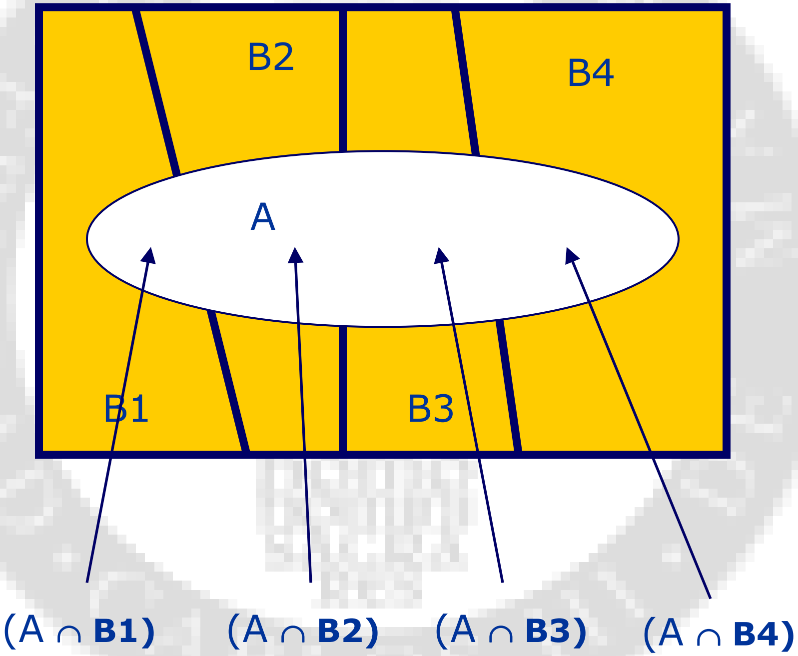
III. PROBABILITY



$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{J} P(A|B_j)P(B_j)}$$

Chapter III: Probability

1. Introduction

Probability: a measure of uncertainty of an event, will it happen or not?

EVENT: Result in an experiment

Example A: rolling a 2 on a dice.

B: getting tail after tossing a coin

C: computer C1 takes more than 10 seconds to complete task T

D: material M1 supports weight P

Before doing the experiment:

Will we observe this event?

Probability = a measure of uncertainty of this event

Important concepts

EXPERIMENT: Any process to obtain information, given the **experimental conditions**

If we get new information, keeping constant the experimental conditions, we are **REPEATING** the experiment

Example: Testing if a sample of material M_1 supports weight P

Example: Timing how long the computer C_1 takes to complete the task T

Example: Measuring the length of a piece P produced by the machine M

Example: Calculating the result of adding 10 and 4

Types of experiments

Deterministic experiment: If we repeat the experiment, we **will always obtain the same result** each time.

¿Why?

Because the experimental conditions contain ALL influential factors.

Example: calculating the result of adding 10 and 4

Random experiment: If we repeat the experiment, we **will NOT always obtain the same result = UNCERTAINTY**

¿Why?

Because the experimental conditions DO NOT contain ALL influential factors.

Example: Testing if a sample of material M1 supports weight P

Example: Timing how long the computer C1 takes to complete the task T

Example: Measuring the length of a piece P produced by the machine M

Important concepts

ELEMENTARY EVENT: Each elementary result of random experiment

Example: If we roll a dice, the elementary events are 1,2,3,4,5,6

If we measure the time a computer takes to complete the same task, the elementary events are infinite.

COMPOUND EVENT: The union of some elementary events

Example: If we roll a dice, a possible compound event is to get an even number A:{2,4,6}

COMPLEMENTARY EVENT:

The complementary event of an event A, $\bar{A}$, is the event that occurs when A does not occur

Example: If we roll a dice, the event A is to get an even number A:{2,4,6}

Then, the complementary event is to get an odd number

$\bar{A}$:{1,3,5}

Important concepts

SAMPLE SPACE: It is the set of every possible elementary event of an experiment. We will use the symbol E to denote it. It is also called the **Certain Event**

