
# Chapter 6
# Optimal Control

# Outlines

- **Basic concepts**
- **Problem formulation**
- **Linear quadratic optimal control**

# Basic concepts

**Optimal control is the key concept in Modern Control Theory**

**Given state equations and certain constraints, try to find the optimal control law so that the system has optimal performance.**

# Problem formulation

## Four important factors

### (1) General form of state equation

$$\dot{x} = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}), \boldsymbol{t}), x\ (t_0) = x_0$$

**(2) Target set: The state can be some point set S.**

$$S = \{x_{tf} | \Psi[x(t_f)] = 0\}$$

**(3) Control u is bounded, that is** $u \in \Omega$

**(4) Performance criteria:** $J = \varphi\left[\boldsymbol{x}\left(t_f\right)\right] + \int_{t_0}^{t_f} L\left[\boldsymbol{x}(t), u(t), t\right] \mathrm{d}t$

**Restriction on final state**

**Restriction on state and control**

# Problem formulation

$$J=\varphi\left[\boldsymbol{x}\left(t_f\right)\right]+\int_{t_0}^{t_f} L\left[\boldsymbol{x}(t),u(t),t\right]\mathrm{d}t$$

**(1) If $\varphi[x(t_f)]$=0, we have Integral performance**

$$J=\int_{t_0}^{t_f} L\left[\boldsymbol{x}(t),u(t),t\right]\mathrm{d}t$$

**(2) If $L[x(t),u(t),t]$=0, we have end performance**

$$J=\varphi[x(t_f)]$$

**(3) Otherwise, it has the general form as shown above.**

# Linear quadratic optimal control

**Suppose the dynamic equation** $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t)$$

**where** $\boldsymbol{x}(t) \in R^n, \boldsymbol{u}(t) \in R^p, \boldsymbol{y}(t) \in R^q$

**The functional performance is**

$$J = \frac{1}{2}\boldsymbol{x}^T(t_f)\boldsymbol{F}\boldsymbol{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{x}^T(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\right]\mathrm{d}t$$

$$\boldsymbol{F} \in \boldsymbol{R}^{n\times n} \geq 0, F = F^T, Q \in \boldsymbol{R}^{n\times n} \geq 0, Q = Q^T, R \in \boldsymbol{R}^{r\times r} > 0, R = R^T,$$

**Try to find** $\boldsymbol{u}(t) \Rightarrow J_{\min}$

# Linear quadratic optimal control

$$J = \frac{1}{2}\boldsymbol{x}^T\left(t_f\right)\boldsymbol{F}\boldsymbol{x}\left(t_f\right) + \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{x}^T\left(t\right)\boldsymbol{Q}\boldsymbol{x}\left(t\right) + \boldsymbol{u}^T\left(t\right)\boldsymbol{R}\boldsymbol{u}\left(t\right)\right]\mathrm{d}t$$

**The meanings of each term:**

$\frac{1}{2}\boldsymbol{x}^T\left(t_f\right)\boldsymbol{F}\boldsymbol{x}\left(t_f\right)$ — **It is related to final state, and each state is controlled by F**

$\frac{1}{2}\boldsymbol{x}^T\left(t\right)\boldsymbol{Q}\boldsymbol{x}\left(t\right)$ — **It is related to the process, and each state is controlled by Q**

$\frac{1}{2}\boldsymbol{u}^T\left(t\right)\boldsymbol{R}\boldsymbol{u}\left(t\right)$ — **It is related to control that minimize the energy consumption**

# Linear quadratic optimal control

**State regulator:**

$$J = \frac{1}{2}\boldsymbol{x}^T(t_f)\boldsymbol{F}\boldsymbol{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{x}^T(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\right]\mathrm{d}t$$

**Output regulator:**

$$J = \frac{1}{2}\boldsymbol{y}^T(t_f)\boldsymbol{F}\boldsymbol{y}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{y}^T(t)\boldsymbol{Q}\boldsymbol{y}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\right]\mathrm{d}t$$

