Exercise 2

Summary

- Controllability and observability
- State feedback and state observer

Controllability and observability

State Space Description:

$$\dot{x} = Ax + Bu \text{ ---} \textbf{State equation}$$
$$y = Cx + Du \text{ ---} \textbf{Output equation}$$

Controllability ---- can the input control the state

Observability ---- can the output reflect the changes of state

Controllability and observability

Controllability criteria

For any LTI continuous system with n dimension state

$$\dot{x} = Ax + Bu$$

For Controllability matrix $\boldsymbol{U_c} = [\boldsymbol{B}\ \boldsymbol{AB}\ \boldsymbol{A^2B}\ \cdots\ \boldsymbol{A^{n-1}B}]$
The necessary and sufficient condition of system being completely controllable is

$$\mathbf{rank}\ \boldsymbol{U_c} = n$$

Output controllable

For any LTI continuous system with *m* dimension output

The necessary and sufficient condition of system output being completely controllable is

$$\mathrm{rank}\begin{bmatrix} \boldsymbol{CB} & \boldsymbol{CAB} & \ldots & \boldsymbol{CA}^{n-1}\boldsymbol{B} & \boldsymbol{D} \end{bmatrix} = m$$

Controllability and observability

Controllability criteria: PBH criteria

For any LTI continuous system $\dot{x} = Ax + Bu$

The necessary and sufficient condition of system being completely controllable is

$$\operatorname{rank}\left[\lambda_i I - A \;\; B\right] = n, \quad i = 1, 2, \cdots, n$$

or

$$\operatorname{rank}\left[sI - A \quad B\right] = n$$

Controllability and observability

Controllability criteria: PBH criteria

For any LTI continuous system $\dot{x} = Ax + Bu$

The necessary and sufficient condition of system being completely controllable is

$$\operatorname{rank}\begin{bmatrix}\lambda_i I - A & B\end{bmatrix} = n, \quad i = 1, 2, \cdots, n$$

or

$$\operatorname{rank}\begin{bmatrix}sI - A & B\end{bmatrix} = n$$

Controllability and observability

Observability criteria:

1) For linear system $\dot{x} = Ax + Bu$**,** $\mathbf{y} = \boldsymbol{C}x + Du$

the necessary and sufficient condition of system being completely observable is

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Duality principle

For linear system $S_1 : \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},$ $S_2 : \dot{\boldsymbol{z}} = \boldsymbol{A}^*\boldsymbol{z} + \boldsymbol{B}^*\boldsymbol{v}$

$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x};$ $\boldsymbol{w} = \boldsymbol{C}^*\boldsymbol{z}$

If $$\boxed{\boldsymbol{A}^* = \boldsymbol{A}^T, \boldsymbol{B}^* = \boldsymbol{C}^T, \boldsymbol{C}^* = \boldsymbol{B}^T}$$

System S_1 and S_2 are called dual systems

3-1 Determine the controllability of system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} u$$

The system state controllability matrix is $[\boldsymbol{B} \quad \boldsymbol{AB} \quad \cdots \quad \boldsymbol{A}^{n-1}\boldsymbol{B}]$

$$[\boldsymbol{A} \quad \boldsymbol{AB} \quad \boldsymbol{A}^2\boldsymbol{B}] = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 & -7 \\ -1 & 1 & 1 & -7 & 1 & 15 \end{bmatrix}$$

$$rank[\boldsymbol{A} \quad \boldsymbol{AB} \quad \boldsymbol{A}^2\boldsymbol{B}] = 3$$

The system is controllable

3-2 Determine the output controllability of system

$$\dot{x}=\begin{bmatrix}-3 & 1 & 0\\ 0 & -3 & 0\\ 0 & 0 & -1\end{bmatrix}x+\begin{bmatrix}1 & -1\\ 0 & 0\\ 2 & 0\end{bmatrix}u,$$

$$y=\begin{bmatrix}1 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}x$$

3-13 Let the system transfer function be

$$g(s) = \frac{s+4}{s^3 + 6s^2 + 11s + 6}$$

(1) Create a system controllable standard realization

(2) Create a system observable standard realization

9-22 Determine the observability of the system

(1) $$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u} \qquad \boldsymbol{y} = [1 \quad 1 \quad 0] \boldsymbol{x}$$

