§ 10.2 平面简谐波的波函数

一、波函数

x 处质元 t 时刻的位移:

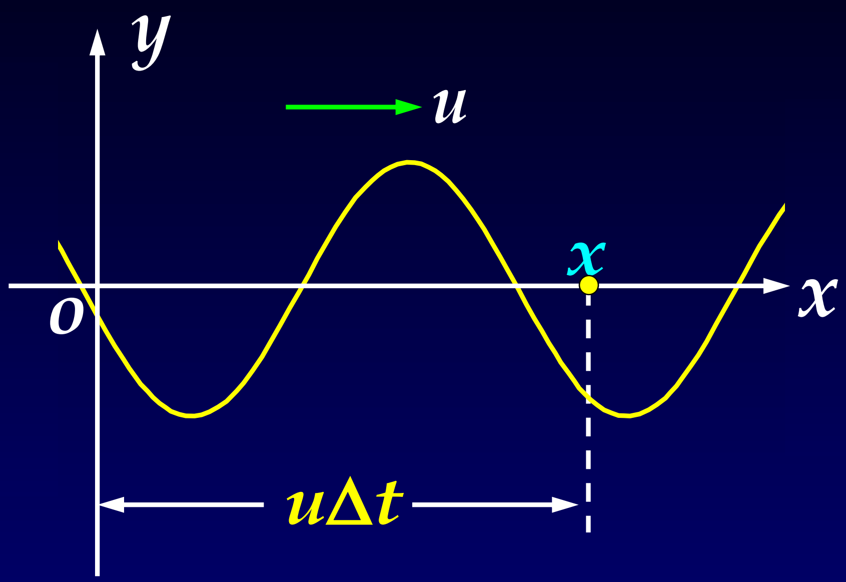
$$y = y(x,\ t) = ?$$

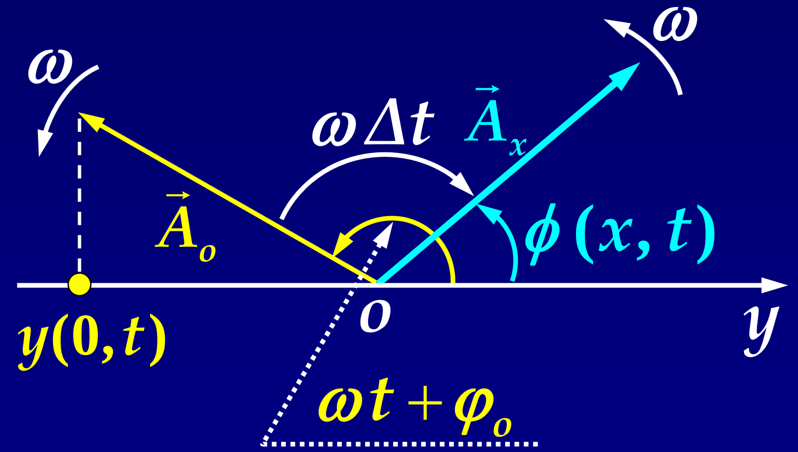
设：原点处质元的振动方程

$$y(0,\ t) = A\cos(\omega t + \varphi_o)$$

x 处质元的振动落后于 o 点

位相： $\omega \Delta t = \omega \dfrac{x}{u}$



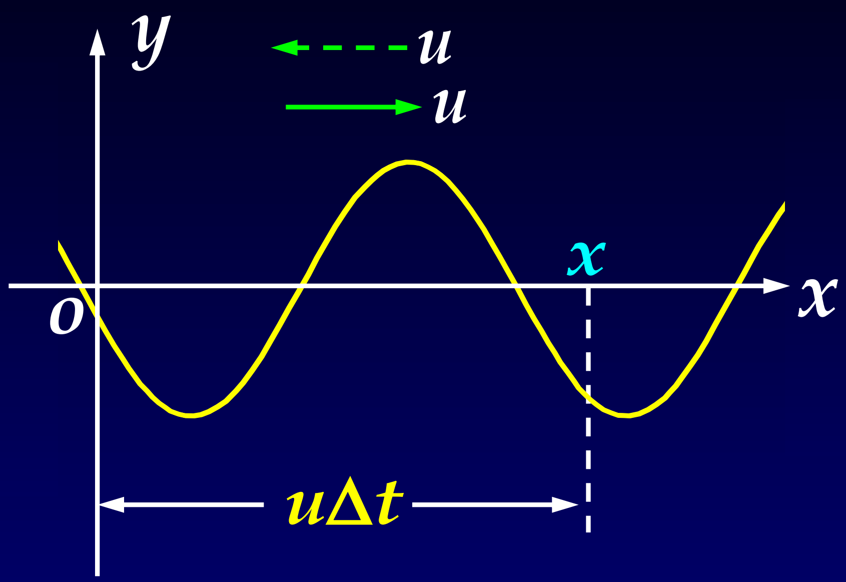


$$\phi(x,\ t) = (\omega t + \varphi_o) - \omega\Delta t$$

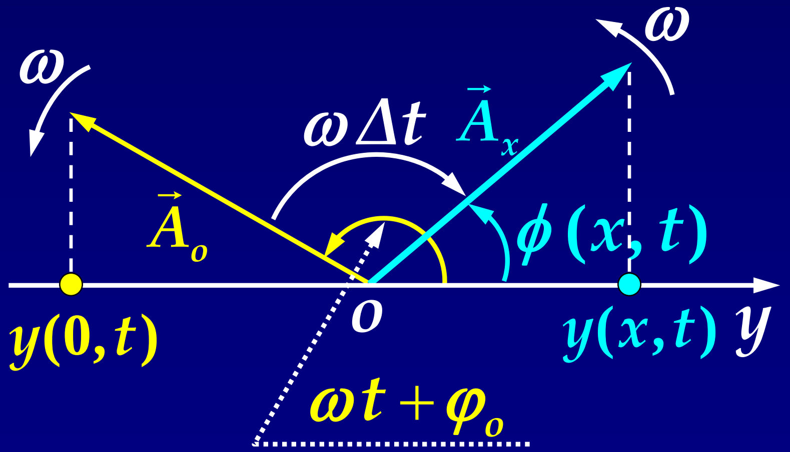
$$= \omega(t - \underbrace{\Delta t}_{\frac{x}{u}}) + \varphi_o$$

x 处质元的振动方程：

$$y(x,\ t) = A\cos\phi(x,\ t)$$



