Solutions to Problem Set 10

Problem 1. There are 4 different coins in a box. The probability of Heads when flipping the ith coin is $1/i$ for $1 \le i \le 4$. A coin is selected from the box randomly, and gets tossed until a Head appears.

(a) Write down a probability space for the experiment. Be sure to verify that the sum of the probabilities of the sample points is 1.

Solution. A possible sample space is

$$\{1, 2, 3, 4\} \times \{H, TH, TTH, TTTH, \dots\},$$

indicating the number of the coin was selected and the flip outcomes where until a head appeared. Define

$$\Pr\{(i, T^n H)\} ::= \frac{1}{4} \cdot \frac{1}{i} \cdot \left(\frac{i-1}{i}\right)^n.$$

Then

$$\begin{aligned}
\sum_{i \in \{1,2,3,4\}, n \ge 0} \Pr\{(i, T^n H)\} &= \sum_{i \in 1,2,3,4} \sum_{n \ge 0} \frac{1}{4} \cdot \frac{1}{i} \cdot \left(\frac{i-1}{i}\right)^n \\
&= \sum_{i \in 1,2,3,4} \frac{1}{4} \cdot \frac{1}{i} \sum_{n \ge 0} \left(\frac{i-1}{i}\right)^n \\
&= \sum_{i \in 1,2,3,4} \frac{1}{4} \cdot \frac{1}{i} \cdot \frac{1}{1 - \frac{i-1}{i}} \\
&= \sum_{i \in 1,2,3,4} \frac{1}{4} \cdot \frac{1}{i} \cdot \frac{1}{\frac{1}{i}} \\
&= 4 \cdot \frac{1}{4} = 1.
\end{aligned}$$

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(b) What is the probability that a Head is first seen in the 2nd toss?

Solution. For this part and the following part, let H_i be the event that "a Head was first seen on the ith toss". Let C_i be the event that "the ith coin was drawn from the box".

$$\begin{aligned}
\Pr\{H_2\} &= \Pr\{H_2 \mid C_1\}\Pr\{C_1\} + \Pr\{H_2 \mid C_2\}\Pr\{C_2\} + \Pr\{H_2 \mid C_3\}\Pr\{C_3\} + \Pr\{H_2 \mid C_4\}\Pr\{C_4\} \\
&= 0 \cdot \frac{1}{4} + \underbrace{\frac{1}{2}\cdot\frac{1}{2}}_{\Pr\{H_2 \mid C_2\}} \cdot \frac{1}{4} + \underbrace{\frac{2}{3}\cdot\frac{1}{3}}_{\Pr\{H_2 \mid C_3\}} \cdot \frac{1}{4} + \underbrace{\frac{3}{4}\cdot\frac{1}{4}}_{\Pr\{H_2 \mid C_4\}} \cdot \frac{1}{4} \\
&= \frac{95}{576}
\end{aligned}$$

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(c) Given that a Head is first seen in the 2nd toss, what is the probability that ith coin was selected from the box?

Solution.

$$\begin{aligned}
\Pr\{C_i \mid H_2\} &= \frac{\Pr\{C_i \cap H_2\}}{\Pr\{H_2\}} \\
&= \frac{\Pr\{H_2 \mid C_i\}\Pr\{C_i\}}{\Pr\{H_2\}} \\
&= \frac{\frac{i-1}{i}\cdot\frac{1}{i}\cdot\frac{1}{4}}{\frac{95}{576}} \\
&= \frac{144(i-1)}{95i^2}
\end{aligned}$$

■

(d) Check your answer to part (c) to be sure it satisfies:

$$\Pr\{C_1 \mid H_2\} + \Pr\{C_2 \mid H_2\} + \Pr\{C_3 \mid H_2\} + \Pr\{C_4 \mid H_2\} = 1.$$

Solution.

$$\begin{aligned}
&\Pr\{C_1 \mid H_2\} + \Pr\{C_2 \mid H_2\} + \Pr\{C_3 \mid H_2\} + \Pr\{C_4 \mid H_2\} \\
&= 0 + \frac{144}{95\cdot 4} + \frac{144\cdot 2}{95\cdot 9} + \frac{144\cdot 3}{95\cdot 16} \\
&= \frac{144\cdot 4\cdot 9}{95\cdot 16\cdot 9} + \frac{144\cdot 2\cdot 16}{95\cdot 16\cdot 9} + \frac{144\cdot 3\cdot 9}{95\cdot 16\cdot 9} \\
&= \frac{4\cdot 9}{95} + \frac{2\cdot 16}{95} + \frac{3\cdot 9}{95} \\
&= \frac{95}{95} = 1.
\end{aligned}$$

