# Solutions to Quiz 1

**Problem 1 (20 points).** Suppose $S(n)$ is a predicate on natural numbers, $n$, and suppose

$$\forall k \in \mathbb{N}\ S(k) \longrightarrow S(k+2). \tag{1}$$

If (1) holds, some of the assertions below must *always* (**A**) hold, some *can* (**C**) hold but not always, and some can *never* (**N**) hold. Indicate which case applies for each of the assertions by **circling** the correct letter.

**(a) (2 points)** **A C N** $\forall n \geq 0\ S(n)$

**Solution.** **C**. The assertion means that $S$ is always true. So $S(k+2)$ is always true, and therefore $S(k) \longrightarrow S(k+2)$ is always true. So this case is possible. But 1 also holds when $S$ is always false, so the asertion does not always hold when 1 does. ∎

**(b) (2 points)** **A C N** $\neg S(0) \wedge \forall n \geq 1\ S(n)$

**Solution.** **C**. This time $S$ is false at 0, but true everywhere else. So $S(k) \longrightarrow S(k+2)$ still always holds because $S(k+2)$ is still always true. So this assertion can hold, but not always, since (1) can hold when $S(0)$ is true. ∎

**(c) (2 points)** **A C N** $\forall n \geq 0\ \neg S(n)$

**Solution.** **C.** Now $S$ is always false. So $S(k) \longrightarrow S(k+2)$ is always true because $S(k)$ is false. So this case is possible, but again does not always hold. ∎

**(d) (2 points)** **A C N** $(\forall n \leq 100\ S(n)) \wedge (\forall n > 100\ \neg S(n))$

**Solution.** **N**. In this case, $S$ is true for $n$ up to 100 and false from 101 on. So $S(99)$ is true, but $S(101)$ is false. That means that $S(k) \not\longrightarrow S(k+2)$ for $k = 99$. This case is impossible. ∎

**(e) (2 points)** **A C N** $(\forall n \leq 100\ \neg S(n)) \land (\forall n > 100\ S(n))$

**Solution.** **C**. In this case, $S$ is false for $n$ up to 100 and true from 101 on. So $S(k) \longrightarrow S(k+2)$ for $k \leq 100$ because $S(k)$ is false, and $S(k) \longrightarrow S(k+2)$ for $k \geq 99$ because $S(k+2)$ is true. So this case is possible, but again does not always hold ■

**(f) (2 points)** **A C N** $S(0) \longrightarrow \forall n\ S(n+2)$

**Solution.** **C**. If $S(n)$ is always true this assertion holds. So this case is possible. If $S(n)$ is true only for even $n$ (1) still holds, but $S(1+2)$ is false. So this case does not always hold. ■

**(g) (2 points)** **A C N** $S(1) \longrightarrow \forall n\ S(2n+1)$

**Solution.** **A**. This assertion says that if $S(1)$ holds, then $S(n)$ holds for all odd $n$. This case is always true. ■

**(h) (2 points)** **A C N** $[\exists n\ S(2n)] \longrightarrow \forall n\ S(2n+2)$

**Solution.** **C**. If $S(n)$ is always true, this assertion holds. So this case is possible. If $S(n)$ is true only for even $n$ greater than 4, (1) holds, but this assertion is false. So this case does not always hold. ■

**(i) (2 points)** **A C N** $\exists n\, \exists m > n\, [S(2n) \land \neg S(2m)]$

**Solution.** **N**. This assertion says that $S$ holds for some even number ,$2n$, but not for some other larger even number, $2m$. However, if $S(2n)$ holds, we can apply (1) $n - m$ times to conclude $S(2m)$ also holds. This case is impossible. ■

**(j) (2 points)** **A C N** $[\exists n\ S(n)] \longrightarrow \forall n\, \exists m > n\ S(m)$

**Solution.** **A**. This assertion says that if $S$ holds for some $n$, then for every number, there is a larger number, $m$, for which $S$ also holds. Since (1) implies that if there is one $n$ for which $S(n)$ holds, there are an infinite, increasing chain of $k$'s for which $S(k)$ holds, this case is always true. ■

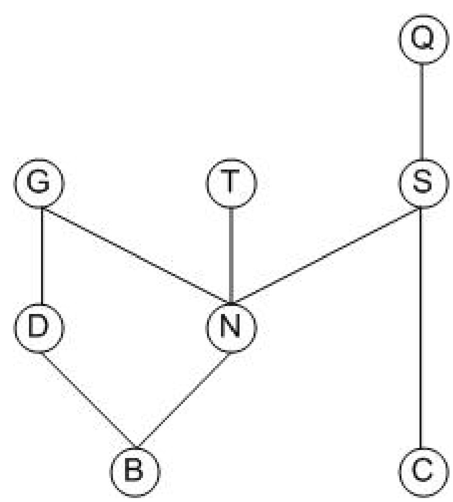
**Problem 2 (15 points).** Attila the Hun is planning another excursion into a Roman village. This requires a number of tasks, each of which takes him one minute to complete. The prerequisites associated with these tasks are listed below.

