Chapter 5

Lyapunov Stability

Outlines

- Lyapunov Stability
- Lyapunov first method
- Preliminaries
- Lyapunov second method

Lyapunov Stability

1. External stability. $(x(0)=0)$

For any system which is casual and relaxed at time t_0, the relationship between input and output is

$$y(t) = \int_{t_0}^{t} G(t,\tau)\,u(\tau)d\tau, \quad t \geq t_0$$

If the input and output are bounded

$$\|u(t)\| < k, \ \forall\, t \geq t_0 \quad \|y(t)\| < \alpha k$$

It is external stable or BIBO（Bounded Input Bounded Output）stable.

Lyapunov Stability

2. Inner stability. (*u*=0)

Consider system

$$\dot{x} = A(t)x, x(t_0) = x_0$$

The solution is $x(t) = \Phi(t, t_0)x(t_0)$

If the solution $x(t)$ **could satisfy** $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \Phi(t, t_0)x_0 = 0$

Then we call the system is inner stable or asymptotically stable

Lyapunov Stability

Remarks: relationship between outer and inner stable

1) **For LTI system, if the system is inner stable, then the system must be outer stable (BIBO stable)**

2) **The outer stable LTI system may not be inner stable.**
   System transfer function matrix reflects only the controllable and observable part of the whole system. The rest part, e.g. the uncontrollable and unobservable subsystem can not be reflected.

3) **If the system is controllable and observable, then the system inner stable is equivalent to outer stable**

Lyapunov Stability

Lyapunov Stability studies state stability with zero input, it is more generally defined, and can be applied to linear system, nonlinear system and time-varying system.

- Lyapunov first method
- Lyapunov second method

Lyapunov Stability

Lyapunov Stability

Consider $\dot{x} = f(x, t)$ $\quad x(t_0) = x_0$ **without input (autonomous system)**

$$x(t, x_0, t_0)$$

If there is a state x_e, **that makes the state equation**

$$\boxed{\dot{x}_e = f(x_e, t) = 0}$$

Then x_e **is called balanced state (equilibrium).**

Lyapunov Stability

Remarks:

(1) The study of stability in sense of Lyapunov is to study the stability of system balanced point stability--------whether a disturbance movement which departures from the balanced state could go back or be restricted within a limited neighborhood around the balanced state point.

(2) The disturbance movement is system free motion----movement based on only system structure property and initial state.

For linear system $\dot{x} = Ax$ $\dot{x}_e = Ax_e = 0$**, if *A* is nonsingular matrix, then** $x_e = 0$ **, there is unique equilibrium state and it is original point;**

If *A* is singular matrix, then there are multiple states of equilibrium. There is more than one state of equilibrium for nonlinear system.

Lyapunov Stability

- **Vector norm:** Considering n-dimension vector, the vector norm is defined as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = x^T x$$

$$\|x - x_e\| = \sqrt{\left(x_1 - x_{e_1}\right)^2 + \cdots + (x_n - x_{en})^2} = (x - x_e)^T (x - x_e)$$

The geometrical meaning of vector norm is distance.

e.g. $\|x - x_e\| \leq \varepsilon$

Super-sphere in state space with the center x_e **and radius** ε

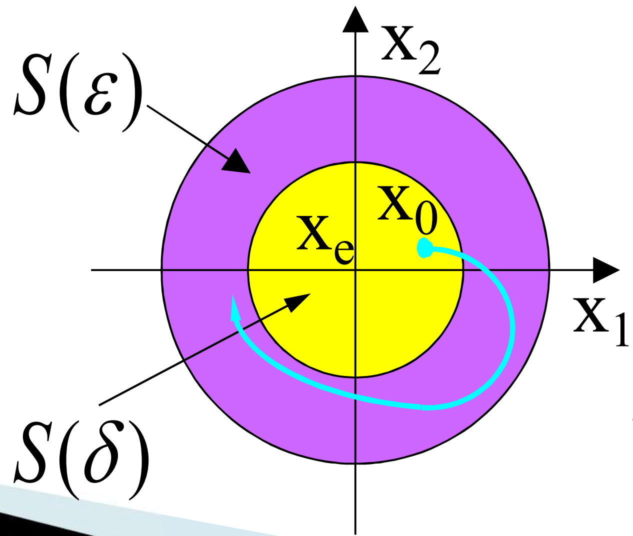
Lyapunov Stability

(1) Lyapunov stable：

For any positive real number ε, if there is always corresponding number $\delta(\varepsilon, t_0) > 0$

the initial disturbance satisfies $\|x_0 - x_e\| \leq \delta(\varepsilon, t_0)$

and the movements caused by the initial disturbance satisfies $\|x(t) - x_e\| \leq \varepsilon \quad \forall t \geq t_0$



