Exercise 1

Summary

- Basic concepts of system models
- State-space representations
- Linear transform of state-space model
- Transfer Function Matrix
- Solutions of linear state-space model

Selection of State Variables

Case 1--- Differential equation without derivative of input variable

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_0 u$$

$$y = x_1 \qquad \left.\begin{array}{c} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + b_0u \end{array}\right\}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$c = [1 \quad 0 \quad \cdots \quad 0]$$

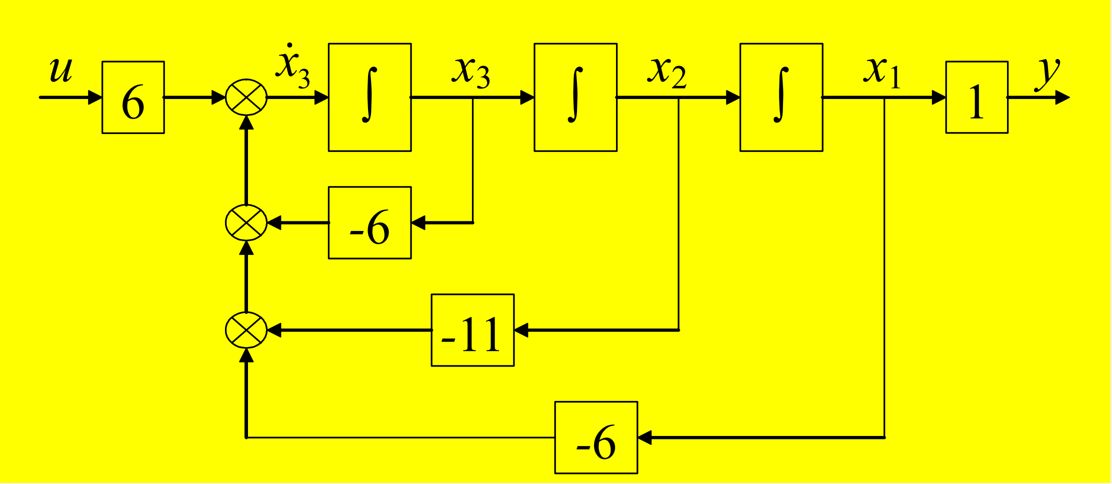
Example 1 Try to transform the differential equation into state space form, and draw the state variable graph

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

$$a_0 = 6,\ a_1 = 11, a_2 = 6, \qquad \mathrm{b} = 6$$

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \mathrm{u}$$

$$\mathrm{y} = [1 \quad 0 \quad 0]\mathrm{x}$$



Selection of State Variables

Case 2--- Differential equation with derivative of input variable

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0y=\boxed{b_nu^n+b_{n-1}u^{n-1}+\cdots+b_1\dot{u}+b_0u}$$

Select state variable as $x_1=y-h_0u \qquad x_i=\dot{x}_{i-1}-h_{i-1}u$

$$h_0=b_n$$

$$h_1=b_{n-1}-a_{n-1}h_0$$

$$h_2=b_{n-2}-a_{n-1}h_1-a_{n-2}h_0$$

$$\vdots$$

$$h_{\mathrm{n}-1}=b_1-a_{n-1}h_{n-2}-a_{n-2}h_{n-3}-\cdots-a_1h_0$$

$$h_n=b_0-a_{n-1}h_{n-1}-a_{n-2}h_{n-2}-\cdots-a_1h_1-a_0h_0$$

$$A=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&1\\-a_0&-a_1&-a_2&\cdots&-a_{n-1}\end{bmatrix},B=\begin{bmatrix}h_1\\h_2\\\vdots\\h_{n-1}\\h_n\end{bmatrix}$$

$$c=[1\quad 0\quad\cdots\quad 0],\ D=h_0$$

Example 2. Try to transform the differential equation into state space form and draw the state variable graph

$$\dddot{y} + 5\ddot{y} + 8\dot{y} + 4y = 2\ddot{u} + 14\dot{u} + 24u$$

$$a_0 = 4,\ a_1 = 8, a_2 = 5,$$

$$b_0 = 24,\ b_1 = 14, b_2 = 2, b_3 = 0$$

$$x_1 = y - h_0 u \qquad x_i = \dot{x}_{i-1} - h_{i-1} u$$

$$h_0 = b_3 = 0$$

$$h_1 = b_2 - a_2 h_0 = 2$$
$$h_2 = b_1 - a_2 h_1 - a_1 h_0 = 4$$
$$h_3 = b_0 - a_2 h_2 - a_1 h_1 - a_0 h_0 = -12$$

