Chapter 3

Controllability and observability

Outlines

- **Problem formulation**
- **Controllability**
- **Observability**

Problem formulation

Conventional control theory deals with input-output relationship in the form of transfer function.

In other word, output is controllable if the system is stable

State Space Description:

$\dot{x} = Ax + Bu$ --- **State equation**

$y = Cx + Du$ --- **Output equation**

Input $u(t)$ controls the state $x(t)$

Output $y(t)$ is affected by the state $x($

Controllability ---- can the input control all the states

Observability ---- can the output reflect the changes of all states

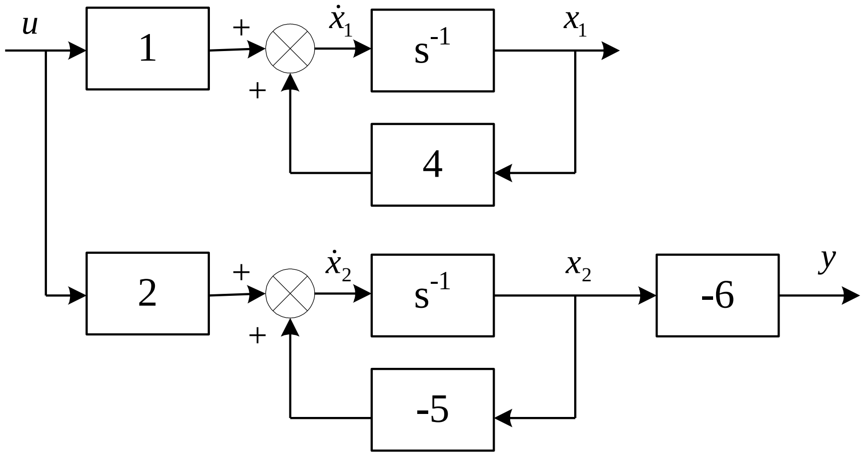
Problem formulation

Example 1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}_1 = 4x_1 + u$$
$$\dot{x}_2 = -5x_2 + 2u$$
$$y = -6x_2$$



the input u can control the state x_2 and x_1, thus state is controllable

the output y can reflect the state x_2, thus state is not totally observable

Problem formulation

Example 2

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

Controllability ---- whether the input can control all the states

Observability ---- whether the output can reflect the changes of all states

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = 2x_2 + 2u \\ y = x_1 \end{cases}$$

the input u can control the state x_2 , not x_1 , thus state x_1 is not controllable

the output y can reflect the state x_1 , thus state x_1 is observable

Problem formulation

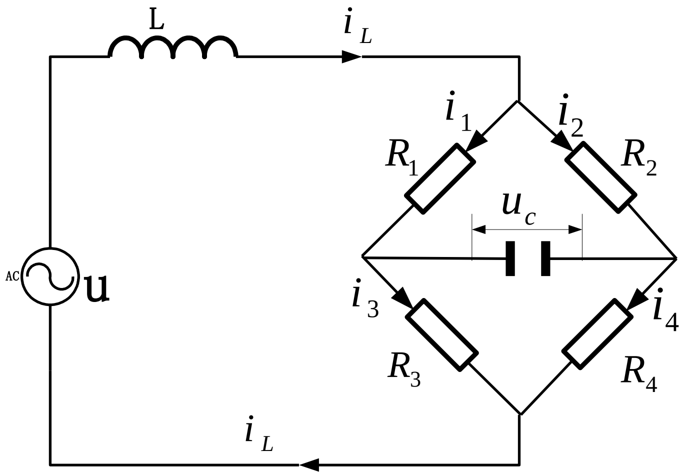
Example 3 Consider bridge-shaped circuit

$$i_L = i_1 + i_2 = i_3 + i_4$$

$$R_4 i_4 + u_c = R_3 i_3$$

$$R_1 i_1 + u_c = R_2 i_2$$

$$L\frac{di_L}{dt} + R_1 i_1 + R_3 i_3 = u$$



By selecting $\boldsymbol{x_1}$=$\boldsymbol{i_L}$, $\boldsymbol{x_2}$=$\boldsymbol{u_c}$, we have state equations

$$u_c(t) = \frac{1}{C}\int i dt \qquad \dot{x}_1 = -\frac{1}{L}\left(\frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_3+R_4}\right)x_1 + \frac{1}{L}\left(\frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4}\right)x_2 + \frac{1}{L}u$$

$$\dot{x}_2 = \frac{1}{C}\left(\frac{R_2}{R_1+R_2} - \frac{R_4}{R_3+R_4}\right)x_1 - \frac{1}{C}\left(\frac{1}{R_1+R_2} - \frac{1}{R_3+R_4}\right)x_2$$

When it is unbalanced, $\boldsymbol{R_1R_4 \neq R_2R_3}$, it is controllable and observable.