The system state observability matrix is $[C \quad \boldsymbol{CA} \quad \cdots \quad \boldsymbol{CA}^{n-1}]$

$$rank[\boldsymbol{C} \quad \boldsymbol{CA} \quad \boldsymbol{CA}^2]^T = 3$$

The system is observable

9-22 Determine the observability of the system

(2)

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \boldsymbol{x} \qquad \boldsymbol{y} = [1 \quad 1 \quad 1]\boldsymbol{x}$$

The system state observability matrix is

$$rank[\boldsymbol{C} \quad \boldsymbol{CA} \quad \boldsymbol{CA}^2]^T = 2$$

The system is unobservable

9-22 Determine the observability of the system

(3)

$$\dot{x} = \left[\begin{array}{cc|cc} -1 & 1 & & 0 \\ & -1 & & \\ \hline & & -2 & 1 \\ 0 & & & -2 \end{array}\right] x$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} x$$

The system state observability matrix is

$$rank[C \quad CA \quad CA^2 \quad CA^3]=4$$

The system is observable

Or there are two Jordan blocks, corresponding to two different eigenvalues, and first column of every Jordan block corresponding to matrix $\overline{C}$ are not all 0. So the system is observable.

Controllability and observability

9-22 Determine the observability of the system

(4)

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x \qquad y = [0 \quad 1 \quad 1]x$$

The system state observability matrix is

$$rank[\boldsymbol{C} \quad \boldsymbol{CA} \quad \boldsymbol{CA}^2]^T = 2$$

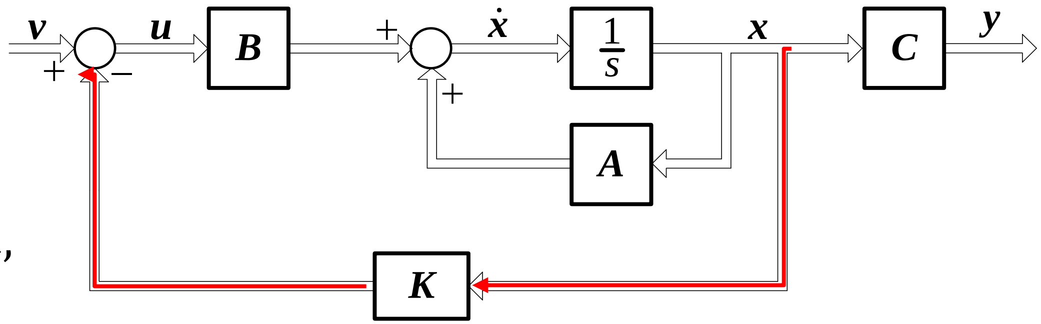
The system is unobservable

Or there is a Jordan block, but the first column of Jordan block corresponding to matrix C is 0. So the system is unobservable.

State feedback and output feedback

Feedback control is the most widely used control strategy

(1) State feedback



Consider a system

$$\dot{x} = \boldsymbol{A}x + Bu,$$
$$y = Cx$$

Introducing the state variable to the input terminal gives the state feedback control law

$$u = v - Kx \qquad v \in p \times 1 \qquad K \in p \times n$$

The closed-loop system is

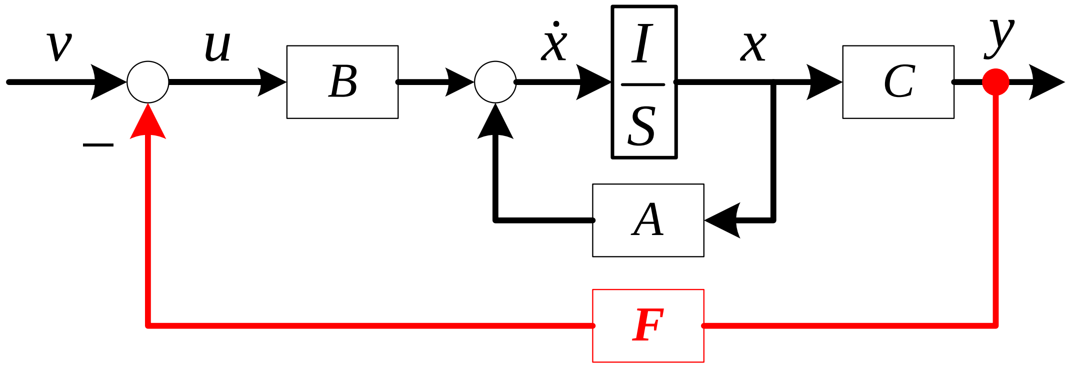
$$\dot{x} = (\boldsymbol{A} - \boldsymbol{BK})x + Bu,$$
$$y = Cx$$