**Output tracking regulator**

$$J = \frac{1}{2}\boldsymbol{e}^T(t_f)\boldsymbol{F}\boldsymbol{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{e}^T(t)\boldsymbol{Q}\boldsymbol{e}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\right]\mathrm{d}t$$

# Linear quadratic optimal control

## (1) When $t_f$ is fixed

**Linear time-varying state space model**
$$\dot{x}(\mathrm{t}) = \boldsymbol{A}(\boldsymbol{t})x(t) + B(t)u(t), \quad x(t_0) = x_0$$
$$y(t) = C(t)x(\mathrm{t})$$

**The functional performance** $$J = \frac{1}{2}\boldsymbol{x}^T(t_f)\boldsymbol{F}\boldsymbol{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{x}^T(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\right]\mathrm{d}t$$

**where** $\boldsymbol{Q}(t) \geq 0,\ \boldsymbol{R}(t) > 0,\ \boldsymbol{F} \geq 0$

**Try to find** $\boldsymbol{u}^*$ $\quad J \to \min$

**There exists the optimal control law** $$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{P}(t)\boldsymbol{x}(t)$$

Optimal performance $$J^* = \frac{1}{2}\boldsymbol{x}^T(t_0)\boldsymbol{P}(t_0)\boldsymbol{x}(t_0)$$

where $\boldsymbol{P}(t)$ satisfies Ricatti equation

$$-\dot{\boldsymbol{P}}(t) = \boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{A}^T(t)\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}(t)\boldsymbol{R}^{-1}\boldsymbol{B}^T(t)\boldsymbol{P}(t) + \boldsymbol{Q}(t)$$

# Linear quadratic optimal control

**Try to find** $\boldsymbol{u}^*$ $\quad J \to \min$

**There exists the optimal control law** $\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{P}(t)\boldsymbol{x}(t)$

Optimal performance $\quad J^* = \frac{1}{2}\boldsymbol{x}^T(t_0)\boldsymbol{P}(t_0)\boldsymbol{x}(t_0)$

where $\boldsymbol{P}(t)$ satisfies Ricatti equation

$$-\dot{\boldsymbol{P}}(t) = \boldsymbol{P}(t)\boldsymbol{A}(t) + \boldsymbol{A}^T(t)\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{B}(t)\boldsymbol{R}^{-1}\boldsymbol{B}^T(t)\boldsymbol{P}(t) + \boldsymbol{Q}(t)$$

Boundary condition $\quad \boldsymbol{P}(t_f) = \boldsymbol{F}$

Optimal trajectory $\boldsymbol{x}^*(t)$

$$\dot{\boldsymbol{x}}(t) = \left[\boldsymbol{A}(t) - \boldsymbol{B}(t)\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{P}(t)\right]\boldsymbol{x}(t)$$

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# Linear quadratic optimal control

**(2) When $t_f$ is infinite ($t_f \to \infty$)**

**Linear Time-Invariant state space model** $\dot{x}(t) = \boldsymbol{A}x(t) + Bu(t),\quad x(t_0) = x_0$
$y(t) = Cx(t)$

**The functional performance** $J = \frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{x}^T(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)\right]\mathrm{d}t$

**where** $\boldsymbol{Q} \geq 0,\ \boldsymbol{R} > 0,$ **Try to find** $\boldsymbol{u}^*$ $J \to \min$

**If (A,B) controllable, and (A,D) observable ($DD^T = Q$),**

**then there exists the optimal control law** $\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}(t)$

Optimal performance $J^* = \frac{1}{2}\boldsymbol{x}^T(0)\boldsymbol{P}\boldsymbol{x}(0)$

where $\boldsymbol{P}$ satisfies Ricatti equation

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} - P\boldsymbol{B}\boldsymbol{r}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = 0$$

