Spamegg's Commentary on "In-Class Problems Week 1, Wed."

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Contents

1 Facts and definitions from mathematics that are used

1.1 Chicken vs egg problem, again

This section requires a lot of stuff that are taught in later sections for some reason. Like Logic, propositions (Chapter 3 of the textbook).

1.2 Facts and definitions

Here we will be using:

The definition of logarithm: for positive real numbers $a \neq 1$ and y, we define $log_a(y)$ to be the real number x such that $a^x = y$.

The definition of a rational number (which is detailed later).

The definition of square root: For a nonnegative real number x, the positive square root $\sqrt{x}$ of x is defined as the POSITIVE real number y such that $y^2 = x$.

Some basic laws of exponents: like $(a^b)^c = a^{bc}$ and $(ab)^c = a^c b^c$.

The Fundamental Theorem of Arithmetic (which is thousands of years old, it's from Euclid's Elements and is proved later in the course):

Every integer $n > 1$ can be uniquely represented as a product of prime powers: if $n > 1$ then there exist unique primes $p_1 < \ldots < p_k$ and unique positive integers $a_1, \ldots, a_k$ such that

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$

Basic logical connectives: like AND $\wedge$, OR $\vee$, NOT $\neg$, IMPLIES $\implies$ where (T being True and F being False):

$$T \wedge T = T, \;\; T \wedge F = F, \;\; F \wedge T = T, \;\; F \wedge F = F,$$
$$T \vee T = T, \;\; T \vee F = T, \;\; F \vee T = T, \;\; F \vee F = F,$$
$$(T \implies T) = T, \;\; (T \implies F) = F, \;\; (F \implies T) = T, \;\; (F \implies F) = T$$

Some laws about logical connectives, such as de Morgan's Laws:

$$\neg(A \wedge B) = (\neg A) \vee (\neg B), \;\; \neg(A \vee B) = (\neg A) \wedge (\neg B),$$

or the transitivity of implication: if $A \implies B$ and $B \implies C$ then $A \implies C$.

2 Problem 1.

2.1 Hidden assumptions

In the "picture proofs" of the Pythagorean Theorem, Prof. Meyer mentions "hidden assumptions".

THIS IS EXTREMELY IMPORTANT! For the rest of the course, you should ALWAYS watch out for hidden assumptions you might be making.

You might have learned a lot of things in high school math and geometry that you take for granted, or you can "intuitively see" that something is correct. Quite often IT WILL NOT BE. Your intuition will lie to you!

If you interact with me on Discord I might keep pushing you to such a level of rigor and detail that you might find annoying, or even get angry about it. THAT'S EXCELLENT.

This is what I see when I tutor learners on Discord. Their proofs are unconvincing to me, because I take a very skeptical view and try not to assume anything. In effect I try to act like a baby who knows nothing but can understand basic English and can follow basic explanations.

One way to detect, avoid or justify hidden assumptions is to do just that: in your mind, always act like you are trying to explain what you are learning to an extremely curious and patient baby who challenges you on EVERYTHING.

2.2 Chicken vs egg problem

One issue with early proof courses is that, it's too difficult to teach you purely theoretical proofs using symbolic logic right off the bat. But the course has to show you proofs of SOMETHING. That something usually ends up being some math you learned in high school, involving numbers, maybe square roots or logarithms, or some geometry.

But you were probably not shown proofs when you learned that stuff. So you don't know how to explain or justify "what you already need to know" to learn these proofs. You cannot tell what is a hidden assumption and what is not; what needs more explanation and what doesn't. So there is a bit of a chicken vs egg problem there.

When you try to justify all the hidden assumptions, eventually you might hit a point where some high school math knowledge, or maybe even Calculus knowledge, is required. Mark those points in your proofs and annotate them, like "from high school math we know that..." or "from basic Single Variable Calculus it follows..." Make it abundantly clear what fact is being used.