Example If we roll a dice, $E=\{1,2,3,4,5,6\}$

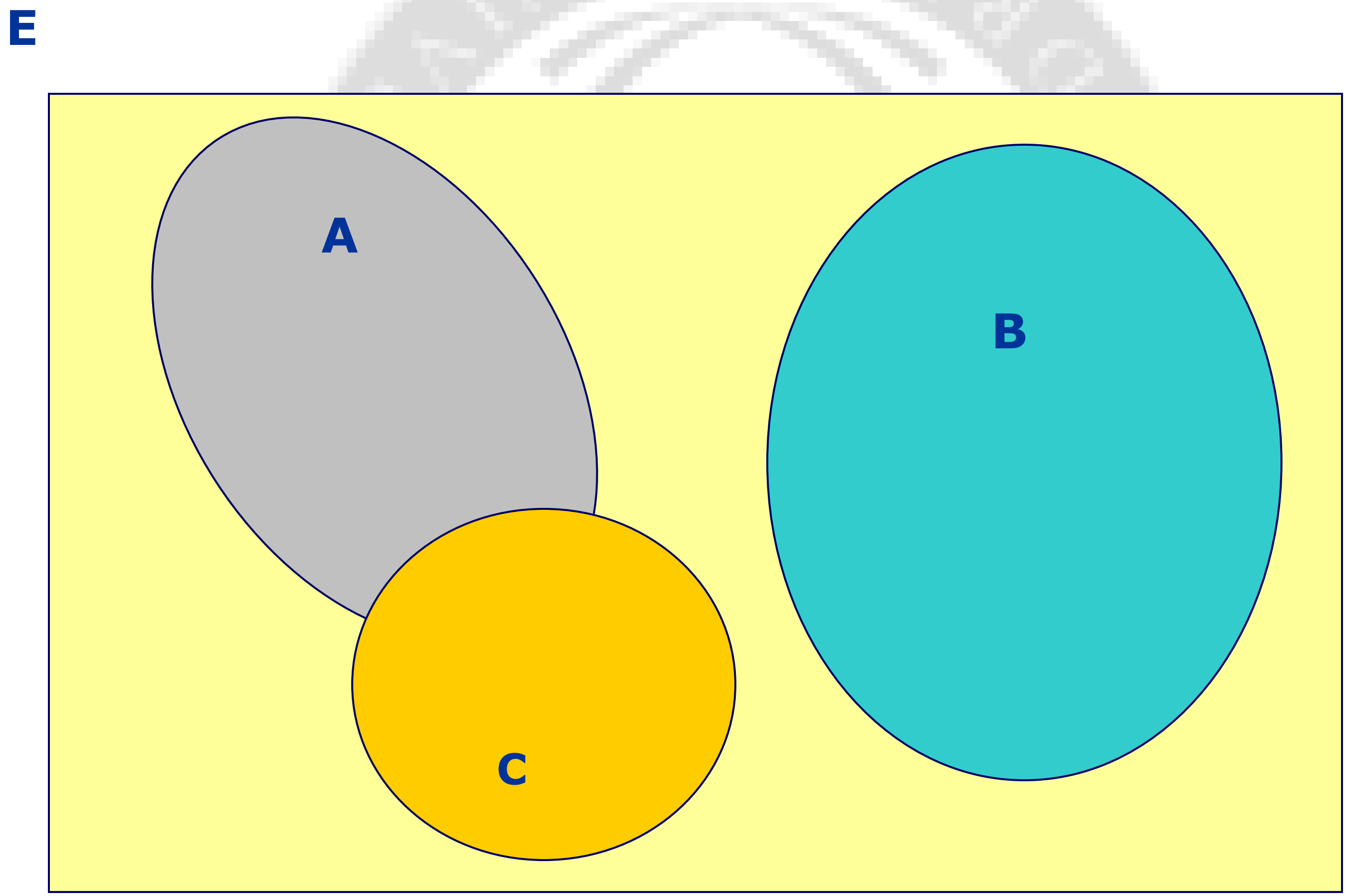
If we measure the time a computer takes to complete a task $E=\{t\geq 0\}$

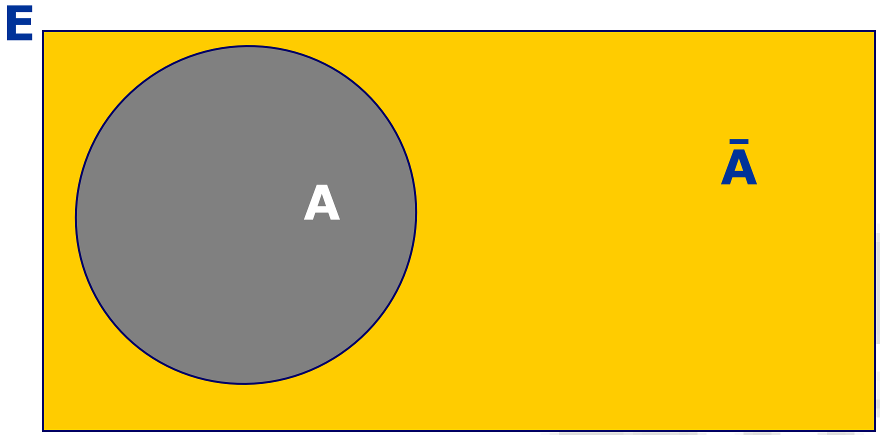
IMPOSSIBLE EVENT: It is the event that never occurs, $\varnothing$

Example If we measure the time a computer takes to complete a task $\varnothing =\{t<0\}$