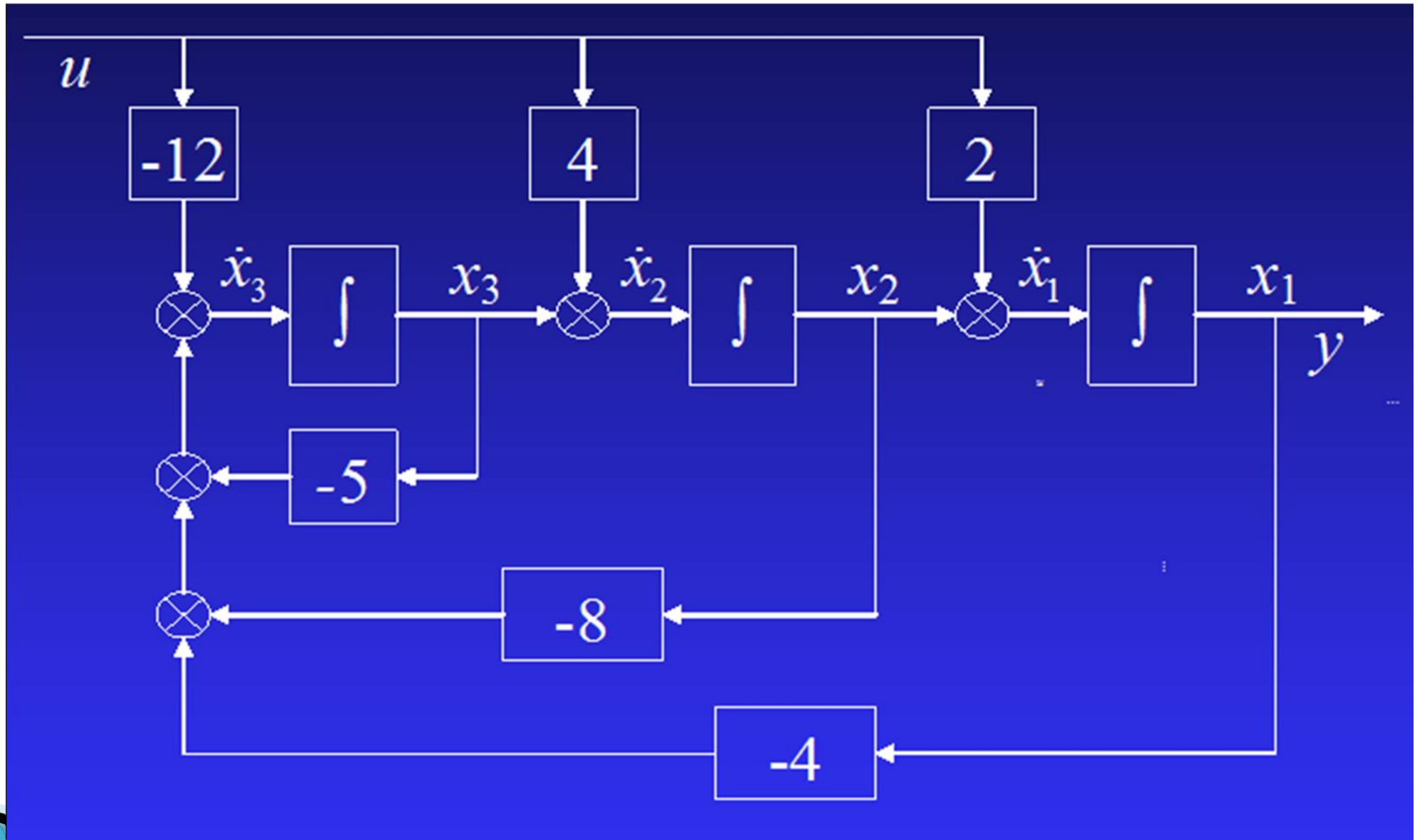
$$\begin{cases} x_1 = y - h_0 u = y \\ x_2 = \dot{y} - h_1 u - h_0 \dot{u} = \dot{y} - 2u \\ x_3 = \ddot{y} - h_2 u - h_1 \dot{u} - h_0 \ddot{u} = \ddot{y} - 4u - 2\dot{u} \end{cases}$$

$$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 4 \\ -12 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = [1 \quad 0 \quad 0]\boldsymbol{x}$$

$$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 4 \\ -12 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = [1 \quad 0 \quad 0]\boldsymbol{x}$$



Selection of State Variables

3. State variables description from transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= b_n + \boxed{\frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \cong b_n + \boxed{\frac{N(s)}{D(s)}}$$

Controllable Canonical Form

$$\dot{x} = A_c x + B_c u$$
$$y = C_c x$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1} \quad]$$

