Recurrence relations

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1 Linear homogeneous recurrence relations

Like we've seen in class, a linear and homogeneous recurrence relation of order k is a relation like this one:

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \dots c_k x_{n-k}$$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$ are constants.

These are solves as follows:

1. Compute the roots (and their multiplicities) of the characteristic equation:
$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_{k-1} x - c_k = 0$$

2. We obtain the solution related to each one of the different roots. If r is a root, solutions have the following shape:

    - Ar^n, when the multiplicity is 1,
    - $(A + Bn)r^n$, when the multiplicity is 2,
    - $(A + Bn + Cn^2)r^n$, when the multiplicity is 3,
    - $(A + Bn + Cn^2 + Dn^3)r^n$, when the multiplicity is 4,
    - etc...

    Notice that $A, B, C, D, \dots$ are constants related to a particular roots[1].

[1] That is, we shall not use the same constants for different roots.

3. The **general solution**, a_n, will be the sum of the solutions related to each different root.

4. Compute the values for the constants using the initial conditions (if there are any).

1.1 Example 1

Let

$$\begin{cases} x_n = 7x_{n-1} - 16x_{n-2} + 12x_{n-3} \\ x_0 = 2,\, x_1 = 5,\, x_2 = 11 \end{cases}$$

We obtain the characteristic equation using the following substitution: $x_n \longmapsto x^n$, then dividing by x^{n-3}. with this, we obtain the equation $x^3 = 7x^2 - 16x + 12$. That is,

$$x^3 - 7x^2 + 16x - 12 = 0$$

which we can factorize into $x^3 - 7x^2 + 16x - 12 = (x-2)^2(x-3)$. Thus,

- $r_1 = 2$ is a root with multiplicity 2, so the solution related to it has the shape
$$(A + Bn)2^n$$

- $r_1 = 3$ is a root with multiplicity 1, so the solution related to it has the shape
$$C3^n$$

We conclude, then, that the general solution of the recurrence is

$$\boxed{a_n = (A + Bn)2^n + C3^n}$$

where A, B, C will be determined using the initial conditions:

$$\begin{cases} 2 = a_0 = A + C \\ 5 = a_1 = 2(A + B) + 3C \\ 11 = a_2 = 4(A + 2B) + 9C \end{cases}$$

From here we get $\boxed{A = 3,\, B = 1,\, C = -1}$.

1.2 Ejemplo 2

Let

$$\begin{cases} x_n = 12x_{n-1} - 54x_{n-2} + 108x_{n-3} - 81x_{n-4} \\ x_0 = 5,\ x_1 = 15,\ x_2 = 45,\ x_3 = -27 \end{cases}$$

We obtain the characteristic as follows: $x_n \longmapsto x^n$, then divide by x^{n-4}. With this, we have the following characteristic equation

$$x^4 - 12x^3 + 54x^2 - 108x + 81 = 0$$

which factorizes as $x^4 - 12x^3 + 54x^2 - 108x + 81 = (x-3)^4$. This means $r = 3$ is the only root and has multiplicity 4.

The general solution to this recurrence, hence, will be

$$\boxed{a_n = (A + Bn + Cn^2 + Dn^3)3^n}$$

where A, B, C, D will be obtained from the initial conditions:

$$\begin{cases} 5 = a_0 = A \\ 15 = a_1 = 3(A + B + C + D) \\ 45 = a_2 = 9(A + 2B + 4C + 8D) \\ -27 = a_3 = 27(A + 3B + 9C + 27D) \end{cases}$$

which, once solved, gives us $\boxed{A = 5,\ B = -2,\ C = 3,\ D = -1}$.

2 Linear non-homogeneous recurrence relations

Now we deal with recurrence relations with the following shape

a) $x_n = 4x_{n-1} - 4x_{n-2} + 4$

b) $x_n = 3x_{n-2} - n^2$

c) $x_n = 3x_{n-2} + 3^n$

d) $x_n = 3x_{n-2} + n4^n$

These are **non-homogeneous** lineal recurrence relations, given that we find certain terms in the equations (colored red) that do not depend on the sequence terms (x_n, x_{n-1}, x_{n-2}, etc.).

Important: These four recurrence relations will be used as an example during this section. We will not set initial conditions as we will emphasize other things.

We can solve these following these steps. Further details will be given after this.

1. We forget about the non-homogeneous term, and solve the linear homogeneous recurrence relation that comes out from that, like we did previously. The solution we obtain doing this is called **solution to the associated homogeneous recurrence relation**, and we will call it a_n^h.

2. Now we seek a **particular solution** to the original recurrence relation that resembles somehow the non-homogeneous term. If the non-homogeneous term has a certain shape, the particular solution will resemble it. The particular solution will be called a_n^p.