Venn diagram

Graphical representation of events

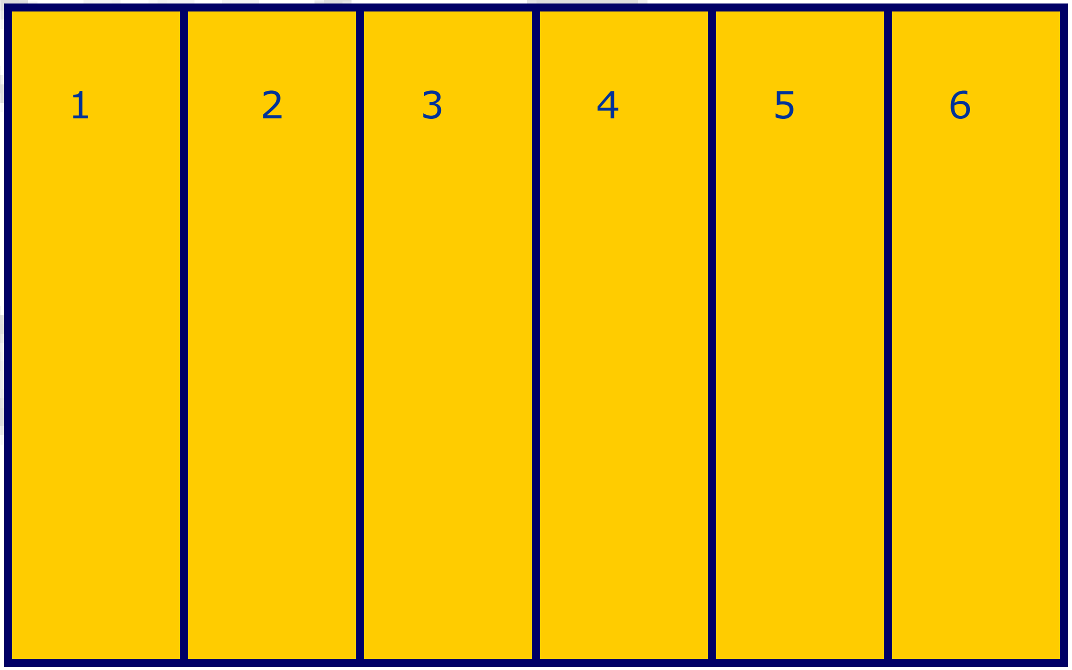




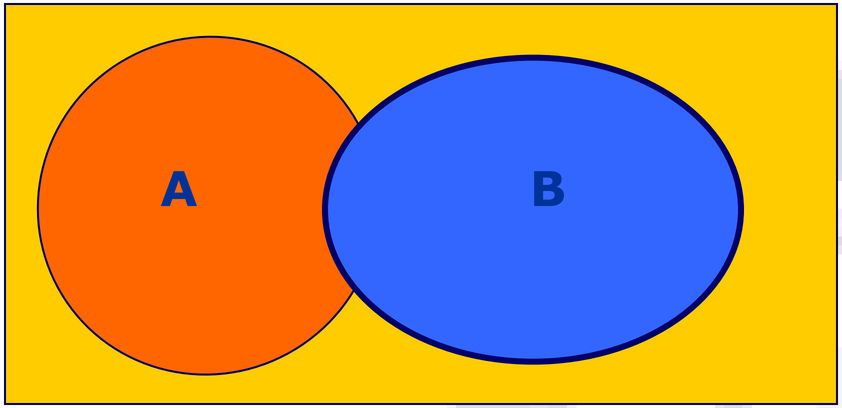
$$A \cup \bar{A} = E$$

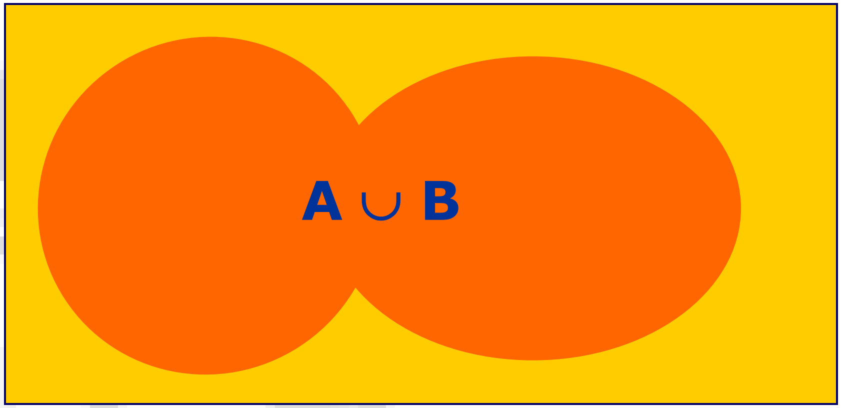
Example: elementary events if we roll a dice

1 U **2** U **3** U **4** U **5** U **6**=**E**



Union of events: it is an event that contains all results that are contained in either of the events which we are joining.