2.3 Detailed example of hidden assumptions

Here is an example: a learner is trying to prove that $\log_4(6)$ is irrational. Here we can assume that the reader knows the definition of $\log_4(x)$: $y = \log_4(x)$ if $4^y = x$.

This learner actually has fairly good grasp of argumentation and logic. Here is what they say:

1. We use proof by contradiction. *Good! Many people don't write this!*

2. Suppose $\log_4(6)$ is rational. *OK.*

3. Then we can write $\log_4(6)$ as a fraction $\frac{n}{m}$ in its lowest terms.

Hmm... what are these n and m? Are they integers, natural numbers, real numbers, are they positive, negative, can they be 0? Why are we able to write it like that? What's "lowest terms?"

There is an implicit hidden assumption of common knowledge between the author and the reader... The author assumes that the reader knows the definition of a rational number. It would be better if the author mentioned this definition properly.

4. Let $\log_4(6) = n/m$, then using logarithmic identities we can find $4^{n/m} = 6$, exponentiating both sides by m, equation becomes $4^n = 6^m(*)$.

Umm... how are we LETTING $\log_4(6)$ *be equal to n/m? What are these n and m anyway? Do they have specific values? Did we make them up? Where did they come from? I like the explicit mention of "logarithmic identities", that's excellent!*

5. Then as $4^1 = 4$, $4^2 = 16$ n/m must be $1 < n/m < 2$.

Umm... what? I don't get it at all. How is that related to $4^n = 6^m$? I think the author is trying to say something like this: we know that 4^x is a monotone increasing function. We know that 6 is in between $4^1 = 4$ and $4^2 = 16$. We know that $4^{n/m} = 6$. Since $4^1 = 4 < 6 = 4^{n/m} < 16 = 4^2$ and since 4^x is increasing, we can get rid of the bases and the same inequality holds for the exponents: $1 < n/m < 2$. We are also using the fact that strictly monotone increasing functions must be one-to-one.

See all the hidden assumptions and explanations there? For this proof the fact that 4^x is a strictly monotone increasing one-to-one function is actually totally unnecessary. And also see how it's not directly related to the conclusion of the previous step. The previous step should have stopped at $4^{n/m} = 6$.

6. Also $m \neq 0$ because it would make the fraction $\frac{n}{0}$ which is undefined.

OK... I understand the intent, we all know from high school "YOU CAN'T DIVIDE BY ZERO!!!" but once again where did n and m come from? I thought we LET them be whatever we wanted. Like, we had choice over what they are. Shouldn't this have been mentioned at the beginning, when n and m came into existence?

Are you annoyed and angry yet?

7. And $n \neq 0$ because if $n = 0$ then $m = 0$ for (*) which would contradict (6).

OK... but it's actually possible, in theory at least, that $n = 0$ and $m \neq 0$ because $\log_4(6)$ COULD be zero, or positive, or negative (unless we plug it into a calculator and look at the result). We know from Calculus that all the logarithm functions cross the x axis at $x = 1$, where they take on the value 0 (because $a^0 = 1$ for all positive a). The author should have mentioned somewhere that $\log_4(6)$ cannot be zero. Once again all of this should have been cleared at the beginning regarding n and m.

I can let this one slide since it's not too bad... but once again there are a lot of hidden assumptions, like the author assumes we know $\log_4(6)$ is not zero. In a proof where n and m not being zero has a lot of importance, this should have been mentioned.

8. Rewrite $4^n = 6^m$ as $2^{2n} = 2^m \cdot 3^m$ and $2^{2n-m} = 3^m$ (**). *OK...*

9. 3 and 2 are prime numbers so it's impossible to get 3^a by exponentiating 2 for $a \neq 0$. *WHY???*

Here we are using another hidden assumption: Fundamental Theorem of Arithmetic! We are using the Uniqueness part of the theorem. But... we are not even doing that correctly. Because... 2^{2n-m} and 3^m MIGHT NOT EVEN BE NATURAL NUMBERS! If the exponents are negative, they would be 1 / something.