The transfer function matrix

$$\boldsymbol{G} = \boldsymbol{C}(\boldsymbol{sI} - \boldsymbol{A} + \boldsymbol{BK})^{-1}\boldsymbol{B}$$

State feedback and output feedback

(2) Output feedback



Consider system

$$\dot{x} = \boldsymbol{A}x + Bu, \quad y = Cx$$

Introducing the output to the input terminal gives the output feedback control law

$$u = v - Fy \qquad F : p \times q$$

The closed-loop system is

$$\dot{x} = (\boldsymbol{A} - \boldsymbol{BFC})x + Bv, \quad y = Cx$$

The transfer function matrix $\boldsymbol{G} = \boldsymbol{C}(\boldsymbol{sI} - \boldsymbol{A} + \boldsymbol{BFC})^{-1}\boldsymbol{B}$

State Observer

- State feedback could improve system performance
  - State variables contain abundant system internal information
  - System stabilization
  - System pole placement

System stabilization

Theorem 3: State feedback system is asymptotically stable if $A - BK$ has negative eigenvalues.

Theorem 4: If and only if the uncontrollable part is asymptotically stable, that the system can be stabilized by state feedback.

Pole placement by state feedback:

Theorem 5: All poles of LTI system $\dot{x} = Ax + Bu, \quad y = Cx$ could be assigned arbitrarily by using a linear state feedback, if and only if the system is completely controllable.