$$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$$



波函数亦称 波动方程 。

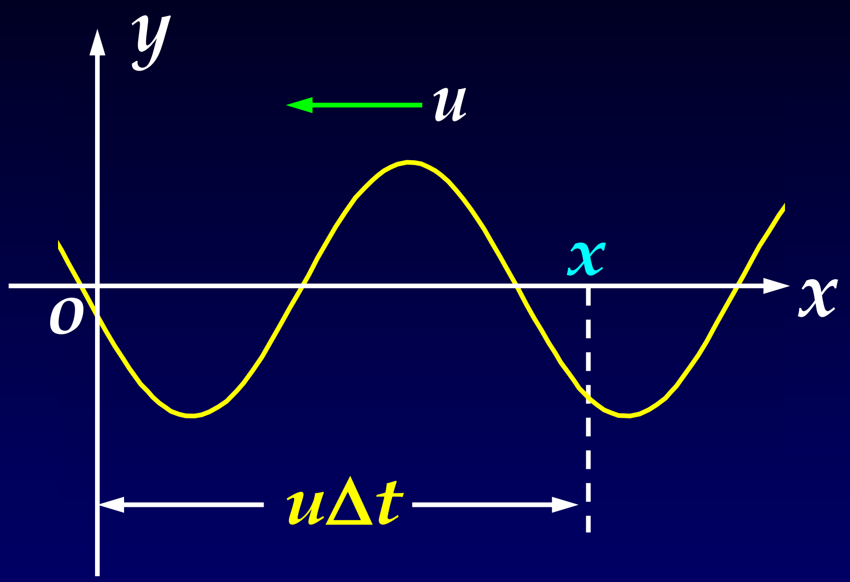
波动方程 的几种标准形式:

$$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$$

$$y = A\cos\left[\frac{2\pi}{T}\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$$

$$y = A\cos\left[2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right) + \varphi_o\right]$$

$$y = A\cos\left[\frac{2\pi}{\lambda}\left(ut \mp x\right) + \varphi_o\right]$$



右行波：取 — 号;

左行波：取 + 号。

即 ： x 处质元的振动方程，或在 t 时刻的位移!