3. The **general solution** to the recurrence relation will be what comes from the sum of the particular solution and the associated homogeneous solution. That is, $a_n = a_n^h + a_n^p$.

4. Finally, if we have initial conditions, we use them on the general solution in order to compute the values for the constants in a_n (that come from a_n^h) that satisfy the initial conditions.

Now we go into details for each step. Emphasizing the second one.

2.1 ASSOCIATED homogeneous recurrence relation

This is what we have left when we take out the non homogenous term from the recurrence.

Specifically, we get:

a) $x_n = 4x_{n-1} - 4x_{n-2}$

b,c,d) $x_n = 3x_{n-1}$

whose characteristic equations and roots are:

a) $x^2 - 4x + 4 = 0$, with root $r_1 = 2$ and multiplicity 2.

b,c,d) $x^2 - 3 = 0$, with roots $r_1 = 3$ and $r_2 = -3$.

Hence, the solution to the associated homogeneous recurrence relation is:

a) $\boxed{a_n^h = (A + nB) \times 2^n}$

b,c,d) $\boxed{a_n^h = A \times 3^n + B \times (-3)^n}$

where A, B are unknown constants (which might be different for each different example and initial conditions).

2.2 Particular solution

If the non homogenous term has the following shape

$$\boxed{s^n(a + bn + cn^2 + \ldots)}$$

that is, it is a polynomial on n, with a certain degree, multiplied by a constant s, to the power of n. Then the particular solution will have this shape

$$\boxed{a_n^p = n^m s^n(\hat{a} + \hat{b}n + \hat{c}n^2 + \ldots)}$$

that is, it'll be a *new* polynomial *with a degree no larger than the original polynomial*, multiplied by the same constant s to the power of n, but now also multiplied by n to a fixed power m. This m is the multiplicity of s as a root to the characteristic equation of the associated homogeneous linear recurrence relation (notice that, if s is not a root, we consider $m = 0$).

The constants in the new polynomial $(\hat{a} + \hat{b}n + \hat{c}n^2 + \ldots)$ **will be** computed right now. This is done by replacing $x_n \longmapsto a_n^p$ in the recurrence relation and solving for the constants, since the relation has to true for all n.

Lets see the examples:

a) The non-homogeneous term is 4, which we want to rewrite into the shape described above. That is,

$$4 = 1^n(4 + 0n + 0n^2 + \dots) = 1^n(4)$$

that's a degree 0 polynomial multiplied by the constant $s = 1$ to the power n. Now, remember the root for the characteristic eq. is 2, hence $s = 1$ is not a root. Thus $m = 0$, and

$$a_n^p = 1^n(\hat{a}) = \hat{a}$$

with $\hat{a} \in \mathbb{R}$ (constants) we shall determine from the recurrence relation. Being a_n^p a particular solution to $x_n = 4x_{n-1} - 4x_{n-2} + 4$, we get

$$a_n^p = 4a_{n-1}^p - 4a_{n-2}^p + 4$$

or just

$$\hat{a} = 4\hat{a} - 4\hat{a} + 4$$

from which we can tell $\hat{a} = 4$. Ergo, $\boxed{a_n^p = 4, \ \forall n}$.

Hence, the general solution, if we forget about initial conditions, is:

$$\boxed{a_n = (A + nB) \times 2^n + 4}$$

b) The non-homogeneous term is $-n^2$, which we rewrite as

$$-n^2 = 1^n(0 + 0n + (-1)n^2 + \dots) = 1^n(-n^2)$$

which is a degree 2 polynomial times a constant $s = 1$ to the power of n. The characteristic equation roots were -3 and 3, hence $s = 1$ is not a root. Thus $m = 0$, and

$$a_n^p = 1^n(\hat{a} + \hat{b}n + \hat{c}n^2) = \hat{a} + \hat{b}n + \hat{c}n^2$$

where $\hat{a}, \hat{b}, \hat{c} \in \mathbb{R}$ (constants) will be determined from the recurrence relation. As a_n^p is a particular solution of $x_n = 3x_{n-2} + n^2$, we have

$$a_n^p = 3a_{n-2}^p + n^2$$

or, if we replace with the expression of a_n^p,

$$\hat{a} + \hat{b}n + \hat{c}n^2 = 3\left(\hat{a} + \hat{b}(n-2) + \hat{c}(n-2)^2\right) + n^2$$

