Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

Solutions to Problem Set 2

Problem 1. Use induction to prove that the following equation holds for all $n \geq 2$.

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

Solution. *Proof.* The proof is by induction on n. Let $P(n)$ be the proposition that the above equation holds. In the base case, $P(2)$ is true because $(1 - \frac{1}{2}) = \frac{1}{2}$. In the inductive step, for $n \geq 2$ assume $P(n)$ to prove $P(n+1)$ as follows.

$$\begin{aligned}
\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n+1}\right) &= \frac{1}{n}\left(1 - \frac{1}{n+1}\right) \\
&= \frac{1}{n} \cdot \frac{n}{n+1} \\
&= \frac{1}{n+1}
\end{aligned}$$

The first step uses the induction hypothesis, and the second and third steps are simplifications. This shows that $P(n)$ implies $P(n+1)$, and the claim is proved by induction. □

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Problem 2. Suppose you take a piece of paper and draw a bunch of straight lines, no one exactly on top of another, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line are different colors. Regions that share only a boundary point may have the same color.

Solution. We assume that all lines are straight and completely cross the paper. Also, when we say that the regions are colored, we always mean they are colored using at most two colors, *e.g.*, red and green. We say that two regions are *adjacent* if they share a boundary line. Note that two regions cannot share two distinct boundary lines.

Let $P(n)$ be the predicate that the regions formed by any set of n distinct lines can be colored so that any two adjacent regions are colored with different colors. We prove that $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base case: The base case is $P(0)$. There is only one region, namely the whole page, and thus there are no adjacent regions. The region may be colored either red or green.

Inductive step: We must show that for all $n \in \mathbb{N}$ it is the case that $(P(n) \longrightarrow P(n+1))$; that is, if the regions formed by any n distinct lines can be two-colored so that adjacent regions have different colors, then the regions formed by any $n+1$ distinct lines can be two-colored so that adjacent regions have different colors.

1. Assume that the regions formed by any n distinct lines can be colored so that adjacent regions have different colors.
2. Given a paper containing $n+1$ distinct lines that completely cross the paper, choose any single line l and erase it.
3. By the inductive hypothesis, the regions formed by the n remaining lines can be colored so that adjacent regions are different colors.

   Call this coloring of the regions the original coloring and redraw line l.
4. Line l splits the paper into two sides. Select one of the sides and flip all the colors of the regions on that side (a region that was previously red should be green and vice versa). Leave the other side of line l exactly the way it was in the original coloring. We will prove that, using the new coloring, any adjacent regions have different colors.
5. Choose any two adjacent regions in the new coloring.

   Case 1: The regions share boundary line l. The two regions had the same color in the original coloring. The two regions are on opposite sides of l. Therefore, the color in exactly one of the two regions was flipped and they have different colors in the new coloring.

   Case 2: The regions do not share boundary line l. The two regions had different colors in the original coloring, and they are on the same side of l. Thus, the color of either none or both regions was flipped and the two regions have different colors in the new coloring.
6. The regions formed by any $n+1$ distinct lines can be colored so that adjacent regions have different colors.

QED (induction)

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Problem 3. The 6.042 course information sheet states:

> Tutorials will be devoted to solving interesting problems, with students working in teams of three or four.

Using strong induction, prove that if a recitation contains at least 6 students, then the class can be divided into teams, each consisting of either 3 or 4 students.

Solution. *Proof.* The proof is by strong induction. Let $P(n)$ be the proposition that a recitation with n students can be divided into teams of 3 or 4. In the base cases, $P(6)$ is true because there could be two teams of 3, $P(7)$ is true because there could be a team of 3 and a team of 4, and $P(8)$ is true because there could be two teams of 4. In the inductive step, for $n \geq 8$, assume $P(6), \ldots, P(n)$ to prove $P(n+1)$. If a recitation has $n+1$ students, then three students can form a team of 3, and the remaining $n-2$ students can be divided into teams of 3 or 4 by induction. This shows that $P(6), \ldots, P(n)$ imply $P(n+1)$, and the claim is proved by induction. □

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Problem 4. Let the function, g, be defined on the natural numbers recursively as follows: $g(0) = 0$, $g(1) = 1$, and $g(n) = 5g(n-1) - 6g(n-2)$, for $n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $g(n) = 3^n - 2^n$.

Solution. Let $P(n)$ be the predicate that $g(n) = 3^n - 2^n$. We prove $P(n)$, for all $n \in \mathbb{N}$, by strong induction on n.

Base Case: In order to simplify the strong induction, we use two base cases, in particular $n = 0$ and $n = 1$. By the definition of the sequence it is the case that $g(0) = 0$. For $n = 0$, it is the case that $3^n - 2^n = 3^0 - 2^0 = 1 - 1 = 0$; that is, it follows that $P(0)$ holds. Similarly, by the definition of the sequence it is the case that $g(1) = 1$. For $n = 1$, it is the case that $3^n - 2^n = 3^1 - 2^1 = 3 - 2 = 1$; that is, it follows that $P(1)$ holds.

