Final exam (English)

Problem 1. Let $x_0 = 1$ and, for $n > 0$, we define $x_n = x_{n-1} + x_{n-2} + \ldots + x_1 + x_0 + 1$. Prove, using induction, that $x_n = 2^n$.

Solution. (Using **weak** induction). Check the initial condition: $x_0 = 1 = 2^0$. Now assume that $x_k = 2^k$. Notice that

$$\begin{aligned} x_{k+1} &= x_k + x_{k-1} + \ldots x_1 + x_0 + 1 = \\ &= x_k + (x_{k-1} + \ldots x_1 + x_0 + 1) = \\ &= x_k + x_k = \\ &= 2x_k \,. \end{aligned}$$

Using the induction hypthosesis, $x_{k+1} = 2 \cdot 2^k = 2^{k+1}$.

By the induction principle, we can claim that $x_n = 2^n$.

Solution. (Using **strong** induction). Check the initial condition: $x_0 = 1 = 2^0$. For a given $k \geq 0$, assume that $x_m = 2^m$ for all $0 \leq m \leq k$. Then

$$\begin{aligned} x_{k+1} &= x_k + x_{k-1} + \ldots x_1 + x_0 + 1 = \\ &= 2^k + 2^{k-1} + \ldots 2^1 + 2^0 + 1 = \\ &= 1 + \sum_{m=0}^{k} 2^m = 1 + \frac{1 - 2^{k+1}}{1-2} = 1 - 1 + 2^{k+1} = \\ &= 2^{k+1} \,. \end{aligned}$$

By the induction principle, we can claim that $x_n = 2^n$.

Problem 2. We have a communication channel through which we can send 10 types of signals. Two of these need 1 second to be fully transmitted, while the other eight require 2 seconds. During a time-lapse of n seconds, how many different messages can be transmitted?

Solution. Let a_n be the number of different messages that can be transmitted in an interval of n seconds. Classifying a message by the last transmitted signal, by the sum principle we can stablish that $a_n = 2a_{n-1} + 8a_{n-2}$. Thus we need two initial conditions, which are $a_1 = 2$ and $a_2 = 12$.

This is an homogeneous second order linear recurrence relation. The characteristic equation is $x^2 = 2x + 8$, with solutions $x = 4$ and $x = -2$, hence $a_n = A4^n + B(-2)^n$, where $A, B \in \mathbb{R}$.

Initial conditions will set the values for A and B:

$$\begin{cases} 2 = a_1 = 4A - 2B \ , \\ 12 = a_2 = 16A + 4B \ . \end{cases}$$

From which, $A = 2/3 = 4/6$ and $B = 1/3 = -(-2)/6$. Thus we conclude that

$$a_n = \frac{1}{6}\left(4^{n+1} - (-2)^{n+1}\right) \ . \tag{1}$$

Problem 3. Let $V = \{18, 30, 44, 75, 105, 175\}$. Let $G = (V, E)$ be the simple graph where two vertices $a, b \in V$ are adjacent if, and only if, $a \neq b$ and $\gcd(a, b) \neq 1$. Let $\overline{G} = (V, \overline{E})$ be its edge-complement graph.

1. Find $|E|$ and compute, in a justified way, $|\overline{E}|$. Find a spanning tree of $\overline{G}$.

2. Find the number of perfect matchings in G.

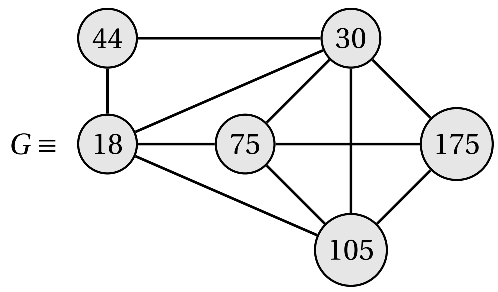
Solution. We factorize the numbers: $18 = 2\cdot 3^2$, $30 = 2\cdot 3\cdot 5$, $44 = 2^2\cdot 11$, $75 = 3\cdot 5^2$, $105 = 3\cdot 5\cdot 7$, and $175 = 5^2\cdot 7$.

1. The vertices' degrees are: $d(18) = 4$, $d(30) = 5$, $d(44) = 2$, $d(75) = 4$, $d(105) = 4$, and $d(175) = 3$.

   By the Handshake theorem, we can tell that $|E| = 22/2 = 11$. Moreover, since $|E| + |\overline{E}| = 6\cdot 5/2 = 15$ (as we can make a K_6), we have that $|\overline{E}| = 4$.