When it is balanced, $\boldsymbol{R_1R_4 = R_2R_3}$,

Controllability

Definition of controllability:

For linear system $\dot{x} = Ax + Bu$, given the initial state x_0 at t_0,

if there exists $t_f > t_0$, and input u(t) that could transit $x(t_0)$ to any state

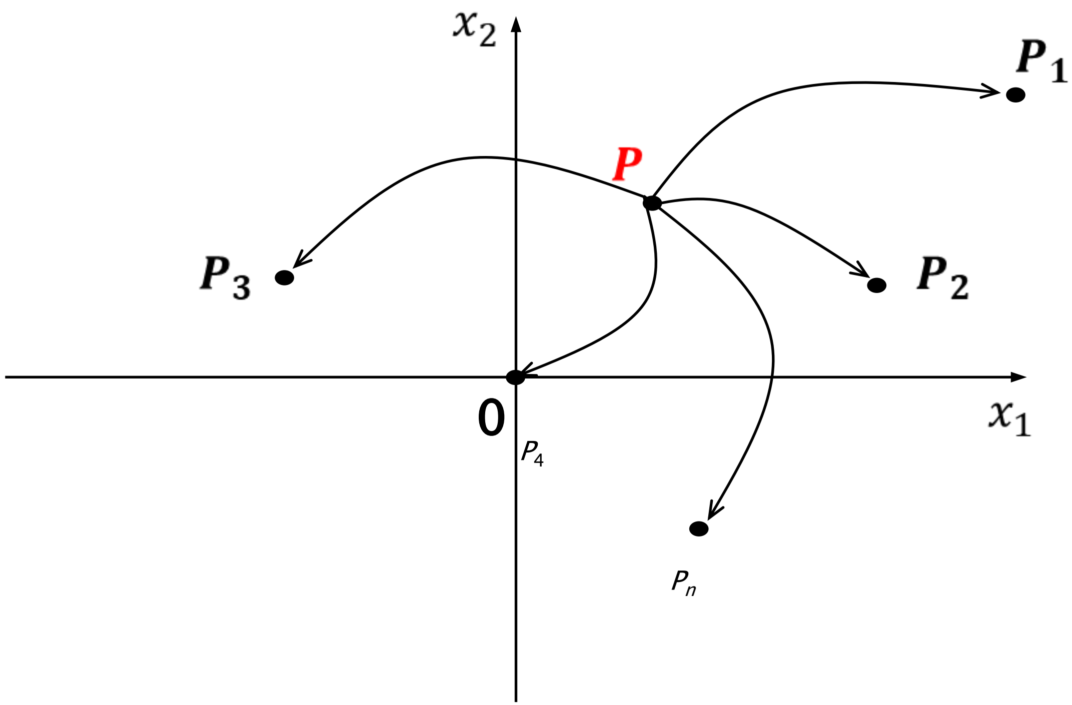
$x(t_f)$ within time $t_f - t_0$, then the system is controllable at t_0

Remarks:

1) **Input-affected state is controllable**
2) ***u*(t) satisfies unique solution condition**
3) **Definition domain is finite interval** $t_f - t_0$

Controllability

Definition of controllability:



Controllability

Caylay-Hamilton theorem:

For matrix $A \in R^n$, the Eigen polynomial is

$$f(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0$$

Then matrix $A \in R^n$ satisfies

$$\boldsymbol{f}(\boldsymbol{A}) = \boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \cdots a_1\boldsymbol{A} + a_0\boldsymbol{I} = \boldsymbol{0}$$

Proof:

$$(\lambda I - A)^{-1} = \frac{B(\lambda)}{|\lambda I - A|} = \frac{B(\lambda)}{f(\lambda)}$$

$$B(\lambda) = \lambda^{n-1}B_{n-1} + \lambda^{n-2}B_{n-2} \cdots + \lambda B_1 + B_0$$

$$B(\lambda)(\lambda I - A) = f(\lambda)I$$

Controllability

$$\lambda^n B_{n-1} + \lambda^{n-1}(B_{n-2} - B_{n-1}A) + \lambda^{n-2}(B_{n-3} - B_{n-2}A) + \cdots + \lambda(B_0 - B_1 A) - B_0 \mathrm{A}$$

