Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 35: Q-valuations

Thanks to "spamegg" (https://github.com/spamegg1/) for draft solutions!

(a) *Take a QL language, with a couple of proper names, a couple of unary predicates, and a binary predicate, and consider the following q-valuation:*

The domain is {Romeo, Juliet, Benedick, Beatrice}

m: Romeo
n: Juliet

F: {Romeo, Benedick}
G: {Juliet, Beatrice}
L: {⟨Romeo, Juliet⟩, ⟨Juliet, Romeo⟩, ⟨Benedick, Beatrice⟩, ⟨Beatrice, Benedick⟩, ⟨Benedick, Benedick⟩}.

Then what are the truth values of the following wffs?

(1) ∃xLmx

True, intuitively: Romeo loves someone.

True, formally. According to our rule for evaluating existential wffs, ∃xLmx is true on the given valuation q just if there is an expanded valuation q_{a}, assigning some object in the domain as reference to the dummy name a, which makes Lma true. Which there is – take the expansion which assigns Juliet as reference to a.

(2) ∀xLxm

False, intuitively: not everyone loves Romeo.

False, formally. According to our rule for evaluating universal wffs, ∀xLxm is false on the given valuation q just if there is an expanded valuation q_{a}, assigning some object in the domain as reference to the dummy name a, which makes Lam false. Which there is – take the expansion which assigns Benedick as reference to a.

(3) (∃xLmx → Lmn)

True. A material conditional with a true antecedent (as we've shown), and a true consequent (obviously, since the pair ⟨Romeo, Juliet⟩ is in the extension of L).

(4) ∀x(Fx → ¬Gx)

True intuitively. Putting it sloppily, the Fs are the men in the domain, the Gs are the women, and none of the men are women.

True, formally. According to our rule for evaluating universal wffs, ∀x(Fx → ¬Gx) is true on the given valuation q just if *every* expanded valuation q_{a} makes (Fa → ¬Ga). Which it does. When a is assigned Romeo or Benedick, the antecedent and consequent of the conditional are both true; when a is assigned Juliet or Beatrice, the antecedent and consequent of the conditional are both false.

(5) ∀x(Gx → (Lxm ∨ ¬Lmx))

True intuitively. Any woman in the domain is either loved by Romeo or doesn't love him.

True, formally. ∀x(Gx → (Lxm ∨ ¬Lmx)) is true on the given valuation q just if *every* expanded valuation q_{a} makes (Ga → (Lam ∨ ¬Lma). Which it does. When a is assigned Romeo or Benedick, the antecedent of this conditional is false, so the whole conditional is true; when a is assigned Juliet, Ga and Lam are true, making the whole conditional true; when a is assigned Beatrice, Ga and ¬Lma are true, making the whole conditional true again.

(6) $\forall x(Gx \to \exists yLxy)$

True intuitively: every woman in the domain loves someone.

True, formally. Because $\forall x(Gx \to \exists yLxy)$ is true on the given valuation q just if *every* expanded valuation q_a makes (*) $(Ga \to \exists yLay)$ true. Which it does:

i. When a is assigned Romeo or Benedick, the antecedent of this conditional is false, so (*) is true.

ii. When a is assigned Juliet, the antecedent of (*) is true, and the consequent is true if $\exists yLay$ is true on q_a, i.e. if there is some expanded valuation q_{ab} which makes Lab true – which there is, because we can assign Romeo to b.

iii. Similarly when a is assigned Beatrice, the antecedent of (*) is true, and the consequent is true if $\exists yLay$ is true on q_a, i.e. if there is some expanded valuation q_{ab} which makes Lab true – which there is, because we can this time assign Benedick to b.

(7) $\exists x(Fx \wedge \forall y(Gy \to Lxy))$

False intuitively: it's not the case that, in this domain, some man loves every woman.

Formally, $\exists x(Fx \wedge \forall y(Gy \to Lxy))$ is true on q just if *some* expanded valuation q_a makes (*) $(Fa \wedge \forall y(Gy \to Laxy))$ true.

i. But obviously an expanded valuation that assigns Juliet or Beatrice to a and hence makes Fa false can't make (*) true.

ii. What about the valuation which assigns Romeo to a? Then (*) is true just if $\forall y(Gy \to Lay)$ is true on q_a, i.e. just if (**) $(Gb \to Lab)$ is true on *every* further expanded valuation q_{ab}. But assign Beatrice of b, and (**) is false. So (*) isn't true on q_a when a is assigned Romeo.

iii. Similarly when a is assigned Benedick.

So (*) comes on true on no expanded valuation q_a, so the wff (7) is false on q.

(b) *Now take the following q-valuation*:

The domain is $\{4, 7, 8, 11, 12\}$

m: 7
n: 12
F: the set of even numbers in the domain
G: the set of odd numbers in the domain
L: the set of pairs $\langle m, n\rangle$ where m and n are in the domain and $m < n$.

What are the truth values of the wffs (1) to (7) now?

(1) True

(2) False

(3) True

The antecedent is true by (1), the consequent Lmn is true because 7 ¡ 12.

