Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

Solutions to Problem Set 6-7

Problem 1. We can generalize win-lose 2-player terminating games of perfect information to games with "payoff" amounts. In these games, two players called the *max-player* and the *min-player* alternate moves until the game ends with the min-player paying some payoff amount to the max-player. How much the min-player pays depends on how the game ends. Negative payoffs mean the max-player pays the min-player. The max-player moves first.

Such games are defined by finite-path trees with leaves labelled with real numbers. These are the payoff amounts. The max-player tries to arrive at a leaf with as large a payoff as possible, and the min-player tries to minimize the payoff to the max-player.

Definition. The set of payoff-game trees, PayT, can be defined recursively as follows:

1. If T is a graph with one vertex, v, and no edges, then T is a PayT and $\text{root}(T) ::= v$.
2. If $\mathcal{S}$ is a set of PayT's such that no vertex occurs in more than one tree in $\mathcal{S}$, and v is a "new" element that is not a vertex of any tree in $\mathcal{S}$, then T is in PayT where $\text{root}(T) = v$ and the edges of T are the edges of all the trees in $\mathcal{S}$ along with edges connecting $\text{root}(T)$ to the roots of each of the trees in $\mathcal{S}$. The trees in $\mathcal{S}$ are called the *children* of T.

We define functions $\text{max-value}(T)$ and $\text{min-value}(T)$ on payoff-game trees, $T \in \text{PayT}$, recursively on the definition of PayT:

1. If T is a single node labelled r, then

$$\text{max-value}(T) = \text{min-value}(T) ::= r.$$

2. If the nonempty set, $\mathcal{S}$, is the set of children of T, then[1]

$$\begin{aligned}\text{max-value}(T) &::= \text{lub}\,\{\text{min-value}(S) \mid S \in \mathcal{S}\}\\ \text{min-value}(T) &::= \text{glb}\,\{\text{max-value}(S) \mid S \in \mathcal{S}\}\,.\end{aligned}$$

(a) Suppose a payoff-game tree, T, is *finite*. Prove that

[1] glb = "greatest-lower-bound" and lub = "least-upper-bound", *cf.*, Rosen, p. 423.

1. If the max-player is the first player to move in T, then he has a strategy that guarantees his payoff will be at least max-value(T), no matter how the min-player behaves.
2. If the max-player is the second player to move in T, then he has a strategy that guarantees his payoff will be at least min-value(T).
3. Likewise, if the min-player is the first player to move in T, then she has a strategy that guarantees the payoff to max-player will be at most min-value(T).
4. If the min-player is the second player to move in T, then she has a strategy that guarantees the payoff to max-player will be at most max-value(T).

(So the players may as well skip playing and just have the min-player pay max-value(T) to the max-player.)

Solution. The max-player's strategy is to move on his turn to the subtree with the largest min-value. The min-player's strategy is to move to the subtree with the smallest max-value.

By structural induction on the definition of max-value(T) and min-value(T), we prove 1. and 2.

Base case (T is a leaf): The game is over with no moves, and the max-player's payoff is the payoff at the the leaf. This is max-value(T) by definition of max-value.

Induction case: (T has a nonempty set, $\mathcal{S}$, of (direct) subtrees)

Suppose the max-player moves first in T. Using his strategy, he moves to $S \in \mathcal{S}$ with the largest min-value. Now in the game defined by S, he moves second. By induction hypothesis 2., he has a strategy in S guaranteeing at payoff at least min-value(S). Hence, he is guaranteed a payoff of at least $\max \{$min-value$S \mid S \in \mathcal{S}\}$. But by definition, this is precisely max-value(T).

A corresponding proof using induction hypothesis 1 shows his strategy ensures payoff at least min-value(T) if he moves second in T. A similar proof confirms 3. and 4. ■

(b) Now generalize the previous part to arbitrary PayT's. *Hint:* It might be helpful to assume the payoff amounts at the leaves are bounded above and below by particular numbers. After settling this case, try it without assuming bounds. Note that in the unbounded case, max-value(T) may be $+\infty$ and min-value(T) may be $-\infty$.

Solution. In the bounded case, we can say that for any $\epsilon > 0$,

1. the max-player has a strategy that guarantees his payoff will be *at least* max-value$(T) - \epsilon$,
2. the min-player has a strategy that guarantees the max-player's payoff will be *at most* max-value$(T) + \epsilon$.