$$\mathbf{A \cup B} \text{ or } \mathbf{A + B}$$

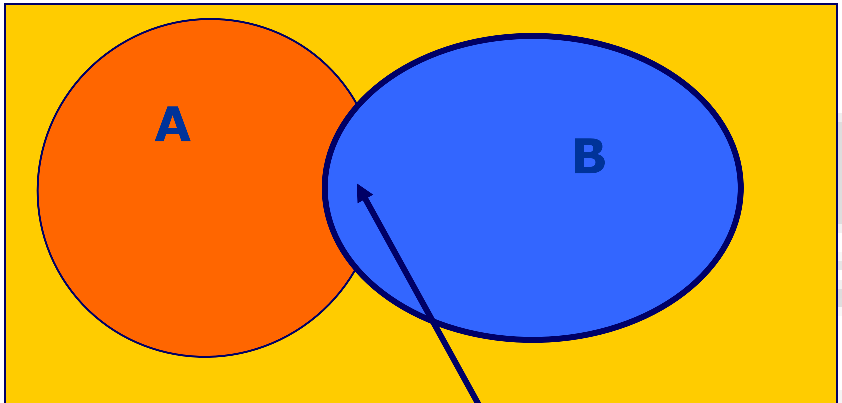
Example:

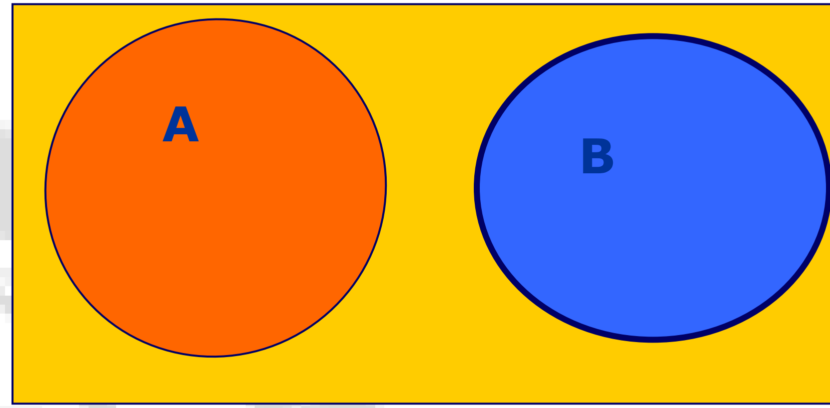
A: rolling 2 or 3 on a dice; A={2,3}

B: rolling an even number; B={2,4,6}

A+B={2,3,4,6}

Intersection of events: it is an event that occurs when all the intersecting events occur simultaneously





$A \cap B$

or **AB**

$A \cap B = \emptyset$

(mutually exclusive events)

Example:

A: rolling 2 or 3 on a dice; A={2,3}

B: rolling an even number; B={2,4,6}

AB={2}

Chapter III: Probability

1. Introduction
2. Probability: definition and properties
3. Conditional and total probability
4. Independence of events
5. Bayes' Theorem

2. Probability: definition and properties

A : event which we are interested in.

Example, 1: getting head in a coin tossing
Example, 2: It takes more than 10 seconds to complete a task
Example, 3: tomorrow it is a rainy day
Example, 4: pass the subject

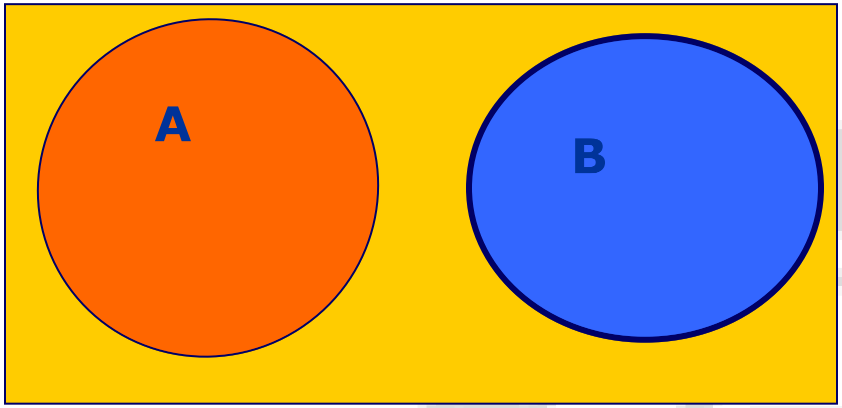
We do the experiment and ... Do we observe the event A?

P(A) ⟶ Measure of uncertainty of an event A

Probability of observe the event A in the next repetition of the random experiment

Probability of an event A: It is the relative frequency of occurrence of the event A if we indefinitely repeat the experiment.

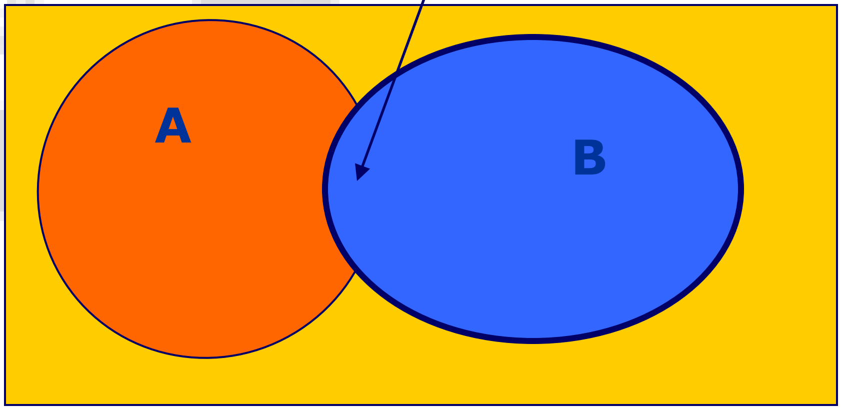
- $0 < P(A) < 1$
- $P(E)=1$
- $P(\emptyset)=0$
- $P(\bar{A})=1-P(A)$



For mutually exclusive events

P(A+B)=P(A)+P(B)

If we do not substract P(AB), we are adding the intersection of these events two times.