(4) True

(5) False.

$\forall x(Gx \to (Lxm \vee \neg Lmx))$ is true on the given valuation q just if *every* expanded valuation q_a makes (*) $(Ga \to (Lam \vee \neg Lma)$ true.

But take the valuation which assigns 11 to a. Then Ga is true, Lam is false and $\neg$Lma is false. Which makes (*) false.

(6) True.

In our domain, every odd number is less than some number.

(7) True.

In our domain, there is an even number (4, in fact) which is less than every odd number.

(c) *Take the language* QL_3 *of Exercises 30(b) whose non-logical vocabulary comprises the name* n*, the one-place predicates* $\mathsf{F}, \mathsf{G}, \mathsf{H}$*, the two-place* L*, and three-place predicate* R*. Consider the following q-valuation (the natural one suggested by the given glossary for the language):*

The domain is the set of natural numbers (the integers from zero up)
n: one
F: the set of odd numbers
G: the set of even numbers
H: the set of prime numbers
L: the set of pairs $\langle m, n\rangle$ such that $m < n$
R: the set of triples $\langle l, m, n\rangle$ such that $l = m + n$.

Carefully work out the values of the wffs (1), (2), (4) and (5) from Exercises 30(b), i.e. the wffs

(1) $\forall\mathsf{x}\forall\mathsf{y}\exists\mathsf{zRzxy}$

True.

Informally this says, given any two numbers, they have a sum. ("The natural numbers are closed under addition.")

Formally, (1) is true on the given valuation q if for every expansion q_{ab} – i.e. whatever values we assign a and b – (*) $\exists\mathsf{zRzab}$ is true. Which it is!

(2) $\exists\mathsf{y}\forall\mathsf{xLxy}$

False.

Informally it translates to: "there is a number bigger than all others."

Formally, we could argue by contradiction. So assume (2) evaluates as true on the given q-valuation q. Then there exists an expanded q-valuation q_{a}, where the dummy name a is assigned the natural number a, such that $\forall\mathsf{xLxa}$ is true.

That means that on every further expanded q-valuation q_{ab}, whatever b is assigned Lba is true on q_{ab}. But that's absurd. Suppose b is assigned $a + 1$!

(4) $\forall\mathsf{x}(\mathsf{Hx} \to \exists\mathsf{y}(\mathsf{Lxy} \wedge \mathsf{Hy}))$

This is true on the given valuation q is true iff for any extended valuation q_{a} (where a now denotes the number a) the wff (*) $(\mathsf{Ha} \to \exists\mathsf{y}(\mathsf{Lay} \wedge \mathsf{Hy}))$ is true.

But the conditional (*) is true on any q_{a} because its consequent is true on any q_{a} – there is aways an extension q_{ab} (where b now denotes the number b) which makes $\mathsf{Lab} \wedge \mathsf{Hb}$ true. Because for any a there is a larger prime b.

(There a famous and elegant ancient Greek proof from Euclid's *Elements* for any number there is a larger prime. You can look it up on the web if you don't know it!)

(5) $\forall\mathsf{x}\forall\mathsf{y}((\mathsf{Fx} \wedge \mathsf{Ryxn}) \to \neg\mathsf{Fy})$

True if and only if on every extended valuation q_{ab} (whatever those dummy names a, b pick out) the wff $((\mathsf{Fa} \wedge \mathsf{Rban}) \to \neg\mathsf{Fb})$ is true. Which it is, because if a is odd, and $b = a + 1$, then b is not odd.

(d*) *Show that if the wff* α doesn't *contain the dummy name* δ*, then* α *is true on the valuation* q *if and only if it is also true on any expansion* q_δ*.*

Obviously, the idea is that if α *doesn't* contain the dummy name δ then what value we give δ on a valuation can't affect the value of α.

And that's basically right, but the devil is the details. I have to confess a mess-up here. Although the basic ideas are in good order, the particular implementation in the book of the idea of a δ-expansion q_δ gets into a bit of trouble in some special cases.

For example, how would the story go for the value of $\forall \mathsf{x}\mathsf{F}\mathsf{x}$ on the extended valuation q_{a}?

Remember we say that $\forall \mathsf{x}\mathsf{F}\mathsf{x}$ is true on a valuation if Fa (plugging in the first available dummy name) is true on every expansion of that valuation assigning a value to a, whatever value we choose. So, applied to the present case, $\forall \mathsf{x}\mathsf{F}\mathsf{x}$ is true on the extended valuation q_{a} if it is true on every expansion of *that* valuation which assigns a value to a. But hold on! By definition q_{a} already assigns an object to a. 'Expansions' here is the wrong word – to get what we want, we must here take ourselves to be considered variations of the valuation q_{a} which spin the interpretation of a. But this doesn't really tally with the strict letter of what is in the book! Oooops.

We could fiddle with the definition of a δ-expansion at the top of p. 35. But in fact, looking at this chapter, I think some rewriting is really needed both to tidy up the glitch but also to make things clearer overall.