10. Then there is no solution for (**) which means there is also no solution for (*). Which is a contradiction.

NO!!! By Step 9 there COULD BE a solution: we can take $a = 0$ which gives: $2n - m = m = 0$.

Again I get what the author is trying to say... since they showed that $n \neq 0$ and $m \neq 0$ there are no solutions... but it's wrong to state that there are no solutions, because there are. The contradiction here would be to show that $n = m = 0$ which contradicts $n \neq 0$, $m \neq 0$ (actually we only need ONE of those contradictions).

You are probably SUPER angry with me now. GOOD. Go punch a pillow and come back.

2.4 A better proof

Let's do a better version of this proof. The main problems are with invoking definitions or known facts/theorems to clarify everything to the reader.

1. Argue by contradiction and assume $\log_4(6)$ is a rational number.

2. *By definition of rationality, THERE EXIST* natural numbers n and m such that $m \neq 0$, $\log_4(6) = \frac{n}{m}$. Without loss of generality we may assume $m > 0$ and n and m have no common factors greater than 1 (also known as "being in lowest terms").

As you can see we cannot LET anything. The definition of a rational number gives the existence of some natural numbers but we don't get to choose them, or even know what they might be. The issue is once again your previous knowledge. You learned the overall idea of a rational number, but never the formal definition. It's the chicken-egg problem again. I don't blame you.

I did a weird thing here: "without loss of generality"... what does that mean? Not losing generality means covering all possible cases. Technically $\frac{4}{6}$, $\frac{2}{3}$ *and* $\frac{-2}{-3}$ *are all the same, and they are all rational, representing the same number, but without losing generality we may assume* n *and* m *have no common factors, this will still cover all possible cases.*

Moreover, here we don't know if n and m are positive or negative or what.

If $\log_4(6)$ is positive, then either n and m are both positive or they are both negative.

If they are both negative we can replace them with $-n$ and $-m$ so they are both positive.

If $\log_4(6)$ is negative, then either n is negative and m is positive, or n is positive and m is negative, in which case we can replace them with $-n$ and $-m$ again.

It should be OK to summarize these "definitional issues" in a "WLOG" statement... That's right, mathematicians have a handy abbreviation for everything! Because they are SUPER LAZY.

OR... using some Calculus knowledge about either $\log_4(x)$ *or* 4^x *we can argue that* $\log_4(6)$ *must be positive. It's up to you how much previous knowledge you want to assume for your reader. I try to remove all assumptions that I can.*

3. By definition of $\log_4$, we have $4^{n/m} = 6$.

4. Exponentiating both sides to the exponent m we obtain $4^n = 6^m$.

5. Using the laws of exponents and rewriting, we get $4^n = \left(2^2\right)^n = 2^{2n}$ and $6^m = (2 \cdot 3)^m = 2^m \cdot 3^m$, therefore $2^{2n} = 2^m \cdot 3^m$.

6. Dividing both sides by 2^m we get $2^{2n-m} = 3^m$.

7. Since $m > 0$, we know that 3^m is a natural number greater than 1.

8. Therefore 2^{2n-m} is also a natural number greater than 1.

9. By the Fundamental Theorem of Arithmetic, the prime factorization of any natural number must be unique. But 2^{2n-m} and 3^m give us two different prime factorizations of the same natural number. This is a contradiction!

10. Therefore our initial assumption was false, so $\log_4(6)$ must be rational. QED

Here is another mathematician thing: QED is short for "quod erat demonstrandum" which means "what was to be shown" in Latin. Feel cool yet?

Now there is another meta-issue here: how did I know that Calculus knowledge was not necessary, but the Fundamental Theorem of Arithmetic is unavoidable in this proof? This comes from intuition built upon years of proof practice. Don't worry too much about this yet, you'll grasp it eventually.

2.5 Keep this meta-model in mind

What is the take away here? First of all, you are always writing a proof for a BABY. A reader who is stupider and less knowledgeable than you. You need to make sure they can understand your proof. Secondly, you are writing a proof for your superior, like your instructor: someone who is smarter and more knowledgeable than you.

