# CALCULUS

**Bachelor in Computer Science and Engineering**

Course 2021–2022

## Taylor polynomial

**Problem 7.1.** Approximate the following values within an error bound given by the indicated $\varepsilon$, using suitable Taylor polynomials.

- $\sin(1)\,, \quad \varepsilon = 10^{-5}\,.$
- $\sqrt[5]{\dfrac{3}{2}}\,, \quad \varepsilon = 10^{-2}\,.$

**Problem 7.2.** Write the Maclaurin polynomial of the indicated degree $n$ for the following functions.

1. $f(x) = \sqrt{1+x}\,, \quad n = 3\,.$
2. $f(x) = \sin(3x^2)\,, \quad n \in \mathbb{N}$ (generic).
3. $f(x) = \tan(x)\,, \quad n = 5\,.$
4. $f(x) = e^{-x^2}\cos(x)\,, \quad n = 3\,.$
5. $f(x) = (1+e^x)^2\,, \quad n \in \mathbb{N}$ (generic).

**Problem 7.3.** Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in terms of powers of $x - 4$.

**Problem 7.4.** Write the Taylor formula of degree $n \in \mathbb{N}$ about point $a = -1$ for the function $f(x) = 1/x$.

**Problem 7.5.** Find the Maclaurin polynomial of degree 5 for $f(x) = e^x \sin(x)$.

**Problem 7.6.** Compute the coefficient of $x^4$ in the Maclaurin polynomial for the function $f(x) = \ln(\cos(x))$.

**Problem 7.7.** Calculate the Taylor polynomial of degree 3 about point $a = 0$ for the following functions.

$$
\begin{aligned}
f(x) &= \sin(2x)\,. \\
f(x) &= e^{3x}\,. \\
f(x) &= x\,e^{-x}\,. \\
f(x) &= e^{x}\ln(1-x)\,. \\
f(x) &= \sin^2(x)\,. \\
f(x) &= \frac{\sqrt{1+x^2}\sin(x)}{1+\ln(1+x)}\,.
\end{aligned}
$$

**Problem 7.8.** Write the Maclaurin polynomial of degree $n \in \mathbb{N}$ for the following functions ($a \in \mathbb{R}$).

$$
\begin{aligned}
f(x) &= \cos(ax)\,. \\
f(x) &= \frac{e^{\,ax} - e^{-ax}}{2}\,. \\
f(x) &= e^{\,ax^2}\,. \\
f(x) &= \frac{1+x}{1-x}\,.
\end{aligned}
$$

**Problem 7.9.** The Taylor polynomial of degree 4 about point $a = 1$ for a function $f(x)$ is given by $P_{4,1}(x) = 2(x-1)^3 - 3(x-1)^4$.

- Find an equation for the tangent line to the graph of $f(x)$ at $x = 1$.
- Calculate $\displaystyle \operatorname{lím}_{x\to 1} \frac{f(x)}{(x-1)^3}$.
- Compute $f^{(4)}(1)$.

**Problem 7.10.** Prove what follows.

$$
\begin{aligned}
&\forall a < 1: \quad \sin(x) = o(x^a)\,, \quad \text{as } x \to 0\,. \\
&\ln(1+x^2) = o(x)\,, \quad \text{as } x \to 0\,. \\
&\tan(x) - \sin(x) = o(x^2)\,, \quad \text{as } x \to 0\,. \\
&\ln(x) = o(x)\,, \quad \text{as } x \to +\infty\,.
\end{aligned}
$$

**Problem 7.11.** Find a polynomial $P(x)$ such that

$$\operatorname{lím}_{x\to 0} \frac{\sqrt{1-x^4} - P(x)}{x^7} = 0\,.$$

**Problem 7.12.** Approximate $f(x) = \cos(x) + e^x$ by means of a polynomial of degree 3 about $a = 0$ and estimate the involved error when such approximation is used for $x \in [-1/4, 1/4]$.

**Problem 7.13.** How many terms should you consider in the Maclaurin polynomial for $f(x) = e^x$, with $x \in [-1, 1]$, in order to get an approximation with three exact decimal places?

**Problem 7.14.** Using a Taylor polynomial of degree 3, approximate the value

$$\frac{1}{\sqrt{1.1}}$$

and find an upper bound of the involved error.

**Problem 7.15.** Using suitable Taylor polynomials, find an approximation within an error smaller than $10^{-3}$ for the following values.

- $\sin(2)$.
- $\ln(4/5)$.
- $\cos(1)$.
- $e^{-2}$.
- $\ln(2)$.

**Problem 7.16.** Find how many terms of the Taylor series for $f(x) = \sin(x)$, about $a = 0$, you need to consider to approximate $\sin(1/2)$ within an error smaller than $10^{-12}$.

**Problem 7.17.** Calculate the given limits by using appropriate Taylor polynomials.

$$\text{(a)} \quad \lim_{x\to 0} \frac{e^x - \sin(x) - 1}{x^2}.$$

$$\text{(b)} \quad \lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{x^5}.$$

$$\text{(c)} \quad \lim_{x\to 0} \frac{\cos(x) - \sqrt{1-x}}{\sin(x)}.$$

$$\text{(d)} \quad \lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x^3}.$$

(e) $\displaystyle \lim_{x\to 0} \frac{x-\sin(x)}{x\,[1-\cos(3x)]}\,.$

(f) $\displaystyle \lim_{x\to 0} \frac{\cos(x)+e^{x}-x-2}{x^3}\,.$

(g) $\displaystyle \lim_{x\to 0} \left(\frac{1}{x}-\frac{1}{\sin(x)}\right)\,.$

(h) $\displaystyle \lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x}-\frac{\cos(x)}{\sin(x)}\right)\,.$

(i) $\displaystyle \lim_{x\to +\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)\,.$

(j) $\displaystyle \lim_{x\to +\infty} \left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]\,.$

**Problem 7.18.** Calculate the given limits by using either Taylor polynomials or l'Hôpital's rule.

(a) $\displaystyle \lim_{x\to 0^+} x\ln(e^{x}-1)\,.$

(b) $\displaystyle \lim_{x\to +\infty} \frac{e^{x}-\arctan(x)}{\ln(1+x)}\,.$

(c) $\displaystyle \lim_{x\to 0^+} x^{x}\,.$

(d) $\displaystyle \lim_{x\to +\infty} \frac{\ln(x)}{x^{a}}\,,\quad a>0\,.$

(e) $\displaystyle \lim_{x\to +\infty} \frac{x}{e^{ax}}\,,\quad a>0\,.$

(f) $\displaystyle \lim_{x\to +\infty} x^{1/x}$ $\quad$ (use the change of variable $t=1/x$) .

(g) $\displaystyle \lim_{x\to 0} (1+x)^{1/x}\,.$