二、波动方程的物理含义

$$y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$$

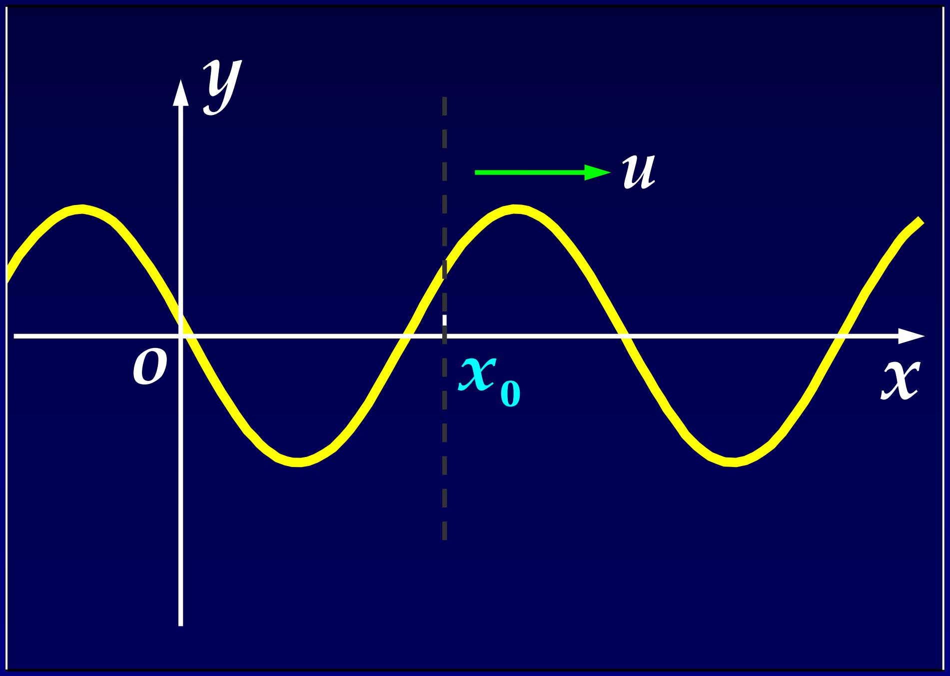
1. x 一定：$x = x0$ ，$y(x_0, t) = A\cos\left[\omega\left(t - \frac{x_0}{u}\right) + \varphi_o\right]$

$$y(x_0, t) = A\cos(\omega t + \varphi_x)$$

$$\varphi_x = -\frac{\omega x_0}{u} + \varphi_o$$

$$v = \frac{dy}{dt} = -\omega A\sin(\omega t + \varphi_x)$$

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \varphi_x)$$



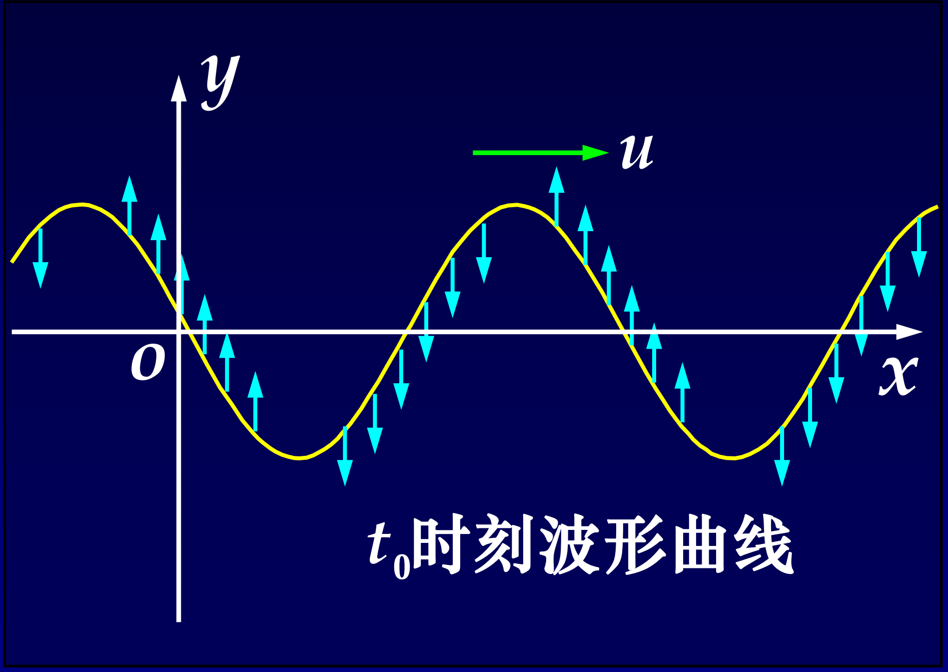
$$y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$$

2. t 一定： $t = t_0$ ， $y(x,t_0) = A\cos\left[\omega\left(t_0 - \frac{x}{u}\right) + \varphi_o\right]$

