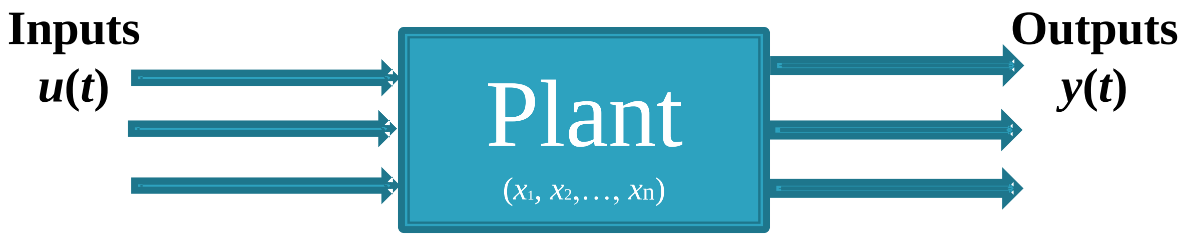
# Chapter 1

## State Space Description of Linear Control System

# Outlines

- **Basic concepts of system models**
- **State-space representations**
- **Linear transform of state-space model**
- **Transfer Function Matrix**
- **Representations and solutions of Discrete linear state-space model**

# Basic concepts of system models




- **Input variable** $\boldsymbol{u} = [u_1, u_2, \cdots u_p]^T$
- **Output variable** $\boldsymbol{y} = [y_1, y_2, \cdots y_q]^T$
- **State variable** $\boldsymbol{x} = [x_1, x_2, \cdots x_n]^T$

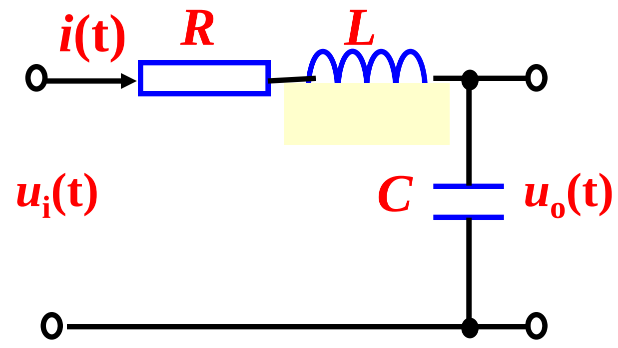
**1.Input-output description**

**2.State-space description**

**The state of a system is a set of variables whose values, together with the input signals and the equations describing the dynamics, will provide the future state and output of the system.**

# Transfer function $\dfrac{U_c(s)}{U_r(s)}$ of RC circuit

$$i = C\frac{\mathrm{d}u_o(t)}{\mathrm{d}t} \quad \text{or} \quad u_o(t) = \frac{1}{C}\int i\mathrm{d}t$$




**1).Differential Equation:**

$$LC\frac{d^2u_o(t)}{dt^2} + RC\frac{du_o(t)}{dt} + u_o(t) = u_i(t)$$

**2).Laplace transform with initial value zero：**

$$LCs^2U_o(s) + RCsU_o(s) + U_o(s) = U_i(s)$$

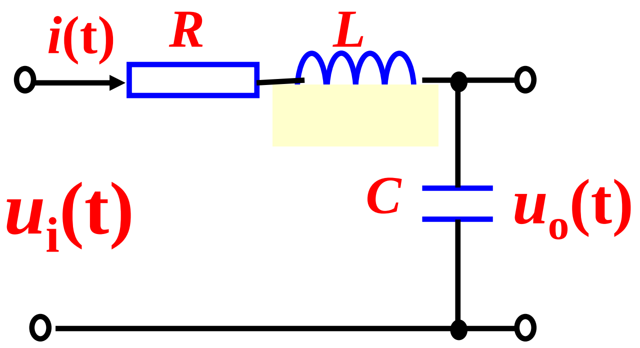
**3).Transfer function:**

$$G(s) = \frac{U_c(s)}{U_r(s)} = \frac{1}{LC\ s^2 + RCs + 1}$$

# State-space model

- **State Variable Description**

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt = u_i$$




$$u_o = \frac{1}{C}\int i dt$$

**set** $x_1 = i, \quad x_2 = u_o(t), \qquad y = x_2$

$$\dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}$$

# State-space model

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}$$

**State Variable: describe the present configuration of a system and can be used to determine the future response, given the excitation inputs and the equations describing the dynamics.**

A minimum set of variables to describe the system's behavior in time domain

**State Space:** State space is defined n-dimensional space in which the state variables represent its coordinate axes

**State Space Description:**

$\dot{x} = Ax + Bu$ --- **State equation**

$y = Cx + Du$ --- **Output equation**

# State-space model

**set** $x_1 = i,\ x_2 = u_o, y = x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \longrightarrow$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**State Space Description:**

$\dot{x} = Ax + Bu$ --- **State equation**

$y = Cx + Du$ --- **Output equation**

$$G(s) = \frac{U_c(s)}{U_r(s)} = \frac{1}{LC_2 s^2 + RCs + 1}$$

$A - n \times n$ **system matrix**

$B - n \times r$ **control matrix**

$C - m \times n$ **output matrix**

$D - m \times r$ **Direct transmission matrix**

$u \in R^r$ **r dimesion input vector**

$y \in R^m$ **m dimesion output vector**

# State-space model

**Linear continuous system**

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\ \boldsymbol{x}(t) + B(t)\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C(t)\ \boldsymbol{x}(t) + D(t)\ \boldsymbol{u}(t) \end{cases}$$

**Linear discrete system**

$$\begin{cases} \boldsymbol{x}(k+1) = G(k)\ \boldsymbol{x}(k) + H(k)\ \boldsymbol{u}(k) \\ \boldsymbol{y}(k) = C(k)\ \boldsymbol{x}(k) + D(k)\ \boldsymbol{u}(k) \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\ \boldsymbol{x}(t) + B\ \boldsymbol{u}(t) \\ \boldsymbol{y}(t) = C\ \boldsymbol{x}(t) + D\ \boldsymbol{u}(t) \end{cases}$$

$$\begin{cases} \boldsymbol{x}(k+1) = G\boldsymbol{x}(k) + H\boldsymbol{u}(k) \\ \boldsymbol{y}(k) = C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{cases}$$

