Chapter 2

Solutions of Linear State-space Control System

Outlines

- **Solution to homogeneous state equation**

$$\dot{x}(t) = Ax(t), \quad u = 0$$

- **Solution to nonhomogeneous state equation**

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Solution to homogeneous state equation

Homogeneous state equation

$$\dot{x}(t) = ax(t) \qquad\qquad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

(1) Exponential method

Assume that the solution is in the form of a power series in *t*,

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \ldots$$

By substituting this assumed solution into the state equation, we could obtain

$$b_1 + 2b_2 t + \cdots + k b_k t^{k-1} + \ldots = a(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \ldots)$$

$$b_1 = ab_0, \qquad b_2 = \frac{1}{2} ab_1 = \frac{1}{2} a^2 b_0, \qquad \cdots \qquad b_k = \frac{1}{k!} a^k b_0$$

Solution to homogeneous state equation

The value of b_0 is determined by substituting t=0 into,

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \ldots$$

$$b_0 = x(0)$$

Thus the solution to the scalar homogeneous state equation is

$$x(t) = (1 + at + \frac{1}{2!}a^2t^2 + \cdots + \frac{1}{k!}a^k t^k + \ldots)x(0)$$

$$= \mathrm{e}^{at}\ x(0)$$

For vector homogeneous state equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

Solution to homogeneous state equation

Homogeneous state equation

$$\dot{x} = Ax \qquad u = 0$$

(2) Laplace transform

$$sX(s) - x(0) = AX(s)$$
$$X(s) = (sI - A)^{-1}x(0)$$

We could obtain

$$x(t) = L^{-1}[(sI - A)^{-1}]x(0) = e^{At}x(0)$$

$$\Phi(t) = \mathrm{e}^{\boldsymbol{A}t} = L^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}]$$

Solution to homogeneous state equation

Example 1 $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{x} \qquad \boldsymbol{x}(0)$

Please get the solution to the equation

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t} \cdot \boldsymbol{x}(0)$$

$$s\boldsymbol{I} - \boldsymbol{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$\Phi(t) = \mathrm{e}^{\boldsymbol{A}t} = L^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}] = \begin{bmatrix} 2\mathrm{e}^{-t} - \mathrm{e}^{-2t} & \mathrm{e}^{-t} - \mathrm{e}^{-2t} \\ -2\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} \end{bmatrix}$$

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t} \cdot \boldsymbol{x}(0) = \begin{bmatrix} 2\mathrm{e}^{-t} - \mathrm{e}^{-2t} & \mathrm{e}^{-t} - \mathrm{e}^{-2t} \\ -2\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} \end{bmatrix} \boldsymbol{x}(0)$$

Solution to homogeneous state equation

Homogeneous state equation

$$\dot{x} = Ax \qquad u = 0$$

$$x(t) = e^{At}x(0) = \Phi(t)x(0)$$

$$Or \;\; x(t) = \Phi(t - t_0)x(t_0)$$

$\Phi(t)$ is the state transition matrix

Solution of free motion can be represented uniform form with state transition matrix

The solution is determined by the state transition matrix

Solution to homogeneous state equation

Property of state transition matrix $\Phi(t) \; = e^{\boldsymbol{A}t}$

(1) $x(t) = \Phi(t - t_0)x(t_0)$

if $\Phi(t - t_0)$ is nonsingular, *then* $\Phi(t - t_0)^{-1} = \Phi(t_0 - t)$

(2) $\Phi(0) = e^{\boldsymbol{A}0} = I$

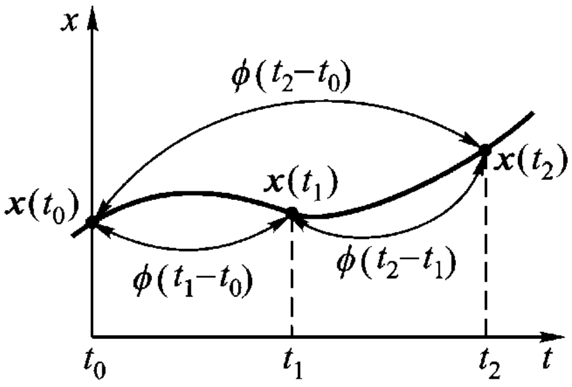
(3) $\dot{\Phi}(t) = A\Phi(t) \qquad \dot{\Phi}(0) = A$

(4) $[\Phi(t)]^k = \Phi(kt)$

Solution to homogeneous state equation

Property of state transition matrix $\Phi(t)$

$$(5)\ \Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$$



Proof

$$\boldsymbol{x}(t_2) = \Phi(t_2 - t_0)\boldsymbol{x}(t_0)$$

$$x(t_1) = \Phi(t_1 - t_0)x(t_0)$$

$$x(t_2) = \Phi(t_2 - t_1)x(t_1) = \Phi(t_2 - t_1)\Phi(t_1 - t_0)x(t_0)$$

then $\Phi(t_2 - t_0) = \Phi(t_2 - t_1)\Phi(t_1 - t_0)$

Solution to homogeneous state equation

Property of state transition matrix $\Phi(t)$

(6)

$$\textit{if } A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\textit{then } \Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

$$\textit{if } A = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \textit{ Jordan form}$$

$$\textit{then } \Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda t} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & te^{\lambda t} \\ 0 & 0 & 0 & \cdots & e^{\lambda t} \end{bmatrix}$$