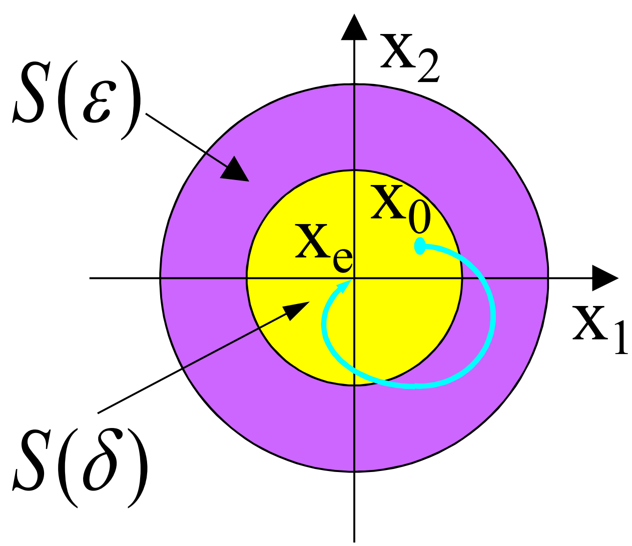
x_e **is Lyapunov stable**

Lyapunov Stability

(2) Asymptotically stable:

If $\boldsymbol{x}_e$ is Lyapunov stable, and $\boxed{\lim_{t\to\infty} \| \boldsymbol{x}(t;\boldsymbol{x}_0,t_0) - \boldsymbol{x}_e \| = 0}$

Then $\boldsymbol{x}_e$ is asymptotically Lyapunov stable

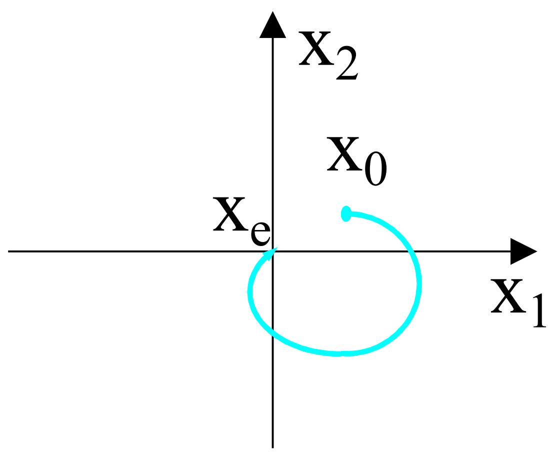


Lyapunov Stability

(3) Global asymptotically stable (asymptotically stable in a whole):

For any initial state x0, if $\boxed{\lim_{t \to \infty} \| \boldsymbol{x}(t; \boldsymbol{x}_0, t_0) - \boldsymbol{x}_e \| = 0}$

Then $\boldsymbol{x}_e$ **is global asymptotically Lyapunov stable**



Lyapunov Stability

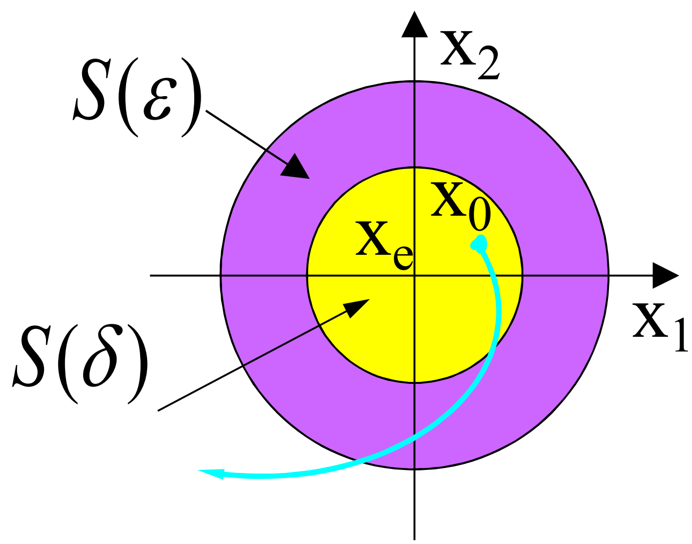
(4) Unstable:

$\forall\ \varepsilon > 0, \exists\ \delta(\varepsilon, t_0)$, such that if $\|\boldsymbol{x}_0 - \boldsymbol{x}_e\| \le \delta$

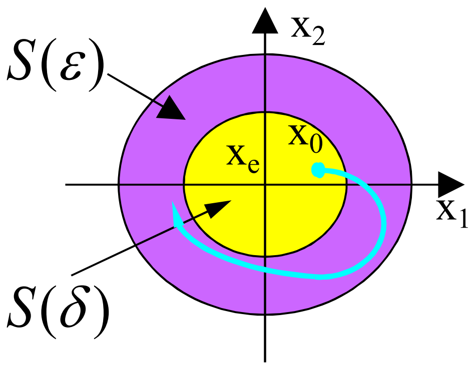
no matter how small δ and ε are,

we have $\|\boldsymbol{x}(t; \boldsymbol{x}_0, t_0) - \boldsymbol{x}_e\| > \varepsilon$

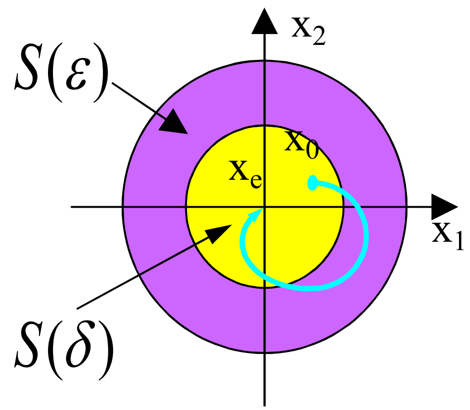
Then $\boldsymbol{x}_e$ is unstable



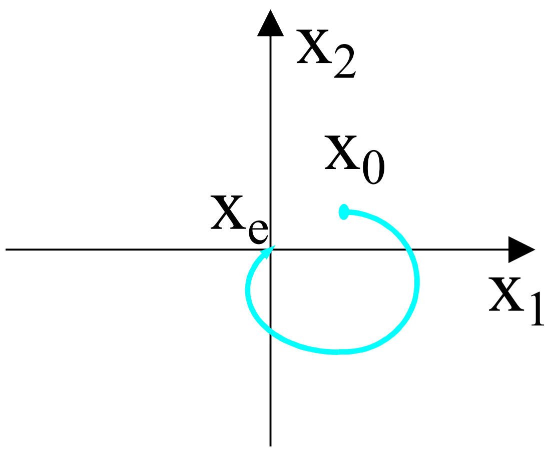
Lyapunov Stability



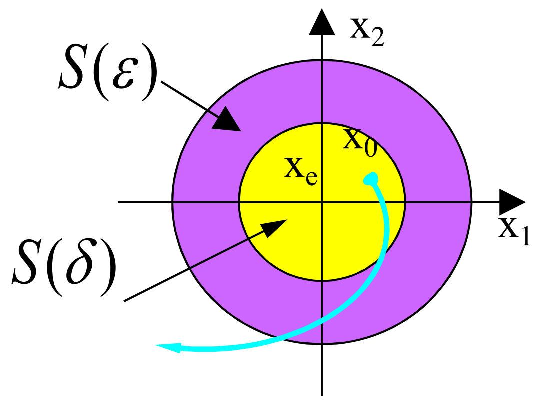
$\boldsymbol{x}_e$ is Lyapunov stable



$\boldsymbol{x}_e$ is asymptotically Lyapunov stable



$\boldsymbol{x}_e$ is asymptotically stable in a whole



$\boldsymbol{x}_e$ is Unstable

Lyapunov Stability

Remarks:

(1) The asymptotically stability is more significant than stability.

(2) The asymptotically stability is equivalent with the stability definition in classic automatic control domain.

(3) For LTI system, if the matrix A is nonsingular and the sole balanced state at original is asymptotically stable, then the balanced state is asymptotically stable in a whole.

