Summary: Series

Notation

A **partial sum** S_N is the **finite** sum

$$S_N = \sum_{n=0}^{N} a_n.$$

A **series** S is the **infinite sum** $\displaystyle\sum_{n=0}^{\infty} a_n.$

If the limit of the partial sum $\displaystyle\lim_{N\to\infty} S_N$ exists, then

$$S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$$

and we say that the series S **converges**.

If the limit does not exist, we say the series S **diverges**.

Note that a divergent series does not have to tend to ∞, and we have already seen the different divergent behaviors of the geometric series.

Short hands for the summation notation: We also sometimes use the following abbreviated notation:

- $\displaystyle\sum_{0}^{\infty} a_n$ for $\sum_{n=0}^{\infty} a_n$ when it is clear which index to sum over,
- $\displaystyle\sum^{\infty} a_n$ for $\sum_{n=N}^{\infty} a_n$ when we are concerned with only the tail of the series.

The geometric series

A **geometric series** is defined as

$$\sum_{n=0}^{\infty} a^n = \lim_{N\to\infty} \sum_{n=0}^{N} a^n \qquad (a \text{ is any number}).$$

Notice that each term (except the first) is a times the previous term. In other words, a is the ratio of consecutive terms.

Here is a formula for the partial sum:

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a} \qquad (a \text{ is any number}).$$

When $|a| < 1$, the geometric series is convergent and converges to

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad (|a| < 1).$$

When $|a| \geq 1$, the geometric series is divergent.

Divergence test

One of the first and simplest tests on a series is the **divergence test**:

If the sequence of numbers $a_1, a_2, a_3, \ldots$ does not tend to 0, that is, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

This is very intuitive. For an infinite sum to approach a finite number, the terms being added had better approach 0.

Figure 1: All tiles in a domino falls with a push on the first tile if each tile is placed close enough to the one before.

Mathematical induction

Sometimes we can guess a formula for the partial sum S_N, but how do we know that our guess is correct for all N?

One way to show that the formula indeed works for all N is by **mathematical induction**.

Mathematical induction consists of two steps:

Base case: Show the formula is true for the $N = 1$,

Induction step: Show that **if** the formula is true for S_N, then formula would also be true for S_{N+1}.

If both statements are true, then the formula works for all S_N.

Mathematical induction works like the domino.

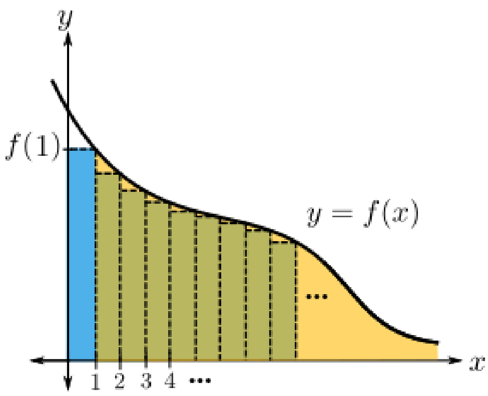
Showing a formula is true for N is analogous to having the N^{th} tile fall. The base case is analogous to the push on the first tile. The induction step is analogous to making sure that if one tile falls, it pushes the next one down.

Integral comparison test

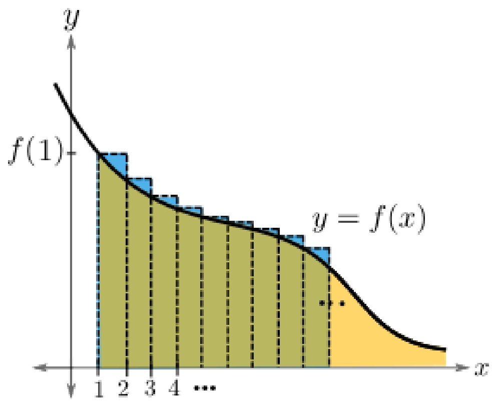
If $f(x) > 0$ and is decreasing, then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\, dx$ either **both converge** or **both diverge**.
Moreoever, we have the following inequality

$$\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)\, dx \right| < f(1).$$

This inequality is equivalent to the pair of inequalities shown in the figure below.



$$\int_1^{\infty} f(x)\, dx > \sum_{n=1}^{\infty} f(n) - f(1)$$



$$\sum_{n=1}^{\infty} f(n) > \int_1^{\infty} f(x)\, dx$$

Direct comparison test

Let $0 \le a_n \le b_n$ for all $n \ge N$.

Then

- $\sum_{n=N}^{\infty} b_n$ converges implies $\sum_{n=N}^{\infty} a_n$ converges;
- $\sum_{n=N}^{\infty} a_n$ diverges implies $\sum_{n=N}^{\infty} b_n$ diverges.

Limit comparison

If

1. $\dfrac{f(n)}{g(n)} \to c$ where $c \neq 0$ is finite,
2. $g(n) > 0$ for all $n > N$ for some $N > 0$

then $\displaystyle\sum_{n=1}^{\infty} f(n)$ and $\displaystyle\sum_{n=1}^{\infty} g(n)$ either **both converge** or **both diverge**.

