Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 33: More QL proofs

(a) *As a warm-up exercise, consider which of the following* QL *arguments ought to be valid (assume the wffs are interpreted). Give proofs warranting the valid inferences.*

(1) $\exists x\, Sxxx \;\therefore\; \exists x \exists y \exists z\, Sxyz$

Remember, the role of variables is to link quantifier-prefixes to slots in predicates. When we have one quantifier in the scope of another, we use different variables to make the linkages clear. But the use of different variables does *not* indicate that they need to be instantiated in different ways.

Suppose for a moment that S expresses the numerical relation which holds between three numbers when the first number is the sum of the second and third. Then $\exists x\, Sxxx$ is true, because zero is the sum of zero and zero. But then there is indeed a number such that there is a number such that there is a number such that the first is the sum of the second and the third, i.e. $\exists x \exists y \exists z\, Sxyz$ is true. Just take zero as the instance each time! Generalizing, we see that argument (1) is going to be valid, however we interpret S.

Formally, we can set up an argument using ($\exists$E) like this:

(1)	$\exists x\, Sxxx$	(Prem)
(2)	$\quad Saaa$	(Supp)
(3)	$\quad \exists z Saaz$	($\exists$I 2)
(4)	$\quad \exists y \exists z Sayz$	($\exists$I 3)
(5)	$\quad \exists x \exists y \exists z Sxyz$	($\exists$I 4)
(6)	$\exists x \exists y \exists z Sxyz$	($\exists$E 2, 3–5)

Note again the key point that, in applying ($\exists$I) by quantifying on a term, we do *not* have to quantify on every occurrence of that term.

(2) $\exists x \forall y \forall y\, Sxyy \;\therefore\; \exists x\, Sxxx$

The premiss here isn't even a wff in our syntax. So this isn't a *well-formed* argument and so it can't be a *valid* argument.

(3) $\forall x \exists y \forall z\, Sxyz \;\therefore\; \exists x \forall y \exists z\, Sxzy$

When can we swap the order of $\forall$ and $\exists$ quantifiers? We can't validly argue from $\forall x \exists y\; xRy$ to $\exists y \forall x\; xRy$. But we *can* argue in the opposite direction from $\exists y \forall x\; xRy$ to $\forall x \exists y\; xRy$ (if there is someone who is loved by everyone, then everyone loves someone [that universal beloved!]).

Formally, we *can* validly argue from $\exists y \forall z Sayz$ to $\forall z \exists y Sayz$, and that is trivially equivalent to $\forall y \exists z Sazy$ by relabelling of variables. Generalizing on a, then, we can validity argue from $\forall x \exists y \forall z Sxyz$ to $\forall x \forall y \exists y Sxzy$. Then use the general point that a wff the form $\forall x \varphi$ entails the corresponding $\exists x \varphi$ to make the further step to $\exists x \forall y \exists z\, Sxzy$. So (3) is valid.

Let's outline a proof:

$\forall x \exists y \forall z\, Sxyz$	(Prem)
$\exists y \forall z\, Sayz$	($\forall$E 1)
$\quad \forall z\, Sabz$	(Supp)
$\quad \vdots$	
$\quad \forall y \exists z\, Sazy$	
$\forall y \exists z\, Sazy$	($\exists$E)
$\exists x \forall y \exists z\, Sxzy$	($\exists$I)

What's happening here? First we instantiate the universal quantification with a dummy name at the beginning of the premiss (with a view to quantifying on that dummy name later). That gives us an existential quantification on the second line. How are we going to use that. By instantiating it in a supposition, of course, working towards a (∃E) inference as shown. Filling in the rest of the proof is then easy:

(1)	$\forall x\exists y\forall z\,Sxyz$	(Prem)
(2)	$\exists y\forall z\,Sayz$	(∀E 1)
(3)	$\quad \forall z\,Sabz$	(Supp)
(4)	$\quad Sabc$	(∀E 3)
(5)	$\quad \exists z\,Sazc$	(∃I 4)
(6)	$\quad \forall y\exists z\,Sazy$	(∀I 5)
(7)	$\forall y\exists z\,Sazy$	(∃E 2, 3–6)
(8)	$\exists x\forall y\exists z\,Sxzy$	(∃I 7)

(4*) $\neg\exists x\forall y\forall z\,Sxyz \;\therefore\; \forall x\exists z\exists y\neg Sxyz$

We know that $\neg\forall\xi$ is equivalent $\exists\xi\neg$, and $\neg\exists\xi$ is equivalent to $\forall\xi\neg$. Push a negation past a quantifier (in either direction) and the quantifer flips. So, in three steps, $\neg\exists x\forall y\forall z\,Sxyz$ should be equivalent $\forall x\exists y\exists z\neg Sxyz$. But we also know that the order of adjacent quantifiers of the same flavour doesn't matter: $\exists y\exists z$ comes to the same as $\exists z\exists y$. So (4*) is valid.

How is a proof from the basic rules going to go? Two ideas:

1. We've seen before that working from a wff of the form $\neg\exists x\alpha(x)$ it can be a good plan to (i) suppose $\alpha(a)$, (ii) infer $\exists x\alpha(x)$ to get a contradiction, and (iii) conclude the simpler $\neg\alpha(a)$.
2. At the end of the proof we'll presumably want to infer the universally quantified conclusion from an instance.

Which suggests the following shape to the proof:

$\neg\exists x\forall y\forall z\,Sxyz$	(Prem)
$\quad \forall y\forall z\,Sayz$	(Supp)
$\quad \exists x\forall y\forall z\,Sxyz$	(∃I 2)
$\quad \bot$	(Abs 3, 1)
$\neg\forall y\forall z\,Sayz$	(RAA 2–4)
$\vdots$	
$\exists z\exists y\neg Sayz$	
$\forall x\exists z\exists y\neg Sxyz$	(∀I)

Which leaves us with an easier proof in the middle. How are we going to fill the gap? Again as we've seen before in similar cases, we often prove an existential quantification by assuming its negation and aiming for a contradiction. And how do we deal with the assumed negated existential – we use the same trick that we've just used: assume an instance and derive the negation of *that*. So we'll to now insert a expect a proof segment like this ....

$\quad \neg\exists z\exists y\neg Sayz$	(Supp)
$\quad\quad \exists y\neg Sayc$	(Supp)
$\quad\quad \exists z\exists y\neg Sayz$	(∃I 2)
$\quad\quad \bot$	(Abs)
$\quad \neg\exists y\neg Sayc$	
$\quad \vdots$	
$\quad \bot$	(Abs)
$\neg\neg\exists z\exists y\neg Sayz$	(RAA 2–4)

Ahah! Another negated existential quantification to work from. Well, let's use the same trick a how to finish the proof (it's pretty obvious what to do when we notice that we've as yet made no use of our derivation of $\neg\forall y\forall z\,Sayz$):

(1)	$\neg\exists x\forall y\forall z\, Sxyz$	(Prem)
(2)	$\quad \forall y\forall z\, Sayz$	(Supp)
(3)	$\quad \exists x\forall y\forall z\, Sxyz$	($\exists$I 2)
(4)	$\quad \bot$	(Abs 3, 1)
(5)	$\neg\forall y\forall z\, Sayz$	(RAA 2–4)
(6)	$\quad \neg\exists z\exists y\neg Sayz$	(Supp)
(7)	$\quad\quad \exists y\neg Sayc$	(Supp)
(8)	$\quad\quad \exists z\exists y\neg Sayz$	($\exists$I 7)
(9)	$\quad\quad \bot$	(Abs 8, 6)
(10)	$\quad \neg\exists y\neg Sayc$	(RAA 7–9)
(11)	$\quad\quad \neg Sabc$	(Supp)
(12)	$\quad\quad \exists y\neg Sayc$	($\exists$I 11)
(13)	$\quad\quad \bot$	(Abs 12, 10)
(14)	$\quad \neg\neg Sabc$	(RAA 11–13)
(15)	$\quad Sabc$	(DN 14)
(16)	$\quad \forall zSabz$	($\forall$I 15)
(17)	$\quad \forall y\forall zSayz$	($\forall$I 16)
(18)	$\quad \bot$	(Abs 17, 5)
(19)	$\neg\neg\exists z\exists y\neg Sayz$	(RAA 6–18)
(20)	$\exists z\exists y\neg Sayz$	(DN 19)
(21)	$\forall x\exists z\exists y\neg Sxyz$	($\forall$I 20)

Which is a bit painful for such an intuitively simple entailment!