   Since a tree requires one less edge than the number of vertices, and $\overline{G}$ has 6 vertices but only 4 edges, there's no way to find a spanning tree.

2. We can represent the graph as follows:



If a perfect matching includes the edge $18-44$, the remaining vertices (30, 75, 105, 175) make a K_4, which allows for 3 perfect matchings.

If a perfect matching includes the edge $30-44$, the remaining vertices (18, 75, 105, 175) make a C_4 with an additional edge ($75-105$), and this allows for 2 perfect matchings.

By the sum principle, G allows for 5 perfect matchings.

Problem 4. We have built a rudimentary calculator with a 7-bit memory, and we have programmed the basic integer operations in it. After some testings, we observe that, for this calculator, $127+1=0$, $50+100=22$, and $50-100=78$.

1. Justify which value will the calculator show if we compute 21^{76806} with it.
2. Justify if the equation $28x = 20$ has a solution under the "point of view" of this calculator. If so, find all the solutions that can be represented by this calculator.

Solution. With 7 bits we can represent 128 different values. From the tests, we can tell the calculator represents values in $\mathbb{Z}_{128}$.

1. As $128 = 2^7$ is not prime, we could only use Euler's theorem. As $\gcd(128, 21) = 1$, we can apply it. Since $\phi(128) = \phi(2^7) = 2^6 = 64$, we have that $21^{64} \equiv 1 \pmod{128}$.

   Dividing, we get $76\,806 = 64 \cdot 1200 + 6$. From which, $21^{76\,806} \equiv 21^6 \pmod{128}$.

Whatever is left, we must compute it by hand:

$$\begin{aligned}
21^2 &= (16+5)^2 = 16^2 + 2\cdot 16\cdot 5 + 5^2 \equiv \\
&\equiv 0 + 32\cdot 5 + 25 \equiv 32 + 25 \equiv \\
&\equiv 57 \pmod{128} . \\
21^4 &= 57^2 = (64-7)^2 = 64^2 - 2\cdot 64\cdot 7 + 7^2 \equiv \\
&= 0 - 0 + 49 \equiv \\
&\equiv 49 \pmod{128} . \\
21^6 &= 21^4 \cdot 21^2 \equiv 49\cdot 57 = (64-15)(64-7) \equiv \\
&\equiv 64\cdot 64 - 15\cdot 64 - 7\cdot 64 + (-15)(-7) \equiv \\
&\equiv 0 - 64 - 64 + 105 \equiv \\
&\equiv 105 \pmod{128} .
\end{aligned}$$

Thus,

$$21^{76\,806} \equiv 105 \pmod{128}.$$

2. We need to solve $28x \equiv 20 \pmod{128}$.

   Let $d = \gcd(28, 128) = 4$. Since d divides 20, there are solutions. To be more precise, there exist $d = 4$ solutions in $\mathbb{Z}_{128}$.

   First we solve $28\tilde{x} \equiv 4 \bmod 128$, equation that we rewrite as the Bézout's Identity $28\tilde{x} + 128\tilde{y} = 4$. By using Euclid's lemma several times, we get the following list of equations:

$$\begin{aligned}
128 &= 28\cdot 4 + 16 , \\
28 &= 16\cdot 1 + 12 , \\
16 &= 12\cdot 1 + 4 .
\end{aligned}$$

   From these we can say that:

$$\begin{aligned}
4 &= 16 + (-1)12 = 16 + (-1)\,[28 + (-1)16] = \\
&= (-1)28 + (2)16 = (-1)28 + (2)\,[128 + (-4)28] = \\
&= 28(-9) + 128(2),
\end{aligned}$$

which means $\tilde{x} \equiv -9 \equiv 119 \pmod{128}$.

Since $20 = 5d$, we have that $x \equiv 5\tilde{x} \equiv 5 \cdot (-9) \equiv -45 \equiv 83 \pmod{128}$ is a solution to the original equation. Now we need to find the remaining solutions.

Let $x_0 = x = 83$. The remaining solutions are give by the expression $x_k = x_0 + \frac{128}{d} k = 83 + 32k$, with $k \in \mathbb{Z}$. We are only interested in the solutions that check $0 \leq x_k \leq 127$. In this case, thee are x_{-2}, x_{-1}, x_0, x_1.

All in all, the four solutions are:

$$x \equiv 19,\ 51,\ 83,\ 115 \pmod{128}.$$