Spamegg's Commentary on "In-Class Problems Week 1, Fri. (Session 2)"

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1 Facts used

In addition to previous sections, we are using:

The Quotient-Remainder Theorem: For all pairs of integers $a, b > 0$ there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < b$.

2 Problem 1.

Prof. Meyer asks us:

Prove that if $a \cdot b = n$, then either a or b must be $\leq n$, where a, b, and n are nonnegative real numbers. Hint: by contradiction, Section 1.8 in the course textbook.

Let's do some "scratch work" first. Just to make sure you are completely clear about everything.

First notice that the statement we want to prove has the form: IF A THEN B (in logical notation: $A \implies B$).

Here A is: "a, b, n are nonnegative real numbers, and $a \cdot b = n$". This is what we will be assuming. (Because we are trying to prove an "if... then..." statement so we assume the IF part.)

B is: "$a \leq \sqrt{n}$ or $b \leq \sqrt{n}$."

B has the form: X or Y (in logical notation: $X \vee Y$) where X is: "$a \leq \sqrt{n}$" and Y is: "$b \leq \sqrt{n}$".

The negation of $X \vee Y$ is: $\neg X \wedge \neg Y$ (or: "not X and not Y"). Why is this? It is due to **De Morgan's Laws**.

So when we do our proof by contradiction we will be assuming $\neg X \wedge \neg Y$ which is: "not $a \leq \sqrt{n}$ and not $b \leq \sqrt{n}$," which is: "$a > \sqrt{n}$ and $b > \sqrt{n}$."

Proof. 1. Assume a, b, n are nonnegative real numbers, and $a \cdot b = n$.

2. Argue by contradiction.

3. Assume $a > \sqrt{n}$ and $b > \sqrt{n}$.

4. Since all the numbers involved $a, b, n, \sqrt{n}$ are nonnegative, we can multiply the two inequalities in (3) to get: $a \cdot b > \sqrt{n} \cdot \sqrt{n}$.

5. Using (1) we can replace $a \cdot b$ with n, so (4) gives us: $n > n$, a contradiction.

6. Our assumption in (3) must be false, therefore either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. $\square$

3 Problem 2.

Prof. Meyer asks us:

Generalize the proof of Theorem 1.8.1 repeated below that $\sqrt{2}$ is irrational in the course textbook. For example, how about $\sqrt{3}$?

Then Prof. Meyer repeats the proof that $\sqrt{2}$ is irrational. First, let's talk about that proof, as I think some things need to be clarified about it. It starts with:

The proof is by contradiction: assume that $\sqrt{2}$ is rational, that is, $\sqrt{2} = \frac{n}{d}$ where n and d are integers. Now consider the smallest such positive integer denominator, d.

3.1 Smallest element with a certain property?

OK, where did this "smallest such positive d" come from? The only thing we know is that d is an integer.

We know that $\sqrt{2}$ is positive by definition of the positive square root. So $\frac{n}{d}$ is also positive. So n and d are either both positive, or both negative. WLOG (without loss of generality) we may assume they are both positive (if they are both negative, replace them with $-n$ and $-d$).

But how do we know there is a smallest such d? Let's think about it. Consider an example where $\frac{n}{d}$ is equal to $\frac{8}{20}$. Then the set of all possible choices for (both positive) n and d would be:

$$\left\{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \frac{10}{25}, \frac{12}{30}, \dots\right\}$$

The choice with the smallest positive d is the one where n and d have no common prime divisors. So when Prof. Meyer says "consider smallest such positive d", this is equivalent to saying "assume that n and d are both positive and they have no common prime divisors" (which is sometimes expressed as "where n and d are in lowest terms").

Then Prof. Meyer later contradicts the minimality of d by showing that both n and d are divisible by 2, and obtaining $\dfrac{n/2}{d/2}$ where $d/2$ is a smaller positive denominator. This is equivalent to contradicting "n and d have no common prime divisors".

Here Prof. Meyer is actually *foreshadowing* another proof technique that will be taught later: using the Well-Ordering Principle (every nonempty subset of natural numbers must have a least element) and arguments by contradicting the minimality of the least element of a set that is defined by a property. (The set I wrote above is non-empty and it's linearly ordered and bounded from below, so it must have a least element.)