$A - n \times n$ **system matrix**

$B - n \times r$ **control matrix**

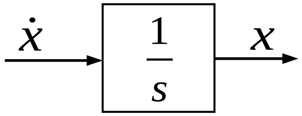
$C - m \times n$ **output matrix**

$D - m \times r$ **Direct transmission matrix**

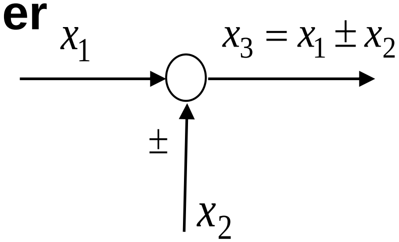
# State block diagram

## Three basic units

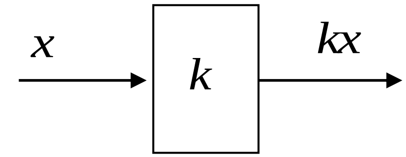
**（1）Integrator**



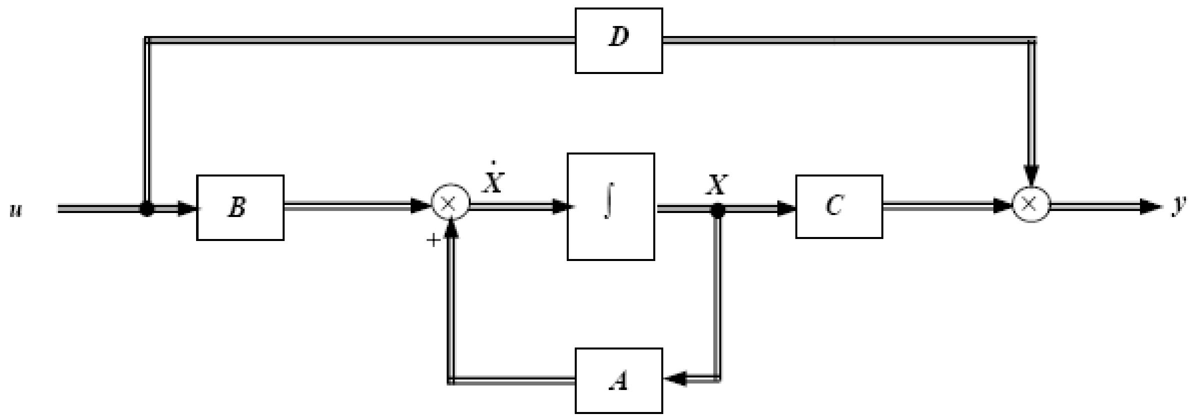

**（2）Adder**




**（3）Scaler**




$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

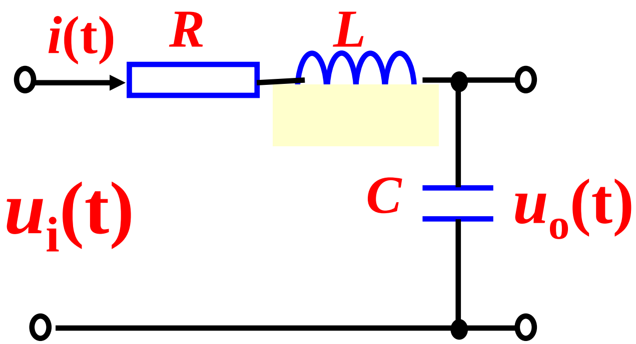



**Remark: *D* goes outside the states, and is considered as one of outer effect.**

# Example:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \boldsymbol{u}$$

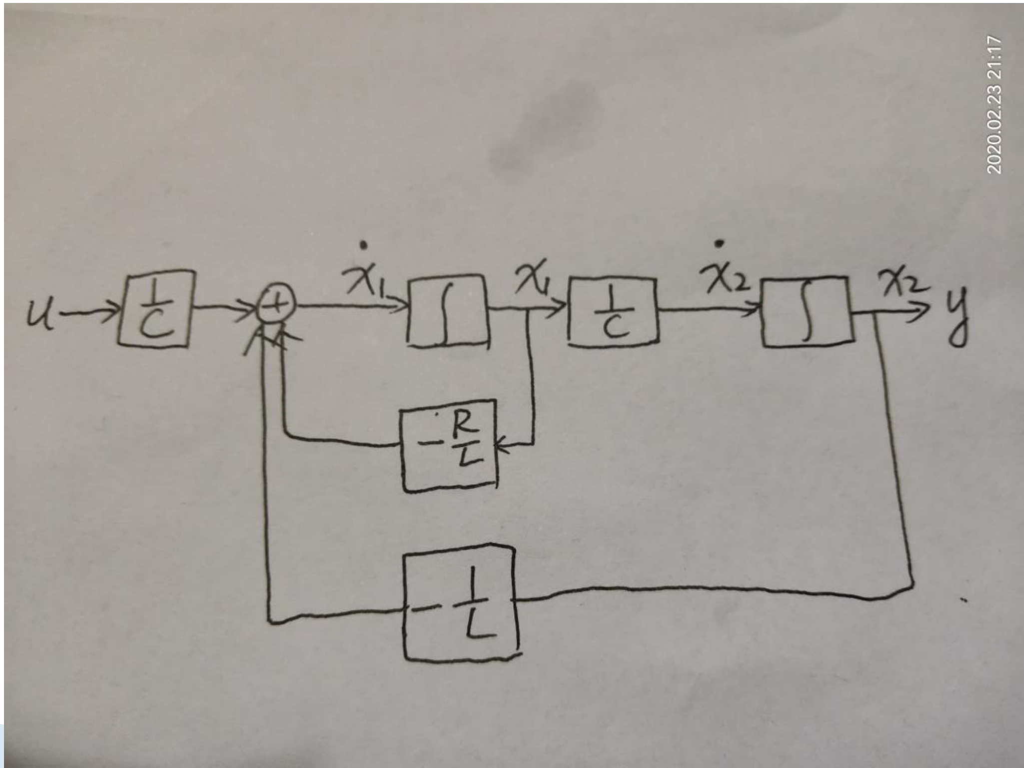
$$\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}$$




$$\dot{x}_1 = -\frac{R}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u,$$

$$\dot{x}_2 = \frac{1}{C} x_1$$

$$y = x_2$$

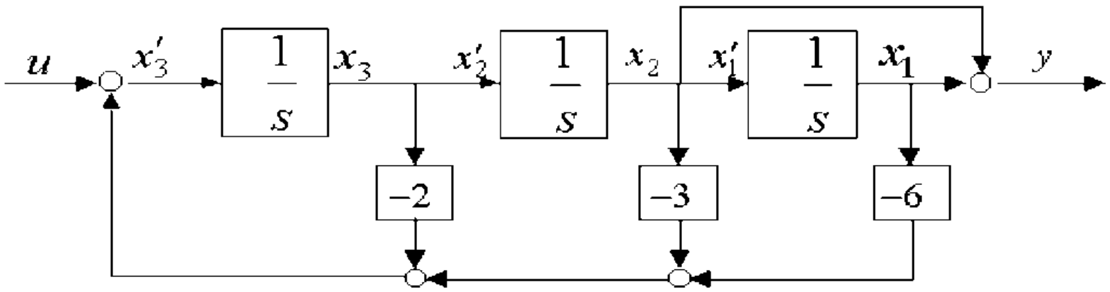


# State block diagram

**Case 1**

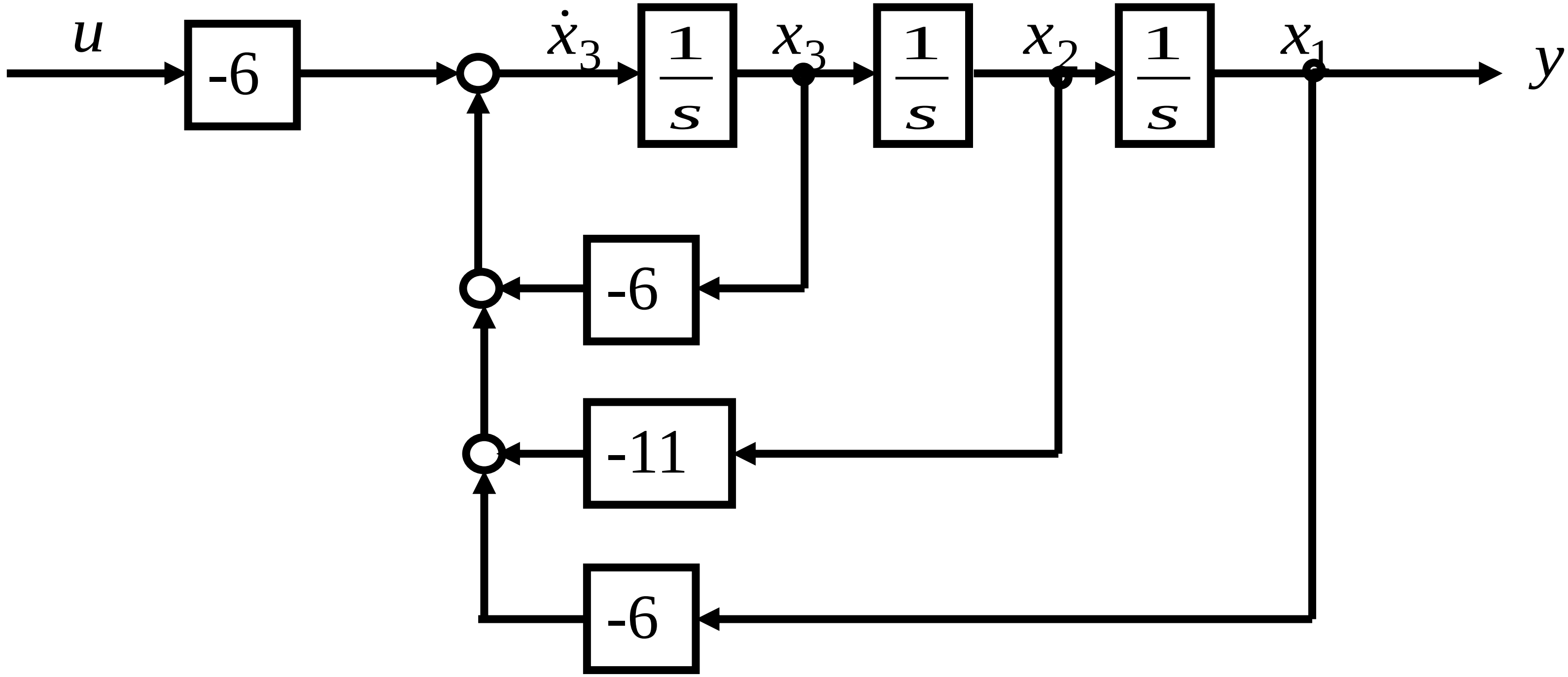
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -3 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