Lyapunov First Method

Indirect method: **Discriminate the system stability based on the system characteristic roots or eigenvalues**

Theorem 1:

For LTI system $\dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad t \geq 0$

The sufficient and necessary condition of equilibrium x_e being Lyapunov stable is all the eigenvalues of matrix A have negative real part

$$\mathrm{Re}[\mathrm{eig}(A)] \leq 0$$

And eigenvalue with zero real part is the unique root of minimal polynomial of A

Lyapunov First Method

Example:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{x}$$

Try to determine the stability

$$\det(sI - A) = s^2(s+1) = 0 \qquad \lambda_1 = -1 \;, \lambda_2 = \lambda_3 = 0$$

$$\left(sI - A\right)^{-1} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} = \frac{1}{s^2(s+1)} \begin{bmatrix} s(s+1) & 0 & 0 \\ 0 & s(s+1) & 0 \\ 0 & 0 & s^2 \end{bmatrix}$$

$$= \frac{1}{s(s+1)} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s \end{bmatrix}$$

$f(s)=s(s+1)$

It is Lyapunov stable.

Lyapunov First Method

Theorem 2:

For LTI system $\dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad t \geq 0$

The sufficient and necessary condition of equilibrium x_e being asymptotically Lyapunov stable is

$$\mathrm{Re}[\mathrm{eig}(A)] < 0$$

Theorem 3:

The sufficient and necessary condition of LTI system being BIBO stable is

The poles of TF are located in left part of S domain.

Lyapunov First Method

Example:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \boldsymbol{x}$$

Try to determine the stability

$$\det(sI - A) = s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) = 0$$

$$\lambda_1 = -1,\quad \lambda_2 = -2,\quad \lambda_3 = -3$$

x_e=0 is asymptotically Lyapunov stable.

Lyapunov First Method

Example: $\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & -6 \\ 1 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}$

Try to determine the stability

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = \lambda(\lambda+1) - 6 = 0. \qquad \lambda_1 = 2 \qquad \lambda_2 = 3$$

it is not asymptotically Lyapunov stable.

$$G(s) = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{1}{s+3}$$

Poles $s = -3$， it is BIBO stable.

Remark： BIBO stable $\not\Leftrightarrow$ asymptotically stable

Preliminaries

(1) Euclidean norm

1) Euclidean norm of vectors (length)

$$\| \boldsymbol{x} \| = | x | \qquad \boldsymbol{x} \in R$$

$$\| \boldsymbol{x} \| = \sqrt{x_1^2 + x_2^2} \qquad \boldsymbol{x} \in R^2$$

$$\| \boldsymbol{x} \| = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad \boldsymbol{x} \in R^3$$

$$\vdots$$

Generally

$$\| \boldsymbol{x} \| = (| x_1 |^2 + | x_2 |^2 + \cdots + | x_n |^2)^{\frac{1}{2}} \qquad \boldsymbol{x} \in R^n$$

Preliminaries

2) Euclidean norm of matrices

For matrix $\boldsymbol{A} \in \mathrm{R}^{\boldsymbol{n}\times\boldsymbol{m}}$, its Euclidean norm is defined as

$$\|\boldsymbol{A}\| = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2\right)^{\frac{1}{2}} = \sqrt{\operatorname{tr}\left(\boldsymbol{A}^T\boldsymbol{A}\right)}$$

For example

$$\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix} \qquad \boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 9 & 12 \\ 12 & 20 \end{pmatrix}$$

$$\|\boldsymbol{A}\| = \sqrt{\operatorname{tr}(\boldsymbol{A}^T\boldsymbol{A})} = \sqrt{29}$$

Preliminaries

(2) Quadratic function and its expression of matrix

1) Quadratic function

A quadratic function consists of *n* variables $x_1, x_2, \cdots, x_n$

$$V(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} x_i x_j$$

2) Matrix form of Quadratic function

$$V(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

***P* is a symmetric matrix**

Preliminaries

3) The sign of Quadratic function

For quadratic function $V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x}.$

a. Positive Definite : $\begin{cases} V(x) > 0 & x \neq 0 \\ V(x) = 0 & x = 0 \end{cases}$ $\quad e.g.\ V(x) = x_1^2 + x_2^2$

b. Positive Semi-definite $\begin{cases} V(x) \geq 0 & x \neq 0 \\ V(x) = 0 & x = 0 \end{cases}$ $\quad e.g.\ V(x) = x_1^2(1 + x_2)^2$

Then $V(\boldsymbol{x})$ **is Positive Semi-definite,** ***P is*** **Positive Semi-definite Matrix** $(\boldsymbol{P} \geqslant 0)$

c. Negative definite: if -*V*(*x*) is positive definite, then *V*(*x*) is negative definite.

