Summary: Singularities and improper integrals

Definition of singularity

A **singularity** of a function $f(x)$ is a point $x = s$ such that the function $f(x)$ does not exist at $x = s$.

There are three main ways that the function can fail to exist at a point:

- $\lim_{x\to s^+} |f(x)|$ and/or $\lim_{x\to s^-} |f(x)|$ tends to ∞. This is the case of most interest in this section.
- $\lim_{x\to s^\pm} f(x)$ does not exist. In this case the function f may oscillate, or have a jump discontinuity.
- $\lim_{x\to s^\pm} f(x)$ exists and is finite. In this case, the function f has a removable discontinuity.

Definition of improper integrals of the 2nd type

An **improper integral of the 2nd type** is an integral $\int_a^b f(x)\,dx$ such that the function $f(x)$ has a singularity at $x = s$ for some s with $a \leq s \leq b$.

For example, if $f(x)$ has a singularity at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{C\to b^-} \int_a^C f(x)\,dx.$$

We say

- the integral **converges** if the limit exists and is finite.
- the integral **diverges** if the limit does not exist (which includes the case that the limit is $\pm\infty$.)

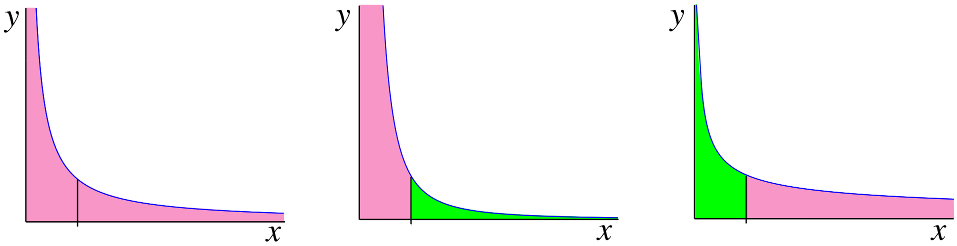

Figure 1: From left to right, we see the areas for $0 \leq x \leq 1$ and $1 \leq x < \infty$ under the graphs of $\frac{1}{x}$, $\frac{1}{x^2}$, and $\frac{1}{\sqrt{x}}$. The areas shaded in pink are infinite. The areas shaded in green are finite.

Overview of improper integrals

The improper integral $\displaystyle\int_a^\infty \frac{dx}{x^p}$ $\begin{cases} \text{diverges} & \text{if } p \leq 1 \\ \text{converges to } \frac{a^{1-p}}{p-1} & \text{if } p > 1 \end{cases}$.

The improper integral $\displaystyle\int_0^a \frac{dx}{x^p}$ $\begin{cases} \text{diverges} & \text{if } p \geq 1 \\ \text{converges to } \frac{a^{1-p}}{1-p} & \text{if } p < 1 \end{cases}$

Comparison tests for improper integrals of 2nd type

Suppose that $f(x)$ and $g(x)$ both have a singularity at $x = s$.

Suppose $f(x) > g(x) \geq 0$ for all $a \leq x \leq b$ except at $x = s$.

If $\displaystyle\int_a^b f(x)\,dx$ converges, then $\displaystyle\int_a^b g(x)\,dx$ converges also.

If $\displaystyle\int_a^b g(x)\,dx$ diverges, then $\displaystyle\int_a^b f(x)\,dx$ diverges also.

More notation

Suppose that $f(x)$ and $g(x)$ have a singularity at $x = s$. ($f,\ g \longrightarrow \pm\infty$ as $x \to s^+$ and/or as $x \to s^-$.)

We say that $f(x)$ is **similar** to $g(x)$, and write $f(x) \sim g(x)$ as $x \to s^+$ or $x \to s^-$ if

$$\frac{f(x)}{g(x)} \longrightarrow \qquad 1 \qquad \text{as} \quad x \longrightarrow s^{\pm}. \tag{1}$$

We say that $f(x)$ grows faster than $g(x)$ as x tends towards s, and write

$$f(x) >> g(x) \text{ as } x \to s^{\pm}, \text{ if } \begin{cases} f(x) \longrightarrow \infty \\ g(x) \longrightarrow \infty \\ \frac{g(x)}{f(x)} \longrightarrow 0 \end{cases} \quad \text{as} \quad x \longrightarrow s^{\pm}.$$

Limit comparison tests for improper integrals of 2nd type

Suppose that $f(x)$ and $g(x)$ both have a singularity at $x = s$.
Suppose $f(x), g(x) \geq 0$ for all $a \leq x \leq b$ except at $x = s$.

1. If $f(x) \sim g(x)$ as $x \to s^{\pm}$,
   then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either **both converge** or **both diverge**.

2. Suppose that $f(x)$ grows faster than $g(x)$ as x tends towards $s^{\pm}$. In other words, $f(x) >> g(x)$ as $x \to s^{\pm}$.

   - If $\int_a^b f(x)\,dx$ converges, then $\int_a^b g(x)\,dx$ converges.
   - If $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.

Note that this notation is exactly the same notation that we had before. The only difference is that instead of having $x \to \infty$, we have $x \to s^+$ or $x \to s^-$, where s is a finite number that is a singularity of the function of interest.