**There are three state variables,** $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$



# State block diagram

**Case 2**




$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} u, y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

# Selection of State Variables

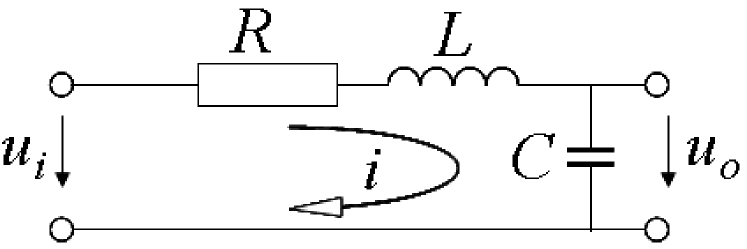
**The number of state variables chosen to represent this system should be as small as possible in order to avoid redundant state variables.**

**The selection of state variables is not unique,**
**but the number of system state variables is unique.**

1. **Choose  those variables that determine the future behavior of a system**
2. **Choose system output and its n-order derivatives**
3. **Choose state variables that can make the state-space model standard**

# Selection of State Variables

**1. Choose those variables that determine the future behavior of a system**




$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = u_i$$

$$u_o = \frac{1}{C}\int i\,dt$$

**a）Choose** $x_1 = i,\quad x_2 = \frac{1}{C}\int i\,dt = u_0$ **,**

**then:**

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}\boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}\boldsymbol{x}$$

**b）Choose，** $x_1 = i,\quad x_2 = \int i\,dt$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -R/L & -1/(CL) \\ 1 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix}\boldsymbol{u}$$

**then:**

$$\boldsymbol{y} = \begin{bmatrix} 0 & 1/C \end{bmatrix}\boldsymbol{x}$$

**The state variables may be any two independent linear combinations of** $x_1$ **and** $x_2$

# Remarks:

- For a passive RLC network, the number of state variables required is equal to the number of independent energy-storage elements.
- It is usual to choose a set of state variables that can be readily measured.

# Selection of State Variables

## 2. Choose system output and its n-order derivatives

**Case 1--- Differential equation without derivative of input variable**

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_0 u$$

$$y = x_1 \qquad \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + b_0u \end{aligned}\right\}$$

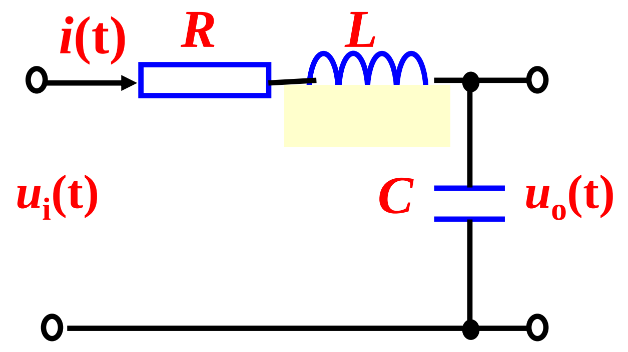
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

$$c = [1 \quad 0 \quad \cdots \quad 0]$$

**controller canonical form**

# Selection of State Variables

**1).Differential Equation:**




$$LC\frac{d^2u_o(t)}{dt^2}+RC\frac{du_o(t)}{dt}+u_o(t)=u_i(t)$$

**2). Set variables as follows:**

$$x_1=u_o=y,\qquad x_2=\dot{x}_1=\frac{du_o(t)}{dt}$$

**3).State space form:** $\dot{x}=\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{LC}\end{bmatrix}u_o$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

**4).Transfer function：**

$$G(s)=\frac{U_c(s)}{U_r(s)}=\frac{1}{LC\ s^2+RCs+1}$$

# Selection of State Variables

## 2. Choose system output and its n-order derivatives

**Case 1--- Differential equation without derivative of input variable**

**Example1: System differential equation**

$$\dddot{y} + 6\ddot{y} + 41\dot{y} + 7y = 6u$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -41 & -6 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

# Selection of State Variables

**2. Choose system output and its n-order derivatives**

**Case 2--- Differential equation with derivative of input variable**

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = \boxed{b_n u^n + b_{n-1} u^{n-1} + \cdots + b_1 \dot{u} + b_0 u}$$

原则：使状态方程不含$u$的导数。

Step 1: Select state variable as

$$x_1 = y - h_0 u \qquad x_i = \dot{x}_{i-1} - h_{i-1} u$$

$$\left.\begin{aligned} x_1 &= y - h_0 u \\ x_2 &= \dot{x}_1 - h_1 u = \boxed{\dot{y} - h_0 \dot{u}} - h_1 u \\ x_3 &= \dot{x}_2 - h_2 u = \ddot{y} - h_0 \dot{u} - h_1 \dot{u} - h_2 u \\ &\vdots \\ x_n &= \dot{x}_{n-1} - h_{n-1} u = y^{(n-1)} - h_0 u^{(n-1)} - h_1 u^{(n-2)} - \dots - h_{n-1} u \end{aligned}\right\}$$

where $h_0, h_1, \cdots, h_{n-1}$ are to be determined

# Selection of State Variables

**2. Choose system output and its n-order derivatives**

**Case 2--- Differential equation with derivative of input variable**

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = b_n u^n + b_{n-1}u^{n-1} + \cdots + b_1\dot{u} + b_0 u$$

Step 2: Output equation $\quad y = x_1 + h_0 u$

Step 3: State equation

$$\begin{gathered} \dot{x}_1 = x_2 + h_1 u \\ \dot{x}_2 = x_3 + h_2 u \\ \vdots \\ \dot{x}_{n-1} = x_n + h_{n-1} u \end{gathered}$$

$$\begin{aligned} \dot{x}_n &= y^{(n)} - h_0 u^{(n)} - h_1 u^{(n-1)} - \cdots - h_{n-1}\dot{u} \\ &= \underline{(-a_{n-1}y^{(n-1)} - \cdots - a_1\dot{y} - a_0 y + b_n u^{(n)} + \cdots + b_0 u)} \\ &\quad - h_0 u^{(n)} - h_1 u^{(n-1)} - \cdots - h_{n-1}\dot{u} \end{aligned}$$

# Selection of State Variables

## 2. Choose system output and its n-order derivatives

**Case 2--- Differential equation with derivative of input variable**

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1\dot{y}+a_0y=b_nu^n+b_{n-1}u^{n-1}+\cdots+b_1\dot{u}+b_0u$$

**System state variable description**

$$\begin{cases}\dot{x}=Ax+Bu\\ y=Cx+Du\end{cases}$$

$$A=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix}, B=\begin{bmatrix}h_1\\ h_2\\ \vdots\\ h_{n-1}\\ h_n\end{bmatrix}$$

$$c=[1\quad 0\quad \cdots\quad 0],\ D=h_0$$

$$h_0=b_n$$

$$h_1=b_{n-1}-a_{n-1}h_0$$

$$h_2=b_{n-2}-a_{n-1}h_1-a_{n-2}h_0$$

$$\vdots$$

$$h_{n-1}=b_1-a_{n-1}h_{n-2}-a_{n-2}h_{n-3}-\cdots-a_1h_0$$

$$h_n=b_0-a_{n-1}h_{n-1}-a_{n-2}h_{n-2}-\cdots-a_1h_1-a_0h_0$$

# Selection of State Variables

**2. Choose system output and its n-order derivatives**

**Case 2--- Differential equation with derivative of input variable**

Example 2 2-order system $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = T\dot{u} + u$

Step 1: Select state variable as

$$x_1 = y - h_0 u$$
$$x_2 = \dot{x}_1 - h_1 u = \dot{y} - h_0\dot{u} - h_1 u$$

Thus

$$y = x_1 + h_0 u$$
$$\dot{x}_1 = \dot{y} - h_0\dot{u} = x_2 + h_1 u$$

$$\dot{x}_2 = \ddot{y} - h_0\ddot{u} - h_1\dot{u}$$
$$= (-\omega_n{}^2 y - 2\zeta\omega_n\dot{y} + T\dot{u} + u) - h_0\ddot{u} - h_1\dot{u}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} T \\ 1 - 2\zeta\omega_n T \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# Selection of State Variables