Solution to nonhomogeneous state equation

Nonhomogeneous state equation

$$\dot{x} = Ax + Bu$$

(1) Direct integral method

$$\mathrm{e}^{-\boldsymbol{A}t}(\dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x}) = \mathrm{e}^{-\boldsymbol{A}t} \cdot \boldsymbol{B}\boldsymbol{u}$$

for $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\boldsymbol{A}t} \cdot \boldsymbol{x}) = \mathrm{e}^{-\boldsymbol{A}t}(\dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x})$

then $\dfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\boldsymbol{A}t} \cdot \boldsymbol{x}) = \mathrm{e}^{-\boldsymbol{A}t} \cdot \boldsymbol{B}\boldsymbol{u}$

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{x}(\tau))\mathrm{d}\tau = \int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{e}^{-\boldsymbol{A}\tau}x(\tau))\mathrm{d}\tau = \mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{x}(\tau)\Big|_0^t = \mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0)$$

$$\mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0) + \mathrm{e}^{\boldsymbol{A}t}\int_0^t \mathrm{e}^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)} \cdot \boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

$$\mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \mathrm{e}^{-\boldsymbol{A}t_0}\boldsymbol{x}(t_0) = \int_{t_0}^t \mathrm{e}^{-\boldsymbol{A}\tau} \cdot \boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^t \mathrm{e}^{\boldsymbol{A}(t-\tau)} \cdot \boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

Solution to nonhomogeneous state equation

Nonhomogeneous state equation

$$\dot{x} = Ax + Bu$$

(2) Laplace transform

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}[x(0) + BU(s)]$$

We could obtain

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$x(t) = \boxed{e^{At}x(0)} + \boxed{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}$$

free motion caused by initial state **motion controled by *input***

Solution to nonhomogeneous state equation

Nonhomogeneous state equation

$$\dot{x} = Ax + Bu$$

When the input is impulse signal $\delta(t)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau = e^{At}x(0) + e^{At}B$$

When the input is step signal $\mathrm{u}(t)$

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau \\ &= e^{At}x(0) + \int_0^t e^{At} Bu(t-\tau)d\tau \\ &= e^{At}x(0) + \int_0^t e^{At}B\, d\tau \end{aligned}$$

Solution to nonhomogeneous state equation

Example 3

For state space model $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$ **, input** $u(t) = 1(t)$

Initial state $\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ **Please solve this nonhomogeneous equation**

$$\boldsymbol{x}(t) = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)} \cdot \boldsymbol{Bu}(\tau)\mathrm{d}\tau$$

$$\mathrm{e}^{\boldsymbol{A}t} = \begin{bmatrix} 2\mathrm{e}^{-t} - \mathrm{e}^{-2t} & \mathrm{e}^{-t} - \mathrm{e}^{-2t} \\ -2\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + 2\mathrm{e}^{-2t} \end{bmatrix}$$

$$\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)} \cdot \boldsymbol{Bu}(\tau)\mathrm{d}\tau = \int_0^t \begin{bmatrix} 2\mathrm{e}^{-(t-\tau)} - \mathrm{e}^{-2(t-\tau)} & \mathrm{e}^{-(t-\tau)} - \mathrm{e}^{-2(t-\tau)} \\ -2\mathrm{e}^{-(t-\tau)} + 2\mathrm{e}^{-2(t-\tau)} & -\mathrm{e}^{-(t-\tau)} + 2\mathrm{e}^{-2(t-\tau)} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathrm{d}\tau$$

Solution to nonhomogeneous state equation

$$\int_0^t e^{\boldsymbol{A}(t-\tau)} \cdot \boldsymbol{Bu}(\tau)\mathrm{d}\tau = \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \mathrm{d}\tau$$

$$= -\begin{bmatrix} -e^{-(t-\tau)} + \frac{1}{2}e^{-2(t-\tau)} \\ e^{-(t-\tau)} - e^{-2(t-\tau)} \end{bmatrix}\Bigg|_0^t = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$+ \begin{bmatrix} -e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Description of discrete system

Continuous system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Discrete system

$$x(k+1) = G(k)x(k) + H(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

where k is the kth sample moment

$$G = e^{AT}$$
$$H = \int_0^T e^{At} B dt$$

C, D **remain the same**

Description of discrete system

Example 4 **LTI continuous system state equation as following**

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Write out the discrete state equation

$$G = e^{AT} = L^{-1}[(sI - A)^{-1}] = L^{-1} \begin{bmatrix} \frac{1}{s} & \frac{-1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} + \frac{1}{2} e^{-2\mathrm{T}} \\ 0 & e^{-2\mathrm{T}} \end{bmatrix}$$

$$H = \int_0^T e^{At} B dt = \int_0^T \begin{bmatrix} 1 & -\frac{1}{2} + \frac{1}{2} e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{T}{2} + \frac{1}{4} - \frac{1}{4} e^{-2T} \\ \frac{1}{2} - \frac{1}{2} e^{-2T} \end{bmatrix}$$