We could have speeded things up if we'd allowed ourselves to use the derived rules ($\neg\exists$) and ($\neg\forall$)? Then we could have argued like this:

(1)	$\neg\exists x\forall y\forall z\, Sxyz$	(Prem)
(2)	$\forall x\neg\forall y\forall z\, Sxyz$	($\neg\exists$ 1))
(3)	$\neg\forall y\forall z\, Sayz$	($\forall$E 3))
(4)	$\quad \neg\exists z\exists y\neg Sayz$	(Supp – as before, beginning a reductio argument)
(5)	$\quad \forall z\neg\exists y\neg Sxyz$	($\neg\exists$ 1))
(6)	$\quad \neg\exists y\neg\, Sayc$	($\forall$E 5)
(7)	$\quad \forall y\neg\neg Sayc$	($\neg\forall$I 8 – careful, don't let negations go astray!)
(8)	$\quad \neg\neg Sabc$	($\forall$E 7)
(9)	$\quad Sabc$	(DN 14)
(10)	$\quad \forall zSabz$	($\forall$I 9)
(11)	$\quad \forall y\forall zSayz$	($\forall$I 10)
(12)	$\quad \bot$	(Abs 11, 3)
(13)	$\neg\neg\exists z\exists y\neg Sayz$	(RAA 4–12)
(14)	$\exists z\exists y\neg Sayz$	(DN 13)
(15)	$\forall x\exists z\exists y\neg Sxyz$	($\forall$I 14)

(5*) $\neg\exists x(Fx \wedge \exists y(Gy \wedge Lxy)) \;\therefore\; \forall x\forall y(Fx \rightarrow (Gy \rightarrow \neg Lxy))$

Both wffs are possible renditions of *No F is L to any G*, so we should be able to infer one from the other!

How is the conclusion going to be arrived at? Presumably by final steps which look like this:

$(\mathsf{Fa} \to (\mathsf{Gb} \to \neg\mathsf{Lab}))$	
$\forall\mathsf{y}(\mathsf{Fa} \to (\mathsf{Gy} \to \neg\mathsf{Lay}))$	($\forall$I)
$\forall\mathsf{x}\forall\mathsf{y}(\mathsf{Fx} \to (\mathsf{Gy} \to \neg\mathsf{Lxy}))$	($\forall$I)

And how are we going to prove that first conditional? The obvious thing to try is assuming Fa and aiming for $(\mathsf{Gb} \to \neg\mathsf{Lab})$, which we can hope to prove by assuming Gb and aiming for $\neg\mathsf{Lab}$, we can prove by assuming Lab and aiming for contradiction. So let's do all that!

(1)	$\neg\exists\mathsf{x}(\mathsf{Fx} \land \exists\mathsf{y}(\mathsf{Gy} \land \mathsf{Lxy}))$	(Prem)
(2)	$\quad \mathsf{Fa}$	(Supp)
(3)	$\quad\quad \mathsf{Gb}$	(Supp)
(4)	$\quad\quad\quad \mathsf{Lab}$	(Supp)
(5)	$\quad\quad\quad (\mathsf{Gb} \land \mathsf{Lab})$	($\land$I 3, 4)
(6)	$\quad\quad\quad \exists\mathsf{y}(\mathsf{Gy} \land \mathsf{Lay})$	($\exists$I 5)
(7)	$\quad\quad\quad (\mathsf{Fa} \land \exists\mathsf{y}(\mathsf{Gy} \land \mathsf{Lay}))$	($\land$I 2, 6)
(8)	$\quad\quad\quad \exists\mathsf{x}(\mathsf{Fx} \land \exists\mathsf{y}(\mathsf{Gy} \land \mathsf{Lxy}))$	($\exists$I 7)
(9)	$\quad\quad\quad \bot$	(Abs 8, 1)
(10)	$\quad\quad \neg\mathsf{Lab}$	(RAA 4–9)
(11)	$\quad (\mathsf{Gb} \to \neg\mathsf{Lab})$	(CP 3–10)
(12)	$(\mathsf{Fa} \to (\mathsf{Gb} \to \neg\mathsf{Lab}))$	(CP 2–11)
(13)	$\forall\mathsf{y}(\mathsf{Fa} \to (\mathsf{Gy} \to \neg\mathsf{Lay}))$	($\forall$I 12)
(14)	$\forall\mathsf{x}\forall\mathsf{y}(\mathsf{Fx} \to (\mathsf{Gy} \to \neg\mathsf{Lxy}))$	($\forall$I 13)

(b) *Render the following inferences into suitable* QL *languages and provide derivations of the conclusions from the premisses in each case:*

(1) *Some people are boastful. No one likes anyone boastful. Therefore some people aren't liked by anyone.*

Take the domain to be the relevant people. Then the first premiss can be rendered simply as

$$\exists\mathsf{xBx}$$

But we have [equivalent!] equally natural options for the second premiss (as with translating other 'no' propositions), including

$$\forall\mathsf{x}\forall\mathsf{y}(\mathsf{By} \to \neg\mathsf{Lxy}) \text{ and } \neg\exists\mathsf{x}\exists\mathsf{y}(\mathsf{By} \land \mathsf{Lxy}).$$

There are two equally naturally options [equivalent!] for the conclusion too (as is often the way when translating 'anyone' propositions):

$$\exists\mathsf{x}\forall\mathsf{y}\neg\mathsf{Lyx} \text{ and } \exists\mathsf{x}\neg\exists\mathsf{yLyx}$$

Rendering the argument with the *first* options for the second premiss and the conclusion, we are therefore looking for a derivation of

$$\exists\mathsf{xBx},\ \forall\mathsf{x}\forall\mathsf{y}(\mathsf{By} \to \neg\mathsf{Lxy}) \ \therefore\ \exists\mathsf{x}\forall\mathsf{y}\neg\mathsf{Lyx}$$

Evidently, to make use of the first premiss, we will want to suppose an instance of it and aim for an argument by ($\exists$E), giving a proof of this shape:

$\exists\mathsf{xBx}$	(Prem)
$\forall\mathsf{x}\forall\mathsf{y}(\mathsf{By} \to \neg\mathsf{Lxy})$	(Prem)
$\quad \mathsf{Ba}$	(Supp)
$\quad \vdots$	
$\quad \exists\mathsf{x}\forall\mathsf{y}\neg\mathsf{Lyx}$	
$\exists\mathsf{x}\forall\mathsf{y}\neg\mathsf{Lyx}$	($\exists$E)

We now need to proceed by instantiating the universal quantifiers in the second premiss. We will plainly want eventually to instantiate y with the dummy name a, so we can set up a modus ponens inference with Ba. But we first have to instantiate the initial quantifier with x. What with? *Not* a again (think why not!). So we use a different name, giving us

$$
\begin{array}{ll}
\mid\ \exists x Bx & \text{(Prem)} \\
\mid\ \underline{\forall x \forall y(By \to \neg Lxy)} & \text{(Prem)} \\
\mid\ \quad \mid\ \underline{Ba} & \text{(Supp)} \\
\mid\ \quad \mid\ \forall y(By \to \neg Lby) & (\forall\text{E } 2) \\
\mid\ \quad \mid\ (Ba \to \neg Lba) & (\forall\text{E } 2) \\
\mid\ \quad \mid\ \vdots & \\
\mid\ \quad \mid\ \exists x \forall y \neg Lyx & \\
\mid\ \exists x \forall y \neg Lyx & (\exists\text{E})
\end{array}
$$

Does it need saying again at this stage – in applying ($\forall$) to a quantified wff, we can only instantiate the *initial* quantifier. If we want to instantiate a doubly quantified wff as in the second premiss, we have to do it strictly one quantifier at a time, and from the outside in.

Now, however, it is easy to finish the proof. Do the modus ponens inference that we have set up, and then restore the quantifiers. (NB, we *can* apply ($\forall$I) at line (7) because b doesn't appear in any live assumption! We *couldn't* apply ($\forall$I) again at line (8) because a does appear in the undischarged supposition (3).)

$$
\begin{array}{lll}
(1) & \mid\ \exists x Bx & \text{(Prem)} \\
(2) & \mid\ \underline{\forall x \forall y(By \to \neg Lxy)} & \text{(Prem)} \\
(3) & \mid\ \quad \mid\ \underline{Ba} & \text{(Supp)} \\
(4) & \mid\ \quad \mid\ \forall y(By \to \neg Lby) & (\forall\text{E } 2) \\
(5) & \mid\ \quad \mid\ (Ba \to \neg Lba) & (\forall\text{E } 2) \\
(6) & \mid\ \quad \mid\ \neg Lba & (\text{MP } 3, 5) \\
(7) & \mid\ \quad \mid\ \forall y \neg Lya & (\forall\text{I } 6) \\
(8) & \mid\ \quad \mid\ \exists x \forall y \neg Lyx & (\exists\text{I } 7) \\
(9) & \mid\ \exists x \forall y \neg Lyx & (\exists\text{E } 1, 2\text{–}8)
\end{array}
$$

What would have happened if we'd used the alternative translations of the second premiss and the conclusion? This time we will look for a derivation shaped like this:

$$
\begin{array}{ll}
\mid\ \exists x Bx & \text{(Prem)} \\
\mid\ \underline{\neg \exists x \exists y(By \wedge Lxy)} & \text{(Prem)} \\
\mid\ \quad \mid\ \underline{Ba} & \text{(Supp)} \\
\mid\ \quad \mid\ \vdots & \\
\mid\ \quad \mid\ \exists x \neg \exists y Lyx & \\
\mid\ \exists x \neg \exists y Lyx & (\exists\text{E})
\end{array}
$$

And now, a good guess is that the wff at the end of the subproof is going to be derived by existential generalization from the wff $\neg\exists\mathsf{yLya}$ (why a?). Since we are now aiming for that negative proposition, that suggests using a reductio argument:

$\mid\ \exists x Bx$	(Prem)
$\mid\ \neg\exists x\exists y(By \wedge Lxy)$	(Prem)
$\mid\ \mid\ Ba$	(Supp)
$\mid\ \mid\ \mid\ \exists y Lya$	(Supp)
$\mid\ \mid\ \mid\ \vdots$	
$\mid\ \mid\ \mid\ \bot$	
$\mid\ \mid\ \neg\exists y Lya$	(RAA)
$\mid\ \mid\ \exists x\neg\exists y Lyx$	
$\mid\ \exists x\neg\exists y Lyx$	($\exists$E)

We'll now need to instantiate the new existential supposition to set up an argument by $\exists$-elmination, and then we can quickly complete the derivation.

