4. Cycle of length 5.

Set 4. Graph theory II

Question 4.1

1. $n(n-1)(n-2)$ cycles.

2. $n-2$ triangles.

Question 4.2 `Hint`: Note that paraffins can be represented as simple connected graphs with two types of vertices: carbon atoms (of degree 4) and hydrogen atoms (of degree 1).

Question 4.3 `Hint`: Use that $V = V_1 \cup V_3$ where V_1 (resp. V_3) is the set of vertices of degree 1 (resp. 3).

Question 4.4 `Hint`: Use the hand-shake lemma and Euler's theorem.

Question 4.5 They are not isomorphic.

Question 4.6

1. C_{2n}.
2. There does not exist such graph.
3. A graph with two connected components: K_{n-1} and a isolated vertex.
4. Two vertices joined by four edges.
5. P_4.
6. C_2.

Question 4.7 11 trees.

Question 4.8

- $|V| = 8$ and $|E| = 19$.
- $d(\varnothing) = d(X) = 7$. The other vertices have $d(v) = 4$. G is not regular.
- G is not planar.
- G is not bipartite.

Question 4.9 `Hint`: Which is the simple graph with n vertices and $\binom{n}{2}$ edges?

Question 4.10 `Hint`: The average degree of a graph $G = (V, E)$ is defined as follows:

$$\bar{d} = \frac{1}{|V|} \sum_{v \in V} d(v) \, .$$

Use also the following result: Let G be a connected graph containing a cycle. Then, if we remove any edge of that cycle, the resulting graph is also connected.

Question 4.11

a) G is simple.

b) $|V| = 8$ and $|E| = 12$.

c) G is regular with $d = 3$.

d) $n_{ij} \in \{0, 6, 7\}$.

e) The length of the shortest cycle in G is $\ell_{\min} = 4$.

Set 5. Graph theory III

Question 5.1

1. It is not bipartite, but it is planar.
2. There are no spanning trees.

Question 5.2 There is no spanning tree of weight ≤ 12.

Question 5.3

1. The path is (A, B, E, G) with length 78.
2. The same as above.

Question 5.4 $d(a, h) = 16$, $d(a, e) = 17$, $d(d, a) = 19$, $d(d, g) = 11$, and $d(b, e) = 13$.

Question 5.5

1. One possible minimum-weight path is (A, B, G, H, K, M) with weight $\omega = 17$.
2. One possible minimum-weight spanning tree has the edge set

$$E = \{\{B, E\}, \{H, K\}, \{B, G\}, \{K, M\}, \{h, L\}, \{F, E\}, \{G, J\},$$
$$\{H, D\}, \{A, B\}, \{F, I\}, \{C, G\}, \{C, H\}\} .$$

Its weight is $\omega = 43$.

Question 5.6 $0 < x \leq 3$.

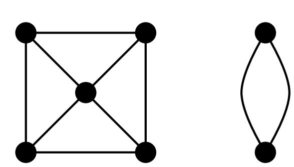
Question 5.7

- The minimum cost from A to I is equal to 19. The total cost of the constructor's design depends on the found tree: it can be in the range from 40 to 43.
- The cost of the alternative design is 39.

Set 6. Graph theory IV

Question 6.1

1. Using Kruskal's algorithm, a spanning tree $T = (V, E)$ would be given by $E = \{\{a, b\}, \{e, g\}, \{b, e\}, \{e, c\}, \{f, g\}, \{f, d\}\}$ with weight $\omega = 10$.
2. It is not bipartite.
3. There are no Euler tours/trails, and there is a Hamilton cycle.
4. A possible example is:



Question 6.2 From left to right:

1. It not an Eulerian/Hamiltonian graph. It contains an Euler trail and a Hamilton path.
2. It does not have any Euler tour/trail. It does not have any Hamilton cycle/path.
3. It does not have any Euler tour/trail. It is a Hamiltonian graph.

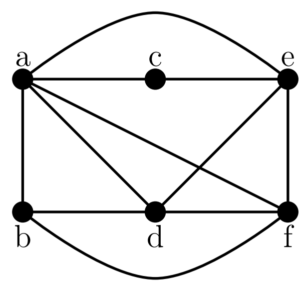
Question 6.3 It contains an Euler trail. One Euler trail would be:

$$(h, d, e, h, i, e, a, d, f, g, c, f, b, e, f, i)\,.$$

Question 6.4 6 hours.