$$y(x,t_0) = A\cos\left(\frac{2\pi}{\lambda}x + \varphi^*\right)$$

判断：

右图中各点的速度方向或运动趋势。

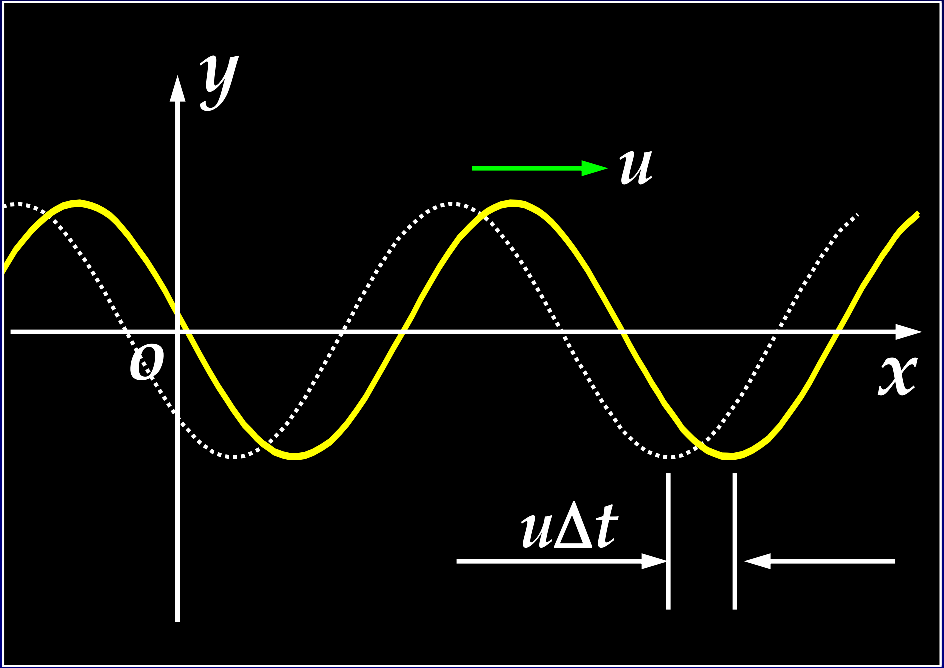


$$y = A\cos\left[\omega\left(t-\frac{x}{u}\right)+\varphi_o\right]$$

3. x、t 都不定： $y = A\cos\left[\omega\left(t-\frac{x}{u}\right)+\varphi_o\right]$

$$\omega\left(t-\frac{x}{u}\right)+\varphi_o = \omega\left(t+\Delta t-\frac{x+\Delta x}{u}\right)+\varphi_o$$

