Chapter 4

Feedback and state observer

Outlines

- State feedback and output feedback
- Pole placement for LTI system
- State observer design
- Closed-loop system with state observer

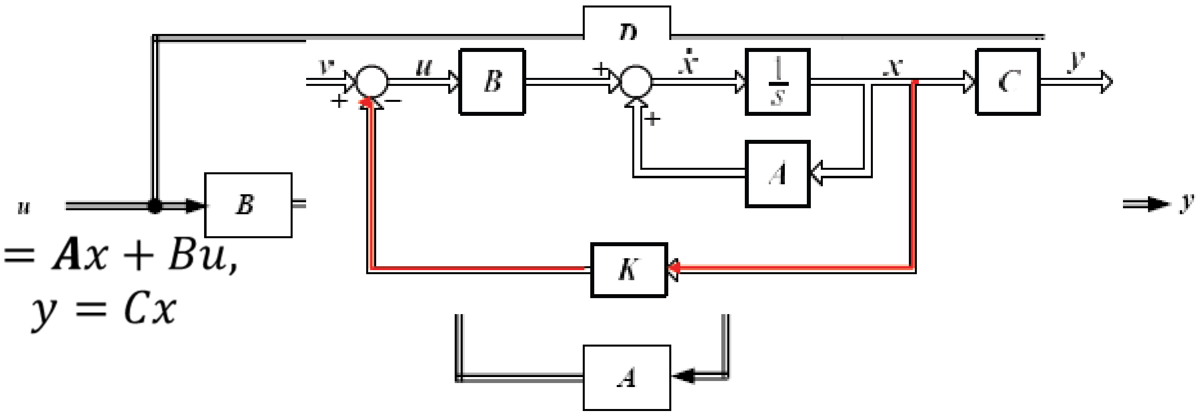
State feedback and output feedback

Feedback control is the most widely used control strategy

(1) State feedback

Consider system

$$\dot{x} = Ax + Bu, \quad y = Cx$$



Introducing the state variable to the input terminal gives the state feedback control law

$$u = v - Kx, \qquad v \in R^{p\times 1}, K \in R^{p\times n}$$

The closed-loop system is

$$\dot{x} = (A - BK)x + Bv, \quad y = Cx$$

The transfer function matrix

$$G = C(sI - A + BK)^{-1}B$$

State feedback and output feedback

The closed-loop system is $\dot{x} = (A - BK)x + Bv, \quad y = Cx$

The transfer function matrix $\boldsymbol{G} = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{B}$

Remarks:

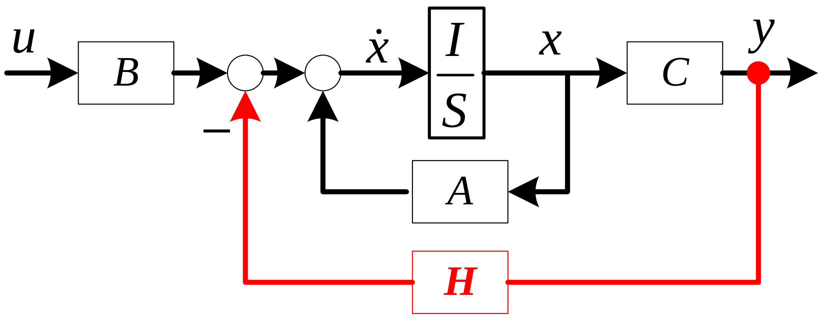
a) State feedback does not change system dimension.

b) State feedback may change system matrix eigenvalue—improve system performance by state feedback

c) State feedback requires the state variables measurable

State feedback and output feedback

(2) Output feedback



Consider system

$$\dot{x} = \boldsymbol{A}x + Bu,$$
$$y = Cx$$

① Introducing the output to the state differential signal

The closed-loop system is

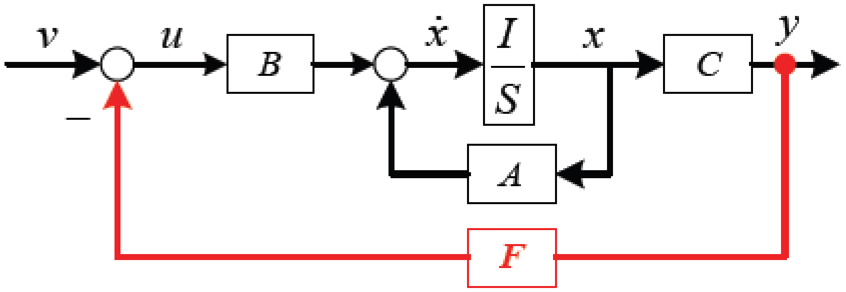
$$\dot{x} = \boldsymbol{A}x + Bu - Hy = (A - HC)x + Bu,$$
$$y = Cx$$

The transfer function matrix

$$\boldsymbol{G} = \boldsymbol{C}(\boldsymbol{sI} - \boldsymbol{A} + \boldsymbol{HC})^{-1}\boldsymbol{B}$$

State feedback and output feedback

(2) Output feedback



Consider system

$$\dot{x} = \boldsymbol{A}x + Bu,$$
$$y = Cx$$

② Introducing the output to the input terminal gives **the output feedback control law**

The closed-loop system is

$$\dot{x} = (A - BFC)x + Bv, \quad y = Cx$$

The transfer function matrix

$$G = C(sI - A + BFC)^{-1}B$$

State feedback and output feedback

Remarks:

a) **Output feedback does not change system dimension.**

b) **Output feedback may change system matrix eigenvalue—improve system performance by output feedback (classical control)**

The closed-loop system is

$$\dot{x} = (A - BK)x + Bv, \quad y = Cx$$

$$\dot{x} = (\boldsymbol{A} - \boldsymbol{BFC})x + Bv, \quad y = Cx$$

$$\boldsymbol{x} \in \boldsymbol{R}^n, \boldsymbol{u} \in \boldsymbol{R}^p, \ \boldsymbol{y} \in \boldsymbol{R}^q \qquad \boldsymbol{A} \in \boldsymbol{R}^{n \times n}, \boldsymbol{B} \in \boldsymbol{R}^{n \times p}, \ \boldsymbol{C} \in \boldsymbol{R}^{q \times n}$$

$$\boldsymbol{K} \in \boldsymbol{R}^{p \times n}, \boldsymbol{F} \in \boldsymbol{R}^{p \times q},$$