The max-player's strategy is to move on his turn to a subtree, S_0, with

$$\text{min-value}(S_0) \geq \text{lub}\,\{\text{min-value}(S) \mid S \in \mathcal{S}\} - \epsilon/2,$$

and then to use his strategy in S_0 to guarantee a payoff $\geq$ min-value$(S) - \epsilon/2$.

The min-player's strategy is to move on his turn to a subtree, S_1, with

$$\text{max-value}(S_1) \leq \text{glb}\,\{\text{max-value}(S) \mid S \in \mathcal{S}\} + \epsilon/2,$$

and then to use his strategy in S_1 to guarantee a payoff $\leq$ max-value$(S) + \epsilon/2$.

The rest of the proof by structural induction is essentially the same as for the finite case.

In the unbounded case, it may be, for example, that max-value$(T) = \infty$. In this case, the generalization of part (a) is that for *any* real number, r, the max-player has a strategy that guarantees him a payoff of at least r. We omit the details. ■

Problem 2. Week 6 Notes describes various functional equations, some of which serve as function definitions because there is only one function satisfying the equations, others of which are *ambiguous* because more than one function satisfies the equations, and some of which are *inconsistent* and aren't satisfied by any function at all. It can be hard to tell which case applies. In this problem, we'll consider some odd-ball equations which *do* turn out to define a function uniquely, though it's not obvious from the form of the equations that this is the case.

Suppose a function $f : \mathbb{N} \to \mathbb{N}$ satisfies the following equations:

$$f(n) = n - 3 \qquad \text{for } n \geq 10000, \tag{1}$$
$$f(n) = f(f(n+5)) \qquad \text{for } n < 10000. \tag{2}$$

(a) What must the values of $f(n)$ be for n = 9996, 9997, 9998, 9999, and 10000?

Solution.

$$\begin{aligned} f(9996) &= 9997 \\ f(9997) &= 9998 \\ f(9998) &= 9997 \\ f(9999) &= 9998 \\ f(10000) &= 9997. \end{aligned}$$

■

(b) Prove that there is a function satisfying equations (1) and (2) by giving a simple, explicit definition for such an f and proving that the function you defined satisfies (1) and (2).

Solution. Let

$$f_0(n) ::= n - 3 \qquad \text{when } n \geq 10000, \tag{3}$$
$$f_0(n) ::= 9997 \qquad \text{when } n \text{ is even and } n < 10000, \tag{4}$$
$$f_0(n) ::= 9998 \qquad \text{when } n \text{ is odd and } n < 10000. \tag{5}$$

We need to show that f_0 satisfies equations (1) and (2).

Now f_0 clearly satisfies (1), since equation (3) in the definition of f_0 is the same as (1).

It remains to show that f_0 satisfies (2), namely, that

$$f_0(n) = f_0(f_0(n+5)) \tag{6}$$

for $n < 10,000$.

First, consider $n < 9995$ so that $n + 5 < 10000$.

Case 1: n is odd. Then

$$\begin{aligned} f_0(n) &= 9998 && \text{by (5)} \\ &= f_0(9997) && \text{by (5)} \\ &= f_0(f_0(n+5)) && \text{by (4) because } n+5 \text{ is even.} \end{aligned}$$

Case 2: n is even. Then

$$\begin{aligned} f_0(n) &= 9997 && \text{by (4)} \\ &= f_0(9998) && \text{by (4)} \\ &= f_0(f_0(n+5)) && \text{by (5) because } n+5 \text{ is odd.} \end{aligned}$$

So it remains only to check (6) for $9995 \le n < 10000$. In particular, for $n = 9995$ we have

$$\begin{aligned} f_0(9995) &= 9998 && \text{by (5)} \\ &= f_0(9997) && \text{by (5)} \\ &= f_0(f_0(10000)) && \text{by (3)} \\ &= f_0(f_0(9995-5)). \end{aligned}$$

For $n = 9996$ we have

$$\begin{aligned} f_0(9996) &= 9997 && \text{by (4)} \\ &= f_0(9998) && \text{by (4)} \\ &= f_0(f_0(10001)) && \text{by (3)} \\ &= f_0(f_0(9996+5)). \end{aligned}$$

The arguments for $n = 9997, 9998$ are essentially the same. Finally, for $n = 9999$ we have

$$\begin{aligned} f_0(9999) &= 9998 && \text{by (5)} \\ &= f_0(10001) && \text{by (3)} \\ &= f_0(f_0(10004)) && \text{by (3)} \\ &= f_0(f_0(9999+5)). \end{aligned}$$

■

(c) Prove that equations (1) and (2) *uniquely* determine f. That is, if f is a function satisfying (1) and (2), then $f(n)$ equals the value you specified in part (b) for all $n \in \mathbb{N}$.