(1)	$\mid\ \exists x Bx$	(Prem)
(2)	$\mid\ \neg\exists x\exists y(By \wedge Lxy)$	(Prem)
(3)	$\mid\ \mid\ Ba$	(Supp)
(4)	$\mid\ \mid\ \mid\ \exists y Lya$	(Supp)
(5)	$\mid\ \mid\ \mid\ \mid\ Lba$	(Supp)
(6)	$\mid\ \mid\ \mid\ \mid\ (Ba \wedge Lba)$	($\wedge$ 3, 5)
(7)	$\mid\ \mid\ \mid\ \mid\ \exists y(By \wedge Lby)$	($\exists$I 6)
(8)	$\mid\ \mid\ \mid\ \mid\ \exists x\exists y(By \wedge Lxy)$	($\exists$I 7)
(9)	$\mid\ \mid\ \mid\ \mid\ \bot$	(Abs 8, 1)
(10)	$\mid\ \mid\ \mid\ \bot$	($\exists$E 4, 5–9)
(11)	$\mid\ \mid\ \neg\exists y Lya$	(RAA 4–10)
(12)	$\mid\ \mid\ \exists x\neg\exists y Lyx$	($\exists$I 11)
(13)	$\mid\ \exists x\neg\exists y Lyx$	($\exists$E 1, 3–12)

(2) *There's someone such that if* they *admire some philosopher, then I'm a Dutchman. So if* everyone *admires some philosopher, then I'm a Dutchman.*

This is a sneaky example. You might be tempted to translate the argument along the following lines, with one quantifier embedded inside the other:

$$\exists x(\exists y(Py \wedge Axy) \to D)\ \therefore\ (\forall x\exists y(Py \wedge Axy) \to D)$$

But in fact, when you look at it, you see that the internal structure of the predicate 'admires some philosopher' does no work. So instead of $\exists y(Py \wedge Axy)$ we only need a simple expression like Fx, giving us

$$\exists x(Fx \to D)\ \therefore\ (\forall x Fx \to D)$$

To make use of the existential premiss we need to suppose an instance of it; and then, since we are aiming for a conditional conclusion, we suppose its antecedent . . . and everything very easily falls into place:

(1)	$\mid\ \exists x(Fx \to D)$	(Prem)
(2)	$\mid\ \mid\ (Fa \to D)$	(Supp)
(3)	$\mid\ \mid\ \mid\ \forall x Fx$	(Supp)
(4)	$\mid\ \mid\ \mid\ Fa$	($\forall$E 3)
(5)	$\mid\ \mid\ \mid\ D$	(MP 4, 2)
(6)	$\mid\ \mid\ (\forall x Fx \to D)$	(CP 3–6)
(7)	$\mid\ (\forall x Fx \to D)$	($\exists$E 1, 2–7)

(3) *Some good philosophers admire Frank; all wise people admire any good philosopher; Frank is wise; hence there is someone who both admires and is admired by Frank.*

Translation: $\exists \mathsf{x}(\mathsf{Gx} \land \mathsf{Axf}),\ \forall \mathsf{x}(\mathsf{Wx} \to \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Axy})),\ \mathsf{Wf}\ \therefore\ \exists \mathsf{x}(\mathsf{Axf} \land \mathsf{Afx})$

Informally, the argument goes, take someone (arbitrary Alex) who is a good philosopher admiring Frank. Since Frank is wise and admires all good philosophers, Frank admires Alex. So there is indeed someone (Alex) who admires and is admired by Frank. So we just need to regiment this simple line of reasoning:

$$\begin{array}{rlll}
(1) & \mid & \exists \mathsf{x}(\mathsf{Gx} \land \mathsf{Axf}) & \text{(Prem)} \\
(2) & \mid & \forall \mathsf{x}(\mathsf{Wx} \to \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Axy})) & \text{(Prem)} \\
(3) & \mid & \underline{\mathsf{Wf}} & \text{(Prem)} \\
(4) & \mid & \quad \mid \underline{(\mathsf{Ga} \land \mathsf{Aaf})} & \text{(Supp)} \\
(5) & \mid & \quad \mid \mathsf{Ga} & (\land\text{E } 4) \\
(6) & \mid & \quad \mid \mathsf{Aaf} & (\land\text{E } 4) \\
(7) & \mid & \quad \mid (\mathsf{Wf} \to \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Afy})) & (\forall\text{E } 2) \\
(8) & \mid & \quad \mid \forall \mathsf{y}(\mathsf{Gy} \to \mathsf{Afy}) & (\text{MP } 3, 7) \\
(9) & \mid & \quad \mid (\mathsf{Ga} \to \mathsf{Afa}) & (\forall\text{E } 8) \\
(10) & \mid & \quad \mid \mathsf{Afa} & (\text{MP } 5, 9) \\
(11) & \mid & \quad \mid (\mathsf{Aaf} \land \mathsf{Afa}) & (\land\text{I } 6, 10) \\
(12) & \mid & \quad \mid \exists \mathsf{x}(\mathsf{Axf} \land \mathsf{Afx}) & (\exists\text{I } 12) \\
(13) & \mid & \exists \mathsf{x}(\mathsf{Axf} \land \mathsf{Afx}) & (\exists\text{E } 1, 3\text{–}12)
\end{array}$$

(4) *Everyone loves themself if there's someone who loves them or whom they love. There's someone who is loved. Therefore someone loves themself.*

The first premiss says that (everyone x is such that)(there's someone y such that)[if y loves x or x loves y, then x loves x]. For the second premiss, we note that to be loved is to be loved by someone. So we can render the argument like this:

$\forall \mathsf{x}(\exists \mathsf{y}(\mathsf{Lyx} \lor \mathsf{Lxy}) \to \mathsf{Lxx}),\ \exists \mathsf{x} \exists \mathsf{y} \mathsf{Lyx}\ \therefore\ \exists \mathsf{x} \mathsf{Lxx}$

To use the second premiss we'll need to instantiate it (setting up a ∃E inference); but that still leaves an existential quantification, and it looks as if we'll have to instantiate that too (setting up a ∃E inference nested inside the outer one). So that suggests our derivation will have the following overall shape:

$$\begin{array}{lll}
\mid & \forall \mathsf{x}(\exists \mathsf{y}(\mathsf{Lyx} \lor \mathsf{Lxy}) \to \mathsf{Lxx}) & \text{(Prem)} \\
\mid & \underline{\exists \mathsf{x} \exists \mathsf{y} \mathsf{Lyx}} & \text{(Prem)} \\
\mid & \quad \mid \underline{\exists \mathsf{y} \mathsf{Lya}} & \text{(Supp)} \\
\mid & \quad \mid \quad \mid \underline{\mathsf{Lba}} & \text{(Supp)} \\
\mid & \quad \mid \quad \mid \vdots & \\
\mid & \quad \mid \quad \mid \exists \mathsf{x} \mathsf{Lxx} & \text{(the conclusion we want)} \\
\mid & \quad \mid \exists \mathsf{x} \mathsf{Lxx} & (\exists\text{E using the first supposition and inner subproof}) \\
\mid & \exists \mathsf{x} \mathsf{Lxx} & (\exists\text{E using the second premiss and outer subproof})
\end{array}$$

Pause to make sure you understand this strategy! Then filling in the dots is pretty easy. We'll obviously need to instantiate the initial premiss – with either a or b. Let's try the first.

(1)	$\forall x(\exists y(Lyx \lor Lxy) \to Lxx)$	(Prem)
(2)	$\exists x \exists y Lyx$	(Prem)
(3)	$\quad \exists y Lya$	(Supp)
(4)	$\quad\quad Lba$	(Supp)
(5)	$\quad\quad (\exists y(Lya \lor Lay) \to Laa)$	(∀E)
(6)	$\quad\quad \vdots$	
(7)	$\quad\quad \exists x Lxx$	
(8)	$\quad \exists x Lxx$	(∃E)
(9)	$\exists x Lxx$	(∃E)

To proceed, we need to prove the antecedent of that newly derived conditional. How can we get it? Like this – and then the proof finishes itself!