**2. Choose system output and its n-order derivatives**

**Case 2--- Differential equation with derivative of input variable**

Example 3

$$\dddot{y} + 18\ddot{y} + 192\dot{y} + 640y = 160\dot{u} + 640u$$

$$a_0 = 640,\ a_1 = 192, a_2 = 18$$
$$b_0 = 640,\ b_1 = 160,\ b_2 = b_3 = 0$$

$$x_1 = y - h_0 u$$
$$x_2 = \dot{x}_1 - h_1 u = \dot{y} - h_0 \dot{u} - h_1 u$$
$$x_3 = \dot{x}_2 - h_2 u = \ddot{y} - h_0 \ddot{u} - h_1 \dot{u} - h_2 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -192 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 160 \\ -2240 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Selection of State Variables

**3. State variables description from transfer function**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= b_n + \boxed{\frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \cong b_n + \boxed{\frac{N(s)}{D(s)}}$$

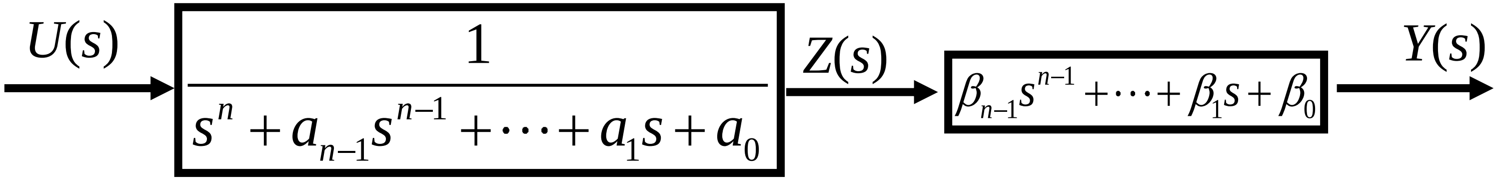
where
$$\beta_0 = b_0 - a_0 b_n$$
$$\beta_1 = b_1 - a_1 b_n$$
$$\vdots$$
$$\beta_{n-1} = b_{n-1} - a_{n-1} b_n$$

**Case 1** **The series decomposition of** $\frac{N(s)}{D(s)}$

# Selection of State Variables

## 3. State variables description from transfer function

### Case 1 The series decomposition of $\dfrac{N(s)}{D(s)}$




Introduce variable $z$

$$z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_1\dot{z} + a_0 z = u$$
$$y = \beta_{n-1}z^{(n-1)} + \cdots + \beta_1\dot{z} + \beta_0 z$$

Select state variable

$$x_1 = z, \quad x_2 = \dot{z}, \quad \cdots, \quad x_n = z^{(n-1)}$$

State equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_n = -a_0 z - a_1\dot{z} - \cdots - a_{n-1}z^{(n-1)} + u$$
$$= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + u$$

$$y = \beta_0 x_1 + \beta_1 x_2 + \cdots + \beta_{n-1}x_n$$

# Selection of State Variables

**3. State variables description from transfer function**

**Case 1** **The series decomposition of** $\dfrac{N(s)}{D(s)}$

State equation

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= -a_0 z - a_1 \dot{z} - \cdots - a_{n-1} z^{(n-1)} + u\\ &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u\end{aligned}$$

$$y = \beta_0 x_1 + \beta_1 x_2 + \cdots + \beta_{n-1} x_n$$

$$\begin{aligned}\dot{x} &= A_c x + B_c u\\ y &= C_c x\end{aligned}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix}, B_c = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}$$

$$C_c = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1} \quad\quad ]$$

# Selection of State Variables

**3. State variables description from transfer function**

## Controllable Canonical Form

$$\dot{x} = A_c x + B_c u$$
$$y = C_c x$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1} \quad ]$$

## Observable Canonical Form

$$\dot{x} = A_o x + B_o u$$
$$y = C_o x$$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}, B_o = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{bmatrix}$$

$$C_o = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

# Selection of State Variables

## 3. State variables description from transfer function

$$\dot{x} = A_c x + B_c u$$
$$y = C_c x$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1} \quad ]$$

$$\dot{x} = A_o x + B_o u$$
$$y = C_o x$$

$$A_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -a_{n-1} \end{bmatrix}, B_o = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{bmatrix}$$

$$C_o = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

$$\boldsymbol{A_c = A_o^T, B_c = C_o^T, C_c = B_o^T}$$

# Selection of State Variables

## 3. State variables description from transfer function

**Example 4**

$$G(s)=\frac{s^2+6s+8}{s^2+4s+3}$$

$$A_c=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B_c=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$G(s)=1+\frac{2s+5}{s^2+4s+3}$$

$$C_c=[\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1} \quad ]$$

1） **Controllable Canonical Form:**

$$\begin{cases} \dot{\boldsymbol{x}}=\boldsymbol{A}_c\boldsymbol{x}+\boldsymbol{b}_c u \\ \boldsymbol{y}=\boldsymbol{c}_c\boldsymbol{x}+d_c u \end{cases}$$

with $\boldsymbol{A}_c=\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ $\quad \boldsymbol{b}_c=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \boldsymbol{c}_c=\begin{bmatrix} 5 & 2 \end{bmatrix}$

2） **Observable Canonical Form:**

$$\begin{cases} \dot{\boldsymbol{x}}=\boldsymbol{A}_o\boldsymbol{x}+\boldsymbol{b}_o u \\ y=\boldsymbol{c}_o\boldsymbol{x}+d_o u \end{cases}$$

with $\boldsymbol{A}_o=\boldsymbol{A}_c^T=\begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}$ $\quad \boldsymbol{b}_o=\boldsymbol{c}_c^T=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\quad \boldsymbol{c}_o=\boldsymbol{b}_c^T=\begin{bmatrix} 0 & 1 \end{bmatrix}$

# Selection of State Variables

**3. State variables description from transfer function**

**Case 2** $\frac{N(s)}{D(s)}$ **has distinct poles**

Assuming $D(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \sum_{i=1}^{n} \frac{c_i}{s-\lambda_i},$$

where $c_i = \left[\frac{N(s)}{D(s)}(s-\lambda_i)\right]\Big|_{s=\lambda_i}$

**Select state variable** $X_i(s) = \frac{1}{s-\lambda_i}U(s), i = 1,2,\cdots,n$

$$\Rightarrow \begin{cases} \dot{x}_i(t) = \lambda_i x_i(t) + u(t) \\ y(t) = \sum_{i=1}^{n} c_i x_i(t) \end{cases}$$

# Selection of State Variables

**3. State variables description from transfer function**

## Case 2 $\frac{N(s)}{D(s)}$ has **distinct** poles

**If select state variable** $X_i(s) = \frac{1}{s-\lambda_i}U(s), i = 1,2,\cdots,n$

$$\Rightarrow \begin{cases} \dot{x}_i(t) = \lambda_i x_i(t) + u(t) \\ y(t) = \sum_{i=1}^{n} c_i x_i(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

**Matrix A could be a diagonal matrix**

# Selection of State Variables

**3. State variables description from transfer function**

**Case 2** $\frac{N(s)}{D(s)}$ **has distinct poles**

**If select state variable**

$$X_i(s) = \frac{c_i}{s-\lambda_i}U(s), i = 1,2,\cdots,n,$$

$$\Rightarrow \begin{cases} \dot{x}_i(t) = \lambda_i x_i(t) + c_i u(t) \\ y(t) = \sum_{i=1}^{n} x_i(t) \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

# Selection of State Variables

**3. State variables description from transfer function**

## Case 2 $\frac{N(s)}{D(s)}$ has **distinct poles**

**Example 5**

$$G(s) = \frac{s^2+8s+15}{s^3+7s^2+14s+8}$$

$$G(s) = \frac{s^2+8s+15}{s^3+7s^2+14s+8} = \frac{c_1}{s+1} + \frac{c_2}{s+2} + \frac{c_3}{s+4}$$

$$c_1 = G(s)(s+1)|_{s=-1} = 8/3,$$

$$c_2 = G(s)(s+2)|_{s=-2} = -3/2,$$

$$c_3 = G(s)(s+4)|_{s=-4} = -1/6$$

**State space model**

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} \frac{8}{3} & -\frac{3}{2} & -\frac{1}{6} \end{bmatrix} \boldsymbol{x}$$

