AUTOMATA THEORY AND FORMAL LANGUAGES 2022-23

UNIT 5 – PART 2: REGULAR LANGUAGES

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Regular Expressions. Bibliography

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- John E. Hopcroft, Rajeev Motwani, Jeffrey D.Ullman. Introduction to Automata Theory, Languages, and Computation (3rd edition). Ed, Pearson Addison Wesley. Unit 3.

- Manuel Alfonseca, Justo Sancho, Miguel Martínez Orga. Teoría de Lenguajes, Gramáticas y Autómatas. Publicaciones R.A.E.C. 1997. Unit 7.

OUTLINE

Unit 5. Part 2: Regular Expressions

- Definition of a Regular Expression (RE)
- Regular Expressions and Regular Languages
- Equivalence of Regular Expressions
- Analysis Theorem and Kleene's Synthesis Theorem
  - Solution of the Analysis Problem. Characteristic Equations
    - Solution of the Characteristic Equations
    - Algorithm to Solve the Analysis Problem
  - Synthesis Problem: Recursive Algorithm
    - Synthesis Problem: Derivatives of Regular Expressions

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Definition of Regular Expression

***Kleene, 1956*:**

"Metalanguage for expressing the set of words accepted by a FA (i.e. to express Type-3 or regular languages)"

Example: given the alphabet $\Sigma = \{0,1\}$

0*10* is a word of the metalanguage representing the infinite words which consist of a 1, preceded and followed by none, one or infinite zeros.

Definition of Regular Expression

- Regular expressions: rules that define exactly the set of words that are included in the language.
- Main operators:
  - **Concatenation:** xy
  - **Alternation:** x+y *also x | y* (x or y)
  - **Repetition:** x^* (x repeated 0 or more times)

    x^+ (x repeated 1 or more times)

Definition of Regular Expression

- Given an alphabet Σ, the rules that define regular expressions of Σ are:
  - $\forall a \in \Sigma$ is a regular expression.
  - λ is a regular expression.
  - Φ is a regular expression.
  - If **r** and **s** are regular expressions, then

    (r)   r·s   r+s   r*

    are regular expressions.
- Nothing else is a regular expression.

$$r^* = \bigcup_{i=0}^{\infty} r^i$$

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Definition of Regular Expression

- Valid RE are those obtained after applying the previous rules a finite number of times over symbols of Σ, Φ, λ

- The priority of the different operations is the following:

$$*, \bullet, +$$

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Regular expressions and Regular Languages

Each RE describes a regular language

- Each RE α has a set of Σ^* associated, $L(\alpha)$, that is the RL described by α. This language is defined by:

  - If $\alpha = \Phi$, $L(\alpha) = \Phi$
  - If $\alpha = \lambda$, $L(\alpha) = \{\lambda\}$
  - If $\alpha = a$, $a \in \Sigma$, $L(\alpha) = \{a\}$
  - If α and β are RE $\Rightarrow L(\alpha + \beta) = L(\alpha) \cup L(\beta)$
  - If α and β are RE $\Rightarrow L(\alpha \bullet \beta) = L(\alpha)\, L(\beta)$
  - If α^* is a RE $\Rightarrow L(\alpha^*) = L(\alpha)^* = [L(\alpha)]^*$

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Regular Expressions. Examples

Write the regular languages described by the following RE:

1) Given $\Sigma = \{a,b,...,z\}$ and $\alpha = (a+b+...+z)^*$, what is $L(\alpha)$?
2) Given $\Sigma = \{0,1\}$ and $\alpha = 0^*10^*$, what is $L(\alpha)$?
3) Given $\Sigma = \{0,1\}$ and $\alpha = 01+000$, what is $L(\alpha)$?
4) Given $\Sigma = \{a,b,c\}$ and $\alpha = a\,(a+b+c)^*$, what is $L(\alpha)$?
5) Given $\Sigma = \{a,b,c\}$ and $\alpha = a+bc+bba$, what is $L(\alpha)$?

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Equivalence of Regular Expressions

- Two RE are equivalent, **$\alpha = \beta$**, if they describe the same regular language, $L(\alpha) = L(\beta)$. <u>Properties</u>:
  1) $(\alpha + \beta) + \sigma = \alpha + (\beta + \sigma)$ (+ is associative)
  2) $\alpha + \beta = \beta + \alpha$ (+ is commutative)
  3) $(\alpha \bullet \beta) \bullet \sigma = \alpha \bullet (\beta \bullet \sigma)$ (• is associative)
  4) $\alpha \bullet (\beta + \sigma) = (\alpha \bullet \beta) + (\alpha \bullet \sigma)$ (+ is distributive
     $(\beta + \sigma) \bullet \alpha = (\beta \bullet \alpha) + (\sigma \bullet \alpha)$ regarding •)
  5) $\alpha \bullet \lambda = \lambda \bullet \alpha = \alpha$ (• has a neutral element)
  6) $\alpha + \Phi = \Phi + \alpha = \alpha$ (+ has a neutral element)
  7) $\lambda^* = \lambda$
  8) $\alpha \bullet \Phi = \Phi \bullet \alpha = \Phi$

Equivalence of Regular Expressions

9) $\Phi^* = \lambda$
10) $\alpha^* \bullet \alpha^* = \alpha^*$
11) $\alpha \bullet \alpha^* = \alpha^* \bullet \alpha$
12) $(\alpha\ ^*)^* = \alpha^*$ (IMPORTANT)
13) $\alpha^* = \lambda\ + \alpha + \alpha^2 + .. + \alpha^n + \alpha^{n+1}.\ \alpha^*$
14) $\alpha^* = \lambda \mid \alpha \bullet \alpha^*$ (13 with n=0) (IMPORTANT)
15) $\alpha^* = (\lambda\ + \alpha\ )^{n-1} + \alpha^n \bullet \alpha^*$
16) Given a function f, $f{:}E^n_\Sigma \rightarrow E_\Sigma$ then:
$f(\alpha, \beta, \ldots, \sigma) + (\alpha + \beta + \ldots + \sigma)^* = (\alpha + \beta + \ldots + \sigma)^*$
17) Given a function, $f{:}E^n_\Sigma \rightarrow E_\Sigma$ then:
$(f(\alpha^*, \beta^*, \ldots, \sigma^*))^* = (\alpha + \beta + \ldots + \sigma)^*$

