(contributed by spamegg)

Exercises 35:

(a) Take a QL language, with a couple of proper names, a couple of unary predicates, and a binary predicate, and consider the following q-valuation q:

The domain is: {Romeo, Juliet, Benedick, Beatrice}

m: Romeo

n: Juliet

F: {Romeo, Benedick}

G: {Juliet, Beatrice}

L: {⟨Romeo, Juliet⟩, ⟨Juliet, Romeo⟩, ⟨Benedick, Beatrice⟩,

⟨Beatrice, Benedick⟩, ⟨Benedick, Benedick⟩}

Then what are the truth values of the following wffs?

(1) $\exists \mathsf{xLmx}$

True.

Informally translates to "Romeo loves someone."

⟨Romeo, Juliet⟩ is in the extension of L, so Lmn is true. Informally Juliet is an x that fulfills the existential quantifier. Formally, we can expand the valuation q to q_a where dummy name a is assigned Juliet and Lma is true. By the definition (Q6) $\exists \mathsf{xLmx}$ is true.

(2) $\forall \mathsf{xLxm}$

False.

Informally translates to "everyone loves Romeo". False since, for example,

$$\langle \text{Benedick, Romeo} \rangle$$

is not in the extension of L. Formally, there is an expanded valuation q_a, where dummy name a is assigned Benedick, for which Lam is false. So by the definition (Q5), $\forall \mathsf{xLxm}$ is false.

I will be less formal for the rest.

(3) $(\exists \mathsf{xLmx} \rightarrow \mathsf{Lmn})$

True.

Both the antecedent and the consequent are true. By (1) above, the antecedent $\exists \mathsf{xLmx}$ is true. The consequent Lmn is also true because $\langle$Romeo, Juliet$\rangle$ is in the extension of L.

(4) $\forall \mathsf{x}(\mathsf{Fx} \rightarrow \neg \mathsf{Gx})$

True.

There are 4 people in the domain, so let's verify $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ for all of them:

When a is Romeo, Fa is true (Romeo is in F's extension), $\neg \mathsf{Ga}$ is true (Romeo is not in G's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

When a is Juliet, Fa is false (Juliet is not in F's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

When a is Benedick, Fa is true (Benedick is in F's extension), $\neg \mathsf{Ga}$ is true (Benedick is not in G's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

When a is Beatrice, Fa is false (Beatrice is not in F's extension), so $\mathsf{Fa} \rightarrow \neg \mathsf{Ga}$ is true.

Therefore by (Q5) $\forall \mathsf{x}(\mathsf{Fx} \rightarrow \neg \mathsf{Gx})$ is true.

(5) $\forall \mathsf{x}(\mathsf{Gx} \to (\mathsf{Lxm} \lor \neg \mathsf{Lmx}))$

True.

Let's approach this like (4) but take shortcuts. We won't consider the a's for which Ga is false, because the implication $\mathsf{Ga} \to \ldots$ is automatically true in that case.

Let's only consider those in the extension of G: Juliet and Beatrice.

When a is Juliet, Ga is true, and $\mathsf{Lam} \lor \neg \mathsf{Lma}$ is true because Lam is true, since $\langle$Juliet, Romeo$\rangle$ is in the extension of L. So $\mathsf{Gx} \to (\mathsf{Lam} \lor \neg \mathsf{Lma})$ is true.

When a is Beatrice, Ga is true, and $\mathsf{Lam} \lor \neg \mathsf{Lma}$ is true, because $\neg \mathsf{Lma}$ is true, since $\langle$Romeo, Beatrice$\rangle$ is not in the extension of L. So $\mathsf{Gx} \to (\mathsf{Lam} \lor \neg \mathsf{Lma})$ is true.

By (Q5) $\forall \mathsf{x}(\mathsf{Gx} \to (\mathsf{Lxm} \lor \neg \mathsf{Lmx}))$ is true.

(6) $\forall \mathsf{x}(\mathsf{Gx} \to \exists \mathsf{yLxy})$

True.

Let's take the shortcuts again. We need to check the antecedent $\exists \mathsf{yLay}$ only for the two cases when Ga is true: Juliet and Beatrice.