# Selection of State Variables

**3. State variables description from transfer function**

**Case 3** $\frac{N(s)}{D(s)}$ **has repeated poles**

Assuming $D(s) = (s-\lambda_1)^3(s-\lambda_4)\cdots(s-\lambda_n)$

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{c_{11}}{(s-\lambda_1)^3} + \frac{c_{12}}{(s-\lambda_1)^2} + \frac{c_{13}}{s-\lambda_1} + \sum_{i=4}^{n} \frac{c_i}{s-\lambda_i}$$

where $c_{1i} = \lim_{s\to\lambda_1} \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[ \frac{N(s)}{D(s)} (s-\lambda_1)^3 \right]$

# Selection of State Variables

**3. State variables description from transfer function**

**Case 3** $\frac{N(s)}{D(s)}$ **has repeated poles**

Assuming $D(s) = (s-\lambda_1)^3(s-\lambda_4)\cdots(s-\lambda_n)$

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{c_{11}}{(s-\lambda_1)^3} + \frac{c_{12}}{(s-\lambda_1)^2} + \frac{c_{13}}{s-\lambda_1} + \sum_{i=4}^{n} \frac{c_i}{s-\lambda_i}$$

**If select state variable**

$$X_{11}(s) = \frac{1}{(s-\lambda_1)^3}U(s) = \frac{1}{(s-\lambda_1)}X_{12}(s)$$

$$X_{12}(s) = \frac{1}{(s-\lambda_1)^2}U(s) = \frac{1}{(s-\lambda_1)}\frac{1}{(s-\lambda_1)}U(s) = \frac{1}{(s-\lambda_1)}X_{13}(s)$$

$$X_{13}(s) = \frac{1}{s-\lambda_1}U(s)$$

$$X_i(s) = \frac{1}{s-\lambda_i}U(s), i = 4, \cdots, n$$

# Selection of State Variables

**3. State variables description from transfer function**

**Case 3** $\frac{N(s)}{D(s)}$ **has repeated poles**

Assuming $D(s) = (s-\lambda_1)^3(s-\lambda_4)\cdots(s-\lambda_n)$

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{c_{11}}{(s-\lambda_1)^3} + \frac{c_{12}}{(s-\lambda_1)^2} + \frac{c_{13}}{s-\lambda_1} + \sum_{i=4}^{n} \frac{c_i}{s-\lambda_i}$$

**Output equation**

$$Y(s) = \left( \frac{c_{11}}{(s-\lambda_1)^3} + \frac{c_{12}}{(s-\lambda_1)^2} + \frac{c_{13}}{s-\lambda_1} + \sum_{i=4}^{n} \frac{c_i}{s-\lambda_i} \right) U(s)$$

$$= c_{11}x_1 + c_{12}x_2 + c_{12}x_3 + \sum_{i=4}^{n} c_i\, x_i$$

# Selection of State Variables

$$X_{11}(s) = \frac{1}{(s-\lambda_1)^3} = \frac{1}{s-\lambda_1} X_{12}(s) \qquad X_{12}(s) = \frac{1}{(s-\lambda_1)^2} = \frac{1}{s-\lambda_1} X_{13}, \qquad X_{13}(s) = \frac{1}{s-\lambda_1} U(s),$$

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} = \frac{c_{11}}{(s-\lambda_1)^3} + \frac{c_{12}}{(s-\lambda_1)^2} + \frac{c_{13}}{s-\lambda_1} + \sum_{i=4}^{n} \frac{c_i}{s-\lambda_i}$$

**State equation**

$$\begin{aligned} \dot{x}_{11} &= \lambda_1 x_{11} + x_{12} \\ \dot{x}_{12} &= \lambda_1 x_{12} + x_{13} \\ \dot{x}_{13} &= \lambda_1 x_{13} + u \\ &\vdots \\ \dot{x}_n &= \lambda_i x_n + u \end{aligned}$$

Jordan canonical form

$$\begin{cases} \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \boxed{\begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & 0 & \\ & & \lambda_1 & & & \\ & & & \lambda_4 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix}} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \\ y = [c_{11} \quad c_{12} \quad c_{13} \quad c_4 \quad \cdots \quad c_n] x \end{cases}$$

# Selection of State Variables

**3. State variables description from transfer function**

**Case 3** $\frac{N(s)}{D(s)}$ **has repeated poles**

**Example 6** $G(s)=\frac{2s^2+5s+1}{(s-2)^3}$

$$G(s)=\frac{c_{11}}{(s-2)^3}+\frac{c_{12}}{(s-2)^2}+\frac{c_{13}}{(s-2)}=\frac{19}{(s-2)^3}+\frac{13}{(s-2)^2}+\frac{2}{(s-2)}$$

with $c_{11}=[G(s)\cdot(s-2)^3]\big|_{s=2}=19$

$$c_{12}=\frac{\mathrm{d}}{\mathrm{d}s}[G(s)\cdot(s-2)^3]\big|_{s=2}=13 \qquad c_{13}=\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}s^2}[G(s)\cdot(s-2)^3]\big|_{s=2}=2$$

**State space model**

$$\dot{\boldsymbol{x}}=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}u \qquad y=\begin{bmatrix}19&13&2\end{bmatrix}\boldsymbol{x}$$

# Linear transform of state-space model

**The state variable set is not exclusive, yielding to different state-space models which represent the same dynamic system.**

**Consider the LIT system** $\dot{x} = Ax + Bu, y = Cx$

**Given a nonsingular matrix** $P$

**Define** $\bar{x} = P^{-1}x, \quad x = P\bar{x}$

**The system can be defined using** $\bar{x}$ **as the state** $\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \\ y = \bar{c}\ \bar{x} \end{cases}$

with $\bar{A} = P^{-1}AP \quad \bar{b} = P^{-1}b \quad \bar{c} = cP$

Remark: 1) Two systems are called algebraically equivalent.

2) Corresponding map $\bar{x} = Px$ is called similarity transformation or equivalence transformation

# Linear transform of state-space model

**Example 7**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

**Take** $$\boldsymbol{P} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\bar{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \quad \bar{\boldsymbol{b}} = \boldsymbol{P}^{-1}\boldsymbol{b}$$

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix} u$$

Remark: Matrix A is a diagonal form, leading the states are decoupled.

# Linear transform of state-space model

$$\dot{x} = Ax + Bu, y = Cx$$

$$\boldsymbol{\bar{x} = Px \quad x = P^{-1}\bar{x}}$$

$$\begin{cases} \dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}u \\ y = \bar{\boldsymbol{c}}\ \bar{\boldsymbol{x}} \end{cases}$$

$$|\lambda I - \bar{A}| = |\lambda I - P^{-1}AP| = |\lambda P^{-1}P - P^{-1}AP| = |P^{-1}(\lambda I - A)P|$$
$$= |P^{-1}||\lambda I - A||P| = |\lambda I - A|$$

**Equivalence transformation does not change system eigenvalues**

# Linear transform of state-space model

**Special case 1: Diagonal form of Matrix A**

$$\dot{x} = Ax + Bu$$

**(1) If matrix A has distinct eigenvalues** $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$

**Exist nonsingular matrix *P***

$$\begin{cases} \dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{b}}u \\ y = \bar{\boldsymbol{c}}\ \bar{\boldsymbol{x}} \end{cases}$$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$P = [p_1 \quad p_2 \quad \cdots \quad p_n]$, $p_i$ **is eigenvectors according to** $\lambda_i$

# Linear transform of state-space model

**Special case 1: Diagonal form of Matrix A**

$$\dot{x} = Ax + Bu$$

**If matrix A is companion matrix with distinct eigenvalues** $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Which satisfies $\bar{A} = P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

# Linear transform of state-space model

**Special case 1: Diagonal form of Matrix A**

$$\dot{x} = Ax + Bu$$

**(2) If matrix A has *m* repeat eigenvalues** $\lambda_1 = \lambda_2 = \cdots = \lambda_m$

**And (*n*-*m*) distinct eigenvalues** $\lambda_{m+1},\ \lambda_{m+2},\ \cdots,\ \lambda_n$

**Exist nonsingular matrix *P*** $P = [p_1 \ \cdots \ p_m \ \ p_{m+1} \ \cdots \ p_n]$,

$$\overline{\boldsymbol{A}} = \left[\begin{array}{ccc|ccc} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ \hline & & & \lambda_{m+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{array}\right]$$