Inductive Step: We must show that for all $n \in \mathbb{N}, n \geq 1$, it is the case that $P(n+1)$ holds, assuming $P(k)$ holds for $0 \leq k < n+1$.

1. Assume that for some $n \in \mathbb{N}, n \geq 1$, the elements of the sequence $g(x)$, for $0 \leq x \leq n$, can be expressed as $g(x) = 3^x - 2^x$.
2. By the definition of the sequence we have $g(n+1) = 5g(n) - 6g(n-1)$.
3. By our induction hypothesis, it is the case that $g(n) = 3^n - 2^n$ and $g(n-1) = 3^{n-1} - 2^{n-1}$
4. Thus, $g(n+1)$
   $= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1})$
   $= 5 * 3^n - 5 * 2^n - 2 * (3)3^{n-1} + 3 * (2)2^{n-1}$
   $= 5 * 3^n - 5 * 2^n - 2 * 3^n + 3 * 2^n$
   $= 3 * 3^n - 2 * 2^n$
   $= 3^{n+1} - 2^{n+1}$

   Working through the algebra, we conclude that $g(n+1) = 3^{n+1} - 2^{n+1}$, as needed for the inductive step.

5. QED (implication and UG)

QED (induction)

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Problem 5. Ackermann's function, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, satisfies the following equations:

$$f(m,n) = 2n, \qquad \text{if } m = 0 \text{ or } n \leq 1, \tag{1}$$
$$f(m,n) = f(m-1, f(m,n-1)), \qquad \text{if } m > 0 \text{ and } n > 1. \tag{2}$$

These equations actually determine Ackermann's function uniquely. That is,

Lemma. *If $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ also satisfies the same equations as f, namely,*

$$g(m,n) = 2n, \qquad \textit{if } m = 0 \textit{ or } n \leq 1, \tag{3}$$
$$g(m,n) = g(m-1, g(m,n-1)), \qquad \textit{if } m > 0 \textit{ and } n > 1, \tag{4}$$

then $f = g$.

(a) Prove this Lemma *by induction*. *Hint:* Try the induction hypothesis

$$P(m) ::= \forall n\ f(m,n) = g(m,n).$$

Solution. *Proof.* We must prove that

$$\forall m \, \forall n\ f(m,n) \quad = \quad g(m,n). \tag{5}$$

The proof is by induction on m. More precisely, the main induction hypothesis, $P(m)$, is

$$P(m) ::= \ \forall n\ f(m,n) = g(m,n).$$

Base case: $m = 0$. It follows directly from equation(s) (1), (3) that $f(m,n) = g(m,n)$ for all n, confirming that $P(0)$ is true.

Induction step: Suppose $P(m)$ holds for some fixed $m \geq 0$. We must prove that $P(m+1)$ also holds.

Namely, assuming $\forall n\ f(m,n) = g(m,n)$, we must prove

$$\forall n f(m+1,n) = g(m+1,n) \tag{6}$$

We prove (6) by another induction, this time on n, using the induction hypothesis,

$$Q_m(n) ::= \ f(m+1,n) = g(m+1,n).$$

Base case: $n = 0$. $Q_m(0)$ follows immediately from equation(s) (1), (3).

Induction step: Suppose $Q_m(n)$ holds for some $n \geq 0$. We must prove that $Q_m(n+1)$ also holds.

Let $k ::= f(m+1, n)$, so

$$f(m+1, n+1) = f(m, k) \tag{7}$$

by (2).

Then also

$$k = g(m+1, n), \tag{8}$$

because $Q_m(n)$ holds, and so

$$g(m+1, n+1) = g(m, k) \tag{9}$$

by (4).

But

$$f(m, k) = g(m, k), \tag{10}$$

since $P(m)$ holds. Now combining equations (7), (9), and (10), we obtain

$$f(m+1, n+1) = g(m+1, n+1).$$

That is, $Q_m(n+1)$ holds.

This completes the proof by induction on n that (6) holds.

So we have also completed the proof by induction on m that $\forall m\, P(m)$.

That is, we have proved (5), as required.

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(b) Now prove this Lemma *using the Least Number Principle*. *Hint:* Consider the least m such that $\exists n\ f(m, n) \neq g(m, n)$.