$$\Delta x = u\Delta t$$



☻波速即为相位传播速度（相速）。

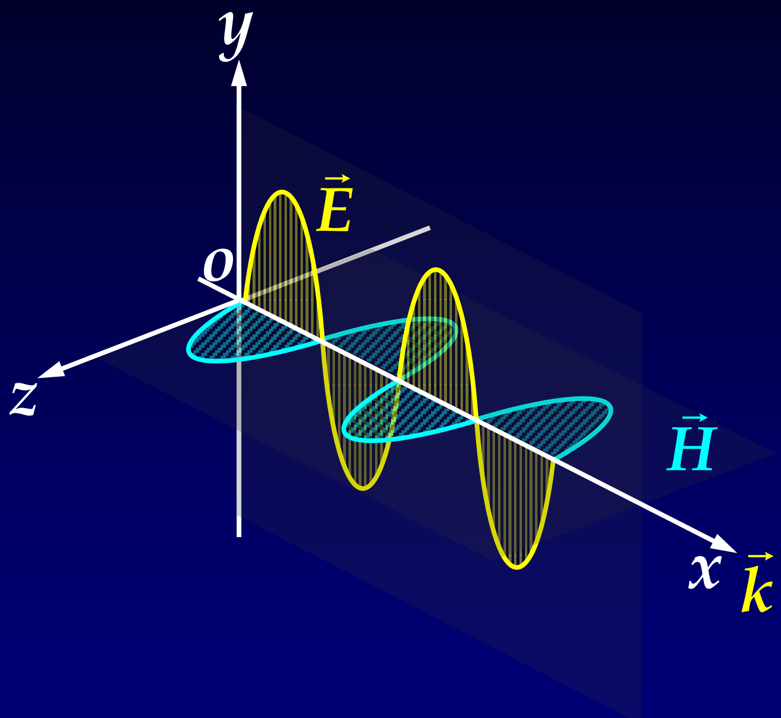
☻行波或前进波。

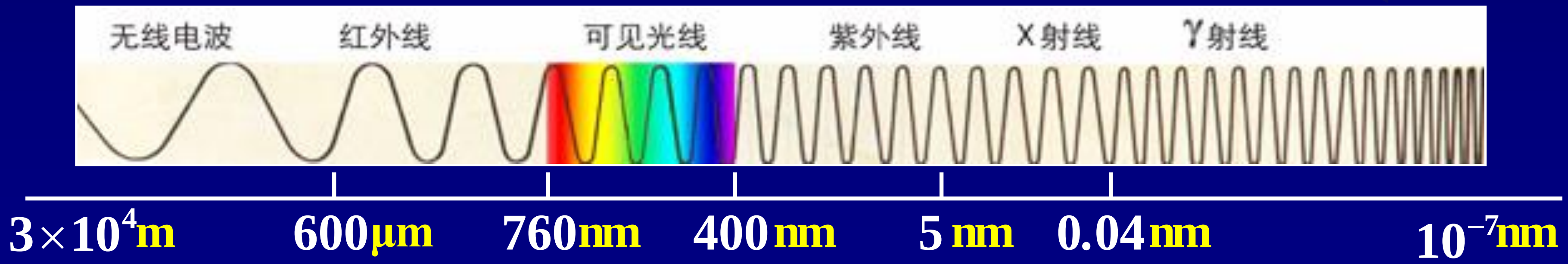
$$\frac{1}{u^2}=\mu\varepsilon \longrightarrow$$ 电磁波波速：$$u=1/\sqrt{\mu\varepsilon}$$

平面电磁波波函数：

$$\begin{cases} E=E_0\cos(\omega t-kx) \\ H=H_0\cos(\omega t-kx) \end{cases}$$

波矢：$$\left|\vec{k}\right|=\frac{2\pi}{\lambda}$$





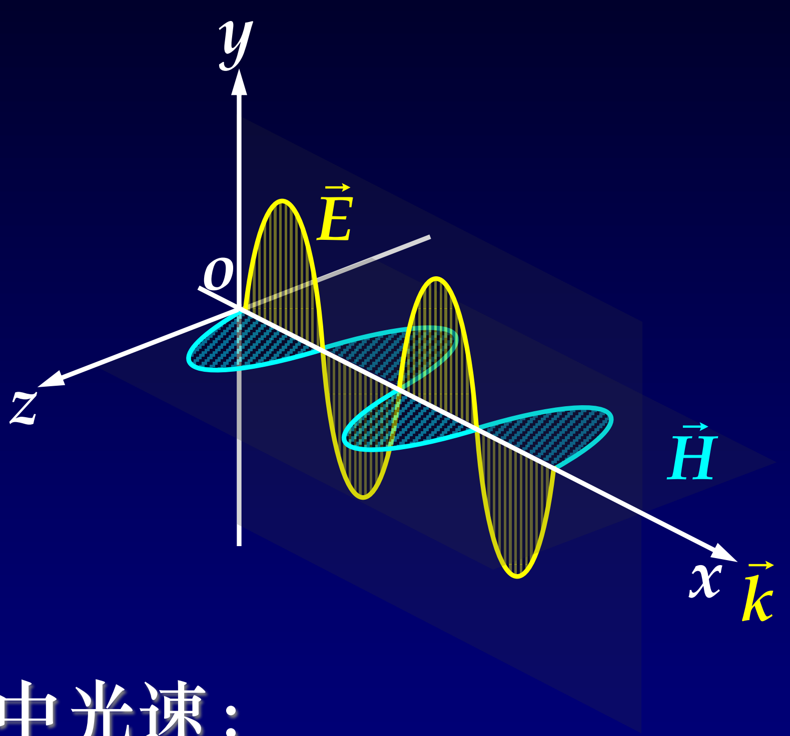
电磁波的主要特性:

1. 横波特性: $\vec{E} \perp \vec{H} \perp \vec{k}$

2. $\vec{E}$和$\vec{H}$同位相。

3. $\vec{E}$和$\vec{H}$的数值成比例:

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} \quad \text{or} \quad \sqrt{\varepsilon}E = \sqrt{\mu}H$$



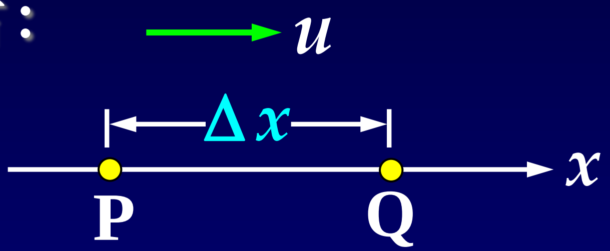
4. 真空中电磁波波速＝真空中光速:

$$u = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 2.998 \times 10^8 \text{ m/s} = c$$

四、波函数的求解

简谐波波函数：$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$

- 波函数与坐标系的建立有关。
- 沿波的传播方向各点相位依次落后：



$$\phi_Q(t) - \phi_P(t) = \omega\Delta t = \omega\frac{\Delta x}{u} = \frac{2\pi}{\lambda}\Delta x$$