For NO mutually exclusive events

P(A+B)=P(A)+P(B)-P(AB)

How to calculate probabilities?

(it can be complex)

- Using the preceding rules
- If we have N equipossible elementary events (example: dice or coin)

P(elementary event)=1/N

- If A is the union of k among N equipossible elementary events

P(A)=k/N

- Using the rules that we will learn in the next sections
- Empirically: repeating the experiment a lot of times (next chapters)

Example

A machine has produced 50 pieces of Type I and 200 of Type II. Each piece can be defective or acceptable. The bivariate distribution is:

	Type I	Type II	TOTAL
Acceptable	46	184	230
Defective	4	16	20
TOTAL	50	200	250

If we choose a piece at random, what is the probability that the piece chosen is defective? → P(Defective)=20/250=0.08

If we choose a piece at random, what is the probability that the piece chosen is of Type II? → P(Tipo II)=200/250=0.80

Example

A machine has produced 50 pieces of Type I and 200 of Type II. Each piece can be defective or acceptable. The bivariate distribution is:

	Type I	Type II	TOTAL
Acceptable	46	184	230
Defective	4	16	20
TOTAL	50	200	250

A buyer wants an *acceptable* piece of *Type II*. We choose a piece at random

What is the probability that the piece chosen is a **not good** for the buyer?

Solution 1:

$$
\begin{aligned}
P(\text{not good}) &= 1 - P(\text{good}) \\
&= 1 - P(\text{Acceptable} \cap \text{Type II}) \\
&= 1 - \frac{184}{250} = 0.264 = 26.4\%
\end{aligned}
$$

Example

A machine has produced 50 pieces of Type I and 200 of Type II. Each piece can be defective or acceptable. The bivariate distribution is:

	Type I	Type II	TOTAL
Acceptable	46	184	230
Defective	4	16	20
TOTAL	50	200	250

A buyer wants an *acceptable* piece of *Type II*. We choose a piece at random

What is the probability that the piece chosen is a **not good** for the buyer?

Solution 2:

$$\begin{aligned} P(\text{not good}) &= P(\text{Defective} \cup \text{Type I}) \\ &= P(\text{Defective}) + P(\text{Type I}) - P(\text{Defective} \cap \text{Type I}) \\ &= \frac{20}{250} + \frac{50}{250} - \frac{4}{250} = 0.264 = 26.4\% \end{aligned}$$

Chapter III: Probability

3. Conditional and total probability

The conditional probability of an event depends on a previously known information

Event A: rolling a 2 on a dice → P(A)=1/6

If we know that the outcome is an even number → P(A knowing that the outcome is an even number)

=1/3>1/6

Conditional probability

Notation:

We need to define the event that we **have not yet** observed yet and the event that we **have already** observed

- Event that we do not know if it will occur: A – for example, Event A: rolling a 2
- Known event: B – for example, Event B: the got number is even

Probability of A assuming that B has occurred

Probability of A given B

$$P(A|B)$$

Example

There are 300 people in a room classified in the following way:

	Women	Men	TOTAL
Smoker	15	15	30
Nonsmoker	105	165	270
TOTAL	120	180	300

If we choose a person at random, what is the probability that the person is a smoker? → P(Smoker)=30/300=0.10

If we choose a person at random and we observe he is a man, what is the probability that he is smoker? → P(Smoker|Man)=15/180=0.083

We know that he is a man

$$P(\text{Smoker} \mid \text{Man}) = \frac{15}{180} = 8.3\%$$

Example

There are 300 people in a room classified in the following way:

	Women	Men	TOTAL
Smoker	15	15	30
Nonsmoker	105	165	270
TOTAL	120	180	300

$$P(\text{Smoker}|\text{Man}) = \frac{15/300}{180/300} = \frac{\#\text{ of men who smoke}/\#\text{ of people}}{\#\text{ of men }/\#\text{ of people}}$$

$$= \frac{P(\text{Smoker} \cap \text{Man})}{P(\text{Man})}$$

$$\boxed{P(\text{Smoker}|\text{Man}) = \frac{P(\text{Smoker} \cap \text{Man})}{P(\text{Man})}}$$