$$= \lambda^n I + a_{n-1}\lambda^{n-1} I + \cdots + a_1 \lambda I + a_0 I$$

$$B_{n-1} = I$$

$$B_{n-2} - B_{n-1}A = a_{n-1}I$$

$$\vdots$$

$$B_0 - B_1 A = a_1 I$$

$$-B_0 \mathrm{A} = a_0 I$$

Multiplying above equations with A^n, A^{n-1}, …, A, respectively yields

$$B_{n-1}A^n = A^n$$

$$B_{n-2}A^{n\text{-}1} - B_{n-1}A^n = a_{n-1}A^{n\text{-}1}$$

$$\vdots$$

$$B_0 A - B_1 A^2 = a_1 A$$

$$-B_0 \mathrm{A} = a_0 I$$

$$\boldsymbol{f}(\boldsymbol{A}) = \boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \cdots a_1 \boldsymbol{A} + a_0 \boldsymbol{I} = \boldsymbol{0}$$

Controllability

Corollary 1 $A^k = \sum_{m=0}^{n-1} \alpha_m A^m \ (k \geq n)$

Proof:

$$\because A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_1 A - a_0 I$$

$$\therefore A^{n+1} = AA^n = -a_{n-1}A^n - a_{n-2}A^{n-1} - \cdots a_1 A^2 - a_0 A$$

$$= -a_{n-1}(-a_{n-1}A^{n-1} - \cdots a_1 A - a_0 I) - a_{n-2}A^{n-1} - \cdots - a_1 A^2 - a_0 A$$

$$= (a_{n-1}^2 - a_{n-2})A^{n-1} + (a_{n-1}a_{n-2} - a_{n-3})A^{n-2} + \cdots$$
$$+ (a_{n-1}a_2 - a_1)A^2 + (a_{n-1}a_1 - a_0)A + a_{n-1}a_0 I$$

Controllability

Corollary 2

$$\mathrm{e}^{\boldsymbol{A}t} = \sum_{m=0}^{n-1} \alpha_m(t)\boldsymbol{A}^m$$

Proof:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots$$

$$= I + At + \cdots + \frac{1}{n!}(-a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_0 I)t^n$$

$$+ \frac{1}{(n+1)!}[(a_{n-1}^2 - a_{n-2})A^{n-1} + (a_{n-1}a_{n-2} - a_{n-3})A^{n-2} + \cdots + a_{n-1}a_0 I]t^{n+1} + \cdots$$

Controllability

$$\alpha_0(t) = 1 - \frac{1}{n!}a_0 t^n + \frac{1}{(n+1)!}a_{n-1}a_0 t^{n+1} + \cdots$$

$$\alpha_1(t) = t - \frac{1}{n!}a_1 t^n + \frac{1}{(n+1)!}(a_{n-1}a_1 - a_0) t^{n+1} + \cdots$$

$$\alpha_2(t) = \frac{1}{2!}t^2 - \frac{1}{n!}a_2 t^n + \frac{1}{(n+1)!}(a_{n-1}a_2 - a_1) t^{n+1} + \cdots$$

$$\vdots$$

$$\alpha_{n-1}(t) = \frac{1}{(n-1)}t^{n-1} - \frac{1}{n!}a_{n-1} t^n + \frac{1}{(n+1)!}(a_{n-1}{}^2 - a_{n-2}) t^{n+1} + \cdots$$

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \cdots + \alpha_{n-1}(t)A^{n-1}$$

$$= \sum_{m=0}^{n-1} \alpha_m(t)A^m$$

Controllability

Example 4 $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, **please try to calculate A^{100}**

$$f(\lambda)=|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2\\0 & \lambda-1\end{vmatrix}=\lambda^2-2\lambda+1$$

$$f(A)=A^2-2A+I=0$$

$$A^2=2A-I$$

$$A^3=AA^2=2A^2-A=2(2A-I)-A=3A-2I$$

$$A^4=AA^3=3A^2-2A=3(2A-I)-2A=4A-3I$$

$$\vdots$$

$$A^k=kA-(k-1)I$$

$$\therefore A^{100}=100A-99I=\begin{bmatrix}100 & 200\\0 & 100\end{bmatrix}-\begin{bmatrix}99 & 0\\0 & 99\end{bmatrix}=\begin{bmatrix}1 & 200\\0 & 1\end{bmatrix}$$