# Linear transform of state-space model

**Example 8**

$$\dot{x} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

**The eigenvalues of matrix *A* are** $\lambda_1 = 2 \quad \lambda_2 = -1 \quad \lambda_3 = 1$

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Since** $\bar{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \Rightarrow \boxed{\bar{\boldsymbol{A}}\boldsymbol{P}^{-1} = \boldsymbol{P}^{-1}\boldsymbol{A}}$

**Let** $\boldsymbol{P}^{-1} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & {}_{32} & p_{33} \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & {}_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & {}_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

**Then we get** $\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \boldsymbol{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \bar{\boldsymbol{b}} = \boldsymbol{P}^{-1}\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \quad \bar{\boldsymbol{c}} = \boldsymbol{c}\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$

# Linear transform of state-space model

## Special case 2: Jordan form of Matrix A

**If matrix A is companion matrix with repeat eigenvalues** $\lambda_1 = \lambda_2 = \cdots = \lambda_m$

**And (*n*-*m*) distinct eigenvalues** $\lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_n$

***m*-order Jordan Block**

$$\overline{A} = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_{m+1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n \end{bmatrix}$$

$$P = [\, p_1 \;\; \frac{\partial p_1}{\partial \lambda_1} \;\; \frac{\partial^2 p_1}{\partial \lambda_1^{\,2}} \;\; \cdots \;\; \frac{\partial^{m-1} p_1}{\partial \lambda_1^{\,m-1}} \;\; p_{m+1} \;\; \cdots \;\; p_n \,]$$

$$p_1 = [1 \;\; \lambda_1 \;\; \lambda_1^{\,2} \;\; \cdots \; \lambda_1^{\,n-1}]^T$$

# Transfer function matrix

SISO system-----Transfer Function (TF)
MIMO system----Transfer Function Matrix

**Consider SISO LTI system**

$$\dot{x} = Ax + bu \qquad x(0) = 0$$
$$y = cx + du$$

**Taking Laplace transform on both sides of the above equation**

$$sX(s) = AX(s) + bU(s) \qquad Y(s) = cX(s) + dU(s)$$

**Solving for $\mathbf{X}(s)$, we obtain** $$(sI - A)X(s) = bU(s)$$

$$X(s) = (sI - A)^{-1}bU(s)$$

$$Y(s) = [c(sI - A)^{-1}b + d]U(s)$$

**Transfer function is** $$g(s) = c(sI - A)^{-1}b + d = \frac{c\,\mathrm{adj}(sI-A)b + d|sI-A|}{|sI-A|}$$

**Remark: 1) Characteristic polynomial of *A*= Denominator polynomial of TF**

**2) Eig(*A*) =poles of TF**

# Transfer function matrix

SISO system-----Transfer Function (TF)
MIMO system----Transfer Function Matrix (TFM)

**Consider MIMO LTI system** $\dot{x} = Ax + Bu \qquad x(0) = 0$
$y = Cx + Du$

**Taking Laplace transform on both sides of the above equation**

$$sX(s) = AX(s) + BU(s) \qquad Y(s) = AX(s) + DU(s)$$

**Solving for $\boldsymbol{x(s)}$, we obtain** $(sI - A)X(s) = BU(s)$

$$\mathrm{X}(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = [C(sI - A)^{-1}\mathrm{B} + D]U(s)$$

**Transfer function is** $G(s) = C(sI - A)^{-1}B + D$

**Remark: Equivalence transformation does not change system transfer matrix**

$$\bar{C}(sI - \bar{A})^{-1}\bar{B} = CP^{-1}(sI - PAP^{-1})^{-1}PB = C(sI - A)B$$

# Transfer function matrix

**Example 9** $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ **Please get the transfer function matrix.**

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\boldsymbol{A}=\begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}\qquad \boldsymbol{B}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad \boldsymbol{C}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad \boldsymbol{D}=\boldsymbol{0}$$

**thus** $(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\begin{bmatrix}s & -1\\ 0 & s+2\end{bmatrix}^{-1}=\begin{bmatrix}\frac{1}{s} & \frac{1}{s(s+2)}\\ 0 & \frac{1}{s+2}\end{bmatrix}$

$$\boldsymbol{G}(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{s} & \frac{1}{s(s+2)}\\ 0 & \frac{1}{s+2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{s} & \frac{1}{s(s+2)}\\ 0 & \frac{1}{s+2}\end{bmatrix}$$

# Transfer function matrix

**Interconnection block diagrams of Transfer Function Matrix**

$$S_1: \begin{cases} \dot{\boldsymbol{x}}_1 = A_1 \boldsymbol{x}_1 + B_1 \boldsymbol{u}_1 \\ \boldsymbol{y}_1 = C_1 \boldsymbol{x}_1 + D_1 \boldsymbol{u}_1 \end{cases} \qquad S_2: \begin{cases} \dot{\boldsymbol{x}}_2 = A_2 \boldsymbol{x}_2 + B_2 \boldsymbol{u}_2 \\ \boldsymbol{y}_2 = C_2 \boldsymbol{x}_2 + D_2 \boldsymbol{u}_2 \end{cases}$$

**1) Parallel connection** $G(s) = G_n(s) + G_{n-1}(s) + \cdots + G_1(s)$

**2) Serial connection** $G(s) = G_n(s) G_{n-1}(s) \cdots G_1(s)$

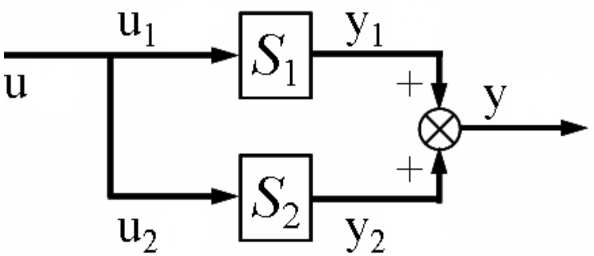
**3) Negative feedback** $G(s) = [I + G_0(s)H(s)]^{-1} G_0(s)$

# Transfer function matrix

**Interconnection block diagrams of Transfer Function Matrix**

$$S_1: \begin{cases} \dot{\boldsymbol{x}}_1 = A_1\boldsymbol{x}_1 + B_1\boldsymbol{u}_1 \\ \boldsymbol{y}_1 = C_1\boldsymbol{x}_1 + D_1\boldsymbol{u}_1 \end{cases} \qquad S_2: \begin{cases} \dot{\boldsymbol{x}}_2 = A_2\boldsymbol{x}_2 + B_2\boldsymbol{u}_2 \\ \boldsymbol{y}_2 = C_2\boldsymbol{x}_2 + D_2\boldsymbol{u}_2 \end{cases}$$

**1) Parallel connection**




$$\dim(\boldsymbol{u}_1) = \dim(\boldsymbol{u}_2)$$

$$\dim(\boldsymbol{y}_1) = \dim(\boldsymbol{y}_2)$$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}$$

$$\boldsymbol{y}_1 + \boldsymbol{y}_2 = \boldsymbol{y}$$

$$\dot{\boldsymbol{x}}_1 = A_1\boldsymbol{x}_1 + B_1\boldsymbol{u}$$

$$\dot{\boldsymbol{x}}_2 = A_2\boldsymbol{x}_2 + B_2\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 = C_1\boldsymbol{x}_1 + C_2\boldsymbol{x}_2 + D_1\boldsymbol{u} + D_2\boldsymbol{u}$$

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} D_1 + D_2 \end{bmatrix} \boldsymbol{u}$$

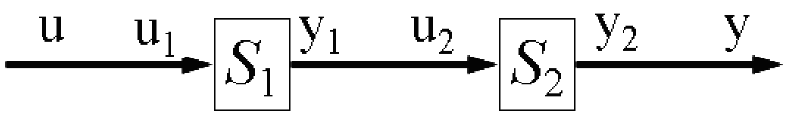
$$G(s) = G_n(s) + G_{n-1}(s) + \cdots + G_1(s)$$