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Problem 2. We consider a variation of Monty Hall's game. The contestant still picks one of three doors, with a prize randomly placed behind one door and goats behind the other two. But now, instead of always opening a door to reveal a goat, Monty instructs Carol to *randomly* open one of the two doors that the contestant hasn't picked. This means she may reveal a goat, or she may reveal the prize. If she reveals the prize, then the entire game is *restarted*, that is, the prize is again randomly placed behind some door, the contestant again picks a door, and so on until Carol finally picks a door with a goat behind it. Then the contestant can choose to stick with his original choice of door or switch to the other unopened door. He wins if the prize is behind the door he finally chooses at this point; otherwise he loses.

To analyze this setup, define two events:

GP**:** The event that the contestant **g**uesses the door with the **p**rize behind it on his first guess.

OP**:** The event that the game is restarted at least once. Another way to describe this is as the event that the door Carol first **o**pens has a **p**rize behind it.

(a) What is $\Pr\{OP\}$? . . . $\Pr\left\{GP \mid \overline{OP}\right\}$?

Solution.

$$\begin{aligned}
\Pr\{GP\} &= \frac{1}{3}, \\
\Pr\{OP \mid GP\} &= 0, \\
\Pr\left\{OP \mid \overline{GP}\right\} &= \frac{1}{2}, \\
\Pr\{OP\} &= \Pr\{OP \mid GP\}\Pr\{GP\} + \Pr\left\{OP \mid \overline{GP}\right\}\Pr\left\{\overline{GP}\right\} \\
&= 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}, \\
\Pr\left\{GP \mid \overline{OP}\right\} &::= \frac{\Pr\left\{GP \cap \overline{OP}\right\}}{\Pr\left\{\overline{OP}\right\}} \\
&= \frac{\Pr\{GP\}}{\Pr\left\{\overline{OP}\right\}}, \qquad \text{(since } \overline{OP} \subset GP\text{)} \\
&= \frac{1/3}{(1-1/3)} = \frac{1}{2}.
\end{aligned}$$

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(b) What is the probability the game will end after *exactly* $n \geq 1$ guesses by the contestant?

Solution. The game ends after *exactly* n guesses by the contestant providing Carol opens the prize door $n-1$ times, and then does not open it after the nth guess. So

$$\Pr\{\text{end after exactly } n \text{ guesses}\} = \Pr\{OP\}^{n-1} \cdot \Pr\left\{\overline{OP}\right\} = (1/3)^{n-1} \cdot \frac{2}{3} = 2(1/3)^n.$$

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(c) Is it possible for the game never to end? What is the probability that the game will end?

Solution. The game could go on forever if the contestant keeps not guessing the prize and Carol does keep guessing it. However, the probability the game will end is 1. That's because by (b)

$$\Pr\{\text{game ends}\} = \sum_{n=1}^{\infty} 2(1/3)^n = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{2}{3}\cdot\frac{1}{1-1/3} = \frac{2}{3}\cdot\frac{3}{2} = 1.$$

■

When Carol finally picks the goat, the contestant has the choice of sticking or switching. The contestant figures that since this is a 6.042 problem with the Monty Hall rules changed to mislead him `:-)`, it's a good guess that the better strategy in the original game will not be better in this game. So he decides to use the *sticking strategy*: when Carol gives him a chance to switch, he will stick with the door he last guessed.

(d) Describe a simple probability space modelling this game, and verify that the sum of the probabilities of the sample points is 1. The space should have enough sample points to describe the event, W_n, that the game contestant wins after making *at least* n guesses, for every $n \geq 1$. Describe the outcomes in W_n.