$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} < 0,$ ***P is*** **Negative definite Matrix** $(\boldsymbol{P} < 0)$

$e.g.\ V(x) = -(x_1^2 + x_2^2)$

d. Negative Semi-definite $\forall \boldsymbol{x} \neq 0 \quad V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} \leq 0,$

Preliminaries

e. unfixed :

$\forall \boldsymbol{x} \neq 0 \quad V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} > 0$

$\forall \boldsymbol{x} \neq 0 \quad V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} < 0,$

Then $V(\boldsymbol{x})$ **is unfixed**

$e.g.\ v(x) = x_1 x_2 + x_2^2$

Some properties of positive matrix

If A is a positive matrix, then A^T, A^{-1} *are positive matrix.*

If A and B are positive matrices, then $A + \boldsymbol{B}$ *is positive matrix.*

- Identity matrix I is positive definite: $x^T I_n x = x_1^2 + \cdots + x_n^2$

- Diagonal matrix D=diag{$d_1, \ldots, d_n$} is positive definite, iff $d_i > 0$

$$f(x) = x^T D x = d_1 x_1^2 + \cdots + d_n x_n^2 > 0$$

Preliminaries

Example: $\boldsymbol{x} = [x_1, x_2]^T$**, try to figure out the sign the following functions**

1) $V(\boldsymbol{x}) = x_1^2 + x_2^2$

2) $V(\boldsymbol{x}) = (x_1 + x_2)^2$

3) $V(\boldsymbol{x}) = -x_1^2 - x_2^2$

4) $V(\boldsymbol{x}) = -(3x_1 + 2x_2)^2$

5) $V(\boldsymbol{x}) = x_1 x_2 - x_2^2$

Preliminaries

Sylvester criterion: The sufficient and necessary condition of $V(x)$ being positive definite is

The Order Master determinant of matrix P are greater than 0, that is

$$p_{11} > 0 \; , \quad \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} > 0 \; , \quad \cdots, \quad \begin{vmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{vmatrix} > 0$$

then $V(x)$ **is positive definite,** $\boldsymbol{P}$ **is a positive definite matrix**

The sign of Order Master determinant of matrix P is altering , that is

$$p_{11} < 0 \; , \quad \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} > 0 \; , \; \ldots, \quad (-1)^n \begin{vmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{vmatrix} > 0$$

then $V(x)$ **is negative definite,** $\boldsymbol{P}$ **is a negative definite matrix**

Preliminaries

Example: try to figure out the sign the following functions

$$V(x) = 10x_1^2 + 2x_1x_2 + 4x_2^2$$
$$= 10x_1^2 + x_1x_2 + x_2x_1 + 4x_2^2$$

$$P = \begin{bmatrix} 10 & 1 \\ 1 & 4 \end{bmatrix} \quad \Delta_1 = 10 > 0 \quad \Delta_2 = \begin{vmatrix} 10 & 1 \\ 1 & 4 \end{vmatrix} > 0$$
$$V(x)\, positive\, definite$$

Preliminaries

Example: $\boldsymbol{x} = [x_1, x_2, x_3]^T$**, try to verify that the following function is positive definite.**

$$V(\boldsymbol{x}) = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_2x_3 + 2x_1x_3$$

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$p_1 = 1$$

$$p_2 = 4$$

$$p_3 = 2$$

Lyapunov Second Method

- **Without solving system characteristic roots.**
- **Construct a scalar system energy (Lyapunov) function and analyze the function and its first order differential coefficient sign to get the correlative information of system stability.**
- **Basic idea: if a system is asymptotical stable, the energy of system will decay with time and approach to a (certain) minimum value.**

$$x(t) \xrightarrow[t\to\infty]{} x_e \ \ or \ \ \lim_{t\to\infty} \|x(t) - x_e\| = 0$$

Lyapunov Second Method

Theorems of Lyapunov second method

1. For any system with state equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{t})$ and balanced state $\boldsymbol{x_e} = \mathbf{0}$,

if there is a scalar function $\mathrm{V}(\boldsymbol{x}, \boldsymbol{t})$ satisfying

(1) $V(x,\, t) > 0$ **(2)** $\dot{V}(x,\, t) < 0$

then we could say the system is asymptotically stable at $\boldsymbol{x_e}$.

Additionally, (3) $V(x, t) \to \infty$ **,** when $\|x\| \to \infty$

then we could say the system is asymptotically stable in a whole at $\boldsymbol{x_e}$.

