2nd Midterm – 88

Problem 1. Solve the following recurrence relation:

$$\begin{cases} a_{n+1} = \pi a_n + k^{n+1}, \\ a_1 = \pi, \end{cases}$$

where $k = \frac{355}{113} \approx 3.141592\ldots$.

Solution. This is a first order linear non-homogeneous recurrence relation, and can be written as

$$a_n = \pi a_{n-1} + k^n.$$

First, we solve the related homogeneous recurrence relation: $b_n = \pi b_{n-1}$. The characteristic equation is $x = \pi$. Hence we have a single root of algebraic multiplicity one. This makes

$$a_n^h = A\pi^n.$$

Second, we find a particular solution. The non-homogeneous term is $t_n = k^n$, where $k = \frac{355}{113}$. Notice that k is a fraction, hence k is a rational number. On the contrary, π is known to be irrational. Thus $k \neq \pi$. In consequence, we know that $a_n^p = k^n p_0$. Substitute that into the recurrence: $k^n p_0 = \pi k^{n-1} p_0 + k^n$. Dividing by k^{n-1} we get $k p_0 = \pi p_0 + k$. Hence $p_0 = \frac{k}{k-\pi}$. All in all,

$$a_n^p = \frac{k^{n+1}}{k-\pi}.$$

Third, as $a_n = a_n^h + a_n^p = A\pi^n + \frac{k^{n+1}}{k-\pi}$ has to check the initial condition, we set $\pi = a_1 = A\pi + \frac{k^2}{k-\pi}$, and we get that $A = 1 - \frac{k^2}{\pi(k-\pi)}$.

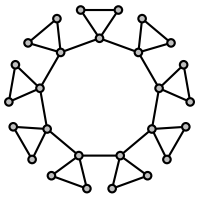
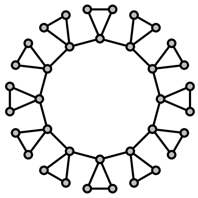
In conclusion,

$$\boxed{a_n = \left(1 - \frac{k^2}{\pi(k-\pi)}\right)\pi^n + \frac{k^{n+1}}{k-\pi}}$$

Problem 2. Let $G_n = (V_n, E_n)$, $n \geq 3$, where

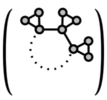
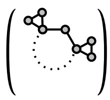
- $V_n = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\} \cup \{c_1, c_2, \ldots, c_n\}$, hence $|V_n| = 3n$, and
- $E_n = \{\{a_i, a_{i+1}\}, 1 \leq i \leq n-1\} \cup \{\{a_n, a_1\}\} \cup \{\{a_i, b_i\}, 1 \leq i \leq n\} \cup \{\{a_i, c_i\}, 1 \leq i \leq n\} \cup$
  $\cup \{\{b_i, c_i\}, 1 \leq i \leq n\}$, hence $|E_n| = 4n$.

For instance,

$G_9 \equiv$  or $G_{12} \equiv$ 

Find, using suitable properties of the chromatic polynomial, $P_{G_n}(q)$ and $\chi(G_n)$, for $n \geq 3$.

Solution. G_n is a C_n with n extra triangles sticking out. Those triangles are a K_3, and overlap with the graph forming a K_1. Using the factorization theorem, we know:

$P_{G_n}(q) = P\Bigg($  $\Bigg) = \dfrac{P_{K_3}(q)}{P_{K_1}(q)} P\Bigg($  $\Bigg).$

Doing the same for the remaining triangles, we end up with a stripped down graph that is a C_n. Since G_n has a total of n triangles, we end up with:

$$P_{G_n}(q) = \left(\frac{P_{K_3}(q)}{P_{K_1}(q)}\right)^n P_{C_n}(q) = (q-1)^n (q-2)^n P_{C_n}(q).$$

Now our problem is to find $P_{C_n}(q)$. This has been done in class, and if $a_n = P_{C_n}(q)$, then we have to solve the recurrence

$$\begin{cases} a_n = q(q-1)^{n-1} - a_{n-1}, \\ a_3 = q(q-1)(q-2). \end{cases}$$

Once solved, we know $P_{C_n}(q) = (q-1)^n + (-1)^n (q-1)$.

In conclusion,

$$\boxed{P_{G_n}(q) = (q-1)^n (q-2)^n \left[(q-1)^n + (-1)^n (q-1)\right]}.$$

This also means that $P_{G_n}(q)$ has as roots $q=1$, $q=2$, and the ones from $P_{C_n}(q)$. Since the roots for $P_{C_n}(q)$ are always $q \le 2$, we can conclude that the largest integer root for $P_{G_n}(q)$ is $q=2$. Hence,

$$\boxed{\chi(G_n) = 3, \text{ for all } n \ge 3}.$$

Problem 3. Solve, using generating functions techniques, this recurrence relation:

$$\begin{cases} a_{n+2} = -6a_{n+1} - 9a_n, \\ a_0 = 0; \quad a_1 = 1. \end{cases}$$

Solution. The recurrence relation is equivalent to $a_n = -6a_{n-1} - 9a_{n-2}$. Encode the solution sequence $(a_n)_{n=0}^{\infty}$ into $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} F(x) &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 0 + x + \sum_{n=2}^{\infty} (-6a_{n-1} - 9a_{n-2}) x^n = \\ &= x - 6 \sum_{n=2}^{\infty} a_{n-1} x^n - 9 \sum_{n=2}^{\infty} a_{n-2} x^n = \\ &= x - 6x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 9x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = \\ &= x - 6x\,(F(x) - a_0) - 9x^2 F(x) = x - 6xF(x) - 9x^2 F(x). \end{aligned}$$

That is,

$$\begin{aligned} F(x) &= \frac{x}{1+6x+9x^2} = x\frac{1}{(1+3x)^2} = x \sum_{n=0}^{\infty} \binom{n+1}{n} (-3x)^n \\ &= x \sum_{n=0}^{\infty} \binom{n+1}{n} (-1)^n 3^n x^n = x \sum_{n=0}^{\infty} (n+1)(-1)^n 3^n x^n = \\ &= \sum_{n=0}^{\infty} (n+1)(-1)^n 3^n x^{n+1} = \sum_{n=1}^{\infty} n(-1)^{n-1} 3^{n-1} x^n. \end{aligned}$$

From which, we can conclude that

$$\boxed{a_n = n(-1)^{n-1} 3^{n-1}},$$

expression that is also valid for $n = 0$.