Question 6.5

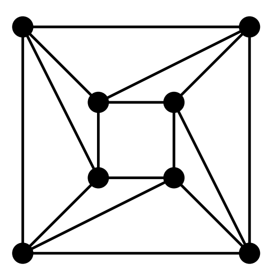
1. The interval graph is the following:



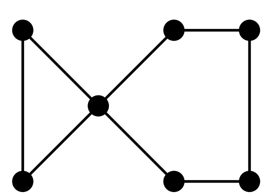
2. It is not bipartite/Eulerian. It is a Hamiltonian graph.

Question 6.6

- C_7.
- One possible example is:



- W_7.
- One possible example is:



- There is no graph satisfying these conditions.

Question 6.7

1. $\chi(G) = 3$.
2. There is no such spanning tree.
3. There is no Euler trail. I need to add at least three edges so that G becomes Eulerian.

Question 6.8

- $|V_n| = 2^{n-1}$.
- G_n is regular with degree $d = \binom{n}{2}$.
- $|E_n| = 2^{n-2}\binom{n}{2}$.
- The values of n for which G_n is Eulerian are:

$$n = \begin{cases} 4p & \text{con } p \in \mathbb{N}, \\ 4p+1 & \text{con } p \in \mathbb{N}, \end{cases}$$

Set 7. Basic combinatorics II

Question 7.1

1. $10 \times 9 \times 8 \times 7$.
2. $\binom{10}{4}$.
3. $\binom{13}{4}$.
4. 10^4.

Question 7.2 $\binom{11}{6}$.

Question 7.3 $\binom{a+1}{k+1}\binom{b-1}{k}$.

Question 7.4 $\binom{12}{4}$.

Question 7.5 $\binom{n-p+1}{p}$.

Question 7.6 $\binom{19}{2}$.

Question 7.7 $\binom{16}{2}$.

Question 7.8 $\binom{19}{2} - 3\binom{12}{2} + 3\binom{5}{2} = 3$.

Question 7.9 $\binom{17}{6} - 7\binom{12}{6} + \binom{7}{2}\binom{7}{1} = 6055$.

Question 7.10

1. 4^8.
2. $\binom{r-1}{n-1}$.
3. $\binom{19}{n-1}$.

Question 7.11

1. $\dfrac{20!}{(4!)^5\, 5!}$.
2. $\dfrac{20!}{(4!)^5}$.

Set 8. Recurrences

Question 8.1

- $a_n = a_{n-1} + n$ for all $n \geq 2$ and $a_1 = 2$.
- $a_n = (n^2 + n + 2)/2$ for all $n \geq 1$.

Question 8.2 29525.

Question 8.3 $a_n = 2a_{n-1} + 3a_{n-2}$ for all $n \geq 2$ with $a_0 = 1$ y $a_1 = 2$.

Question 8.4 $a_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+2}\right]$ for all $n \geq 1$.

Question 8.5 $a_n = \frac{1}{2}(3^n - 1)$ for all $n \geq 1$.

Question 8.6 $a_n = (3-n)\, 2^{n-2}$ for all $n \geq 1$.

Question 8.7 $a_n = \dfrac{1}{2}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right]$ for all $n \geq 1$.

Question 8.8 $a_n = 2^n + 2(-1)^n$ for all $n \geq 1$.

Question 8.9

- $b_1 = 0$, $b_2 = 1$, $b_3 = 2$ y $b_4 = 3$.
- If $n = 2p$ is even $(p \geq 1)$, $b_{2p} = 2b_p + 1$. If $n = 2p-1$ is odd $(p \geq 1)$, $b_{2p-1} = b_p + b_{p-1} + 1$.
- $b_{2^k} = 2^k - 1$ for all $k \geq 0$.
- You need to use the strong version of the induction principle you have seen in Calculus. However, you can go back to this problem after Section 13 (order relations).