Physical meaning: $\boldsymbol{V}(\boldsymbol{x},\, \boldsymbol{t}) > \mathbf{0}$ system energy is always positive.

$\dot{\boldsymbol{V}}(\boldsymbol{x},\, \boldsymbol{t}) < \mathbf{0}$ the energy decays with time.

Lyapunov Second Method

Example: For state equation: $\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

Try to figure out whether it is stable.

(x_1=0, x_2=0) is the only balanced state.

Choose a Lyapunov function $V(\boldsymbol{x}) = x_1^2 + x_2^2$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1x_2 - 2x_1^2(x_1^2 + x_2^2) - 2x_1x_2 - 2x_2^2(x_1^2 + x_2^2) = -2(x_1^2 + x_2^2)^2$$

is negative definite. When $\|\boldsymbol{x}\| \to \infty$ $V(\boldsymbol{x}) \to \infty$

Lyapunov Second Method

Theorems of Lyapunov second method

2 For any system with state equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{t})$ and balanced state $\boldsymbol{x_e} = \mathbf{0}$,

if there is a scalar function $\mathrm{V}(\boldsymbol{x}, \boldsymbol{t})$, satisfying

(1) $V(x,\ t) > 0$ **(2)** $\dot{V}(x,\ t) \leq 0$ **(3)** $\dot{V}(x,\ t) \not\equiv 0$, when $x \neq \mathbf{0}$

then we could say the system is asymptotically stable at $\boldsymbol{x_e}$.

Additionally, (4) $V(x, t) \to \infty$ **,** when $\|x\| \to \infty$

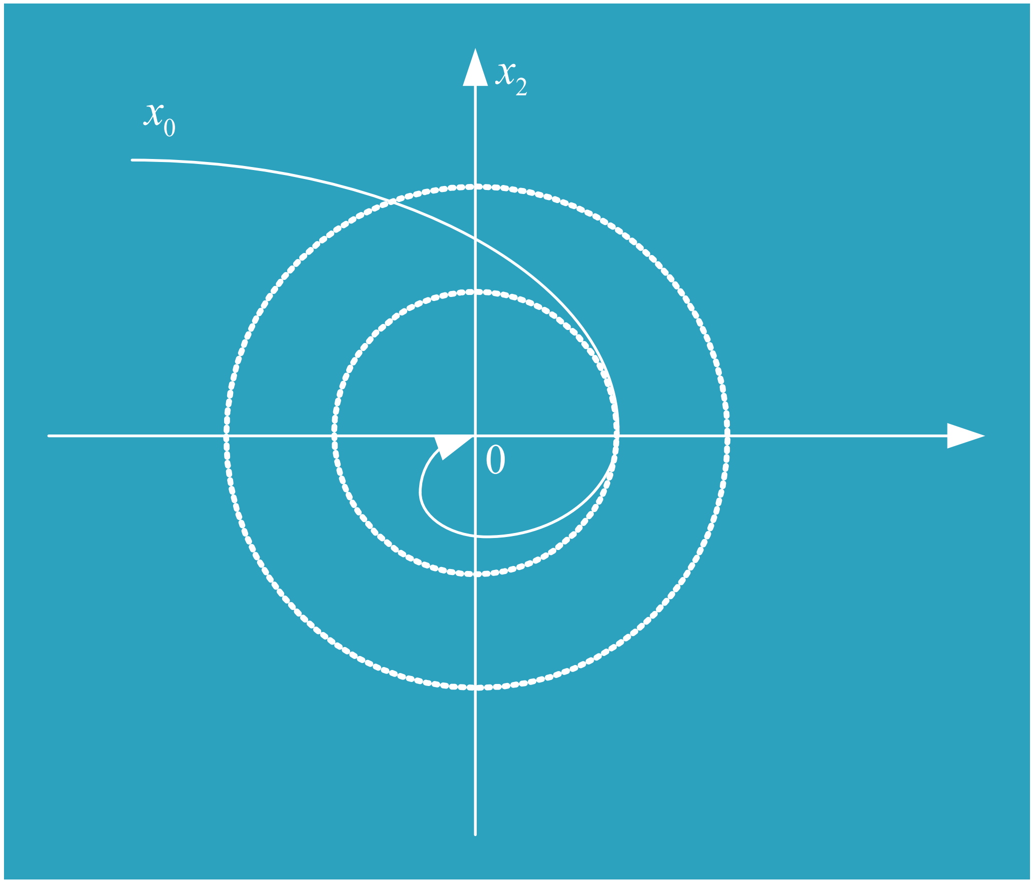
then we could say the system is asymptotically stable in a whole at $\boldsymbol{x_e}$.