Equivalence of Regular Expressions

18) $(\alpha^* + \beta^*)^* = (\alpha^* \cdot \beta^*)^* = (\alpha + \beta)^*$ (IMPORTANT)
19) $(\alpha \cdot \beta)^* \cdot \alpha = \alpha \cdot (\beta \cdot \alpha)^*$
20) $(\alpha^* \cdot \beta)^* \cdot \alpha^* = (\alpha + \beta)^*$
21) $(\alpha^* \cdot \beta)^* = \lambda + (\alpha + \beta)^* \cdot \beta$
22) Inference Rules:

given three regular expressions R,T and S:

$$R = S^* \cdot T \Rightarrow R = S \cdot R + T$$

If $\lambda \notin S$, then:

$$R = S \cdot R + T \Rightarrow R = S^* \cdot T$$

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Analysis and Kleene’s Synthesis Theorems

1) Analysis Theorem:

Every language accepted by a FA is a regular language.

L(FA) => L regular

Solution to the problem of analysis: To find the language associated to a specific FA: **“Given a FA, A, find a RE that describes L(A)”.**

2) Synthesis Theorem:

Every regular language is a language accepted by a FA.

L regular => L(FA)

Solution to the problem of synthesis: To find a recognizer for a given regular language: **“Given a RE representing a regular language, build a FA that accepts that regular language”.**

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Solution of the Analysis Problem. Characteristic Equations

ANALYSIS PROBLEM (AF→RE): Given a FA, write the characteristic equations of each one of its states, solve them and obtain the requested RE.

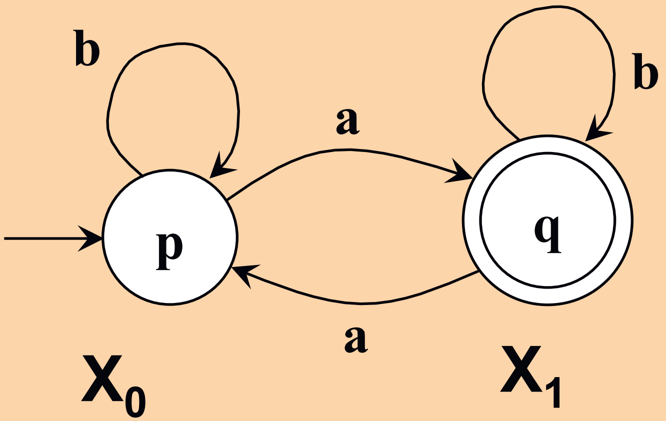
- **CHARACTERISTIC EQUATIONS:** They describe all the strings that can be recognized from a given state:
  - An equation x_i is written for each state q_i
    - First member x_i;
    - The second member has a term for each branch from q_i
      - Branches has the format $a_{ij} \bullet x_j$ where a_{ij} is the label of the branch that joins qi with q_j, x_j is the variable corresponding to q_j
      - A term a_{ij} is added for each branch that joins q_i with a final state.
      - λ is added is q_i is a final state.
      - If there is not an output branch for a state, the second member will be:

