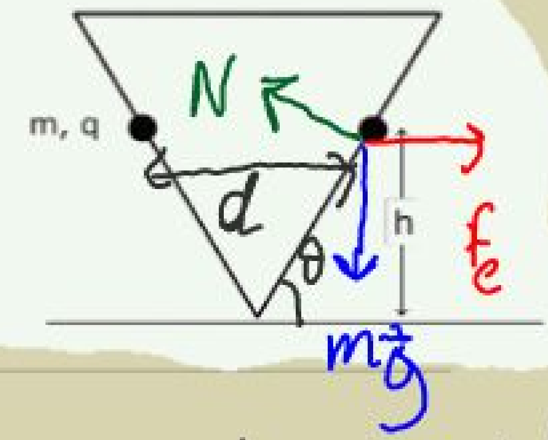
P1 Two identical particles of mass m and charge q are at rest inside an equilateral, triangle-shaped ramp construction. Calculate the height h of each particle with respect to the ground in this equilibrium configuration (use two decimal points and SI units).



There are 3 forces acting on each particle:
The gravitational force $m\vec{g}$, the electric force $\vec{F_e}$ and the normal force N that is perpendicular to the ramp.
Each particle is in equilibrium $\Rightarrow \Sigma \vec{F} = 0$

$\Rightarrow \vec{F_e} + \vec{N} + m\vec{g} = 0$

Along the ramp axis: $\vec{F}_{e_{\parallel}} + m\vec{g}_{\parallel} = 0$

$\Rightarrow F_e \cos\theta - mg\sin\theta = 0 \Rightarrow F_e = mg\tan\theta$

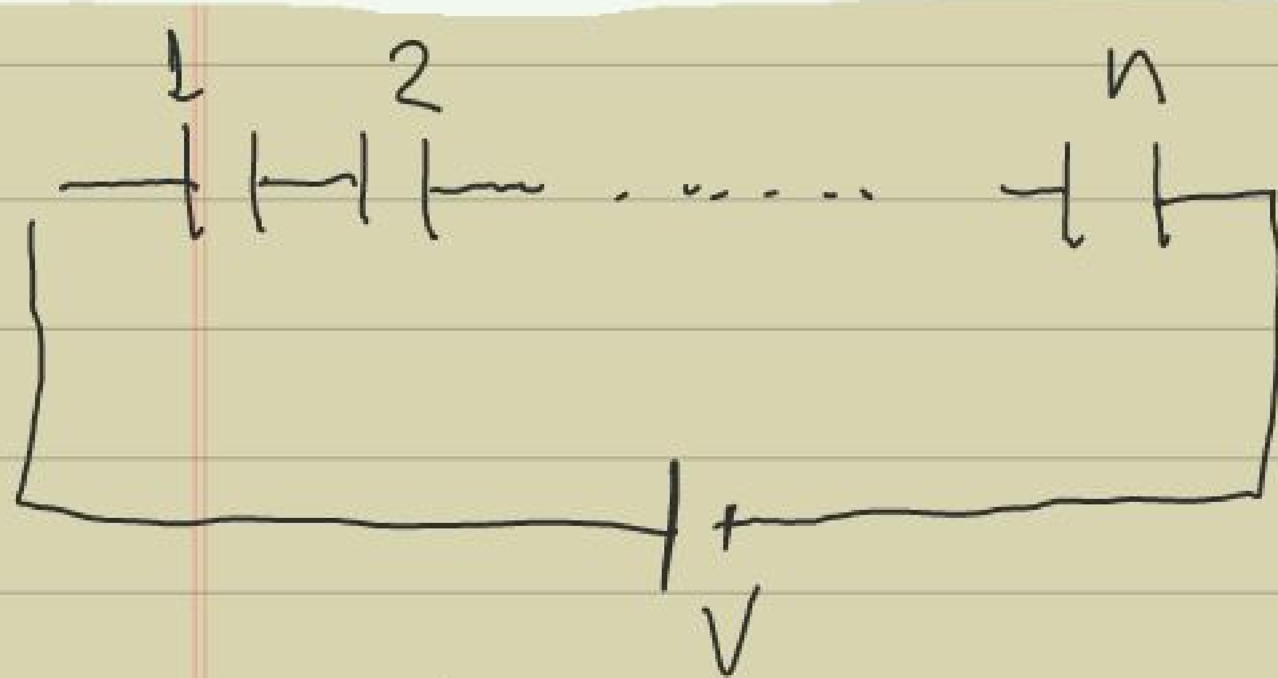
$\Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d^2} = mg\tan\theta \Rightarrow d = \frac{q}{\sqrt{4\pi\varepsilon_0 mg\tan\theta}}$