(1)	$\forall x(\exists y(Lyx \lor Lxy) \to Lxx)$	(Prem)
(2)	$\exists x \exists y Lyx$	(Prem)
(3)	$\quad \exists y Lya$	(Supp)
(4)	$\quad\quad Lba$	(Supp)
(5)	$\quad\quad (\exists y(Lya \lor Lay) \to Laa)$	(∀E 1)
(6)	$\quad\quad (Lba \lor Lab)$	(∨I 4)
(7)	$\quad\quad \exists y(Lya \lor Lay)$	(∃I 5)
(8)	$\quad\quad Laa$	(MP 7, 5)
(9)	$\quad\quad \exists x Lxx$	(∃I 8)
(10)	$\quad \exists x Lxx$	(∃E 3, 4–9)
(11)	$\exists x Lxx$	(∃E 2, 3–10)

(5*) *Only rational people with good judgement are logicians. Those who take some creationist myth literally lack good judgement. So logicians do not take any creationist myth literally.*

The premisses and the conclusion are all universal claims. *Only As are Bs* gets rendered as *All Bs are As* (why?). With Txy for *x takes y literally*, we can render the argument like this:

Translation: $\forall x(Lx \to (Rx \land Gx)),\ \forall x(\exists y(My \land Txy) \to \neg Gx))\ \therefore\ \forall x(Lx \to \forall y(My \to \neg Txy))$

Here's an informal argument: Suppose Alex is a logician. By the first premiss, they have good judgement. But by the second premiss, if Alex takes a creationist myth literally, they lack good judgement. So given Alex is a logician, they don't take any creationist myth literally. But Alex was arbitrarily chosen so we can generalize.

Putting that formally, our derivation is going to start by instantiating the two universal premisses. with the same dummy name (think Alex!). So our proof will start:

$\forall x(Lx \to (Rx \land Gx))$	(Prem)
$\forall x(\exists y(My \land Txy) \to \neg Gx))$	(Prem)
$(La \to (Ra \land Ga))$	(∀E 1)
$(\exists y(My \land Tay) \to \neg Ga))$	(∀E 2)

And our proof will presumably end with an instance of the quantification in the conclusion which we can then generalize on:

$(La \to \forall y(My \to \neg Tay))$	
$\forall x(Lx \to \forall y(My \to \neg Txy))$	(∀I)

And how are we going to get that penultimate conditional? By a conditional proof, of course:

$$
\begin{array}{lll}
 & \vert\ \forall x(Lx \to (Rx \land Gx)) & \text{(Prem)} \\
 & \vert\ \underline{\forall x(\exists y(My \land Txy) \to \neg Gx))} & \text{(Prem)} \\
 & \vert\ (La \to (Ra \land Ga)) & (\forall\text{E } 1) \\
 & \vert\ (\exists y(My \land Tay) \to \neg Ga)) & (\forall\text{E } 2) \\
 & \vert\ \quad \vert\ \underline{La} & \text{(Supp)} \\
 & \vert\ \quad \vert\ \vdots & \\
 & \vert\ \quad \vert\ \forall y(My \to \neg Tay) & \\
 & \vert\ (La \to \forall y(My \to \neg Tay)) & \text{(CP)} \\
 & \vert\ \forall x(Lx \to \forall y(My \to \neg Txy)) & (\forall\text{I})
\end{array}
$$

How are we going to derive $\forall y(My \to \neg Tay)$? Presumably by generalizing on an instance, which is a conditional to be proved by another conditional proof:

$$
\begin{array}{lll}
(1) & \vert\ \forall x(Lx \to (Rx \land Gx)) & \text{(Prem)} \\
(2) & \vert\ \underline{\forall x(\exists y(My \land Txy) \to \neg Gx))} & \text{(Prem)} \\
(3) & \vert\ (La \to (Ra \land Ga)) & (\forall\text{E } 1) \\
(4) & \vert\ (\exists y(My \land Tay) \to \neg Ga)) & (\forall\text{E } 2) \\
(5) & \vert\ \quad \vert\ \underline{La} & \text{(Supp)} \\
(6) & \vert\ \quad \vert\ \vdots & \\
(7) & \vert\ \quad \vert\ \quad \vert\ \underline{Mb} & \text{(Supp)} \\
(8) & \vert\ \quad \vert\ \quad \vert\ \vdots & \\
(9) & \vert\ \quad \vert\ \quad \vert\ \neg Tab & \\
(10) & \vert\ \quad \vert\ (Mb \to \neg Tab) & \text{(CP)} \\
(11) & \vert\ \quad \vert\ \forall y(My \to \neg Tay) & (\forall\text{I}) \\
(12) & \vert\ (La \to \forall y(My \to \neg Tay)) & \text{(CP)} \\
(13) & \vert\ \forall x(Lx \to \forall y(My \to \neg Txy)) & (\forall\text{I})
\end{array}
$$

We use a new dummy name b so we can generalize on it. So can we fill in the dots? Given the likely way of showing $\neg Tab$ is going to be assuming Tab and aiming for absurdity, the rest of the proof almost writes itself!

$$
\begin{array}{lll}
(1) & \vert\ \forall x(Lx \to (Rx \land Gx)) & \text{(Prem)} \\
(2) & \vert\ \underline{\forall x(\exists y(My \land Txy) \to \neg Gx))} & \text{(Prem)} \\
(3) & \vert\ (La \to (Ra \land Ga)) & (\forall\text{E } 1) \\
(4) & \vert\ (\exists y(My \land Tay) \to \neg Ga)) & (\forall\text{E } 2) \\
(5) & \vert\ \quad \vert\ \underline{La} & \text{(Supp)} \\
(6) & \vert\ \quad \vert\ (Ra \land Ga) & \text{(MP 5, 3)} \\
(7) & \vert\ \quad \vert\ Ga & (\land\text{E } 6) \\
(8) & \vert\ \quad \vert\ \quad \vert\ \underline{Mb} & \text{(Supp)} \\
(9) & \vert\ \quad \vert\ \quad \vert\ \quad \vert\ \underline{Tab} & \text{(Supp)} \\
(10) & \vert\ \quad \vert\ \quad \vert\ \quad \vert\ (Mb \land Tab) & (\land\text{I } 7, 8) \\
(11) & \vert\ \quad \vert\ \quad \vert\ \quad \vert\ \exists y(My \land Tay) & (\exists\text{I } 9) \\
(12) & \vert\ \quad \vert\ \quad \vert\ \quad \vert\ \neg Ga & \text{(MP 10, 4)} \\
(13) & \vert\ \quad \vert\ \quad \vert\ \quad \vert\ \bot & \text{(Abs 7, 12)} \\
(14) & \vert\ \quad \vert\ \quad \vert\ \neg Tab & \text{(RAA 9–13)} \\
(15) & \vert\ \quad \vert\ (Mb \to \neg Tab) & \text{(CP 8–14)} \\
(16) & \vert\ \quad \vert\ \forall y(My \to \neg Tay) & (\forall\text{I } 15) \\
(17) & \vert\ (La \to \forall y(My \to \neg Tay)) & \text{(CP 5–16)} \\
(18) & \vert\ \forall x(Lx \to \forall y(My \to \neg Txy)) & (\forall\text{I } 17)
\end{array}
$$

(6*) *Given any two people, if the first admires Gödel and Gödel admires the second, then the first admires the second. Gödel admires anyone who has understood* Principia*. There's someone who has understood* Principia *who admires Gödel. Therefore there's someone who has understood* Principia *who admires everyone who has understood* Principia*!*

The translation is straightforward:

$$\forall x \forall y((Axg \wedge Agy) \to Axy),\ \forall x(Px \to Agx),\ \exists x(Px \wedge Axg),\ \therefore\ \exists x(Px \wedge \forall y(Py \to Axy))$$

Informally, take Alex who has understood *Principia* and admires Gödel (the third premiss tells us that there is such a person). Now take anyone you like, Beth as it might be: If Beth has understood *Principia*, Gödel admires her. So Alex admires Gödel who admires Beth, so by the first premiss Alex admires Beth. So: Alex has understood *Principia*, and admires anyone like Beth who has understood *Principia*. So there is indeed someone like that!

Now let's recast that as a formal proof. So first 'take Alex' – i.e. instantiate the existential premiss (following that usual rule of thumb, instantiate existential premisses first):

$$\begin{array}{ll}
|\ \forall x \forall y((Axg \wedge Agy) \to Axy) & \text{(Prem)} \\
|\ \forall x(Px \to Agx) & \text{(Prem)} \\
|\ \underline{\exists x(Px \wedge Axg)} & \text{(Prem)} \\
|\quad |\ \underline{(Pa \wedge Aag)} & \text{(Supp)} \\
|\quad |\ \vdots & \\
|\quad |\ \exists x(Px \wedge \forall y(Py \to Axy)) & \\
|\ \exists x(Px \wedge \forall y(Py \to Axy)) & (\exists \mathrm{E})
\end{array}$$

How are going to reach the wff at the end of the subproof? – presumably by existentially generalizing an instance, and given that we already have Pa available, we surely want to get to the following instance:

$$\begin{array}{ll}
|\ \forall x \forall y((Axg \wedge Agy) \to Axy) & \text{(Prem)} \\
|\ \forall x(Px \to Agx) & \text{(Prem)} \\
|\ \underline{\exists x(Px \wedge Axg)} & \text{(Prem)} \\
|\quad |\ \underline{(Pa \wedge Aag)} & \text{(Supp)} \\
|\quad |\ \vdots & \\
|\quad |\ (Pa \wedge \forall y(Py \to Aay)) & \\
|\quad |\ \exists x(Px \wedge \forall y(Py \to Axy)) & (\exists \mathrm{I}) \\
|\ \exists x(Px \wedge \forall y(Py \to Axy)) & (\exists \mathrm{E})
\end{array}$$