If it is a final state: $x_i = \lambda$

If it not a final state: $x_i = \Phi$

Solution of the Analysis Problem. Characteristic Equations

Exercise 1

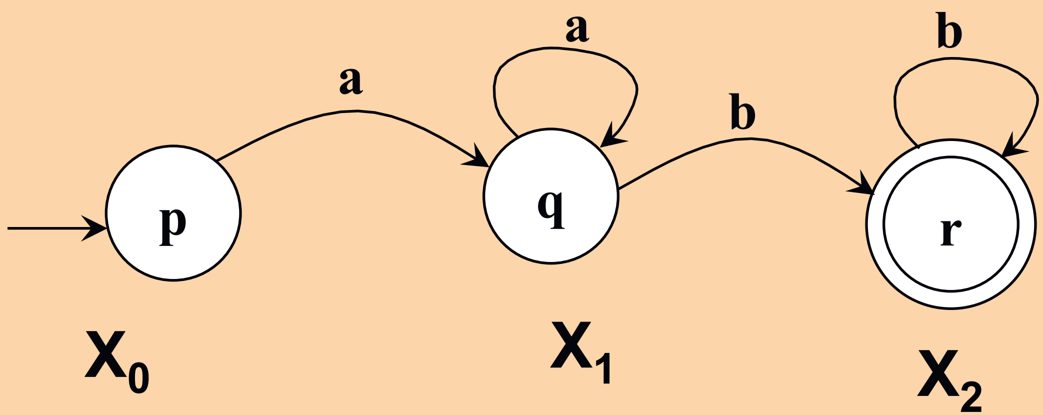


$$\mathbf{X_0} = b\,X_0 + a\,X_1 + a$$

$$\mathbf{X_1} = b\,X_1 + a\,X_0 + b$$

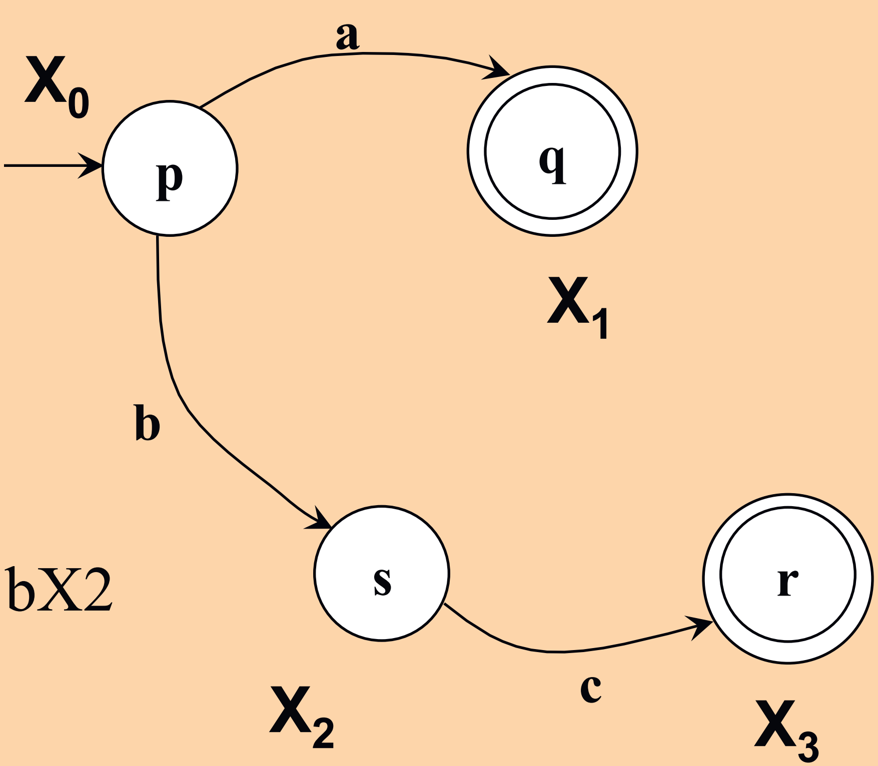
Solution of the Analysis Problem. Characteristic Equations

Exercise 2



Solution of the Analysis Problem. Characteristic Equations

Exercise 3



$X0 = aX1 + a + bX2$

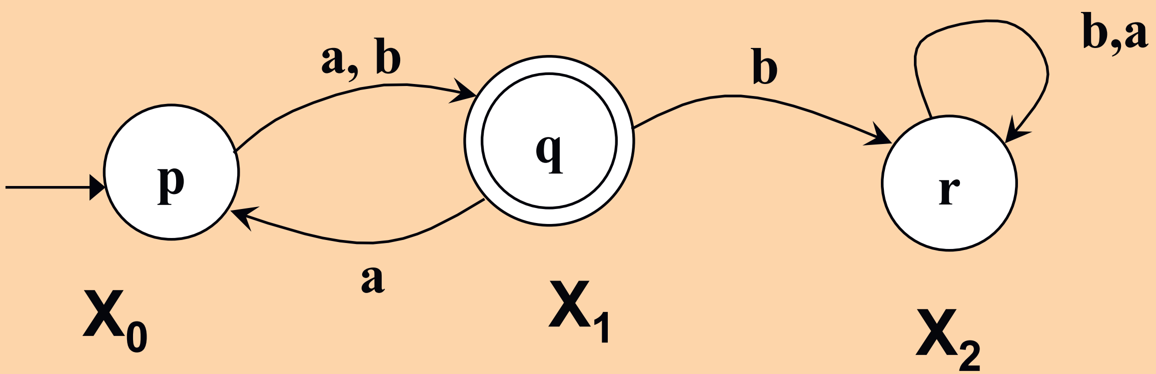
$X1 = \lambda$

$X2 = cX3 + c$

$X3 = \lambda$

Solution of the Analysis Problem. Characteristic Equations

Exercise 4



$X0 = aX1 + bX1 + a + b$

$X1 = aX0 + \cancel{bX2}$

$X2 = aX2 + bX2$

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Solution of the Characteristic Equations

They have the form: **X = AX + B**

You must get this form for them

where:

X: set of strings that allow transitting from q_i to $q_f \in F$

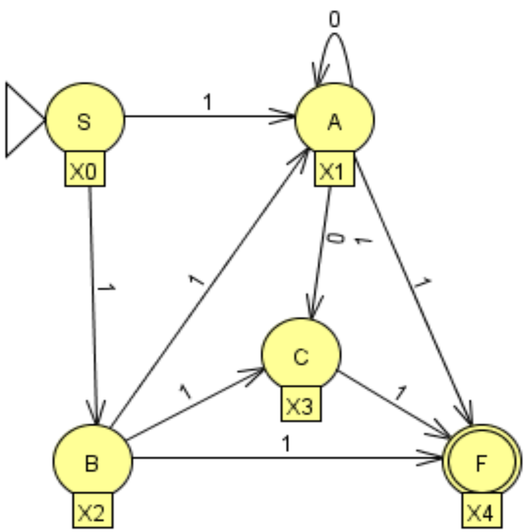
A: set of strings that allows reaching a state q from q.

B: set of strings that allows reaching a final state, without reaching again the leaving state q_i.

$\Downarrow$ (Arden solution or proof by contradiction)

The solution is: $\mathbf{X = A^* \cdot B}$

Solution of the Characteristic Equations



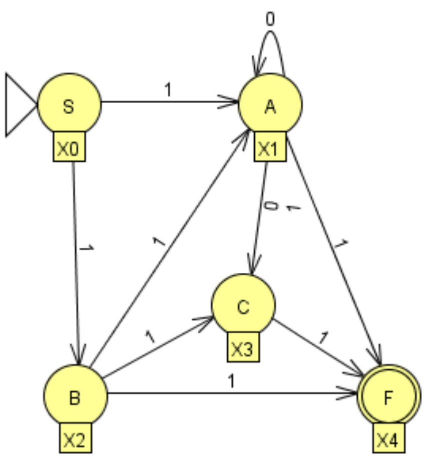
$$X_0 = 1 \cdot X_1 + 1 \cdot X_2$$

$$X_1 = 0 \cdot X_1 + 0 \cdot X_3 + 1 \cdot X_3 + 1 \cdot X_4 + 1$$

$$X_2 = 1 \cdot X_1 + 1 \cdot X_3 + 1 \cdot X_4 + 1$$

$$\begin{cases} X_3 = 1X_4 + 1 \Rightarrow \quad X_3 = 1 \\ X_4 = \lambda \end{cases}$$

