(contributed by spamegg)

Exercises 36: QL-validity

(a) Find countervaluations to show that the following inferences are not q-valid:

(1) $\forall \mathsf{xLxx} \therefore \forall \mathsf{y} \forall \mathsf{zLyz}$

To make things easier for ourselves, let's continue thinking along the lines of our QL_1 example from before. So L is a predicate that translates to "loves". The premise means: "everybody loves themselves" and the conclusion means: "everyone loves everyone else".

So, all we have to do is, come up with a situation where everyone loves themselves, but not everyone loves everyone else. We only need one person who does not love another person. Let us formalize this idea.

Let's take a language with just the following non-logical vocabulary:

Binary predicate: L.

And let q be the following q-valuation for this language:

The domain: {Socrates, Plato}

Extensions:

L : $\{\langle \text{Socrates, Socrates}\rangle, \langle \text{Plato, Plato}\rangle, \langle \text{Socrates, Plato}\rangle\}$

We see that $\forall \mathsf{xLxx} \coloneqq_q \mathrm{T}$, since both $\langle$Socrates, Socrates$\rangle$ and $\langle$Plato, Plato$\rangle$ belong to the extension of L (so Laa is true under all possible expanded evaluations q_a, no matter what object from the domain is assigned to the dummy name a, whether Socrates or Plato).

However $\forall \mathsf{y} \forall \mathsf{zLyz} \coloneqq_q \mathrm{F}$.

To see this, notice that $\forall \mathsf{y} \forall \mathsf{zLyz} \coloneqq_q \mathrm{T}$ if and only if $\forall \mathsf{zLaz} \coloneqq_{q_\mathsf{a}} \mathrm{T}$ on every possible expanded evaluation q_a; otherwise $\forall \mathsf{y} \forall \mathsf{zLyz} \coloneqq_q \mathrm{F}$.

Let q_{a} be a valuation which augments q by assigning Plato as reference to the dummy name a.

Now notice that $\forall \mathsf{zLaz} \coloneqq_{q_{\mathsf{a}}} \mathrm{T}$ if and only if $\mathsf{Lab} \coloneqq_{q_{\mathsf{ab}}} \mathrm{T}$ on every possible expanded evaluation q_{ab}; otherwise $\forall \mathsf{zLaz} \coloneqq_{q_{\mathsf{a}}} \mathrm{F}$.

Let q_{ab} be a valuation which augments q_{a} by assigning Socrates as reference to the dummy name b.

We see that $\mathsf{Lab} \coloneqq_{q_{\mathsf{ab}}} \mathrm{F}$ since $\langle$Plato, Socrates$\rangle$ does not belong to the extension of L. So we found one possible expanded valuation q_{ab} of q_{a} under which $\mathsf{Lab} \coloneqq_{q_{\mathsf{ab}}} \mathrm{F}$. This means $\forall \mathsf{zLaz} \coloneqq_{q_{\mathsf{a}}} \mathrm{F}$.

Similarly, we just found an expanded valuation q_{a} of q under which $\forall \mathsf{zLaz} \coloneqq_{q_{\mathsf{a}}} \mathrm{F}$. This means $\forall \mathsf{y} \forall \mathsf{zLyz} \coloneqq_{q} \mathrm{F}$.

For the next two, we will not go into as much detail (with expanded valuations and such), but the arguments are similar.

(2) $\forall \mathsf{x}(\mathsf{Fx} \vee \mathsf{Gx}) \therefore (\forall \mathsf{xFx} \vee \forall \mathsf{xGx})$

Let's think of F as "is a philosopher" and G as "is a logician". We need an example where everyone is either a philosopher or a logician, but not everyone is a philosopher, and not everyone is a logician.

Let's take a language with just the following non-logical vocabulary:

Unary predicates: F, G.

And let q be the following q-valuation for this language:

The domain: {Socrates, Plato}

Extensions: F : {Socrates} and G : {Plato}

We can see that the premise $\forall \mathsf{x}(\mathsf{Fx} \vee \mathsf{Gx})$ is true under this valuation. When the object is Socrates, we see that predicate F is satisfied, so it holds that either F or G is satisfied. Similarly for when the object is Plato (since G is satisfied).

However the conclusion $(\forall x Fx \vee \forall x Gx)$ is false under this valuation: $\forall x Fx$ is false since Plato does not satisfy F; and $\forall x Gx$ is false since Socrates does not satisfy G.

(3) $\forall x \exists y \forall z Rxyz \therefore \forall z \exists y \forall x Rxyz$

Again, to help ourselves, let's think of the predicate R as the "1 prefers 2 to 3" relation. The premise says: "everyone has someone whom they prefer over everyone", the conclusion says "for every person z, there is a person y who is preferred over z by everyone." To satisfy the premise, we need an example world where everyone has a person that they prefer over everyone.

