ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

CALCULUS – Second Partial Exam

Bachelor in Computer Science and Engineering

November 2022 | maximum 3 points

Name		GROUP	87

1. *[1 point]* Approximate the function

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

by a Taylor polynomial of degree $n = 2$. Then, estimate the involved error for $x \in [-1/5, 1/5]$.

SOLUTION

Note that the Maclaurin polynomial of degree $n = 2$ of e^{x} is given by $1 + x + \frac{x^2}{2!}$. Hence, the given function, $f(x) = \cosh(x)$, can be expressed by means of the Taylor's theorem as

$$f(x) = \frac{1 + x + \frac{x^2}{2!} + 1 - x + \frac{x^2}{2!}}{2} + R_2(x)\,,$$

where the remainder $R_2(x)$ is given by

$$R_2(x) = \frac{f'''(c)}{3!}\,x^3\,,$$

with $c \in (0, x)$ or $(x, 0)$. Thus, we get the approximation

$$\cosh(x) \approx 1 + \frac{x^2}{2}\,.$$

On the other hand, we can estimate the involved approximation error for $x \in [-1/5, 1/5]$ as

$$|R_2(x)| = \frac{|e^{c} - e^{-c}|}{2}\,\frac{|x|^3}{6} \le \frac{|e^{c} - e^{-c}|}{12 \cdot 5^3} \le \frac{e^{c} + e^{-c}}{1500} \le \frac{e^{1/5}}{750}\,.$$

2. *[1 point]* Let

$$f(x) = \begin{cases} x^2 \arctan\left(\dfrac{1}{x}\right) - \sqrt{1+x^2} & \text{if } x \neq 0\,, \\ -1 & \text{if } x = 0\,. \end{cases}$$

(a) Study the continuity of $f(x)$ in the domain.

(b) Calculate $f'(x)$ for $x \neq 0$.

(c) Study whether $f(x)$ is differentiable at $x = 0$.

SOLUTION

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0$, the function is continuous as given in terms of continuous elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $f(0) = -1$ and we have

$$\lim_{x \to 0} f(x) = -1\,.$$

(b) For $x \neq 0$, the derivative of the given function is

$$f'(x) = 2x \arctan\left(\frac{1}{x}\right) - \frac{x^2}{x^2+1} - \frac{x}{\sqrt{1+x^2}}\,.$$

(c) The derivative of $f(x)$ at $x = 0$ is given by

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \left[2x \arctan\left(\frac{1}{x}\right) - \frac{x^2}{x^2+1} - \frac{x}{\sqrt{1+x^2}}\right] = 0\,,$$

where the l'Hôpital's rule has been applied. Thus, $f'(0) = 0$ and the function is differentiable at $x = 0$.

3. *[1 point]* Calculate

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \frac{\cos(x)}{\sin(x)}\right)$$

using appropriate Taylor polynomials for the involved functions.

SOLUTION

In the given limit, we can approximate all involved elementary functions by Maclaurin polynomials of suitable degree. Indeed, we can write

$$\lim_{x\to 0} \frac{\sin(x) - x\,\cos(x)}{x^2 \sin(x)} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + o(x^3) - x\left[1 - \frac{x^2}{2!} + o(x^2)\right]}{x^2\,[x + o(x)]} = \lim_{x\to 0} \frac{\frac{x^3}{3} + o(x^3)}{x^3 + o(x^3)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} = \frac{1}{3}\,.$$