Solution of the Characteristic Equations



$$\mathbf{X_2 = 1 \cdot X_1 + 1 \cdot 1 + 1 + 1 = = 1 \cdot X_1 + 1 \cdot 1 + 1}$$

$$\mathbf{X_1 = 0 \cdot X_1 + 0 \cdot 1 + 1 \cdot 1 + 1 \quad = 0 \cdot X_1 + (0 \cdot 1 + 1 \cdot 1 + 1)}$$

$$\mathbf{A = 0,\ B = (0 \cdot 1 + 1 \cdot 1 + 1) \ \Rightarrow}$$

$$\boxed{\mathbf{X_1 = 0^*(01 + 11 + 1)}}$$

$X_2 = \mathbf{10^*(0 \cdot 1 + 1 \cdot 1 + 1) + 1 \cdot 1 + 1}$ (solution)

$$X_0 = \mathbf{1 \cdot 0^*(0 \cdot 1 + 1 \cdot 1 + 1) + 1 \cdot (1 \cdot 0^*(0 \cdot 1 + 1 \cdot 1 + 1) \ + 1 \cdot 1 + 1) =}$$

$$= \underbrace{10^*01}_{\mathbf{a1}} + \underbrace{10^*11}_{b1} + \underbrace{10^*1}_{a2} + \underbrace{110^*01}_{\mathbf{c1}} + \underbrace{110^*11}_{b2} + \underbrace{110^*1}_{c2} + \underbrace{111}_{\mathbf{b3}} + \underbrace{11}_{\mathbf{a3}}$$

$a1, a3 \in a2$
$b3 \in b1$
$c1 \in c2$

$$\mathbf{ER = \underbrace{10^*11}_{b1} + \underbrace{10^*1}_{a2} + \underbrace{110^*11}_{b2} + \underbrace{110^*1}_{c2}}$$

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Solution of the Analysis Problem. Algorithm

1. Write the characteristic equations of the FA.

2. Resolve them.

3. If the initial state is q_0, X_0 gives us the set of strings that leads from q_0 to q_f and, therefore, the language accepted by the FA.

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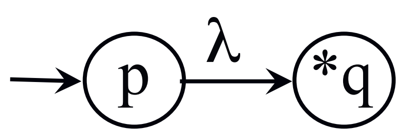
Synthesis Problem: Recursive Algorithm

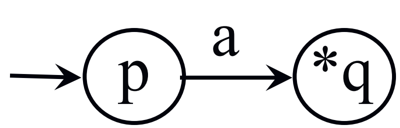
<u>SYNTHESIS PROBLEM (RE→FA):</u> "Given an RE representing a regular language, build a FA that accepts that regular language.

- Given a regular expression α:
  - If $\alpha = \Phi$, the automaton is:

- If $\alpha = \lambda$, the automaton is:

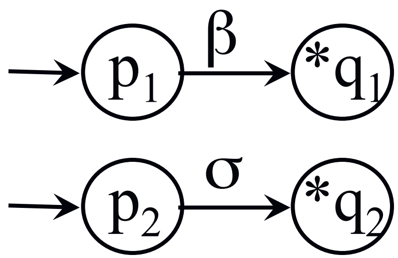


- If $\alpha =$ a, a$\in\Sigma$, the automaton is:

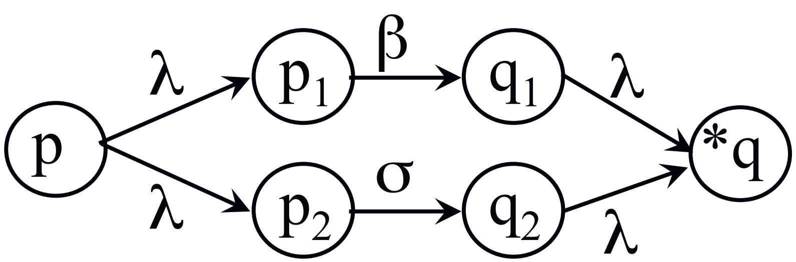


Synthesis Problem: Recursive Algorithm

- If $\alpha=\beta+\sigma$, using the automata β and σ



the result is:



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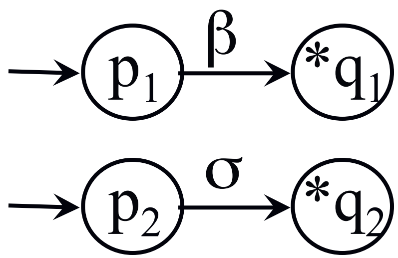
- If $\alpha = \beta \bullet \sigma$, using the automata β and σ



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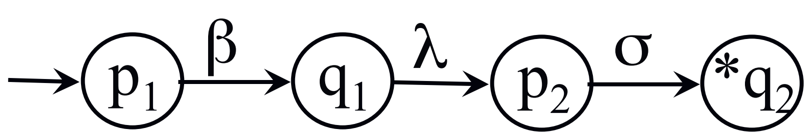
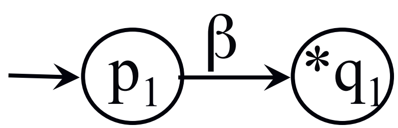
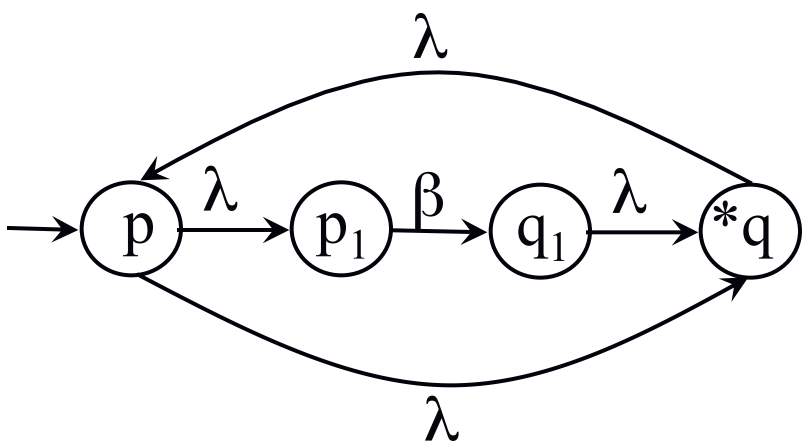


