1.1 Integration of trigonometric functions

Integrals of the form

$$\int \sin^m(x)\cos^n(x)\,dx\,,$$

with m, $n \in \mathbb{N} \cup \{0\}$, can be solved taking into account the following.

- Let m and/or n be odd. Apply the change of variable $u = \cos(x)$ and write $\sin^{2k+1}(x) = \sin(x)[\,1-\cos^2(x)\,]^k$ (if $m = 2k+1$, for some $k = 0, 1, \ldots$), or apply the change of variable $u = \sin(x)$ and write $\cos^{2k+1}(x) = \cos(x)[\,1-\sin^2(x)\,]^k$ (if $n = 2k+1$, for some $k = 0, 1, \ldots$).

  Example. Calculate $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$. We have that

  $$\int \sin^2(x)\cos^3(x)\,dx = \int \sin^2(x)[1-\sin^2(x)]\cos(x)\,dx$$
  $$= \int u^2(1-u^2)\,du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + c\,,$$

  with $c \in \mathbb{R}$, where the change of variable $u = \sin(x)$ has been applied.

- Let m and n be even. Consider the identities $\sin^2(x) = [\,1-\cos(2x)\,]/2$ and $\cos^2(x) = [\,1+\cos(2x)\,]/2$.

  Example. Calculate $\displaystyle\int \cos^4(x)\,dx$. We have that

  $$\int \cos^4(x)\,dx = \frac{1}{4}\int [1+\cos(2x)]^2\,dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos(2x)\,dx$$
  $$+ \frac{1}{8}\int [1+\cos(4x)]\,dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c\,,$$

  with $c \in \mathbb{R}$.

In order to integrate

$$\int R(\sin(x), \cos(x))\,dx\,,$$

where R is a rational function of sine and cosine, there exists a universal change of variable, $u = \tan(x/2)$, which transforms the given integral into the integral of a rational function. In this case, we have

$$\sin(x) = \frac{2u}{1+u^2}\,, \quad \cos(x) = \frac{1-u^2}{1+u^2}\,, \quad dx = \frac{2}{1+u^2}\,du\,.$$

Such change of variable typically yields an integral that is difficult to solve. Thus, it is recommended to consider the following, more specific changes of variables.

(1) Let R be odd with respect to $\sin(x)$, namely $R(-\sin(x), \cos(x)) = -R(\sin(x), \cos(x))$. Apply the change of variable $u = \cos(x)$.

(2) Let R be odd with respect to $\cos(x)$, namely $R(\sin(x), -\cos(x)) = -R(\sin(x), \cos(x))$. Apply the change of variable $u = \sin(x)$.

(3) Let R be even with respect to $\sin(x)$ and $\cos(x)$, namely $R(-\sin(x), -\cos(x)) = R(\sin(x), \cos(x))$. Apply the change of variable $u = \tan(x)$ and note that

$$\cos^2(x) = \frac{1}{1+u^2}\,, \quad dx = \frac{1}{1+u^2}\,du\,.$$

Example. Calculate $\displaystyle\int \frac{\sin(x)}{1-\cos^2(x)}\,dx$.

Observe that $R(\sin(x), \cos(x)) = \sin(x)/[1-\cos^2(x)]$ is odd with respect to $\sin(x)$, since

$$R(-\sin(x), \cos(x)) \;=\; -\frac{\sin(x)}{1-\cos^2(x)} \;=\; -R(\sin(x), \cos(x))\,.$$

Hence, according to case (1), let us apply the change of variable $u = \cos(x)$. Substituting into the integral and taking into account that $du = -\sin(x)\,dx$, we get

$$\begin{aligned}\int \frac{\sin(x)}{1-\cos^2(x)}\,dx \;&=\; \int \frac{du}{u^2-1} = \frac{1}{2}\int \frac{1}{u-1}\,du - \frac{1}{2}\int \frac{1}{u+1}\,du \\ &=\; \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c \;=\; \frac{1}{2}\ln\left|\frac{\cos(x)-1}{\cos(x)+1}\right| + c\,,\end{aligned}$$

with $c \in \mathbb{R}$.

Example. Calculate $\displaystyle\int \frac{dx}{\sin(x)\cos^3(x)}$.

