Summary: Parametric Curves

Parametric curves

A parametric curve (in the plane) is a curve defined by two equations

$$x = x(t),$$
$$y = y(t),$$

where t is called a **parameter**. For each real number t, the point $(x(t), y(t))$ is a point on the curve.

Eliminating parameters

To find the underlying curve, try eliminating the parameter using algebra and/or trig identities.

Tangent lines of parametric curves

The slope of a parametric curve $x = x(t)$, $y = y(t)$ is

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

In particular, to find the slope of the tangent line to the curve at $t = t_0$, we compute $\dfrac{y'(t_0)}{x'(t_0)}$.

Arc length of parametric curves

Consider a particle moving along a trajectory. The motion is described by the parametric curve

$$x = x(t)$$
$$y = y(t).$$

The speed of the particle is given by

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2}$$

The differential arc length element is given by

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

Consider the arc length, or distance travelled by the particle from time t_0 to time t_1. The letter s is customarily used to denote arc length. You should think of $s = s(t)$ as a function of time, where $s(t)$ is the distance travelled by the particle since some starting time. If $s_0 = s(t_0)$ and $s_1 = s(t_1)$, then the distance traveled by the particle from time t_0 to t_1 can be calculated by

$$s_1 - s_0 = \int_{s_0}^{s_1} ds = \int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

A note about notation

$$\begin{aligned} ds^2 &= dx^2 + dy^2 &\quad \text{means} \quad& (ds)^2 = (dx)^2 + (dy)^2 \\ ds &= \sqrt{dx^2 + dy^2} &\quad \text{means} \quad& ds = \sqrt{(dx)^2 + (dy)^2} \end{aligned}$$

In particular, $dx^2 = (dx)^2$. This is the square of a differential, not the differential of the square. The differential of the square is $d(x^2) = 2x\,dx$, which is not the same.

Position and speed along a parametric curve

We have been thinking of a parametric curve as the description of a particle's position over time. So what is the velocity? And what about the acceleration?

- The derivative $x'(t)$ is the velocity in the direction of the x-axis.
- The derivative $y'(t)$ is the velocity in the direction of the y-axis.
- The speed along the curve is given by $\dfrac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2}$.
- The notion of velocity along the curve requires considering x' and y' together in what is known as a vector. Both velocity and acceleration are vectors and you will see them in multivariable calculus.

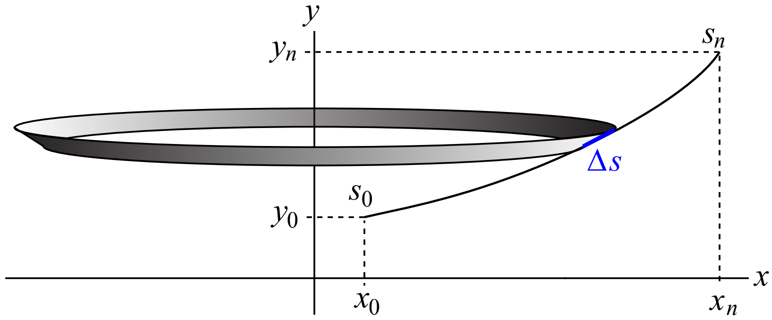

Figure 1: Rotating a curve about the y-axis.

Surface area

Consider the parametric curve

$$\begin{aligned} x &= x(t) \\ y &= y(t). \end{aligned}$$

Consider the surface formed by rotating the curve about the y-axis. The differential surface area element is given by

$$\begin{aligned} dA &= 2\pi x \, ds \\ &= 2\pi x(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt. \end{aligned}$$

The the surface area from time t_0 to time t_1 is given by the integral:

$$\text{Surface area of parametric curve} = \int_{t_0}^{t_1} 2\pi x(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

Consider the surface formed by rotating the curve about the x-axis. The differential surface area element is given by

$$\begin{aligned} dA &= 2\pi y \, ds \\ &= 2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt. \end{aligned}$$

The the surface area from time t_0 to time t_1 is given by the integral:

$$\text{Surface area of parametric curve} = \int_{t_0}^{t_1} 2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

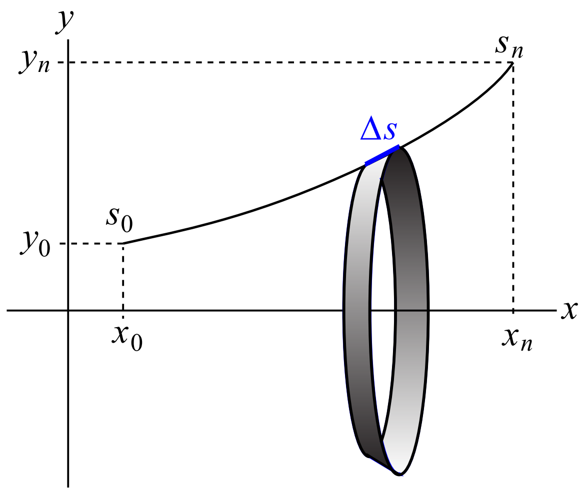

Figure 2: Rotating a curve about the x-axis.

The (signed) area under a curve $y = f(x)$ between $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.

Suppose that this curve is parameterized by the equations

$$\begin{aligned} x &= x(t) \\ y &= y(t). \end{aligned}$$

Moreover suppose that $x(t_0) = a$ and $x(t_1) = b$. Then the (signed) area under the curve is also equal to

$$\int_a^b f(x)\,dx = \qquad \int_{t_0}^{t_1} f(x(t))\,x'(t)\,dt.$$

by change of variables (or by substitution and applying the chain rule).

But note that

$$\begin{aligned} y &= f(x) \\ y &= y(t) \\ \implies y(t) &= f(x(t)) \end{aligned}$$

hence

$$\int_a^b f(x)\,dx = \int_{t_0}^{t_1} y(t)\,x'(t)\,dt.$$

This formula for the (signed) area under a parametric curve holds in general!

General result for parametric curves

Given a parametric curve:

$$\begin{aligned} x &= x(t) \\ y &= y(t), \end{aligned}$$

The signed area of the region bounded between the curve and the x-axis for $t_0 < t < t_1$ is given by the integral

$$\text{Signed Area} = \int_{t_0}^{t_1} y(t)\, x'(t)\, dt.$$