Summary: Polar Coordinates

Definition

The **polar coordinates** of a point P are an ordered pair (r, θ) such that

$$x = r\cos(\theta) \tag{1}$$
$$y = r\sin(\theta), \tag{2}$$

where the ordered pair (x, y) give the rectangular coordinates of the point P.

In other words, given the polar coordinates (r, θ) of a point, we can find its x- and y-coordinates using these formulas.

The usual rectangular coordinates are also called **Cartesian coordinates**. To find polar coordinates from Cartesian coordinates, we use

$$r = \pm\sqrt{x^2 + y^2} \tag{3}$$
$$\theta = \arctan\left(\frac{y}{x}\right). \tag{4}$$

However, r and θ are not unique for any given point, as explained below.

Polar coordinates are motivated by the fact that we can locate a point on a plane by specifying:

r :the distance from the origin to the point,

θ :the angle of the ray from the origin to the point with the positive x-axis.

Ambiguities in polar coordinates

The polar coordinates describing a point are not unique. First,

$$(r, \theta) = (r, \theta + 2\pi n) \qquad (n \text{ any integer}).$$

That is , knowing the x- and y- coordinates only determines θ up to 2π-periodicity. We frequently use conventions such as:

$$0 \leq \theta < 2\pi$$
$$\text{or} - \pi < \theta \leq \pi.$$

Second,

$$(-r, \theta) = (r, \theta \pm \pi) \qquad -\infty < r < \infty$$
$$\text{and equivalently } (r, \theta) = (-r, \theta \pm \pi) \qquad -\infty < r < \infty.$$

Finding theta

To find θ, we first find

$$\theta_0 = \arctan\left(\frac{|y|}{|x|}\right).$$

Then we find θ using θ_0, by considering which quadrant it lies in, which is best

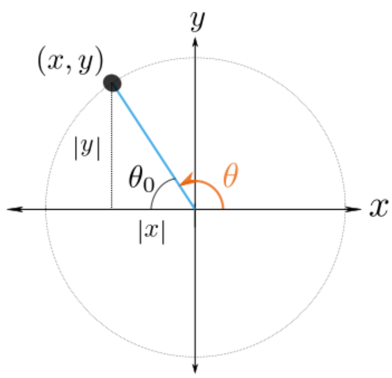


Figure 1: Here, (x, y) lies in the second quadrant, $\theta = \pi - \theta_0$.

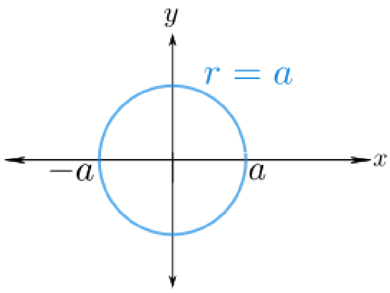
done using a picture like the one above.

Circles and rays in polar coordinates

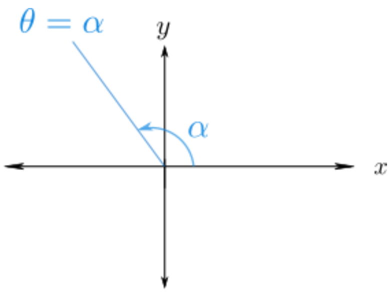
An equation in polar coordinates is called a **polar equation**. We will mostly be dealing with polar equations of the form $r = r(\theta)$.

The simplest examples are:

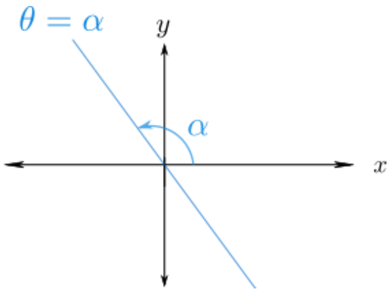
- $r = a$:



- $\theta = \alpha$ $(r \geq 0)$:



- If we use the convention $-\infty < r < \infty$, then $\theta = \alpha$ is a line through the origin:



Rotation about the origin

The graph of $r = r(\theta - \alpha)$ is obtained by rotating the graph of $r = r(\theta)$ **about the origin** by the angle $+\alpha$.

If $\alpha > 0$ the rotation is counterclockwise.

if $\alpha < 0$ the rotation is clockwise.