Solution. *Proof.* Assume (for contradiction) that f is a function satisfying (1) and (2), but $f(n) \neq f_0(n)$ for some $n \in \mathbb{N}$.

We know that $f_0(n) = n - 3 = f(n)$ for $n \geq 10000$, by equation (3) in the definition of f_0 and equation (1) for f. So it must be that $f(9999 - m) \neq f_0(9999 - m)$ for some natural number $m \leq 9999$. Let k be the least such m.

We already observed in part (a) that $k > 3$. We can show that this implies $k \neq 4$, which, in term, implies $k > 4$

$$\begin{aligned}
f(9999-4) &= f(9995) \\
&= f(f(9995+5)) && \text{by (2)} \\
&= f(f(10000)) \\
&= f(9997) && \text{by (1)} \\
&= f(9999-2) \\
&= f_0(9999-2) && \text{since } k > 2 \\
&= f_0(9997) \\
&= 9998 && \text{by (5)} \\
&= f_0(9995) && \text{by (5)} \\
&= f(9999-4).
\end{aligned}$$

But now if $k \geq 5$ is odd, we have

$$\begin{aligned}
f(9999-k) &= f(f((9999-k)+5)) && \text{by (2)} \\
&= f(f((9999-(k-5)))) \\
&= f(f_0((9999-(k-5)))) && \text{since } 0 \leq k-5 < k \\
&= f(9998) && \text{by (5)} \\
&= f(9999-1) \\
&= f_0(9999-1) && \text{since } k > 1 \\
&= f_0(9998) \\
&= 9997 && \text{by (4)} \\
&= f_0(9999-k) && \text{by (4) since } 9999-k \text{ is even,}
\end{aligned}$$

contradicting the fact that $f(9999 - k) \neq f_0(9999 - k)$.

A similar argument applies if $k \geq 5$ is even. Hence there is no k such that $f(9999 - k) \neq f_0(9999 - k)$, which implies $f = f_0$.

□

■

Problem 3. Closed Form Summations

Find a simplified closed form for each of the expressions listed below.

(a) $\sum_{i=x}^{y} 2i + 1$

Solution. There are two ways of doing this problem. In the first, we re-form the sum so the lower limit is 0 instead of x and apply the arithmetic sum formula.

$$\begin{aligned}
\sum_{i=x}^{y} 2i + 1 &= \sum_{i=0}^{y-x} 2(i+x) + 1 \\
&= \sum_{i=0}^{y-x} 2i + 2x + 1 \\
&= (y-x+1)(2x+1) + 2\sum_{i=0}^{y-x} i \\
&= (y-x+1)(2x+1) + 2\frac{(y-x)(y-x+1)}{2} \\
&= (y-x+1)(2x+1+y-x) = (y-x+1)(y+x+1) = ((y+1)-x)((y+1)+x) \\
&= (y+1)^2 - x^2.
\end{aligned}$$

In the second, we express this sum as the difference of two summations, one involving only y and one involving only x.

$$\sum_{i=x}^{y} 2i + 1 = \sum_{i=0}^{y} 2i + 1 - (\sum_{i=0}^{x-1} 2i + 1).$$

Now, since

$$\sum_{i=0}^{n} 2i + 1 = n + 1 + 2\frac{n(n+1)}{2} = n + 1 + n(n+1) = n^2 + 2n + 1 = (n+1)^2,$$

we can evaluate this to

$$(y+1)^2 - x^2.$$

■

(b) $\displaystyle\sum_{i=0}^{\infty}\sum_{j=1}^{n}\left(\frac{j}{j+2}\right)^i$

Solution. First, we reorder the summations to get

$$\sum_{j=1}^{n}\sum_{i=0}^{\infty}\left(\frac{j}{j+2}\right)^i.$$

We can do this reordering because the value of j never depends on the value of i, we just use every possible pair of values for a nonnegative i and a j between 1 and n.