Let's take a language with just the following non-logical vocabulary:

Ternary predicate: R.

And let q be the following q-valuation for this language:

The domain: {Socrates, Plato, Aristotle}

Extensions:

$$\begin{aligned}\mathsf{R} : \{&\langle \text{Socrates, Plato, Socrates}\rangle,\\ &\langle \text{Socrates, Plato, Plato}\rangle,\\ &\langle \text{Socrates, Plato, Aristotle}\rangle,\\ &\langle \text{Plato, Aristotle, Socrates}\rangle,\\ &\langle \text{Plato, Aristotle, Plato}\rangle,\\ &\langle \text{Plato, Aristotle, Aristotle}\rangle,\\ &\langle \text{Aristotle, Socrates, Socrates}\rangle,\\ &\langle \text{Aristotle, Socrates, Plato}\rangle,\\ &\langle \text{Aristotle, Socrates, Aristotle}\rangle\}\end{aligned}$$

We can see that the premise $\forall\mathsf{x}\exists\mathsf{y}\forall\mathsf{z}\mathsf{Rxyz}$ is true under this valuation: Socrates prefers Plato, Plato prefers Aristotle, and Aristotle prefers Socrates, over everyone.

To see that the conclusion $\forall\mathsf{z}\exists\mathsf{y}\forall\mathsf{x}\mathsf{Rxyz}$ is false, temporarily let the dummy names a correspond to Socrates, b to Plato and c to Aristotle respectively.

Notice that Raac is false, therefore $\forall\mathsf{x}\mathsf{Rxac}$ is false.

Similarly Rbbc is false, therefore $\forall\mathsf{x}\mathsf{Rxbc}$ is false.

Similarly Racc is false, therefore $\forall\mathsf{x}\mathsf{Rxcc}$ is false.

Putting these three together, we see that $\exists\mathsf{y}\forall\mathsf{x}\mathsf{Rxyc}$ is false.

Finally, from this we can conclude that $\forall\mathsf{z}\exists\mathsf{y}\forall\mathsf{x}\mathsf{Rxyz}$ is false.

Use informal reasoning in the style of §36.4 to decide which of these inferences is q-valid:

(4) $\exists\mathsf{x}\mathsf{Fx}, \forall\mathsf{y}(\mathsf{Gy} \to \neg\mathsf{Fy}) \therefore \exists\mathsf{z}\neg\mathsf{Gz}$

This inference is q-valid. Here is an informal argument:

First, for convenience, change all the variables in the premises to z.

We can use Modus Tollens on the "inside" of the second premise to turn it into $\forall \mathsf{z}(\mathsf{Fz} \to \neg \mathsf{Gz})$.

From the first premise we know that there is an object, say m, which satisfies F.

From the second premise, we know that every object that satisfies F must satisfy $\neg \mathsf{G}$.

So $\neg \mathsf{Gm}$ is true, which gives us the conclusion.

(5) $\neg \exists \mathsf{x}(\mathsf{Px} \wedge \mathsf{Mx}), \neg \exists \mathsf{x}(\mathsf{Sx} \wedge \neg \mathsf{Mx}) \therefore \neg \exists \mathsf{x}(\mathsf{Sx} \wedge \mathsf{Px})$

This inference is also q-valid. Here is an informal argument:

First, for convenience, in our premises, let's push all the negations inside, use De Morgan's Laws, and eliminate double negations. This gives us

$$\forall \mathsf{x}(\neg \mathsf{Px} \vee \neg \mathsf{Mx}), \forall \mathsf{x}(\neg \mathsf{Sx} \vee \mathsf{Mx}) \therefore \neg \exists \mathsf{x}(\mathsf{Sx} \wedge \mathsf{Px})$$

Argue by contradiction and assume there exists an object, say m, which satisfies both S and P at the same time.

The second premise says that every object must either satisfy $\neg \mathsf{S}$ or satisfy M. Since m satisfies S, this forces Mm.

The first premise says that every object must either satisfy $\neg \mathsf{P}$ or satisfy $\neg \mathsf{M}$. Since m satisfies M, this forces $\neg \mathsf{Pm}$.

But this is a contradiction! Because m also satisfies P.

(6) $\forall \mathsf{x}(\mathsf{Fx} \to \mathsf{Gx}), \forall \mathsf{y}(\neg \mathsf{Gx} \to \mathsf{Hx}) \therefore \forall \mathsf{z}(\neg \mathsf{Hz} \to \mathsf{Fx})$

Invalid, since the second premise and the conclusion are not even wffs! They have dangling, unquantified x variables.

