Summary: Improper integrals

Consider $f, g > 0$. We say that g grows faster than f as x tends towards ∞, and write this as

$$0 < f(x) << g(x),$$

if $\quad f, g \longrightarrow \infty \quad$ and $\quad \dfrac{f(x)}{g(x)} \longrightarrow 0 \quad$ as $\quad x \to \infty.$

Examples

For $p > 0$,

$\ln x << x^p << e^x << e^{x^2}$.

Note that $e^{x^2} = e^{(x^2)}$, not $(e^x)^2 = e^{2x}$.

Rate of Decay (as $x \to \infty$)

Consider $f, g > 0$. We say that f decays faster than g as x tends towards ∞, and write this as

$$g(x) >> f(x) > 0,$$

if $\quad f, g \longrightarrow 0 \quad$ and $\quad \dfrac{f(x)}{g(x)} \longrightarrow 0 \quad$ as $\quad x \to \infty.$

Examples

For $p > 0$,

$$\frac{1}{\ln x} >> \frac{1}{x^p} >> e^{-x} >> e^{-x^2}.$$

Improper integrals definition

An **improper integral** is defined by $\displaystyle\int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx.$

This improper integral **converges** if the limit exists and is finite.
This improper integral **diverges** if the limit does not exist (this includes when the limit is $\pm\infty$).

Conclusion of example: powers of x

$$\int_a^\infty \frac{dx}{x^p} \qquad \begin{cases} \text{diverges} & \text{if } \; p \le 1 \\ \text{converges to } \frac{a^{-p+1}}{p-1} & \text{if } \; p > 1 \end{cases}$$

Notation

We say that $f(x) \sim g(x)$ as $x \to \infty$ if $\dfrac{f(x)}{g(x)} \underset{x\to\infty}{\longrightarrow} 1$.

In words, we say that $f(x)$ and $g(x)$ are **similar** as $x \to \infty$. (The idea is that $f(x)$ and $g(x)$ have the same asymptotic behavior as x tends to infinity.)

Limit comparison

The idea behind limit comparison is that if the asymptotic behavior is the same, then the improper integrals have the same behavior.

If $f(x) \sim g(x)$ as $x \to \infty$,

then the two integrals $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ (for large a) either **both converge** or **both diverge**.

This also works in the case where one function decays faster than the other as x tends towards infinity.

Suppose that $g(x)$ decays faster than $f(x)$ as $x \to \infty$. That is $f(x) >> g(x)$ as $x \to \infty$.

- If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges.
- If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Comparison

Suppose $f(x) \ge g(x) > 0$ for $x \ge a$.

If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges also.

If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges also.