CALCULUS EXTRAORDINARY EXAM

Bachelor in Computer Science and Engineering

June 2022

SURNAME			
NAME		GROUP	

Problem 1. *[2 points]* Study the convergence of the series

$$\sum_{n=1}^{\infty} \sqrt{n}\,(1+|x|)^n\, e^{-xn}$$

in terms of $x \in \mathbb{R}$.

SOLUTION

Let $a_n = \sqrt{n}\,(1+|x|)^n\, e^{-xn}$ and $x > 0$. Then, according to the *ratio test*, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}\,(1+|x|)^{n+1}\, e^{-x(n+1)}}{\sqrt{n}\,(1+|x|)^n\, e^{-xn}}\right| = \frac{\sqrt{n+1}}{\sqrt{n}}\,(1+x)\, e^{-x} \;\longrightarrow\; (1+x)\, e^{-x},$$

as $n \to \infty$. Now, the function $f(x) = (1+x)\,e^{-x}$ is decreasing if $x \geq 0$ and $f(0) = 1$. Hence, $f(x) < 1$ and the series is absolutely convergent for all $x > 0$.

On the other hand, if $x \leq 0$, the series is divergent since $\lim\limits_{n\to\infty} a_n = +\infty$.

Problem 2. *[2 points]* Approximate $\cos(x)$ by means of a Taylor polynomial of degree 4 for all $x \in [-1/6, 1/6]$. Then, find an upper bound for the involved approximation error.

SOLUTION

The function $\cos(x)$ can be expressed by using the *Taylor theorem* as

$$\cos(x) \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \;+\; R_4(x)\,,$$

where the remainder $R_4(x)$ verifies

$$|R_4(x)| \;=\; \left|\, \frac{\cos(c)}{6!}\; x^6 \,\right|\,,$$

with $c \in (0, x)$ or $(x, 0)$. Thus, we can approximate the function by

$$\cos(x) \;\approx\; 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in [-1/6, 1/6]$. Finally, an upper bound for the involved error on the same interval can be found as

$$|R_4(x)| \;=\; \left|\, \frac{\cos(c)}{6!}\; x^6 \,\right| \;\leq\; \frac{1}{6!\,6^6}\,.$$

Problem 3. *[2 points]* Let $f(x) = x - \dfrac{\ln(x^2+1)}{x} + x\,\arctan(x^2)$.

(a) Find the local behavior of $f(x)$ close to $x = 0$.

(b) Calculate $\displaystyle\lim_{x\to 0} \frac{f(x)}{\sin(x^3)}$.

SOLUTION

(a) By using appropriate Maclaurin polynomials for the involved elementary functions, we can write

$$f(x) \;=\; \frac{3}{2}\,x^3 + o(x^3)\,.$$

Hence, $f(x)$ is *locally* increasing close to $x = 0$, which is an inflection point.

(b) Using the result in (a) and taking into account that $\sin(x^3) = x^3 + o(x^3)$, we get

$$\lim_{x\to 0} \frac{f(x)}{\sin(x^3)} \;=\; \lim_{x\to 0} \frac{\frac{3}{2}\,x^3 + o(x^3)}{x^3 + o(x^3)} \;=\; \frac{3}{2}\,.$$

Problem 4. *[1.5 points]* Calculate the indefinite integral

$$\int x^n \ln(x^n)\, dx \,,$$

with $n \neq -1$.

SOLUTION

Integration *by parts* yields

$$\int x^n \ln(x^n)\, dx = \frac{x^{n+1}}{n+1} \ln(x^n) - \frac{n}{n+1} \int x^n\, dx = \frac{x^{n+1}}{n+1} \ln(x^n) - \frac{n}{(n+1)^2}\, x^{n+1} + k \,,$$

with $k \in \mathbb{R}$.

Problem 5. *[2.5 points]* Find the global extrema of the function

$$F(x) = \int_{5-2x}^{1} e^{-t^4}\,dt$$

in the interval $x \in [1,3]$. In addition, prove that the maximum value of $F(x)$ on this interval is larger than $2/3$. Finally, calculate the values of $a, b \in \mathbb{R}$ such that

$$F(ax+b) \;=\; -\int_{1}^{x} e^{-t^4}\,dt\,.$$

SOLUTION

Thanks to the *Fundamental Theorem of Calculus*, we can write

$$F'(x) \;=\; 2\,e^{-(5-2x)^4},$$

which is strictly positive for all $x \in \mathbb{R}$. Thus, the function $F(x)$ is increasing in $\mathbb{R}$, hence in the interval $x \in [1,3]$. As a consequence, its global minimum is located at $x = 1$ and its global maximum is attained at $x = 3$.

On the other hand, we have

$$F(3) \;=\; \int_{-1}^{1} e^{-t^4}\,dt \;=\; 2\int_{0}^{1} e^{-t^4}\,dt \;\geq\; 2\int_{0}^{1} \frac{1}{e}\,dt \;=\; \frac{2}{e} \;>\; \frac{2}{3}\,.$$

Finally, note that

$$F(ax+b) \;=\; \int_{5-2b-2ax}^{1} e^{-t^4}\,dt \;=\; -\int_{1}^{5-2b-2ax} e^{-t^4}\,dt\,.$$

Hence, the given equality holds if $5-2b=0$ and $-2a=1$, which means $a=-1/2$ and $b=5/2$.