Synthesis Problem: Recursive Algorithm

- If $\alpha=\beta^*$, using the automata β



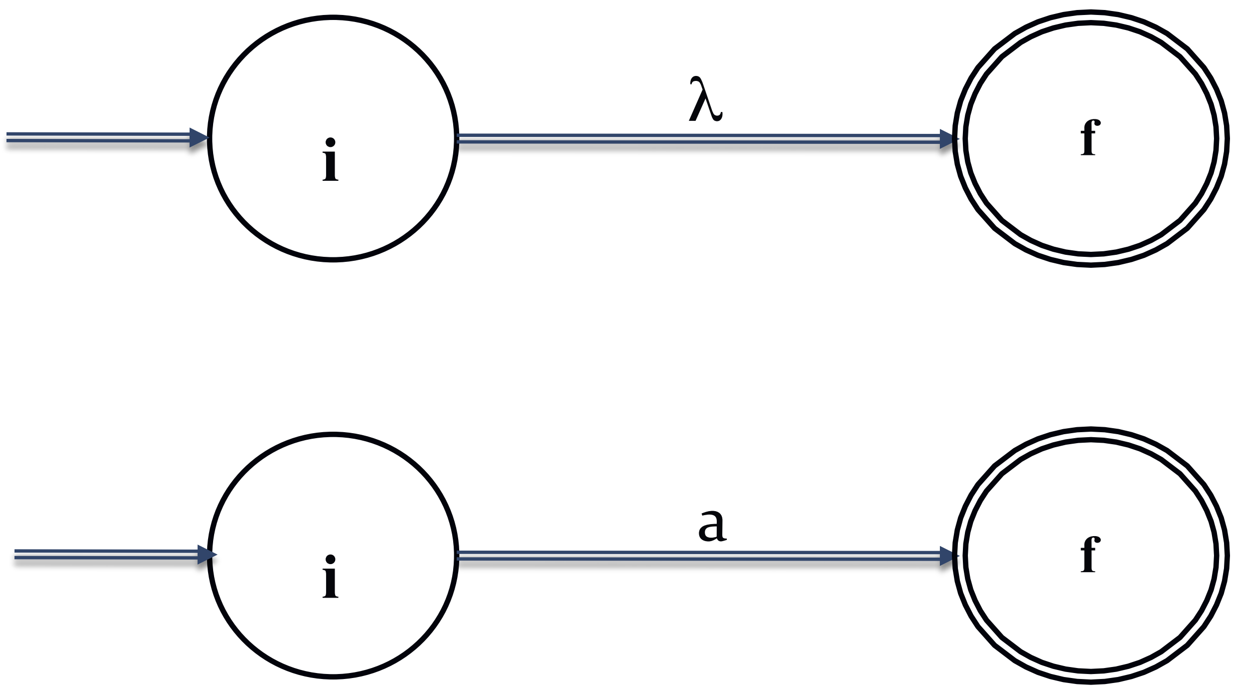
the result is:



Synthesis Problem: Recursive Algorithm

Summary

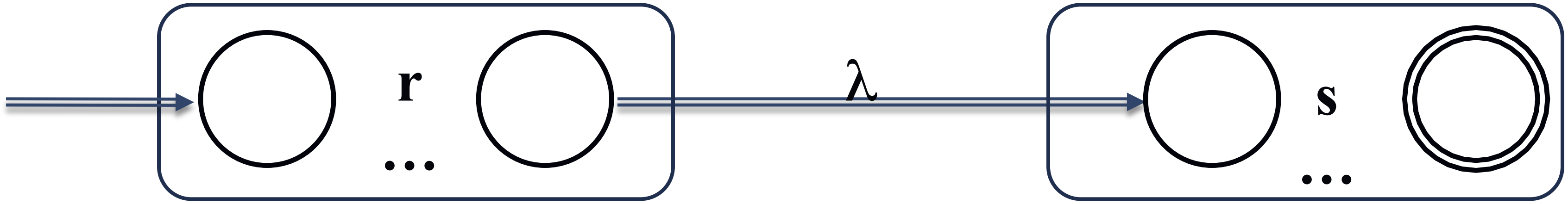
Basic Regular expressions (λ, a):



Synthesis Problem: Recursive Algorithm

Summary

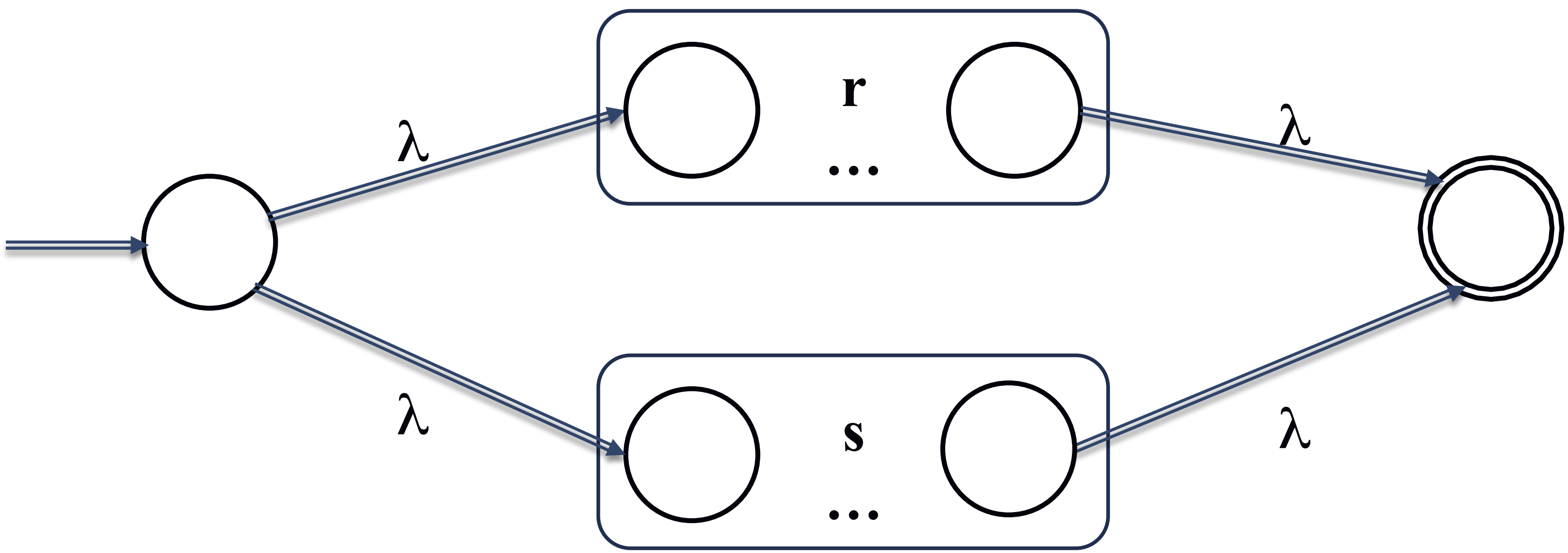
Concatenation rs:



