Spamegg's Commentary on "Intro to Proofs: Chapter 1.1 - 1.6"

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Contents

1 Facts and definitions from mathematics that are used

1.1 Chicken vs egg problem

In this chapter Prof. Meyer introduces a lot of stuff that are taught in later sections for some reason. Like Logic, propositions and quantifiers (in Chapter 3 of the book). So you might struggle with concepts that you haven't learned yet, which you are supposed to learn later. To show you any proofs at all, Prof. Meyer has to assume you have some past experience/understanding of these concepts. In the textbook he says things like "you probably did a lot of geometry proofs in high school" even though this is not true in the sad state of today's world and math education.

1.2 Facts and definitions

Here we will be using:

The definition of the set of all integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

The definition of logarithm: for positive real numbers $a \neq 1$ and y, we define $log_a(y)$ to be the real number x such that $a^x = y$.

The definition of a rational number (which is detailed later).

The definition of square root: For a nonnegative real number x, the positive square root $\sqrt{x}$ of x is defined as the POSITIVE real number y such that $y^2 = x$.

Some basic laws of exponents: like $(a^b)^c = a^{bc}$ and $(ab)^c = a^c b^c$.

The Fundamental Theorem of Arithmetic (which is thousands of years old, it's from Euclid's Elements and is proved later in the course):

Every integer $n > 1$ can be uniquely represented as a product of prime powers: if $n > 1$ then there exist unique primes $p_1 < \ldots < p_k$ and unique positive integers $a_1, \ldots, a_k$ such that

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$

Basic logical connectives: like AND $\wedge$, OR $\vee$, NOT $\neg$, IMPLIES $\implies$, IFF (or biconditional, $\iff$) where (T being True and F being False):

$$T \wedge T = T,\ \ T \wedge F = F,\ \ F \wedge T = T,\ \ F \wedge F = F,$$

$$T \vee T = T,\ \ T \vee F = T,\ \ F \vee T = T,\ \ F \vee F = F,$$

$$(T \implies T) = T,\ \ (T \implies F) = F,\ \ (F \implies T) = T,\ \ (F \implies F) = T$$

$$(A \iff B) = (A \implies B) \wedge (B \implies A),$$

$$(T \iff T) = T,\ \ (T \iff F) = F,\ \ (F \iff T) = F,\ \ (F \iff F) = T$$

Some laws about logical connectives, such as de Morgan's Laws:

$$\neg(A \wedge B) = (\neg A) \vee (\neg B), \;\; \neg(A \vee B) = (\neg A) \wedge (\neg B),$$

and the transitivity of implication: if $A \implies B$ and $B \implies C$ then $A \implies C$, and the transitivity of the biconditional: if $A \iff B$ and $B \iff C$ then $A \iff C$.

2 1.1 Propositions

2.1 Layers upon layers of DEFINITIONS!

It's too bad this material is taught properly in later sections but thrown at you here. See the Chicken vs egg problem for reference.

Notice that the first two pages are chock full of DEFINITIONS. There is only one proper, formal definition, whereas the other definitions are given in parentheses or in footnotes at the bottom of pages. There is this one:

Definition 1. *A proposition is a statement (communication) that is either true or false.*

Then there are definitions of:

what it means for an integer to be *prime* (inside parentheses),

what the symbol ::= means (in a footnote),

what $\forall$ means, what $\in$ means, what $\mathbb{N}$ means,

how a universally quantified statement is defined,

how a multiple universally quantified statement is defined, and so on.

Prof. Meyer tends to keep it casual and avoid defining things formally, and assumes that you either learned these definitions in high school and have a good grasp of them, or you have enough intuition and smarts to grasp them and remember them, but later this ends up being an issue.

I've worked with learners who have a lot of trouble remembering definitions or invoking them properly in proofs. There is also quite a lot of confusion about quantified statements and predicates (propositions that depend on a variable).

In undergrad, I had a Real Analysis professor who would force us to memorize the complex definitions found in real analysis, and in fact he would put them on the midterms and the final! Say, a midterm is out of 100 points, and 20 points of that would be just correctly writing down a formal definition from the textbook! That was his solution to students' universal problem with remembering definitions. When grades were on the line, people memorized them like their lives depended on it!

