CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Functions: derivative

Problem 5.1. Consider the function

$$f(x) \;=\; \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

- Is $f(x)$ continuous at $x = 0$?
- Is $f(x)$ differentiable at $x = 0$?

Problem 5.2. Study the continuity and differentiability of the function

$$f(x) = \sqrt{x+2}\,\arccos(x+2)\,.$$

Problem 5.3. Find the first derivative of the following functions.

1. $f(x) \;=\; \sqrt{3x^2 - 7x - 2}\,.$
2. $f(x) \;=\; x^2\,\sin(x)\,\tan(x)\,.$
3. $f(x) \;=\; \sqrt[3]{\dfrac{x-1}{x+1}}\,.$
4. $f(x) \;=\; \sin\left(\sqrt{1+\cos(x)}\right).$
5. $f(x) \;=\; \ln\left(\dfrac{x^2\sin(x)}{\sqrt{1+x}}\right).$

Problem 5.4. Find an equation for the *tangent line* to the graph of the function

$$g(x) = \frac{x+1}{x-1}$$

at $x = 2$.

Problem 5.5. Study the differentiability of the following functions and calculate the corresponding derivatives.

1. $f(x) = x^{1/3}$.
2. $f(x) = \ln|x|$.

Problem 5.6. Consider the function

$$f(x) = \begin{cases} \cos(x) & \text{if } x \leq 0, \\ 1 - x^2 & \text{if } 0 < x < 1, \\ \arctan(x) & \text{if } x \geq 1, \end{cases}$$

and say whether it is differentiable in $\mathbb{R}$. In addition, find an expression for $f'(x)$ where it exists.

Problem 5.7. Let f, g be differentiable functions in $\mathbb{R}$. Then, write an expression for the first derivative of the functions below.

1. $h(x) = f(g(x))\, e^{f(x)}$.
2. $h(x) = \dfrac{1}{\ln\left(f(x) + g^2(x)\right)}$.
3. $h(x) = \sqrt{f^2(x) + g^2(x)}$.
4. $h(x) = \arctan\left(\dfrac{f(x)}{g(x)}\right)$.
5. $h(x) = \ln\big(g(x)\,\cos(f(x))\big)$.

Problem 5.8. Let c, c_1, and c_2 be constants. For each case, prove that the function $f(x)$ is solution of the corresponding *differential* equation.

1. $f(x) = c/x\,;$ $\qquad xf' + f = 0\,.$
2. $f(x) = x\tan(x)\,;$ $\qquad xf' - f - f^2 = x^2\,.$
3. $f(x) = c_1\sin(3x) + c_2\cos(3x)\,;$ $\qquad f'' + 9f = 0\,.$
4. $f(x) = c_1e^{3x} + c_2e^{-3x}\,;$ $\qquad f'' - 9f = 0\,.$
5. $f(x) = c_1e^{2x} + c_2e^{5x}\,;$ $\qquad f'' - 7f' + 10f = 0\,.$
6. $f(x) = \ln\left(c_1e^{x} + c_2e^{-x}\right) + c_2\,;$ $\qquad f'' + (f')^2 = 1\,.$

Problem 5.9. Prove the following equalities.

1. $\arctan(x) + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ $\qquad$ (for $x > 0$).
2. $\arctan\left(\dfrac{1+x}{1-x}\right) - \arctan(x) = \dfrac{\pi}{4}$ $\qquad$ (for $x < 1$).
3. $2\arctan(x) + \arcsin\left(\dfrac{2x}{1+x^2}\right) = \pi$ $\qquad$ (for $x > 1$).

HINT: *calculate the first derivative of the function on the left of each equality.*

Problem 5.10. Calculate the angle formed by the tangent lines from the right and from the left, at $x = 0$, to the graph of the function

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$