# Transfer function matrix

**Interconnection block diagrams of Transfer Function Matrix**

$$S_1\!: \begin{cases} \dot{\boldsymbol{x}}_1 = A_1\boldsymbol{x}_1 + B_1\boldsymbol{u}_1 \\ \boldsymbol{y}_1 = C_1\boldsymbol{x}_1 + D_1\boldsymbol{u}_1 \end{cases} \qquad S_2\!: \begin{cases} \dot{\boldsymbol{x}}_2 = A_2\boldsymbol{x}_2 + B_2\boldsymbol{u}_2 \\ \boldsymbol{y}_2 = C_2\boldsymbol{x}_2 + D_2\boldsymbol{u}_2 \end{cases}$$

**2) Serial connection**




$$\dim(\boldsymbol{y}_1) = \dim(\boldsymbol{u}_2)$$

$$\boldsymbol{u} = \boldsymbol{u}_1 \qquad \boldsymbol{u}_2 = \boldsymbol{y}_1 \qquad \boldsymbol{y}_2 = \boldsymbol{y}$$

$$\dot{\boldsymbol{x}}_1 = A_1\boldsymbol{x}_1 + B_1\boldsymbol{u}$$

$$\dot{\boldsymbol{x}}_2 = A_2\boldsymbol{x}_2 + B_2\boldsymbol{u}_2 = A_2\boldsymbol{x}_2 + B_2(C_1\boldsymbol{x}_1 + D_1\boldsymbol{u})$$

$$= A_2\boldsymbol{x}_2 + B_2C_1\boldsymbol{x}_1 + B_2D_1\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{y}_2 = C_2\boldsymbol{x}_2 + D_2\boldsymbol{u}_2 = C_2\boldsymbol{x}_2 + D_2(C_1\boldsymbol{x}_1 + D_1\boldsymbol{u})$$

$$= C_2\boldsymbol{x}_2 + D_2C_1\boldsymbol{x}_1 + D_2D_1\boldsymbol{u}$$

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} &= \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} D_2D_1 \end{bmatrix} \boldsymbol{u} \end{aligned}}$$

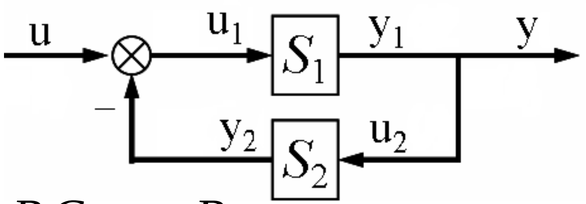
$$G(s) = G_n(s)G_{n-1}(s)\cdots G_1(s)$$

# Transfer function matrix

**Interconnection block diagrams of Transfer Function Matrix**

$$S_1: \begin{cases} \dot{\boldsymbol{x}}_1 = A_1\boldsymbol{x}_1 + B_1\boldsymbol{u}_1 \\ \boldsymbol{y}_1 = C_1\boldsymbol{x}_1 + D_1\boldsymbol{u}_1 \end{cases} \qquad S_2: \begin{cases} \dot{\boldsymbol{x}}_2 = A_2\boldsymbol{x}_2 + B_2\boldsymbol{u}_2 \\ \boldsymbol{y}_2 = C_2\boldsymbol{x}_2 + D_2\boldsymbol{u}_2 \end{cases}$$

**3) Negative feedback**




$$\dim(\boldsymbol{u}_1) = \dim(\boldsymbol{y}_2)$$

$$\dim(\boldsymbol{u}_2) = \dim(\boldsymbol{y}_1)$$

$$\boldsymbol{u}_1 = \boldsymbol{u} - \boldsymbol{y}_2$$

$$\boldsymbol{y}_1 = \boldsymbol{y} = \boldsymbol{u}_2$$

$$\dot{\boldsymbol{x}}_1 = A_1\boldsymbol{x}_1 + B_1\boldsymbol{u} - B_1\boldsymbol{y}_2 = A_1\boldsymbol{x}_1 - B_1C_2\boldsymbol{x}_2 + B_1\boldsymbol{u}$$

$$\dot{\boldsymbol{x}}_2 = A_2\boldsymbol{x}_2 + B_2\boldsymbol{y}_1 = A_2\boldsymbol{x}_2 + B_2C_1\boldsymbol{x}_1$$

$$\boldsymbol{y} = C_1\boldsymbol{x}_1$$

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}$$

$$G(s) = [I + G_1(s)G_2(s)]^{-1}G_1(s)$$

# Transfer function matrix

**Example 10**

$$\begin{cases}\dot{x}_1=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}x_1+\begin{bmatrix}0\\ 1\end{bmatrix}u_1\\ y_1=[\,1 \quad 0\,]x_1\end{cases}\qquad \begin{cases}\dot{x}_2=\begin{bmatrix}0 & 2\\ 1 & 3\end{bmatrix}x_2+\begin{bmatrix}1\\ 0\end{bmatrix}u_2\\ y_2=[\,0 \quad 1\,]x_2\end{cases}$$

**Please derive the parallel, serial and feedback system model：**

**1）Parallel**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u$$

$$y=\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}D_1+D_2\end{bmatrix}u$$

$$\Rightarrow\quad \dot{x}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & -2 & 0 & 0\\ \hline 0 & 0 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right]x+\left[\begin{array}{c}0\\ 1\\ \hline 1\\ 0\end{array}\right]u$$

$$y=\left[\begin{array}{cc|cc}1 & 0 & 0 & 1\end{array}\right]x$$

**2）Serial**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_1 & 0\\ B_2C_1 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}B_1\\ B_2D_1\end{bmatrix}u$$

$$y=\begin{bmatrix}D_2C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}D_2D_1\end{bmatrix}u$$

$$\Rightarrow\quad \dot{x}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & -2 & 0 & 0\\ \hline 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right]x+\left[\begin{array}{c}0\\ 1\\ \hline 0\\ 0\end{array}\right]u$$

$$y=\left[\begin{array}{cc|cc}0 & 0 & 0 & 1\end{array}\right]x$$

**3）Feedback**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_1 & -B_1C_2\\ B_2C_1 & A_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}B_1\\ 0\end{bmatrix}u$$

$$y=\begin{bmatrix}C_1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\Rightarrow\quad \dot{x}=\left[\begin{array}{cc|cc}0 & 1 & 0 & 0\\ -1 & -2 & 0 & -1\\ \hline 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 3\end{array}\right]x+\left[\begin{array}{c}0\\ 1\\ \hline 0\\ 0\end{array}\right]u$$

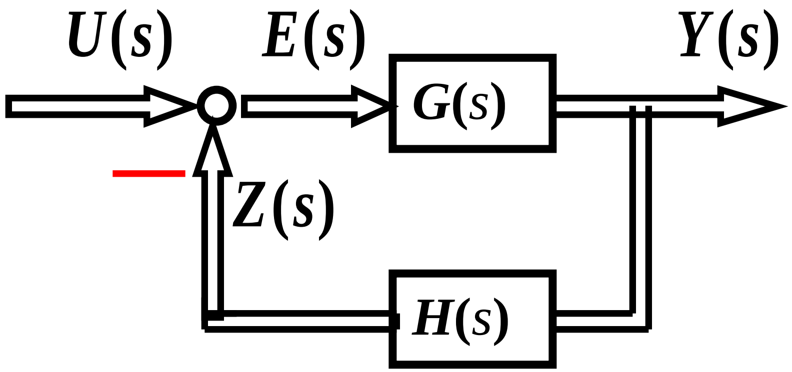
$$y=\left[\begin{array}{cc|cc}1 & 0 & 0 & 0\end{array}\right]x;$$

# Transfer function matrix

**Open-loop and closed-loop TFM**

**Open-loop TFM**

-- $H(s)G(s)$




**Closed-loop TFM**

$$\boldsymbol{\Phi}(s) = [\boldsymbol{I} + \boldsymbol{H}(s)\boldsymbol{G}(s)]^{-1}\boldsymbol{G}(s)$$