My recommendation is: turn every one of these definitions into a "formal" one like the very first one, and WRITE THEM DOWN. Do not read passively. For example:

Definition 2. *A prime number is an integer greater than 1 that is not divisible by any other integer greater than 1 (except itself).*

2.2 Hunt down the hidden definitions!

But this has some HIDDEN DEFINITIONS (right, we have to worry about those now!) For example, what does "divisible" mean? You should write down the definition of that too, because later you will have to invoke it in proofs. HUNT DOWN ALL THE DEFINITIONS and write them down!

Once again Prof. Meyer assumes you already know the definition from high school, but the chances are: you have some vague, informal, intuitive idea of what "divisible" means, but you haven't learned, or learned but forgot, the formal definition and you cannot write it down. I get responses from learners like: "well, when you divide it, there is no remainder" or "it divides completely without a fraction or decimal part" or things like that. These are totally useless in proofs unfortunately.

Well, what *is* the formal definition of "divisible"? If you have the entire textbook, you can do a text search (Ctrl+F) for "divisibility" and we find the definition **on page 243, in Chapter 8: Number Theory, under section 8.1: Divisibility:**

Definition 3. *a divides b (notation $a \mid b$) iff there is an integer k such that $ak = b$.*

You must be getting angry now. "Wait a minute! How am I supposed to know a definition that is in Chapter 8?" Once again, as I've mentioned before, there is a chicken vs egg issue with teaching elementary proofs. We can't teach you abstract symbolic logic proofs right off the bat, so we have to teach you proofs of *something*, and that something is usually what you are already (supposed to be) familiar with from high school. Once again, the chicken vs egg problem.

2.3 iff? What?

There are even more problems with this definition.

First of all what the hell is "iff"? It's the pinnacle of mathematicians' laziness: it's short for "if and only if". (STOP BEING LAZY! Mathematicians are evil, don't be like them.) What does that mean? It wasn't defined. It's something assumed to be intuitively clear: most humans intuitively know "if" and know "only if", even though later we will see that humans have a lot of trouble with both of these when it comes to doing proper logic with them.

In English we have a weird way of saying things out of order: "p if q" means the logical statement $q \implies p$, and "p only if q" means the logical statement $p \implies q$. Here is what we call a "truth table" for these statements:

p	q	$p \implies q$	$q \implies p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

So, "p if and only if q" means the logical statement: "$p \implies q$ and $q \implies p$". We denote "and" with the symbol $\wedge$ and "iff" with the symbol $\iff$ (even more definitions!) This means that, for "p if and only if q" to be true, p and q must have the same truth value: they are either both true, or both false:

p	q	$(p \implies q) \wedge (q \implies p)$	$p \iff q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Many math professors have a habit of using "iff" in definitions. For example, for the definition of divisibility, this means that:

IF (not iff) there exists an integer k such that $ak = b$ THEN a and b "satisfy the definition of $a \mid b$" so we get to say "a divides b" as much as we want; similarly

IF (not iff) $a \mid b$ THEN we can INVOKE THE DEFINITION to conlude that there exists an integer k such that $ak = b$.

So a definition goes both ways: it completely *identifies a "name"* (like "a divides b") with *a property* (like "there exists an integer k such that $ak = b$). The name implies the property, and the property implies the name.

2.4 A good definition

Second, what are a and b? A GOOD DEFINITION SHOULD CLEARLY INDICATE THE *types* OF THINGS IN IT! We have to read the text above the definition in Chapter 8. Prof. Meyer has a convention for that chapter: *Since we'll be focusing on properties of the integers, we'll adopt the default convention in this chapter that variables range over the set, $\mathbb{Z}$, of integers.*

OK, now what is $\mathbb{Z}$? It is the set of all integers, positive, negative or zero: $\mathbb{Z} ::= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (notice I used Prof. Meyer's ::= notation.)

So... after tracking down everything we need, we have the following:

Definition 4. *The set of all integers, $\mathbb{Z}$, is defined as the infinite set:*

$\mathbb{Z} ::= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Definition 5. *Assume a and b are integers. We say that* ***a divides b**, or **b is divisible by a**, denoted with the notation $a \mid b$, if and only if there exists an integer k such that $ak = b$.*

Definition 6. *An integer p is called a **prime number** if and only if it is greater than 1 and it is not divisible by any integer greater than 1 (other than itself).*

This is part and parcel of the daily life of mathematicians and math students. We need to justify everything, and that involves knowing and using definitions. Without commonly agreed upon definitions, proofs don't really mean anything.

