

CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Functions: properties and continuity

Problem 4.2.

- 1) The domain is \mathbb{R} , the image is \mathbb{Z} , and f is continuous in $\mathbb{R} \setminus \mathbb{Z}$.
- 2) The domain is \mathbb{R} , the image is $[0, 1)$, and f is continuous in $\mathbb{R} \setminus \mathbb{Z}$.
- 3) The domain is \mathbb{R} , the image is $[0, 1)$, and f is continuous in $\mathbb{R} \setminus \mathbb{Z}$.
- 4) The domain is \mathbb{R} , the image is \mathbb{R} , and f is continuous in \mathbb{R} .
- 5) The domain is $\mathbb{R} \setminus \{0\}$, the image is \mathbb{Z} , and f is continuous in $\mathbb{R} \setminus \{0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \dots\}$.

Problem 4.3.

- 1) The function f is continuous in \mathbb{R} .
- 2) The function f is continuous in \mathbb{R} .
- 3) The function f is continuous in $\mathbb{R} \setminus \{0\}$.
- 4) The function f is continuous in $[-1, 1]$.
- 5) The function f is continuous in $(4/9, +\infty)$.
- 6) The function f is continuous in $(4/9, 1]$.

Problem 4.4.

- The function f is continuous in $\mathbb{R} \setminus \{0\}$ as product of continuous functions ($\cos(1/x)$ is the continuous composition of continuous functions). At $x = 0$, the function f is also continuous because $\lim_{x \rightarrow 0} f(x) = f(0) = 0$ (the limit can be calculated taking into account that $\cos(1/x)$ is bounded).

- The function g is continuous in $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as composition of continuous functions. At $x = 1$, the function g is also continuous because $\lim_{x \rightarrow 1^+} g(x) = \lim_{x \rightarrow 1^-} g(x) = g(1) = 0$ (here, we need to use lateral limits as $g(x)$ is defined by two different expressions on the left and the right of $x = 1$). At $x = 0$, the function g is not continuous since $\lim_{x \rightarrow 0^+} g(x) = 0$ and $\lim_{x \rightarrow 0^-} g(x) = 1$, hence the limit of g when $x \rightarrow 0$ does not exist (again, we need to use lateral limits).
- The function h is continuous in $\mathbb{R} \setminus \{0\}$ being a sum of continuous functions. At $x = 0$, the function h is not continuous since we have that $\lim_{x \rightarrow 0^+} h(x) = 1$ and $\lim_{x \rightarrow 0^-} h(x) = 5$, hence the limit of h when $x \rightarrow 0$ does not exist (we need to use lateral limits).

Problem 4.5. In order to prove that f is bounded in the interval $[-7, 5]$ (closed and bounded), we can show that it is continuous. Then, f is continuous in $[-7, 0) \cup (0, 5]$ as composition of continuous functions. At $x = 0$, f is also continuous because $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) = 0$ (we need to use lateral limits). Thus, we can conclude that f is continuous, hence bounded, in $[-7, 5]$.

Problem 4.6. Apply Bolzano's theorem to the function $f(x) = \cos x - x$ on, for instance, $[0, 1]$.

Problem 4.7. (A) Apply Bolzano's theorem to the function $F(x) = f(x) - x$ on $[0, 1]$.
(B) Apply Bolzano's theorem to the function $F(x) = f(x) - g(x)$ on $[x_1, x_2]$.

Functions: derivative

Problem 5.1. The function f is continuous at $x = 0$ because $\lim_{x \rightarrow 0} f(x) = f(0) = 0$ (the limit can be calculated taking into account that $\sin(1/x)$ is bounded). Also, f is differentiable at $x = 0$ because $f'(0) = \lim_{h \rightarrow 0} [f(0 + h) - f(0)] / h = 0$.

Problem 5.2. The function f is continuous in $[-2, -1]$ and differentiable in $(-2, -1)$.