Set 9. Generating functions

Question 9.1

- $\binom{19}{2} - 3\binom{12}{2} + 3\binom{5}{2} = 3.$
- $\binom{8}{2} = 28.$
- $\binom{9}{2} = 36.$

Question 9.2

1. $a_n = \frac{1}{2}\left(3^n + 1\right), \quad n \geq 0.$
2. $a_n = 4\binom{n+1}{1} - 5\binom{n+2}{2} + 2\binom{n+3}{3}, \quad n \geq 0.$
3. $a_n = \frac{1}{4}\left(5^n + 3\right), \quad n \geq 0.$
4. $a_n = 5 \times 2^n - 4, \quad n \geq 0.$
5. $a_n = 2^n, \quad n \geq 0.$
6. $a_n = -2^n - \frac{1}{\sqrt{2}}(1-\sqrt{2})^{n+1} + \frac{1}{\sqrt{2}}(1+\sqrt{2})^{n+1}, \quad n \geq 0.$

Question 9.3 The two generating functions are:

- $f_1(x) = \displaystyle\prod_{n=1}^{\infty}(1 + x^n).$
- $f_2(x) = \displaystyle\prod_{n=1}^{\infty}\frac{1}{(1 - x^{2n-1})}.$

Question 9.4 $F'(x) = 2(1+x)\,F(x).$

Question 9.5 $\binom{2+N}{2}.$

Set 10. Graph theory V

Question 10.1

- It is not bipartite. It is planar.
- The rest of the results are included in the exercise.

Question 10.2

- $P_{C_n}(q) = -P_{C_{n-1}}(q) + q(q-1)^{n-1}$ for any $n \geq 4$ with $P_{C_3}(q) = P_{K_3}(q) = q(q-1)(q-2).$
- $P_{C_n}(q) = (q-1)^n + (-1)^n(q-1).$
- $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3.$

Question 10.3

- $P_G(q) = q(q-1)(q-2)^2$.
- $\chi(G) = 3$.

Question 10.4 quad $a_n = a_{n-1} + a_{n-3} + 3a_{n-4}$ for any $n \geq 5$ and initial conditions $a_1 = a_2 = 1$, $a_3 = 2$, and $a_4 = 6$.

Question 10.5

- $P_G(q) = q(q-1)(q-2)(q^2 - 5q + 7)$.
- $\chi(G) = 3$.

Question 10.6 The solution depends on the parity of n:

$$N_{\text{e.p.}} = n\,(n \mod 2) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Question 10.7 $P_{B_{n,k}}(q) = q^k\,(q-1)^{n-k}$.

Question 10.8 $P_{G_n} = (q-1)(q-2)[(q-2)(q-1)^{n-2} + 2(-1)^n]$ for any $n \geq 3$.

Set 11. Equivalence relations

Question 11.1 The quotient set is isomorphic to $\mathrm{Im}(f)$.

Question 11.2 It is an equivalence relation (use Question 11.1), and the equivalence classes are $[6]_{\mathcal{R}} = \{6, 12, 18\}$, $[10]_{\mathcal{R}} = \{10, 40\}$, and $[21]_{\mathcal{R}} = \{21, 441, 1323\}$.

Question 11.3

1. Use Question 11.1.
2. $\mathcal{P}(A)/\mathcal{R} = \{[C]_{\mathcal{R}} \colon C \in \mathcal{P}(B)\}$, and it is isomorphic to $\mathcal{P}(B)$.

Question 11.4

1. Use Question 11.1.
2. $\mathbb{N} \times \mathbb{N}/\mathcal{R} = \{[(N,1)]_{\mathcal{R}} \colon N \geq 1\}$, and it is isomorphic to $\mathbb{N}$. $\mathbb{N}$ is obviously isomorphic to $\mathbb{N} \setminus \{1\}$: use the bijective function $f \colon \mathbb{N} \to \mathbb{N} \setminus \{1\}$ such that $f(n) = n + 1$.

Question 11.5

1. Use Question 11.1.
2. The equivalence classes are $[1]_{\mathcal{R}} = \{1, 2, 3\}$, $[4]_{\mathcal{R}} = \{4, 5, 6, 7, 8\}$, and $[9]_{\mathcal{R}} = \{9, 10, 11, 12, 13, 14, 15\}$.
3. $A/\mathcal{R} = \{[1]_{\mathcal{R}}, [4]_{\mathcal{R}}, [9]_{\mathcal{R}}\}$.