with $h = d\sin\theta \Rightarrow$

$$h = \frac{q\sin\theta}{\sqrt{4\pi\varepsilon_0 mg\tan\theta}}, \quad \theta = 60^\circ$$

where we made use of the fact that the triangle defined by each charge and the bottom of the ramp is equilateral.

P2 Consider a set of n identical capacitors that are connected in series to a battery with potential a difference V. All the capacitors are filled with a dielectric liquid of dielectric constant ε_r. We disconnect the battery and then we drain completely the liquid from m of these capacitors. Calculate the potential difference across the set of capacitors (use two decimal points and SI units).



The equivalent capacitance before draining m capacitors:

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{C} + \cdots \frac{1}{C} = \frac{n}{C} \Rightarrow C_{eq} = \frac{C}{n} = \frac{Q}{V}$$

where C the capacitance of each capacitor filled with dielectric. The total charge of the system Q remains constant after disconnecting the battery and draining some of the capacitors.

After draining m capacitors:

$$\frac{1}{C'_{eq}} = \underbrace{\frac{1}{C} + \frac{1}{C} + \cdots + \frac{1}{C}}_{n-m \text{ times}} + \underbrace{\frac{1}{C'} + \frac{1}{C'} + \cdots \frac{1}{C'}}_{m \text{ times}}$$

where C' is the capacitance of each individindual capacitor C after removing the dielectric liquid:

$$C' = \frac{C}{\varepsilon_r}$$

Therefore $\dfrac{1}{C'_{eq}} = \dfrac{n-m}{C} + \dfrac{m\varepsilon_r}{C}$

$$\Rightarrow C'_{eq} = \frac{C}{n-m+m\varepsilon_r}$$

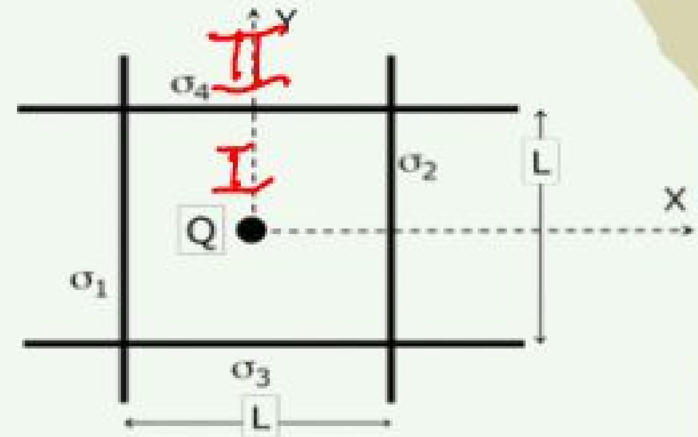
But $\dfrac{C_{eq}}{C'_{eq}} = \dfrac{\cancel{Q}/V}{\cancel{Q}/V'} = \dfrac{V'}{V}$ where V'

the potential difference across the set of capacitors after draining m of the capacitors.

Finally we obtain $V' = V\dfrac{C_{eq}}{C'_{eq}}$

$$\Rightarrow \boxed{V' = \frac{n-m+m\varepsilon_r}{n} \cdot V}$$

P3 Consider four uniformly charged infinite planes, with charge densities as indicated in the figure. Two planes are parallel to the YZ plane and the other two planes are parallel to the XZ plane. A point charge Q is fixed at the origin of the coordinate system (which coincides with the centre of the square determined by the cross sections of the planes with the XY plane).



a) Calculate the general expression of the electric field vector (in Cartesian coordinates) for a generic point along the Y axis with y>0. Divide space into as many regions as necessary.

b) Given the points A(L/7,0,0) and B(L/4,0,0) calculate the potential difference (VA-VB) (use two decimal points and SI units).

a) We have to calculate the electric field for generic points along the positive y-axis. We have to define the area in two regions: region I, with $0<y<\frac{L}{2}$ and region II with $y>\frac{L}{2}$ because the direction of the electric field created by plane 4 changes once we cross the plane.

