(contributed by spamegg)

Exercises 35:

(a) Take a QL language, with a couple of proper names, a couple of unary predicates, and a binary predicate, and consider the following q-valuation q:

The domain is: {Romeo, Juliet, Benedick, Beatrice}

m: Romeo

n: Juliet

F: {Romeo, Benedick}

G: {Juliet, Beatrice}

L: {⟨Romeo, Juliet⟩, ⟨Juliet, Romeo⟩, ⟨Benedick, Beatrice⟩,

⟨Beatrice, Benedick⟩, ⟨Benedick, Benedick⟩}

Then what are the truth values of the following wffs?

(1) $\exists\mathsf{xLmx}$

True.

Informally translates to "Romeo loves someone."

⟨Romeo, Juliet⟩ is in the extension of L, so Lmn is true. Informally Juliet is an x that fulfills the existential quantifier. Formally, we can expand the valuation q to q_a where dummy name a is assigned Juliet and Lma is true. By the definition (Q6) $\exists\mathsf{xLmx}$ is true.

(2) $\forall\mathsf{xLxm}$

False.

Informally translates to "everyone loves Romeo". False since, for example,

$$\langle \text{Benedick, Romeo} \rangle$$

is not in the extension of L. Formally, there is an expanded valuation q_{a}, where dummy name a is assigned Benedick, for which Lam is false. So by the definition (Q5), $\forall \mathsf{xLxm}$ is false.

I will be less formal for the rest.

(3) $(\exists \mathsf{xLmx} \rightarrow \mathsf{Lmn})$

True.

Both the antecedent and the consequent are true. By (1) above, the antecedent $\exists \mathsf{xLmx}$ is true. The consequent Lmn is also true because $\langle$Romeo, Juliet$\rangle$ is in the extension of L.

(4) $\forall \mathsf{x}(\mathsf{Fx} \rightarrow \neg \mathsf{Gx})$

True.

There are 4 people in the domain, so let's verify $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ for all of them:

When a is Romeo, Fa is true (Romeo is in F's extension), $\neg \mathsf{Ga}$ is true (Romeo is not in G's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

When a is Juliet, Fa is false (Juliet is not in F's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

When a is Benedick, Fa is true (Benedick is in F's extension), $\neg \mathsf{Ga}$ is true (Benedick is not in G's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

When a is Beatrice, Fa is false (Beatrice is not in F's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

Therefore by (Q5) $\forall \mathsf{x}(\mathsf{Fx} \rightarrow \neg \mathsf{Gx})$ is true.

(5) $\forall \mathsf{x}(\mathsf{Gx} \to (\mathsf{Lxm} \lor \neg \mathsf{Lmx}))$

True.

Let's approach this like (4) but take shortcuts. We won't consider the a's for which Ga is false, because the implication $\mathsf{Ga} \to \ldots$ is automatically true in that case.

Let's only consider those in the extension of G: Juliet and Beatrice.

When a is Juliet, Ga is true, and $\mathsf{Lam} \lor \neg \mathsf{Lma}$ is true because Lam is true, since $\langle$Juliet, Romeo$\rangle$ is in the extension of L.

Therefore $\mathsf{Ga} \to (\mathsf{Lam} \lor \neg \mathsf{Lma})$ is true.

When a is Beatrice, Ga is true, and $\mathsf{Lam} \lor \neg \mathsf{Lma}$ is true, because $\neg \mathsf{Lma}$ is true, since $\langle$Romeo, Beatrice$\rangle$ is not in the extension of L. So $\mathsf{Gx} \to (\mathsf{Lam} \lor \neg \mathsf{Lma})$ is true.

By (Q5) $\forall \mathsf{x}(\mathsf{Gx} \to (\mathsf{Lxm} \lor \neg \mathsf{Lmx}))$ is true.

(6) $\forall \mathsf{x}(\mathsf{Gx} \to \exists \mathsf{yLxy})$

True.

Let's take the shortcuts again. We need to check the antecedent $\exists \mathsf{yLay}$ only for the two cases when Ga is true: Juliet and Beatrice.

When a is Juliet, Ga is true, and $\exists \mathsf{yLay}$ is true, because $\langle$Juliet, Romeo$\rangle$ is in the extension of L (informally y = Romeo).

When a is Beatrice, Ga is true, and since $\langle$Benedick, Beatrice$\rangle$ is in the extension of L, $\exists \mathsf{yLay}$ is true (informally y = Benedick).

So by (Q5) $\forall \mathsf{x}(\mathsf{Gx} \to \exists \mathsf{yLxy})$ is true.

(7) $\exists \mathsf{x}(\mathsf{Fx} \land \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Lxy}))$

False.