Physical meaning: $\boldsymbol{V}(\boldsymbol{x},\ \boldsymbol{t}) > \mathbf{0}$ system energy is always positive.

$\dot{\boldsymbol{V}}(\boldsymbol{x},\ \boldsymbol{t}) \leq \mathbf{0}$ the energy decays with time,

$\dot{\boldsymbol{V}}(\boldsymbol{x},\ \boldsymbol{t}) \not\equiv \mathbf{0}$, when $\boldsymbol{x} \neq \mathbf{0}$ the energy remain temporarily the same at certain state but will not stop decaying.

Lyapunov Second Method



Lyapunov Second Method

Example: For state equation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$$

Try to figure out whether it is stable.

(x_1=0, x_2=0) is the only balanced state.

Choose a Lyapunov function $V(\boldsymbol{x}) = x_1^2 + x_2^2$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

$$= 2x_1x_2 - 2x_2(x_1 + x_2) = -2x_2^2$$

is negative semi-definite.

$$\dot{V}(\boldsymbol{x}, \boldsymbol{t}) \not\equiv \mathbf{0}, \quad \text{when} \quad \boldsymbol{x} \neq \mathbf{0}$$

When $\|\boldsymbol{x}\| \to \infty$ $\quad V(\boldsymbol{x}) \to \infty$

Lyapunov Second Method

Choose another Lyapunov function

$$V(\boldsymbol{x}) = \frac{1}{2}[(x_1 + x_2)^2 + 2x_1^2 + x_2^2]$$

$$\dot{V}(x) = (x_1 + x_2)(\dot{x}_1 + \dot{x}_2) + 2x_1\dot{x}_1 + x_2\dot{x}_2 = -(x_1^2 + x_2^2)$$

When $\|\boldsymbol{x}\| \to \infty$ $\quad V(\boldsymbol{x}) \to \infty$

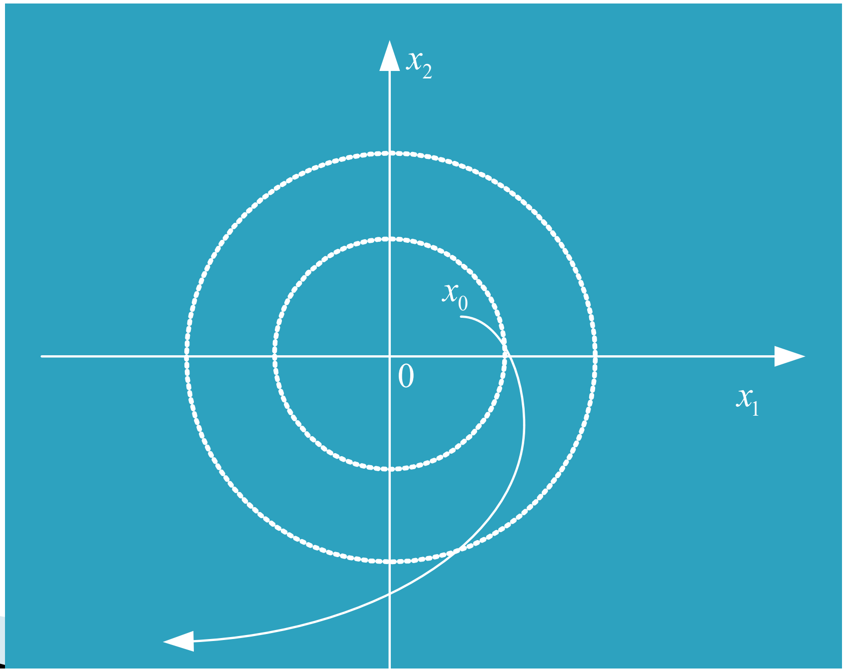
Lyapunov Second Method

3 For any system with state equation $\dot{x} = f(x, t)$ and balanced state $x_e = 0$, if there is a scalar function $V(x, t)$ satisfying

(1) $V(x,\ t) > 0$ (2) $\dot{V}(x,\ t) > 0$

then we could say the system is instable at x_e in sense of Lyapunov.

