Solutions to Problem Set 9

Problem 1. A state Senate has fifteen Republicans and ten Democrats. They are creating a committee, which must have six people in it. Since the Republicans have a majority in the Senate, they insist on having a majority on the committee. How many ways are there to form a committee that satisfies the above conditions?

Solution. We do these by breaking down the cases according to how many Democrats are on the committee.

- No Democrats: There are $\binom{15}{6} = 5005$ such committees, since we are choosing 6 Republicans out of 15.
- One Democrat: There are are $\binom{15}{5}\binom{10}{1} = 30030$ committees.
- Two Democrats: There are are $\binom{15}{4}\binom{10}{2} = 61425$ committees.

The total number of possible committees is the sum of the above three numbers, which is 96460. ■

Problem 2. How many positive integers less than $1,000,000$ have exactly one digit equal to 9 and have a sum of digits equal to 13.

Solution. There are six choices for the position where 9 appears. After that choice has been made, the problem reduces to figuring out how many ways are there to select a sequence of five positive integers summing to 4 (since $13 - 9 = 4$). This is the same as counting permutations of four stars and four bars; the first integer in the sequence corresponds to the number of stars before the first bar, the second to the number of stars between first and second bars, etc. The total number of ways to permute four stars and four bars is $\binom{8}{4}$. Therefore th number of positive integers less than $1,000,000$ that have exactly one digit equal to 9 and the sum of digits equal to 13 is $6\binom{8}{4}$. ■

Problem 3. Let $B(n)$ denote the number of equivalence relations on n elements.

(a) Show that $B(n) \leq n!$.

Solution. Without loss of generality, we assume that the n-element set in question is $\{1 \ldots n\}$.

Let's start with noticing a simpler upper bound - we can bound $B(n)$ by $2^{\frac{n^2-n}{2}}$ because an equivalence relation is reflexive and symmetric. However $n!$ is much tighter bound.

What else do we know about equivalence relations? They partition a set into "equivalence classes". In fact every distinct partition of the set of n elements corresponds to an equivalence relation. However it is hard to count the number of partitions exactly, there is no simple closed form solution for this. To deal with this, we associate a function to each equivalence relation. To be more precise for any equivalence relation $\equiv$ on $\{1 \ldots n\}$, define a function $f_{\equiv}$ by

$$f_{\equiv}(i) = \min\{j \mid j \equiv i\}.$$

The function $f_{\equiv}$ is essentially providing a naming scheme for the equivalence classes, where each class is named by the smallest member. Now we can bound the number of ways this function can be constructed.

For any $\equiv$, given an element i, $f_{\equiv}(i)$ must be less than or equal to i, because if $f_{\equiv}(i)$ is in the same equivalence class as i, and it is the smallest element of that equivalence class, so it cannot exceed i. Hence$1 \leq f_{\equiv}(i) \leq i$, so there is at most one choice for $f_{\equiv}(1)$, at most two choices for $f_{\equiv}(2)$, $\ldots$, and at most n choices for $f_{\equiv}(n)$. Thus we have at most $n(n-1)\cdots 1 = n!$ possibile $f_{\equiv}$'s and therefore at most $n!$ possibilities for a function $f_{\equiv}$.

We conclude the proof by showing that every equivalence relation $\equiv$ yields a different function $f_{\equiv}$. To see this, suppose that $\equiv_1$ and $\equiv_2$ are such that $f_{\equiv_1} = f_{\equiv_2}$. Given any i, j, if $i \equiv_1 j$ we have $f_{\equiv_1}(i) = f_{\equiv_1}(j) = f_{\equiv_2}(j) = f_{\equiv_2}(i)$ so $i \equiv_2 j$. Similarly, for any i, j, such that $i \equiv_2 j$, we can conlude that $i \equiv_1 j$. Hence $\equiv_1$ and $\equiv_2$ are the same equivalence relation.

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(b) Show that $B(n) \geq 2^{n-1}$.

