CALCULUS

Bachelor in Computer Science and Engineering

Course 2022–2023

Integrals: Fundamental Theorem of Calculus

Problem 9.1. Find the area enclosed between the graph of $f(x) = \sin(x)$, the x-axis, and the lines $x = \pi/2$ and $x = \pi$.

Problem 9.2. Find the area enclosed between the graph of $f(x) = x/(x+1)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Problem 9.3. Consider the function

$$f(x) = \begin{cases} \cos(x) & \text{for } \; 0 \le x \le \pi/2\,, \\ -1 & \text{for } \; \pi/2 < x \le \pi\,. \end{cases}$$

Then, calculate

$$F(x) = \int_0^x f(t)\,dt$$

with $x \in [0, \pi]$ and compare $F'(x)$ with $f(x)$ for $x \in (0, \pi)$, where $F'(x)$ exists.

Problem 9.4. Calculate the equation of the tangent line at $x = 1$ to the graph of

$$F(x) = \int_{-1}^x \frac{t^3}{t^4 - 4}\,dt\,.$$

Problem 9.5. Find the values of $x \in \mathbb{R}$ for which the function

$$F(x) = \int_1^x \arctan(e^t)\,dt$$

is one-to-one.

Problem 9.6. Calculate the following limits.

$$\text{(a)} \;\; \lim_{x\to 0} \frac{1}{x} \int_0^x \frac{|\cos(t^3)|}{t^2+1}\,dt \qquad\qquad \text{(b)} \;\; \lim_{x\to +\infty} \frac{1}{x} \int_0^x \frac{|\cos(t^3)|}{t^2+1}\,dt$$

Problem 9.7. Calculate the first and second derivatives of the function

$$H(x) = x \int_{2x}^{3x} e^{-t^2}\, dt\,.$$

Problem 9.8. Prove that the function

$$H(x) = \int_{1-x}^{1+x} \ln(t)\, dt$$

is decreasing for $x \in [0, 1/2]$.

Problem 9.9. Find the global extrema of the function

$$H(x) = \int_{5-2x}^{1} e^{-t^4}\, dt$$

in the interval $[1, 3]$. In addition, prove that the maximum value of $H(x)$ is larger than $2/3$.

Problem 9.10. Calculate the following limits.

$$(a)\ \lim_{x\to 0^+} \frac{1}{x^{3/2}} \int_0^{x^2} \sin(t^{1/4})\, dt \qquad (b)\ \lim_{x\to 0} \frac{\int_0^x e^{t^2}\, dt - x}{x^3}$$

Problem 9.11.

- Prove that
$$F(x) = \int_0^x \Big(1 + \sin\big(\sin(t)\big)\Big)\, dt$$
is one-to-one and show that $F(0) = 0$. Then, calculate $(F^{-1})'(0)$.

- Consider the function
$$G(x) = \int_1^x \sin\big(\sin(t)\big)\, dt\,.$$
Prove that G is even, namely $G(x) = G(-x)$. Then, use this result to justify that G^{-1} does not exist.

Problem 9.12. Write the Taylor polynomial of degree 3 about $a = 0$ for

$$F(x) = \int_0^x t^2 \cos(t^2)\, dt$$

and use it to calculate

$$\lim_{x \to 0} \frac{F(x)}{x^3}.$$

Problem 9.13. Find the first derivative of the following functions.

$$\text{(a)} \quad H(x) = \int_3^{\left(\int_1^x \sin^3(t)\, dt\right)} \frac{dt}{1 + t^2 + \sin^6(t)}.$$

$$\text{(b)} \quad K(x) = \sin\left(\int_0^x \sin\left(\int_0^t \sin^3(s)\, ds\right) dt\right).$$