Controllability

Controllability criteria

1) For any LTI continuous system with n dimension state

$$\dot{x} = Ax + Bu$$

For Controllability matrix $\boldsymbol{U_c} = [\boldsymbol{B}\ \boldsymbol{AB}\ \boldsymbol{A^2B}\ \cdots\ \boldsymbol{A^{n-1}B}]$
The necessary and sufficient condition of system being completely controllable is

$$\text{rank}\ \boldsymbol{U_c} = n$$

Controllability

Proof: $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$

$$x(t_f) = \Phi(t_f - t_o)x(t_0) + \int_{t_0}^{t_f} \Phi(t_f - \tau)Bu(\tau)\mathrm{d}\tau$$

By assuming that $x(t_f) = 0$, there hold

$$\boldsymbol{x}(t_0) = -\Phi^{-1}(t_f - t_0)\int_{t_0}^{t_f} \Phi(t_f - \tau)Bu(\tau)\mathrm{d}\tau = -\int_{t_0}^{t_f} \Phi(t_0 - \tau)Bu(\tau)\mathrm{d}\tau$$

$$\Phi(t_0 - \tau) = \mathrm{e}^{\boldsymbol{A}(t_0 - \tau)} = \sum_{m=0}^{n-1} \alpha_m(t_0 - \tau)\boldsymbol{A}^m$$

$$\boldsymbol{x}(t_0) = -\int_{t_0}^{t_f} \sum_{m=0}^{n-1} \alpha_m(t_0 - \tau)\boldsymbol{A}^m \boldsymbol{B}\boldsymbol{u}(\tau)\mathrm{d}\tau$$

$$\boldsymbol{x}(t_0) = -\sum_{m=0}^{n-1} \boldsymbol{A}^m \boldsymbol{B}\int_{t_0}^{t_f} \alpha_m(t_0 - \tau)u(\tau)\mathrm{d}\tau$$

$$= -[\boldsymbol{B}\int_{t_0}^{t_f} \alpha_0(t_0 - \tau)u(\tau)\mathrm{d}\tau + \boldsymbol{A}\boldsymbol{B}\int_{t_0}^{t_f} \alpha_1(t_0 - \tau)u(\tau)\mathrm{d}\tau$$

$$+ \cdots \boldsymbol{A}^{n-1}\boldsymbol{B}\int_{t_0}^{t_f} \alpha_{n-1}(t_0 - \tau)u(\tau)\mathrm{d}\tau$$

Controllability

$$\boldsymbol{x}(t_0) = -(\boldsymbol{B} \quad \boldsymbol{AB} \quad \cdots \quad \boldsymbol{A}^{n-1}\boldsymbol{B})\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} \qquad \operatorname{rank}(\boldsymbol{B} \quad \boldsymbol{AB} \quad \cdots \quad \boldsymbol{A}^{n-1}\boldsymbol{B}) = n$$

Example 5

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\boldsymbol{u},$$

try to determine the controllability

$$U_c = [\boldsymbol{b} \quad \boldsymbol{Ab}] = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\operatorname{rank} U_c = 1 < n$$

$\therefore$ it is uncontrollable

Controllability

Example 6　Try to determine the controllability

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$S = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \left[\begin{array}{cc:cc:cc} 2 & 1 & 3 & 2 & 5 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ -1 & -1 & -2 & -2 & -4 & -4 \end{array}\right]$$

rank$S$=2<3,

∴it is uncontrollable

Controllability

Controllability criteria

2) For any LTI continuous system with n dimension state

$$\dot{x} = Ax + Bu$$

If the system has distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$
The necessary and sufficient condition of system being completely controllable is

$\bar{\boldsymbol{B}}$ **does not contain row with All 0 element in diagonal canonical form obtained by equivalent transform**

$$x = P\bar{x} \Rightarrow \begin{cases} \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \\ \bar{B} = P^{-1}B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \boxed{b_i \neq 0} \end{cases}$$

Controllability

Example 7　Try to determine the controllability

（1）
$$\dot{x}=\begin{bmatrix}-7&0&0\\0&-5&0\\0&0&-1\end{bmatrix}x+\begin{bmatrix}2\\5\\7\end{bmatrix}u$$

（2）
$$\dot{x}=\begin{bmatrix}-7&0&0\\0&-5&0\\0&0&-1\end{bmatrix}x+\begin{bmatrix}0\\5\\7\end{bmatrix}u$$

（3）
$$\dot{x}=\begin{bmatrix}-7&0&0\\0&-5&0\\0&0&-1\end{bmatrix}x+\begin{bmatrix}0&1\\4&0\\7&5\end{bmatrix}u$$

（4）
$$\dot{x}=\begin{bmatrix}-7&0&0\\0&-5&0\\0&0&-1\end{bmatrix}x+\begin{bmatrix}0&1\\0&0\\7&5\end{bmatrix}u$$

(2) (4) are uncontrollable

Controllability

Controllability criteria

3) For any LTI continuous system with n dimension state

$$\dot{x} = Ax + Bu$$

If the system has repeat eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_m$
And only a Jordan block corresponding to each repeated eigenvalue, the necessary and sufficient condition of system being Completely controllable is

The elements of all these rows in matrix B which are corresponding to the last row of every Jordan block J are not all 0.