Solution. Again, we assume that the n-element set in question is $U ::= \{1 \ldots n\}$. Consider a subset S of $\{1 \ldots n-1\}$. Then the partition of U into S and $U \setminus S$ defines an equivalence relation. Since different choices of S yield different partitions, and hence different equivalence relations, we conclude that there are at least as many equivalence relations on U as subsets of $\{1 \ldots n-1\}$. Therefore there are at least 2^{n-1} distinct equivalence relations on U. ■

Problem 4. Prove the following identity both by (a) algebraic manipulation and by (b) giving a combinatorial argument:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

Solution. By algebra,

$$\begin{aligned}\binom{2n}{2} &= \frac{2n(2n-1)}{2} \\ &= n(2n-1) \\ &= n(n-1) + n^2 \\ &= 2\binom{n}{2} + n^2\end{aligned}$$

The left hand side is the number of ways to choose two elements out of $2n$. Counting in another way, we first divide the $2n$ elements into two sets of n elements. Either we choose both elements out of first n element set, both out of the second n elements set, or one element out of each set. The number of ways we can do this is: $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. The sum of these is the right hand side. ■

Problem 5. George's 6.042 table has 12 students, who are supposed to break up into 4 groups of 3 students each. George has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

How many *different* group assignments are possible? (Clarification: There is no "numbering" of the groups. All that matters is who ends up with whom. Two group assignments are different if they differ in at least one group.)

Solution. First note that given a list of 12 names, there is a simple way to break it up into four groups of three students each; namely take first three names as first group, next three as second, and so on.

For instance, say the list is ABCDEFGHIJKL. The above method gives the following grouping: $\boxed{\text{ABC}}\boxed{\text{DEF}}\boxed{\text{GHI}}\boxed{\text{JKL}}$. There are 12! different permutations of the list. However, not all permutations result in a different group assignment. For example, if B and C change places, the list is different but the assignment is the same. We will try to see how many different lists represent the *same* group assignment.

There are 3! ways to move the people in $\boxed{\text{ABC}}$ around without changing the assignment. The same holds for all other groups. This means that there are $(3!)^4$ ways to move people around within their group.

Then, if $\boxed{\text{ABC}}$ switches places with $\boxed{\text{JKL}}$ we still get the same assignment. There are $4!$ ways to permute the groups without actually affecting the assignment.

So, there are $4!(3!)^4$ different lists that end up producing the same assignment.

There are $12!$ possible lists, and each assignment is represented by *exactly* $4!(3!)^4$ different lists so according to the *division rule* there are

$$\frac{12!}{4! \cdot (3!)^4} = 15400$$

different assignments.

Alternative solution Another way to see this is as follows:

We need to compute the number of partitions of a set of 12 students into 4 groups of 3 students each. We first compute the number of partitions given that the groups are numbered. The first group can be selected in $\binom{12}{3}$ ways, the second in $\binom{9}{3}$ ways, the third in $\binom{6}{3}$ ways, and the fourth in $\binom{3}{3}$ ways. Therefore, the number of different partitions is

$$\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = \frac{12!}{(3!)^4} = 369600.$$

Now, to account for the fact that the groups are not numbered, we need to divide 369600 by the number of permutations of the numbered groups, $4! = 24$. To conclude, the number of tutorials we can have in 6.042 is at most $369600/24 = 15400$. ■

Problem 6. The *king positioning* in an arrangement of a deck of 52 playing cards is the sequence of numerical positions in the deck, from one to 52, of the four successive kings. For example, the king positioning $(1, 2, 3, 4)$ means all the kings come at the beginning of the deck. The king positioning $(1, 18, 35, 52)$ describes the situation in which the kings are spaced uniformly—with exactly 16 cards between successive kings.

(a) How many king positionings are there?

Solution. $\binom{52}{4}$. ■

(b) How many king positionings are there in which *no two kings are adjacent*?

Solution. Think of kings as bars and the remaining 48 cards as stars. Place one star between each two adjacent kings (bars). This requires three stars and leaves 45 stars that can be placed anywhere; there are $\binom{45+4}{4}$ ways to do this. ■

(c) Of the $52!$ possible arrangements of the deck, how many have no two kings adjacent?

Solution. Now in each of the previous patterns of stars and bars, the kings can be permuted in $4!$ ways and the remaining cards can be permuted in $48!$ ways, so the total number of desired arrangements is

$$\binom{45+4}{4} 48!\,4! = 49!\,48!/45! = 48!\,P(49,4).$$

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