CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Taylor polynomial

Problem 7.1.

- We get $\sin(1) \approx 0.8415$ by using a Maclaurin polynomial of degree 7 for $\sin(x)$ evaluated at $x = 1$.
- We get $\sqrt[5]{\frac{3}{2}} \approx 1.08$ by using a Maclaurin polynomial of degree 2 for $(1+x)^{1/5}$ evaluated at $x = 1/2$.

Problem 7.2.

1. $P_3(x) \, = \, 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \, .$
2. $P_n(x) \, = \, 3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} + \ldots + (-1)^k \frac{3^{2k+1}}{(2k+1)!} \, x^{4k+2} \, , \;\; n = 2k+1 \, .$
3. $P_5(x) \, = \, x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \, .$
4. $P_3(x) \, = \, 1 - \frac{3}{2}x^2$ (the coefficient of x^3 is equal to zero).
5. $P_n(x) \, = \, 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \ldots + \frac{2^n + 2}{n!} \, x^n \, .$

Problem 7.3. The given polynomial can be written (with no error) as its Taylor polynomial of degree 4 about $a = 4$, namely

$$x^4 - 5x^3 + x^2 - 3x + 4 \, = \, -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4 \, .$$

Problem 7.4. By the method of induction, we can prove that

$$f(x) \, = \, -1 - (x+1) - (x+1)^2 - \ldots - (x+1)^n + \frac{1}{c} \left(-\frac{x+1}{c} \right)^{n+1} \, ,$$

where the last term is the remainder and $c \in (-1, x)$ or $(x, -1)$.

Problem 7.5. We get $P_5(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$.

Problem 7.6. The desired coefficient is $\frac{f^{(4)}(0)}{4!} = -\frac{1}{12}$.

Problem 7.7.

- $P_3(x) = 2x - \frac{4}{3}x^3$.
- $P_3(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3$.
- $P_3(x) = x - x^2 + \frac{1}{2}x^3$.
- $P_3(x) = -x - \frac{3}{2}x^2 - \frac{4}{3}x^3$.
- $P_3(x) = x^2$ (the coefficient of x^3 is equal to zero).
- $P_3(x) = x - x^2 + \frac{11}{6}x^3$.

Problem 7.8.

- $P_n(x) = 1 - \frac{a^2}{2!}x^2 + \frac{a^4}{4!}x^4 - \frac{a^6}{6!}x^6 + \ldots + (-1)^k \frac{a^{2k}}{(2k)!} x^{2k}, \; n = 2k$.
- $P_n(x) = ax + \frac{a^3}{3!}x^3 + \frac{a^5}{5!}x^5 + \ldots + \frac{a^{2k+1}}{(2k+1)!} x^{2k+1}, \; n = 2k+1$.
- $P_n(x) = 1 + ax^2 + \frac{a^2}{2!}x^4 + \frac{a^3}{3!}x^6 + \ldots + \frac{a^k}{k!}x^{2k}, \; n = 2k$.
- $P_n(x) = 1 + 2x + 2x^2 + \ldots + 2x^n$.

Problem 7.9.

- An equation for the tangent line is $y = 0$.
- The value of the limit is 2.

- $f^{(4)}(1) = -72$.

Problem 7.10. In each case, prove that the indicated limit of the function on the left divided by the power of x in the $o(\cdot)$ is zero (use suitable Taylor polynomials in the first three cases and l'Hôpital's rule in the last case).

Problem 7.11. The polynomial $P(x)$ is not unique. For instance, $P(x) = 1 - x^4/2$ can be used.

Problem 7.12. The desired polynomial is $P_3(x) = 2 + x + x^3/6$. In addition, the approximation error can be estimated as

$$|R_3(x)| = \left|\frac{\cos(c) + e^c}{4!} x^4\right| \leq \frac{1 + e^{1/4}}{4!} \left(\frac{1}{4}\right)^4 ,$$

as $c \in (-1/4, 1/4)$.

Problem 7.13. A Maclaurin polynomial of degree $n = 7$ (at least) should be used.

Problem 7.14. We get $1/\sqrt{1.1} \approx 0.9534375$ by using the Maclaurin polynomial of degree 3 for $(1+x)^{-1/2}$ evaluated at $x = 0.1$. An upper bound for the involved error is given by

$$\frac{35\,(0.1)^4}{2^7} \approx 0.000027 .$$

Problem 7.15.