Problem 5.3.

$$1. f'(x) = \frac{6x - 7}{2\sqrt{3x^2 - 7x - 2}}.$$

$$2. f'(x) = x \sin(x) \left(2 \tan(x) + x + \frac{x}{\cos^2(x)} \right).$$

$$3. f'(x) = \frac{2}{3(x-1)^{2/3}(x+1)^{4/3}}.$$

$$4. f'(x) = \frac{-\sin(x) \cos(\sqrt{1+\cos(x)})}{2\sqrt{1+\cos(x)}}.$$

$$5. f'(x) = \frac{2}{x} + \frac{1}{\tan(x)} - \frac{1}{2x+2}.$$

Problem 5.4. An equation for the desired tangent line is $y = -2x + 7$.

Problem 5.5.

1. For $x \neq 0$, f is differentiable and we have $f'(x) = 1/(3x^{2/3})$. At $x = 0$, f is not differentiable.
2. For $x \neq 0$, f is differentiable and we have $f'(x) = 1/x$. At $x = 0$, f is not differentiable.

Problem 5.6. The function is differentiable in $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ as defined in terms of differentiable functions (an expression for $f'(x)$ in the three intervals is obtained by differentiating the corresponding elementary functions). At $x = 0$, we have that $\lim_{h \rightarrow 0^+} [f(0+h) - f(0)]/h = \lim_{h \rightarrow 0^-} [f(0+h) - f(0)]/h = 0$, thus $f(x)$ is differentiable and $f'(0) = 0$ (we need to use lateral derivatives). At $x = 1$, the function is not continuous (thus not differentiable) as $\lim_{x \rightarrow 1^+} f(x) = \pi/4$ and $\lim_{x \rightarrow 1^-} f(x) = 0$, hence the limit of $f(x)$ when $x \rightarrow 1$ does not exist.

Problem 5.7. By applying the chain rule we get the following expressions.

$$1. h'(x) = f'(g(x)) g'(x) e^{f(x)} + f(g(x)) f'(x) e^{f(x)}.$$

$$2. h'(x) = \frac{-f'(x) - 2g(x)g'(x)}{(f(x) + g^2(x)) \ln^2(f(x) + g^2(x))}.$$

$$3. h'(x) = \frac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}.$$

$$4. h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{f^2(x) + g^2(x)}.$$

$$5. h'(x) = \frac{g'(x)}{g(x)} - f'(x) \tan(f(x)).$$

Problem 5.8. For each case, calculate the necessary derivatives, substitute their expressions into the left-hand side of the equation, and verify that an equality is obtained.

Problem 5.9. For each case, calculate the first derivative of the function appearing in the left-hand side of the equality and realize that such derivative is equal to zero for all given values of x . As a consequence, the value of the involved function must be the same for all indicated values of x . Thus, in order to prove the equality, we can evaluate it at some convenient x .

Problem 5.10. The slope of the desired tangent line from the right is given by $\lim_{h \rightarrow 0^+} [f(0+h) - f(0)]/h = 0$, thus such line is parallel to the x -axis. The slope of the desired tangent line from the left is given by $\lim_{h \rightarrow 0^-} [f(0+h) - f(0)]/h = 1$, thus such line is parallel to the line $y = x$. As a consequence, the two tangent lines form an angle equal to $\pi/4$.

Applications of the derivative

Problem 6.1.

- For $x \neq 0$, we have $f'_1(x) = kx|x|^{k-2}$ and $f'_2(x) = k|x|^{k-1}$.
- Using the definition of differentiability, we get $f'_1(0) = f'_2(0) = 0$.
- From $0 \leq |f(x)| \leq |x|^k$ for $x = 0$, we deduce that $f(0) = 0$. In addition, we have $0 \leq |f(x)/x| \leq |x|^{k-1}$ for each $x \neq 0$ in a neighborhood of $x_0 = 0$, which implies that $f'(0) = \lim_{x \rightarrow 0} f(x)/x = 0$ (by the sandwich theorem, noting that $k > 1$).