State feedback and output feedback

3) Closed-loop system controllability and observability

Theorem 1: **State feedback will not change system controllability while probably change system observability (pole-zero cancellations).**

$$\dot{x} = (A - BK)x + Bv, \qquad \text{rank}(\text{ctrb}(A, B)) = \text{rank}(\text{ctrb}(A - BK, B))$$

$$y = Cx \qquad \text{rank}(obsv(A, C)) \neq \text{rank}(obsv(A - BK, C))$$

Theorem 2: **Output feedback (to the state differential) will not change system observability while probably change system controllability**

$$\dot{x} = (A - HC)x + Bu, \qquad \text{rank}(\text{ctrb}(A, B)) \neq \text{rank}(\text{ctrb}(A - HC, B))$$

$$y = Cx \qquad \text{rank}(obsv(A, C)) = \text{rank}(obsv(A - HC, C))$$

Theorem 3: **Output feedback (to the input) will not change system controllability and observability.**

$$\dot{x} = (A - BFC)x + Bv, \qquad \text{rank}(\text{ctrb}(A, B)) = \text{rank}(\text{ctrb}(A - BFC, B))$$

$$y = Cx \qquad \text{rank}(obsv(A, C)) = \text{rank}(obsv(A - BFC, C))$$

State feedback and output feedback

Example 1 $\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ **state feedback matrix** $K = \begin{bmatrix} 3 & 1 \end{bmatrix}$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

Try to determine the controllability and observability of both open-loop and state feedback control system.

open-loop system controllability and observability

$$\boldsymbol{rank(B, AB) = 2}$$

$$\boldsymbol{rank\begin{pmatrix} C \\ CA \end{pmatrix} = 2}$$

system is controllable and observable

Closed-loop system with state feedback $K = \begin{bmatrix} 3 & 1 \end{bmatrix}$

$$\dot{x} = [A - BK]x + Bv = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$rank(B, A^* B) = 2 \qquad rank\begin{pmatrix} C \\ CA^* \end{pmatrix} = 1$$

System is still controllable but not observable

State feedback and output feedback

4) System stabilization

The closed-loop system is $$\dot{x} = (\boldsymbol{A} - \boldsymbol{BK})x + Bu, \quad y = Cx$$

Theorem 3: State feedback system is asymptotically stable if $\boldsymbol{A} - \boldsymbol{BK}$ has negative eigenvalues.

Theorem 4: If and only if the uncontrollable part is asymptotically stable, that the system can be stabilized by state feedback.

A system whose unstable modes are controllable, then it is stabilizable.

$$(1) \quad \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}$$

$$(2) \quad \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boldsymbol{u}$$

Pole placement for LTI system

- The closed-loop system poles determine the system performance and stability.
- The closed-loop system poles should be assigned according to the system performance requirement.

Traditional design:

(1) Adjusting gain or parameters of the system;

(2) Adding a compensator.

Pole placement:

Assigning the closed-loop system poles to expected position by state feedback or output feedback

Pole placement for LTI system

Pole placement by state feedback:

Theorem 5: All poles of LTI system $\dot{x} = Ax + Bu, \quad y = Cx$ could be assigned arbitrarily by using a linear state feedback, if and only if the system is completely controllable.

Sufficiency: If the system is controllable, then there exists a nonsingular linear transform, $\bar{x} = P^{-1}x$, that transform the system to a controllable canonical form with

$$\bar{A} = P^{-1}AP = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \qquad \bar{b} = P^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Introducing state feedback $\quad u = v - kx = v - kP\bar{x} = v - \bar{k}\bar{x}$

Pole placement for LTI system

$$\bar{\boldsymbol{k}} = \boldsymbol{k}\mathbf{P} = [\bar{k}_0 \quad \bar{k}_1 \quad \cdots \quad \bar{k}_{n-1}]$$

$$\bar{\boldsymbol{A}} - \bar{\boldsymbol{b}}\bar{\boldsymbol{k}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ (-a_0 - \bar{k}_0) & (-a_1 - \bar{k}_1) & (-a_2 - \bar{k}_2) & \cdots & (-a_{n-1} - \bar{k}_{n-1}) \end{bmatrix}$$

$$f(\lambda) = \left| \lambda \boldsymbol{I} - (\bar{\boldsymbol{A}} - \bar{\boldsymbol{b}}\bar{\boldsymbol{k}}) \right|$$

$$= \lambda^n + (a_{n-1} + \bar{k}_{n-1})\lambda^{n-1} + \cdots + (a_1 + \bar{k}_1)\lambda + (a_0 + \bar{k}_0)$$

Assuming the expected system poles are $\lambda_1, \lambda_2, \cdots, \lambda_n$

The expected closed-loop system characteristic polynomial is

$$f^*(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \lambda^n + a_{n-1}^*\lambda^{n-1} + \cdots + a_1^*\lambda + a_0^*$$

Pole placement for LTI system

$$f(\lambda) = f^*(\lambda)$$

$$\begin{cases} a_{n-1} + \bar{k}_{n-1} = a_{n-1}^* \\ \vdots \\ a_1 + \bar{k}_1 = a_1^* \\ a_0 + \bar{k}_0 = a_0^* \end{cases}$$

$$\bar{\boldsymbol{k}} = \boldsymbol{k}\mathbf{P} = [\bar{k}_0 \quad \bar{k}_1 \quad \cdots \quad \bar{k}_{n-1}] \quad = [a_0^* - a_0, a_1^* - a_1, \cdots, a_{n-1}^* - a_{n-1}]$$

Necessary **Assuming system (A,B,C) is not controllable, then there exists a nonsingular linear transform $\boldsymbol{x} = \boldsymbol{P}_c\bar{\boldsymbol{x}}$, transforming the system to**

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [\bar{C}_1 \quad \bar{C}_2] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}$$

Introducing state feedback $u = v - Kx = v - K\boldsymbol{P}_c\,\bar{\boldsymbol{x}} = v - \bar{\boldsymbol{K}}\bar{\boldsymbol{x}}$