(7) $\exists \mathsf{x}(\mathsf{Rx} \wedge \neg \mathsf{Px}), \forall \mathsf{x}(\mathsf{Rx} \to \mathsf{Sx}) \therefore \forall \mathsf{x}(\mathsf{P} \to \neg \mathsf{Sx})$

I am assuming that there is a typo in the conclusion, and that it should read $\forall \mathsf{x}(\mathsf{Px} \to \neg \mathsf{Sx})$ instead.

The first premise gives us an object, say m, that satisfies R and $\neg \mathsf{P}$. The second premise implies that this object should also satisfy S. So we have Rm and $\neg \mathsf{Pm}$ and Sm.

Do these facts force the conclusion $\forall \mathsf{x}(\mathsf{Px} \to \neg \mathsf{Sx})$? No, we have information about only one object. Possible domains may have many more objects.

Or do they provide a counterexample that disproves the conclusion? No, a counterexample would require Pm and Sm.

But these observations do tell us that the inference is invalid, and that we can manufacture a countervaluation. All we need to do is create a domain with two objects, one of which fulfils the role of m, and another object which does not contradict the premises, but is a counterexample for the conclusion. Let's do it.

Let's take a language with just the following non-logical vocabulary:

Unary predicates: $\mathsf{P}, \mathsf{R}, \mathsf{S}$.

And let q be the following q-valuation for this language:

The domain: $\{\mathsf{m}, \mathsf{n}\}$

Extensions: $\mathsf{P} : \{\mathsf{n}\}$, $\mathsf{R} : \{\mathsf{m}\}$, $\mathsf{S} : \{\mathsf{m}, \mathsf{n}\}$.

Here are all the atomic formulas, and negations of atomic formulas, which are true: $\neg \mathsf{Pm}, \mathsf{Pn}, \mathsf{Rm}, \neg \mathsf{Rn}, \mathsf{Sm}, \mathsf{Sn}$.

We see that the first premise is true (fulfilled by m). The second premise is also true: Rm and Sm are both true. But the conclusion is false: we have Pn but Sn.

(8) $\forall \mathsf{x}(\mathsf{Px} \to \mathsf{Mx}), \exists \mathsf{x}(\mathsf{Sx} \wedge \neg \mathsf{Mx}) \therefore \exists \mathsf{x}(\mathsf{Sx} \wedge \neg \mathsf{Px})$

This inference is q-valid. Here is an informal argument:

The second premise gives us an object, say m, such that Sm and $\neg\mathsf{Mm}$.

We can use Modus Tollens on the "inside" of the first premise, to get $\forall \mathsf{x}(\neg \mathsf{Mx} \to \neg \mathsf{Px})$. Since we already have $\neg\mathsf{Mm}$, this gives us $\neg\mathsf{Pm}$.

Thus we have Sm and $\neg\mathsf{Pm}$, which gives us the conclusion.

(9) $\forall \mathsf{x}(\mathsf{Lxm} \vee \neg \mathsf{Lxn}), \neg \mathsf{Lnm} \therefore \neg \exists \mathsf{xLxn}$

This inference is invalid, and here is a countervaluation. Consider a domain with only two objects m, n and a binary predicate L with the following extension:

$$\{\langle \mathsf{m}, \mathsf{m}\rangle, \langle \mathsf{m}, \mathsf{n}\rangle\}$$

For convenience let's write down all 4 atomic (and negation of atomic) formulas that are true: $\mathsf{Lmm}, \mathsf{Lmn}, \neg\mathsf{Lnm}, \neg\mathsf{Lnn}$.

Let's verify the first premise: when x is replaced by m, $\mathsf{Lmm} \vee \neg\mathsf{Lmn}$ is in fact true. When x is replaced by n, $\mathsf{Lnm} \vee \neg\mathsf{Lnn}$ is also true.

The conclusion is false, because Lmn is true.

(10) $(\exists \mathsf{xLox} \to \mathsf{Ho}), (\exists \mathsf{xLxn} \to \mathsf{Lon}), \mathsf{Lmn} \therefore \mathsf{Ho}$

This inference is q-valid. Here is an informal argument:

From the third premise Lmn, by existentially quantifying over m we can get $\exists \mathsf{xLxn}$.

Use Modus Ponens on the second premise with this. We get Lon. Existentially quantify over n this time, to get ∃xLox.

Now use Modus Ponens on the first premise with this, to get Ho.

(b*) Find out more about systematically looking for countervaluations and perhaps discovering that there can't be one by looking at the online supplement on quantifier truth trees.

No solutions available for this one.