Conditional Probability

$$\boxed{P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})}}$$

Using this rule we can obtain

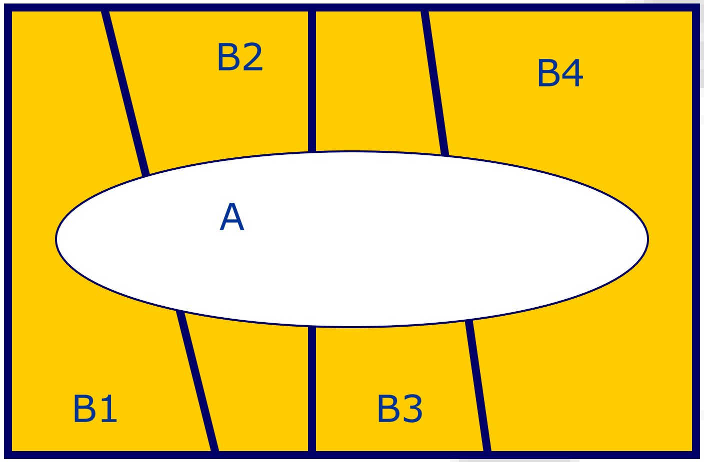
$$\boxed{P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{A}|\mathrm{B})P(\mathrm{B})}$$

Analogously:

$$\boxed{P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{A})}} \longrightarrow \boxed{P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{B}|\mathrm{A})P(\mathrm{A})}$$

Total probability theorem

Let $B_1, B_2, \ldots, B_n$ events of an experiment whose union is the certain event E



$$\bigcup_{i=1}^{N} B_i = E$$

Let A be an event that we can observe at the same time as the events B_i

Example:

B_1: to be male

B_2: to be female

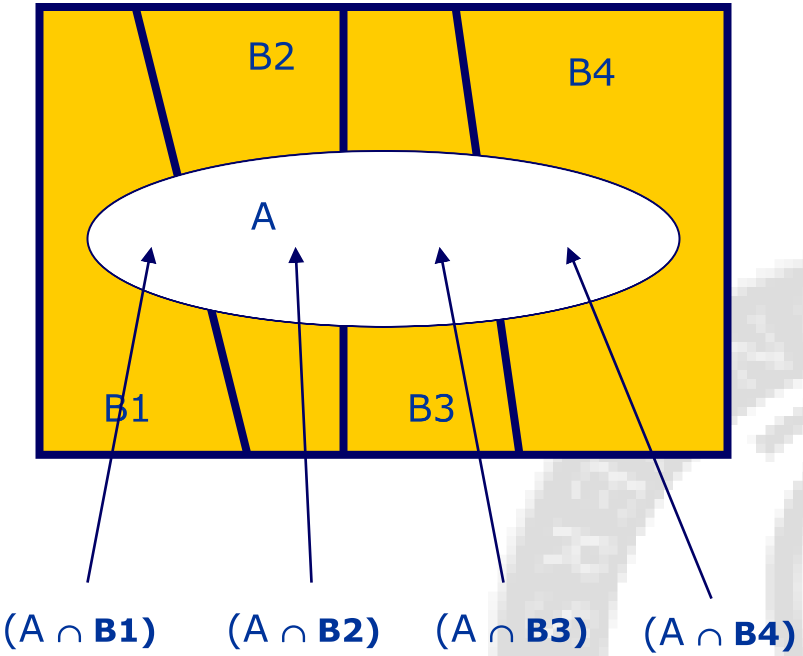
A: to be smoker

$$B_1 + B_2 = E$$

Problem

We want to know P(A), but we only know $P(A|B_1)$ and $P(A|B_2)$

How can we compute **the total probability**?



$$P(\mathrm{A}) = P(\mathrm{A} \cap \mathrm{E})$$

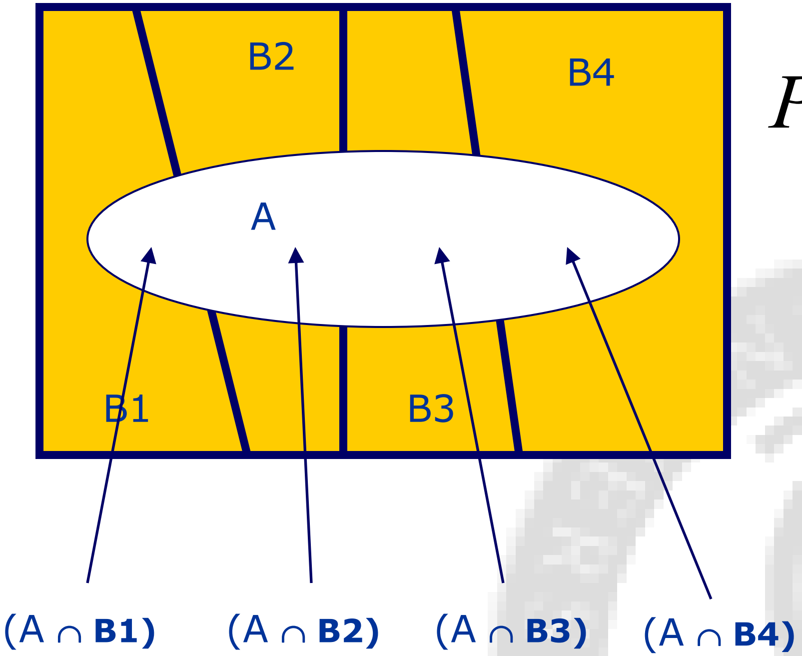
since $\bigcup_{i=1}^{N} \mathrm{B}_i = \mathrm{E}$