Observable Canonical Form

$$\dot{x} = A_o x + B_o u$$
$$y = C_o x$$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}, B_o = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{bmatrix}$$

$$C_o = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

Selection of State Variables

3. State variables description from transfer function

$$\frac{Y(s)}{U(s)}=\frac{N(s)}{D(s)}=\sum_{i=1}^{n}\frac{c_i}{s-\lambda_i},$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}u,\ y=\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

$$\frac{Y(s)}{U(s)}=\frac{N(s)}{D(s)}=\frac{c_{11}}{(s-\lambda_1)^3}+\frac{c_{12}}{(s-\lambda_1)^2}+\frac{c_{13}}{s-\lambda_1}+\sum^{n}\frac{c_i}{s-\lambda_i}$$

$$\begin{cases}\begin{bmatrix}\dot{x}_{11}\\ \dot{x}_{12}\\ \dot{x}_{13}\\ \dot{x}_4\\ \vdots\\ \dot{x}_n\end{bmatrix}=\boxed{\left[\begin{array}{ccc|ccc}\lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ \hline & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n\end{array}\right]}\begin{bmatrix}x_{11}\\ x_{12}\\ x_{13}\\ x_4\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\\ 1\\ \vdots\\ 1\end{bmatrix}u\\ y=\begin{bmatrix}c_{11} & c_{12} & c_{13} & c_4 & \cdots & c_n\end{bmatrix}x\end{cases}$$

Selection of State Variables

3. State variables description from transfer function

Case 3 $\frac{N(s)}{D(s)}$ **has repeat poles**

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{c_{11}}{(s-\lambda_1)^3} + \frac{c_{12}}{(s-\lambda_1)^2} + \frac{c_{13}}{s-\lambda_1} + \sum_{i=4}^{n} \frac{c_i}{s-\lambda_i}$$

State equation

$$\begin{aligned} \dot{x}_{11} &= \lambda_1 x_{11} + x_{12} \\ \dot{x}_{12} &= \lambda_1 x_{12} + x_{13} \\ \dot{x}_{13} &= \lambda_1 x_{13} + u \\ &\vdots \\ \dot{x}_n &= \lambda_i x_n + u \end{aligned}$$

Jordan canonical form

$$\begin{cases} \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \boxed{\begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_4 & \cdots & c_n \end{bmatrix} x \end{cases}$$

Linear transform of state-space model

Special case 1: Diagonal form of Matrix A

$$\dot{x} = Ax + Bu$$

(1) If matrix A has distinct eigenvalues $\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_n$

Exist nonsingular matrix *P*

$$\begin{cases} \dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}u \\ y = \bar{\boldsymbol{c}}\ \bar{\boldsymbol{x}} \end{cases}$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$P = [p_1 \quad p_2 \quad \cdots \quad p_n]$, p_i **is eigenvectors according to λ_i**

Linear transform of state-space model

Special case 1: Diagonal form of Matrix A

$$\dot{x} = Ax + Bu$$

If matrix A is companion matrix with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Which satisfies $\bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Example 5.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$|\lambda I-A|=\begin{vmatrix}\lambda&-1&0\\0&\lambda&-1\\6&11&\lambda+6\end{vmatrix}=(\lambda+1)(\lambda+2)(\lambda+3)=0$$

Try to get the diagonal form

$$\lambda_1=-1,\ \lambda_2=-2,\lambda_3=-3 \qquad P=\begin{bmatrix}1&1&1\\ \lambda_1&\lambda_2&\lambda_3\\ \lambda_1^2&\lambda_2^2&\lambda_3^2\end{bmatrix}=\begin{bmatrix}1&1&1\\-1&-2&-3\\1&4&9\end{bmatrix},P^{-1}=\begin{bmatrix}3&\frac{5}{2}&\frac{1}{2}\\-3&-4&-1\\1&\frac{3}{2}&\frac{1}{2}\end{bmatrix}$$

$$\widetilde{A}=P^{-1}AP=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}$$

$$\widetilde{B}=P^{-1}B=\begin{bmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix}$$

$$\widetilde{C}=CP=\begin{bmatrix}1&1&1\end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix}\dot{\widetilde{x}}_1\\ \dot{\widetilde{x}}_2\\ \dot{\widetilde{x}}_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}\widetilde{x}_1\\\widetilde{x}_2\\\widetilde{x}_3\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\\-1\\\frac{1}{2}\end{bmatrix}u$$

$$y=\begin{bmatrix}1&1&1\end{bmatrix}\begin{bmatrix}\widetilde{x}_1\\\widetilde{x}_2\\\widetilde{x}_3\end{bmatrix}$$

Linear transform

$$\ddot{x} + 3\dot{x} + 2x = u$$

(1)Let $x_1 = x, x_2 = \dot{x}$, try to derive the state equation

(2)Let $x_1 = \overline{x_1} + \overline{x_2}, x_2 = -\overline{x_1} - 2\overline{x_2}$, try to derive the linear transform matrix P and the transformed state equation

Solution to homogeneous state equation

Homogeneous state equation

$$\dot{x} = Ax \qquad u = 0$$

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0) = e^{At}x(0)$$

$$\Phi(t) = \mathrm{e}^{\boldsymbol{A}t} = L^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}]$$

