# CALCULUS – Final Exam

**Bachelor in Computer Science and Engineering**

January 2022

| SURNAME | | | |
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| NAME | | GROUP | |

**Problem 1.** *[1 point]* Prove that 5 is an upper bound of the recursive sequence

$$a_1 = 0; \quad a_{n+1} = 4 + \frac{1}{5}a_n\,, \text{ for } n \geq 1\,.$$

Then, for $n \geq 1$, check that

$$a_n = 5 - \frac{1}{5^{n-2}}.$$

**SOLUTION**

Let us prove by the method of induction that the given sequence is bounded above by $5$. First, we have $a_1 = 0 \leq 5$. Then, supposing that $a_k \leq 5$ for a generic $k \in \mathbb{N}$ ($k > 1$), we get

$$\frac{1}{5}a_k \leq 1 \implies 4 + \frac{1}{5}a_k \leq 5 \implies a_{k+1} \leq 5\,.$$

Hence, we can conclude that 5 is an upper bound for $(a_n)_{n\in\mathbb{N}}$.

Now, from $a_n = 5 - \frac{1}{5^{n-2}}$, we get

$$a_1 = 5 - \frac{1}{5^{-1}} = 5 - 5 = 0$$

for $n = 1$. In addition, we have

$$a_{n+1} = 5 - \frac{1}{5^{n-1}}$$

and

$$4 + \frac{1}{5}a_n = 5 - \frac{1}{5^{n-1}}$$

for $n \geq 1$. Thus, we can conclude that the indicated explicit expression for $a_n$ is the solution of the recursive sequence for $n \geq 1$.

**Problem 2.** *[1 point]* Determine the number of real solutions of the equation

$$e^{-x} - e^{x} - \ln(x) = 0\,, \quad \text{for} \;\; x \in (0, +\infty)\,.$$

**SOLUTION**

Let us define $f(x) = e^{-x} - e^{x} - \ln(x)$, which is continuous and differentiable for $x > 0$. Thus, we have

$$f'(x) = -e^{-x} - e^{x} - \frac{1}{x} < 0$$

for $x \in (0, +\infty)$, which means that $f(x)$ is decreasing on the definition interval. Finally, since

$$\lim_{x \to 0^+} f(x) = +\infty, \qquad \lim_{x \to +\infty} f(x) = -\infty,$$

we can guarantee that the equation $f(x) = 0$ has a unique real solution for $x \in (0, +\infty)$.

**Problem 3.** *[1 point]* Let

$$F(x) = \int_0^x e^{-t^2} dt \, .$$

Use a Taylor polynomial of degree 3 to estimate the value $F(1/10)$ and find an upper bound of the involved approximation error.

**SOLUTION**

After writing $e^{-t^2} = 1 - t^2 + o(t^2)$, we get

$$F(x) = \int_0^x e^{-t^2} dt \; = \int_0^x [\, 1 - t^2 + o(t^2) \,] dt \; = \; x - \frac{x^3}{3} + o(x^3) \, ,$$

for values of $x$ 'close' to zero. Hence, the Taylor polynomial of degree 3 about $a = 0$ for $F(x)$ is given by $P_3(x) = x - x^3/3$, which can be used to estimate the desired value as

$$F(1/10) \approx P_3(1/10) = \frac{1}{10} - \frac{1}{3000} = \frac{299}{3000} \, .$$

On the other hand, noting that $F^{(4)}(x) \; = 4x(3 - 2x^2)e^{-x^2}$, the remainder associated with $P_3(x)$ is given by

$$R_3(x) = \frac{4c(3 - 2c^2)e^{-c^2}}{4!} \, x^4 \, ,$$

with $c \in (0, x)$. Thus, an upper bound of the involved approximation error at $x = 1/10$ can be found as

$$|R_3(1/10)| = \left| \frac{c(3 - 2c^2)e^{-c^2}}{6} \, \frac{1}{10^4} \right| \; \leq \; \frac{1}{6 \cdot 10^4} (3c + 2c^3) e^{-c^2} \; \leq \; \frac{1}{6 \cdot 10^4} \left( \frac{3}{10} + \frac{2}{10^3} \right) ,$$

where the last inequality holds as $e^{-c^2} < 1$, being $c \in (0, 1/10)$.

**Problem 4.** *[1.5 points]* Find all differentiable functions $F : (0, +\infty) \to \mathbb{R}$ that satisfy

$$F'(x) = \ln^2(x)\,, \quad F(1) = 0\,.$$

**SOLUTION**

Successively integrating by parts two times, we get

$$F(x) = \int \ln^2(x)dx \; = \; x \ln^2(x) - 2\int \ln(x)dx \; = \; x \ln^2(x) - 2x \ln(x) + 2\int dx$$

$$= x \ln^2(x) - 2x \ln(x) + 2x + k\,,$$

with $k \in \mathbb{R}$. Finally, after imposing $F(1) = 0$, which yields $k = -2$, we obtain

$$F(x) = x \ln^2(x) - 2x \ln(x) + 2x - 2\,.$$

**Problem 5.** *[1.5 points]* Find all values of $a, b \in \mathbb{R}$ that make the function

$$f(x) = \begin{cases} a + \displaystyle\int_0^{2x} \frac{\sin(t)}{t} dt\,, & \text{if } x < 0\,, \\ \sqrt{2} + b\cos(2x)\ln(1+3x)\,, & \text{if } x \geq 0\,, \end{cases}$$

continuous and differentiable.

**SOLUTION**

For $x < 0$, $f(x)$ is continuous and differentiable thanks to the Fundamental Theorem of Calculus (all assumptions are readily seen to be satisfied). Also, for $x > 0$, $f(x)$ is continuous and differentiable as given in terms of continuous and differentiable elementary functions.

Now, continuity of $f(x)$ at $x = 0$ holds if $\lim_{x\to 0} f(x) = f(0) = \sqrt{2}$. Since

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[ a + \int_0^{2x} \frac{\sin(t)}{t} dt \right] = a\,,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left[ \sqrt{2} + b\cos(2x)\ln(1+3x) \right] = \sqrt{2}\,,$$

we need $a = \sqrt{2}$ to ensure the continuity of $f(x)$ at $x = 0$, hence on its domain.

On the other hand, taking $a = \sqrt{2}$, $f(x)$ is differentiable at $x = 0$ if the following lateral limits

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{\sin(2x)}{x} = 2\,,$$

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{b\cos(2x)\ln(1+3x)}{x} = 3b\,,$$

provide the same finite result. Note that in the first limit, the l'Hôpital's rule has been applied, together with the Fundamental Theorem of Calculus. Thus, $b = 2/3$ ensures the differentiability of $f(x)$ at $x = 0$, hence on its domain.