State Observer

(1) Full dimensional observer

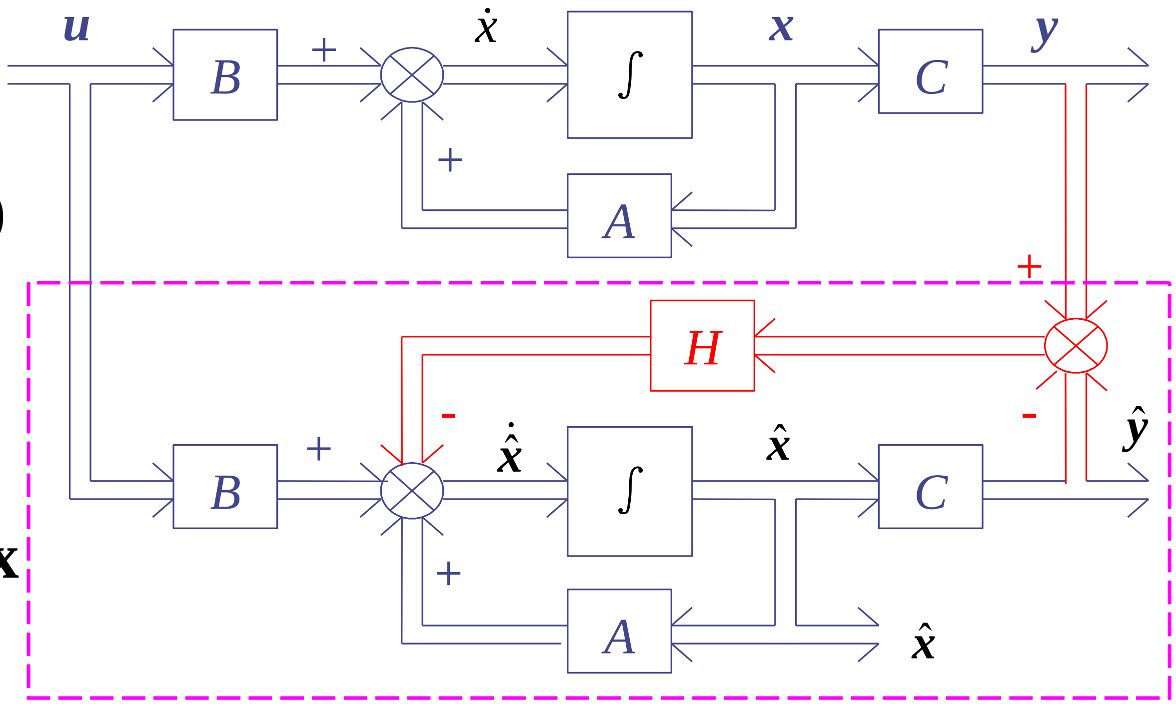
$\dim(\hat{x}) = \dim(x) = n$

$$\dot{\hat{x}} = A\hat{x} + Bu - H(\hat{y} - y)$$

$$= A\hat{x} + Bu - H(C\hat{x} - y)$$

$$\dot{\hat{x}} = (A - HC)\hat{x} + Bu + Hy$$

where $A\text{-}HC$ **is observer system matrix**



Theorem: If LTI system (A,B,C) is observable, then the state of observer could be estimated by $\dot{\hat{x}}(t) = (A\text{-}HC)\hat{x}(t) + Bu(t) + Hy$ **and H can be determined by arbitrarily assigned poles.**

Pole placement for LTI system

How to determine the feedback gain matrix K

(1) Determine if the system is controllable

(2) Derive the closed-loop system characteristic polynomial

$$|sI - A + BK|$$

(3) Derive the system characteristic polynomial with expected poles

$$\prod_{i=1}^{n}(s - s_i)$$

(4) The two polynomial should be the same, so the corresponding coefficients should be the same. Solving linear equation set to get the matrix.

State observer for LTI system

How to determine the observer matrix H

(1) Determine if the system is observable

(2) Derive the closed-loop system characteristic polynomial

$$|sI - A + HC|$$

(3) Derive the expected observer poles

$$\prod_{i=1}^{n}(s - s_i)$$

(4) The two polynomial should be the same, so the corresponding coefficients should be the same. Solving linear equation set to get the matrix.

5-1 Determine whether the system

can be arbitrarily configured eigenvalues by state feedback.

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$$

3) For any LTI continuous system with n dimension state

$$\dot{x} = Ax + Bu$$

If the system has repeat eigenvalues
And only a Jordan block corresponding to each repeated eigenvalue, the necessary and sufficient condition of system being Completely controllable is
The elements of all these rows in matrix B which are corresponding of last row of every Jordan block J are not all 0.

9-29 Given the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

Try to use state feedback to make the eigenvalues of closed-loop system as -10, $-1 \pm j\sqrt{3}$

Matlab related functions

Transfer function G(s)

$$G(s)=\frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \quad (m \le n)$$

G=tf(num,den)

State space model (A,B,C,D)

$$\dot{x} = \boldsymbol{A}x + Bu,$$
$$y = Cx + Du$$

G=ss(A, B, C, D)