$$P(\mathrm{A}) = P\left(\mathrm{A} \cap \bigcup_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{B}_{\mathrm{i}}\right)$$

$$P(\mathrm{A}) = P((\mathrm{A} \cap \mathrm{B}_1) \cup (\mathrm{A} \cap \mathrm{B}_2) \cup \cdots \cup (\mathrm{A} \cap \mathrm{B}_{\mathrm{N}}))$$

mutually exclusive

$$P(\mathrm{A}) = P(\mathrm{A} \cap \mathrm{B}_3) + P(\mathrm{A} \cap \mathrm{B}_2) + \cdots + P(\mathrm{A} \cap \mathrm{B}_{\mathrm{N}})$$



$$P(\mathrm{A}) = P(\mathrm{A} \cap \mathrm{B}_3) + P(\mathrm{A} \cap \mathrm{B}_2) + \ldots$$
$$\cdots + P(\mathrm{A} \cap \mathrm{B}_N)$$

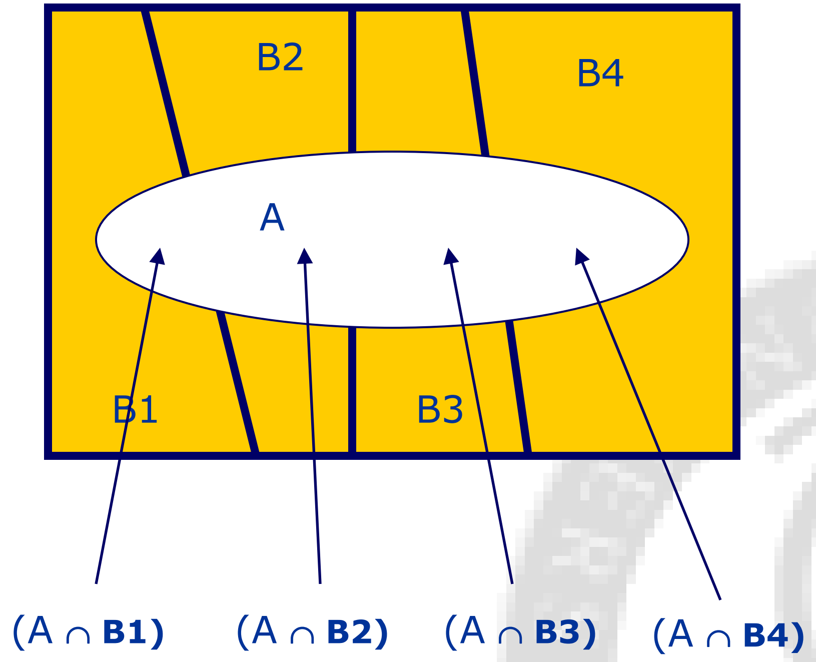
using

$$P(\mathrm{A} \cap \mathrm{B}_i) = P(\mathrm{A} | \mathrm{B}_i) P(\mathrm{B}_i)$$

$$P(\mathrm{A}) = P(\mathrm{A} | \mathrm{B}_1) P(\mathrm{B}_1) + P(\mathrm{A} | \mathrm{B}_2) P(\mathrm{B}_2) + \cdots + P(\mathrm{A} | \mathrm{B}_N) P(\mathrm{B}_N)$$

Total probability theorem

$$P(\mathrm{A}) = \sum_{i=1}^{N} P(\mathrm{A} | \mathrm{B}_i) P(\mathrm{B}_i)$$



Total probability theorem

$$P(\mathrm{A}) = \sum_{i=1}^{N} P(\mathrm{A}|\mathrm{B_i})P(\mathrm{B_i})$$

Example: B1: to be male B2: to be female

A: to be smoker

We want to know P(A: smoker), but we only know P(A|man) y P(A|woman)

P(smoker)=P(smoker|woman)P(woman)+P(smoker|man)P(man)

Example

There are 300 people in a room classified in the following way:

	Women	Men	TOTAL
Smoker	15	15	30
Nonsmoker	105	165	270
TOTAL	120	180	300

P(smoker|woman)=15/120=0.125

P(woman)=120/300=0.40

P(smoker|man)=15/180=0.0833

P(man)=180/300=0.60

P(smoker)=P(smoker|woman)P(woman)+P(smoker|man)P(man)

$$= 0.125 \times 0.40 + 0.0833 \times 0.60 = 0.10$$

Chapter III: Probability

1. Introduction
2. Probability: definition and properties
3. Conditional and total probability
4. **Independence of events**
5. Bayes' Theorem

4. Independence of events

What does it mean that two events A and B are **independent**?