To calculate the net electric field at some point P along the y-axis we need to add the electric fields created by planes $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ as well as the electric field created by the point charge Q

In region I:

$$\vec{E}(P_I) = \vec{E}_1(P_I) + \vec{E}_2(P_I) + \vec{E}_3(P_I) + \vec{E}_4(P_I) + \vec{E}_Q(P_I)$$

$$= \frac{\sigma_1}{2\varepsilon_0}\vec{i} - \frac{\sigma_2}{2\varepsilon_0}\vec{i} + \frac{\sigma_3}{2\varepsilon_0}\vec{j} - \frac{\sigma_4}{2\varepsilon_0}\vec{j} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2}\vec{j}$$

$$\Rightarrow \boxed{\vec{E}(P_I) = \left(\frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\right)\vec{i} + \left(\frac{\sigma_3 - \sigma_4}{2\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 y^2}\right)\vec{j}}$$

In region II:

$$\vec{E}(P_{II}) = \vec{E}_1(P_{II}) + \vec{E}_2(P_{II}) + \vec{E}_3(P_{II}) + \vec{E}_4(P_{II}) + \vec{E}_Q(P_{II})$$

$$= \frac{\sigma_1}{2\varepsilon_0}\vec{i} - \frac{\sigma_2}{2\varepsilon_0}\vec{i} + \frac{\sigma_3}{2\varepsilon_0}\vec{j} + \frac{\sigma_4}{2\varepsilon_0}\vec{j} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2}\vec{j}$$

$$\Rightarrow \boxed{\vec{E}(P_{II}) = \left(\frac{\sigma_1 - \sigma_2}{2\varepsilon_0}\right)\vec{i} + \left(\frac{\sigma_3 + \sigma_4}{2\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 y^2}\right)\vec{j}}$$

b) Points A and B are located along the x-axis in region I

The potential difference

$$V_A - V_B = -\int_B^A \vec{E}_I(x)\,d\vec{x} \quad (1)$$

with $\vec{E}_I(x)$ the electric field at points $a < x < \frac{L}{2}$ located on the x-axis

$$\vec{E}_t(x) = \vec{E}_1(x) + \vec{E}_2(x) + \vec{E}_3(x) + \vec{E}_4(x) + \vec{E}_Q(x)$$

$$= \frac{\sigma_1}{2\varepsilon_0}\vec{i} - \frac{\sigma_2}{2\varepsilon_0}\vec{i} + \frac{\sigma_3}{2\varepsilon_0}\vec{j} - \frac{\sigma_4}{2\varepsilon_0}\vec{j} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{x^2}\vec{i}$$

$$= \left(\frac{\sigma_1 - \sigma_2}{2\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 x^2}\right)\vec{i} + \left(\frac{\sigma_3 - \sigma_4}{2\varepsilon_0}\right)\vec{j}$$

Substituting in (1) we obtain

$$V_A - V_B = -\int_B^A \left[\left(\frac{\sigma_1 - \sigma_2}{2\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 x^2}\right)\vec{i} + \left(\frac{\sigma_3 - \sigma_4}{2\varepsilon_0}\right)\vec{j}\right] d\vec{x}$$

$$= -\int_B^A \left(\frac{\delta_1 - \delta_2}{2\varepsilon_0} + \frac{Q}{4\pi\varepsilon_0 x^2}\right) dx$$