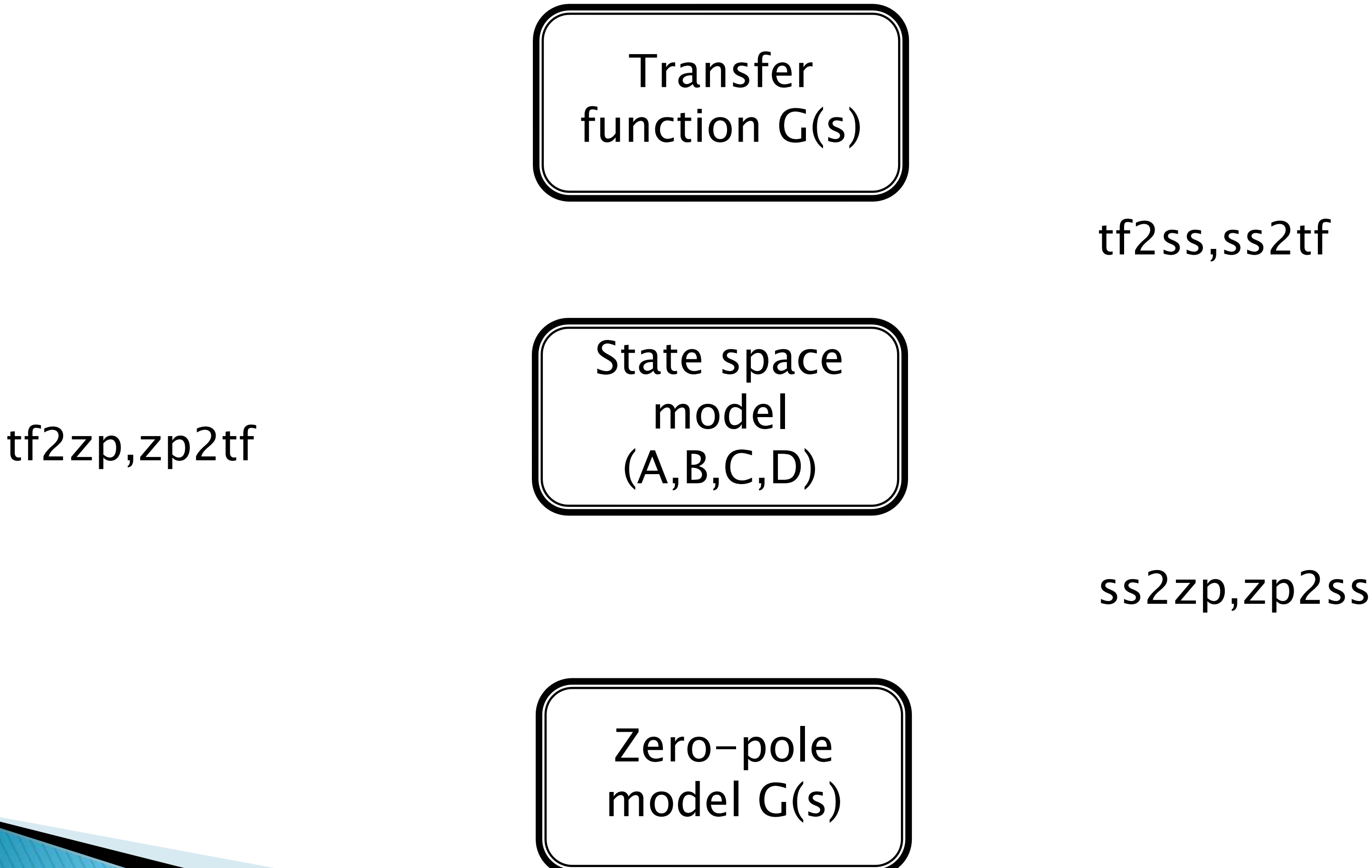
Zero-pole model G(s)

$$G(s)=\frac{K\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_j)}$$

G=zpk(Z,P,K)

Matlab related functions

P438, Example 9-6



Matlab related functions

Solutions of state space model

(1) Response of system with step signal as input：

```
G=ss(A,B,C,D);
[y,t,x]=step(G);
plot(t,x).
```

(2) u=0, homogenous state equation

```
[y,t,x]=initial(G,x0)
```

P435 Example 9-4

Matlab related functions

Controllability and observability of state space model

co=ctrb(A，B)

rank(co)

P458 Example 9-15,16

ob=obsv(A，C)

rank(ob)

Matlab related functions

Linear transform

[At，Bt，Ct，Dt]=ss2ss(A，B，C，D，T)

$$z = Tx$$

Matlab related functions

Diagonal form:

[At，Bt，Ct，Dt，T]=canon(A，B，C，D，'modal')

Observable canonical form:

[At，Bt，Ct，Dt，T]=canon(A，B，C，D，'companion')

Controllable canonical form:

[At，Bt，Ct，Dt，T]=canon(A，B，C，D，'companion')

[At'，Ct'，Bt'，Dt']

P512 exercise 9-6

Matlab related functions

Pole placement	**K=place(A,B,P)**
	K=acker(A, b, p)
State observer	**h=(acker(A',c',P_o))'**
	H=(place(A',c',P_o))'