2.5 A bad definition

Remember I told you how mathematicians are evil lazy bastards? Out there in the wild you will see a definition that looks like this:

$a \mid b \Longleftrightarrow \exists k\,(ak = b)$.

Wow. Great.

There are many issues with this: first of all, it's written like this out of laziness. Second, it uses formal logical notation (with quantifiers, symbols and the like) for something that should be defined using the meta-language (English). Confusing or blending the formal language with the meta-language creates a lot of problems for learners. Third, it obscures the types of the variables. I don't know if these are integers or real numbers or graphs or trees or what.

2.6 Quantifiers, free vs. quantified variables

Prof. Meyer introduces the universal quantifier "for all": $\forall$ and gives examples of it in mathematical statements, like:

$$\forall a, b, c, d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4$$

First of all I don't like this DOT notation, I've never learned it or seen it anywhere else before. We normally use parentheses, which do a much better job of indicating the SCOPE of a quantifier:

$$\forall a, b, c, d \in \mathbb{Z}^+(a^4 + b^4 + c^4 \neq d^4)$$

This means that all the OCCURRENCES of the variables a, b, c, d INSIDE parentheses are "affected" by the universal quantifier $\forall$. If they occurred outside of the parentheses they would not be affected by the quantifier. Let me give a made-up example:

$$(\forall a, b, c, d \in \mathbb{Z}^+(a^4 + b^4 + c^4 \neq d^4)) \implies a + b = \frac{c}{d}$$

Here the occurrences of a, b, c, d in $a + b = \frac{c}{d}$ are NOT affected by "for all". The "for all" does not apply to them.

Unfortunately Prof. Meyer does not teach us about the two kinds of occurrences of a variable: **free** occurrences and **quantified** occurrences.

Definition 7. *A variable in a statement is called free if it does not occur within the scope of any quantifier. It's called quantified if it occurs within the scope of a quantifier.*

In the above made-up example I gave, the occurrences of a, b, c, d inside the parentheses are QUANTIFIED, while the occurrences in $a + b = \frac{c}{d}$ are FREE.

A free occurrence means: we can "plug-in" a value for that variable. A quantified occurrence means: we cannot plug-in a value for that variable. Imagine "plugging-in" $a = 1$ in Prof. Meyer's example:

$$\forall 1, b, c, d \in \mathbb{Z}^+(1^4 + b^4 + c^4 \neq d^4)$$

Here, the "for all 1" is complete nonsense! So this kind of operation is illegal.

But it would make sense to "plug-in" $a = 1$ in the second part of my made up example:

$$(\forall a, b, c, d \in \mathbb{Z}^+(a^4 + b^4 + c^4 \neq d^4)) \implies 1 + b = \frac{c}{d}$$

This still makes logical sense. But wait! How can we plug-in a value for one a and not the other? Shouldn't we replace all occurrences of a with the value 1? This is a valid complaint. That's why we would first RENAME THE VARIABLE in order to avoid a CLASH between free and quantified occurrences of the same variable: first

$$(\forall a, b, c, d \in \mathbb{Z}^+(a^4 + b^4 + c^4 \neq d^4)) \implies x + b = \frac{c}{d}$$

and then plug-in $x = 1$ safely:

$$(\forall a, b, c, d \in \mathbb{Z}^+(a^4 + b^4 + c^4 \neq d^4)) \implies 1 + b = \frac{c}{d}$$

3 1.2 Predicates

A common confusion I see with learners regards predicates. Prof. Meyer says:

If P is a predicate, then P(n) is either true or false, depending on the value of n.

This is true but confusing! This means that:

$P(1)$, $P(2)$, $P(3), \ldots$ all have truth values: they are either true or false;

if we add quantifiers, say $\forall n.P(n)$ is either true or false;

BUT $P(n)$ has no truth value! It's neither true nor false!

Unfortunately Prof. Meyer does not mention this. I see learners make the mistake of trying to give something like $P(n)$ a truth value in proofs, and getting stuck or doing wrong things.

Think about it from your algebra knowledge. It would be like saying: $x^2 + 3 < 5$ is true or false. FOR WHICH x?

3.1 Formulas vs Sentences

First, if $P(n)$ (without a specific value "plugged-in" for n) is not true or false, then WHAT IS IT? It's just a syntactic expression.