Pole placement for LTI system

Necessary Assuming system (A,B,C) is not controllable, then there exists a nonsingular linear transform $x = P_c\bar{x}$, transforming the system to

$$\begin{bmatrix}\dot{\bar{x}}_1(t)\\ \dot{\bar{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\bar{A}_c & \bar{A}_{12}\\ 0 & \bar{A}_{\bar{c}}\end{bmatrix}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix} + \begin{bmatrix}\bar{B}_1\\ 0\end{bmatrix}u(t)$$

$$y(t) = [\bar{C}_1 \quad \bar{C}_2]\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix}$$

Introducing state feedback $u = v - Kx = v - K\boldsymbol{P_c}\,\boldsymbol{\bar{x}} = v - \boldsymbol{\bar{K}\bar{x}}$

$$\bar{K} = KP_c = [\bar{K}_c \quad \bar{K}_{\bar{c}}]$$

The closed-loop system is

$$\begin{bmatrix}\dot{\bar{x}}_1(t)\\ \dot{\bar{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\bar{A}_c - \bar{B}_1\bar{K}_c & \bar{A}_{12}\bar{B}_1\bar{K}_{\bar{c}}\\ 0 & \bar{A}_{\bar{c}}\end{bmatrix}\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix} + \begin{bmatrix}\bar{B}_1\\ 0\end{bmatrix}u(t)$$

$$y(t) = [\bar{C}_1 \quad \bar{C}_2]\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix}$$

$$|sI - (\bar{A} - \bar{B}\bar{K})| = \begin{vmatrix}s - \bar{A}_c + \bar{B}_1\bar{K}_c & -\bar{A}_{12}\bar{B}_1\bar{K}_{\bar{c}}\\ 0 & s - \bar{A}_{\bar{c}}\end{vmatrix}$$

$$= |s - \bar{A}_c + \bar{B}_1\bar{K}_c||s - \bar{A}_{\bar{c}}|$$

Pole placement for LTI system

Consider that the pole placement is to design a gain vector K

by state feedback $u = -K\boldsymbol{x}$ $(\boldsymbol{v}=0)$

so that the system poles are $\lambda_1, \cdots, \lambda_n$

Then closed-loop system is

$$\dot{x} = (A - BK)x, \; y = Cx$$

$$\left|\lambda I - (A - BK)\right| = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$$

Pole placement for LTI system

Example 2 Transfer function is $G(s)=\dfrac{10}{s(s+1)(s+2)}$

Try to design state feedback gain matrix K, which makes the poles of the closed-loop system be $(-2,-1\pm j)$

Controllable canonical form

$$\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}x+\begin{bmatrix}0\\0\\1\end{bmatrix}\mathrm{u},\qquad y=\begin{bmatrix}10 & 0 & 0\end{bmatrix}x$$

$$K=\begin{bmatrix}k_0 & k_1 & k_2\end{bmatrix}$$

$$f(\lambda)=\left|\lambda I-(A-bK)\right|=\lambda^3+(3+k_2)\lambda^2+(2+k_1)\lambda+k_0$$

$$f^*(\lambda)=(\lambda+2)(\lambda+1-j)(\lambda+1+j)=\lambda^3+4\lambda^2+6\lambda+4$$

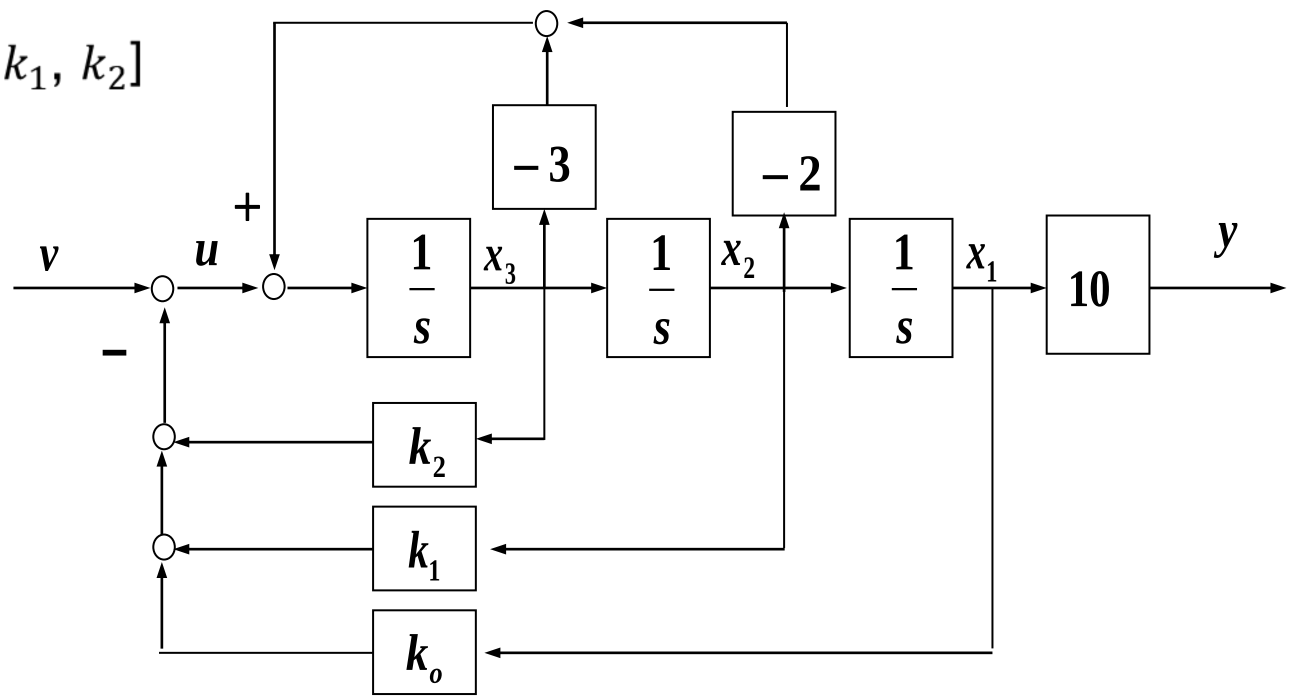
$$f^*(\lambda)=f(\lambda)\qquad 3+k_2=4,\qquad 2+k_1=6,\qquad k_0=4$$

$$K=\begin{bmatrix}4 & 4 & 1\end{bmatrix}$$

Pole placement for LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} x$$

$K=[k_0,\ k_1,\ k_2]$



Pole placement for LTI system

How to determine the feedback gain matrix K

(1) Determine if the system is controllable

(2) Derive the closed-loop system characteristic polynomial

$$|sI - A + BK|$$

(3) Derive the system characteristic polynomial with expected poles

$$\prod_{i=1}^{n}(s - s_i)$$

(4) The two polynomial should be the same, so the corresponding coefficients should be the same. Solving linear equation set to get the matrix.