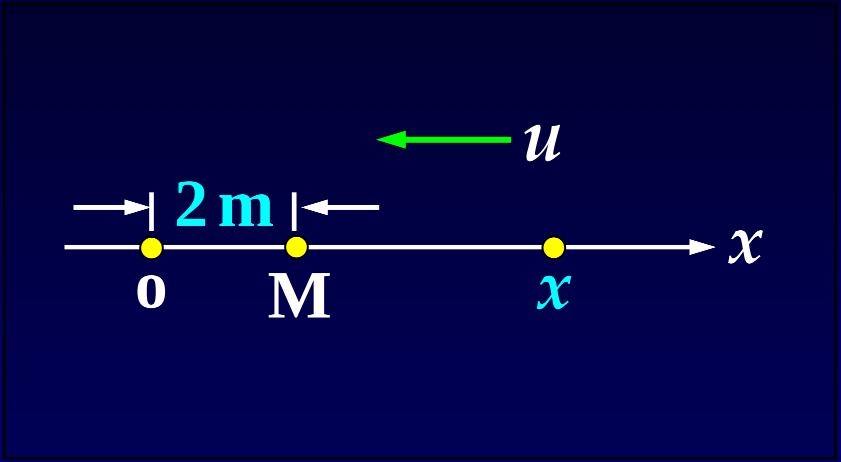
- 求波函数 ← 求 x 处质点的振动方程 ← 求 $\phi(x, t) = ?$
- x_0 处质元在 t_0 时刻的 v、a：

$$v(x_0,\ t_0) = \left.\frac{\partial y}{\partial t}\right|_{x_0, t_0} \qquad a(x_0,\ t_0) = \left.\frac{\partial^2 y}{\partial t^2}\right|_{x_0, t_0}$$

例 如图，波沿 $-x$方向传播，oM=2m，M点的振动方程:

$y=10\cos(\pi t+\frac{\pi}{2})$，$u$=2m/s，求波动方程(即波函数)。

解 x 处相位超前 M 点:



$$\phi(x,t)-\phi(x_M,t)=\omega\frac{x-x_M}{u}$$

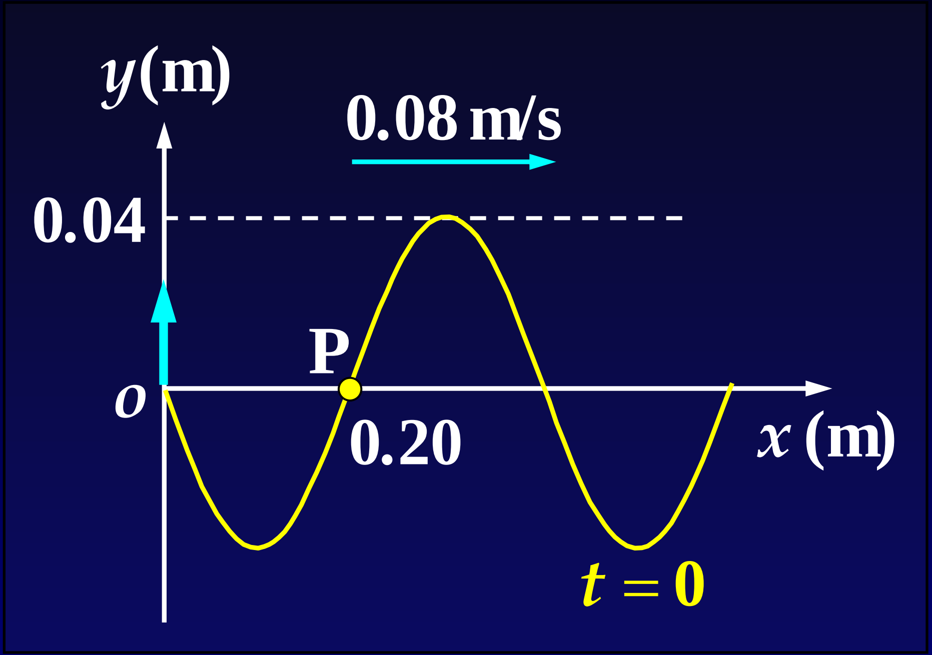
$$\phi(x_M,t)=\pi t+\frac{\pi}{2}$$

$$\phi(x,t)=(\pi t+\frac{\pi}{2})+\pi\frac{x-2}{2}=2\pi(\frac{t}{2}+\frac{x}{4})-\frac{\pi}{2}$$

$$y=10\cos[2\pi(\frac{t}{2}+\frac{x}{4})-\frac{\pi}{2}]$$

(the end)

