Summary: Polar Coordinates

Definition

The **polar coordinates** of a point P are an ordered pair (r, θ) such that

$$x = r\cos(\theta) \tag{1}$$
$$y = r\sin(\theta), \tag{2}$$

where the ordered pair (x, y) give the rectangular coordinates of the point P.

In other words, given the polar coordinates (r, θ) of a point, we can find its x- and y-coordinates using these formulas.

The usual rectangular coordinates are also called **Cartesian coordinates**. To find polar coordinates from Cartesian coordinates, we use

$$r = \pm\sqrt{x^2 + y^2} \tag{3}$$
$$\theta = \arctan\left(\frac{y}{x}\right). \tag{4}$$

However, r and θ are not unique for any given point, as explained below.

Polar coordinates are motivated by the fact that we can locate a point on a plane by specifying:

r :the distance from the origin to the point,

θ :the angle of the ray from the origin to the point with the positive x-axis.

Ambiguities in polar coordinates

The polar coordinates describing a point are not unique. First,

$$(r, \theta) = (r, \theta + 2\pi n) \qquad (n \text{ any integer}).$$

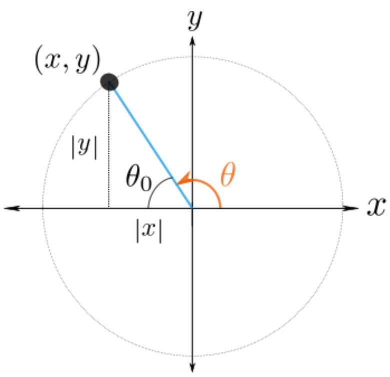


Figure 1: Here, (x, y) lies in the second quadrant, $\theta = \pi - \theta_0$.

That is , knowing the x- and y- coordinates only determines θ up to 2π-periodicity. We frequently use conventions such as:

$$\begin{gathered} 0 \leq \theta < 2\pi \\ \text{or} - \pi < \theta \leq \pi. \end{gathered}$$

Second,

$$\begin{aligned} (-r, \theta) &= (r, \theta \pm \pi) \qquad - \infty < r < \infty \\ \text{and equivalently } (r, \theta) &= (-r, \theta \pm \pi) \qquad - \infty < r < \infty. \end{aligned}$$

Finding theta

To find θ, we first find

$$\theta_0 = \arctan\left(\frac{|y|}{|x|}\right).$$

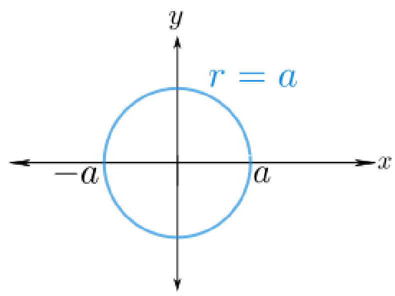
Then we find θ using θ_0, by considering which quadrant it lies in, which is best done using a picture like the one above.

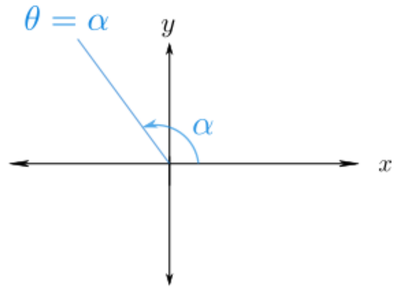
Circles and rays in polar coordinates

An equation in polar coordinates is called a **polar equation**. We will mostly be dealing with polar equations of the form $r = r(\theta)$.

The simplest examples are:

- $r = a$:





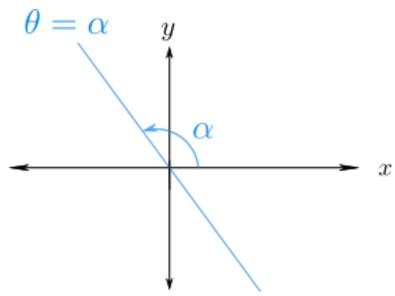
- $\theta = \alpha$ $(r \geq 0)$:
  If we use the convention $-\infty < r < \infty$, then $\theta = \alpha$ is a line through the origin:

We use the circles and rays as the grid of the polar coordinate systems.

Rotation about the origin

The graph of $r = r(\theta - \alpha)$ is obtained by rotating the graph of $r = r(\theta)$ **about the origin** by the angle $+\alpha$.

If $\alpha > 0$ the rotation is counterclockwise.



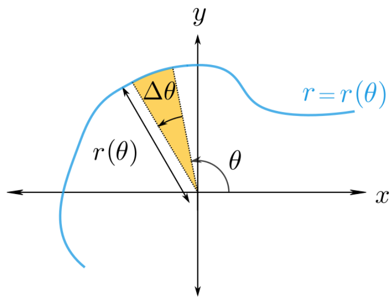
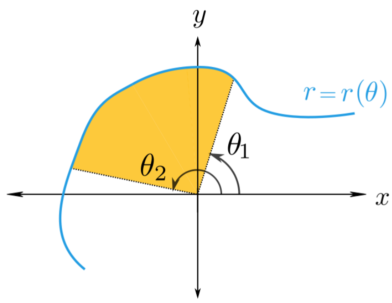


Figure 2: $\Delta A \approx \dfrac{1}{2} r^2 \Delta\theta$



if $\alpha < 0$ the rotation is clockwise.

The area element

Passing to the differential, the area element dA in polar coordinates is the area of the infinitesimal region spanned by $d\theta$, given by

$$dA = \frac{1}{2} r(\theta)^2 d\theta. \tag{5}$$

$$\tag{6}$$

The area bounded a curve $r = r(\theta)$, and the two rays $\theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \int_{\theta_1}^{\theta_2} dA = \frac{1}{2} \int_{\theta_1}^{\theta_2} r(\theta)^2 d\theta$$

Examples of polar curves

Lines:	$y = b:$ $r = \frac{b}{\sin(\theta)}$ $(0 < \theta < \pi)$ $x = b:$ $r = \frac{b}{\cos(\theta)}$ $(-\pi/2 < \theta < \pi/2)$ $y = mx + b:$ $r = \frac{b}{\sin(\theta) - m\cos(\theta)}$ $(\arctan(m) < \theta < \arctan(m) + \pi)$
Circles: (touching origin)	centered on x-axis: $r = 2b\cos(\theta)$ $(-\pi/2 \leq \theta \leq \pi/2)$ centered on y-axis: $r = 2b\sin(\theta)$ $(0 \leq \theta < \pi)$ centered on $\theta = \alpha$ ray: $r = 2b\cos(\theta - \alpha)$ $(\alpha - \pi/2 \leq \theta < \alpha + \pi/2)$
Roses Limacons Cardioids	$r = A + B\cos(n\theta)$ (Domain varies) $r = A + B\sin(n\theta)$ (Domain varies)
Spirals:	$r = n\theta$ $(0 \leq \theta)$
Conics: (with one focus at the origin)	Hyperbolas: $r = \frac{1}{1+b\cos(\theta)}$, $\lvert b\rvert > 1$ $(0 \leq \theta < 2\pi, 1 + b\cos(\theta) \neq 0)$ Parabolas: $r = \frac{1}{1+b\cos(\theta)}$, $\lvert b\rvert = 1$ $(0 \leq \theta < 2\pi, 1 + b\cos(\theta) \neq 0)$ Ellipses: $r = \frac{1}{1+b\cos(\theta)}$, $\lvert b\rvert < 1$ $(0 \leq \theta < 2\pi)$