Question 11.6

1. Use Question 11.1.
2. $\mathbb{R}_2/\mathcal{R} = \{[(K,1)]_{\mathcal{R}} \colon K \in \mathbb{R}\}$.

Question 11.7 You have to show the implications in both directions:

1. If $\mathcal{R}$ is an equivalence relation, then $\mathcal{R}$ is circular and reflexive.
2. If $\mathcal{R}$ is a circular and reflexive relation, then $\mathcal{R}$ is an equivalence relation.

Question 11.8 1) False. 2) True.

Question 11.9 $a = b = 0$, and $c = 1$.

Question 11.10

1. It is an equivalence relation by Question 1.1 with $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x \mod 2$. As $\operatorname{Im} f = \{0,1\}$, there are two classes of equivalence:

$$\begin{aligned} [0]_{\mathcal{R}} &= \{x \in \mathbb{Z} \colon x \text{ is even}\}, \\ [1]_{\mathcal{R}} &= \{x \in \mathbb{Z} \colon x \text{ is odd}\}. \end{aligned}$$

   and the quotient set is

$$\mathbb{Z}/\mathcal{R} = \{[0]_{\mathcal{R}}, [1]_{\mathcal{R}}\}.$$

2. It is an equivalence relation by Question 1.1 with $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2 - x$. If v, w belongs to the same equivalence class, they should satisfy

$$v^2 - w^2 = (v-w)(v+w) = v - w \Rightarrow \begin{cases} v = w \\ v + w = 1 & \text{if } v \neq w \end{cases}$$

   Therefore, the equivalence classes have two elements:

$$[n]_{\mathcal{R}} = \{n, 1-n\}.$$

   The quotient set is

$$\mathbb{Z}/\mathcal{R} = \{[n]_{\mathcal{R}} \colon n \in \mathbb{N}\},$$

   and it is isomorphic to $\mathbb{N}$. The equivalence class containing 2005 is

$$[2005]_{\mathcal{R}} = \{2005, -2004\}.$$

3. It is an equivalence relation by Question 1.1 with $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = xy$. The equivalence classes are given by those points in $\mathbb{R}^2$ satisfying $xy = \alpha$, where α is a constant: each equivalence class is a 45°-rotated standard hyperbola. In particular, $\alpha > 0$ corresponds to a hyperbola on the first and third quadrants; $\alpha = 0$ to the coordinate axes; and $\alpha < 0$ to a hyperbola on the second and fourth quadrants. The equivalence classes are:

$$[(1,\alpha)]_{\mathcal{R}} = \{(x,y) \in \mathbb{R}^2 \colon xy = \alpha\}.$$

   The quotient set is:

$$\mathbb{R}^2/\mathcal{R} = \{[(1,\alpha)]_{\mathcal{R}} \colon \alpha \in \mathbb{R}\},$$

   therefore, such set is isomorphic to $\mathbb{R}$.

4. It is an equivalence relation by Question 1.1 with $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = x^2 - y^2$. The equivalence classes are given by those points in $\mathbb{R}^2$ satisfying $x^2 - y^2 = \alpha$, where α is a constant. The equivalence classes are standard hyperbolas with asymptotes $y = \pm x$. In particular, $\alpha > 0$ corresponds to a hyperbola that crosses the x-axis at $\pm\sqrt{\alpha}$. If $\alpha = 0$, the equivalence class consists precisely in the lines $y = \pm x$. Finally, if $\alpha < 0$, we have a 90°-rotated hyperbola that crosses the y-axis at $y = \pm\sqrt{-\alpha}$. The equivalence classes are:

$$\begin{aligned} [(\sqrt{\alpha},0)]_{\mathcal{R}} &= \{(x,y) \in \mathbb{R}^2 \colon x^2 - y^2 = \alpha\}\,, \quad \alpha \geq 0 \\ \left[(0,\sqrt{-\alpha})\right]_{\mathcal{R}} &= \{(x,y) \in \mathbb{R}^2 \colon x^2 - y^2 = \alpha\}\,, \quad \alpha < 0 \end{aligned}$$

The quotient set is:

$$\mathbb{R}^2/\mathcal{R} \;=\; \{[(\sqrt{\alpha},0)]_{\mathcal{R}} \colon \alpha \geq 0\} \cup \{[(0,\sqrt{-\alpha})]_{\mathcal{R}} \colon \alpha < 0\}\,,$$

therefore, such set is isomorphic to $\mathbb{R}$.