例 图示为 $t=0$ 时的波形图，求波动方程及此时P点 v 。

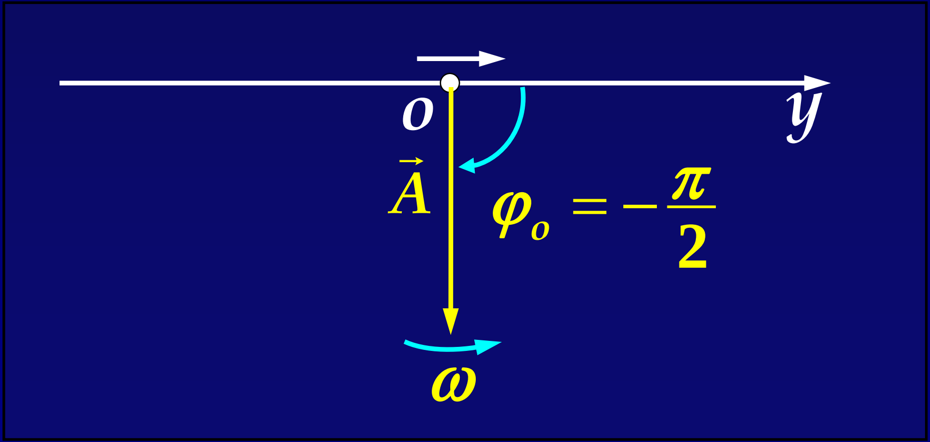


解　$\lambda = 0.40\,\mathrm{m}$

$$\omega = \frac{2\pi}{\lambda}u = \frac{2\pi}{5}$$

由旋转矢量图可知：

$$\varphi_o = -\frac{\pi}{2}$$



波函数的标准形式：

$$y = A\cos\left[\omega\left(t - \frac{x}{u}\right) + \varphi_o\right]$$

$$y = 0.04\cos\left[\frac{2\pi}{5}\left(t - \frac{x}{0.08}\right) - \frac{\pi}{2}\right]$$

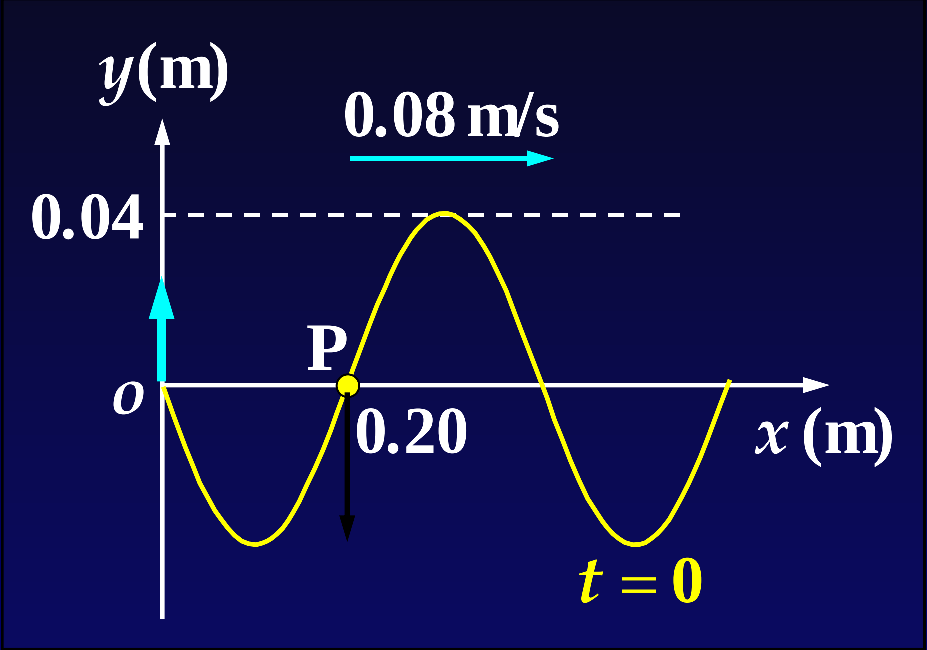
$t = 0$ 时 P 点的速度:

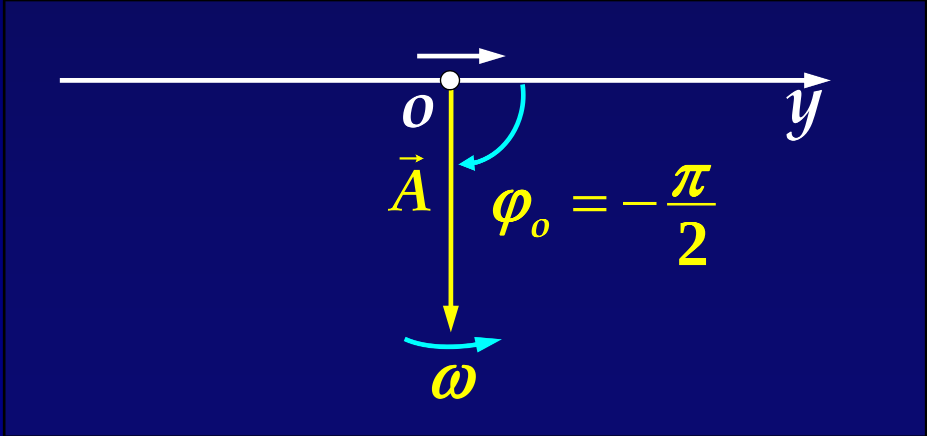
$$v_P = \left.\frac{\partial y}{\partial t}\right|_{\substack{x=0.20\\ t=0}}$$