Controllability

The elements of all these rows in matrix B which are corresponding to the last row of every Jordan block J are not all 0.

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_{m+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}, \qquad \bar{\mathrm{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \\ b_{m+1} \\ \vdots \\ b_n \end{bmatrix}$$

Controllability

Example 8　Try to determine the controllability

(1)
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

(2)
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

(3)
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 1 & & \\ 0 & -4 & \multicolumn{2}{c}{0} \\ & & -3 & 1 \\ \multicolumn{2}{c}{0} & 0 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \boldsymbol{u}$$

(4)
$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 1 & & \\ 0 & -4 & \multicolumn{2}{c}{0} \\ & & -3 & 1 \\ \multicolumn{2}{c}{0} & 0 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{u}$$

(2) (4) are uncontrollable

Controllability

Controllability criteria

Output controllable

For linear system $\dot{x} = Ax + Bu$, there exists input u(t) that could transit any given $y(t_0)$ to $y(t_f)$ within finite time interval $t_f - t_0$, then the system is output controllable.

For any LTI continuous system with *m* dimension output

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

The necessary and sufficient condition of system being completely controllable is

$$\operatorname{rank}\begin{bmatrix} CB & CAB & \dots & CA^{n-1}B & D \end{bmatrix} = m$$

Controllability

Proof: $\dot{x} = Ax + Bu, \quad x(t_0) = x_0, \quad t \in [t_0, t_1]$

$$y = Cx + Du$$

$$x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau), t \in [t_0, t_1]$$

$$y(t_1) = Ce^{At_1}x(0) + C\int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau) + Du(t_1)$$

Without loss of generality, set $y(t_1)=0$， and apply Caylay-Hamilton theorem,

Controllability

$$Ce^{At_1}x(0) = -C\int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau - Du(t_1)$$

$$= -C\int_0^{t_1} (\sum_{m=0}^{n-1} a_m(\tau)A^m)Bu(\tau)d\tau - Du(t_1)$$

$$= -C\sum_{m=0}^{n-1} A^m B[\int_0^{t_1} a_m(\tau)u(\tau)d\tau - Du(t_1)]$$

set $\quad u_m(t_1) = \int_0^{t_1} a_m(\tau)u(\tau)d\tau$

$$Ce^{At_1}x(0) = -C\sum_{m=0}^{n-1} A^m Bu_m(t_1) - Du(t_1)$$

$$= -CBu_0(t_1) - CABu_1(t_1) - CA^2Bu_2(t_1) - \cdots - CA^{n-1}Bu_{n-1}(t_1) - Du(t_1)$$

$$= -\begin{bmatrix} CB & CAB & CA^2B & \cdots & CA^{n-1}B & D \end{bmatrix} \begin{bmatrix} u_0(t_1) \\ u_1(t_1) \\ \vdots \\ u_{n-1}(t_1) \\ u(t_1) \end{bmatrix}$$

Controllability

Example 9 Try to determine the controllability of the state and output

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

① state controllable matrix S

$$S = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

det[S]=0， rank S < 2 **The state is uncontrollable**

② Output controllability matrix S_0

$$S_0 = \begin{bmatrix} cb & cAb & d \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

rank $S_o = 1 = m$

The output is controllable

Controllability

Controllability criteria: PBH criteria

For any LTI continuous system with *m* dimension output

$$\dot{x} = Ax + Bu$$

The necessary and sufficient condition of system being completely controllable is

$$\operatorname{rank}\left[\lambda_i I - A \ B\right] = n, \quad i = 1, 2, \cdots, n$$

or

$$\operatorname{rank}\left[sI - A \quad B\right] = n$$

Controllability

Example 10 **Try to determine the controllability of the system**

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ -2 & 0 \end{bmatrix} u, \quad n = 4$$

$$\begin{bmatrix} sI - A & B \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & 0 & 0 & 1 \\ 0 & s & 1 & 0 & 1 & 0 \\ 0 & 0 & s & -1 & 0 & 1 \\ 0 & 0 & -5 & s & -2 & 0 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 0, \lambda_3 = \sqrt{5}, \lambda_4 = -\sqrt{5},$$

When $\lambda_1 = \lambda_2 = 0$

$$rank\begin{bmatrix} sI - A & B \end{bmatrix} = rank \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -5 & 0 & -2 \end{bmatrix} = 4$$

Controllable

Observability

Definition of observability:

For linear system $\dot{x} = Ax + Bu$, given $t_f > t_0$, if the initial state $x(t_0)$ could be uniquely determined according to the measured output y(t) of (t_f, t_0) then the system is observable.