5. It is an equivalence relation by Question 1.1 with $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = x^2 + y^2$. The equivalence classes are given by those points in $\mathbb{R}^2$ satisfying $x^2 + y^2 = \alpha^2 \geq 0$, where α is a constant. Therefore, the equivalence classes are circumferences of radius $\alpha \geq 0$:

$$[(\alpha,0)]_{\mathcal{R}} \;=\; \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = \alpha^2 \quad \text{and} \quad \alpha \geq 0\}\,.$$

The quotient set is

$$\mathbb{R}^2/\mathcal{R} \;=\; \{[(\alpha,0)]_{\mathcal{R}} \colon \alpha \geq 0\}\,,$$

therefore, such set is isomorphic to $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geq 0\}$.

Set 12. Modular arithmetic

Question 12.1 $\gcd(92,84) = 4 = 11 \cdot 84 - 10 \cdot 92$.

Question 12.2 The set of solutions is:

$$\{(2,630),(10,126),(14,90),(18,70),(630,2),(126,10),(90,14),(70,18)\}\,.$$

Question 12.3 1) 1728. 2) 576. 3) 0.

Question 12.4 Use the fundamental theorem of arithmetic.

Question 12.5 You only need to prove that $p \nmid 101$ for $p = 2,3,5,7$. Why?

Question 12.6 As any integer a can be written as $a = 6q + r$ with $r \in \{0,1,2,3,4,5\}$, it suffices to prove the statement for each possible remainder r.

Question 12.7

- The solutions are

$$\begin{aligned} x_k &= 11 \cdot 4 + \frac{36}{4}k \;=\; 44 + 9k\,, \\ y_k &= 11 \cdot (-3) - \frac{28}{4}k \;=\; -33 - 7k\,, \end{aligned}$$

for any $k \in \mathbb{Z}$.

- The solutions are

$$\begin{aligned} x_k &= 4 \cdot (-8) + \frac{550}{22}k = -32 + 25k\,, \\ y_k &= 4 \cdot 1 - \frac{66}{22}k = 4 - 3k\,, \end{aligned}$$

  for any $k \in \mathbb{Z}$.

Question 12.8

1. $x \equiv 6 \pmod{13}$.
2. There are two solutions: $x \equiv 4 \pmod{10}$ and $x \equiv 9 \pmod{10}$.
3. There are no solutions.
4. There are three solutions: $x \equiv 3 \pmod{15}$, $x \equiv 8 \pmod{15}$, and $x \equiv 13 \pmod{15}$.

Question 12.9 6.

Question 12.10 `Hint:` First prove that $p \mid (a^{25} - a)$ for $p = 2, 3, 5$.

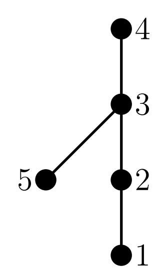
Question 12.11 41.

Question 12.12 2.

Set 13. Order relations

Question 13.1

1. $\mathrm{Dom}(\mathcal{R}) = \mathrm{Im}(\mathcal{R}) = A$.
2. The Hasse diagram is:



3. $1 \preceq 2 \preceq 5 \preceq 3 \preceq 4$.

Question 13.2

1. Set of maximal elements $= \{(1,8), (1,5), (0,8), (2,8), (2,5)\}$.
2. Set of minimal elements $= \{(1,2), (0,2), (0,5), (2,5)\}$.
3. $\max(A)$ and $\min(A)$ do not exist.

