CALCULUS

Bachelor in Computer Science and Engineering

Course 2021–2022

Sequences of real numbers

Problem 2.1. Consider the following sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers and study whether they are bounded, monotone, and convergent.

$$\begin{aligned}
&\text{a)}\quad a_n = \frac{1+(-1)^n}{2}\,.\\
&\text{b)}\quad a_n = \frac{(-1)^{n+1}}{n}\,.\\
&\text{c)}\quad a_n = \frac{n}{n+2}\,.\\
&\text{d)}\quad a_n = \frac{\lfloor n/2 \rfloor}{n}\,.\\
&\text{e)}\quad a_n = \frac{\lfloor nx \rfloor}{n}\,, \quad x\in\mathbb{R}\,.\\
&\text{f)}\quad a_n = \frac{n+\sin(\pi n/2)}{2n+1}\,.\\
&\text{g)}\quad a_n = \sqrt[n]{\pi^n + (\sqrt{7})^n}\,.\\
&\text{h)}\quad a_n = \frac{1}{n^2+1}\sum_{k=1}^{n} k\,.\\
&\text{i)}\quad a_n = \sqrt[n]{x^n+y^n}\,, \quad 0<y\le x\,.
\end{aligned}$$

Problem 2.2. Calculate the limit as $n\to\infty$ of the following sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers.

$$\begin{aligned}
&\text{a)}\quad a_n = \frac{n^2}{(n-7)!} \quad (n\ge 7)\,.\\
&\text{b)}\quad a_n = \frac{n!}{n^n}\,.\\
&\text{c)}\quad a_n = \sqrt{n^3-1}-n\,.\\
&\text{d)}\quad a_n = \frac{\sqrt{n^3-1}-n}{5n^2-7\sqrt{n}}\,.\\
&\text{e)}\quad a_n = \frac{3^n+2^{n+1}}{3^{n+1}+2^n}\,.
\end{aligned}$$

Problem 2.3. Prove that the given *recursive* sequences of real numbers are bounded and monotone. Then, calculate their limit as $n \to \infty$.

$$\text{a)}\quad a_1 = \sqrt{3},\ a_2 = \sqrt{3\sqrt{3}},\ a_3 = \sqrt{3\sqrt{3\sqrt{3}}},\ \ldots .$$

$$\text{b)}\quad a_n = 5 + \frac{a_{n-1}}{4}\quad \forall n \geq 2\,,\quad a_1 = 0\,.$$

$$\text{c)}\quad a_n = \frac{1 + 3a_{n-1}^2}{4}\quad \forall n \geq 2\,,\quad 1/3 \leq a_1 < 1\,.$$

$$\text{d)}\quad a_{n+1} = \sqrt{2a_n + 3}\quad \forall n \geq 1\,,\quad a_1 = 1\,.$$

Problem 2.4. Calculate the following limits.

$$\text{a)}\quad \lim_{n \to \infty} n^{1/(n-1)}\,.$$

$$\text{b)}\quad \lim_{n \to \infty} (7n^3 - 1)^{1/n}\,.$$

$$\text{c)}\quad \lim_{n \to \infty} \left(\frac{3n^2 + 1}{3n^2 + 2}\right)^{-n^2}\,.$$