But if you found that confusing, you can go the "no common divisors" route of argument. If you remember from my previous commentaries, I gave an alternate definition of a rational number that incorporated this.

3.2 Imitate the argument for square root of 3

OK, let's get on with the task Prof. Meyer assigned us. Repeat the proof for $\sqrt{3}$. Let's think ahead of the problems we will face:

We will be arguing by contradiction. So we will have to contradict the minimality of the positive denominator like before (or contradict that the numerator and denominator were not supposed to have common divisors).

In the previous proof we needed the fact that if n^2 is even then n is even. This time we will need: "if n^2 is divisible by 3 then n is divisible by 3."

That can be done by yet another proof by contradiction in itself.

Theorem 1. *$\sqrt{3}$ is an irrational number.*

Proof. 1. Argue by contradiction and assume $\sqrt{3}$ is rational.

2. By the definition of a rational number, there exist integers n and d such that $\sqrt{3} = \frac{n}{d}$ where $d \neq 0$ and n and d have no common divisors greater than 1. WLOG we may assume that both n and d are both positive, since $\sqrt{3}$ is positive.

3. Squaring both sides we get $3 = \frac{n^2}{d^2}$.

4. Multiplying both sides by d^2 we get $3d^2 = n^2$.

5. From this equation we notice that 3 divides n^2. (Because there exists an integer $k = d^2$ such that $n^2 = 3k$, which is the definition of divisibility).

Claim 1. *Assume n is a positive integer. If 3 divides n^2, then 3 divides n.*

This is actually Problem 1.10 part (b) in the textbook! Prof. Meyer tells us to do it in the proof of $\sqrt{2}$ is irrational.

Proof. (of the Claim.) Assume n is a positive integer and 3 divides n^2.

By definition of divisibility there exists an integer k such that $3k = n^2$ (we will need this later below).

Argue by contradiction and assume that 3 does not divide n.

By the Quotient-Remainder Theorem (from high school) there exist integers q, r such that $n = 3q + r$ where $0 \leq r < 3$.

Since 3 does not divide n, r cannot be 0. So r must be 1 or 2.

Case 1 (of the Claim). $r = 1$.

Then $n = 3q + 1$. So $n^2 = (3q + 1)^2 = 9q^2 + 6q + 1$.

So $3k = 9q^2 + 6q + 1$, dividing by 3 we get $k = 3q^2 + 2q + \frac{1}{3}$.

Moving terms, we get $k - 3q^2 - 2q = \frac{1}{3}$. This is a contradiction! Because the left-hand side $k - 3q^2 - 2q$ is an integer, but the right-hand side $\frac{1}{3}$ is not an integer.

Case 2 (of the Claim). $r = 2$.

Then $n = 3q + 2$. So $n^2 = (3q + 2)^2 = 9q^2 + 12q + 4$.

So $3k = 9q^2 + 12q + 4$, dividing by 3 we get $k = 3q^2 + 4q + \frac{4}{3}$.

Moving terms, we get $k - 3q^2 - 4q = \frac{4}{3}$. This is a contradiction! Because the left-hand side $k - 3q^2 - 4q$ is an integer, but the right-hand side $\frac{4}{3}$ is not an integer.

These cases are exhaustive of all possibilities, and in all cases we had a contradiction.

Therefore our assumption must have been false, therefore 3 divides n, **finishing the proof of the Claim.** □

Continuing the proof of the Theorem:

6. By (5) and the Claim, 3 divides n.

7. By definition of divisibility, there exists an integer m such that $3m = n$.

8. Substituting (7) into (4) we get $3d^2 = (3m)^2 = 9m^2$.

9. Dividing by 3, we get $d^2 = 3m^2$. This means d is divisible by 3, which is a contradiction to the fact that n and d have no common divisors greater than 1.

10. Therefore our initial assumption was false, hence $\sqrt{3}$ is irrational. $\square$

3.3 Generalizing even further

This subsection is fairly hard and is optional.