Pole placement for LTI system

Example 3 $\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -6 & 0 \\ 0 & 1 & -12 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$ **Try to design state feedback gain matrix**

K, **which makes the poles of the closed-loop system be** $(-2, -1 \pm j)$

Step 1: Is it controllable?

$$U_c = \begin{bmatrix} b & Ab & A^2 b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}, \quad \operatorname{rank}(U_c) = 3 = n$$

Yes, it is controllable.

Step 2: the closed-loop system characteristic polynomial

$$K = \begin{bmatrix} k_0 & k_1 & k_2 \end{bmatrix}$$

$$|sI - (A - bK)| = \begin{vmatrix} s + k_0 & k_1 & k_2 \\ -1 & s+6 & 0 \\ 0 & -1 & s+12 \end{vmatrix}$$

$$= s^3 + (18 + k_0)s^2 + (18k_0 + k_1 + 72)s + (72k_0 + 12k_1 + k_2)$$

Pole placement for LTI system

Step 3: the system characteristic polynomial with expected poles

$$(s+2)(s+1+j)(s+1-j) = s^3 + 4s^2 + 6s + 4$$

Step 4: the system characteristic polynomial with expected poles

$$s^3 + (k_0 + 18)s^2 + (18k_0 + k_1 + 72)s + (72k_0 + 12k_1 + k_2) = s^3 + 4s^2 + 6s + 4$$

$$K = [-14, 186, -1220]$$

Pole placement for LTI system

Pole placement by output feedback:

Theorem 6: All poles of LTI system $\dot{x} = Ax + Bu, \quad y = Cx$ could be assigned arbitrarily by using output feedback to state differential, if and only if the system is observable.

Theorem 7: All poles of LTI system $\dot{x} = Ax + Bu, \quad y = Cx$ could not be assigned arbitrarily by using output feedback to input terminal

$$\dot{x} = (A - HC)x + Bu,$$
$$y = Cx$$

Pole placement for LTI system

State feedback effect on observability.

Example 4:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, y = \begin{bmatrix} 1 & 2 \end{bmatrix} \boldsymbol{x}$$

It is controllable and observable

Introduce control law $u = v - \begin{bmatrix} 3 & 1 \end{bmatrix} \boldsymbol{x}$

then $$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 2 \end{bmatrix} \boldsymbol{x}$$

Now the system is controllable but unobservable

Pole placement for LTI system

State feedback effect on zeros and poles of system.

Consider controlled canonical form $\dot{\bar{\mathbf{x}}} = \bar{A}\bar{\mathbf{x}} + \bar{b}\mathbf{u},\ \mathbf{y} = \bar{c}\bar{\mathbf{x}}$

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\bar{c} = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-1}]$$

$$G(s) = \bar{G}(s) = \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{D(s)}$$

$$\bar{A} - \bar{\mathrm{b}}\bar{\mathrm{k}} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 1 \\ (-a_0 - \bar{k}_0) & (-a_1 - \bar{k}_1) & \ldots & (-a_{n-1} - \bar{k}_{n-1}) \end{bmatrix}$$

$$G_k(s) = \frac{\beta_{n-1}s^{n-1} + \ldots + \beta_1 s + \beta_0}{D_k(s)}$$

State feedback does not change the zeros but the poles.

Pole placement for LTI system

Useful MATLAB commands

K=acker(A, b, p) **is based on Ackermann's formula. This command applies to single-input systems only**

K=place(A,B,P) **command place requires that there be no multiple poles in the set of desired closed loop poles.**

The commands acker and place yield the same K.
(for single input system)

State Observer

- State feedback could improve system performance
  - State variables contain abundant system internal information
  - System stabilization
  - System pole placement

- State variable may not be measurable directly
- State observer is designed to reconstruct the state variables based on the original system measurable input and output variables
- The reconfiguration of state variables from observer could be used as the original states to improve system characteristics

State Observer

(1) Full dimensional observer

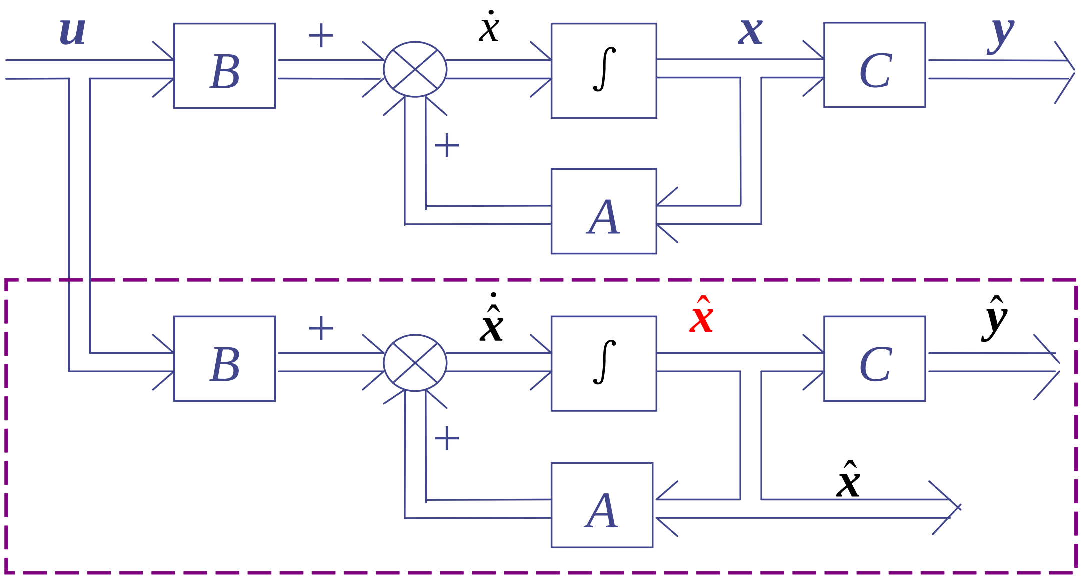
If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement

(2) Reduced-dimensional observer

An observer that estimates fewer than *n* state variables, where *n* is the dimension of the state vector

