# Summary: Convergence Tests for Series

## Direct comparison test

Let $0 \leq a_n \leq b_n$ for all $n \geq N$.

Then

- $\sum_{n=N}^{\infty} b_n$ converges implies $\sum_{n=N}^{\infty} a_n$ converges;
- $\sum_{n=N}^{\infty} a_n$ diverges implies $\sum_{n=N}^{\infty} b_n$ diverges.

## Limit comparison

If

1. $\dfrac{f(n)}{g(n)} \to c$ where $c \neq 0$ is finite,
2. $g(n) > 0$ for all $n > N$ for some $N > 0$

then $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ either **both converge** or **both diverge**.

In other words, if $f(n)$ and $g(n)$ decay at the same rate as $n$ tends to $\infty$, then the series $\sum_{n=1}^{\infty} f(n)$ and $\sum_{n=1}^{\infty} g(n)$ converge or diverge together.

This is analogous to limit comparison for improper integrals.

**Note:** The condition $\dfrac{f(n)}{g(n)} \to c \neq 0$ is equivalent to

$$f(n) \sim cg(n), \text{ that is, } \frac{f(n)}{cg(n)} \to 1 \text{ as } n \to \infty.$$

## More on limit comparison

We can also use limit comparison in the following way.

Suppose $f(n), g(n) > 0$ for all $n \geq N$ for some large $N$.

If $\dfrac{f(n)}{g(n)} \to 0$ as $n \to \infty$, that is, $f(n)$ decays faster than $g(n)$, then

- $\displaystyle\sum_{n=N}^{\infty} g(n)$ converges implies $\displaystyle\sum_{n=N}^{\infty} f(n)$ converges,
- $\displaystyle\sum_{n=N}^{\infty} f(n)$ diverges implies $\displaystyle\sum_{N}^{\infty} g(n)$ diverges.

## Absolute convergence versus conditional convergence

So far, we have focused on series whose terms are positive (with the exception of the geometric series and the divergence test). For general series, including series with both positive and negative terms, there are two notions of convergence.

Consider the series

$$S = \sum_{n=1}^{\infty} a_n \qquad (a_n \text{ can be positive or negative}).$$

The series $S$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

The series $S$ is **conditionally convergent** if it converges but is **not absolutely convergent**.

For series with only positive terms, the two notions are the same.

In general, absolute convergence of a series implies convergence.

Because absolute convergence concerns the convergence of $\displaystyle\sum_{n=1}^{\infty} |a_n|$, we can apply all of the techniques we have learned to determine absolute convergence.

## Ratio test

The **ratio test** is another way to determine convergence of a series.

Consider $\sum_{n=1}^{\infty} a_n$ .

Define

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

There are three cases:

1. If $L < 1$, then the series **absolutely converges**;
2. If $L > 1$, then the series **diverges**;
3. If $L = 1$, then there is **no conclusion**.

When we talk about Taylor series, we will use the ratio test to find what is known as the radius of convergence.

## Root test

The **root test** is yet another test to determine the convergence of a series.

Consider $\sum_{n=1}^{\infty} a_n$ .

Define $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ . The conclusions are the same as for the ratio test, that is,

1. If $L < 1$, then the series **absolutely converges**;
2. If $L > 1$, then the series **diverges**;
3. If $L = 1$, then there is **no conclusion**.

When using the root test, we often need to evaluate the $n$-th root of expression. Here are some examples.

A. For any constant $b > 0$, $\lim_{n\to\infty} b^{\frac{1}{n}} = 1$

B. For any power $p > 0$, $\lim_{n\to\infty} (n^p)^{\frac{1}{n}} = 1$

C. $\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$

To evaluate these limits, we use

$$\ln\left(\lim_{n\to\infty} b(n)\right) = \lim_{n\to\infty} \ln(b(n))$$

provided $\lim_{n\to\infty} b(n)$ is positive or $+\infty$.

## Alternating series

An **alternating series** is a series whose terms alternate in signs. That is, an alternating series takes the form

$$\pm\sum_{n=1}^{\infty}(-1)^n c_n \qquad \text{where } c_n \geq 0.$$

There is a simple test for convergence of an alternating series.

If for all $n$ large enough, $\begin{cases} \lim_{n\to\infty} c_n = 0, \\ c_n \text{ decreases as } n \text{ increases,} \end{cases}$

then $\pm\sum_{n=1}^{\infty}(-1)^n c_n$, where $c_n \geq 0$, converges.

## Examples of series

In all the series below, the subscripts and superscripts of the summation notation is suppressed. That is, $\sum$ is the abbreviation of $\sum_{n=N}^{\infty}$ for some $N$.

$$\sum x^n \qquad \text{(Geometric series)} \qquad \begin{cases} \text{converges absolutely} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases}$$

$$\sum \frac{1}{n^p} \qquad \text{($p$-series)} \qquad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

$$\sum \frac{1}{n\,(\ln(n))^p} \qquad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

$$\sum \frac{\ln(n)}{n^p} \qquad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

$$\sum \frac{x^n}{n!} \qquad \text{converges absolutely for all } x$$

$$\sum \frac{x^n}{n^p} \qquad \begin{cases} \text{converges absolutely for all } p & \text{if } |x| < 1 \\ \text{diverges for all } p & \text{if } |x| > 1 \end{cases}$$

$$\sum \frac{n!}{n^n} \qquad \text{converges}$$