You need to demonstrate to your superior that you understand how much detail needs to be shown to the baby.

A good instructor will play the role of the baby for you, to make sure you understand hidden assumptions, to make sure you have good judgment of how much detail needs to be shown. These are meta-cognitive skills.

Instructor
||
V
You
||
V
Reader

You write ONE proof simultaneously for both of these audiences, satisfying both of their needs. The tension and push-pull between these two, with you in the middle, makes your proofs better.

In my years of grading students' proofs, many times I came across proofs that were technically true, but the steps skipped a ton of details. I, the instructor, after thinking quite a bit about them, could see that the steps were correct, but the student did not demonstrate enough detail that they understand why it's correct. It's possible for students to guess a step without understanding why, without seeing the mini-steps leading to it; or even bullshit their way through a proof by hiding a lot of details on purpose. Think about it: let's say I start with A and want to prove B. I do it in ONE STEP: I just jump to B! I proved it! My proof is impenetrable! There is nothing wrong with my proof that you can point out, and it's correct! I'm so smart I can immediately see how to go from A to B! You see the problem.

3 Problem 2.

This regards the bogus proof:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (\sqrt{-1})^2 = -1$$

The issue is with the step $\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$. We are assuming some sort of hidden property of the square root symbol $\sqrt{}$ (whatever $\sqrt{}$ means): $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. (We'll get back to this property later.)

3.1 Definitions, definitions

Indeed, what DOES $\sqrt{}$ mean?

There are a lot of learners who are confused by square roots, because they are not properly taught in schools. The issue, and this will be a universal, permanent issue throughout the course, is one of DEFINITIONS.

Quite often *the square root function* is confused with *solutions to a quadratic equation.*

The positive square root function $f(x) = \sqrt{x}$ is DEFINED on nonnegative real numbers x and its output is also nonnegative.

The negative square root function $f(x) = -\sqrt{x}$ is also DEFINED on nonnegative real numbers x and its output is nonpositive.

These are matters of DEFINITION. We simply define these functions to be like that. Otherwise they would not fulfill the definition of being a function. The square root operation, or at least as we intuitively understand it, without involving complex numbers, would not accept negative input, and also a function must have only one output for each input. (That is the DEFINITION of being a function! See? IT'S ALL ABOUT DEFINITIONS.)

"Taking the square root of both sides" of an equation like $x^2 = 4$ to get solutions $x = \pm 2$ is totally different than that. We are being lazy: instead of doing:

$$x^2 = 4 \implies x^2 - 4 = 0 \implies (x-2)(x+2) = 0 \implies x = \pm 2$$

we are taking the shortcut of "taking the square root of both sides". Here the "square root" cannot be the positive function we mentioned above: because it cannot output negative values. So we are actually implicitly applying the two different, separate square root functions to x^2 and to 4. Hidden assumptions that lead to issues!

We are using a hidden assumption here even in the non-lazy solution... can you see what it is?

We are using the fact that if $a \cdot b = 0$ then either $a = 0$ or $b = 0$. This is actually an axiom called "no zero divisors" that only SOME mathematical structures satisfy, called *integral domains*! For example the mathematical structure known as "integers modulo 6": $\mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$ does not satisfy this property: $2 \cdot 3 = 0$ but neither 2 nor 3 is equal to 0. Wild isn't it? It's OK to assume that your reader knows the fact that the real numbers $\mathbb{R}$ is an integral domain.

What is the solution to issues like this? STOP BEING LAZY. Write your proofs IN DETAIL, with English, instead of a bunch of weird vague equations, where the flow of the logic is unclear. If you were taught properly, or somehow brilliantly figured out how to overcome your poor education, you know that you should always pile up all the terms on one side, leaving 0 on the other side, then factor out if possible, then use the implicitly assumed "no zero divisors" axiom that holds for the real numbers $\mathbb{R}$.