State Observer

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad (1)$$



$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu \\ \hat{y} = C\hat{x} \end{cases} \qquad (2)$$

$$(2)-(1) \qquad \begin{cases} \dot{\hat{x}} - \dot{x} = A(\hat{x} - x) \\ \hat{y} - y = C(\hat{x} - x) \end{cases}$$

open-loop state observer

$$\hat{x}(t) - x(t) = e^{At}\left[\hat{x}(0) - x(0)\right]$$

State Observer

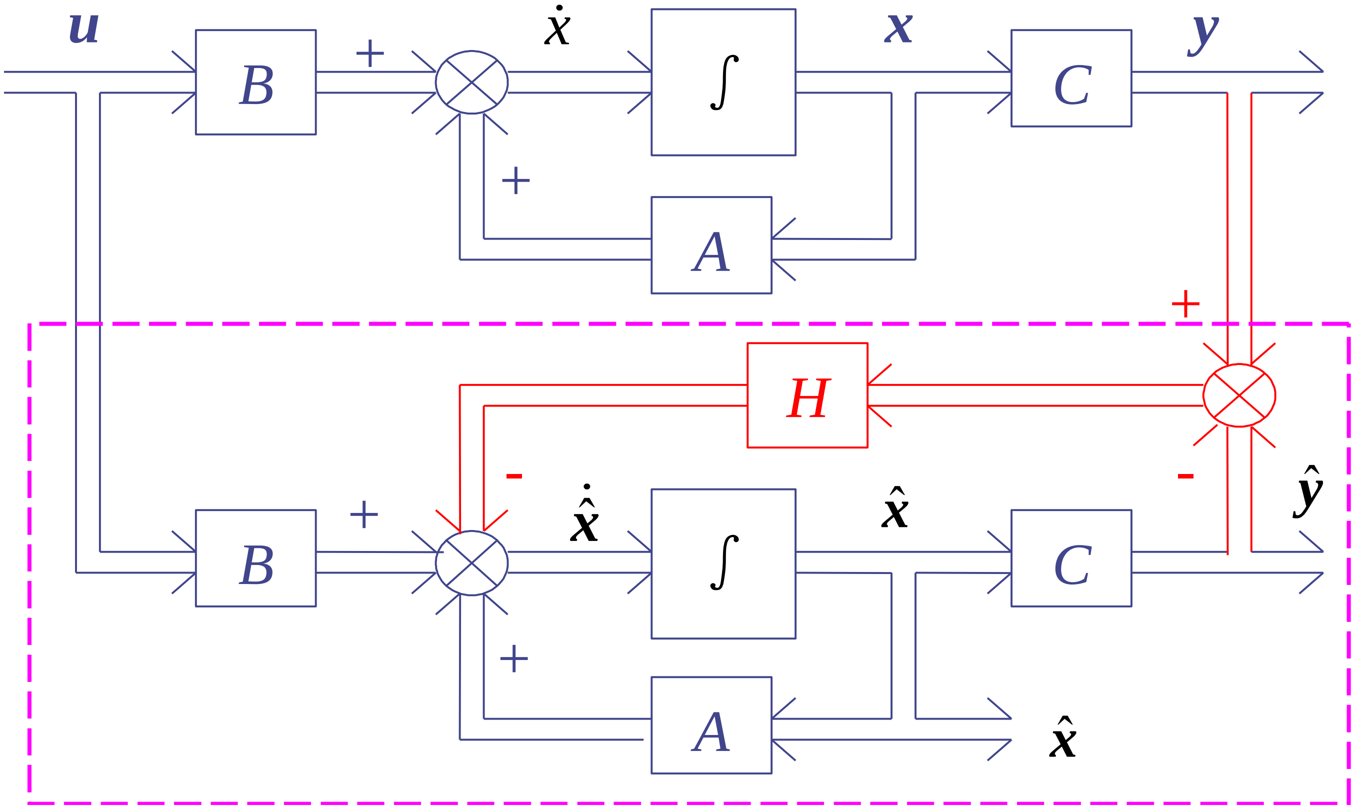
$$\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t) = e^{At}\left[\hat{\boldsymbol{x}}(0) - \boldsymbol{x}(0)\right]$$

(1) If $\hat{\boldsymbol{x}}(0) = \boldsymbol{x}(0)$, **then** $\hat{\boldsymbol{x}}(t) = \boldsymbol{x}(t)$

(2) If $\hat{\boldsymbol{x}}(0) \neq \boldsymbol{x}(0)$, **then** $\hat{\boldsymbol{x}}(t) \neq \boldsymbol{x}(t)$, $\hat{y}(t) \neq y(t)$

However, it is difficult to satisfy $\hat{\boldsymbol{x}}(0) = \boldsymbol{x}(0)$ **, to eliminate the error, error feedback** $y(t) - \hat{y}(t)$ **is introduced.**