Remarks:

1) **Output reflected state is observable**
2) **Only the system free motion is considered when studying observability**

Observability

Observability criteria:

1) **For linear system** $\dot{x} = Ax + Bu, \mathbf{y} = \boldsymbol{C}x + Du$

the necessary and sufficient condition of system being completely observable is

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Observability

Example 11 **Try to determine the observability of the system**

(1) $$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{x}$$

(2) $$\dot{\boldsymbol{x}} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \boldsymbol{x}$$

(1) $$\boldsymbol{S}_o = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{cA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -5 & 5 \end{bmatrix}$$ $$\mathrm{rank}\boldsymbol{S}_o = 1 < 2$$ **Not observable**

(2) $$\boldsymbol{S}_o = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{cA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 2 & -1 \\ -2 & 1 \end{bmatrix}$$ $$\mathrm{rank}\boldsymbol{S}_o = 2$$ **Observable**

Observability

Observability criteria:

2) For linear system $\dot{x} = Ax + Bu, \mathbf{y} = \boldsymbol{C}x + Du$

If the system has distinct eigenvalues, the necessary and sufficient condition of system being completely observable is
$\overline{\boldsymbol{C}}$ does not contain column with all 0 element in diagonal canonical form obtained by equivalent transform

$$x = P\bar{x} \Rightarrow \begin{cases} \bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \\ \bar{C} = CP = [c_1 \quad c_2 \quad \cdots \quad c_m] \; c_i \neq 0 \end{cases}$$

Observability

Example 12 Try to determine the observability of the system

（1）

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & 3 & 2 \end{bmatrix} \boldsymbol{x}$$

Observable

（2）

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & 3 & 0 \end{bmatrix} \boldsymbol{x}$$

Not observable

Duality principle

For linear system

$$S_1 : \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \qquad S_2 : \dot{\boldsymbol{z}} = \boldsymbol{A}^*\boldsymbol{z} + \boldsymbol{B}^*\boldsymbol{v}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}; \qquad \boldsymbol{w} = \boldsymbol{C}^*\boldsymbol{z}$$

If

$$\boxed{\boldsymbol{A}^* = \boldsymbol{A}^T, \boldsymbol{B}^* = \boldsymbol{C}^T, \boldsymbol{C}^* = \boldsymbol{B}^T}$$

System S_1 and S_2 are called dual systems

Duality principle

Controllable Canonical Form

$$\dot{x} = A_c x + B_c u$$
$$y = C_c x$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1} \quad]$$

Observable Canonical Form

$$\dot{x} = A_o x + B_o u$$
$$y = C_o x$$

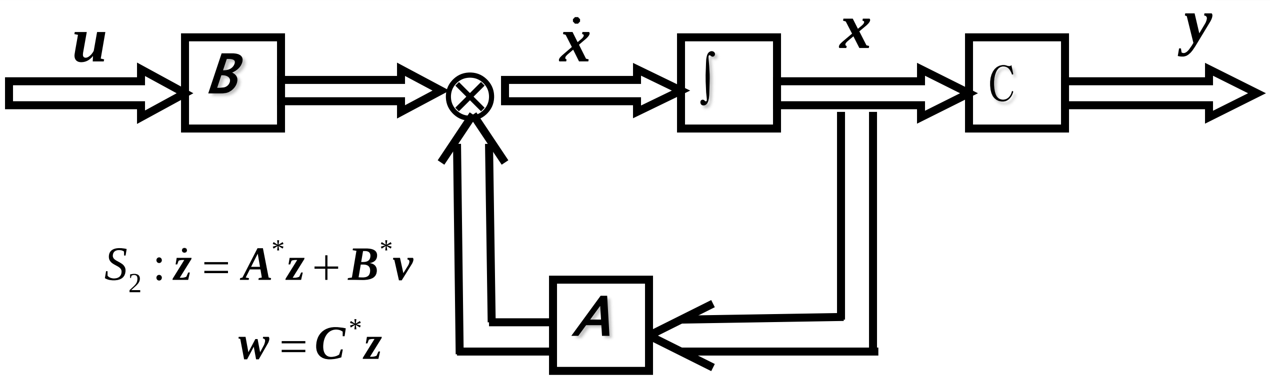
$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}, B_o = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{bmatrix}$$