Question 13.3

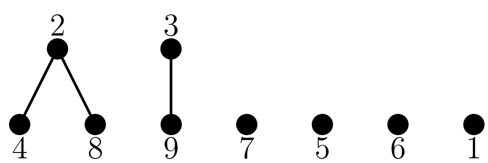
1. Set of maximal elements $= \{(x,y) \in C \colon x, y \geq 0\}$.
2. Set of minimal elements $= \{(x,y) \in C \colon x, y \leq 0\}$.
3. $\sup(C) = (1,1)$ and $\inf(C) = (-1,-1)$.

Question 13.4

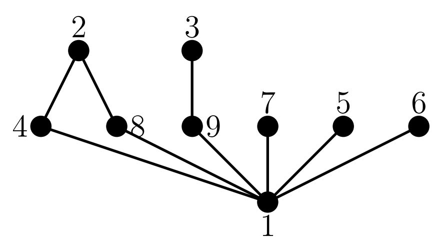
1. Set of maximal elements $= \{8, 9, 10, 12\}$.
2. Set of minimal elements $= \{2\}$.
3. $\max(A)$ does not exist; but $\min(A) = 2$.

Question 13.5

1. The proof is straightforward. $\mathcal{R}_2$ is an order relation. Both relations are partial orders.
2. The Hasse diagram for $\mathcal{R}_1$ is:



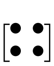
The Hasse diagram for $\mathcal{R}_2$ is:

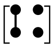


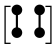
3. $\mathcal{R}_1$: set of maximal elements $= \{1, 2, 3, 5, 6, 7\}$; set of minimal elements $= \{1, 4, 8, 9, 5, 6, 7\}$; $\max(A)$ and $\min(A)$ do not exist.
   $\mathcal{R}_2$: set of maximal elements $= \{2, 3, 5, 6, 7\}$; set of minimal elements $= \{1\}$; $\max(A)$ does not exist; but $\min(A) = 1$.
4. $\mathcal{R}_1$: $\text{major}(A) = \text{minor}(A) = \varnothing$, therefore, $\sup(A)$ and $\inf(A)$ do not exist.
   $\mathcal{R}_2$: $\text{major}(A) = \varnothing$, therefore $\sup(A)$ does not exist; $\text{minor}(A) = \{1\}$, and $\inf(A) = 1$.

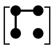
Question 13.6

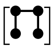
1. $|A| = 16$.
2. There are 6 equivalence classes (for simplicity let us denote them C_j):

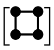
   (a) $C_0 = [$$]_{\mathcal{R}} = \{H = (V_4, \varnothing)\}$.

(b) $C_1 = [$$]_{\mathcal{R}} = \{H = (V_4, E) \colon |E| = 1\}$.

(c) $C_{2a} = [$$]_{\mathcal{R}} = \{H = (V_4, E) \colon |E| = 2 \quad \text{and } H \text{ is a p.m. of } C_4\}$.

(d) $C_{2b} = [$$]_{\mathcal{R}} = \{H = (V_4, E) \colon |E| = 2 \quad \text{and } H \text{ is not a p.m. of } C_4\}$.

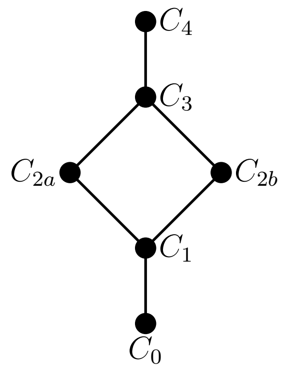
(e) $C_3 = [$$]_{\mathcal{R}} = \{H = (V_4, E) \colon |E| = 3\}$.

(f) $C_4 = [$$]_{\mathcal{R}} = \{H = (V_4, E) \colon |E| = 4\}$.

where p.m. means perfect matching.

3. $C = A/\mathcal{R} = \{C_0, C_1, C_{2a}, C_{2b}, C_3, C_4\}$.

4. The Hasse diagram is:

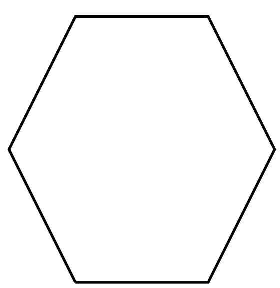


5. It is not a totally ordered set.

6. $\sup(Z) = C_3$ and $\inf(Z) = C_1$.