3.2 Part b. Very Crucial!

In the video and slides, Prof. Meyer gives a correct proof that: if $1 = -1$ then $2 = 1$. It goes like this (here $\implies$ means "implies"):

$$1 = -1 \implies \frac{1}{2} = -\frac{1}{2} \implies \frac{1}{2} + \frac{3}{2} = -\frac{1}{2} + \frac{3}{2} \implies 2 = 1$$

This is an example of universally misunderstood things about implications

Proving $A \implies B$ does not mean you proved B!

If you assume a false A you can prove literally any B! Try proving: if $1 = -1$ then $3 = 1$. Hell, you should even be able to prove $1000000 = 1$!

Proving B from a chain of implications which all look correct does not mean you proved B! If you started from a false assumption then you haven't proved B.

If you started with what needs to be shown, and concluded something true from it, while using all correct implications, you still haven't shown anything! Implications (in general) go only in one direction, not both directions. $A \implies B \implies C \implies D$ where D is true does not mean A, B, C are true! (They MAY be true, but that would be a coincidence; their truth does not follow from the implications proven.)

There is actually some real research about humans' problem with "if A then B". Humans struggle quite a lot universally with the fact that A can be false while $A \implies B$ can be true. Humans also have trouble with simply assuming A in order to prove $A \implies B$.

3.3 Part c. Let's actually prove that square root property!

Prof. Meyer corrects us about the definition of the positive square root function. Then we are given a task:

Assuming familiar properties of multiplication of real numbers, prove that for positive real numbers r, s: $\sqrt{rs} = \sqrt{r}\sqrt{s}$.

Notice how Prof. Meyer mentions the appropriate assumptions, making us aware of hidden assumptions and to watch out for them!

Here is a different task: can you come up with a bogus proof? Think about it before you continue.

OK, here's my bogus proof of this property:

Proof. Assume r and s are positive real numbers. Then:

$$\begin{aligned}
\sqrt{rs} &=^? \sqrt{r}\sqrt{s} && (A)\\
(\sqrt{rs})^2 &=^? (\sqrt{r}\sqrt{s})^2 && (B)\\
rs &=^? (\sqrt{r})^2(\sqrt{s})^2 && (C)\\
rs &= rs && (D)
\end{aligned}$$

□

When we write equational proofs like this, there is an implicit understanding: we are assuming that it means "the first equation implies ($\implies$) the second, the second implies the third, and so on...". Each equality is supposed to do legal operations on the previous equality, either to one side of the equality or sometimes to both sides at the same time, concluding something correct from it. (Later on Prof. Meyer will warn us that **proofs are not equations!**)

So it's logically equivalent to $\sqrt{rs} =^? \sqrt{r}\sqrt{s} \implies (\sqrt{rs})^2 =^? (\sqrt{r}\sqrt{s})^2 \implies rs =^? (\sqrt{r})^2(\sqrt{s})^2 \implies rs = rs$

Although I don't know what $=^?$ means. I guess it means: "is this true? I don't know... let me manipulate it into something else and see if that's true..." It's a bogus proof after all. It has nonsense in it.

This is a perfect illustration of part b. All the implications, going from the first step to the last, are correct, but the proof is wrong! Because we put the cart before the horse. *We assumed what was to be shown, and concluded something true.* But this proof does not work: We proved $A \implies B \implies C \implies D$ where D is true, but we don't know if A is true. So we only know D is true (because it's obvious by itself), but cannot say anything about A, B, C for sure from this chain of implications. (Even if A, B, C happen to be true, it's incidental.) Basically we did everything backwards! We should have tried to show $D \implies C \implies B \implies A$ instead

This method of *arguing backwards* is wrong as a proof, but it can be a very helpful technique for discovering how a proof should go. Just make sure not to actually use it as a proof! It should only be used as a way to develop some proof ideas.