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}(t)$$

# Linear quadratic optimal control

**Example: (P567)** **Consider a system** $\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Try to find the optimal control** $u^*$ **that can achieve optimal performance**

$$J = \int_0^\infty \left[ x_2^2 + \frac{1}{4} u^2 \right] dt$$

**The final time is infinite** $J = \int_0^\infty \left[ x_2^2 + \frac{1}{4} u^2 \right] dt = \frac{1}{2} \int_0^\infty \left[ [x_1, x_2] \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} u^2 \right] dt$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \qquad r = \frac{1}{2} \qquad DD^T = Q = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \qquad D = [0 \ \ \sqrt{2}]$$

**Check (A,B) controllable, and (A,D) observable.**

**then there exists the optimal control law to make the closed-loop system stable.**

# Pole placement for LTI system

$$\boldsymbol{P}=\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

Get $\boldsymbol{P}$ from Ricatti equation

$$\boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P}-P\boldsymbol{B}\boldsymbol{r}^{-1}\boldsymbol{B}^T\boldsymbol{P}+\boldsymbol{Q}=0$$

$$P_{12}\text{-}P_{11}{}^2=0$$
$$P_{22}\text{-}2P_{11}P_{12}=0$$
$$\text{-}2P_{12}{}^2+2=0$$

$$\Rightarrow \boldsymbol{P}=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}>0$$

$$\boldsymbol{u}^*(t)=-r^{-1}\boldsymbol{B}^T\boldsymbol{Px}(t)=-2x_1(t)-2x_2(t)$$

Optimal performance

$$J^*=\frac{1}{2}\boldsymbol{x}^T(0)\boldsymbol{Px}(0)=1$$

# Linear quadratic optimal control

In MATLAB, there are some functions that can solve linear quadratic problem: lqr()、 lqr2()、 lqry()

$$J=\frac{1}{2}\int_{t_0}^{t_f}\left[\boldsymbol{x}^T(t)\boldsymbol{Q}\boldsymbol{x}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)+2\boldsymbol{x}^T(t)\boldsymbol{N}\boldsymbol{u}(t)\right]\mathrm{d}t$$

$$\left[K,S,E\right]=lqr(A,B,Q,R,N)$$

***K*** **is the feedback gain, that u = -Kx ；**
***S*** **the solution P of the associated algebraic Riccati equation;**
***E*** **is the closed-loop eigenvalues, E = EIG(A-B*K).**

$$J=\frac{1}{2}\int_{t_0}^{t_f}\left[y^T(t)\boldsymbol{Q}\boldsymbol{y}(t)+\boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)+2y^T(t)\boldsymbol{N}\boldsymbol{u}(t)\right]\mathrm{d}t$$

$$\left[K,S,E\right]=lqry(A,B,Q,R,N)$$

**linear-quadratic (LQ) state-feedback regulator with output weighting**

# Linear quadratic optimal control

**Example: Try to find the optimal control law by LQR and LQRY**

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

**(1) lqr()**

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt \qquad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R = 1$$

**(2) lqry()**

$$J = \frac{1}{2}\int_0^\infty (y^T Q y + u^T R u)dt, Q = R = 1$$

# Linear quadratic optimal control

**(1) lqr()** $J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R = 1$$

```
A=[0,1,0;0,0,1;-1,-4,-6];
B=[0,0,1]';C=[1,0,0];D=0;
Q=diag([1,1,1]);
R=1;
K=lqr(A,B,Q,R)
Ac=A-B*K;Bc=B;Cc=C;Dc=D;
Step(Ac,Bc,Cc,Dc)
```

ans：

K =0.4142    0.7486    0.2046

# Linear quadratic optimal control

**(2) lqry()**

$$J = \frac{1}{2}\int_0^{\infty}(y^T Q y + u^T R u)dt, Q = R = 1$$

```
A=[0,1,0;0,0,1;-1,-4,-6];
  B=[0,0,1]';C=[1,0,0];D=0;
  Q=1;
  R=1;
  K=lqry(A,B,C,D,Q,R)
 Ac=A-B*K;Bc=B;Cc=C;Dc=D;
  Step(Ac,Bc,Cc,Dc)
```