In other words, if $f(n)$ and $g(n)$ decay at the same rate as n tends to ∞, then the series $\displaystyle\sum_{n=1}^{\infty} f(n)$ and $\displaystyle\sum_{n=1}^{\infty} g(n)$ converge or diverge together.

This is analogous to limit comparison for improper integrals.

Note: The condition $\dfrac{f(n)}{g(n)} \to c \neq 0$ is equivalent to

$$f(n) \sim cg(n), \text{ that is, } \frac{f(n)}{cg(n)} \to 1 \text{ as } n \to \infty.$$

More on limit comparison

We can also use limit comparison in the following way.

Suppose $f(n), g(n) > 0$ for all $n \geq N$ for some large N.

If $\dfrac{f(n)}{g(n)} \to 0$ as $n \to \infty$, that is, $f(n)$ decays faster than $g(n)$, then

- $\displaystyle\sum_{n=N}^{\infty} g(n)$ converges implies $\displaystyle\sum_{n=N}^{\infty} f(n)$ converges,
- $\displaystyle\sum_{n=N}^{\infty} f(n)$ diverges implies $\displaystyle\sum_{N}^{\infty} g(n)$ diverges.

Absolute convergence versus conditional convergence

So far, we have focused on series whose terms are positive (with the exception of the geometric series and the divergence test). For general series, including series with both positive and negative terms, there are two notions of convergence.

Consider the series

$$S = \sum_{n=1}^{\infty} a_n \qquad (a_n \text{ can be positive or negative}).$$

The series S is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

The series S is **conditionally convergent** if it converges but is **not absolutely convergent**.

For series with only positive terms, the two notions are the same.

In general, absolute convergence of a series implies convergence.

Because absolute convergence concerns the convergence of $\displaystyle\sum_{n=1}^{\infty} |a_n|$, we can apply all of the techniques we have learned to determine absolute convergence.

Ratio test

The **ratio test** is another way to determine convergence of a series.

Consider $\displaystyle\sum_{n=1}^{\infty} a_n$.

Define

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

There are three cases:

1. If $L < 1$, then the series **absolutely converges**;
2. If $L > 1$, then the series **diverges**;
3. If $L = 1$, then there is **no conclusion**.

When we talk about Taylor series, we will use the ratio test to find what is known as the radius of convergence.

Root test

The **root test** is yet another test to determine the convergence of a series.

Consider $\sum_{n=1}^{\infty} a_n$.

Define $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$. The conclusions are the same as for the ratio test, that is,

1. If $L < 1,$ then the series **absolutely converges**;
2. If $L > 1,$ then the series **diverges**;
3. If $L = 1,$ then there is **no conclusion**.

When using the root test, we often need to evaluate the n-th root of expression. Here are some examples.

A. For any constant $b > 0,$ $\lim_{n\to\infty} b^{\frac{1}{n}} = 1$

B. For any power $p > 0,$ $\lim_{n\to\infty} (n^p)^{\frac{1}{n}} = 1$

C. $\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$

To evaluate these limits, we use

$$\ln\left(\lim_{n\to\infty} b(n)\right) = \lim_{n\to\infty} \ln(b(n))$$

provided $\lim_{n\to\infty} b(n)$ is positive or $+\infty$.

Alternating series

An **alternating series** is a series whose terms alternate in signs. That is, an alternating series takes the form

$$\pm\sum_{n=1}^{\infty}(-1)^n c_n \qquad \text{where } c_n \geq 0.$$

There is a simple test for convergence of an alternating series.

$$\text{If for all } n \text{ large enough,} \quad \begin{cases} \lim_{n\to\infty} c_n = 0, \\ c_n \text{ decreases as } n \text{ increases,} \end{cases}$$

then $\pm\sum_{n=1}^{\infty}(-1)^n c_n,$ where $c_n \geq 0,$ converges.

Examples of series

In all the series below, the subscripts and superscripts of the summation notation is suppressed. That is, $\sum$ is the abbreviation of $\sum\limits_{n=N}^{\infty}$ for some N.

$$\sum x^n \qquad \text{(Geometric series)} \qquad \begin{cases} \text{converges absolutely} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases}$$

$$\sum \frac{1}{n^p} \qquad \text{(p-series)} \qquad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

$$\sum \frac{1}{n\,(\ln(n))^p} \qquad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

$$\sum \frac{\ln(n)}{n^p} \qquad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

$$\sum \frac{x^n}{n!} \qquad \text{converges absolutely for all } x$$

$$\sum \frac{x^n}{n^p} \qquad \begin{cases} \text{converges absolutely for all } p & \text{if } |x| < 1 \\ \text{diverges for all } p & \text{if } |x| > 1 \end{cases}$$

$$\sum \frac{n!}{n^n} \qquad \text{converges}$$