So let's try to prove *the reverse implications* (notice the removal of the nonsensical $=^?$):

$$\begin{aligned}
rs &= rs && (D)\\
rs &= (\sqrt{r})^2(\sqrt{s})^2 && (C)\\
(\sqrt{rs})^2 &= (\sqrt{r}\sqrt{s})^2 && (B)\\
\sqrt{rs} &= \sqrt{r}\sqrt{s} && (A)
\end{aligned}$$

Why would this work? This time we are starting from a TRUE first assumption: $rs = rs$. If D is true and we show $D \implies C \implies B \implies A$ then we can conclude that C, B, A are all true!

So let's try to verify them. We need to show:

(1) $rs = rs$ implies $rs = (\sqrt{r})^2(\sqrt{s})^2$ ($D \implies C$),

(2) $rs = (\sqrt{r})^2(\sqrt{s})^2$ implies $(\sqrt{rs})^2 = (\sqrt{r}\sqrt{s})^2$ ($C \implies B$),

(3) $(\sqrt{rs})^2 = (\sqrt{r}\sqrt{s})^2$ implies $\sqrt{rs} = \sqrt{r}\sqrt{s}$ ($B \implies A$).

Now it is going to look like I'm going way out of my way to prove really really obvious things, but bear with me. We will keep INVOKING THE DEFINITION of the square root. Get used to invoking definitions: this is your primary way of graduating from amateur non-proofs to actual proofs.

To show (1): ($D \implies C$) we notice that the left-hand sides (LHS) are the same, so we only need to show that the right-hand sides (RHS) are equal: $rs = (\sqrt{r})^2(\sqrt{s})^2$.

BY DEFINITION of the positive square root, $\sqrt{r}$ is the positive real number x such that $x^2 = r$. Therefore $(\sqrt{r})^2 = x^2 = r$.

Similarly, BY DEFINITION of the positive square root, $\sqrt{s}$ is the positive real number y such that $y^2 = s$. Therefore $(\sqrt{s})^2 = y^2 = s$.

Multiplying the two equalities above, we get $(\sqrt{r})^2(\sqrt{s})^2 = rs$. This proves (1): ($D \implies C$).

To prove (2): ($C \implies B$) we need to show that the LHSs are equal to each other and the RHSs are equal to each other. So we need to show: $rs = (\sqrt{rs})^2$ and $(\sqrt{r})^2(\sqrt{s})^2 = (\sqrt{r}\sqrt{s})^2$.

To prove the first one, BY DEFINITION of the positive square root, $\sqrt{rs}$ is the positive real number z such that $z^2 = rs$. Therefore $(\sqrt{rs})^2 = z^2 = rs$. So this proves that the LHSs are equal.

To prove the second one, we have:

$$\begin{aligned}
(\sqrt{r})^2(\sqrt{s})^2 &= \sqrt{r}\cdot\sqrt{r}\cdot\sqrt{s}\cdot\sqrt{s} && \text{(by definition of square)}\\
&= \sqrt{r}\cdot\sqrt{s}\cdot\sqrt{r}\cdot\sqrt{s} && \text{(by commutativity of multiplication)}\\
&= (\sqrt{r}\cdot\sqrt{s})\cdot(\sqrt{r}\cdot\sqrt{s}) && \text{(by associativity of multiplication)}\\
&= (\sqrt{r}\sqrt{s})^2 && \text{(by definition of square)}
\end{aligned}$$

So this proves the RHSs are equal, finishing the proof of (2): ($C \implies B$). As you can see, it was necessary to follow Prof. Meyer's instruction to "assume familiar properties of multiplication of real numbers"!

Finally to prove (3): $B \implies A$ we need to follow a different strategy, because this time the LHSs of B and A are not equal to each other, and the RHSs of B and A are not equal to each other.