$V(x,\ t) > 0$, $\dot{V}(x, t) > 0$ **system energy keeps increasing**



Lyapunov Second Method

Example: For state equation:
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + x_2 \end{cases}$$

Try to figure out whether it is stable.

(x_1=0, x_2=0) is the only balanced state.

Choose a Lyapunov function $V(\boldsymbol{x}) = x_1^2 + x_2^2$

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
$$= 2x_1x_2 \text{-} 2x_2(x_1 \text{-} x_2) = 2x_2^2 \geq 0$$

Instable

Lyapunov Second Method

The Lyapunov stability analysis of linear system

For any linear variant system with state equation$x$ $\dot{x} = Ax$, $x(0) = x_0$, $t \geq 0$,

and A is nonsingular, choose positive quadratic function

$$\boxed{V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}}$$

If the system is asymptotically stable at x_e , according to theorem 1, make

$$\dot{V}(x, t) < 0$$

$$\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x} = x^T (A^T P + PA)x$$

$$\boxed{A^T P + PA = -Q}$$

That means Q must be positive definite.

Lyapunov Second Method

Lyapunov stability theorem of linear system

For linear variant system with state equation $\dot{x} = Ax, \ x(0) = x_0 \ , \ t \geq 0$,

and A is nonsingular, the suffecient and necessary condition that the system is asymptotically stable at x_e is

for any given positive definite matrix Q,
there is a unique symmetry positive-definite matrix P which satisfies

$$\boxed{A^T P + PA = -Q}$$

Lyapunov Second Method

Remarks:

(1) For any given positive definite matrix Q, matrix P which satisfies equation $A^T P + PA = -Q$ is unique. The system is asymptotically stable when P is positive definite.

(2) If the function $\dot{V}(x) = x^T(-Q)x \equiv 0$, then, matrix Q could be just semi-positive definite matrix. And the system is still asymptotically stable.

(3) Generally, we choose Q=I, so the equation is $A^T P + PA = -I$

Lyapunov Second Method

Example: For state equation: $$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

Try to figure out whether it is stable by Lyapunov equation.

$$A^{\mathrm{T}} P + PA = -Q = -\mathrm{I} \ ,$$

$$P = P^T = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Lyapunov Second Method

$$\begin{cases} -2P_{12} = -1 \\ -P_{22} + P_{11} - P_{12} = 0 \\ 2(P_{12} - P_{22}) = -1 \end{cases} \qquad \begin{cases} P_{12} = \dfrac{1}{2} \\ P_{22} = 1 \\ P_{11} = \dfrac{3}{2} \end{cases}$$

$$p_{11} = \frac{3}{2} > 0 \qquad \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} = \begin{bmatrix} \dfrac{3}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & 1 \end{bmatrix} > 0$$

$$V = \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} = \frac{1}{2}(3x_1^2 + 2x_1x_2 + 2x_2^2) > 0$$

$$\dot{V} = -(x_1^2 + x_2^2)$$

Lyapunov Second Method

There is MATLAB function LYAP(A,Q) that can solve the Lyapunov equation

$$A^TP+PA=-Q$$

Derive positive definite matrix P

```
%LYAP example
A=[0 1;-1 -1];
A=A'; %transpose of A
Q=[1 0;0 1];
P=lyap(A,Q)
end
```

results: P =

$$\begin{matrix} 1.5000 & 0.5000 \\ 0.5000 & 1.0000 \end{matrix}$$

Lyapunov Second Method

Example: For state equation: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x$

Try to figure out whether it is stable by Lyapunov equation.

$$A^{\mathrm{T}} P + PA = -Q = -\mathrm{I} \ ,$$

$$P = P^T = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Lyapunov Second Method

$$\begin{bmatrix} 4p_{12} & p_{11} - p_{12} + 2p_{22} \\ p_{11} - p_{12} + 2p_{22} & 2p_{12} - 2p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{cases} 4p_{12} = -1 \\ p_{11} - p_{12} + 2p_{22} = 0 \\ 2p_{12} - 2p_{22} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = -0.75 \\ p_{12} = -0.25 \\ p_{22} = \ \ 0.25 \end{cases} \Rightarrow P = \begin{bmatrix} -0.75 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}$$

$$p_{11} = -0.75 < 0, \quad \det P = -0.25 < 0$$

Instable

Lyapunov Second Method

For comparison, consider Lyapunov first method

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x$$

$$\det(sI - A) = s^2 + s - 2 = (s-1)(s+2) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -2,$$

Lyapunov Second Method