1.1.1 Limits calculation for sequences

Here, we present some useful theorems to calculate the limit of a sequence as $n \to \infty$.

Theorem 3 *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two convergent sequences such that*

$$\lim_{n\to\infty} a_n = a \in \mathbb{R}, \quad \lim_{n\to\infty} b_n = b \in \mathbb{R}.$$

Then, the following properties hold.

- $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = a \pm b\,.$
- $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right) = a\,b\,.$
- *If $b \neq 0$:* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{a}{b}\,.$
- *If $a \neq 0$ or $b \neq 0$:* $\lim\limits_{n\to\infty} (a_n)^{b_n} = \left(\lim\limits_{n\to\infty} a_n\right)^{\left(\lim\limits_{n\to\infty} b_n\right)} = a^b\,.$
- *If $a_n > 0$ for all $n \in \mathbb{N}$ and $a > 0$:* $\lim\limits_{n\to\infty} \ln(a_n) = \ln\left(\lim\limits_{n\to\infty} a_n\right) = \ln(a)\,.$

In the case of divergent sequences with limit equal to $\pm\infty$ (or in cases different from those considered in Theorem 3), the limits calculation is performed according to the next result.

Theorem 4 *The following properties hold.*

- *If $(a_n)_{n\in\mathbb{N}}$ is bounded and $\lim\limits_{n\to\infty} b_n = \pm\infty$:* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0\,;$ $\lim\limits_{n\to\infty} (a_n + b_n) = \pm\infty$ *and* $\lim\limits_{n\to\infty} (a_n - b_n) = \mp\infty.$
- *If $(a_n)_{n\in\mathbb{N}}$ is bounded and $\lim\limits_{n\to\infty} b_n = 0$:* $\lim\limits_{n\to\infty} (a_n\, b_n) = 0.$
- *If $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$ and $\lim\limits_{n\to\infty} b_n = \pm\infty$:* $\lim\limits_{n\to\infty} (a_n\, b_n) = \pm\infty$ *if $a > 0$ and* $\lim\limits_{n\to\infty} (a_n\, b_n) = \mp\infty$ *if $a < 0$.*
- *If $\lim\limits_{n\to\infty} a_n = \pm\infty$ and $\lim\limits_{n\to\infty} b_n = \pm\infty$:* $\lim\limits_{n\to\infty} (a_n + b_n) = \pm\infty\,.$
- *If $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} b_n = \pm\infty$:* $\lim\limits_{n\to\infty} (a_n\, b_n) = \pm\infty\,.$

- *If* $\lim_{n\to\infty} a_n = -\infty$ *and* $\lim_{n\to\infty} b_n = \pm\infty$ *:* $\lim_{n\to\infty} (a_n\, b_n) = \mp\infty$.

- *If* $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ *and* $\lim_{n\to\infty} b_n = 0$*, with* $b_n > 0$ *for all* $n \in \mathbb{N}$ *:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = +\infty$ *if* $a > 0$ *and* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = -\infty$ *if* $a < 0$.

- *If* $\lim_{n\to\infty} a_n = 0$*, with* $a_n > 0$ *for all* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} b_n = \pm\infty$ *:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ *and* $\lim_{n\to\infty} \dfrac{b_n}{a_n} = \pm\infty$.

- *If* $\lim_{n\to\infty} a_n = +\infty$ *and* $\lim_{n\to\infty} b_n = b$ *:* $\lim_{n\to\infty} a_n^{b_n} = +\infty$ *if* $b > 0$ *and* $\lim_{n\to\infty} a_n^{b_n} = 0$ *if* $b < 0$ *;* $\lim_{n\to\infty} b_n^{a_n} = +\infty$ *if* $b > 1$ *and* $\lim_{n\to\infty} b_n^{a_n} = 0$ *if* $0 < b < 1$.

However, we cannot directly assign a value to those limits leading to any of the following *indeterminate forms*:

$$+\infty - \infty\,, \quad 0\,(\pm\infty)\,, \quad \frac{0}{0}\,, \quad \frac{\pm\infty}{\pm\infty}\,, \quad 0^0\,, \quad (\pm\infty)^0\,, \quad 1^{(\pm\infty)}\,.$$

Note that the last one appears whenever we have a limit like

$$\lim_{n\to\infty} a_n^{b_n}$$

with $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} b_n = \pm\infty$. In all these cases, one should rearrange the involved expressions or use known theorems in order to get some conclusion.