Second, what is the reason behind it not having a truth value? Unfortunately Prof. Meyer does not teach us about FORMULAS versus SENTENCES.

Definition 8. *A formula is any syntactically valid, well-formed expression.*

$P(1)$, $P(2)$, $P(3), \ldots$, $P(n)$ and $\forall n(P(n))$ are all examples of FORMULAS. Formulas are simply any syntactically valid expressions. Some formulas do not have truth values. They are neither true nor false: for example: $x^2+3 < 5$ is a formula with a free variable x, which is not true or false.

Definition 9. *A sentence is a formula with no free variables.*

$P(1)$, $P(2)$, $P(3), \ldots$ and $\forall n(P(n))$ are examples of SENTENCES. A sentence is a formula that contains no free variables. Sentences always have a truth value: they are either true or false.

For example, here is a formula that is not a sentence:

$$x + y < 4$$

There are two free variables, x and y. The following IS a sentence:

$$\forall x(x + 2 < 4)$$

y was replaced by 2 and x is now within the scope of a quantifier, so now there are no more any free variables!

4 1.3 The Axiomatic Method

There are lots of definitions here: *axiom, proof, theorem, lemma, corollary.* Pay attention, they are asked in the multiple-choice quiz of the online version of the course.

5 1.4 Our axioms

Prof. Meyer says: *So instead of starting with ZFC, we're going to take a huge set of axioms as our foundation: we'll accept all familiar facts from high school math.*

That's a lot of assumptions! Next is probably the most important statement of the entire book/course:

This will give us a quick launch, but you may find this imprecise spec of the axioms troubling at times. For example, in the midst of a proof, you may start to wonder, "Must I prove this little fact or can I take it as an axiom?" There really is no absolute answer, since what's reasonable to assume and what requires proof depends on the circumstances and the audience. A good general guideline is simply to be up front about what you're assuming.

YES! You've read my mind, sir. That's why I have been obsessively pouring over hidden assumptions and definitions in my commentaries. Because it's hard for new learners to decide "what's reasonable to assume".

5.1 1.5 Logical Deductions

Prof. Meyer says:

As with axioms, we will not be too formal about the set of legal inference rules. Each step in a proof should be clear and "logical"; in particular, you should state what previously proved facts are used to derive each new conclusion.

This is both good and bad news. It would be too much work to write down and use all the rules formally. Prof. Meyer simply assumes that these rules must be intuitively obvious for you. But in my experience for many learners they are not obvious at all. When I work with learners they are not aware what rules they are using in their proofs at all. They keep using UNSOUND rules that they made up in their minds. (Meyer gives an example of an unsound rule.)

Especially when it comes to manipulating quantifiers: learners don't understand the rules of "universal elimination" and "universal introduction":

Universal Elimination: if we have a proof of $\forall x(P(x))$ and v is any value in the domain (not a variable, an ACTUAL value, like 5), this rule gives us a proof of $P(v)$.

This rule allows us to argue the specific from the general. The intuition is that: "if the statement is true for all the values, then the statement should be true for a specific thing too. I can plug-in whatever value I want, and the result would be true."

In more practical terms, let's see an example: say x ranges over the domain of all natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let's say that we have a proof of $\forall x(P(x))$ for some predicate P, like: $P(x) ::= x < x + 1$. Using Universal Elimination we can prove that $P(5) = 5 < 6$ is true, $P(12345) = 12345 < 12346$ is true, $P(10^{10})$ is true, and so on.

In this example, even though this is not legal, you can think of $\forall x(P(x))$ as an "infinite conjunction": $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \ldots$ (remember $\wedge$ means "and".) So if we know this infinite "and chain" is true, then we can conclude that each one of the parts should also be true. That's the idea.

Universal Introduction: if we have proofs of $P(v)$ for all the values v in the domain (actual values), this gives us a proof of $\forall x(P(x))$.

This rule allows us to argue the general from an exhaustive collection of specifics. The intuition is that: "if I know the statement to be true individually for each value in the domain, then I can generalize over all these values and conclude that the statement is true for all the values in the domain. I can "un-plug" the values and replace them with a variable, add a universal quantifier and the result would be true."