| ABBRV. | TASK | PREREQUISITES |
|---|---|---|
| C | Assemble the barbarians | |
| N | Plunder the village | B |
| D | Get shots for his own cat, also named Emilios | B |
| B | Blow the trumpet | |
| T | Sell T-shirts: "I got A LOT more than just this lousy t-shirt." | N |
| Q | Grade the 6.042 quiz | S |
| G | Cook a feast | D,N |
| S | Burn the village | C,N |

**(a) (4 points)** Draw the Hasse diagram for the tasks and their prerequisites.

**Solution.** Students received full credit for drawing a correct Hasse diagram showing the correct dependencies.

Students missing edges or incorrectly drawing reflexive edges lost one point (Hasse diagrams never contain reflexive edges).




■

**(b) (2 points)** Attila has decided that since his barbarians are quite smart and he has so many, he can get as many tasks done at a time as he wishes. What is the minimum amount of time required for him to finish the excursion?

**Solution.** The minimum required time is 4 (length of the critical path B-N-S-Q).

This was an all or nothing problem. Any answer other than 4 received no credit. ■

It turns out that Attila's graph is actually far more complicated than the one above. In fact, he doesn't know what the actual diagram because his Scribe forgot to tell him. All Attila has been told is that he must complete $n$ tasks, each of which takes 1 minute, and that the minimum amount of time required to finish is $t$ minutes. Without knowing anything more about the actual graph, Attila is trying to figure out how many barbarians to recruit. A barbarian can only complete one task in 1 minute, but has the stamina to work for days on end. Let $n$ and $t$ be fixed, and $n > t > 1$.

**(c) (4 points)** Write a simple formula in $n$ and $t$ for the smallest number of barbarians Attila can recruit in order to be *guaranteed* to finish in $t$ minutes.

**Solution.** There are $n$ tasks, $t$ is the longest chain. We are guaranteed that everything can be partitioned into $t$ antichains.

$n-t+1$. In the worst case there could be an antichain of size $n-t+1$ but no larger, which means that if Attila hired that many barbarians he'd be set.

Some students wrote $n - t$ and ended up off by one, losing 2 points.

Nearly all other answers received no credit. ∎

**(d) (5 points)** Write a simple formula in $n$ and $t$ for the smallest number of barbarians he could recruit and still *possibly* finish the job in $t$ minutes. (That is, if he recruits fewer barbarians, he will never be able to finish in the minimum number, $t$, of minutes, no matter how "favorable" graph turns out to be.)

**Solution.** $\lceil n/t \rceil$. Since there is no chain of size $t + 1$, we know that at least one antichain must have $\geq n/t$ elements. If Attila recruits fewer than $\lceil n/t \rceil$ barbarians, this antichain will take at least two seconds, and there will still be $t - 1$ stages to complete, so there will be no chance he can get done in $t$ seconds.

Many students simply wrote $n/t$ without "ceiling" and got 3 out of 5 points.

Some students realized that $n/t$ from Dilworth's theorem is not always an integer but said to "round down" or "round to the nearest number" or to take the "floor plus one." All of these have cases that are incorrect, so these students received 4 out of 5 points.

A few students misunderstood the problem, which says to write a simple formula in $n$ and $t$. The previous paragraph also mentions specifically that $n$ and $t$ are known values and that the only unknown is the Hasse diagram. The students who thought that all of $n$, $t$, and the Hasse diagram were unknown wrote 2, the rationale being that $n > t > 1$ so $n = 3$, $t = 2$ is the most favorable scenario. These students received 1 out of 5 points. ∎

**Problem 3 (15 points).** In this problem, let $R$ be a binary relation on a set $A$, and $S$ be a binary relation from $A$ to a set $B$. Indicate whether each of the following statements is **True** or **False**. For the false ones, ***describe a counterexample***[1].

**Solution.** Comments: Parts (a) and (c) were give 3 points for true answers, 0 for false answers. Parts (b), (d), and (e) were given 0 points for true answers, 1 point for the correct false answer and 2 more points for an appropriate counterexample. ■

**(a) (3 points)** If $R = R^*$, then $R$ is a transitive relation.

**Solution.** True ■

**(b) (3 points)** If $R = R^*$, then $R$ is an equivalence relation.

**Solution.** False. Counterexample: $A ::= \mathbb{N},\ R ::= \leq$ ■

**(c) (3 points)** If $R$ is an equivalence relation, then $R = R^*$.

**Solution.** True. ■

**(d) (3 points)** If $A = B$, then $S \circ R = R \circ S$.

**Solution.** False. Counterexample: $A = B ::= \{1, 2, 3\}, R ::= \{(1, 2)\}, S ::= \{(2, 3)\}$, so $S \circ R = \{(1, 3)\}, R \circ S = \emptyset$.