State Observer



Closed-loop state observer

State Observer

(1) Full dimensional observer

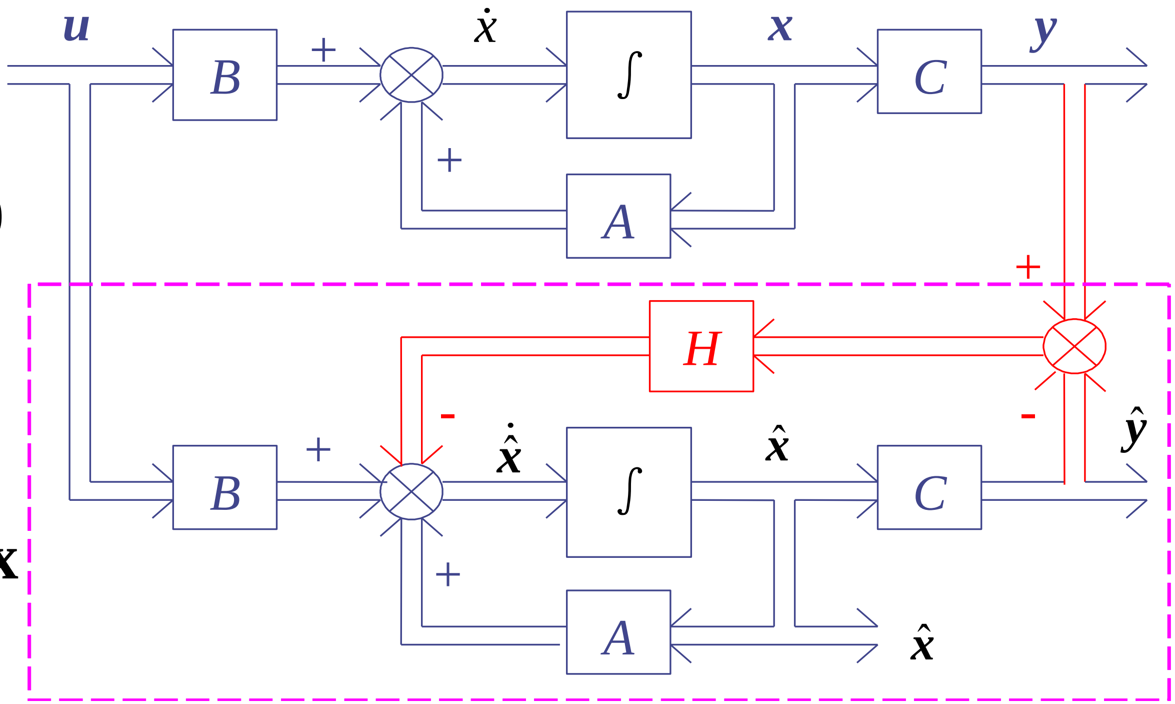
$\dim(\hat{x}) = \dim(x) = n$

$$\dot{\hat{x}} = A\hat{x} + Bu - H(\hat{y} - y)$$

$$= A\hat{x} + Bu - H(C\hat{x} - y)$$

$$\dot{\hat{x}} = (A - HC)\hat{x} + Bu + Hy$$

where $A\text{-}HC$ is observer system matrix



Now the state estimation error $\dot{e} = \dot{x} - \dot{\hat{x}} = (A - HC)(x - \hat{x})$

We could get the solution

$$e = x - \hat{x} = \mathrm{e}^{(A-HC)(t-t_0)}[x(t_0) - \hat{x}(t_0)]$$

State Observer

$$\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}} = \mathrm{e}^{(A-HC)(t-t_0)}[\boldsymbol{x}(t_0) - \hat{\boldsymbol{x}}(t_0)]$$

If $\boldsymbol{x}(t_0) = \hat{\boldsymbol{x}}(t_0)$, **then** $x(t) = \hat{x}(t)$

If $\boldsymbol{x}(t_0) \neq \hat{\boldsymbol{x}}(t_0)$, **then** $x(t) \neq \hat{x}(t), y(t) \neq \hat{y}(t)$

if matrix $A\text{-}HC$ **is stable, or**

the error vector will converge to zero for any initial error vector e(0). That is

$\hat{\boldsymbol{x}}(t)$ **will converge to** $\boldsymbol{x}(t)$ **regardless of the values of** $\boldsymbol{x}(t_0), \hat{\boldsymbol{x}}(t_0)$

Namely, $\lim\limits_{t\to\infty}(\hat{x}(t) - x(t)) = 0$

State Observer

Theorem: For LTI system (A,B,C), its observer poles could be assigned arbitrarily if and only if the system is observable.

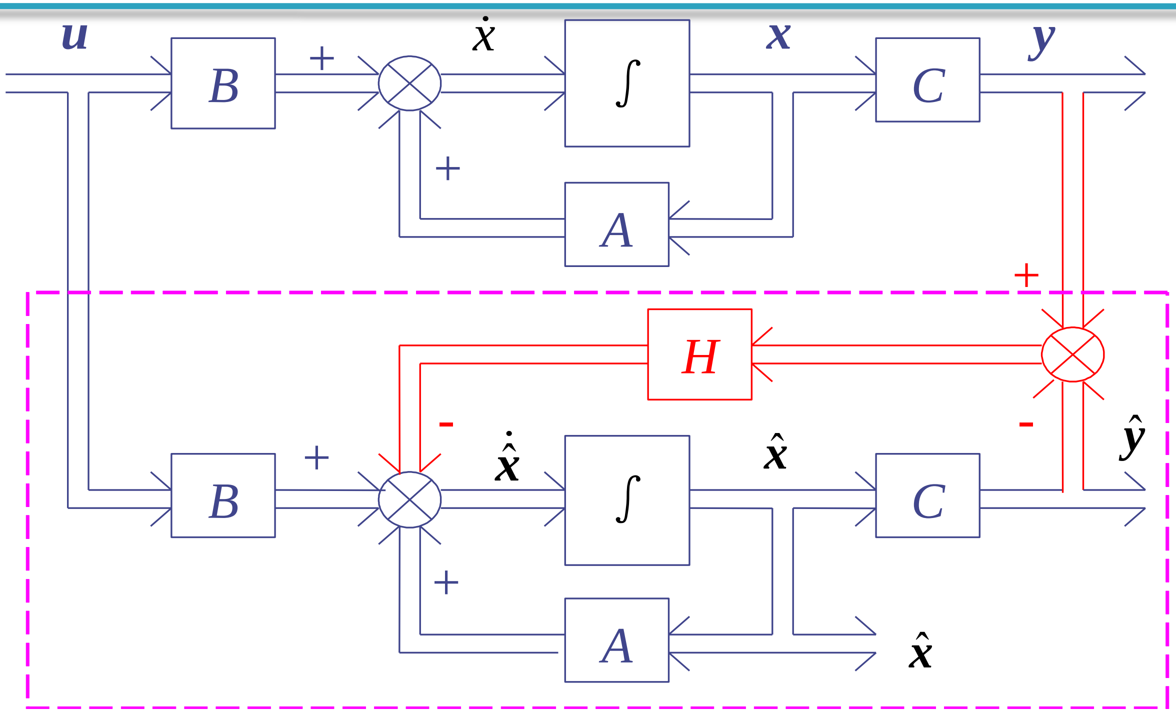
Theorem: If LTI system (A,B,C) is observable, then the state of observer could be estimated by

$$\dot{\hat{\boldsymbol{x}}}(t) = (A\text{-}HC)\hat{\boldsymbol{x}}(t) + B\boldsymbol{u}(t) + Hy$$

and H can be determined by arbitrarily assigned poles.

State observer for LTI system

Closer look:



(1) Relationship between system and observer

(2) How to design H?

As a general rule, the observer poles must be two to five times faster than the controller poles to make sure the observation error (e) converges to zero quickly.