How far can this Theorem be generalized? Is $\sqrt{4}$ irrational too? No, it's equal to 2. Where would the proof go wrong if we tried it on $\sqrt{4}$?

Let m vary over the positive integers, and consider the general statement: "$\sqrt{m}$ is irrational." Intuitively, it seems like this should be true as long as m itself is not a perfect square. If we go through the proof, we end up with a step where $md^2 = n^2$, and we notice m divides n^2. Then we would have to argue the Claim, that is, if m divides n^2 then m divides n, and derive the contradiction similarly.

So, is it true that if m and n are positive integers, m **is not a perfect square**, and m divides n^2, then m divides n? Not quite. We can let $n = pq$ where p and q are two primes that are different from each other, and let $m = p^2q$. Then m divides $n^2 = p^2q^2$ but not $n = pq$. So we cannot use the same argument, with the same Claim, to prove the Theorem for all m that are not perfect squares.

However, the Claim that if m divides n^2 then m divides n *should* hold true for all **prime** m. When $m = 3$ we had to consider two cases: where the remainder of dividing n by m was 1 or 2. In general there will be $m - 1$ cases! We cannot go through them one by one (we don't know how many there are, since we don't know the value of m), so we will have to "parametrize" all the cases and handle them in a generic way.

Lemma 1. *Assume m and n are positive integers and m is prime. If m divides n^2 then m divides n.*

Proof. 1. Assume m and n are positive integers, m is prime, and m divides n^2.

2. By definition of divisibility, there exists an integer k such that $mk = n^2$. (We notice that k must be positive.)

3. By the Quotient-Remainder theorem there exist integers q, r such that $n = qm + r$ where $0 \le r < m$.

4. If $r = 0$ then $n = qm$ so m divides n, and we are done. So now consider the case $r > 0$.

5. Then $n^2 = (qm + r)^2 = q^2m^2 + 2qmr + r^2$.

6. By (2) and (4) we have $q^2m^2 + 2qmr + r^2 = mk$.

7. Dividing by m we get $q^2m + 2qr + \frac{r^2}{m} = k$.

8. Moving terms, we get $q^2m + 2qr - k = -\frac{r^2}{m}$.

9. Since m is prime and $0 < r < m$, r^2 is not divisible by m. **(We need to prove this!)**

10. So the LHS of (8) is an integer, while the RHS of (8) is not an integer (because $r \neq 0$), a contradiction.

11. Our initial assumption must have been false, therefore m divides n. □

Let's prove step (8). We have to use the Fundamental Theorem of Arithmetic and properties of prime numbers.

Claim 2. *Assume m is prime and $0 < r < m$ is an integer. Then m does not divide r^2.*

Proof. By the Fundamental Theorem of Arithmetic, r is a unique product of prime numbers that are all less than or equal to r. Since $0 < r < m$, these primes are all less than m. So m does not divide any of these primes. Since m itself is a prime and it is different than all these primes, m does not divide any products of these primes either. Therefore m does not divide r or r^2 (or any power of r for that matter). □

With Lemma 1, we are able to generalize the Theorem to square roots of any primes (just repeat the proof for $\sqrt{3}$ where m replaces 3, and use the Lemma in the place of Claim 1):

Theorem 2. *Assume m is prime. Then $\sqrt{m}$ is irrational.*

Earlier we said that the theorem should hold not just for prime m, but any m that is not a perfect square itself. However proving this greater generalization would require more work.

4 Problem 3

Prof. Meyer asks: *If we raise an irrational number to an irrational power, can the result be rational? Show that it can, by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.*

This is a very cool exercise! It requires a bit of creativity. Let's think about it: why did Prof. Meyer give us $\sqrt{2}^{\sqrt{2}}$? What is your intuition about this number? Is it rational or irrational? It looks irrational doesn't it? I mean, it would be pretty incredible if it were rational...

Wait a minute. "...if it were rational..." Prof. Meyer told us to consider cases! So this could be one of the cases. What should be the other case(s) in order to be exhaustive? A real number is either rational or irrational, by definition. So the second case would be when it's irrational.

Case 1. $\sqrt{2}^{\sqrt{2}}$ is rational.