# Transfer function matrix

**An MIMO system----each output is influenced by all the input**

**Or---each input can control all the output**

**Consider MIMO LTI system**

$$\dot{x} = Ax + Bu \qquad x(0) = 0$$
$$y = Cx + Du$$

**Transfer function is** $G(s) = C(sI - A)^{-1}B + D$

$$y(s) = [C(sI - A)^{-1}\mathrm{B} + D]u(s)$$

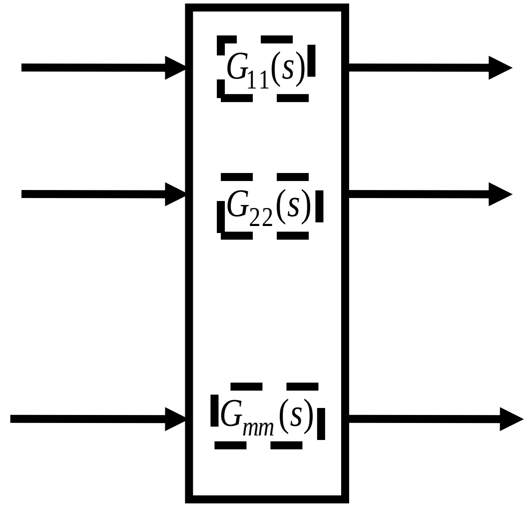
$$y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s) + \cdots + g_{1l}(s)u_l(s)$$
$$y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s) + \cdots + g_{2l}(s)u_l(s)$$
$$y_l(s) = g_{l1}(s)u_1(s) + g_{l2}(s)u_2(s) + \cdots + g_{ll}(s)u_l(s)$$

**Remark: Decoupling the system is a good way to analyze.**

# Transfer function matrix

**Decoupling the MIMO system**

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_m(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & G_{mm}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \\ \vdots \\ u_m(s) \end{bmatrix}$$

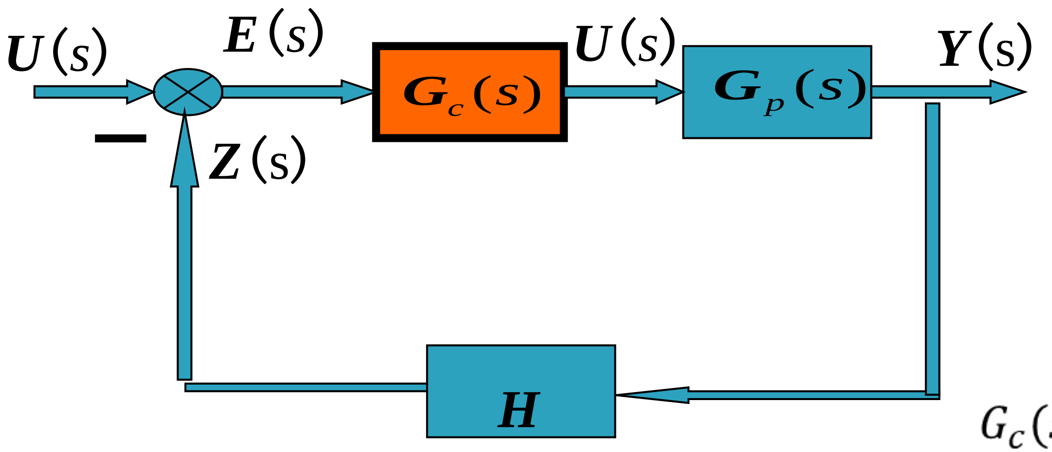



**Decoupling the MIMO system is to make the transfer function matrix diagonal**

**Decoupled system**

# Transfer function matrix

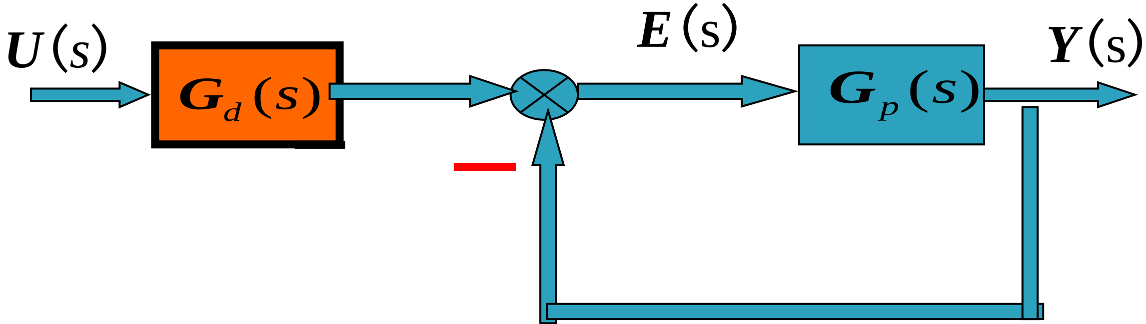
**To decouple the MIMO system, compensators are designed to meet the requirement**




$$\boldsymbol{\Phi}(s) = \begin{bmatrix} \Phi_{11}(s) & & & \\ & \Phi_{22}(s) & & \\ & & \ddots & \\ & & & \Phi_{mm}(s) \end{bmatrix}$$

$$G_c(s) = G_p^{-1}(s)\Phi(s)[I - \mathrm{H}(s)\Phi(s)])$$

**(1) Serial compensator**

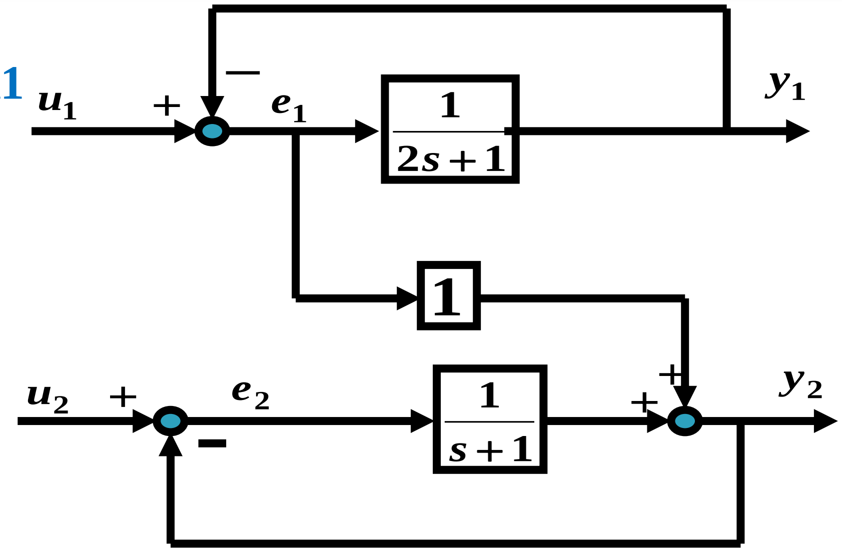



**(2) Feedforward compensator**

$$\boldsymbol{G}_d(s) = \boldsymbol{G}_p^{-1}(s)\left[\boldsymbol{I} + \boldsymbol{G}_p(s)\right]\boldsymbol{\Phi}(s)$$

# Transfer function matrix

Example 11




Please design serial compensator to Make closed-loop TFM

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{5s+1} \end{bmatrix}$$

$$G_c(s) = G_p^{-1}(s)\Phi(s)[I - \mathrm{H}(s)\Phi(s)])$$

$$\boldsymbol{G}_c(s) = \begin{bmatrix} \frac{2s+1}{s} & 0 \\ -\frac{(2s+1)(s+1)}{s} & \frac{s+1}{5s} \end{bmatrix}$$

# State-space expression of discrete system

**Z transform**

- Laplace transform of continuous signal f(t)：

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- discretize f(t)：

$$f^*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t - nT)$$

- Laplace transform of discrete signal f*(t)：

$$F^*(s) = \sum_{n=0}^{\infty} f(nT)e^{-nTs}$$

- Let $e^{Ts} = z$ ， then Z transform of discrete signal f*(t)：

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

# State-space expression of discrete system

## Consider the discrete SISO system

$$\begin{aligned}
&y(k+n)+a_{n-1}y(k+n-1)+\cdots+a_1y(k+1)+a_0y(k) \\
&=b_nu(k+n)+b_{n-1}u(k+n-1)+\cdots+b_1u(k+1)+b_0u(k)
\end{aligned}$$

## Transfer Function

$$\begin{aligned}
G(z)=\frac{Y(z)}{U(z)}&=\frac{b_nz^n+b_{n-1}z^{n-1}+\cdots+b_1z+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0} \\
&=b_n+\frac{\beta_{n-1}z^{n-1}+\cdots+\beta_1z+\beta_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0} \\
&=b_n+\frac{N(z)}{D(z)}
\end{aligned}$$

## State-space model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}u(k)$$

$$y(k)=[\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1}]\mathbf{x}(k)+b_nu(k)$$