Summary: Calculus in Polar Coordinates

The area element

Passing to the differential, the area element dA in polar coordinates is the area of the infinitesimal region spanned by $d\theta$, given by

$$dA = \frac{1}{2} r(\theta)^2 d\theta. \tag{1}$$

$$\tag{2}$$

The area bounded a curve $r = r(\theta)$, and the two rays $\theta = \theta_1$ and $\theta = \theta_2$ is

$$A = \int_{\theta_1}^{\theta_2} dA = \frac{1}{2} \int_{\theta_1}^{\theta_2} r(\theta)^2 d\theta$$

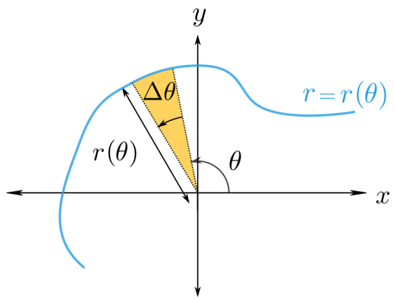
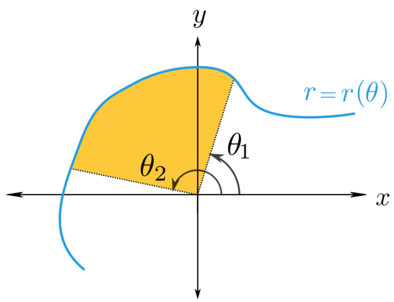


Figure 1: $\Delta A \approx \frac{1}{2} r^2 \Delta\theta$



Examples of polar curves

Lines:	$y = b:\quad r = \frac{b}{\sin(\theta)} \quad (0 < \theta < \pi)$ $x = b:\quad r = \frac{b}{\cos(\theta)} \quad (-\pi/2 < \theta < \pi/2)$ $y = mx + b:\quad r = \frac{b}{\sin(\theta) - m\cos(\theta)} \quad (\arctan(m) < \theta < \arctan(m) + \pi)$
Circles: (touching origin)	centered on x-axis: $r = 2b\cos(\theta) \quad (-\pi/2 \leq \theta \leq \pi/2)$ centered on y-axis: $r = 2b\sin(\theta) \quad (0 \leq \theta < \pi)$ centered on $\theta = \alpha$ ray: $r = 2b\cos(\theta - \alpha) \quad (\alpha - \pi/2 \leq \theta < \alpha + \pi/2)$
Roses Limacons Cardioids	$r = A + B\cos(n\theta)$ (Domain varies) $r = A + B\sin(n\theta)$ (Domain varies)
Spirals:	$r = n\theta \quad (0 \leq \theta)$
Conics: (with one focus at the origin)	Hyperbolas: $r = \frac{1}{1+b\cos(\theta)},\ \lvert b\rvert > 1 \quad (0 \leq \theta < 2\pi, 1 + b\cos(\theta) \neq 0)$ Parabolas: $r = \frac{1}{1+b\cos(\theta)},\ \lvert b\rvert = 1 \quad (0 \leq \theta < 2\pi, 1 + b\cos(\theta) \neq 0)$ Ellipses: $r = \frac{1}{1+b\cos(\theta)},\ \lvert b\rvert < 1 \quad (0 \leq \theta < 2\pi)$