CALCULUS – Second Partial Exam

Bachelor in Computer Science and Engineering

ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

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Name		GROUP	87

1. *[1 point]* Let

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x < 0\,, \\ xe^x + 1 & \text{if} \quad x \geq 0\,. \end{cases}$$

(a) Study the continuity and differentiability of f in the domain.

(b) Prove whether f is bounded in the interval $[-\pi, \pi]$.

SOLUTION

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0$, the given function is continuous and differentiable as defined in terms of continuous and differentiable elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $\lim_{x\to 0} f(x) = f(0) = 1$. In addition, in order for $f(x)$ to be differentiable at $x = 0$, the following lateral limits must provide the same result:

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{(xe^x + 1) - 1}{x} = \lim_{x\to 0^+} e^x = 1\,,$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{\sin(x) - x}{x^2} = \lim_{x\to 0^-} \frac{\cos(x) - 1}{2x} = \lim_{x\to 0^-} \frac{-\sin(x)}{2} = 0\,.$$

Note that for the calculation of $f'_-(0)$ l'Hôpital's rule has been applied twice. Thus, being $f'_+(0) \neq f'_-(0)$, we can conclude that the given function is not differentiable at $x = 0$.

(b) Since $f(x)$ is continuous for all $x \in \mathbb{R}$, it is also continuous in the closed and bounded interval $[-\pi, \pi]$. Hence, by the Weierstrass' theorem, it is bounded in the same interval.

2. *[1 point]* Calculate

$$\lim_{x\to 0} \frac{2x + \sqrt{1+x^2} + x\arctan(x) - e^{3x}\left[1 - \ln(1+x)\right]}{7x^2}$$

using appropriate Taylor polynomials for the involved functions.

SOLUTION

In the given limit, we have $x \to 0$, hence we can approximate all involved elementary functions by suitable Maclaurin polynomials, namely

$$\lim_{x\to 0} \frac{2x + 1 + \frac{1}{2}x^2 + o(x^2) + x\left[\,x + o(x)\,\right] - \left[\,1 + 3x + \frac{9}{2}x^2 + o(x^2)\,\right]\left[\,1 - x + \frac{1}{2}x^2 + o(x^2)\,\right]}{7x^2}\,.$$

Finally, after simplifying the previous expression and retaining terms up to degree 2, we get

$$\lim_{x\to 0} \frac{-\frac{1}{2}x^2 + o(x^2)}{7x^2} \;=\; -\frac{1}{14}\,.$$

3. *[1 point]* Approximate the value

$$\frac{1}{e^{1/3}}$$

using a Taylor polynomial of suitable degree such that the involved error is smaller than 10^{-3}.

SOLUTION

The value $e^{-1/3}$ can be calculated by evaluating the function $f(x) = e^{x}$ at $x = -1/3$. Such function can be expressed by means of the Taylor's theorem as

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n(x)\,,$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{e^{c}}{(n+1)!}\, x^{n+1}\,,$$

with $c \in (x, 0)$. Hence, at $x = -1/3$, we can estimate the involved error as

$$|R_n\,(-1/3)| = \frac{e^{c}}{3^{n+1}\,(n+1)!} < \frac{1}{3^{n+1}\,(n+1)!}\,.$$

Finally, imposing

$$\frac{1}{3^{n+1}\,(n+1)!} < 10^{-3} \iff 3^{n+1}\,(n+1)! > 1000\,,$$

we can deduce that the considered Maclaurin polynomial must have degree $n = 3$, at least. Thus, a proper approximation is

$$e^{-1/3} \approx 1 - \frac{1}{3} + \frac{1}{18} - \frac{1}{162}\,.$$