CALCULUS — FINAL EXAM

January 2021 — **Bachelor in Informatics Engineering**

SURNAME			
NAME		GROUP	

Problem 1. *[1 point]* Study the convergence of the following series of real numbers

$$\sum_{n=1}^{\infty} \frac{n^{\frac{1}{3}} e^{2n}}{(n!)^2} .$$

SOLUTION

Let $a_n = n^{\frac{1}{3}} e^{2n}/(n!)^2$. Then, according to ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{\frac{1}{3}} e^{2n+2}}{((n+1)!)^2} \frac{(n!)^2}{n^{\frac{1}{3}} e^{2n}}\right| = e^2 \frac{1}{(n+1)^2} \left(1+\frac{1}{n}\right)^{\frac{1}{3}} \longrightarrow 0 < 1, \quad \text{as } n \to \infty .$$

Hence, the given series is absolutely convergent.

Problem 2. *[1 point]* Calculate the following limit

$$\lim_{x\to 0}\frac{1}{x^9}\left(x^3-\int_0^{x^3}\frac{\sin(t)}{t}dt\right).$$

SOLUTION

The given limit can be written as

$$\lim_{x\to 0}\frac{1}{x^9}\left(x^3-\int_0^{x^3}\frac{\sin(t)}{t}dt\right)=\lim_{x\to 0}\frac{x^3-F(x)}{x^9},$$

where $F(x)=\displaystyle\int_0^{x^3}\frac{\sin(t)}{t}dt$. Hence, by applying l'Hôpital's rule, we have

$$\lim_{x\to 0}\frac{x^3-F(x)}{x^9}=\lim_{x\to 0}\frac{3x^2-F'(x)}{9x^8}=\lim_{x\to 0}\frac{3x^2-3x^2\frac{\sin(x^3)}{x^3}}{9x^8}=\lim_{x\to 0}\frac{x^3-\sin(x^3)}{3x^9},$$

where the Fundamental Theorem of Calculus has been used in the last-but-one equality. Finally, using the Taylor polynomial for $\sin(x^3)$ about $a=0$ of degree 9, we get

$$\lim_{x\to 0}\frac{x^3-\sin(x^3)}{3x^9}=\lim_{x\to 0}\frac{x^9/6+o(x^9)}{3x^9}=\frac{1}{18}.$$

Problem 3. *[1.5 points]* Let

$$f(x) = \begin{cases} 1 + A\,e^{x}\sin(x) & \text{if} \quad x < 0\,, \\ B + 3\tan(x) - x^3 - x^4 & \text{if} \quad 0 \le x < \pi/2\,, \end{cases}$$

where A, B are real constants.

(a) Find the values of A, B such that $f(x)$ is continuous and differentiable on its domain.

(b) Take A, B as found in (a). Then, study the local behavior of $f(x)$ close to $x = 0$.

SOLUTION

(a) For $x \in (-\infty, \pi/2)$, with $x \neq 0$, the given function is continuous and differentiable as composition of continuous and differentiable elementary functions. In addition, $f(x)$ is continuous at $x = 0$ if $\lim_{x\to 0} f(x) = f(0) = B$. Since

$$\lim_{x\to 0^-} f(x) \;=\; \lim_{x\to 0^-} \big[\,1 + A\,e^{x}\sin(x)\,\big] = 1\,,$$

$$\lim_{x\to 0^+} f(x) \;=\; \lim_{x\to 0^+} \big[\,B + 3\tan(x) - x^3 - x^4\,\big] = B\,,$$

we need $B = 1$ to ensure the continuity of $f(x)$ at $x = 0$, hence on its domain. On the other hand, taking $B = 1$, $f(x)$ is differentiable at $x = 0$ if the following lateral limits

$$f'_-(0) \;=\; \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0^-} \frac{A\,e^{x}\sin(x)}{x} \;=\; A\,,$$

$$f'_+(0) \;=\; \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0^+} \frac{3\tan(x) - x^3 - x^4}{x} \;=\; 3\,,$$

provide the same finite result. Thus, $A = 3$ ensures the differentiability of $f(x)$ at $x = 0$, hence on its domain.

(b) According to Taylor theory, we have that

$$f(x) \;=\; 1 + 3x + 3x^2 + o(x^2)\,,\ \text{ as } x \to 0^-\,,$$

$$f(x) \;=\; 1 + 3x - x^4 + o(x^4)\,,\ \text{ as } x \to 0^+\,.$$

Hence, we can conclude that $f(x)$ is locally concave up at the left of $x = 0$ and locally concave down at the right of $x = 0$.

Problem 4. *[1.5 points]* Let

$$F(x) = \int_0^x t^9 \ln(1+t)dt\,.$$

(a) Write the Taylor polynomial for $F(x)$ about $a = 0$ of generic degree $n \in \mathbb{N}$.

(b) Prove that $F(x)$ has an inflection point at $x = 0$.

SOLUTION

(a) We can write

$$\begin{aligned}
F(x) &= \int_0^x \left[t^9 \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots + (-1)^{n-1}\,\frac{t^n}{n} + o(t^n) \right) \right] dt \\
&= \int_0^x \left[t^{10} - \frac{t^{11}}{2} + \frac{t^{12}}{3} - \frac{t^{13}}{4} + \ldots + (-1)^{n-1}\,\frac{t^{n+9}}{n} + o(t^{n+9}) \right] dt \\
&= \left[\frac{t^{11}}{11} - \frac{t^{12}}{24} + \ldots + (-1)^{n-1}\,\frac{t^{n+10}}{n(n+10)} + o(t^{n+10}) \right]_{t=0}^{t=x} \\
&= \frac{x^{11}}{11} - \frac{x^{12}}{24} + \ldots + (-1)^{n-1}\,\frac{x^{n+10}}{n(n+10)} + o(x^{n+10})\,,
\end{aligned}$$

with $n \in \mathbb{N}$.

(b) According to the Taylor polynomial calculated in (a), the first non-zero derivative of $F(x)$ at $x = 0$ has order $p = 11$ (odd). Hence, $F(x)$ has an inflection point at $x = 0$.

Problem 5. *[1 point]* Calculate $\displaystyle\int_{e^2}^{e^4} \ln(\sqrt{x})dx.$

SOLUTION

Using the change of variable

$$t = \sqrt{x}\,, \quad x = t^2\,; \quad dx \;=\; 2\,t\,dt\,,$$

the given integral becomes

$$\int_{e^2}^{e^4} \ln(\sqrt{x})dx \;=\; 2\int_{e}^{e^2} t\ln(t)dt\,.$$

Now, using integration by parts, we get

$$2\int_{e}^{e^2} t\ln(t)dt \;=\; \left[\,t^2\ln(t)\,\right]_{t=e}^{t=e^2} \;-\; \int_{e}^{e^2} t\,dt \;=\; \frac{e^2}{2}(3e^2-1)\,.$$