Problem 6.2. The function $f(x)$ is continuous in \mathbb{R} with $f(1) = 1$ and differentiable in \mathbb{R} with $f'(1) = -1$. In the interval $[0, 2]$, all assumptions of Lagrange's mean-value theorem are satisfied and the points of the theorem statement (namely, points $c \in (0, 2)$ where $f'(c) = -1/2$) are $c = 1/2, \sqrt{2}$.

Problem 6.3. The function $f(x)$ satisfies all assumptions of Rolle's theorem except for the differentiability in the whole interval $(-1, 1)$. Indeed, $f'(0)$ does not exist, hence the mentioned theorem cannot be applied.

Problem 6.4. The desired values are $h(0) = 0$, $h'(0) = 0$, and $h''(0) = 2$. All of them are obtained by first noting that $\lim_{x \rightarrow 0} h(x)/x^2 = f(0) = 1$ (since $f(x)$ is continuous at $x = 0$) and then using this limit in the definition of continuity and (twice) differentiability of $h(x)$ at $x = 0$ (in the calculation of $h''(0)$ it may be convenient to apply l'Hôpital's rule).

Problem 6.5. As $f(x)$ is continuous at $x = 0$, we have $\lim_{x \rightarrow 0} f(2x^3) = f(0)$. Then, since the given limit is finite, it must be $f(0) = 0$. The value $f'(0) = 5/2$ is obtained by using the given limit in the definition of differentiability of $f(x)$ at $x = 0$. Finally, we have

$$\lim_{x \rightarrow 0} \frac{f(f(2x))}{3f^{-1}(x)} = \lim_{x \rightarrow 0} \frac{f(f(2x))}{f(2x)} \lim_{x \rightarrow 0} \frac{f(2x)}{x} \lim_{x \rightarrow 0} \frac{x}{3f^{-1}(x)},$$

where each one of the limits on the right can be calculated after transformation (by a suitable change of variable) into $\lim_{t \rightarrow 0} f(t)/t = f'(0) = 5/2$. The value of the above limit is then $125/12$.

Problem 6.6.

THEOREM 1. Let $f(x)$ vanish at $k \geq 2$ points in $[x_1, x_2]$, say $\bar{x}_1, \dots, \bar{x}_k$. Thus, we get $k - 1$ intervals $[\bar{x}_1, \bar{x}_2], [\bar{x}_2, \bar{x}_3], \dots, [\bar{x}_{k-1}, \bar{x}_k]$ where $f(x)$ satisfies the assumptions of Rolle's theorem. Hence, there are $k - 1$ points in $[x_1, x_2]$ (one per interval) where $f'(x)$ must be zero.

THEOREM 2. The statement is proved by repeatedly applying THEOREM 1 above to $f(x), f'(x), f''(x), \dots, f^{(k-1)}(x)$.

Problem 6.7. Write the given equations as $f(x) = 0$, where $f(x)$ is a function to be conveniently defined in each case. Then, analyze the sign of $f'(x)$ in the indicated intervals.

- a) 1 real solution.
- b) 1 real solution.
- c) 2 real solutions.
- d) 1 real solution.
- e) No real solutions.

Problem 6.8.

- The value of the limit is $1/2$ (use l'Hôpital's rule two times).
- The value of the limit is 1 (use l'Hôpital's rule two times).

Extra problem. Consider $f(x) = x^{1+\frac{1}{x}}$. Then, by using the Lagrange's mean-value theorem, the given limit can be calculated as

$$\lim_{x \rightarrow +\infty} f'(x) = \lim_{x \rightarrow +\infty} x^{\frac{1}{x}} \left(1 + \frac{1}{x} - \frac{\ln(x)}{x} \right) = \lim_{x \rightarrow +\infty} x^{\frac{1}{x}} \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x} - \frac{\ln(x)}{x} \right) = 1.$$