State observer for LTI system

How to determine the observer matrix H

(1) Determine if the system is observable

(2) Derive the closed-loop system characteristic polynomial

$$|sI - A + HC|$$

(3) Derive the expected observer poles

$$\prod_{i=1}^{n}(s - s_i)$$

(4) The two polynomial should be the same, so the corresponding coefficients should be the same. Solving linear equation set to get the matrix.

State Observer

Example 5: A controlled system with transfer function system

$$\frac{Y(s)}{U(s)} = \frac{2}{(s+1)(s+2)}$$

Please design a full-dimensional observer with poles $\lambda_1 = \lambda_2 = -10$

1) The controllable canonical form from the transfer function is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t)\,,\ y = \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{x}(t)$$

$$Q_c = rank \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 = n,$$

It is observable

State Observer

2) Observer $n=2, q=1$ $H=\begin{bmatrix} h_0 \\ h_1 \end{bmatrix}$

$$A\text{-}HC=\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}-\begin{bmatrix} h_0 \\ h_1 \end{bmatrix}[2 \quad 0]=\begin{bmatrix} -2h_0 & 1 \\ -2-2h_1 & -3 \end{bmatrix}$$

Observer:

$$\det(\lambda I-A+HC)=\begin{vmatrix} \lambda+2h_0 & -1 \\ 2+2h_1 & \lambda+3 \end{vmatrix}$$