Taking shortcuts, let us consider only those a in the domain for which Fa is true (otherwise $(\mathsf{Fa} \wedge \ldots)$ is false anyway), namely Romeo and Benedick. For those, we will check the statement $\forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lay})$.

For checking $\forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lay})$, take another shortcut: we only need to look at those b in the domain for which Gb is true (else $(\mathsf{Gb} \rightarrow \mathsf{Lab})$ is true anyway), namely Juliet and Beatrice.

When a is Romeo, let's check $\forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lay})$.

When b is Juliet, Gb is true and Lab is true (check the extension of L, Romeo loves Juliet!).

When b is Beatrice, Gb is true and Lab is false (check the extension of L, Romeo does not love Beatrice!). Uh oh! So $\mathsf{Gb} \rightarrow \mathsf{Lab}$ is false, so $\forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lay})$ is false by (Q5).

So the existential quantifier in $\exists \mathsf{x}(\mathsf{Fx} \wedge ...)$ is not satisfied by Romeo. We have one more candidate, Benedick.

When a is Benedick, let's check $\forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lay})$.

When b is Juliet, Gb is true and Lab is false (check the extension of L, Benedick does not love Juliet!). Uh oh! So $\mathsf{Gb} \rightarrow \mathsf{Lab}$ is false, so $\forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lay})$ is false by (Q5).

Therefore $\exists \mathsf{x}(\mathsf{Fx} \wedge \forall \mathsf{y}(\mathsf{Gy} \rightarrow \mathsf{Lxy}))$ is false.

(b) Now take the following q-valuation:

The domain is: {4, 7, 8, 11, 12}

m: 7

n: 12

F: the set of even numbers in the domain

G: the set of odd numbers in the domain

L: the set of pairs $\langle m, n \rangle$ where m and n are in the domain and $m < n$

What are the truth values of the wffs (1) to (7) now?

(1) $\exists \mathsf{xLmx}$

True.

Since $7 < 8$, the pair $\langle 7, 8 \rangle$ is in the extension of L. Formally there is an expanded q-valuation q_{a} where dummy name a is assigned the object 8, for which Lma is true, therefore by the definition (Q6) $\exists \mathsf{xLmx}$ is true.

Being less formal and taking shortcuts from now on.

(2) $\forall \mathsf{xLxm}$

False.

Informally it translates to "everything is less than 7" which is not true, since the domain contains 8, 11 and 12 which are all greater than 7.

(3) $(\exists \mathsf{xLmx} \to \mathsf{Lmn})$

True.

The antecedent is true by (1), the consequent Lmn is true because $7 < 8$.

(4) $\forall \mathsf{x}(\mathsf{Fx} \to \neg \mathsf{Gx})$

True.

Because if a number is even, it is not odd.

(5) $\forall \mathsf{x}(\mathsf{Gx} \to (\mathsf{Lxm} \lor \neg \mathsf{Lmx}))$

False.

Informally translates to: "if a number is odd, then either it's less than 7, or 7 is not less than it." This statement is not true for 11, which is odd, but 11 is not less than 7, and 7 is less than 11.

(6) $\forall x(Gx \to \exists y Lxy)$

True.

Translates to: "if a number is odd, it is less than another number." True for 7 because $7 < 8$, true for 11 because $11 < 12$.

(7) $\exists x(Fx \land \forall y(Gy \to Lxy))$

True.

Translates to: "there is an even number less than all the odd numbers." This is satisfied by 4.

(c) Take the language QL_3 of Exercises 30(b) (copied below):

This is the language QL_3 whose quantifiers range over the positive integers, with the following glossary:

n: one

F: ① is odd

G: ① is even

H: ① is prime

L: ① is less than ②

R: ① is the sum of ② and ③

Consider the following q-valuation (the natural one suggested by the given glossary for the language):

The domain is the set of natural numbers (the integers from zero up) n: one

F: the set of odd numbers

G: the set of even numbers

H: the set of prime numbers

L: the set of pairs $\langle m, n \rangle$ such that $m < n$

R: the set of triples $\langle l, m, n \rangle$ such that $l = m + n$.

Carefully work out the values of the wffs (1) to (8) from Exercises 30(b) (copied from that exercise):

(1) ∀x∀y∃zRzxy

True.

Informally it says, for every pair, they add up to a third number. This is another way of saying the more mathematical: "the natural numbers are closed under addition."

Being careful, let's prove this. Argue by contradiction and assume that there is an expanded q-valuation q_{ab} with natural numbers a, b (where dummy name a is assigned a and dummy name b is assigned b) such that ∃zRzab is false.

So Rcab is false for all expanded q-valuations q_{abc} of q_{ab} (where, the dummy name c is assigned the natural number, say c).

So $c = a + b$ is false for all natural numbers c. This means $a + b$ must be negative. But we know this is impossible: if both a and b are at least zero, their sum is also at least zero! Contradiction.