The information that we have about one of the events **is not useful** to improve our knowledge about the other event.

The probability of an event **does not change** if we observe the other event

$P(A|B)=P(A)$; $P(B|A)=P(B)$

Example:

A: rolling a 2 on a dice

B: getting tail on a coin tossing

$P(A)=1/6$; $P(A|B)=1/6=P(A)$
Independent

B: rolling an even number

$P(A)=1/6$; $P(A|B)=1/3\neq P(A)$
Dependent

B: rolling an odd number

$P(A)=1/6$; $P(A|B)=0\neq P(A)$
Dependent

4. Independence of events

Two events A and B are **independent** if the probability of one of them does not change after observe or not the other event

P(A|B)=P(A) ; P(B|A)=P(B)

Using conditional probability events

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A}\cap\mathrm{B})}{P(\mathrm{B})} = P(\mathrm{A})$$

$$P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{A}\cap\mathrm{B})}{P(\mathrm{A})} = P(\mathrm{B})$$

Independence:

$$\boxed{P(\mathrm{A}\cap\mathrm{B}) = P(\mathrm{A})P(\mathrm{B})}$$

Dependence:

$$P(\mathrm{A}\cap\mathrm{B}) = P(\mathrm{A}|\mathrm{B})P(\mathrm{B})$$

or

$$P(\mathrm{A}\cap\mathrm{B}) = P(\mathrm{B}|\mathrm{A})P(\mathrm{A})$$

Example

Some cylindrical pieces can be defective if they have an inappropriate length or an inappropriate diameter, being both types of defect independent. If the percentage of cylinders with inappropriate length is **5%** and the percentage of cylinders with inappropriate diameter is **3%**. What percentage of cylinders are defective?

Event A: inappropriate Length P(A)=0.05

Event B: inappropriate Diameter P(B)=0.03

$$P(\text{Defective}) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\overset{\text{ind}}{=} P(A) + P(B) - P(A)P(B)$$

$$= 0.05 + 0.03 - 0.05 \times 0.03 = 7.85\%$$

Chapter III: Probability

1. Introduction
2. Probability: definition and properties
3. Conditional and total probability
4. Independence of events
5. Bayes' Theorem

Problem: We know P(B|A), how can we calculate P(A|B)?

P(A|B)?

Using the conditional probability rule

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{B})}$$

Moreover we know

$$P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{A})}$$

$$P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{B}|\mathrm{A})P(\mathrm{A})$$

Bayes' theorem

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{B}|\mathrm{A})P(\mathrm{A})}{P(\mathrm{B})}$$

Example A company that administers the net of another company purchases a new antivirus software with the following features. If there is a virus, the software raises the alarm with probability 0.95. Even if there is not a virus, the software may raise a false alarm with probability of 0.08.

If the net usually receives a virus attack to each 1000 access, calculate the probability that when the software raises the alarm that alarm is indeed true.

A: the software raises the alarm

V: the message contains a virus

$$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \quad \begin{array}{l} P(A \mid V) = 0.95 \\ P(A \mid \bar{V}) = 0.08 \\ P(V) = 0.001 \end{array}$$

$$P(V \mid A) = \frac{P(A \mid V) P(V)}{P(A)} \longrightarrow P(V \mid A) = \frac{0.95 \times 0.001}{P(A)}$$

Total probability theorem

$$\boxed{P(\mathrm{A}) = \sum_{i=1}^{N} P(\mathrm{A} \mid \mathrm{B}_{\mathrm{i}}) P(\mathrm{B}_{\mathrm{i}})}$$

Example A company that administers the net of another company purchases a new antivirus software with the following features. If there is a virus, the software raises the alarm with probability 0.95. Even if there is not a virus, the software may raise a false alarm with probability of 0.08.

If the net usually receives a virus attack to each 1000 access, calculate the probability that when the software raises the alarm that alarm is indeed true.

A: the software raises the alarm

V: the message contains a virus

$$\left.\begin{aligned} P(A \mid V) &= 0.95 \\ P(A \mid \overline{V}) &= 0.08 \\ P(V) &= 0.001 \end{aligned}\right.$$

$$P(V \mid A) = \frac{P(A \mid V)P(V)}{P(A)} \longrightarrow P(V \mid A) = \frac{0.95 \times 0.001}{P(A)}$$

$$P(A) = P(A \mid V)P(V) + P(A \mid \overline{V})P(\overline{V})$$
$$= 0.95 \times 0.001 + 0.08 \times 0.999 = 0.08087$$

$$P(V \mid A) = \frac{0.95 \times 0.001}{0.08087} = 0.012$$

Bayes' Theorem

$$\boxed{P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{B}|\mathrm{A})P(\mathrm{A})}{P(\mathrm{B})}}$$

We can write it in the following equivalent way as well

$$P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{B}|\mathrm{A})}{P(\mathrm{B})} P(\mathrm{A})$$

Probability AFTER observing B

'posterior' Probability

Probability BEFORE observing B

'prior' Probability