$$=\lambda^2+(2h_0+3)\lambda+(2+6h_0+2h_1)=0$$

Expected observer: $(\lambda+10)^2=\lambda^2+20\lambda+100=0$

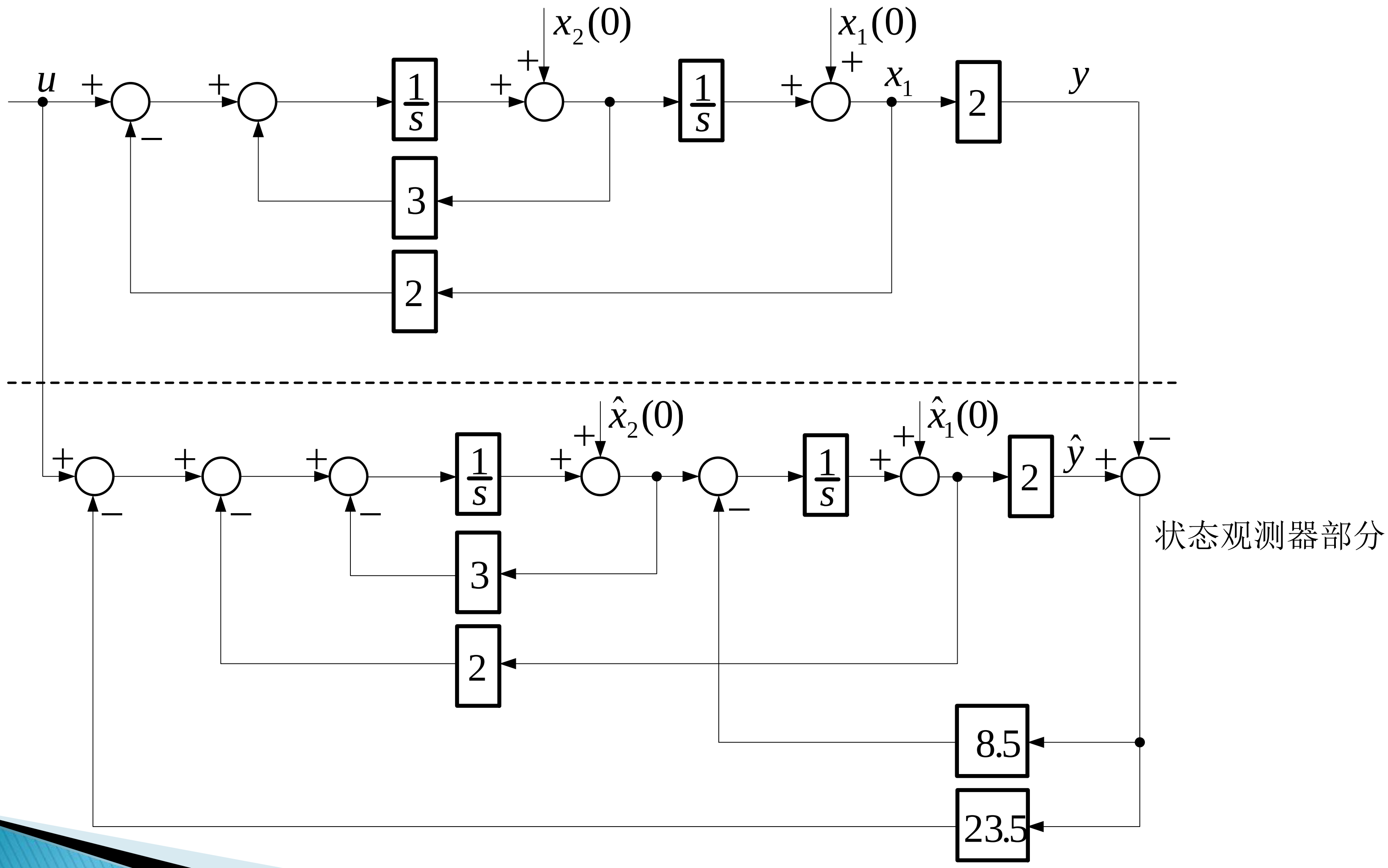
$$2h_0+3=20$$

$$2+6h_0+2h_1=100$$

$$h_0=8.5$$

$$h_1=23.5$$

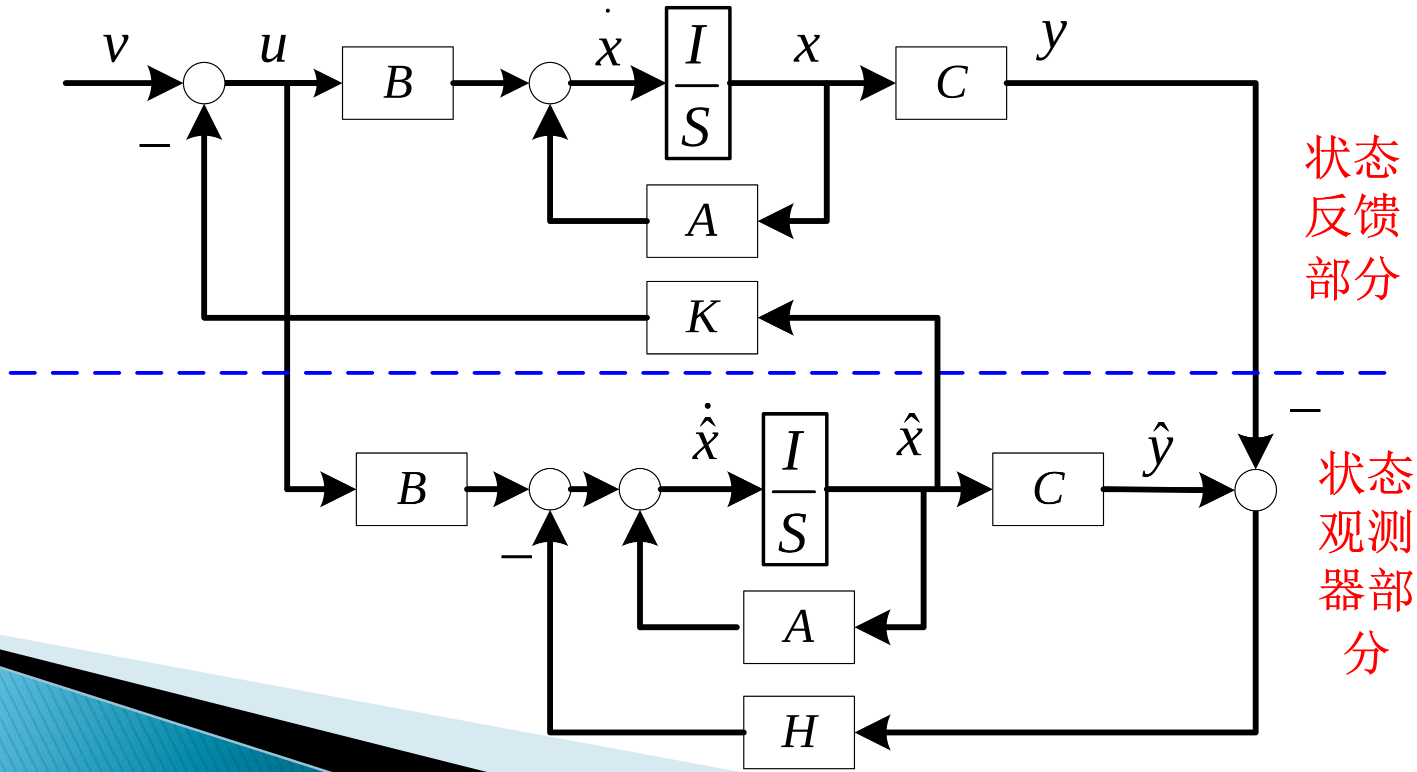
State Observer



State Observer

(3) Closed-loop system with state observer

- **State observer could reconstruct the system state variable based on the measurable input and output.**
- **The estimated state could be used for closed-loop system design**



State Observer

The control law: $\mathbf{u} = \mathbf{v} - K\hat{\mathbf{x}}$

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu = A\mathbf{x} - BK\hat{\mathbf{x}} + B\mathbf{v} \qquad y = C\mathbf{x}$$

Full dimensional state observer:

$$\dot{\hat{x}} = (A - HC)\hat{x} + B\mathbf{u} + Hy = (A - HC - BK)\hat{x} + HCx + B\mathbf{v}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ HC & A - HC - BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} +$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

The above system is 2*n* dimensional. To make it easy for future analysis, we introduce the state error

$$\dot{x} - \dot{\hat{x}} = (A - HC)(x - \hat{x})$$

State Observer

$$\begin{bmatrix} R & S \\ 0 & T \end{bmatrix}^{-1} = \begin{bmatrix} R^{-1} & -R^{-1}ST^{-1} \\ 0 & T^{-1} \end{bmatrix}$$

We have

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ \boldsymbol{0} & A - HC \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x} - \hat{\boldsymbol{x}} \end{bmatrix} + \begin{bmatrix} B \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{v}$$

$$\boldsymbol{y} = [C \quad \boldsymbol{0}] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x} - \hat{\boldsymbol{x}} \end{bmatrix}$$

$$G(s) = [C \quad \boldsymbol{0}] \begin{bmatrix} sI - A + BK & -BK \\ \boldsymbol{0} & sI - A + HC \end{bmatrix}^{-1} \begin{bmatrix} B \\ \boldsymbol{0} \end{bmatrix}$$

$$\therefore \quad G(s) = [C \quad 0] \begin{bmatrix} (sI - A + BK)^{-1} & * \\ 0 & (sI - A + HC)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$= C(sI - A + BK)^{-1} B$$

State observer does not change the closed-loop system TF

State Observer

System characteristic polynomial:

$$\left|sI-\begin{bmatrix}A-BK & BK\\ 0 & A-HC\end{bmatrix}\right| = \left|\begin{bmatrix}sI-(A-BK) & -BK\\ 0 & sI-(A-HC)\end{bmatrix}\right|$$

$$= |sI-(A-BK)|\cdot|sI-(A-HC)|$$

The two groups of poles are separated with each other

Separation principle: If LTI system (A,B,C) is controllable and observable, then the feedback matrix H in the observer and state feedback matrix K could be designed according to the requirements separately.

State observer for LTI system

Example 5:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \boldsymbol{u}, \; y = [\mathbf{0} \quad \mathbf{1}] \boldsymbol{x}$$

Try to design both state feedback K, to make the system poles are -1,-2, and The state observer poles at -6, -6.

State observer for LTI system

Useful MATLAB commands

```
H=(acker(A',c',P_o))'
```

```
H=(place(A',c',P_o))'
```

The commands acker and place yield the same **H.**

State Observer

(2) Reduced-dimensional observer

$$\mathbf{dim}(\hat{\boldsymbol{x}}) = \boldsymbol{n} - \mathrm{rank}(\boldsymbol{C}) < \mathbf{dim}(\boldsymbol{x}) = \mathrm{n}$$

where n is the dimension of the state vector and **m=rank(C)** is the dimension of the output vector.

The state vector x can be partitioned into two parts

$x_1 \in R^m,$ measurable

$x_2 \in R^{n-m},$ unmeasurable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u,$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$