Solution. There are various satisfactory ways to model the game. The simplest is to use part (b) and let the outcomes be the number of guesses in the game, along with an indication of whether the contestant won. That is, $\mathcal{S} ::= \mathbb{N} \times \{\texttt{win}, \texttt{lose}\}$ and,

$$\Pr\{(n, \texttt{win})\} = \Pr\{(n, \texttt{lose})\} ::= (1/3)^n.$$

which in part (c) we already proved sum to 1. In this case, $W_n = \{(j, \texttt{win}) \mid j \geq n\}$.

Assuming the contestant chose each door with equal probability, it would be possible to define a more detailed model indicating at each round exactly which door the goat was behind and which door the contestant chose. For example, the sample point could be all finite sequences of one or more pairs of the form (z, g) where $z \in \{1, 2, 3\}$ describes the number of the door with the prize on some round of the game, and $g \in \{1, 2, 3\}$ is the number of the door guessed by the contestant on that round. Since this sequence could occur only if Carol chose the prize door on every round except on the last one, we require $z \neq g$ in all pairs except possibly for the last one, where $z = g$ is allowed. A sequence describes a win for the contestant iff $z = g$ in the last pair.

Each pair is equally likely to occur on a given round, so the probability of each sequence of length n should be the $(1/9)^n$ times the probability that the sequence describes a game that ends. The probability of ending will $(1/2)^n$ for sequences ending with $z \neq g$, since that is the probability Carol that picks the prize door the first $n-1$ times and the nonprize door on the last round. For sequences ending with $z = g$, it will be $(1/2)^{n-1}$ since she is sure to pick the nonprize door on the final round. So the probability of a sequence of length n that ends with $z = g$ is *defined* to be $(1/9)^n(1/2)^{n-1}$ and is defined to be $(1/9)^n(1/2)^n$ for sequences ending with $z \neq g$. The event W_n would consist of all sequences of length $\geq n$. Using the fact that there are $6^{n-1} \cdot 3$ such sequences ending with $z = g$ and 6^n ending with $z \neq g$, a small calculation using the formula for the sum of a infinite geometric series implies the probabilities sum to 1. ■

(e) Let $w ::= \Pr\{W\}$, where W is the event that the contestant $w$ins with the sticking strategy. Express the following probabilities as simple closed forms in terms of w.

i) $\Pr\left\{W \mid \overline{GP} \cap \overline{OP}\right\}$

Solution. 0 ■

ii) $\Pr\left\{W \mid \overline{GP} \cap OP\right\}$

Solution. w ■

iii) $\Pr\{W \mid GP\}$

Solution. 1 ■

iv) $\Pr\{W\}$

Solution. 1/2.

One argument is based on the values of preceding probabilities:

$$\begin{aligned} w =& \Pr\{W \mid GP\}\Pr\{GP\} + \Pr\left\{W \mid \overline{GP} \cap OP\right\}\Pr\left\{\overline{GP} \cap OP\right\} + \Pr\left\{W \mid \overline{GP} \cap \overline{OP}\right\}\Pr\left\{\overline{GP} \cap \overline{OP}\right\} \quad \text{(La} \\ =& 1 \cdot \frac{1}{3} + w \cdot \Pr\left\{\overline{GP} \mid OP\right\}\Pr\{OP\} + 0 \\ =& \frac{1}{3} + w \cdot 1 \cdot \frac{1}{3} = \end{aligned}$$

So $w(1 - 1/3) = 1/3$ and hence $w = 1/2$.

Alternatively, we can observe that when the game ends exactly when Carol opens a door that does not have the prize. At this point, the contestant wins iff he last guessed the door with the prize. So

$$\Pr\{W\} = \Pr\left\{GP \mid \overline{OP}\right\},$$

and by part (a), $\Pr\{W\} = 1/2$. ■

(f) For any final outcome where the contestant wins with a "stick" strategy, he would lose if he had used a "switch" strategy, and vice versa. In the original Monty Hall game, we concluded immediately that the probability that he would win with a "switch" strategy was $1 - \Pr\{W\}$. Why isn't this conclusion quite as obvious for this new, restartable game? Is this conclusion still sound? Briefly explain.

Solution. Switching strategies turns wins to losses for games that *end*. So the probability of win with switch is $t - \Pr\{W\}$ where t is the probability the game will end. The original Monty Hall game terminates after one door-opening, so t was 1.