where we made use of the fact that

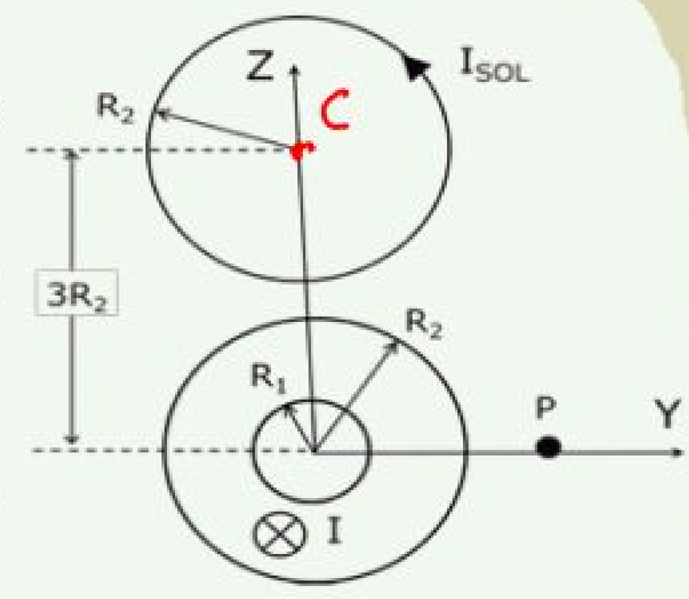
$\vec{i} \cdot d\vec{x} = dx$ and $\vec{j} \cdot d\vec{x} = 0$

Therefore $V_A - V_B = -\left(\frac{\delta_1 - \delta_2}{2\varepsilon_0}\left[x\right]_B^A + \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{x}\right]_B^A\right)$

$$= \frac{\delta_1 - \delta_2}{2\varepsilon_0}\left(\frac{L}{4} - \frac{L}{7}\right) + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\frac{L}{7}} - \frac{1}{\frac{L}{4}}\right)$$

$$\Rightarrow \boxed{V_A - V_B = \frac{3L(\delta_1 - \delta_2)}{56\varepsilon_0} + \frac{3Q}{4\pi\varepsilon_0 L}}$$

P4 An infinite hollow cylinder, with internal radius R_1 and external radius R_2, has its axis coincident with the X axis and carries a uniformly distributed current I along the direction indicated in the figure. Additionally, an ideal solenoid, with radius R_2 carries current I_{SOL} which circulates in the direction indicated in the figure. The number of wire turns per unit length is n and the axis of the solenoid is parallel to the X axis and passes through the point $(0,0,3R_2)$. Calculate the coordinates of a point P, located on the Y axis ($y>R_2$), at which the magnitude of the magnetic field is one-third the magnitude of the magnetic field at the centre of the solenoid (use two decimal points and SI units).



The magnetic field created by the solenoid:

inside $\vec{B}_{sol} = \mu_0 n I_{sol} \vec{i}$

outside $B_{sol} = 0$

The net magnetic field at the centre of the solenoid $\vec{B}_C = \vec{B}_{sol} + \vec{B}_I(C)$ (i)

where $\vec{B}_I(C)$ the magnetic field at C, created by the current I.

To calculate the magnetic field created by I at points outside the pipe (C and P) we have to use

Ampere's law. We consider a circle
A concentric to the pipe
and with radius r.
From Ampere's law:

$$\oint_A \vec{B}\, d\vec{l} = \mu_0 I$$ where I

is the current that passes through
the circular path

Because $\vec{B} \parallel d\vec{l}$ we get

$$\oint_A \vec{B}\, d\vec{l} = \oint_A B\, dl = B \oint_A dl = B \cdot 2\pi r$$

B is constant
along A

Therefore the magnitude of the
magnetic field $B = \frac{\mu_0 I}{2\pi r}$ for points
located outside the pipe

From (i) $\rightsquigarrow$ $\vec{B}_c = \mu_0 n I_{sol} \vec{i} + \frac{\mu_0 I}{2\pi 3 R_2} \vec{j}$

where we made use of the fact that at C the direction of $\vec{B}_I$ is tangenctial.

The magnitude of the magnetic field at the centre of the solenoid

$$|\vec{B}_c| = \sqrt{(\mu_0 n I_{sol})^2 + \left(\frac{\mu_0 I}{6\pi R_2}\right)^2} \quad \text{(ii)}$$

The magnetic field at point P is only due to I

with magnitude $|\vec{B}_p| = \frac{\mu_0 I}{2\pi y_p}$ (iii)

We know that $|\vec{B}_p| = \frac{1}{3} |\vec{B}_c|$

(ii), (iii) $\rightsquigarrow$

$$\boxed{y_p = \frac{3 I}{2\pi \sqrt{n^2 I_{sol}^2 + \frac{I^2}{36\pi^2 R_2^2}}}}$$