Second counterexample:

$$\begin{aligned} R &::= \{(x, x+1) \mid x \in \mathbb{R}\}, \\ S &::= \{(x, 2x) \mid x \in \mathbb{R}\}, \end{aligned}$$

because $2(x + 1) \neq (2x) + 1$. ■

**(e) (3 points)** If $R = R^{-1}$ and $R$ is nonempty, then $R$ is reflexive.

**Solution.** False. Counterexample: $xRy ::= x \neq y$ as long as $|A| \geq 2$. Another counterexample: $R = R^{-1} ::= \{(a, b), (b, a)\}$. ■

[1]No explanation required for statements that are True.

**Problem 4 (25 points).** Recall that a $k$-coloring of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color, and no more than $k$ colors are used. In class we proved

**Lemma.** If a graph has maximum degree $k$, then it is $k + 1$-colorable,

Consider the following variation of this Lemma:

**False Claim.** *A graph with maximum degree $k$ that also has a vertex of degree less than $k$, is $k$-colorable.*

**(a) (5 points)** Give a counterexample to the False Claim when $k = 2$.

**Solution.** One node by itself, and a separate $K_3$. The graph has max degree 2, but is not 2-colorable. ■

Consider the following proof of the False Claim:

*False proof.* Proof by induction on the number $n$ of vertices:

**Induction hypothesis:**

$P(n)$::= "Every graph with $n$ vertices and maximum degree $k$ that also has a vertex of degree less then $k$, is $k$-colorable."

**Base case:** (n=1) The graph has only one vertex of degree zero, so $P(1)$ holds vacuously.

**Inductive step:**

We may assume $P(n)$. To prove $P(n + 1)$, let $G_{n+1}$ be a graph with maximum degree $k$ that has a vertex, $v$, of degree less than $k$.

Remove the vertex $v$ to produce a graph $G_n$. Removing $v$ reduces the degree of all vertices adjacent to $v$ by 1. Therefore $G_n$ must contain at least one vertex with degree less than $k$. Also the maximum degree of $G_n$ is at most $k$. If the maximum degree of $G_n$ is less than $k$, then by the Lemma above, $G_n$ is $k$-colorable. Otherwise $G_n$ has maximum degree $k$ and a vertex of degree less than $k$, so by our induction hypothesis, $G_n$ is $k$-colorable. So in any case, $G_n$ is $k$-colorable.

Now a $k$-coloring of $G_n$ gives a coloring of all the vertices of $G_{n+1}$, except for $v$. Since $v$ has degree less than $k$, there will be fewer than $k$ colors assigned to the nodes adjacent to $v$. So there will be a color left over among the $k$ colors that can be assigned to $v$. Hence $G_{n+1}$ is $k$-colorable. □

**(b) (5 points)** Identify the exact sentence where the proof goes wrong (underlining or circling the sentence is sufficient).

**Solution.** "Therefore $G_n$ must contain at least one vertex with degree less than $k$." ■

**(c) (5 points)** Briefly explain why the sentence is incorrect.

**Solution.** This is only true if $v$ is adjacent to some other vertex, but $v$ might not be connected to anything, *i.e.*, it might be of degree zero. ■

**(d) (5 points)** The False Claim can be made true by adding an additional assumption about the graph. Which of the following is the most general asumption that will make the False Claim true?[2]

1. The graph is a line graph.
2. The graph has only even length cycles.
3. The graph is connected.
4. The graph does not contain a complete graph on $k$ vertices.
5. The graph has no node of degree zero.
6. The graph has a Hamiltonian cycle.
7. $k < 2$.

**Solution.** "The graph is connected" is the most general sufficient assumption among those listed. Being a linegraph or having a Hamiltonian cycle are also sufficient, because both imply connectedness. None of the other assumptions are sufficient. ■

**(e) (5 points)** Assuming that $G_{n+1}$ satisifes the additional assumption from part (d), both the False Claim and the sentence that was incorrect from part (b) become correct. But now another sentence in the proof becomes incorrect and requires fixing. Indicate the new incorrect sentence and briefly explain what's wrong. (You are *not* expected to suggest a fix.)

**Solution.** The new false sentence is, "Otherwise $G_n$ has maximum degree $k$ and a vertex of degree less than $k$, *so by our induction hypothesis*, $G_n$ is $k$-colorable."

The italicized phrase is wrong, because removing $v$ might leave a $G_n$ that is not connected, and such a $G_n$ would not satisfy the corrected induction hypothesis, which requires that the graph is connected. The fix is to consider instead each connected component of $G_n$ separately. ■

[2] By "most general," we mean it is implied by all the other assumptions that verify the False Claim. For example, the assumption that $G$ is a line graph is more general than the assumption that $G$ is a line graph with 3 vertices.

