CALCULUS – Second Partial Exam

Bachelor in Computer Science and Engineering

ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

November 2021 | maximum 3 points

Name		GROUP	88

1. *[1 point]* Let

$$f(x) = \begin{cases} \arctan(x) & \text{if } x \le 0\,, \\ \sin(\pi x) & \text{if } 0 < x < 1\,, \\ \left|x^2 - 5x + \beta\right| & \text{if } x \ge 1\,, \end{cases}$$

with $\beta \in \mathbb{R}$.

(a) Find the value of β for which $f(x)$ is continuous for all $x \in \mathbb{R}$.

(b) Prove whether $f'(x)$ is continuous at $x = 0$.

SOLUTION

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0,\ 1$, the given function is continuous as defined in terms of continuous elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $\lim_{x\to 0} f(x) = f(0) = 0$. Finally, $f(x)$ will be continuous at $x = 1$ if $\lim_{x\to 1} f(x) = f(1) = |\beta - 4|$. Thus, observing that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin(\pi x) = 0\,,$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left|x^2 - 5x + \beta\right| = |\beta - 4|\,,$$

we can conclude that $f(x)$ is continuous at $x = 1$, hence for all $x \in \mathbb{R}$, only if $|\beta - 4| = 0$, namely $\beta = 4$.

(b) We have that

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sin(\pi x)}{x} = \pi\,,$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{\arctan(x)}{x} = 1\,.$$

Since $f'_+(0) \neq f'_-(0)$, we can conclude that the given function is not differentiable at $x = 0$, hence $f'(x)$ cannot be continuous at $x = 0$.

2. *[1 point]* Approximate the value

$$\ln\left(\frac{3}{2}\right)$$

by a Taylor polynomial of suitable degree such that the involved error is smaller than 10^{-2}.

SOLUTION

Note that

$$\ln\left(\frac{3}{2}\right) = \ln\left(1+\frac{1}{2}\right)$$

can be calculated by evaluating the function $f(x) = \ln(1+x)$ at $x = 1/2$. Thanks to the Taylor's theorem, such function can be expressed as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + R_n(x)\,,$$

where the remainder $R_n(x)$ is

$$R_n(x) = (-1)^n \frac{1}{(n+1)(1+c)^{n+1}}\, x^{n+1}\,,$$

with $c \in (0, x)$. Hence, at $x = 1/2$, we can estimate the approximation error as

$$\left|R_n\left(\frac{1}{2}\right)\right| = \frac{1}{2^{n+1}(n+1)(1+c)^{n+1}} < \frac{1}{2^{n+1}\,(n+1)}\,.$$

Finally, after imposing

$$\frac{1}{2^{n+1}\,(n+1)} < 10^{-2},$$

we deduce that the degree of the considered Maclaurin polynomial must be $n = 4$, at least. Thus, a proper approximation is

$$\ln\left(\frac{3}{2}\right) \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64}\,.$$

3. *[1 point]* Calculate

$$\lim_{x\to 0} \frac{-1+\cos(x)\, e^{x-x^3} - x}{2x^3}$$

using appropriate Taylor polynomials for the involved functions.

SOLUTION

In the given limit, we have $x \to 0$, hence we can approximate all involved elementary functions by suitable Maclaurin polynomials. In particular, note that

$$e^{x-x^3} \approx 1 + (x - x^3) + \frac{1}{2}(x - x^3)^2 + \frac{1}{6}(x - x^3)^3 \,.$$

Hence, after retaining terms up to degree 3, we can write

$$\lim_{x\to 0} \frac{-\frac{4}{3}x^3 + o(x^3)}{2x^3} \;=\; -\frac{2}{3} \,.$$