Assume B, in other words assume $(\sqrt{rs})^2 = (\sqrt{r}\sqrt{s})^2$. We want to prove A, in other words we want to prove $\sqrt{rs} = \sqrt{r}\sqrt{s}$. I bet that your first instinct is to "just take square roots of both sides":

$$
\begin{aligned}
(\sqrt{rs})^2 &= (\sqrt{r}\sqrt{s})^2 \\
\sqrt{(\sqrt{rs})^2} &= \sqrt{(\sqrt{r}\sqrt{s})^2} \\
\sqrt{rs} &= \sqrt{r}\sqrt{s}
\end{aligned}
$$

But this argument relies on: "if $x^2 = y^2$ then $x = y$". This is clearly NOT true in general: let $x = -2$ and $y = 2$. However we know that $r, s, \sqrt{r}, \sqrt{s}$ are all positive in our problem. We can deduce $\sqrt{rs}$ and $\sqrt{r}\sqrt{s}$ are also positive. So now we are relying on the fact: "if x and y are both positive and $x^2 = y^2$ then $x = y$".

This is certainly TRUE, but it relies on the fact that the square function $f(x) = x^2$ is one-to-one on the positive half of the real number line $(0, +\infty)$. This requires some Calculus: we could prove that f is strictly increasing on $(0, +\infty)$ by taking its derivative and showing that the derivative is positive; then we invoke a theorem from Calculus that says "a strictly increasing function is one-to-one".

Can we do better? That is, not rely on Calculus to finish our proof?

Let's start with B again, but MOVE EVERYTHING TO ONE SIDE AND FACTOR. We are simply using the algebraic fact that $a^2 - b^2 = (a - b)(a + b)$:

$$
\begin{aligned}
(\sqrt{rs})^2 &= (\sqrt{r}\sqrt{s})^2 \\
(\sqrt{rs})^2 - (\sqrt{r}\sqrt{s})^2 &= 0 \\
(\sqrt{rs} - \sqrt{r}\sqrt{s}) \cdot (\sqrt{rs} + \sqrt{r}\sqrt{s}) &= 0
\end{aligned}
$$

Now we use the fact that for real numbers a, b, if $ab = 0$ then either $a = 0$ or $b = 0$:

$\sqrt{rs} - \sqrt{r}\sqrt{s} = 0$ or $\sqrt{rs} + \sqrt{r}\sqrt{s} = 0$.

Now we argue that the second one is impossible: since r and s are positive, rs is positive, and by definition, the positive square roots $\sqrt{r}$, $\sqrt{s}$ and $\sqrt{rs}$ are all positive too. Therefore $\sqrt{rs} + \sqrt{r}\sqrt{s}$ is positive, therefore it is not zero.

Hence it must be that $\sqrt{rs} - \sqrt{r}\sqrt{s} = 0$, in other words $\sqrt{rs} = \sqrt{r}\sqrt{s}$, which is A (what we were trying to prove).

This finishes the proof of $\sqrt{rs} = \sqrt{r}\sqrt{s}$ finally! We didn't use anything more than basic rules of algebra, definition of square and square root, and properties of multiplication that Prof. Meyer mentioned!

3.4 Are you crazy, Spam?

You might think I went into excruciating detail for a seemingly obvious and trivial proof. But this is exactly the point where you should develop the discipline of being painstakingly meticulous in your proofs: if you don't practice working everything all the way down to their basic assumptions NOW, you will have a lot of trouble with harder proofs later. You need to develop the mental patterns, learn logical rules, grow very sharp reasoning and argumentation abilities on the simpler proofs, or you'll be lost when you have to do Induction, or even worse, Graph Theory proofs.

Remember the "mental model" I mentioned earlier? You write proofs for an audience that is below you (breaking it down enough that they can understand it), and an audience that is above you (convincing them that you understand all the details that need to be shown to a casual reader):

Instructor
||
V
You
||
V
Reader

What I did with the above proof is one step further in this meta-hierarchy: I taught you what's necessary for you to write proofs to satisfy both audiences:

Spamegg
||
V
Instructor
||
V
You
||
V
Reader

Now, can you write proofs that satisfy me?