$$= -0.04 \times \frac{2\pi}{5}\sin\left(-\frac{3\pi}{2}\right)$$

$$= -0.05\ (\mathrm{m/s})$$

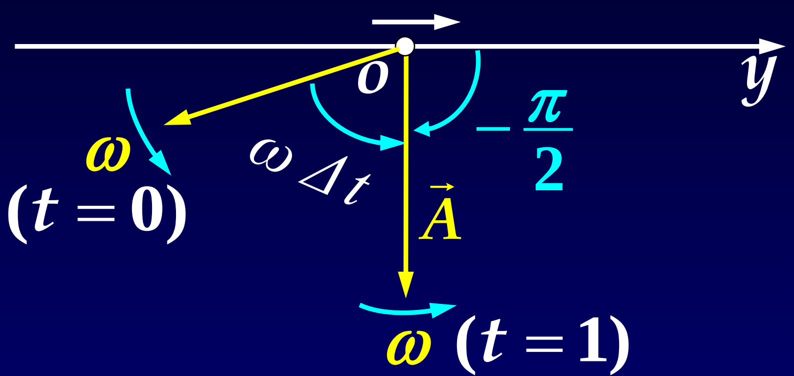
沿$-y$方向。 (the end)

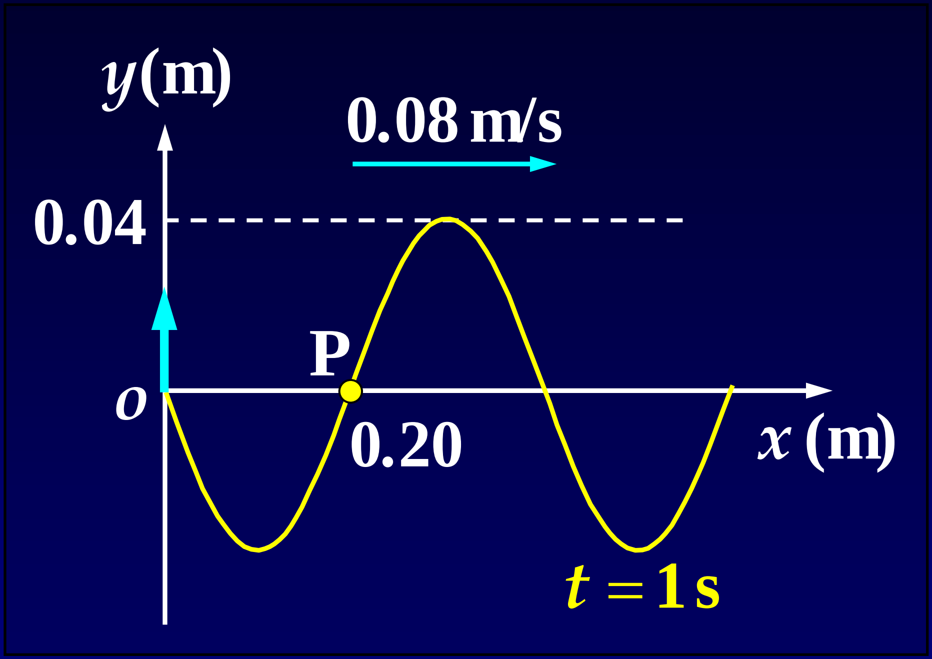




课堂练习 图示为 $t = 1\text{s}$ 时的波形曲线，求波动方程。

提示 关键：求解原点o处质元初位相 φ_o！





$$\omega\Delta t = \frac{2\pi}{5}(1-0) = \frac{2\pi}{5}$$

$$\therefore\ \varphi_o = -\frac{\pi}{2} - \frac{2\pi}{5} = -\frac{9\pi}{10}$$

答案：$y = 0.04\cos\left[\frac{2\pi}{5}\left(t - \frac{x}{0.08}\right) - \frac{9\pi}{10}\right]$

归纳:

1. 简谐波的波函数(波动方程):

$y = A\cos\left[\omega\left(t \mp \frac{x}{u}\right) + \varphi_o\right]$ 及其他几种标准形式。

2. 波动方程的微分形式:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 y}{\partial t^2} \qquad \nabla^2 \psi = \frac{1}{u^2}\frac{\partial^2 \psi}{\partial t^2}$$

3. 波方程的求解。

(The end)