For example, let's say that the domain we are considering is the set $A = \{1, 2, 3\}$ and we know $P(1)$ is true, $P(2)$ is true, $P(3)$ is true. Since that covers everything in the domain we should be able to conclude $\forall x(P(x))$. That's the idea. This rule captures this idea.

Basically: these two rules can be thought of as ways to convert "and chains" into universally quantified sentences, and vice versa.

There are two more, similar rules, for the existential quantifier $\exists$. These are, informally speaking, ways to convert "or chains" into existentially quantified sentences and vice versa. For example if the domain is $A = \{1, 2\}$ and we know $P(1) \vee P(2)$ then we should be able to conclude $\exists x(P(x))$.

5.2 The Inference Rules for your convenience!

Here is a list of sound inference rules that capture most of our thinking. There are many alternative rulesets out there, this is just one flavor. The idea is that, you don't have to memorize these or explicitly say which rule you are using in your proofs, but if you are stuck, or you are not sure of what you are doing, it can be helpful to take a look.

You should be able to convince yourself that these rules are sound; try proving a few of them if you'd like. Most of them are self-explanatory, I added some explanations for a few. Here $\neg$ means: "not". It turns True to False, and False to True.

And Introduction

$$\frac{A \quad B}{A \wedge B}$$

And Elimination: (1)

$$\frac{A \wedge B}{A}$$

And Elimination: (2)

$$\frac{A \wedge B}{B}$$

Or Introduction: (1)

$$\frac{A}{A \vee B}$$

Or Introduction: (2)

$$\frac{B}{A \vee B}$$

Or Elimination (Proof by Cases)

$$\frac{A \vee B \quad A \implies C \quad B \implies C}{C}$$

Negation Introduction (Proof by Contradiction)

$$\frac{A \implies B \quad A \implies \neg B}{\neg A}$$

Negation Elimination

$$\frac{\neg\neg A}{A}$$

Biconditional Introduction

$$\frac{A \implies B \quad B \implies A}{A \iff B}$$

Biconditional Elimination: (1)

$$\frac{A \iff B}{A \implies B}$$

Biconditional Elimination: (2)

$$\frac{A \iff B}{B \implies A}$$

Implication Introduction

$$\frac{\frac{A}{B}}{A \implies B}$$

(Here the idea is that: if we have a proof that by assuming A, we can obtain B, then the rule allows us to conclude $A \implies B$.)

Implication Elimination (Modus Ponens)

$$\frac{A \quad A \implies B}{B}$$

Proof by Contrapositive (Modus Tollens)

$$\frac{A \implies B \quad \neg B}{\neg A}$$

(Here the idea is that $A \implies B$ is logically equivalent to its contrapositive: $\neg B \implies \neg A$ (draw a truth table to convince yourself!), so it's like Modus Ponens with $\neg B$ and $\neg B \implies \neg A$ as antecedents.)

Universal Introduction

$$\frac{P(v) \text{ for all values } v \text{ in the domain}}{\forall x(P(x))}$$

Universal Elimination

$$\frac{\forall x(P(x))}{P(v) \text{ where } v \text{ is any value in the domain}}$$

Existential Introduction

$$\frac{P(v) \text{ for some value } v \text{ in the domain}}{\exists x(P(x))}$$

Existential Elimination

$$\frac{\exists x(P(x))}{P(v) \text{ where } v \text{ is some value in the domain}}$$

6 1.5 Proving an Implication

6.1 1.5.1 Method #1: "Direct Proof"

Here Prof. Meyer mentions doing some "scratch work" to see why an implication is true. THIS IS VERY IMPORTANT!

Proofs require creativity, and selecting *just the right facts* from our background axioms (which, as you know, is a HUGE pool of facts, including all high school math), so you should "sketch" the overall idea of the proof first.

Remember the bogus proofs we did in the In-Class problems? The ones that were correct if done in reverse order? Those bogus proofs were basically scratch work to "discover" the correct proof. Then we applied our thinking meat to realize how the logic should go, and how the proof should be written.

Even more important, is to know that THE SCRATCH IS NOT PROOF! You need to comb through your scratch work carefully, making sure you justify everything, and explicitly state what you are assuming.

