CALCULUS

Bachelor in Informatics Engineering

Course 2022–2023

Integrals: Fundamental Theorem of Calculus

Problem 9.1. The desired area is equal to 1 .

Problem 9.2. The desired area is equal to $2 - \ln(3)$.

Problem 9.3. We get

$$F(x) \;=\; \begin{cases} \sin(x) & \text{for} \quad 0 \le x \le \pi/2\,, \\ 1 + \pi/2 - x & \text{for} \quad \pi/2 < x \le \pi\,. \end{cases}$$

In addition, by the Fundamental Theorem of Calculus we have that $F'(x) = f(x)$ for each $x \in (0, \pi)$ with $x \neq \pi/2$ (indeed $F(x)$ is not differentiable at $x = \pi/2$).

Problem 9.4. The equation of the tangent line is

$$y = F(1) + F'(1)(x-1) \;=\; -\frac{1}{3}\,x + \frac{1}{3}\,.$$

Problem 9.5. The function $F(x)$ is strictly increasing in $\mathbb{R}$ (as $F'(x) > 0$), hence $F(x)$ is one-to-one for any value of $x \in \mathbb{R}$.

Problem 9.6.

(a) The value of the limit is 1 (for instance, by l'Hôpital's rule).

(b) The value of the limit is 0 (for instance, by the sandwich theorem).

Problem 9.7. We have

$$H'(x) \;=\; \int_{2x}^{3x} e^{-t^2}\,dt \;+\; x\,\left\{ 3e^{-9x^2} - 2e^{-4x^2} \right\}\,,$$

$$H''(x) \;=\; 2\,\left\{ 3e^{-9x^2} - 2e^{-4x^2} \right\} \;+\; 2x^2\,\left\{ 8e^{-4x^2} - 27e^{-9x^2} \right\}\,.$$

Problem 9.8. The function $H(x)$ is decreasing in $[0, 1/2]$ as $H'(x) = \ln(1 - x^2) < 0$ in that interval.

Problem 9.9. The global maximum is at $x = 3$ and the global minimum is at $x = 1$. In addition

$$H(3) \;=\; 2\int_0^1 e^{-t^4}\,dt \;>\; 2\int_0^1 e^{-1}\,dt \;=\; \frac{2}{e} \;>\; \frac{2}{3}\,.$$

Problem 9.10. $(a)\;\; 0\,. \quad (b)\;\; 1/3\,.$

Problem 9.11.

- We have that $F'(x) = 1 + \sin(\sin(x)) > 0$ for each $x \in \mathbb{R}$. Thus, the function $F(x)$ is strictly increasing and one-to-one in $\mathbb{R}$. In addition, it's easy to see that $F(0) = 0$, which implies that $F^{-1}(0) = 0$. As a consequence, we get

$$(F^{-1})'(0) \;=\; \frac{1}{F'(0)} \;=\; \frac{1}{1+\sin(\sin(0))} \;=\; 1\,.$$

- Observe that

$$G(x) \;=\; \int_1^0 \sin(\sin(t))\,dt \;+\; \int_0^x \sin(\sin(t))\,dt\,,$$

  where the first integral is independent of x (say, equal to $G_0 \in \mathbb{R}$) and the second integral is a function $H(x)$ that is invariant under replacing x with $-x$ (this can be shown by means of the change of variable $u = -t$). As a consequence, we get $G(-x) = G_0 + H(-x) = G_0 + H(x) = G(x)$, which means that $G(x)$ is even and not one-to-one, namely G^{-1} does not exist.

Problem 9.12. The desired Taylor polynomial is given by $P_3(x) \;=\; x^3/3$ and the value of the limit is $1/3$.

Problem 9.13. We have

$$(a)\quad H'(x) = \sin^3(x)\left\{1 + \left(\int_1^x \sin^3(t)\,dt\right)^2 + \sin^6\left(\int_1^x \sin^3(t)\,dt\right)\right\}^{-1}\,.$$

$$(b)\quad K'(x) = \cos\left(\int_0^x \sin\left(\int_0^t \sin^3(s)\,ds\right)\,dt\right)\,\sin\left(\int_0^x \sin^3(s)\,ds\right)\,.$$

Techniques of integration

Problem 10.1. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

- $I(x) = x\arctan(3x) - \frac{1}{6}\ln(1+9x^2) + k$ (integration by parts)
- $I(x) = \frac{1}{2}e^x(\sin(x) - \cos(x)) + k$ (integration by parts)
- $I(x) = \frac{1}{2}x\left(\cos(\ln(x)) + \sin(\ln(x))\right) + k$ (change of variable $t = \ln(x)$ and integration by parts)
- $I(x) = \frac{x}{2} + \frac{x}{10}\cos(2\ln(x)) + \frac{x}{5}\sin(2\ln(x)) + k$ (change of variable $t = \ln(x)$, identity $\cos(2\alpha) = 2\cos^2(\alpha) - 1$ and integration by parts)
- $I(x) = \arctan\left(\frac{1}{2}\sqrt{e^x - 4}\right) + k$ (change of variable $t = \sqrt{e^x - 4}$)
- $I(x) = \arctan\left(\sqrt{x^2 - 1}\right) + k$ (change of variable $t = \sqrt{x^2 - 1}$)