Question 13.7 `Note:` this statement can be proven by using modular arithmetic (without induction); but we look for a proof that makes explicit use of mathematical induction.

Question 13.8 The base case is $n = 3$: the sum of the internal angles of a triangle is equal to $\pi = (3 - 2)\pi$ (Euclid's axiom). A convex polygon of $n = 6$ sides is the following (hexagon):



Question 13.9 Once the induction hypothesis is used, the final result followed by using the inequalities $3 > 2 > 1$.

Question 13.10 The base case corresponds to the trivial graph of n vertices $G = (V, \varnothing)$ (with $|V| = n \geq 1$). Then, you apply induction on the number of edges $|E|$.

Question 13.11 In the inductive step, it is easier to use a proof by contradiction (*reductio ad absurdum*).

Set 14. Lattices and Boolean algebras

Question 14.1

1. The relation $\mathcal{R}$ can be written as

$$(x, y)\ \mathcal{R}\ (z, t) \ \Leftrightarrow\ f(x, y) \ = \ f(z, t)\,,$$

with $f\colon \mathbb{R}^2 \to [0, \infty)$, therefore $\mathcal{R}$ is an equivalence relation. The equivalence classes are:

$$[(R, 0)]_{\mathcal{R}} \ = \ \left\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = R^2\right\}\,,$$

with $R \geq 0$. The quotient set is

$$\mathbb{R}^2/\mathcal{R} \ = \ \{[(R, 0)]_{\mathcal{R}} \colon R \geq 0\}\,.$$

2. It is trivial:

$$[(x, y)]_{\mathcal{R}} \ \preceq\ [(z, w)]_{\mathcal{R}} \ \Leftrightarrow\ f(x, y) \ \leq\ f(z, w)\,.$$

3. From the previous characterization, it follows that $\preceq$ is an order relation.

4. It suffices to prove that $(\mathbb{R}^2/\mathcal{R}, \preceq)$ is a totally ordered set.

Question 14.2 1) Lattice. 2) Not a lattice. 3) Not a lattice. 4) Lattice.

Question 14.3

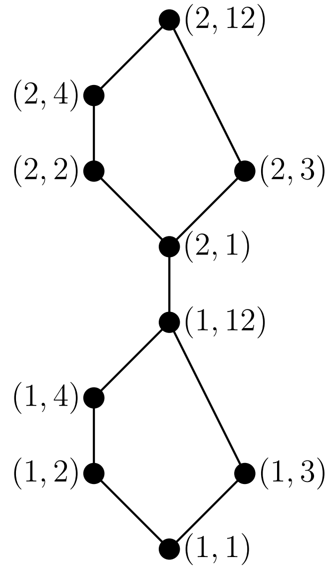
- Use the fundamental theorem of arithmetic and the representation of $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ as products of the prime factors appearing in the prime decomposition of a and b.
- The proof that it is a distributive lattice follows from the proof that $\min(n, \max(m, r)) = \max(\min(n, m), \min(n, r))$ for all integers $n, m, r \geq 0$.
- It is not a Boolean algebra.

Question 14.4 If $\mathcal{A} = (\mathcal{P}(A), \cup, \cap, \setminus, \varnothing, A)$, then

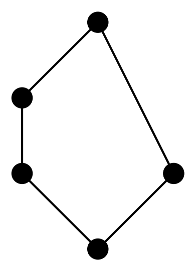
1. $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.
2. S_1 is a Boolean sub-algebra of $\mathcal{A}$.
3. S_2 is not a Boolean sub-algebra of $\mathcal{A}$. It is not a Boolean algebra.
4. S_3 is not a Boolean sub-algebra of $\mathcal{A}$; but it is a Boolean algebra.
5. S_4 s not a Boolean sub-algebra of $\mathcal{A}$; but it is a Boolean algebra.

Question 14.5

- The Hasse diagram is



- It is enough to prove that given two elements a, b, $\sup(a, b)$ and $\inf(a, b)$ both exist.
- It is bounded with $0 = (1, 1)$ and $1 = (2, 12)$.
- It is not a Boolean algebra because it is not a distributive lattice (it contains N_5



  as a sub-lattice).