(2) ∃y∀xLxy

False.

Informally it translates to: "there is a number bigger than all others."

Argue by contradiction, assume there exists an expanded q-valuation q_{a}, where dummy name a is assigned the natural number a, such that $\forall \mathsf{xLxa}$ is true.

Consider the further expanded q-valuation q_{ab} where dummy name b is assigned the natural number $a+1$. Since $a+1 \not< a$, Lba is false. By (Q5) $\forall \mathsf{xLxa}$ is false, contradiction!

Being less formal and careful from now on. In particular, for the sake of brevity, sometimes I will write bits and pieces of a formula with dangling unquantified variables (which are technically not wffs). It simply means that I am skipping the "consider the expanded valuation q_{a} where a is assigned..." type of details (where I would replace the dangling unquantified variables with dummy names which are assigned an object in the domain).

(3) $\forall \mathsf{x} \exists \mathsf{y}(\mathsf{Lxy} \wedge \mathsf{Hy})$

True.

Translates to: "for every number there is a prime bigger than it." Although this is something we cannot prove here. It is equivalent to the existence of infinitely many prime numbers. There is a famous ancient Greek proof from Euclid's *Elements*, which is usually taught in "introduction to math and proofs" courses. You can look it up on the web, it's a very elegant proof.

(4) $\forall \mathsf{x}(\mathsf{Hx} \rightarrow \exists \mathsf{y}(\mathsf{Lxy} \wedge \mathsf{Hy}))$

True.

Similar to (4), it translates to "for every prime, there is another prime bigger than it."

(5) $\forall \mathsf{x} \forall \mathsf{y}((\mathsf{Fx} \wedge \mathsf{Ryxn}) \rightarrow \neg \mathsf{Fy})$

True.

It means "if you add 1 to an odd number, you get an even number."

(6) $\forall \mathsf{x} \exists \mathsf{y}((\mathsf{Gx} \wedge \mathsf{Fy}) \wedge \mathsf{Rxyy})$

False.

It translates to: "for every x there is y such that x is even, and y is odd, and x = 2y."

Simply let a be any odd number, say 1. Then Ga is false, so for every b, $(\mathsf{Ga} \wedge \mathsf{Fb}) \wedge \mathsf{Rabb}$ is false.

(7) $\forall \mathsf{x} \forall \mathsf{y}(\exists \mathsf{z}(\mathsf{Rzxn} \wedge \mathsf{Ryzn}) \rightarrow (\mathsf{Gx} \rightarrow \mathsf{Gy}))$

True.

It says "if you add 2 to an even number, you get an even number." (Remember n stands for the number one.)

Assume a, b are two arbitrary natural numbers such that there exists a third natural c with the properties $c = a + 1$ (i.e. Rzxn) and $b = c + 1$ (i.e. Ryzn).

So $b = a + 2$. Then it's clearly true that if a is even (i.e. Gx) then so is b (i.e. Gy).

The naturals a, b were arbitrary, so we can add the two universal quantifiers and the full statement is true. (Again I was being less formal and swept a lot of the details under the rug.)

(8) $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z}(((\mathsf{Fx} \wedge \mathsf{Fy}) \wedge \mathsf{Rzxy}) \rightarrow \mathsf{Gz})$

True.

It translates to: "the sum of two odd numbers is even."

Assume a, b, c are three arbitrary natural numbers such that a and b

are odd (i.e. $\mathsf{Fx} \wedge \mathsf{Fy}$) and $c = a + b$ (i.e. Rzxy). Then c is even (Gz is true). All three numbers were arbitrary, so we can add 3 universal quantifiers.

(9) $\forall\mathsf{x}((\mathsf{Gx} \wedge \neg\mathsf{Rxnn}) \rightarrow \exists\mathsf{y}\exists\mathsf{z}((\mathsf{Hy} \wedge \mathsf{Hz}) \wedge \mathsf{Rxyz}))$

Unknown!

It translates to: "if x is an even number other than 2, then x is the sum of two primes."

This is known as the *Goldbach Conjecture* and it is currently unknown whether it's true or false! It is one of the most famous unsolved mathematical problems.

(10) $\forall\mathsf{x}\exists\mathsf{y}((\mathsf{Hy} \wedge \mathsf{Lxy}) \wedge \exists\mathsf{w}\exists\mathsf{z}((\mathsf{Rwyn} \wedge \mathsf{Rzwn}) \wedge \mathsf{Hz}))$

Unknown again!

It says: "for every x there is a prime y bigger than x, such that y+2 is also a prime."

This is also a famous unsolved problem called the *Twin Prime Conjecture.*

(d*) Show that if the wff α doesn't contain the dummy name δ, then α is true on the valuation q if and only if it is also true on any expansion q_δ.

No solution available for this one.