Synthesis Problem: Recursive Algorithm

Summary

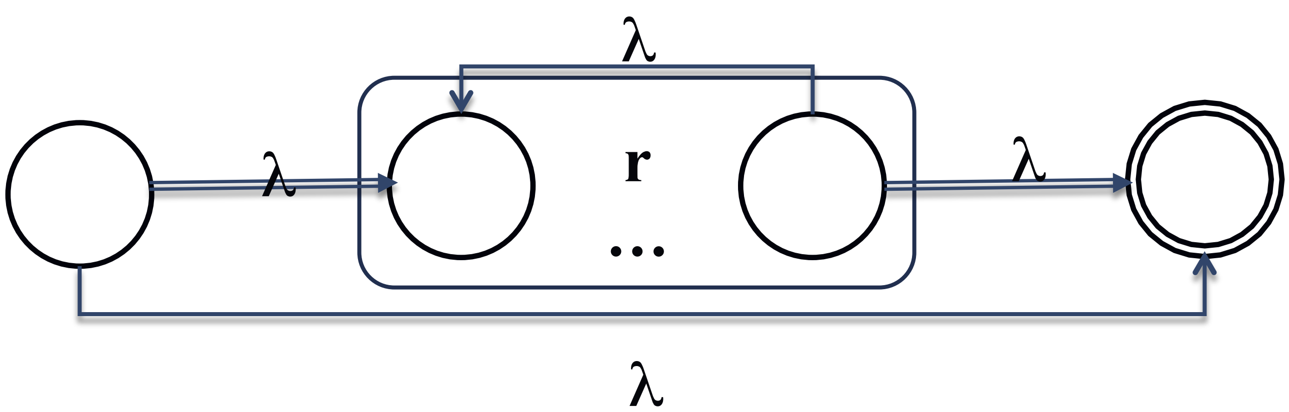
Selection r + s:



Synthesis Problem: Recursive Algorithm

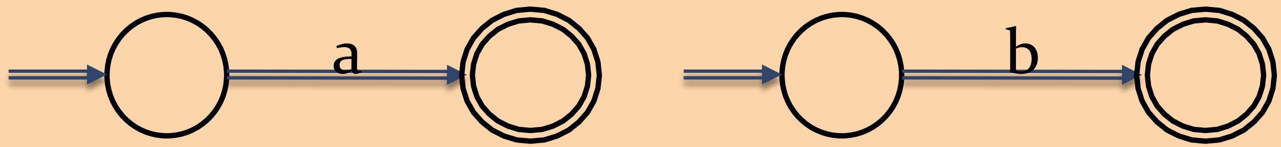
Summary

Repetition r^*:

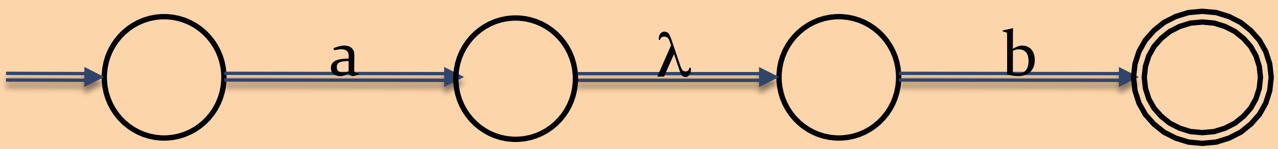


Synthesis Problem: Recursive Algorithm

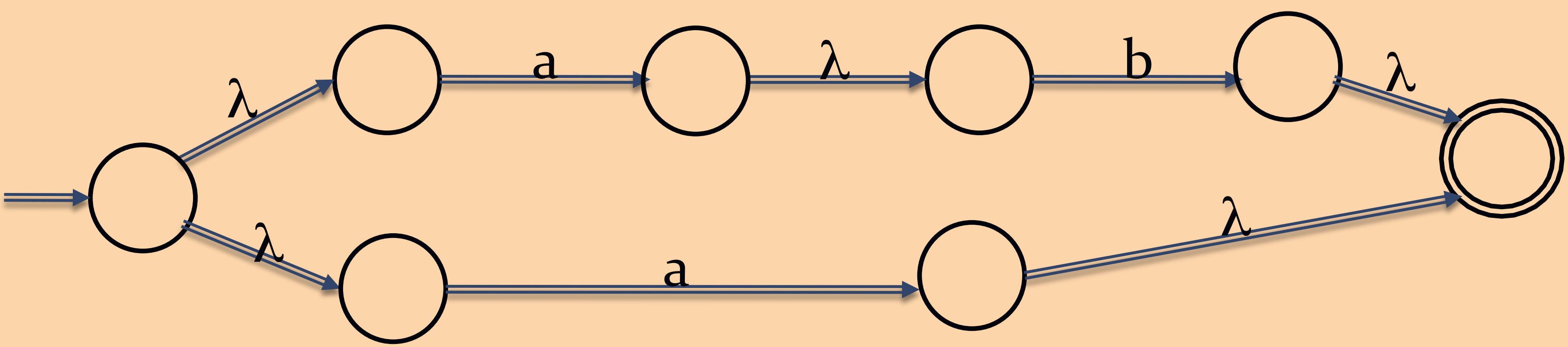
Example 1: ab + a



<u>ab</u>

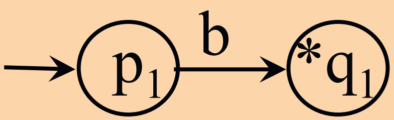


<u>ab+a</u>

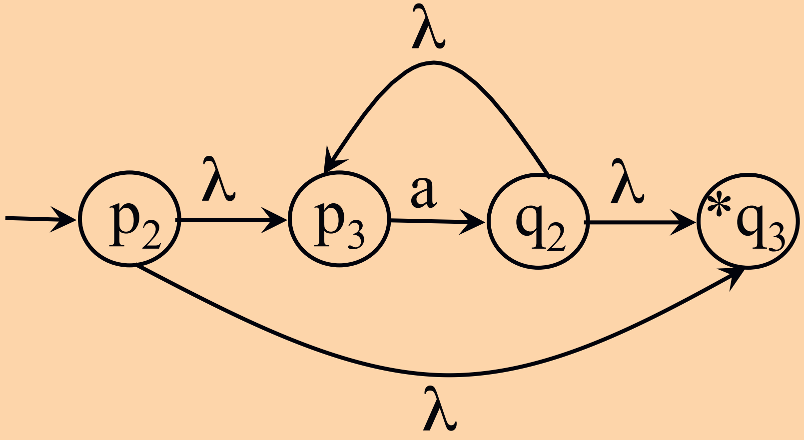


Synthesis Problem: Recursive Algorithm

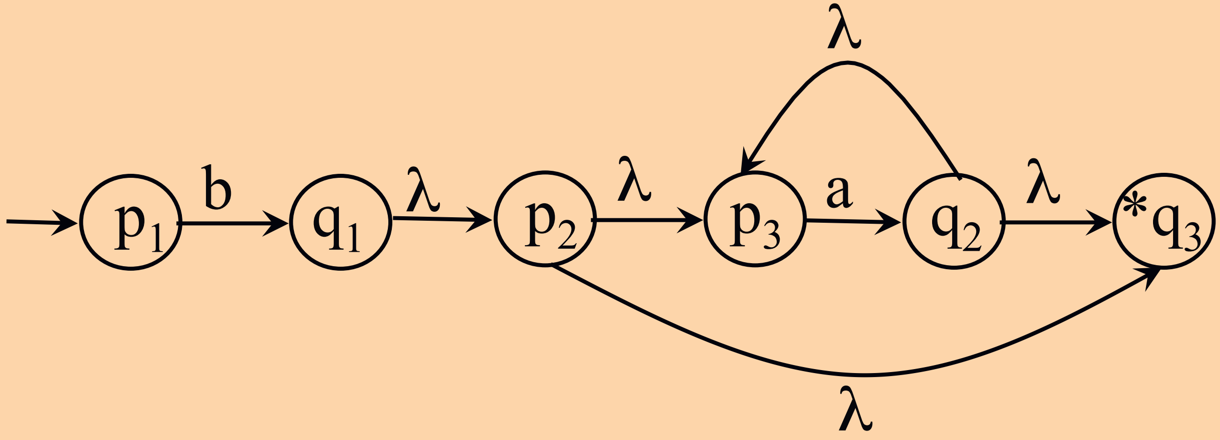
- Example α=(b • a*)*
  - b:



a*:

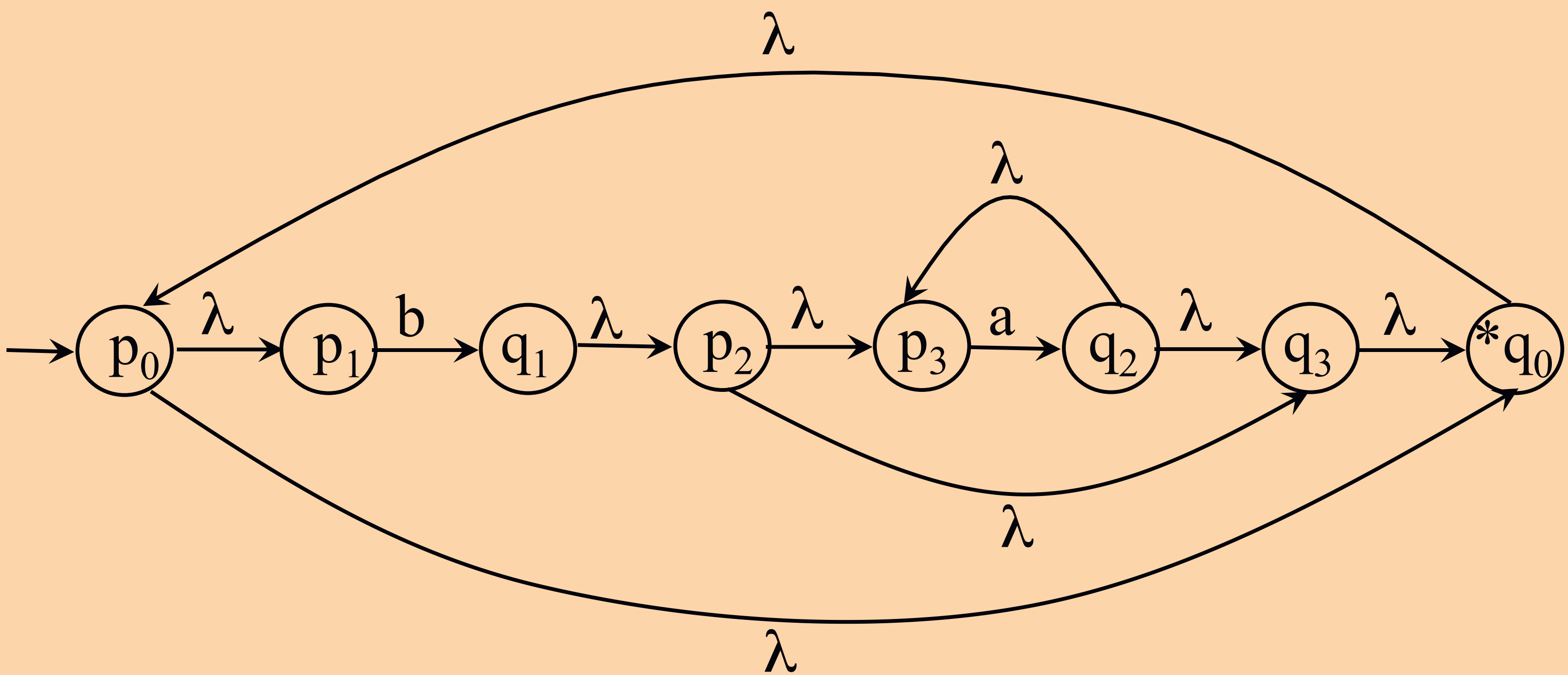


  - b • a*



Synthesis Problem: Recursive Algorithm

- (b • a*)*



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Solution to the synthesis problem: Derivatives of Regular Expressions

- Given a RE, construct a FA which recognizes the language that the RE describes.

  - Derive the RE and obtain a Right-Linear G3 and, from it, a FA.
    - Derivative of a RE?

- Derivative of a RE: $D_a(R) = \{ x \mid a \bullet x \in R \}$.

    - Derivative of a regular expression R with regard an input symbol $a \in \Sigma$ is the set of cues of every word represented by R whose head is a.
    - Let's see a recursive definition.

Solution to the synthesis problem: Derivatives of Regular Expressions

Given an RE → right-linear G3 grammar → FA which recognizes the language that describes the ER.

$$D_a(R) = \{ x \mid a.x \in R \}$$

Derivative of a RE: Recursive definition. $\forall$ a, b $\in \Sigma$ and R, S Reg. Exp.

- $D_a(\Phi) = \Phi$
- $D_a(\lambda) = \Phi$
- $D_a(a) = \lambda, \quad a \in \Sigma$
- $D_a(b) = \Phi, \quad \forall\, b \neq a,\ b \in \Sigma$
- $D_a(R+S) = D_a(R) + D_a(S)$
- $D_a(R \bullet S) = D_a(R) \bullet S + \delta(R) \bullet D_a(S) \quad \forall R$

  $\lambda \in L(R) \Rightarrow \delta(R) = \lambda$

  $\lambda \notin L(R) \Rightarrow \delta(R) = \Phi$
- $D_a(R^*) = D_a(R) \bullet R^*$

Solution to the synthesis problem: Derivatives of Regular Expressions

- **Definition: $D_{ab}(R)=D_b(D_a(R))$**
- **From a derivative** of a RE, obtain the **right-linear G3 grammar**.
    - The number of different derivatives of a RE is finite.
    - Once all have been obtained, you can obtain the G3 grammar:
    - **Given $D_a(R) = S$**, **with $S \neq \Phi$**
        - $S \neq \lambda \Rightarrow$ **$R ::= aS$** $\in P$
        - $S = \lambda \Rightarrow$ **$R ::= a$** $\in P$
    - **Given $\delta(D_a(R)) = S$**
        - $\delta(D_a(R)) = \lambda \Rightarrow$ **$R ::= a$** $\in P$
        - $\delta(D_a(R)) = \Phi \Rightarrow$ no rules included in P
- The axiom is R (starting RE)
- Σ_T = symbols that make up the starting RE.
- Σ_N = letters which distinguish each one of the different derivatives.

Solution to the synthesis problem: Derivatives of Regular Expressions

Obtain the G3 RL grammars that are equivalent to the following RE:

- $R = \mathbf{a \bullet a^* \bullet b \bullet b^*}$ $\Sigma=\{a,b\}$
  - $D_a(R) = D_a(a)\ a^*\ b\ b^* = a^*\ b\ b^*$
  - $D_b(R) = \Phi$
  - $D_{aa}(R) = D_a(a^*\ b\ b^*) = D_a(a^*)\ b\ b^* + \lambda\ D_a(b\ b^*) = a^*bb^* = D_a(R)$
  - $D_{ab}(R) = D_b(a^*\ b\ b^*) = D_b(a^*)\ b\ b^* + \lambda\ D_b(b\ b^*) = b^*$
  - $D_{aba}(R) = D_a(b^*) = \Phi$
  - $D_{abb}(R) = D_b(b^*) = D_b(b)\ b^* = b^* = D_{ab}(R)$

  - $D_a(R)= a^*bb^*$ $\delta(D_a(R))= \Phi$
  - $D_{aa}(R)= a^*bb^*$ $\delta(D_{aa}(R))= \Phi$
  - $D_{ab}(R)= b^*$ $\delta(D_{ab}(R))= \lambda$
  - $D_{abb}(R)= b^*$ $\delta(D_{abb}(R))= \lambda$

Solution to the synthesis problem: Derivatives of Regular Expressions

- $R_0 = aa^*bb^*$ $R_1 = a^*bb^*$ $R_2 = b^*$
  - $D_a(R_0) = R_1$ $\delta(D_a(R_0)) = \Phi$
  - $D_a(R_1) = R_1$ $\delta(D_a(R_1)) = \Phi$
  - $D_b(R_1) = R_2$ $\delta(D_b(R_1)) = \lambda$
  - $D_b(R_2) = R_2$ $\delta(D_b(R_2)) = \lambda$
- $D_a(R) = S \Rightarrow R \rightarrow aS$ $\delta(D_a(R)) = \lambda \Rightarrow R \rightarrow a$
  - $R_0 \rightarrow aR_1$ ----------
  - $R_1 \rightarrow aR_1$ ----------
  - $R_1 \rightarrow bR_2$ $R_1 \rightarrow b$
  - $R_2 \rightarrow bR_2$ $R_2 \rightarrow b$