When a is Juliet, Ga is true, and $\exists \mathsf{yLay}$ is true, because $\langle$Juliet, Romeo$\rangle$ is in the extension of L (informally y = Romeo).

(7) $\exists \mathsf{x}(\mathsf{Fx} \land \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Lxy}))$

(b) Now take the following q-valuation:

The domain is: $\{4, 7, 8, 11, 12\}$

m: 7

n: 12

F: the set of even numbers in the domain

G: the set of odd numbers in the domain

L: the set of pairs $\langle m, n\rangle$ where m and n are in the domain and $m < n$

What are the truth values of the wffs (1) to (7) now?

(1) $\exists \mathsf{xLmx}$

(2) $\forall \mathsf{xLxm}$

(3) $(\exists \mathsf{xLmx} \to \mathsf{Lmn})$

(4) $\forall \mathsf{x}(\mathsf{Fx} \to \neg \mathsf{Gx})$

(5) $\forall \mathsf{x}(\mathsf{Gx} \to (\mathsf{Lxm} \vee \neg \mathsf{Lmx}))$

(6) $\forall \mathsf{x}(\mathsf{Gx} \to \exists \mathsf{yLxy})$

(7) $\exists \mathsf{x}(\mathsf{Fx} \wedge \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Lxy}))$

(c) Take the language QL_3 of Exercises 30(b) (copied from that exercise):

This is the language QL_3 whose quantifiers range over the positive integers, with the following glossary:

n: one

F: ① is odd

G: ① is even

H: ① is prime

L: ① is less than ②

R: ① is the sum of ② and ③

Consider the following q-valuation (the natural one suggested by the given glossary for the language):

The domain is the set of natural numbers (the integers from zero up)

n: one

F: the set of odd numbers

G: the set of even numbers

H: the set of prime numbers

L: the set of pairs $\langle m, n\rangle$ such that $m < n$

R: the set of triples $\langle l, m, n\rangle$ such that $l = m + n$.

Carefully work out the values of the wffs (1) to (8) from Exercises 30(b) (copied from that exercise):

(1) $\forall\mathsf{x}\forall\mathsf{y}\exists\mathsf{zRzxy}$

(2) $\exists\mathsf{y}\forall\mathsf{xLxy}$

(3) $\forall\mathsf{x}\exists\mathsf{y}(\mathsf{Lxy} \wedge \mathsf{Hy})$

(4) $\forall\mathsf{x}(\mathsf{Hx} \to \exists\mathsf{y}(\mathsf{Lxy} \wedge \mathsf{Hy}))$

(5) $\forall\mathsf{x}\forall\mathsf{y}((\mathsf{Fx} \wedge \mathsf{Ryxn}) \to \neg\mathsf{Fy})$

(6) $\forall\mathsf{x}\exists\mathsf{y}((\mathsf{Gx} \wedge \mathsf{Fy}) \wedge \mathsf{Rxyy})$

(7) $\forall\mathsf{x}\forall\mathsf{y}(\exists\mathsf{z}(\mathsf{Rzxn} \wedge \mathsf{Ryzn}) \to (\mathsf{Gx} \to \mathsf{Gy}))$

(8) $\forall\mathsf{x}\forall\mathsf{y}\forall\mathsf{z}(((\mathsf{Fx} \wedge \mathsf{Fy}) \wedge \mathsf{Rzxy}) \to \mathsf{Gz})$

(9) $\forall\mathsf{x}((\mathsf{Gx} \wedge \neg\mathsf{Rxnn}) \to \exists\mathsf{y}\exists\mathsf{z}((\mathsf{Hy} \wedge \mathsf{Hz}) \wedge \mathsf{Rxyz}))$

(10) $\forall\mathsf{x}\exists\mathsf{y}((\mathsf{Hy} \wedge \mathsf{Lxy}) \wedge \exists\mathsf{w}\exists\mathsf{z}((\mathsf{Rwyn} \wedge \mathsf{Rzwn}) \wedge \mathsf{Hz}))$

(d*) Show that if the wff α doesn't contain the dummy name δ, then α is true on the valuation q if and only if it is also true on any expansion q_δ.

No solution available for this one.