ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

CALCULUS – Second Partial Exam

Bachelor in Computer Science and Engineering

November 2021 | maximum 3 points

Name		GROUP	89

1. *[1 point]* Approximate the value

$$e^{-1/2}$$

using a Taylor polynomial of suitable degree such that the involved error is smaller than 10^{-2}.

SOLUTION

The value $e^{-1/2}$ can be calculated by evaluating the function $f(x) = e^x$ at $x = -1/2$. Such function can be expressed by means of the Taylor's theorem as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + R_n(x)\,,$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{e^c}{(n+1)!}\,x^{n+1}\,,$$

with $c \in (x, 0)$. Hence, at $x = -1/2$, we can estimate the involved error as

$$|R_n(-1/2)| = \frac{e^c}{2^{n+1}\,(n+1)!} < \frac{1}{2^{n+1}\,(n+1)!}\,.$$

Finally, imposing

$$\frac{1}{2^{n+1}\,(n+1)!} < 10^{-2} \quad \Longleftrightarrow \quad 2^{n+1}\,(n+1)! > 100\,,$$

we can deduce that the considered Maclaurin polynomial must have degree $n = 3$, at least. Thus, a proper approximation is

$$e^{-1/2} \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48}\,.$$

2. *[1 point]* Let

$$f(x) = \begin{cases} \dfrac{2\sin(x) - \arctan(x)}{x} & \text{if } \; x < 0\,, \\ \alpha^2 x e^x + 1 & \text{if } \; x \geq 0\,, \end{cases}$$

with $\alpha \in \mathbb{R}$.

(a) Study the continuity of f in the domain.

(b) Find the values of α for which the angle formed by the tangent line from the right and the tangent line from the left, at $x = 0$, to the graph of f is equal to $\pi/4$.

SOLUTION

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0$, the given function is continuous as defined in terms of continuous elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $\lim_{x\to 0} f(x) = f(0) = 1$, independently of the value of α.

(b) The slope of the right-tangent line to $f(x)$ at $x = 0$ is given by

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \alpha^2 e^x = \alpha^2\,.$$

On the other hand, the slope of the left-tangent line to $f(x)$ at $x = 0$ is given by

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{2\,\sin(x) - \arctan(x) - x}{x^2} = 0\,.$$

Hence, the desired angle is equal to $\pi/4$ if $\alpha = \pm 1$.

3. *[1 point]* Calculate

$$\lim_{x\to 0} \frac{e^{x-x^3}\cos(x) - 1 - x}{3x^3}$$

using appropriate Taylor polynomials for the involved functions.

SOLUTION

In the given limit, we have $x \to 0$, hence we can approximate all involved elementary functions by suitable Maclaurin polynomials. In particular, note that

$$e^{x-x^3} \approx 1 + (x - x^3) + \frac{1}{2}(x - x^3)^2 + \frac{1}{6}(x - x^3)^3 .$$

Hence, after retaining terms up to degree 3, we can write

$$\lim_{x\to 0} \frac{-\frac{4}{3}x^3 + o(x^3)}{3x^3} = -\frac{4}{9} .$$