And how are going to get *that*? By proving the quantified conditional $\forall y(Py \to Axy)$. And that is presumably going to be proved by generalizing on an instance – which is a conditional we'll prove by conditional proof. So we'll now expect the proof to go

$$\begin{array}{ll}
|\quad |\ \underline{(Pa \wedge Aag)} & \text{(Supp)} \\
|\quad |\ \vdots & \\
|\quad |\quad |\ \underline{Pb} & \text{(Supp)} \\
|\quad |\quad |\ \vdots & \\
|\quad |\quad |\ Aab & \\
|\quad |\ (Pb \to Aab) & \text{(CP)} \\
|\quad |\ \forall y(Py \to Aay) & (\forall \mathrm{I}\ ) \\
|\quad |\ (Pa \wedge \forall y(Py \to Aay)) & (\wedge \mathrm{I}) \\
|\quad |\ \exists x(Px \wedge \forall y(Py \to Axy)) & (\exists \mathrm{I}) \\
|\ \exists x(Px \wedge \forall y(Py \to Axy)) & (\exists \mathrm{E})
\end{array}$$

NB We need to introduce a new dummy name ('Beth'!) with our second supposition, so that we can later generalize on it. Now filling in the dots, we arrive at:

(1)	$\forall x \forall y((Axg \land Agy) \to Axy)$	(Prem)
(2)	$\forall x(Px \to Agx)$	(Prem)
(3)	$\exists x(Px \land Axg)$	(Prem)
(4)	$\quad (Pa \land Aag)$	(Supp)
(5)	$\quad Pa$	($\land$E 4)
(6)	$\quad Aag$	($\land$E 4)
(7)	$\quad\quad Pb$	(Supp)
(8)	$\quad\quad (Pb \to Agb)$	($\forall$E 2)
(9)	$\quad\quad Agb$	(MP 7, 8)
(10)	$\quad\quad \forall y((Aag \land Agy) \to Aay)$	($\forall$E 1)
(11)	$\quad\quad ((Aag \land Agb) \to Aab$	($\forall$E 10)
(12)	$\quad\quad (Aag \land Agb)$	($\land$I 6, 9)
(13)	$\quad\quad Aab$	(MP 12, 11)
(14)	$\quad (Pb \to Aab)$	(CP 7–13)
(15)	$\quad \forall y(Py \to Aay)$	($\forall$I 14)
(16)	$\quad (Pa \land \forall y(Py \to Aay))$	($\land$I 5, 15)
(17)	$\quad \exists x(Px \land \forall y(Py \to Axy))$	($\exists$I 16)
(18)	$\exists x(Px \land \forall y(Py \to Axy))$	($\exists$E 3, 4–17)

(7*) *Any adult elephant weighs more than any horse. Some horse weighs more than any donkey. If a first thing weighs more than a second thing, and the second thing weighs more than a third, then the first weighs more than the third. Hence any adult elephant weighs more than any donkey.*

Using 'Wxy' for 'x weighs more than y', we can render the argument like this:

$$\forall x(Ex \to \forall y(Hy \to Wxy)),\ \exists x(Hx \land \forall y(Dy \to Wxy)),\ \forall x \forall y \forall z((Wxy \land Wyz) \to Wxz)\ \therefore\ \forall x(Ex \to \forall y(Dy \to Wxy))$$

Of course we could equally well have translated the first premiss by $\forall x \forall y((Ex \land Hy) \to Wxy)$, or the second premiss by $\exists x \forall y(Hx \land (Dy \to Wxy))$, or the conclusion by $\forall x \forall y((Ex \land Dy) \to Wxy)$. But these minor variations won't make any difference to our overall proof strategy.

We have two universal premisses and an existential one. The usual rule of thumb (remember!) in that sort of case is: instantiate the existential one first. (Why? Because it introduces a new dummy name you'll typically want to then use to instantiate the universal quantifiers.) But that's not quite the first thing to do in this case.

Look at the conclusion: it's a universally quantified conditional. So we will expect the *final* step of the proof to be an application of ($\forall$I) to an instance of the conditional. And we'll expect that conditional to be the result of an argument by conditional proof. In other words, we'll expect our derivation to have the following overall shape, with the initial supposition Ea

$\forall x(Ex \to \forall y(Hy \to Wxy))$	(Prem)
$\exists x(Hx \land \forall y(Dy \to Wxy))$	(Prem)
$\forall x \forall y \forall z((Wxy \land Wyz) \to Wxz)$	(Prem)
$\quad Ea$	(Supp)
$\quad \vdots$	
$\quad \forall y(Dy \to Way)$	
$(Ea \to \forall y(Dy \to Way))$	(CP)
$\forall x(Ex \to \forall y(Dy \to Wxy))$	($\forall$I)

So after making that first supposition, let's *now* let's instantiate the existential premiss in a new supposition (with a new dummy name, of course):

(1)	$\forall x(Ex \to \forall y(Hy \to Wxy))$	(Prem)
(2)	$\exists x(Hx \land \forall y(Dy \to Wxy))$	(Prem)
(3)	$\forall x \forall y \forall z((Wxy \land Wyz) \to Wxz)$	(Prem)
(4)	$\mid\ Ea$	(Supp)
(5)	$\mid\ \mid\ (Hb \land \forall y(Dy \to Wby))$	(Supp)
(6)	$\mid\ \mid\ \vdots$	
(7)	$\mid\ \mid\ \forall y(Dy \to Way)$	
(8)	$\mid\ \forall y(Dy \to Way)$	$(\exists E)$
(9)	$(Ea \to \forall y(Dy \to Way))$	(CP)
(10)	$\forall x(Ex \to \forall y(Dy \to Wxy))$	$(\forall I)$

Note, an application of $(\exists E)$ here is legimate, since $\forall y(Dy \to Way)$ does not contain the dummy name b introduced at the head of the relevant subproof.

In the subproof, we now need to derive *another* quantified conditional, $\forall y(Dy \to Way)$. So same strategy again: prove an instance of it and generalize, and prove the instance by a conditional proof. If we are going to be able to safely generalize, however, we will need to be using a third, new, dummy name different from the ones already in play. So we will have something that looks like this:

(1)	$\forall x(Ex \to \forall y(Hy \to Wxy))$	(Prem)
(2)	$\exists x(Hx \land \forall y(Dy \to Wxy))$	(Prem)
(3)	$\forall x \forall y \forall z((Wxy \land Wyz) \to Wxz)$	(Prem)
(4)	$\mid\ Ea$	(Supp)
(5)	$\mid\ \mid\ (Hb \land \forall y(Dy \to Wby))$	(Supp)
(6)	$\mid\ \mid\ \vdots$	
(7)	$\mid\ \mid\ \mid\ Dc$	(Supp)
(8)	$\mid\ \mid\ \mid\ \vdots$	
(9)	$\mid\ \mid\ \mid\ Wac$	
(10)	$\mid\ \mid\ (Dc \to Wac)$	(CP)
(11)	$\mid\ \mid\ \forall y(Dy \to Way)$	$(\forall I)$
(12)	$\mid\ \forall y(Dy \to Way)$	$(\exists E)$
(13)	$(Ea \to \forall y(Dy \to Way))$	(CP)
(14)	$\forall x(Ex \to \forall y(Dy \to Wxy))$	$(\forall I)$

So now we just need to fill in the dots. Obviously we can instantiate the first premiss using a, so we can then apply modus ponens. Obviously we can disassemble the second supposition which is a conjunction. Do these easy things first.

We'll thereby get some quantified conditionals into play. Instantiate them in the natural ways (e.g. instantiate $\forall y(Hy \to Way)$ with b because we already have Hb so can use modus ponens). Then at last bring the third premiss into play, and with a bit of labour we get ...

$$
\begin{array}{lll}
(1) & \mid\ \forall x(Ex \to \forall y(Hy \to Wxy)) & \text{(Prem)} \\
(2) & \mid\ \exists x(Hx \wedge \forall y(Dy \to Wxy)) & \text{(Prem)} \\
(3) & \mid\ \forall x\forall y\forall z((Wxy \wedge Wyz) \to Wxz) & \text{(Prem)} \\
(4) & \mid\ \mid\ Ea & \text{(Supp)} \\
(5) & \mid\ \mid\ \mid\ (Hb \wedge \forall y(Dy \to Wby)) & \text{(Supp)} \\
(6) & \mid\ \mid\ \mid\ Hb & (\wedge\text{E } 5) \\
(7) & \mid\ \mid\ \mid\ \forall y(Dy \to Wby) & (\wedge\text{E } 5) \\
(8) & \mid\ \mid\ \mid\ (Ea \to \forall y(Hy \to Way)) & (\forall\text{I } 1) \\
(9) & \mid\ \mid\ \mid\ \forall y(Hy \to Way) & \text{(MP 4, 8)} \\
(10) & \mid\ \mid\ \mid\ \mid\ Dc & \text{(Supp)} \\
(11) & \mid\ \mid\ \mid\ \mid\ (Dc \to Wbc) & (\forall\text{E } 7) \\
(12) & \mid\ \mid\ \mid\ \mid\ Wbc & \text{(MP 10, 11)} \\
(13) & \mid\ \mid\ \mid\ \mid\ (Hb \to Wab) & (\forall\text{E } 9) \\
(14) & \mid\ \mid\ \mid\ \mid\ Wab & \text{(MP 6, 13)} \\
(15) & \mid\ \mid\ \mid\ \mid\ \forall y\forall z((Way \wedge Wyz) \to Waz) & (\forall\text{E}) \\
(16) & \mid\ \mid\ \mid\ \mid\ \forall z((Wab \wedge Wbz) \to Waz) & (\forall\text{E } 3) \\
(17) & \mid\ \mid\ \mid\ \mid\ ((Wab \wedge Wbc) \to Wac) & (\forall\text{E } 15) \\
(18) & \mid\ \mid\ \mid\ \mid\ (Wab \wedge Wbc) & (\wedge\text{I } 14, 12) \\
(19) & \mid\ \mid\ \mid\ \mid\ Wac & \text{(MP 18, 17)} \\
(20) & \mid\ \mid\ \mid\ (Dc \to Wac) & \text{(CP 10–19)} \\
(21) & \mid\ \mid\ \mid\ \forall y(Dy \to Way) & (\forall\text{I } 20) \\
(22) & \mid\ \mid\ \forall y(Dy \to Way) & (\exists\text{E } 2, 5\text{–}21) \\
(23) & \mid\ (Ea \to \forall y(Dy \to Way)) & \text{(CP 4–22)} \\
(24) & \mid\ \forall x(Ex \to \forall y(Dy \to Wxy)) & (\forall\text{I } 23)
\end{array}
$$