Observe that $R(\sin(x), \cos(x)) = 1/[\sin(x)\cos^3(x)]$ is even with respect to $\sin(x)$ and $\cos(x)$, since

$$R(-\sin(x), -\cos(x)) \;=\; \frac{1}{\sin(x)\cos^3(x)} \;=\; R(\sin(x), \cos(x))\,.$$

Hence, according to case (3), let us apply the change of variable $u = \tan(x)$. Substituting into the integral, we get

$$\begin{aligned}\int \frac{dx}{\sin(x)\cos^3(x)} &= \int \frac{(1+u^2)^2}{u(1+u^2)}\,du \\ &= \int \frac{1}{u}\,du + \int u\,du \;=\; \ln|\tan(x)| + \frac{1}{2}\tan^2(x) + c\,,\end{aligned}$$

with $c \in \mathbb{R}$, where the identity $\sin(x)\cos^3(x) = \tan(x)\cos^4(x)$ has been used.

1.2 Integration of irrational functions

Integrating irrational functions is generally difficult and proper changes of variable are known in specific cases only. Let us analyze three types of such integrals.

- $\int R(x, \sqrt[n]{ax+b})\,dx$, where R is a rational function with respect to x and the indicated root ($n \geq 2$ is an integer, a and b are real constants). These integrals are transformed into integrals of rational functions by means of the change of variable $u = \sqrt[n]{ax+b}$.

  Example. Calculate $\displaystyle\int x\,\sqrt[3]{1+x}\;dx$.

  Let us apply the change of variable $u = \sqrt[3]{1+x}$. Thus, we have that $x = u^3 - 1$ and $dx = 3u^2\,du$. Hence

  $$\int x\sqrt[3]{1+x}\,dx = 3\int (u^6 - u^3)\,du = \frac{3}{7}u^7 - \frac{3}{4}u^4 + c = \frac{3}{7}(1+x)^{7/3} - \frac{3}{4}(1+x)^{4/3} + c\,,$$

  with $c \in \mathbb{R}$.

- $\int R(x, \sqrt{a^2 - x^2})\,dx$, where R is a rational function with respect to x and the indicated root (a is a real, positive constant). These integrals are transformed into integrals of trigonometric functions by means of the change of variable $x = a\sin(u)$.

  Example. Calculate $\displaystyle\int \sqrt{9 - x^2}\,dx$.

  Apply the change of variable $x = 3\sin(u)$. Hence, $dx = 3\cos(u)\,du$ and we get

  $$\int \sqrt{9-x^2}\,dx = 9\int \cos^2(u)\,du = \frac{9}{2}\int [1 + \cos(2u)]\,du = \frac{9}{2}u + \frac{9}{4}\sin(2u) + c\,,$$

with $c \in \mathbb{R}$, where the identity $\cos^2(u) = [\,1 + \cos(2u)\,]/2$ has been used. Restoring the original variable, $u = \arcsin(x/3)$, we finally have

$$\int \sqrt{9 - x^2}\, dx = \frac{9}{2} \arcsin(x/3) + \frac{x}{2}\sqrt{9 - x^2} + c\,,$$

after using $\sin(2u) = 2\sin(u)\cos(u) = 2\sin(u)\sqrt{1 - \sin^2(u)}$.

- $\int R(x, \sqrt{x^2 + a})\, dx$, where R is a rational function with respect to x and the indicated root (a is a real, positive constant). These integrals can be solved either by a trigonometric change of variable or by setting $u = x + \sqrt{x^2 + a}$, which transforms them into integrals of rational functions, since $(u - x)^2 = u^2 - 2ux + x^2 = x^2 + a$ implies that $x = (u - a/u)/2$ and $dx = (1 + a/u^2)/2\, du$.

Example. Calculate $\displaystyle\int \frac{dx}{x\sqrt{1 + x^2}}$.

Let us apply the change of variable $u = x + \sqrt{1 + x^2}$. Hence

$$x = \frac{u^2 - 1}{2u}\,, \quad dx = \frac{u^2 + 1}{2u^2}\, du\,.$$

Finally, we get

$$\int \frac{dx}{x\sqrt{1 + x^2}} = 2\int \frac{du}{u^2 - 1} = \ln\left|\frac{u - 1}{u + 1}\right| + c = \ln\left|\frac{x + \sqrt{1 + x^2} - 1}{x + \sqrt{1 + x^2} + 1}\right| + c\,,$$

with $c \in \mathbb{R}$.