The extra complication here is that it is possible for the game to run forever. But this event has probability 0, so t is still 1, and the conclusion is still sound. ■

Problem 3. (a) Suppose that you are looking in your desk for a letter from some time ago. Your desk has eight drawers, and you assess the probability that it is in any particular drawer as 10% (so there is a 20% chance that it is not in the desk at all). Suppose now that you start searching systematically through your desk, one drawer at a time. In addition, suppose that you have not found the letter in the first i drawers, where $0 \leq i \leq 7$. Let p_i denote the probability that the letter will be found in the next drawer, and let q_i denote the probability that the letter will be found in some subsequent drawer (both p_i and q_i are conditional probabilities, since they are based on the assumption that the letter is not in the first i drawers).

Find formulas for p_i and q_i, and conclude that p_i is a strictly increasing function of i, and q_i is strictly decreasing.

Hint: Observe that if event A implies event B then $\Pr\{A \mid B\} = \Pr\{A\} / \Pr\{B\}$

Solution. First, we verify the hint. Saying that event A implies B is the same as saying that that $A \cap B = A$, so

$$\Pr\{A \mid B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}} = \frac{\Pr\{A\}}{\Pr\{B\}}.$$

For $1 \leq i \leq 8$, define the following events

$$\begin{aligned} E_i &::= \text{letter is in drawer } i, \\ N_i &::= \text{letter is not in drawers } 1, 2, \ldots i. \end{aligned}$$

We are given that

$$\Pr\{E_i\} = 0.1.$$

But N_i fails to occur iff the letter is in some drawer $j \leq i$, so

$$\Pr\{N_i\} = 1 - \sum_{1 \leq j \leq i} \Pr\{E_j\} = 1 - (0.1)i.$$

Hence,

$$\begin{aligned} p_i &::= \Pr\{E_{i+1} \mid N_i\} \\ &::= \frac{\Pr\{E_{i+1} \cap N_i\}}{\Pr\{N_i\}} \\ &= \frac{\Pr\{E_{i+1}\}}{\Pr\{N_i\}} && \text{(since } E_{i+1} \text{ implies } N_i\text{)} \\ &= \frac{0.1}{1 - (0.1)i} \\ &= \frac{1}{10 - i}, \end{aligned}$$

which clearly increases as i increases, since the denominator gets smaller while the numerator remains constant.

Now define for $0 \leq i \leq 7$,

$$R_i ::= \text{letter is in one of drawers } i+1, i+2, \ldots 8,$$

so

$$\Pr\{R_i\} = \sum_{i<j\leq 8} \Pr\{E_j\} = 0.1(8-i).$$

Hence,

$$\begin{aligned}
q_i ::= & \Pr\{R_i \mid N_i\} \\
= & \frac{\Pr\{R_i \cap N_i\}}{\Pr\{N_i\}} \\
= & \frac{\Pr\{R_i\}}{\Pr\{N_i\}} && \text{(since } R_i \text{ implies } N_i\text{)} \\
= & \frac{0.1(8-i)}{1-(0.1)i} \\
= & \frac{8-i}{10-i} \\
= & 1 - \frac{2}{10-i},
\end{aligned}$$

which decreases as i increases, since the denominator of the fraction shrinks, so the fraction becomes greater, so the negation of the fraction becomes smaller. ■

(b) The following data appeared in an article in the Wall Street Journal. For the ages 20, 30, 40, 50, and 60, the probability of a woman in the U.S. developing cancer in the next ten years is 0.5%, 1.2%, 3.2%, 6.4%, and 10.8%, respectively. At the same set of ages, the probability of a woman in the U.S. eventually developing cancer is 39.6%, 39.5%, 39.1%, 37.5%, and 34.2%, respectively. This seems strange, but use the previous part of the problem to give an explanation for these data.

Solution. We can use the results from a directly by figuring out what the p_i's and the q_i's are in this situation. In this case, we care about if a woman will develop cancer for the ages of 20, 30, 40, 50 and 60. Therefore, i in this case is such that $i \subseteq \{20, 30, 40, 50, 60\}$.

The p_i's are the probabilities that a woman will develop cancer in the next 10 years. (analogous to the letter being in the next envelope)

The q_i's are the probabilities of a woman in the U.S eventually developing cancer sometime in the future given that she does not have it now. (analogous to the letter being somewhere left in the drawer).

As seen from part a, the p_i's increase while the q_i's decrease. ■