$$C_o = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

$$\boldsymbol{A_c = A_o^T, B_c = C_o^T, C_c = B_o^T}$$

Duality principle

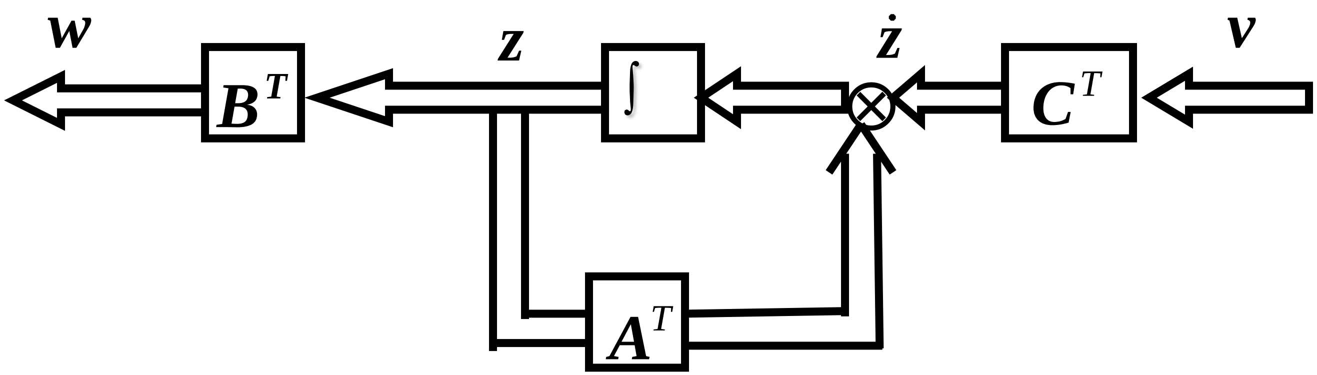
$S1:$



$$S_1 : \dot{x} = Ax + Bu, \qquad S_2 : \dot{z} = A^* z + B^* v$$

$$y = Cx; \qquad w = C^* z$$

$S2:$



The system S1 is completely controllable (observable) if its dual system S2 is completely observable (controllable).

Transfer function matrix of S1 is the transpose of TFM S2.

$$[\, G_2(s)]^T = G_1(s)$$

Structure decomposition

If the LTI system is not completely controllable or observable,

$$\textbf{rank}\left[B\ AB\ A^2B\ \cdots\ A^{n-1}B\right] < n \qquad \textbf{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} < n$$

we could sort the state variable as

$$x = \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}$$

x_{co}: **controllable and observable**
$x_{c\bar{o}}$: **controllable but unobservable**
$x_{\bar{c}o}$: **uncontrollable but observable**
$x_{\bar{c}\bar{o}}$: **uncontrollable and unobservable**

called system structure decomposition.

Structure decomposition

Controllability structure decomposition

if n-dimension system (A, B, C) is not completely controllable

$$\mathrm{rank}\left[\boldsymbol{B}\ \boldsymbol{AB}\ \boldsymbol{A}^2\boldsymbol{B}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{B}\right] = \boldsymbol{r} < \boldsymbol{n}$$

there exists a nonsingular linear transform $\boldsymbol{x} = \boldsymbol{P}^{-1}\overline{\boldsymbol{x}}$, making the system to be

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{12} \\ \hline 0 & \bar{A}_{22} \end{array}\right] \left[\begin{array}{c} \bar{x}_1(t) \\ \hline \bar{x}_2(t) \end{array}\right] + \left[\begin{array}{c} \bar{B}_1 \\ \hline 0 \end{array}\right] u(t)$$

$$y(t) = \left[\begin{array}{c|c} \bar{C}_1 & \bar{C}_2 \end{array}\right] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}$$

The *r*-dimension subsystem is completely controllable

$$\dot{\bar{x}}_1(t) = \bar{A}_{11}\bar{x}_1(t) + \bar{A}_{12}\bar{x}_2(t) + \bar{B}_1 u(t)$$
$$y_1(t) = \bar{C}_1\bar{x}_1(t)$$

The $(\boldsymbol{n}-\boldsymbol{r})$-dimension subsystem is completely uncontrollable