Problem 10.2. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

- $I = \sqrt{3} - \frac{\pi}{3}$ (change of variable $u = \sqrt{t^2 - 1}$)
- $I = 2 - \frac{\pi}{2}$ (change of variable $u = \sqrt{e^t - 1}$)
- $I(x) = 2\arctan\left(\sqrt{1+x}\right) + k$ (change of variable $t = \sqrt{1+x}$)
- $I(x) = -\frac{3}{2}(1-x)^{2/3} + 3(1-x)^{1/3} - 3\ln\left|(1-x)^{1/3} + 1\right| + k$ (change of variable $t = (1-x)^{1/3}$)

Problem 10.3. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

- $I(x) = \frac{1}{\sqrt{2}}\arctan\left(\frac{3}{\sqrt{2}}x + \sqrt{2}\right) + k$
- $I(x) = \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} + \frac{1}{2}x^2 + k$

- $I(x) = \dfrac{1}{x} + \ln|x-1| - \ln|x+1| + k$
- $I(x) = \dfrac{3}{2}\ln(x^2+4x+13) + \dfrac{47}{3}\arctan\left(\dfrac{x+2}{3}\right) + \dfrac{1}{2}x^2 - 4x + k$
- $I(x) = \dfrac{3}{2}\ln|x-1| - \dfrac{1}{2}\ln|x-3| - \dfrac{13}{x-3} + k$

Problem 10.4. In each case, the integral $I(x)$ is given by the indicated expression ($k \in \mathbb{R}$).

1. $I(x) = \sin(x) - \dfrac{1}{3}\sin^3(x) + k$
2. $I(x) = \dfrac{3}{8}x - \dfrac{1}{4}\sin(2x) + \dfrac{1}{32}\sin(4x) + k$
3. $I(x) = \dfrac{1}{2}e^{2x} - 2e^x + \ln\left(e^{2x} + 2e^x + 2\right) + 2\arctan(e^x + 1) + k$
4. $I(x) = \cos(x) - 2\arctan(\cos(x)) + k$
5. $I(x) = -\dfrac{1}{2}\ln|1-\sin(x)| + \dfrac{1}{2}\ln|1+\sin(x)| + k$
6. $I(x) = \dfrac{a^2}{2}\arcsin\left(\dfrac{x}{a}\right) + \dfrac{x}{2}\sqrt{a^2-x^2} + k$

Improper integrals

Problem 11.1.

- Divergent (for instance, by the definition of improper integral) .
- Divergent (for instance, by limit comparison test with $\int_1^{+\infty} dx/x$) .
- Convergent (for instance, by comparison test with $\int_1^{+\infty} dx/x^3$ after using the absolute value of the integrand) .
- Convergent (for instance, by limit comparison test with $\int_1^{+\infty} dx/x^{\alpha+1}$) .
- Convergent (for instance, by limit comparison test with $\int_1^{+\infty} dx/x^{3/2}$; note that the change of variable $t = 1/\sqrt{x}$ may be useful) .

- Divergent (for instance, by limit comparison test with $\int_2^7 dx/(x-2)$).
- It's not an improper integral.
- Convergent (for instance, by limit comparison test with $\int_1^2 dx/(x-1)^{1/2}$).
- Divergent. Indeed, the given integral can be written as $\int_1^2 x/\sqrt{x^4-1}\,dx$ + $\int_2^{+\infty} x/\sqrt{x^4-1}\,dx$, where the first integral converges (for instance, by limit comparison test with $\int_1^2 dx/(x-1)^{1/2}$) but the second integral diverges (for instance, by limit comparison test with $\int_2^{+\infty} dx/x$).
- Convergent. Apply the method of induction together with the definition of improper integral.
- Convergent. First note that e^{-x^2} is even, thus the integral can be written as $2\int_0^{+\infty} e^{-x^2}\,dx$. Then, the latter converges by limit comparison test with $\int_0^{+\infty} e^{-x}\,dx$.
- Convergent. Indeed, the integral can be written as

$$\int_0^{1/2} x^{\alpha-1}(1-x)^{\beta-1}\,dx + \int_{1/2}^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx\,,$$

  where the first integral converges by limit comparison test with $\int_0^{1/2} dx/x^{1-\alpha}$ and the second one as well by limit comparison test with $\int_{1/2}^1 dx/(1-x)^{1-\beta}$.