6.2 1.5.2 Method #2: "Prove the contrapositive"

When we do proofs of this kind, we are implicitly using the Modus Tollens inference rule that we mentioned above. Simply follow Prof. Meyer's suggestion: *Write, "We prove the contrapositive:" and then state the contrapositive.*

6.2.1 Rational definition

Ah, here comes another dreaded definition (again hidden in the text instead of being declared): *A number is rational when it equals a quotient of integers —that is, if it equals m/n for some integers m and n.*

This is such a loaded definition and there are a lot of things we need to clarify; once again Prof. Meyer is being very casual about it. For example, could n be equal to 0? Is m positive or negative? Is n positive or negative? Do m and n have any common factors (so that you could "cancel them out" and "simplify" the fraction)?

On the face of it, the definition is fine. Because for the number to be equal to a quotient of integers, the quotient of integers has to be defined in the first place. So if the number is equal to a quotient of integers, then we can assume that the integer in the denominator is not zero.

Another issue is that the definition does not express its overall meaning: is it an existential statement, is it a universal statement? We have to interpret the English, and realize that this is actually an existential statement: *"when r equals a quotient of integers"* means: **there exist two integers m and n such that $r = \frac{m}{n}$.**

6.2.2 A useful example of case analysis, and WLOG

Another issue is the multiple different ways to represent the same rational number: for example $\frac{1}{2}$, $\frac{-1}{-2}$ and $\frac{2}{4}$ are all equal. But when we are doing a proof, we don't get to pick the m and n. Should we settle down on a "best default choice" with certain properties? For example, if m and n have common factors that can be "cancelled out" that might create difficulties for us when we are trying to give some sort of divisibility argument. Or the sign of m and n can give us trouble if we are trying to prove an inequality.

If m and n have common divisors greater than 1, then we can divide both of them by all those divisors until they have no common divisors left greater than 1.

Assuming m and n have no common divisors greater than 1 for the remainder of the discussion, consider the case when $r = 0$. Then m must be 0, and n could be any non-zero integer. We can replace n with 1.

Consider the case when $r > 0$. There are two subcases: either both $m, n > 0$ or both $m, n < 0$. In the second case, we could replace m and n with $-m$ and $-n$, which puts us back in the first case (where $m, n > 0$).

Consider the case when $r < 0$. There are two subcases: either $m > 0$ and $n < 0$, or $m < 0$ and $n > 0$. Depending on what we are trying to prove, we might want to have either m or n to be always positive, to make things easier. In the first case, we could replace m and n with $-m$ and $-n$, which puts us back in the second case (where $m < 0$ and $n > 0$).

In all cases we can guarantee $n > 0$. All of this is OK because we still maintain $r = \frac{m}{n}$. So we do not lose any generality.

In mathematics you can simplify some cases to other cases, but you have to make sure not to overlook any case, or lose generality because of the assumptions that simplified all possible cases down to just a small few. This type of argument is so common, that lazy mathematicians have an abbreviation for it: "WLOG..." which stands for "without loss of generality..."

Let's settle down on the following definition:

Definition 10. *A real number r is called **rational** if and only if there exist integers m and n such that $r = \frac{m}{n}$ where $n > 0$ and m and n have no divisors in common greater than 1.*

I think this is a much more useful, clarified definition. If you have to INVOKE this definition in one of your proofs, and you need m and n to have particular signs for the sake of your proof, WLOG you can change the assumption $n > 0$ to $n < 0$, or you can give the above case analysis depending on what you want to do.

7 1.6 Proving an "if and only if"

Prof. Meyer suggests two methods: (1) Proving each statement implies the other, and (2) Constructing a chain of "iff"s.

In practice I would recommend that you almost always use (1). I've seen learners struggle quite a bit with method (2), whether using such method themselves, or simply reading a proof that uses method (2).

If you remember the bogus proofs from the In-class problems, they were written in the style of method (2), but they were not chains of iffs. It was not clear if the implications were iffs, we had to prove each implication in reverse, and it took a lot of time and effort. It is very easy to make a mistake in one of the equivalences.

Prof. Meyer warns about method (2): *This method sometimes requires more ingenuity than the first, but the result can be a short, elegant proof.* At this point in your math career, don't worry about elegance, and stay away from ingenuity. Get the basics down solid instead.

Generally speaking, an equivalence (iff) is a much stronger statement than just a one-way implication ($\implies$, or "if... then..."). So it's harder to prove. Using method (1) allows you to break what needs to be proved into smaller, more manageable chunks. The "chain of iffs" is suitable the most when the proof boils down to equations.