So far, so good. But there is a pretty low limit to the additional enlightenment you get from doing more and more complicated QL proofs. If you've managed the proofs so far, that's evidence enough that you have a pretty good grasp of what's going on! There are some more proofs to do under (c) which illustrate some further common proof strategies – but it's up to you whether you are interested in tackling them.

(c) *Why should the following* QL *wffs be logically true (assuming the wffs are interpreted)?*

(1) $\exists x(Fx \to \forall yFy)$

$(\forall xFx \to \forall yFy)$ is a logical truth. And if β is a wff not containing x, $(\forall xFx \to \beta)$ is equivalent to $\exists x(Fx \to \beta)$. Put those thoughts together, and it follows that (1) is a logical truth.
Here's a formal proof – allowing ourselves in these examples to call on the derived rules $(\neg\forall)$ and $(\neg\exists)$ from §32.5 as well as PL inferences whenever we want:

$$
\begin{array}{lll}
(1) & \mid\ & \\
(2) & \mid\ \mid\ \neg\exists x(Fx \to \forall yFy) & \text{(Supp)} \\
(3) & \mid\ \mid\ \forall x\neg(Fx \to \forall yFy) & (\neg\exists\ 2) \\
(4) & \mid\ \mid\ \neg(Fa \to \forall yFy) & (\forall\text{E } 3) \\
(5) & \mid\ \mid\ Fa & \text{(PL 4)} \\
(6) & \mid\ \mid\ \neg\forall yFy & \text{(PL 4)} \\
(7) & \mid\ \mid\ \forall yFy & (\forall\text{I } 5) \\
(8) & \mid\ \mid\ \bot & \text{(Abs 7, 6)} \\
(9) & \mid\ \neg\neg\exists x(Fx \to \forall yFy) & \text{(RAA 2–8)} \\
(10) & \mid\ \exists x(Fx \to \forall yFy) & \text{(DN 9)}
\end{array}
$$

At (5) and (6) we use the fact that, by PL reasoning we can get from $\neg(\alpha \to \beta)$ to α and $\neg\beta$. And at (7) we rely on the fact that the dummy name a doesn't appear in any still-live assumption and so we can universally generalize on it.

(2) $\forall\mathsf{x}\exists\mathsf{y}(\exists\mathsf{zLxz} \to \mathsf{Lxy})$

Again logically true. Fix an arbitrary temporary reference for a; then evidently $(\exists\mathsf{zLaz} \to \exists\mathsf{yLay})$ is a logical truth (change of bound variables makes no difference). And if β doesn't contain y, then $(\beta \to \exists\mathsf{yLay})$ is equivalent to $\exists\mathsf{y}(\beta \to \exists\mathsf{yLay})$. So $\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$ is a logical truth. Now generalize and we get (2)!

We aim to prove $\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$ and then generalize. So assume the negation of that, and then one option is to use the same familiar trick, to get a proof of this shape:

$\quad\neg\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	(Supp)
$\quad\vdots$	(As usual we can argue from $\neg\exists\alpha(\xi)$ to $\neg\alpha(\tau)$)
$\quad\neg(\exists\mathsf{zLaz} \to \mathsf{Lab})$	
$\quad\exists\mathsf{zLaz}$	(PL – we could also derive $\neg\mathsf{Lab}$, but that's not useful)
$\quad\vdots$	
$\quad\bot$	
$\neg\neg\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	(RAA)
$\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	(DN)
$\forall\mathsf{x}\exists\mathsf{y}(\exists\mathsf{zLxz} \to \mathsf{Lxy})$	($\forall$I)

Filling in the rest of the proof is then easy enough:

(1)		
(2)	$\quad\neg\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	(Supp)
(3)	$\quad\forall\mathsf{y}\neg(\exists\mathsf{zLaz} \to \mathsf{Lay})$	($\neg\exists$)
(4)	$\quad\neg(\exists\mathsf{zLaz} \to \mathsf{Lab})$	($\forall$E 3)
(5)	$\quad\exists\mathsf{zLaz}$	(PL 4)
(6)	$\quad\quad\mathsf{Lac}$	(Supp)
(7)	$\quad\quad(\exists\mathsf{zLaz} \to \mathsf{Lac})$	(PL 6)
(8)	$\quad\quad\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	($\exists$I 7)
(9)	$\quad\quad\bot$	(Abs 8, 2)
(10)	$\quad\bot$	($\exists$E 5, 6–9)
(11)	$\neg\neg\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	RAA 2–10)
(12)	$\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	(DN 11)
(13)	$\forall\mathsf{x}\exists\mathsf{y}(\exists\mathsf{zLxz} \to \mathsf{Lxy})$	($\forall$I 12)

Here's a slightly differently structured proof that you might have found:

(1)		
(2)	$\quad\neg\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	(Supp)
(3)	$\quad\quad\exists\mathsf{zLaz}$	(Supp)
(4)	$\quad\quad\quad\mathsf{Lab}$	(Supp)
(5)	$\quad\quad\quad(\exists\mathsf{zLaz} \to \mathsf{Lab})$	(PL 4)
(6)	$\quad\quad\quad\exists\mathsf{y}(\exists\mathsf{zLaz} \to \mathsf{Lay})$	($\exists$I 5)
(7)	$\quad\quad\quad\bot$	(Abs 6, 2)
(8)	$\quad\quad\bot$	($\exists$E 3, 4–7)
(9)	$\quad\quad\mathsf{Lab}$	(EFQ 8)

(10)	$\mid\ \mid\ (\exists z\mathsf{Laz} \to \mathsf{Lab})$	(CP 3–9)
(11)	$\mid\ \mid\ \exists y(\exists z\mathsf{Laz} \to \mathsf{Lay})$	(∃I 10)
(12)	$\mid\ \mid\ \bot$	(Abs 11, 2)
(13)	$\mid\ \neg\neg\exists y(\exists z\mathsf{Laz} \to \mathsf{Lay})$	(RAA 2–12)
(14)	$\mid\ \exists y(\exists z\mathsf{Laz} \to \mathsf{Lay})$	(DN 13)
(15)	$\mid\ \forall x\exists y(\exists z\mathsf{Lxz} \to \mathsf{Lxy})$	(∀I 14)

(3) $\exists x\forall y(\neg \mathsf{F}y \lor \mathsf{F}x)$

This too should be logically true.

1. Suppose some particular thing, we'll dub it a, satisfies F. Then Fa, so $(\neg\mathsf{Fb} \lor \mathsf{Fa})$ is true whatever b could denote, so $\forall y(\neg\mathsf{F}y \lor \mathsf{Fa})$ hence its existential quantification is true.
2. Suppose alternatively that nothing satisfies F, i.e. whatever b could denote, ¬Fb hence $(\neg\mathsf{Fb} \lor \mathsf{Fa})$ for any chosen thing for a to denote. Since b was arbitrary we again have $\forall y(\neg\mathsf{F}y \lor \mathsf{Fa})$ hence its existential quantification is true.

How can we prove it? Let's talk through the proof. We are going to proceed by assuming the negation of the desired theorem and aiming for a reductio – what else? And we'll use our derived rules to reduce the number of quantifiers, until we get this far ...

(1)	$\mid\underline{\quad}$	
(2)	$\mid\ \mid\ \neg\exists x\forall y(\neg\mathsf{F}y \lor \mathsf{F}x)$	(Supp)
(3)	$\mid\ \mid\ \forall x\neg\forall y(\neg\mathsf{F}y \lor \mathsf{F}x)$	(¬∃ 2)
(4)	$\mid\ \mid\ \neg\forall y(\neg\mathsf{F}y \lor \mathsf{Fa})$	(∀E 3)
(5)	$\mid\ \mid\ \exists y\neg(\neg\mathsf{F}y \lor \mathsf{Fa})$	(¬∀ 4)

Obviously enough we are going to need to work with the existential quantification (5) by supposing and instance of it – an instance that can be unpacked by PL reasoning:

(6)	$\mid\ \mid\ \mid\ \neg(\neg\mathsf{Fb} \lor \mathsf{Fa})$	(Supp)
(7)	$\mid\ \mid\ \mid\ \mathsf{Fb}$	(PL 6)
(8)	$\mid\ \mid\ \mid\ [\neg\mathsf{Fa}$	(PL 6 – but it turns out that this won't be of use!)]

