Summary: Infinite Series

Notation

A **partial sum** S_N is the **finite** sum

$$S_N = \sum_{n=0}^{N} a_n.$$

A **series** S is the **infinite sum** $\displaystyle\sum_{n=0}^{\infty} a_n$.

If the limit of the partial sum $\displaystyle\lim_{N\to\infty} S_N$ exists, then

$$S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N$$

and we say that the series S **converges**.

If the limit does not exist, we say the series S **diverges**.

Note that a divergent series does not have to tend to ∞, and we have already seen the different divergent behaviors of the geometric series.

Short hands for the summation notation: We also sometimes use the following abbreviated notation:

- $\displaystyle\sum_{0}^{\infty} a_n$ for $\sum_{n=0}^{\infty} a_n$ when it is clear which index to sum over,
- $\displaystyle\sum^{\infty} a_n$ for $\sum_{n=N}^{\infty} a_n$ when we are concerned with only the tail of the series.

The geometric series

A **geometric series** is defined as

$$\sum_{n=0}^{\infty} a^n = \lim_{N\to\infty} \sum_{n=0}^{N} a^n \qquad (a \text{ is any number}).$$

Notice that each term (except the first) is a times the previous term. In other words, a is the ratio of consecutive terms.

Here is a formula for the partial sum:

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a} \qquad (a \text{ is any number}).$$

When $|a| < 1$, the geometric series is convergent and converges to

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad (|a| < 1).$$

When $|a| \geq 1$, the geometric series is divergent.

Divergence test

One of the first and simplest tests on a series is the **divergence test**:

If the sequence of numbers $a_1, a_2, a_3, \ldots$ does not tend to 0, that is, if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

This is very intuitive. For an infinite sum to approach a finite number, the terms being added had better approach 0.

Figure 1: All tiles in a domino falls with a push on the first tile if each tile is placed close enough to the one before.

Mathematical induction

Sometimes we can guess a formula for the partial sum S_N, but how do we know that our guess is correct for all N?

One way to show that the formula indeed works for all N is by **mathematical induction**.

Mathematical induction consists of two steps:

Base case: Show the formula is true for the $N = 1$,

Induction step: Show that **if** the formula is true for S_N, then formula would also be true for S_{N+1}.

If both statements are true, then the formula works for all S_N.

Mathematical induction works like the domino.

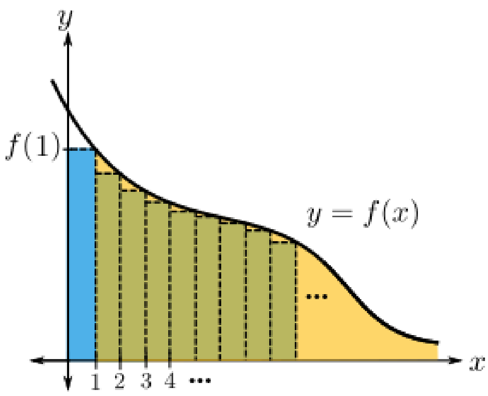
Showing a formula is true for N is analogous to having the N^{th} tile fall. The base case is analogous to the push on the first tile. The induction step is analogous to making sure that if one tile falls, it pushes the next one down.

Integral comparison test

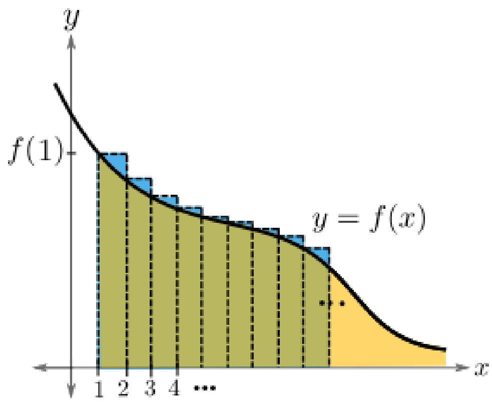
If $f(x) > 0$ and is decreasing, then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ either **both converge** or **both diverge**.
Moreoever, we have the following inequality

$$\left| \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)\,dx \right| < f(1).$$

This inequality is equivalent to the pair of inequalities shown in the figure below.



$$\int_1^{\infty} f(x)\,dx > \sum_{n=1}^{\infty} f(n) - f(1)$$



$$\sum_{n=1}^{\infty} f(n) > \int_1^{\infty} f(x)\,dx$$