4 Problem 3.

4.1 Part a.

More bogus proofs! YAY! At least this one starts with a true statement. It goes like:

$$3 > 2 \tag{1}$$
$$3 \log_{10}(1/2) > 2 \log_{10}(1/2) \tag{2}$$
$$\log_{10}(1/2)^3 > \log_{10}(1/2)^2 \tag{3}$$
$$(1/2)^3 > (1/2)^2 \tag{4}$$
$$1/8 > 1/4 \tag{5}$$

The problem is on Step (2). We are multiplying both sides of a true inequality $3 > 2$ with the same number $\log_{10}(1/2)$. According to the rules of real numbers, if r, s and

t are real numbers with $r > s$, and if $t > 0$ then $rt > st$; if $t < 0$ then $rt < st$.

So the inequality should stay the same if that number is positive; it should reverse if the number is negative. The issue is that $\log_{10}(1/2)$ is actually negative! It's around -0.3. So the inequality should reverse directions, but in the bogus proof it does not.

4.2 Part b.

This one is about UNITS. It goes like:

$$1¢ = \$0.01 = (\$0.1)^2 = (10¢)^2 = 100¢ = \$1$$

Let's try to give an accurate calculation, by using the fact that $\$1 = 100¢$ and squaring the units along with the numbers (like you would in physics). This one is certainly true:

$$1¢ = \$0.01$$

How about $(\$0.1)^2$?

$$(\$0.1)^2 = (100¢ \cdot 0.1)^2 = (10¢)^2 = 100¢^2$$

So as you can see, the second step is already wrong. $1¢$ is not the same as $100¢^2$. Step 3 seems true:

$$(\$0.1)^2 = 100¢^2 = (10¢)^2$$

Step 4 is clearly wrong:

$$(10¢)^2 = 100¢^2 \neq 100¢$$

Step 5 is correct:

$$100¢ = \$1$$

4.3 Part c.

This is one of the greatest hits of all time in bogus proofs: assume $a = b$, then we "prove" $a = 0$.

$$a = b \tag{6}$$
$$a^2 = ab \tag{7}$$
$$a^2 - b^2 = ab - b^2 \tag{8}$$
$$(a-b)(a+b) = b(a-b) \tag{9}$$
$$a + b = b \tag{10}$$
$$a = 0 \tag{11}$$

The only wrong step is going from (9) to (10): we are dividing both sides by zero! Because $a - b = 0$.

5 Problem 4.

This one is the same as our earlier example of my bogus proof of the square root property. It uses the ambiguous nonsensical $\geq^?$. For all nonnegative numbers a, b it says:

$$\frac{a+b}{2} \geq^? \sqrt{ab} \tag{12}$$

$$a+b \geq^? 2\sqrt{ab} \tag{13}$$

$$a^2+2ab+b^2 \geq^? 4ab \tag{14}$$

$$a^2-2ab+b^2 \geq^? 0 \tag{15}$$

$$(a-b)^2 \geq^? 0 \text{ true} \tag{16}$$

So it proves (12) $\implies$ (13) $\implies$ (14) $\implies$ (15) $\implies$ (16), which is true, but we don't know if (12) is true or not. Once again we assumed what we were supposed to prove. All the statements *happen to be true* in this case, but this is not shown by the implications in this bogus proof.

Once again, if we prove all the implications in reverse, the proof is correct!

There is a bit of an issue that we need to be careful with: proving the implication (14) $\implies$ (13) involves taking square roots of both sides. We have to know that both $a^2+2ab+b^2$ and $4ab$ are nonnegative (can't take square root of a negative number), and we also have to use the Calculus fact that the square root function is increasing (so that the inequality $\geq$ is preserved after taking square roots).

6 Problem 5.

This is optional (thankfully) and it's definitely not useful, helpful or educational. It's just crazy nonsense because it mixes in vague non-logical non-mathematical notions like surprise, expectations, future etc. Just skip this and don't waste your time. Thinking about wordy English paradoxes will rot your logical thinking brain.