As this equality holds for all n, one thing we can do is to replace n with different values and then obtain $\hat{a}, \hat{b}$ and $\hat{c}$ from the equations we get. If we set $n = 0, 1, 2$, we achieve the following 3 equations:

$$\begin{cases} \hat{a} = 3\left(\hat{a} - 2\hat{b} + 4\hat{c}\right) \\ \hat{a} + \hat{b} + \hat{c} = 3\left(\hat{a} - \hat{b} + \hat{c}\right) + 1 \\ \hat{a} + 2\hat{b} + 4\hat{c} = 3\left(\hat{a}\right) + 4 \end{cases}$$

or...

$$\begin{cases} 2\hat{a} - 6\hat{b} + 12\hat{c} = 0 \\ \quad 2\hat{a} - 4\hat{b} + 2\hat{c} = -1 \\ \quad 2\hat{a} - 2\hat{b} - 4\hat{c} = -4 \end{cases}$$

whose solution is $\hat{a} = -6$, $\hat{b} = -3$, and $\hat{c} = -1/2$. So, we have $\boxed{a_n^p = -6 - 3n - \frac{1}{2}n^2}$.

Hence, the general solution (up to initial conditions), is:

$$\boxed{a_n = A \times 3^n + B \times (-3)^n - 6 - 3n - \frac{1}{2}n^2}$$

c) The non-homogeneous term is 3^n, which we rewrite as:

$$3^n = 3^n(1 + 0n + 0n^2 + \ldots) = 3^n(1)$$

landing us a degree 0 polynomial times a constant $s = 3$ to the power of n. As before, the characteristic eq. roots are -3 and 3, meaning that $s = 3$ is, in fact, a root, with multiplicity 1. Thus $m = 1$, and

$$a_n^p = n^1 \times 3^n(\hat{a}) = \hat{a}n3^n$$

where $\hat{a} \in \mathbb{R}$ (constant) will be determined from the recurrence relation.

Being a_n^p a particular solution to $x_n = 3x_{n-2} + 3^n$, we have

$$a_n^p = 3a_{n-2}^p + 3^n$$

or, once we replace by the expression of a_n^p,

$$\hat{a}n3^n = 3\hat{a}(n-2)3^{n-2} + 3^n$$

Since this has to hold for all n, we'll do as before and set values for n in order to obtain $\hat{a}$. If we set $n = 0$, we have the following equation:

$$0 = 3\hat{a}(-2)3^{-2} + 1$$

that is, $1 = \frac{6}{9}\hat{a} = \frac{2}{3}\hat{a}$. From which, $\hat{a} = \frac{3}{2}$, and $\boxed{a_n^p = \frac{3}{2}n3^n}$.

In conclusion, the general solution will be

$$\boxed{a_n = A \times 3^n + B \times (-3)^n + \frac{3}{2}n3^n}$$

d) The non-homogenous term, $n4^n$, can be rewritten as

$$n4^n = 4^n(0 + 1n + 0n^2 + \ldots) = 4^n(n)$$

which is a degree 1 polynomial multiplied by a constant $s = 4$ powered to n. The roots to the characteristic equation were -3 and 3, hence $s = 4$ is no root. Thus $m = 0$, and

$$a_n^p = 4^n(\hat{a} + \hat{b}n)$$

with $\hat{a}, \hat{b} \in \mathbb{R}$ (constant) to be determined by the recurrence relation.

As a_n^p is a particular solution to $x_n = 3x_{n-2} + n4^n$, we have

$$a_n^p = 3a_{n-2}^p + n4^n$$

or, once we replace by the expression of a_n^p,

$$4^n(\hat{a} + \hat{b}n) = 3 \times 4^{n-2}(\hat{a} + \hat{b}(n-2)) + n4^n$$

from which we want to obtain $\hat{a}$ and $\hat{b}$. For a change, now we proceed comparing powers of n. That is, we use that two polynomials are the same one if their coefficients are the same. But before that, lets simplify the expression. First, we divide by 4^{n-2},

$$4^2(\hat{a} + \hat{b}n) = 3(\hat{a} + \hat{b}(n-2)) + n4^2$$

and, second, we group into powers of n:

$$4^2\hat{a} + 4^2\hat{b}n = 3(\hat{a} - 2\hat{b}) + (3\hat{b} + 4^2)n$$

from which we obtain the following equations (comparing powers of n):

$$\begin{cases} n^0: & 4^2\hat{a} = 3(\hat{a} - 2\hat{b}) \\ n^1: & 4^2\hat{b} = 3\hat{b} + 4^2 \end{cases}$$

from this we deduce $\hat{a} = -\frac{96}{169}$ and $\hat{b} = \frac{16}{13}$. That is, the particular solution is $\boxed{a_n^p = 4^n\left(-\frac{96}{169} + \frac{16}{13}n\right)}$

Hence, the general solution will be:

$$\boxed{a_n = A \times 3^n + B \times (-3)^n + 4^n\left(-\frac{96}{169} + \frac{16}{13}n\right)}$$