But now where can we go? The only previous wff we can use again is (3). There's no point instantiating it with a again. But we could instantiate it using b. And this is a hopeful thought, because instead of that useless ¬Fa we should get to ¬Fb and hence a contradiction – and we are looking for contradictions for our reductio proof! So let's try that ...

(6)	$\mid\ \mid\ \mid\ \neg(\neg\mathsf{Fb} \lor \mathsf{Fa})$	(Supp)
(7)	$\mid\ \mid\ \mid\ \mathsf{Fb}$	(PL 6)
(8)	$\mid\ \mid\ \mid\ \neg\forall y(\neg\mathsf{F}y \lor \mathsf{Fb})$	(∀E 3)
(9)	$\mid\ \mid\ \mid\ \exists y\neg(\neg\mathsf{F}y \lor \mathsf{Fb})$	(¬∀ 8)
(10)	$\mid\ \mid\ \mid\ \mid\ \neg(\neg\mathsf{Fc} \lor \mathsf{Fb})$	(Supp)
(11)	$\mid\ \mid\ \mid\ \mid\ \neg\mathsf{Fb}$	(PL 10)
(12)	$\mid\ \mid\ \mid\ \mid\ \bot$	(Abs 7, 11)
(13)	$\mid\ \mid\ \mid\ \bot$	(∃E 9, 10–12)
(14)	$\mid\ \mid\ \bot$	(∃E 5, 6–13)
(15)	$\mid\ \neg\neg\exists x\forall y(\neg\mathsf{F}y \lor \mathsf{F}x)$	(RAA 2–14)
(16)	$\mid\ \exists x\forall y(\neg\mathsf{F}y \lor \mathsf{F}x)$	(DN 15)

And we are done! Masochists can now try proving this without using the derived rules or skipping PL inferences!

Also give derivations to warrant the following inferences:

(4) $(\forall x Fx \to \exists y Gy)\ \therefore\ \exists x \exists y(Fx \to Gy)$

The obvious plan is to set off on a reductio argument, like this:

$\mid\ (\forall x Fx \to \exists y Gy)$	(Prem)
$\mid\ \mid\ \neg\exists x \exists y(Fx \to Gy)$	(Supp)
$\mid\ \mid\ \vdots$	
$\mid\ \mid\ \bot$	($\exists$E 11, 12–16)
$\mid\ \neg\neg\exists x \exists y(Fx \to Gy)$	(RAA 2–17)
$\mid\ \exists x \exists y(Fx \to Gy)$	(DN 18)

Here's two thoughts:

1. To use the first premiss it would be good to be able to establish the antecedent of the conditional – i.e. $\forall x Fx$. How can we do that, given just our supposition on the second line? Well, can we prove something like Fa which we can then generalize??
2. We've seen before that a good way of using a negated existential (like our supposition) is to suppose something α that entails the unnegated existential, note we've now got a contradiction, and infer $\neg\alpha$.

Putting those thoughts together, if we can manage things so that $\neg\alpha$ is equivalent to Fa (i.e. α is equivalent to $\neg Fa$, then we'll be in the game! So let's try that, using a fairly obvious trick at line (4):

(1)	$\mid\ (\forall x Fx \to \exists y Gy)$	(Prem)
(2)	$\mid\ \mid\ \neg\exists x \exists y(Fx \to Gy)$	(Supp)
(3)	$\mid\ \mid\ \mid\ \neg Fa$	(Supp)
(4)	$\mid\ \mid\ \mid\ (Fa \to Gb)$	(PL 3)
(5)	$\mid\ \mid\ \mid\ \exists y(Fa \to Gy)$	($\exists$I 4)
(6)	$\mid\ \mid\ \mid\ \exists x \exists y(Fx \to Gy)$	($\exists$I 5)
(7)	$\mid\ \mid\ \mid\ \bot$	(Abs 6, 2)
(8)	$\mid\ \mid\ \neg\neg Fa$	(RAA 1–7)
(9)	$\mid\ \mid\ Fa$	(DN 8)
(10)	$\mid\ \mid\ \forall x Fx$	($\forall$I 9)

So now we apply modus ponens, get an existential quantification, which we'll presumably want to instantiate with a view to an argument by ($\exists$E)

(11)	$\mid\ \mid\ \exists y Gy$	(MP 10, 1)
(12)	$\mid\ \mid\ \mid\ Gb$	(Supp)
(13)	$\mid\ \mid\ \mid\ \vdots$	

And now what? A moment's thought suggests we can use the same trick again. Just as $\neg Fa$ gives us $(Fa \to Gb)$ by PL reasoning, and that leads to absurdity, so too does Gb give us $(Fa \to Gb)$ by PL reasoning, and that will lead to absurdity again. So we can complete our proof

(13)	$\mid\ \mid\ \mid\ (Fa \to Gb)$	(PL 12)
(14)	$\mid\ \mid\ \mid\ \exists y(Fa \to Gy)$	($\exists$I 13)
(15)	$\mid\ \mid\ \mid\ \exists x \exists y(Fx \to Gy)$	($\exists$I 14)
(16)	$\mid\ \mid\ \mid\ \bot$	(Abs 15, 2)
(17)	$\mid\ \mid\ \bot$	($\exists$E 11, 12–16)
(18)	$\mid\ \neg\neg\exists x \exists y(Fx \to Gy)$	(RAA 2–17)
(19)	$\mid\ \exists x \exists y(Fx \to Gy)$	(DN 18)

(5) $(\exists zFz \to \exists zGz)\ \therefore\ \forall x\exists y(Fx \to Gy)$

We might expect the conclusion $\forall x\exists y(Fx \to Gy)$ to be derived from an instance $\exists y(Fa \to Gy)$ by a final step of ($\forall$I). So let's aim to prove that existential quantification by a reductio argument, starting from $\neg\exists y(Fa \to Gy)$. And then push the negation past the quantifier using the derived shortcut rule ($\neg\exists$) which we are allowing ourselves.

That suggests we should be looking for a proof of the following overall shape:

$(\exists zFz \to \exists zGz)$	(Prem)
$\quad \neg\exists y(Fa \to Gy)$	(Supp)
$\quad \forall y\neg(Fa \to Gy)$	($\neg\exists$)
$\quad \vdots$	
$\quad \bot$	(Abs)
$\neg\neg\exists y(Fa \to Gy)$	(RAA)
$\exists y(Fa \to Gy)$	(DN)
$\forall x\exists y(Fx \to Gy)$	($\forall$I)

So now where?

There is no point in instantiating the universal quantifier at line (3) yet, as we have nothing useful to instantiate it with. So let's think about using the first premiss. How can we deploy that? We can assume the antecedent, use modus ponens, extract the consequent $\exists zGz$ and then make use of *that*. Bulldozing forward in an exploratory way, where would that take us?

(1)	$(\exists zFz \to \exists zGz)$	(Prem)
(2)	$\quad \neg\exists y(Fa \to Gy)$	(Supp)
(3)	$\quad \forall y\neg(Fa \to Gy)$	($\neg\exists$ 2)
(4)	$\quad\quad \exists zFz$	(Supp)
(5)	$\quad\quad \exists zGz$	(MP 4, 1)
(6)	$\quad\quad\quad Gb$	(Supp)
(7)	$\quad\quad\quad \neg(Fa \to Gb)$	($\forall$E 3)
(8)	$\quad\quad\quad \neg Gb$	(PL 7)
(9)	$\quad\quad\quad \bot$	(Abs 6, 8)
(10)	$\quad\quad \bot$	($\exists$E 5, 6–9)
(11)	$\quad \neg\exists xFx$	(RAA 4–10)

So how do we now get from $\neg\exists xFx$ to a contradiction? What about using that quantification at line (3) again – as it seems the only useful thing in play? Instantiate it with some dummy name δ.

And now note that getting something of the form $\neg(Fa \to G\delta)$ into play, we can infer Fa by PL reasoning and then get our contradiction with $\neg\exists xFx$! Hooray. Since it doesn't matter what the δ is, use whatever you like – a will do. Then the proof can be finished

(12)	$\quad \neg(Fa \to Ga)$	($\forall$E 3)
(13)	$\quad Fa$	(PL 12)
(14)	$\quad \exists xFx$	($\exists$I 13)
(15)	$\quad \bot$	(Abs 14, 11)
(16)	$\neg\neg\exists y(Fa \to Gy)$	(RAA 2–15)
(17)	$\exists y(Fa \to Gy)$	(DN 16)
(18)	$\forall x\exists y(Fx \to Gy)$	($\forall$I 17)

(6) $\forall x \exists y(Fy \to Gx) \ \therefore\ \exists y \forall x(Fy \to Gx)$

Normally, of course, we can't argue from $\forall x \exists y \alpha$ to $\exists y \forall x \alpha$ – that's in general a quantifier shift fallacy. But we can in *this* special case – the essential point being that there is here no connection between what the two quantifiers are quantifying (they are, so to speak, working quite independently so it doesn't matter which order we consider them in).

