CALCULUS – Second Partial Exam

Bachelor in Computer Science and Engineering

ESCUELA POLITÉCNICA SUPERIOR
UNIVERSIDAD CARLOS III DE MADRID

November 2022 | maximum 3 points

Name		GROUP	88

1. *[1 point]* Approximate the value
$$\left(\frac{8}{7}\right)^{1/5}$$
by a Taylor polynomial of degree $n = 2$. Then, find an upper bound for the involved error.

SOLUTION

Note that
$$\left(\frac{8}{7}\right)^{1/5} = \left(1+\frac{1}{7}\right)^{1/5}$$
can be obtained by evaluating the function $f(x) = (1+x)^{1/5}$ at $x = 1/7$. Thanks to the Taylor's theorem, such function can be expressed as
$$(1+x)^r = 1 + r\,x + \frac{1}{2}r(r-1)x^2 + R_2(x)\,,$$
with $r = 1/5$. The remainder $R_2(x)$ is given by
$$R_2(x) = \frac{f'''(c)}{3!}\,x^3\,,$$
where $f'''(c) = \frac{36}{125}\,(1+c)^{-14/5}$ and $c \in (0, 1/7)$. Thus, we can approximate the desired value as
$$\left(\frac{8}{7}\right)^{1/5} \approx 1 + \frac{1}{35} - \frac{2}{25}\frac{1}{49} \approx 1.02694\,.$$
Finally, an upper bound for the involved approximation error at $x = 1/7$ can be obtained as
$$\left|R_2\left(\frac{1}{7}\right)\right| = \frac{36}{125}\,\frac{1}{(1+c)^{14/5}}\,\frac{(1/7)^3}{3!} < \frac{36}{125}\,\frac{(1/7)^3}{3!} \approx 1.4\cdot 10^{-4}\,,$$
where the inequality holds since $1/(1+c)^{14/5} < 1$.

2. *[1 point]* Let

$$f(x) = \begin{cases} \arctan(\ln(x^2)) & \text{if} \quad x \neq 0\,, \\ -\pi/2 & \text{if} \quad x = 0\,. \end{cases}$$

(a) Study the continuity of $f(x)$ in the domain.

(b) Calculate $f'(x)$ for $x \neq 0$.

(c) Prove that the inverse function of $f(x)$ exists in the interval $(0, +\infty)$.

SOLUTION

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0$, the given function is continuous as composition of continuous elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $f(0) = -\pi/2$ and we have

$$\lim_{x \to 0} f(x) = \arctan(-\infty) = -\pi/2\,.$$

(b) For $x \neq 0$, the derivative of the given function is

$$f'(x) = \frac{2x/x^2}{1 + (\ln(x^2))^2} = \frac{2}{x + 4x\ln^2(x)}\,.$$

(c) Observe that $f'(x) > 0$ for all $x \in (0, +\infty)$, which means that the function is strictly increasing in that interval. Thus, $f(x)$ is one-to-one in $(0, +\infty)$, which implies that its inverse function exists in the same interval.

3. *[1 point]* Calculate

$$\lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x^3}$$

using appropriate Taylor polynomials for the involved functions.

SOLUTION

In the given limit, we can approximate all involved elementary functions by Maclaurin polynomials of suitable degree. Indeed, we can write

$$\lim_{x\to 0} \frac{x + \frac{x^3}{3} + o(x^3) - \left[x - \frac{x^3}{3!} + o(x^3)\right]}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{2} + o(x^3)}{x^3} = \lim_{x\to 0} \left(\frac{1}{2} + \frac{o(x^3)}{x^3}\right) = \frac{1}{2}\,.$$