Evaluating the inner summation, which is just an infinite geometric series, we get

$$\sum_{j=1}^{n} \frac{1}{1 - \frac{j}{j+2}}$$
$$= \sum_{j=1}^{n} \frac{j+2}{2}$$
$$= n + \frac{1}{2}\sum_{j=1}^{n} j$$
$$= n + \frac{n(n+1)}{4}$$

which simplifies to $\frac{1}{4}n^2 + \frac{5}{4}n$.

■

(c) $\prod_{i=1}^{n} 2 \cdot 4^i$.

Solution. We know that $\prod_{i=1}^{n} 2 \cdot 4^i = (2 \cdot 4^1) \cdot (2 \cdot 4^2) \cdot ... \cdot (2 \cdot 4^n)$. Taking $\log_2$ of both sides of the equation gives us

$$\begin{aligned}
\log_2(\prod_{i=1}^{n} 2 \cdot 4^i) &= \log_2(2 \cdot 4^1 \cdot 2 \cdot 4^2 \cdot ... \cdot 2 \cdot 4^n) \\
&= \log_2(2 \cdot 4^1) + \log_2(2 \cdot 4^2) + ... + \log_2(2 \cdot 4^n) \\
&= \sum_{j=1}^{n} \log_2(2 \cdot 4^j) \\
&= \sum_{j=1}^{n} (\log_2(2) + \log_2(4^j)) \\
&= \sum_{j=1}^{n} (1 + 2j) \\
&= n + n(n+1) \\
&= n^2 + 2n.
\end{aligned}$$

Now, exponentiating both sides of the equation gives $\prod_{i=1}^{n} 2 \cdot 4^i = 2^{n^2+2n}$ ■

Problem 4. Let $S_n = \sum_{i=1}^{n} i^{1/3}$. Use the integral method to determine a constant $c \in \mathbb{R}$ such that

$$S_n \sim cn^{\frac{4}{3}}.$$

Solution. $c = 3/4$.

We use the integral method for bounding S_n.

$$\int_0^n x^{\frac{1}{3}}\,dx \leq S_n \leq \int_1^{n+1} x^{\frac{1}{3}}\,dx.$$

That is,

$$\frac{(3/4)n^{4/3}}{n^{4/3}} \leq \frac{S_n}{n^{4/3}} \leq \frac{(3/4)((n+1)^{4/3}-1)}{n^{4/3}}$$

Taking limits as $n \to \infty$ we find that

$$3/4 \leq \lim_{n\to\infty} \frac{S_n}{n^{4/3}} \leq 3/4$$

Therefore, $\lim_{n\to\infty} \frac{S_n}{n^{4/3}} = 3/4$. ■

Problem 5. Use the integral method to find upper and lower bounds for the following summation that differ by at most 0.1.

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

Hint: Try adding the first few terms explicitly and then use integrals to bound the sum of the remaining terms.

Solution. We can bound the summation above as follows:

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{i^2} &\leq \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \int_3^{\infty} \frac{1}{x^2}\,dx \\
&= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \left(-\frac{1}{x}\right)_3^{\infty} \\
&= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{3} \\
&= 1.694\ldots
\end{aligned}$$

We can lower bound the summation similarly:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} \geq \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \int_3^{\infty} \frac{1}{(x+1)^2}\, dx$$
$$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \left(-\frac{1}{x+1}\right)_3^{\infty}$$
$$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{4}$$
$$= 1.611\ldots$$

■

(The actual value of the summation turns out to be $\pi^2/6 = 1.644\ldots$.)

Problem 6. Growing Choices

The expression $\binom{n}{k}$ is read "n choose k" and represents the quantity

$$\frac{n!}{k!\,(n-k)!}.$$

This expression comes up frequently in probability and combinatorics and will be used extensively later in this course. Suppose that n is even. Prove that

$$\binom{n}{n/2} = \Theta(2^n/\sqrt{n}).$$

Use the following form of Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Solution. Though substituting in Stirling's formula initially gives a complicated expression, most of the square roots and the exponentials cancel.