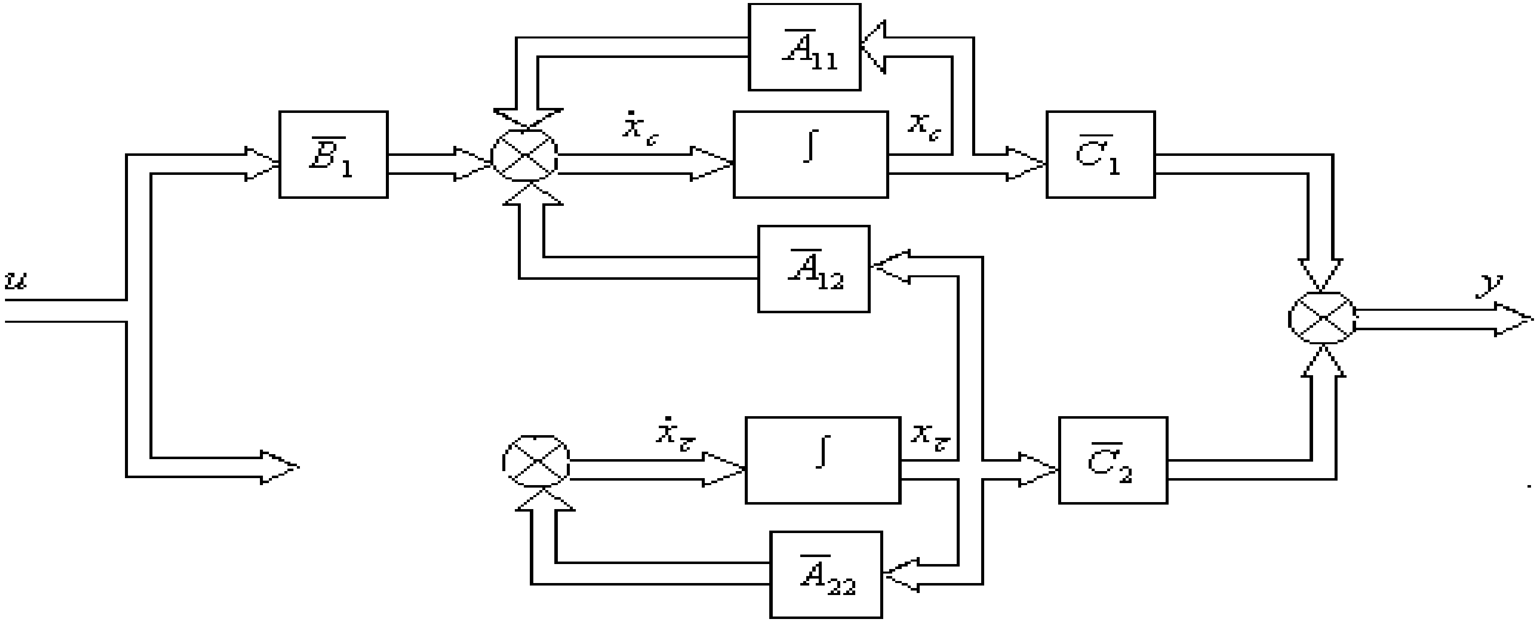
$$\dot{\bar{x}}_2(t) = \bar{A}_{22}\bar{x}_2(t)$$
$$y_2(t) = \bar{C}_2\bar{x}_2(t)$$

Structure decomposition

Controllability structure decomposition

$$\dot{\bar{x}}_1(t) = \bar{A}_{11}\bar{x}_1(t) + \bar{A}_{12}\bar{x}_2(t) + \bar{B}_1 u(t)$$
$$y_1(t) = \bar{C}_1\bar{x}_1(t)$$

$$\dot{\bar{x}}_2(t) = \bar{A}_{22}\bar{x}_2(t)$$
$$y_2(t) = \bar{C}_2\bar{x}_2(t)$$



Structure decomposition

Observability structure decomposition

if n−dimension system (A, B, C) is not completely observable

$$\mathbf{rank}\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA}^2 \\ \vdots \\ \boldsymbol{CA}^{n-1} \end{bmatrix} = \boldsymbol{l} < \boldsymbol{n}$$

there exists a nonsingular linear transform $\boldsymbol{x} = \boldsymbol{P}^{-1}\overline{\boldsymbol{x}}$, making the system to be

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \left[\begin{array}{c:c} \bar{A}_{11} & 0 \\ \hdashline \bar{A}_{21} & \bar{A}_{22} \end{array}\right] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \left[\begin{array}{c} \bar{B}_1 \\ \hdashline \bar{B}_2 \end{array}\right] u(t)$$

$$y(t) = \left[\begin{array}{c:c} \bar{C}_1 & 0 \end{array}\right] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}$$