Nonhomogeneous state equation

$$\dot{x} = Ax + Bu$$

$$x(t) = \boxed{e^{At}x(0)} + \boxed{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

free motion caused by initial state **motion controled by *input***

Example 6 Try to get the inverse matrix of the system state transition matrix and the solution under the initial state.

$$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{x} \qquad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\Phi(t) = \mathrm{e}^{At} = L^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}] = \begin{bmatrix} 2\mathrm{e}^{-t} - \mathrm{e}^{-2t} & \mathrm{e}^{-t} - \mathrm{e}^{-2t} \\ -2\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} \end{bmatrix}$$

$$\Phi^{-1}(t) = \Phi(-t) = \begin{bmatrix} 2\mathrm{e}^{t} - \mathrm{e}^{2t} & \mathrm{e}^{t} - \mathrm{e}^{2t} \\ -2\mathrm{e}^{t} + 2\mathrm{e}^{2t} & -\mathrm{e}^{t} + 2\mathrm{e}^{2t} \end{bmatrix}$$

The solution under initial state

$$\boldsymbol{x}(t) = \mathrm{e}^{At}\boldsymbol{x}_0 = \begin{bmatrix} 2\mathrm{e}^{-t} - \mathrm{e}^{-2t} & \mathrm{e}^{-t} - \mathrm{e}^{-2t} \\ -2\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4\mathrm{e}^{-t} - 3\mathrm{e}^{-2t} \\ -4\mathrm{e}^{-t} + 6\mathrm{e}^{-2t} \end{bmatrix}$$

Example 7. Consider the system defined by

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

1）Obtain the transfer function of the system.

2）Find the state transition matrix.

3）Find system output y in terms of $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $u(t) = 1(t)$

解：令 $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$，$B=\begin{bmatrix}1\\1\end{bmatrix}$，$C=\begin{bmatrix}0&1\end{bmatrix}$则

$$sI-A=\begin{bmatrix}s&0\\0&s\end{bmatrix}-\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}s-1&0\\-1&s-1\end{bmatrix}$$ -- 1 分

$$(sI-A)^{-1}=\frac{adj(sI-A)}{|sI-A|}=\frac{1}{(s-1)^2}\begin{bmatrix}s-1&0\\1&s-1\end{bmatrix}=\begin{bmatrix}\frac{1}{s-1}&0\\\frac{1}{(s-1)^2}&\frac{1}{s-1}\end{bmatrix}$$ ------------------- 2 分

系统的传递函数为：$G(s)=C(sI-A)^{-1}B=\dfrac{s}{s^2-2s+1}$ ------------------------ 2 分

对上述矩阵做 Laplacian 逆变换，得状态转移矩阵

$$\Phi(t)=L^{-1}\left[(sI-A)^{-1}\right]=\begin{bmatrix}e^t&0\\te^t&e^t\end{bmatrix}$$ -- 2 分

由于

$$\int_0^t\Phi(\tau)Bu(t-\tau)d\tau=\int_0^t\begin{bmatrix}e^\tau&0\\\tau e^\tau&e^\tau\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}d\tau=\int_0^t\begin{bmatrix}e^\tau\\\tau e^\tau+e^\tau\end{bmatrix}d\tau=\begin{bmatrix}e^t-1\\te^t\end{bmatrix}$$ ------------- 3 分

得系统状态方程的解为

$$x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}e^t & 0\\te^t & e^t\end{bmatrix}\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}+\begin{bmatrix}e^t-1\\te^t\end{bmatrix}$$

$$=\begin{bmatrix}e^t & 0\\te^t & e^t\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}e^t-1\\te^t\end{bmatrix}$$ -- 4 分

$$=\begin{bmatrix}0\\e^t\end{bmatrix}+\begin{bmatrix}e^t-1\\te^t\end{bmatrix}$$

$$=\begin{bmatrix}e^t-1\\e^t+te^t\end{bmatrix}$$

$$y(t)=Cx(\mathrm{t})=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}e^t-1\\e^t+te^t\end{bmatrix}=e^t+te^t$$

-- 1 分

Solution to homogeneous state equation

Property of state transition matrix $\Phi(t)$

(6)

$$\textit{if } A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\textit{then } \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

$$\textit{if } A = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \textit{ Jordan form}$$

$$\textit{then } \Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & te^{\lambda t} \\ 0 & 0 & 0 & \cdots & e^{\lambda t} \end{bmatrix}$$

Example 8. $$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} u$$

Initial state vector $x(0) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$, input is step signal, try to get the solution $x(t)$

$$e^{At} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= \begin{bmatrix} 1 + te^{-t} \\ 1 + e^{-t} \\ 2 - e^{-2t} \end{bmatrix}$$