Solution. Suppose that $f \neq g$. Then there must be values $m_0, n_0 \in \mathbb{N}$ such that $f(m_0, n_0) \neq g(m_0, n_0)$. So the set

$$M ::= \{m \mid \exists n\, f(m, n) \neq g(m, n)\}$$

is nonempty, because it contains m_0. Therefore, it has a least element, m_1. But since $m_1 \in M$, it follows that the $N = \{n \mid f(m_1, n) \neq g(m_1, n)\}$ is also nonempty, and so must contain a least number, n_1.

Now, $m_1 > 0$ and $n_1 > 1$ because $f(m, n) = g(m, n) = 2n$ for $m = 0$ or $n \leq 1$ by equation (1).

Therefore $m_1 - 1 \notin M$ and $n_1 - 1 \notin N$ and we can assume the following:

$$f(m_1 - 1, k) = g(m_1 - 1, k) \tag{11}$$

and

$$f(m_1, l) = g(m_1, l) \tag{12}$$

$\forall k, l \in \mathbb{N}$, and $l < n_1$.

So letting $l = n_1 - 1$ in (12), we conclude that

$$f(m_1, n_1 - 1) = g(m_1, n_1 - 1).$$

So we can replace the first k in (11) by $f(m_1, n_1 - 1)$, and the second k by $g(m_1, n_1 - 1)$, and conclude that

$$f(m_1 - 1, f(m_1, n_1 - 1)) = g(m_1 - 1, g(m_1, n_1 - 1)). \tag{13}$$

But by (2) and (4), $f(m_1, n_1)$ is equal to the lefthand side of (13), and similarly, $g(m_1, n_1)$ is equal to the righthand side of (13). Hence, we have proved that

$$f(m_1, n_1) = g(m_1, n_1),$$

contradicting our choice of m_1, n_1 as arguments where f and g differ.

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Problem 6. An *n-player tournament* consists of some set of $n \geq 2$ *players*, and, for every two distinct players, a specification that one of the players beats the other. That is, player p beats player q iff q does not beat p, for all players $p \neq q$.

A sequence of distinct players $p_1, p_2, \ldots, p_k$, such that player p_i beats player p_{i+1} for $1 \leq i < k$ is called a *ranking* of these players. If also player p_k beats player p_1, the ranking is called a *k-cycle*.

(a) Prove by induction that in every tournament, either there is a "champion" player that beats every other player, or there is a 3-cycle.

Solution. The induction hypothesis $P(n)$, is "In an n-player tournament, either there is a player that beats every other player, or there is a 3-cycle."

Base case $n = 2$: By definition of a 2-player tournament, one of the players beats the only other player and therefore is the champion.

Inductive step: Assume that $P(n)$ is true in order to show that $P(n+1)$ is true.

Consider an $n+1$-player tournament. If there is a champion, we are done. So we may assume that every player is beaten by some other player. Now remove any player, say player p, and treat the remaining n players as an n-player tournament. There are two cases:

1) Every player in the n-player tournament is beaten by some other player in the tournament.

2) There is some champion in the n-player tournament who beats every other player in the n-player tournament.

In case (1), we have by induction hypothesis a 3-cycle of players in the n-player tournament. Since a 3-cycle in the n-player tournament will also be a 3-cycle in the full $n+1$-player tournament, we are OK in this case.

In case (2), there is some "champion", q, in the n-player tournament. Since everybody in the $(n+1)$-player tournament is beaten by somebody, q must be beaten by a player not in the n-player tournament, that is, p beats q. For the same reason, there is some player r that beats p. Since r is in the n-player tournament, r is beaten by the champion q. So we have found a 3-cycle p, q, r. ■

(b) A *consistent ranking* is a sequence $p_1, p_2, \ldots, p_n$ of all n players in the tournament such that p_i beats p_j iff $i < j$, for $1 \leq i, j \leq n$. Conclude that a tournament has no consistent ranking iff some subset of three of its players has no consistent ranking.

Solution. By definition of tournament, the restriction of the tournament to any subset of two or more of its players is also a (sub)tournament. Also any consistent ranking of a tournament remains consistent when restricted to a subset of players. So if some subset of three players has no consistent ranking, then the whole tournament can't have a consistent ranking either.

So we have only to prove the converse implication: if every three players have a consistent ranking, then the tournament as a whole has one. The proof is by induction on n.

Base case $n = 2$: Every two player tournament obviously has a consistent ranking.

Inductive step: Suppose $n \geq 2$ and there is a consistent ranking of any tournament of n players whose three-player subsets have consistent rankings. Now consider any $n + 1$ player tournament whose three-player subsets have consistent rankings.

This tournament cannot have a 3-cycle, because there is no consistent ranking for the players in the 3-cycle. Hence, by the previous part, the tournament has a champion. The n-player tournament obtained by ignoring the champion also has no 3-cycles, so by induction it has a consistent ranking. Then this ranking of n-nonchampions followed by the champion at then end is a consistent ranking of the whole tournament of $n + 1$ players. ■