ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

CALCULUS – Second Partial Exam

Bachelor in Computer Science and Engineering

November 2022 | maximum 3 points

Name		GROUP	89

1. *[1 point]* Approximate the value

$$\frac{1}{\sqrt{e}}$$

using a Taylor polynomial of suitable degree such that the involved error is smaller than 10^{-2}.

SOLUTION

The value $e^{-1/2}$ can be obtained by evaluating the function $f(x) = e^{x}$ at $x = -1/2$. Such function can be expressed by means of the Taylor's theorem as

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n(x)\,,$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{e^{c}}{(n+1)!}\,x^{n+1}\,,$$

with $c \in (x, 0)$ and $n \in \mathbb{N}$. Hence, at $x = -1/2$, we can estimate the involved approximation error as

$$|R_n(-1/2)| = \frac{e^{c}}{2^{n+1}\,(n+1)!} < \frac{1}{2^{n+1}\,(n+1)!}\,.$$

Finally, imposing

$$\frac{1}{2^{n+1}\,(n+1)!} < 10^{-2} \iff 2^{n+1}\,(n+1)! > 100\,,$$

we can deduce that the considered Maclaurin polynomial must have degree $n = 3$, at least. Thus, a proper approximation is

$$e^{-1/2} \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48}\,.$$

2. *[1 point]* Let

$$f(x) = \begin{cases} \arctan(\ln(x^2)) & \text{if } x \neq 0\,, \\ -\pi/2 & \text{if } x = 0\,. \end{cases}$$

(a) Study the continuity of $f(x)$ in the domain.

(b) Calculate $f'(x)$ for $x \neq 0$.

(c) Study whether $f(x)$ is differentiable at $x = 0$.

SOLUTION

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0$, the given function is continuous as composition of continuous elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $f(0) = -\pi/2$ and we have

$$\lim_{x\to 0} f(x) = \arctan(-\infty) = -\pi/2\,.$$

(b) For $x \neq 0$, the derivative of the given function is

$$f'(x) = \frac{2x/x^2}{1 + (\ln(x^2))^2} = \frac{2}{x + 4x\ln^2(x)}\,.$$

(c) The derivative of $f(x)$ at $x = 0$ is given by

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\arctan(\ln(x^2)) + \frac{\pi}{2}}{x} = \lim_{x\to 0} \frac{2}{x + 4x\ln^2(x)}\,,$$

where the l'Hôpital's rule has been applied. Noting that

$$\lim_{x\to 0} 4x\ln^2(x) = \lim_{x\to 0} \frac{4\ln^2(x)}{1/x} = 0\,,$$

which is obtained by using l'Hôpital's rule twice, we finally get $f'(0) = \infty$. Hence, the function is not differentiable at $x = 0$.

3. *[1 point]* Find the exact number of real solutions of the equation

$$e^{\sin(x)} = x + 1$$

in the interval $\left[\dfrac{\pi}{2}, \pi\right]$.

SOLUTION

The given equation can be written as $f(x) = 0$, where $f(x) = e^{\sin(x)} - x - 1$. Note that

$$f'(x) = \cos(x)\, e^{\sin(x)} - 1 < 0$$

for all $x \in \left[\frac{\pi}{2}, \pi\right]$, which means that the function is strictly decreasing on that interval. On the other hand, $f(\pi/2) > 0$ and $f(\pi) < 0$, hence $f(x)$ intersects the x-axis only once. Thus, we can conclude that the equation has only one real solution in the indicated interval.