- We get an approximation of $\sin(2)$ by using the Maclaurin polynomial of degree $n \geq 9$ for $\sin(x)$ evaluated at $x = 2$.
- We get an approximation of $\ln(4/5)$ by using the Maclaurin polynomial of degree $n \geq 3$ for $\ln(1 + x)$ evaluated at $x = -1/5$.
- We get an approximation of $\cos(1)$ by using the Maclaurin polynomial of degree $n \geq 6$ for $\cos(x)$ evaluated at $x = 1$.
- We get an approximation of e^{-2} by using the Maclaurin polynomial of degree $n \geq 9$ for e^x evaluated at $x = -2$.
- We get an approximation of $\ln(2)$ by the Maclaurin polynomial of degree $n \geq 1000$ for $\ln(1 + x)$ evaluated at $x = 1$.

Problem 7.16. At least all terms of the Taylor series (about $a = 0$) for $\sin(x)$ up to $-x^{11}/11!$ included must be considered (with $x = 1/2$).

Problem 7.17. The values of the indicated limits are the following.

(a) $1/2$.

(b) $1/120$.

(c) $1/2$.

(d) $1/2$.

(e) $1/27$.

(f) $1/6$.

(g) 0.

(h) $1/3$.

(i) $-1/4$ (use the change of variable $t = 1/x$).

(j) $1/2$ (use the change of variable $t = 1/x$).

Problem 7.18. The values of the indicated limits are the following.

(a) 0 (l'Hôpital).

(b) $+\infty$ (l'Hôpital).

(c) 1 (l'Hôpital).

(d) 0 (l'Hôpital).

(e) 0 (l'Hôpital).

(f) 1 (l'Hôpital).

(g) e (Taylor).

Local and global behavior of a function

Problem 8.1.

(a) $x = 2$ is a point of local minimum and $x = -1$ is a point of local maximum.

(b) No local extrema.

(c) $x = 0$ is a point of local minimum and $x = 1$ is a point of local maximum.

Problem 8.2.

- The function $f(x)$ is (strictly) increasing in $(0, 3) \cup (4, +\infty)$ and decreasing in $(-\infty, 0) \cup (3, 4)$.
- $x = 0, 4$ are points of local minima and $x = 3$ is a point of local maximum.
- The equation $f(x) = 0$ has a unique solution as $f(x)$ is strictly increasing in the interval $(0, 1)$ and $f(0) < 0$, $f(1) > 0$.

Problem 8.3. The desired area is $2ab$.

Problem 8.4. The point $x = 0$ is an inflection point and $f(x)$ is concave down / up on the left / right of it.

Problem 8.5.

- $f(x)$ is concave up in $(-2/5, 0) \cup (0, +\infty)$ and concave down in $(-\infty, -2/5)$; in addition, $x = -2/5$ is an inflection point.
- $f(x)$ is concave up in $(2, +\infty)$.
- $f(x)$ is concave up in $\mathbb{R}$.
- $f(x)$ is concave down in $(-\infty, 2) \cup (4, +\infty)$.

Problem 8.6. The point $x = 0$ is of local minimum and $f(x)$ is concave up in a neighborhood of it.

Problem 8.7.

(1) The function $f(x)$ is decreasing in $(-\infty, -1/2)$.

(2) Values $\alpha = 0$ and $\beta = 1$ make $f(x)$ differentiable at $x = 0$ and in $\mathbb{R}$.

(3) The global minimum is $-5/4$ and is attained at $x = -1/2$. On the other hand, there is no global maximum.

Problem 8.8.

(a) The critical points are $x = 1$ (of local minimum) and $x = 0$ (inflection point).

(b) $f(x)$ is increasing in $(1, +\infty)$ and decreasing in $(-\infty, 0) \cup (0, 1)$.

(c) The inflection points are $x = 0, 2/3$.

(d) $f(x)$ is concave down in $(0, 2/3)$ and concave up in $(-\infty, 0) \cup (2/3, +\infty)$.

Problem 8.9.

- The global minimum is $\dfrac{3\pi - 4}{4\sqrt{2}}$ and is attained at points $x = \pm\dfrac{3}{4}\pi$. The global maximum is $\dfrac{\pi + 4}{4\sqrt{2}}$ and is attained at points $x = \pm\dfrac{\pi}{4}$.
- The global minimum is 0 and is attained at point $x = 0$, while the global maximum is 7 and is attained at point $x = 1$.