The proof is surprisingly long (even if we allow ourselves some short-cut rules). Here's my best shot. The overall idea will be a reductio proof, starting with the premiss and negation of the conclusion. So we can get this far more or less on autopilot as we unpack the supposition:

(1)	$\forall x \exists y(Fy \to Gx)$	(Prem)
(2)	$\quad \neg\exists y \forall x(Fy \to Gx)$	(Supp)
(3)	$\quad \forall y \neg\forall x(Fy \to Gx)$	($\neg\exists$ 2)
(4)	$\quad \neg\forall x(Fa \to Gx)$	($\forall$E 3)
(5)	$\quad \exists x \neg(Fa \to Gx)$	($\neg\forall$ 4)

Now we are going to need to suppose an instance of (5) with a view to a ($\exists$E) proof. We can, for example, continue like this, still on autopilot:

(6)	$\quad\quad \neg(Fa \to Gb)$	(Supp)	
(7)	$\quad\quad Fa$	(PL 6)	[In fact, we won't be using this line later!]
(8)	$\quad\quad \neg Gb$	(PL 6)	

But now what? At some point we are going to use the first premiss; so let's instantiate the initial universal quantifier with b to get another occurrence of Gb. This gives us another existential quantification which we'll again want to instantiate with a view to another ($\exists$E) proof nested inside the one we've already started:

(9)	$\quad\quad \exists y(Fy \to Gb)$	($\forall$E 1)
(10)	$\quad\quad\quad (Fc \to Gb)$	(Supp)
(11)	$\quad\quad\quad \neg Fc$	(PL 8, 10)

Now, how we can use this last wff? Well, if we instantiate the initial universal quantifier of (3) by c we'll get another occurrence of Fc. So let's see if we can use that! – and we quickly get *another* existential quantification which once more we'll want to instantiate (once more with a new dummy name):

(12)	$\quad\quad\quad \neg\forall x(Fc \to Gx)$	($\forall$E 3)
(13)	$\quad\quad\quad \exists x \neg(Fc \to Gx)$	($\neg\forall$ 12)
(14)	$\quad\quad\quad\quad \neg(Fc \to Gd)$	(Supp)
(15)	$\quad\quad\quad\quad Fc$	(PL 14)
(16)	$\quad\quad\quad\quad \bot$	(Abs 15, 11)
(17)	$\quad\quad\quad \bot$	($\exists$E 13, 14–16)
(18)	$\quad\quad \bot$	($\exists$E 9, 10–17)
(19)	$\quad \bot$	($\exists$E 5, 6–18)
(20)	$\neg\neg\exists y \forall x(Fy \to Gx)$	(RAA 2–19)
(21)	$\exists y \forall x(Fy \to Gx)$	(DN 29)

Kudos if you got that! (And grateful thanks – and eternal fame in the form of a mention in an update of these answers – to anyone who sends me a significantly snappier/more elegant proof.)

(d*) We are here setting up some little results that won't actually be needed until the Appendix, which you might well be skipping. But still, it is a useful reality check to think through this final exercise anyway!

Say that the wffs Γ (not necessarily all sentences) are QL-consistent *if there is no* QL *proof using wffs Γ as premisses and ending with '$\bot$'; otherwise Γ are* QL-inconsistent – *compare Exercises 22(d*).*

Assuming that the terms mentioned belong to the relevant language, show

(1) If the wffs $\Gamma, \alpha(\tau)$ are QL-inconsistent and the wffs Γ include $\forall\xi\alpha(\xi)$ then those wffs Γ are already QL-inconsistent.

This is elementary. Suppose that there is a proof from $\Gamma, \alpha(\tau)$ to absurdity and the wffs Γ include $\forall\xi\alpha(\xi)$. Then the proof on the left can be turned into the proof on the right!

$$\begin{array}{|ll}
\Gamma & \text{(Prem)} \\
\underline{\alpha(\tau)} & \text{(Prem)} \\
\vdots & \\
\bot &
\end{array}
\quad\Longrightarrow\quad
\begin{array}{|ll}
\underline{\Gamma} & \text{(Prem)} \\
\alpha(\tau) & (\forall\text{E from premiss } \forall\xi\alpha(\xi) \text{ in } \Gamma) \\
\vdots & \\
\bot &
\end{array}$$

and then conclude that

(2) If the wffs Γ are QL-consistent and $\forall\xi\alpha(\xi)$ is one of them, then $\Gamma, \alpha(\tau)$ are also QL-consistent.

We've seen this move in earlier exercises! We just apply propositional reasoning to (1)! For (1) has the form $(\neg\mathsf{P} \land \mathsf{Q}) \to \neg\mathsf{R}$, and this implies $(\mathsf{R} \land \mathsf{Q}) \to \mathsf{P}$ which gives us (2).

Show further that

(2) If the wffs Γ are QL-consistent and $\forall\xi\alpha(\xi)$ is one of them, then $\Gamma, \alpha(\tau_1)$, $\alpha(\tau_2)$, …, $\alpha(\tau_k)$ all together are also QL-consistent (for any terms $\tau_1, \tau_2, \ldots, \tau_k$ of the relevant language).*

Start by showing a more general version of (1):

(1*) If the wffs $\Gamma, \alpha(\tau_1), \alpha(\tau_2), \ldots, \alpha(\tau_k)$ are QL-inconsistent and the wffs Γ include $\forall\xi\alpha(\xi)$, then those wffs Γ are already QL-inconsistent.

We just note that if the wffs Γ include $\forall\xi\alpha(\xi)$, the proof on the left can again be turned into the proof on the right!

$$\begin{array}{|ll}
\Gamma & \text{(Prem)} \\
\alpha(\tau_1) & \text{(Prem)} \\
\alpha(\tau_2) & \text{(Prem)} \\
\vdots & \\
\underline{\alpha(\tau_k)} & \text{(Prem)} \\
\vdots & \\
\bot &
\end{array}
\quad\Longrightarrow\quad
\begin{array}{|ll}
\underline{\Gamma} & \text{(Prem)} \\
\alpha(\tau_1) & (\forall\text{E from premiss } \forall\xi\alpha(\xi) \text{ in } \Gamma) \\
\alpha(\tau_2) & (\forall\text{E from premiss } \forall\xi\alpha(\xi) \text{ in } \Gamma) \\
\vdots & \\
\alpha(\tau_3) & (\forall\text{E from premiss } \forall\xi\alpha(\xi) \text{ in } \Gamma) \\
\vdots & \\
\bot &
\end{array}$$

Then we can infer (2*) from (1*) just as we can infer (2) from (1).

Show similarly that

(3) If the wffs Γ are QL-consistent and $\exists\xi\alpha(\xi)$ is one of them, then $\Gamma, \alpha(\delta)$ are also QL-consistent if δ is a dummy name that doesn't appear in Γ.

Again let's first show the result the other way about:

(3^-) If $\Gamma, \alpha(\delta)$ are QL-inconsistent, where δ is a dummy name that doesn't appear in Γ, and $\exists\xi\alpha(\xi)$ is one of the wffs Γ, then Γ are already QL-inconsistent.

Note this time that, given $\exists\xi\alpha(\xi)$ is one of the wffs Γ, the proof on the left can be turned into the proof on the right:

$$
\begin{array}{|ll}
\Gamma & \text{(Prem)} \\
\alpha(\delta) & \text{(Prem)} \\
\hline
\vdots & \\
\bot &
\end{array}
\quad\Longrightarrow\quad
\begin{array}{|ll}
\Gamma & \text{(Prem)} \\
\hline
\exists\xi\alpha(\xi) & \text{(Iterating a premiss from among } \Gamma\text{)} \\
\quad\begin{array}{|l}
\alpha(\delta) \\
\hline
\vdots \\
\bot
\end{array} & \text{(Supp)} \\
\bot & (\exists\text{E})
\end{array}
$$

where we note the second proof is correctly formed because, by assumption, the dummy name at the top of the subproof is new to the derivation, as is required for an application of our version of ($\forall$E).

Given (3^-), then (3) now follows as before.

Also show that:

(4) If the wffs Γ are QL*-consistent and $\neg\forall\xi\alpha(\xi)$ is one of them, then $\Gamma, \exists\xi\neg\alpha(\xi)$ are* QL*-consistent.*

(5) If the wffs Γ are QL*-consistent and $\neg\exists\xi\alpha(\xi)$ is one of them, then $\Gamma, \forall\xi\neg\alpha(\xi)$ are* QL*-consistent.*

For (4), note that if $\neg\forall\xi\alpha(\xi)$ is one of the wffs Γ, the proof on the left can be turned into the proof on the right:

$$
\begin{array}{|ll}
\Gamma & \text{(Prem)} \\
\exists\xi\neg\alpha(\xi) & \text{(Prem)} \\
\hline
\vdots & \\
\bot &
\end{array}
\quad\Longrightarrow\quad
\begin{array}{|ll}
\Gamma & \text{(Prem)} \\
\hline
\neg\forall\xi\alpha(\xi) & \text{(Iterating a premiss from among } \Gamma\text{)} \\
\exists\xi\neg\alpha(\xi) & \text{(By QL reasoning with quantifiers)} \\
\vdots & \\
\bot &
\end{array}
$$

So if $\Gamma, \exists\xi\neg\alpha(\xi)$ are QL-inconsistent, and $\neg\forall\xi\alpha(\xi)$ is one of Γ, then Γ by themselves are QL-inconsistent. Which (by the now familiar propositional shuffle) entails (4).

(5) is proved exactly similarly.