We know that $\sqrt{2}$ is irrational (earlier Theorem from the lecture). So in this case, an irrational, namely $\sqrt{2}$, raised to an irrational power, namely $\sqrt{2}$, gives us a rational number, namely $\sqrt{2}^{\sqrt{2}}$. Therefore we proved the claim in this case.

Case 2. $\sqrt{2}^{\sqrt{2}}$ is irrational.

OK, here we need some creativity. We need to obtain, like the previous case, an irrational raised to an irrational power that results in a rational number. We already have an irrational number: $\sqrt{2}^{\sqrt{2}}$. Can you see a way of obtaining a rational number out of it?

The base of the number is $\sqrt{2}$. How could we turn this base into rational? We would have to SQUARE it, to get 2, which is rational. So the power should be 2.

How can we turn the power of $\sqrt{2}^{\sqrt{2}}$ into 2? Currently the power is $\sqrt{2}$. That power would have to be multiplied by $\sqrt{2}$ to turn into 2. So we need:

$$\sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

How can we get $\sqrt{2} \cdot \sqrt{2}$ as the power? By the law of exponents, $(a^b)^c = a^{bc}$. So:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

So, in this case, once again we have an irrational, namely $\sqrt{2}^{\sqrt{2}}$, raised to an irrational power, namely $\sqrt{2}$, that results in a rational number, namely 2. So we proved the claim in this case too.

Pretty cool huh? The only issue is that we don't know which case is actually true. So this is what the philosophers (the intuitionists Meyer mentioned in the videos) would call a "non-constructive proof".

5 Problem 4

Prof. Meyer outlines a *constructive* proof that a^b can be rational for irrational a, b. He gives us: $a = \sqrt{2}$ and $b = 2\log_2(3)$. We know a is irrational from the lectures. **We need to prove b is irrational.** Then:

$$a^b = \sqrt{2}^{2\log_2(3)} = \left(2^{1/2}\right)^{2\log_2(3)} = 2^{\frac{1}{2}\cdot 2\log_2(3)} = 2^{\log_2(3)} = 3$$

Here we are using the fact that $\sqrt{2} = 2^{1/2}$, then a law of exponents, and then at the last step the definition of $\log_2$.

To prove b is irrational, how should we argue? By contradiction? By cases? Let's do some "scratch work" first.

If we argue by contradiction and assume b is rational, we can write $b = \frac{n}{d}$ where n and d have no common divisors greater than 1 and WLOG $d > 0$. So

$$2\log_2(3) = \frac{n}{d} \implies \log_2(3) = \frac{n}{2d} \implies 2^{\log_2(3)} = 2^{\frac{n}{2d}} \implies 3 = 2^{\frac{n}{2d}} \implies 3^{2d} = 2^n$$

Now with the last equality $3^{2d} = 2^n$ we can use the Unique Factorization Theorem (which is the name sometimes given to the second clause of the Fundamental Theorem of Arithmetic) to get a contradiction. OK, that should work!

Claim 3. $2\log_2(3)$ *is irrational.*

Proof. 1. Argue by contradiction and assume $2\log_2(3)$ is rational.

2. By the definition of a rational number, there exist integers n and d such that $2\log_2(3) = \dfrac{n}{d}$, where n and d have no common divisors greater than 1, and WLOG $d > 0$.

3. Dividing both sides by 2, we get $\log_2(3) = \dfrac{n}{2d}$.

4. Using exponentiation with base 2 for both sides, we get $2^{\log_2(3)} = 2^{n/2d}$.

5. By the definition of $\log_2$, we get $3 = 2^{n/2d}$.

6. Raising both sides to the power $2d$ we get $3^{2d} = 2^n$.

7. Since $d > 0$, $x = 3^{2d}$ is a natural number greater than 1. Then so is $x = 2^n$.

8. By the Fundamental Theorem of Arithmetic, every natural number greater than 1 has a unique prime factorization. But, since 2 and 3 are different primes, we have two different prime factorizations of x: one is $x = 3^{2d}$ and the other is $x = 2^n$. This is a contradiction!

9. Therefore $2\log_2(3)$ is irrational. $\square$