$$\begin{aligned}
\binom{n}{\alpha n} &= \frac{n!}{(\alpha n)!\,((1-\alpha)n)!} \\
&\sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi\alpha n}\left(\frac{\alpha n}{e}\right)^{\alpha n} \cdot \sqrt{2\pi(1-\alpha)n}\left(\frac{(1-\alpha)n}{e}\right)^{(1-\alpha)n}} \\
&= \frac{1}{\sqrt{2\pi n\alpha(1-\alpha)}} \cdot \frac{1}{\alpha^{\alpha n} \cdot (1-\alpha)^{(1-\alpha)n}}
\end{aligned}$$

Letting $\alpha = 1/2$,

$$\begin{aligned}
\binom{n}{n/2} &\sim \frac{1}{\sqrt{2\pi n(1/2)(1-1/2)}} \cdot \frac{1}{(1/2)^{n/2} \cdot (1-1/2)^{(1-1/2)n}} \\
&= \frac{1}{\sqrt{\pi n/2}} \cdot \frac{1}{(1/2)^{n/2} \cdot (1/2)^{n/2}} \\
&= \frac{1}{\sqrt{\pi/2}} \cdot \frac{2^n}{\sqrt{n}} \\
&= \Theta\left(\frac{2^n}{\sqrt{n}}\right).
\end{aligned}$$

■

Problem 7. Asymptotic Notation

Determine which of the eight choices below best describes each function's asymptotic behavior. Briefly indicate your reasoning; you may appeal to any of the results in the Notes or Rosen.

$\theta(n)$	$\theta(n!)$
$\theta(n \log n)$	$\theta(n^2)$
$\theta(1)$	$\theta(2^n)$
$2^{f(n)}$ where $f = \theta(n)$	None of the above

(a) $g(n) ::= n + \ln n + (\ln n)^2$.

Solution. $\theta(n)$.

We know that $(\ln n)^2 = o(n)$ and $\ln n = o((\ln n)^2)$, so for sufficiently large n

$$n > (\ln n)^2 > \ln n.$$

Thus, for sufficiently large n,

$$n + \ln n + (\ln n)^2 < n + n + n = 3n.$$

Furthermore, as long as $\ln n$ is positive,

$$n + \ln n + (\ln n)^2 > n.$$

Thus, for all sufficiently large n, $n < g(n) < 3n$, so $g(n) = \theta(n)$. ■

(b) $h(n) ::= (n^2 + 2n - 3)/(n^2 - 1)$

Solution. $\theta(1)$.

Dividing the top and bottom by $n-1$ yields

$h(n) = \frac{n+3}{n+1}$.

Now, since the top is greater than the bottom, $h(n) > 1$ for all $n > -1$. Furthermore, solving the equation $\frac{n+3}{n+1} < 2$ yields the solution $n > 1$. So for all $n > 1$, $h(n) < 2$.

Thus, for all $n > 1$, we have $1 < h(n) < 2$ so $h(n) = \theta(1)$. ■

(c) $j(n) ::= \sum_{i=0}^{n} 2^{2i+1}$

Solution. $2^{\theta(n)}$.

Putting the summation in closed form yields $j(n) = 2(4^n)/3$.

Now certainly, $\log_2 j(n) = \log_2(2/3) + n\log_2(4) = \theta(n)$. Thus, $j(n) = 2^{\theta(n)}$. However, as shown in Week 7 Notes, §7.4, $\neg[4^n = O(2^n)]$, so $\neg[j(n) = \theta(2^n)]$. Therefore, $2^{\theta(n)}$ best describes the asymptotic behavior of $j(n)$. ■

(d) $k(n) ::= (2 + \sin(n))2^{n+\sin(n)}$.

Solution. $\theta(2^n)$.

We know that $\sin n \geq -1$ so $k(n) \geq (2-1)2^{n-1} = \frac{1}{2}2^n$.

We also know that $\sin n \leq 1$ so $k(n) \leq (2+1)2^{n+1} = 6 \cdot 2^n$.

Thus, $\frac{1}{2}2^n \leq k(n) \leq 6 \cdot 2^n$, so $k(n) = \theta(2^n)$. ■

(e) $f(n) ::= \ln((n^2)!)$.

Solution. None of the above.

By Stirling's approximation, we know that $f(n) = \theta(n^2 \ln n^2) = \theta(n^2 \ln n)$. So f grows faster than $\theta(1), \theta(n), \theta(n^2)$ and $\theta(n \log n)$, and slower than $\theta(n!), \theta(2^n)$ and $2^{\theta(n)}$, so "none of these" is the correct answer.

■