**Problem 5 (25 points).** One of the three monks working on the famed Towers of Hanoi project recently rubbed his pained back and burst out, "Yo! What are we doing? This is for chumps! Let's punt!" But before wandering off to start up fast food joints, they must evenly divide the monastery's collection of prayer beads.

Initially, monk $A$ has 5 beads, monk $B$ has 3 beads, and monk $C$ has 4 beads. The monastic order has strict rules regarding the exchange of prayer beads. Only the following transactions are allowed.

1. Monk $B$ may give a bead to monk $A$ at any time.
2. If $C$ has an odd number of beads, then monk $A$ may give a bead to monk $B$.
3. If $C$ has an even number of beads, then monk $C$ may give or take a bead from either monk $A$ or monk $B$.
4. If monk $A$ has at least two more beads than monk $B$, then monk $C$ may give or take a bead from either monk $A$ or monk $B$.

**(a) (10 points)** Model the situation with a state machine. Define the set of states, the set of start states, and the set of transitions.

**Solution.** The set of states $Q$ consists of all triples $(a, b, c)$ such that $a, b, c \geq 0$ and $a+b+c = 12$. The set of starts states $Q_0$ consists of the single triple $(5, 3, 4)$. There are six types of transitions:

$$
\begin{array}{lcll}
(a,b,c) & \to & (a+1, b-1, c) & (b > 0) \\
(a,b,c) & \to & (a-1, b+1, c) & (a > 0 \text{ and } c \text{ is odd}) \\
(a,b,c) & \to & (a-1, b, c+1) & (a > 0 \text{ and either } c \text{ is even or } a \geq b+2) \\
(a,b,c) & \to & (a, b-1, c+1) & (b > 0 \text{ and either } c \text{ is even or } a \geq b+2) \\
(a,b,c) & \to & (a+1, b, c-1) & (c > 0 \text{ and either } c \text{ is even or } a \geq b+2) \\
(a,b,c) & \to & (a, b+1, c-1) & (c > 0 \text{ and either } c \text{ is even or } a \geq b+2)
\end{array}
$$

(One point was taken off for omitting the "$> 0$" conditions.) ■

**(b) (5 points)** Describe a sequence of steps (transitions) leading to a state where monk $A$ has 0 beads, monk $B$ has 9 beads, and monk $C$ has 3 beads.

**Solution.** Initially, monk $C$ gives a bead to monk $B$ using rule 3. Then monk $A$ gives his 5 beads to monk $B$ using rule 2. repeatedly. ■

**(c) (4 points)** A clever TA tells you that the following predicate is an invariant.

$$(C \text{ has an odd number of beads}) \vee (A \text{ has more beads than } B)$$

*Assuming* she is correct, prove that the monks can not reach the state where every monk has 4 beads.

**Solution.** The invariant holds in the single start state $Q_0$, since in this case monk $C$ has an even number of beads (4), and monk $A$ has more beads than monk $B$ (5 vs. 3). It does not hold when all three Monks have 4 beads, since $C$ is even and $A$ and $B$ have the same number of beads. So by the Invariant Theorem, the 4-4-4 state cannot be reachable.

It was necessary to show that the invariant holds for the start state, which is key to proving that the state 4-4-4 is not reachable from it. ■

**(d) (6 points)** Now prove that the TA was correct in her assumption that the predicate is an invariant.

**Solution.** We have to show that if the TA's predicate holds before a transition, then it holds after a transition. There are six cases to check, corresponding to the six types of transition.

1. $(a, b, c) \to (a + 1, b - 1, c)$ where $b > 0$. If $c$ is odd, then the invariant immediately holds after the transition. If $c$ is even, then the invariant holds after the transition because $a + 1 > a > b > b - 1$.
2. $(a, b, c) \to (a - 1, b + 1, c)$ where $a > 0$ and $c$ is odd. The invariant holds after the transition because $c$ is unchanged, so the invariant immediately holds.
3. $(a, b, c) \to (a-1, b, c+1)$ where $a > 0$ and either $c$ is even or $a \geq b+2$. If $c$ is even, then the invariant holds after the transition $c$ becomes odd. If $c$ is odd, then the invariant holds after the transition because monk A has (at least two) more beads than monk B.

The argument in the remaining three cases is identical to the argument for case 3.

Remember the definition of an invariant from the course notes:

**Definition.** An *invariant* for a state machine is a predicate, $P$, on states, such that whenever $P(q)$ is true of a state, $q$, and $q \to r$ for some state, $r$, then $P(r)$ holds.

Notice that it is *not* necessary for an invariant to hold for the start states. ■