# Summary: Taylor Series and Power Series

## General power series

A **power series** is an infinite series involving positive powers of a variable $x$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

The **radius of convergence** $R$ of the power series $\sum_{n=0}^{\infty} a_n x^n$, is a real number $0 \leq R < \infty$ such that

- for $|x| < R$, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges (to a finite number);
- for $|x| > R$, the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges;
- for $|x| = R$, the power series may converge or diverge. But we will mostly ignore what happens at the end points of the interval of convergence.

**Examples:**

- Geometric series: $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, radius of convergence is 1.
- Polynomials: $a_0 + a_1 x + a_2 x^2 + \cdots a_N x^N = \sum_{n=0}^{N} a_n x^n$, radius of convergence $\infty$. In other words, the sum converges for all $x$.

## Finding the radius of convergence

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the ratio test implies that the power series converges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1.$$

There are 3 possibilities:

1. There is a finite number $R$ such that
   - $|x| < R \Longrightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1,$
   - $|x| > R \Longrightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| > 1.$

   We say the radius of convergence is $R$.

2. For all $x$ $\quad \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| < 1$. We say the radius of convergence is $\infty$. (All $x$ satisfy $|x| < \infty$.)

3. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| > 1$ for all $x \neq 0$. We say the radius of convergence is 0.

## Remark: Alternative method using ratio test

(Note that in the method that follows, the $n+1$ term is in the denominator and the $n$ term is in the numerator, which is the opposite of the ratio test.)

Given a power series $\sum_{n=0}^{\infty} a_n x^n$,
if $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R \quad$ (where $R$ exists or is $\infty$),
then the radius of convergence for the power series is $R$.

### Example

Consider $\sum_{n=0}^{\infty} 2^n x^n$, then $\lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \right| = \frac{1}{2} \quad \Longrightarrow \quad R = \frac{1}{2}.$

## Root test for radius of convergence

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, the root test implies that the power series converges if

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| < 1.$$

There are 3 possibilities:

1. There is a finite number $R$ such that
   - $|x| < R \Longrightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| < 1,$
   - $|x| > R \Longrightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|}|x| > 1.$

   We say the radius of convergence is $R$.
2. For all $x$ $\lim_{n\to\infty} \sqrt[n]{|a_n|}|x| < 1$. We say the radius of convergence is $\infty$. (All $x$ satisfy $|x| < \infty$.)
3. $\lim_{n\to\infty} \sqrt[n]{|a_n|}|x| > 1$ for all $x \neq 0$. We say the radius of convergence is 0.

### Example

Consider $\sum_{n=0}^{\infty} 2^n x^n$, then $\lim_{n\to\infty} \sqrt[n]{|2^n x^n|} = 2|x| < 1$ when $|x| < \frac{1}{2}$. This implies that the radius of convergence is $R = \frac{1}{2}$.

## Properties of power series

Add, subtract, multiply, divide, differentiate, and integrate convergent power series as one does for polynomials. We will discuss multiplication and division at a later time.

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, which converges for $|x| < A$.

- The derivative $\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ also converges for $|x| < A$.
- The integral $\int \left( \sum_{n=0}^{\infty} a_n x^n \right) dx = c + \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ also converges for $|x| < A$. Note that $c$ is the constant of integration.

Consider another power series $\sum_{n=0}^{\infty} b_n x^n$, which converges for $|x| < B$.

- If $A \neq B$, then $\left(\sum_{n=0}^{\infty} a_n x^n\right) \pm \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ converges for $|x| < \min(A, B)$.
- If $A = B$ , then $\left(\sum_{n=0}^{\infty} a_n x^n\right) \pm \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ has a radius of convergence which is at least $A$, but it *could* have a larger radius of convergence.

## Taylor's formula

Recall that $n! = n(n-1)(n-2)\cdots(3)(2)(1)$ for all integers $n \geq 1$.

We define $0! = 1$. This is a very valuable convention that simplifies many formulas.

**Taylor's formula** says that

$$\begin{aligned} f(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \cdots \\ &= \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned}$$

when $|x| < R$ where $R$ is the radius of convergence of the power series above.

The power series in Taylor's formula is called the **Taylor series** of $f(x)$.

### Important examples

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

We will use Taylor's formula to derive the series for $\sin(x)$ and $\cos(x)$ on the next page.

Notice that the factorial appears in the denominator of all terms in all three power series above.

Using the Taylor series of $e^x$, we find a formula for the number $e$ as the rapidly converging series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

## Known Maclaurin series

So far, we have used Taylor's formula to obtain the following Taylor series:

- $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$.
- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

We have also integrated the geometric series to obtain a power series for $\ln(1-x)$:

- $-\ln(1-x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.