

Math for CS 2015/2019 Problem Set 9 solutions

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1 Problem 1

Assuming the following sum equals a polynomial in n , find the polynomial. Optionally, you might want to use induction to prove that the sum equals the polynomial you find, but no such proof is required for full credit.

$$\sum_{i=1}^n i^3$$

Proof. 1. Assume the above sum equals a polynomial $p(n)$. Let us try to guess a formula for the polynomial by looking at some cases and generalizing.

$$2. p(1) = \sum_{i=1}^1 i^3 = 1$$

$$p(2) = \sum_{i=1}^2 i^3 = 1 + 8 = 9 = 3^2 = 1^2 \cdot 3^2$$

$$p(3) = \sum_{i=1}^3 i^3 = 9 + 27 = 36 = 2^2 \cdot 3^2$$

$$p(4) = \sum_{i=1}^4 i^3 = 36 + 64 = 100 = 2^2 \cdot 5^2$$

$$p(5) = \sum_{i=1}^5 i^3 = 100 + 125 = 225 = 3^2 \cdot 5^2$$

$$p(6) = \sum_{i=1}^6 i^3 = 225 + 216 = 441 = 3^2 \cdot 7^2$$

3. It seems to be always the product of two squares. But the numbers that are squared change.

For $n = 2$ we have the numbers 1 and 3, which are $n - 1$ (or $n/2$) and $n + 1$.

For $n = 3$ we have the numbers 2 and 3, which are $n - 1$ (or $(n + 1)/2$ and n .

For $n = 4$ we have the numbers 2 and 5, which are $n/2$ and $n + 1$.

For $n = 5$ we have the numbers 3 and 5, which are $(n + 1)/2$ and n .

4. So it seems like one of the numbers gets divided by 2, and it keeps alternating between n and $n + 1$ depending on which one is odd/even.

5. In the above cases we see possibilities like $p(n) = n^2((n+1)/2)^2$ or $p(n) = (n/2)^2(n+1)^2$.

6. Let's guess: $p(n) = \left(\frac{n(n+1)}{2}\right)^2$.

7. Let's try to prove this formula by induction to make sure that our guess is correct. We claim: for all $n \geq 1$:

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

8. For the base case $n = 1$ we observe

$$\sum_{i=1}^1 i^3 = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$$

so the base case holds.

9. For the induction step, assume that $n \geq 1$ and assume

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

We want to prove

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

10. Let's write

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^n i^3 + (n+1)^3$$

11. By (10) and the induction hypothesis (9) we get

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

12. Rewriting the right hand side of (11):

$$\left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2 \left(\frac{n^2}{4} + n + 1\right)$$

13. Rewriting the right hand side of (12):

$$(n+1)^2 \left(\frac{n^2}{4} + n + 1 \right) = (n+1)^2 \frac{n^2 + 4n + 4}{4} = (n+1)^2 \frac{(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2} \right)^2$$

14. By (11), (12) and (13) we have

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2} \right)^2$$

so we have proved the induction step. By the Principle of Mathematical Induction our claim in (7) holds for all $n \geq 1$. \square

2 Problem 2

Show that

$$\ln(n^2!) = \Theta(n^2 \ln n)$$

Hint: Stirling's formula for $(n^2)!$.

Proof. 1. By Stirling's formula

$$(n^2)! = \sqrt{2\pi n^2} \left(\frac{n^2}{e} \right)^{n^2} e^{\epsilon(n^2)}$$

where $\frac{1}{12n^2+1} \leq \epsilon(n^2) \leq \frac{1}{12n^2}$. ($\epsilon(n^2) \rightarrow 0$ as $n \rightarrow \infty$.)

2. Applying $\ln(\cdot)$ to both sides of (1) and using the fact that $\ln(xy) = \ln(x) + \ln(y)$:

$$\ln(n^2!) = \ln(\sqrt{2\pi n} \left(\frac{n^2}{e} \right)^{n^2} e^{\epsilon(n^2)}) = \ln(\sqrt{2\pi}) + \ln(n) + \ln\left(\left(\frac{n^2}{e}\right)^{n^2}\right) + \ln(e^{\epsilon(n^2)})$$

3. Using the fact that $\ln(x^y) = y \ln(x)$ on (2) we get

$$\ln(n^2!) = \frac{1}{2} \ln(2\pi) + \ln(n) + n^2 \ln\left(\frac{n^2}{e}\right) + \epsilon(n^2) \ln(e)$$

4. Using $\ln(xy) = \ln(x) + \ln(y)$ and the fact that $\ln(e) = 1$ on (3):

$$\ln(n^2!) = \frac{1}{2} \ln(2\pi) + \ln(n) + n^2(2 \ln(n) - 1) + \epsilon(n^2)$$

5. Divide both sides of (4) by $n^2 \ln(n)$:

$$\frac{\ln(n^2!)}{n^2 \ln(n)} = \frac{\ln(2\pi)}{2n^2 \ln(n)} + \frac{1}{n^2} + 2 - \frac{1}{\ln(n)} + \frac{\epsilon(n^2)}{n^2 \ln(n)}$$

6. Take limits of both sides as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \frac{\ln(n^2!)}{n^2 \ln(n)} = \lim_{n \rightarrow \infty} \frac{\ln(2\pi)}{2n^2 \ln(n)} + \lim_{n \rightarrow \infty} \frac{1}{n^2} + 2 - \lim_{n \rightarrow \infty} \frac{1}{\ln(n)} + \lim_{n \rightarrow \infty} \frac{\epsilon(n^2)}{n^2 \ln(n)} = 2$$

7. (6) proves that $\ln(n^2!) = O(n^2 \ln(n))$ (we can let $c = 2$ and n_0 be sufficiently large in the definition of the Big-O notation). But (6) also proves

$$\lim_{n \rightarrow \infty} \frac{n^2 \ln(n)}{\ln(n^2!)} = \frac{1}{2}$$

so we also have $\ln(n^2 \ln(n)) = O(n^2!)$.

8. With these two facts, by definition of Big-Theta, $\ln(n^2!) = \Theta(n^2 \ln n)$ □

3 Problem 3

Prove that

$$\sum_{k=1}^n k^6 = \Theta(n^7)$$

Hint: One solution uses the Integral Method, and there are other workable approaches that avoid calculus.

Proof. 1. Consider the graph of $y = x^6$ on the interval $[0, n + 1]$. We can use integrals to both under-estimate and over-estimate $\sum_{k=1}^n k^6$. Divide the interval into equal subintervals $[0, 1], \dots, [n, n + 1]$ of width 1.

2. We can place rectangles of width 1 and heights $1^6, 2^6, \dots, k^6, \dots, n^6$ under the graph of $y = x^6$ from 0 to $n + 1$. This gives us

$$\sum_{k=1}^n k^6 \leq \int_0^{n+1} x^6 dx$$

3. We can place rectangles of width 1 and heights $1^6, 2^6, \dots, k^6, \dots, n^6$ to completely cover the area under the graph of $y = x^6$ from 0 to n . This gives us

$$\int_0^n x^6 dx \leq \sum_{k=1}^n k^6$$

4. Putting together (2) and (3) we have

$$\int_0^n x^6 dx \leq \sum_{k=1}^n k^6 \leq \int_0^{n+1} x^6 dx$$

5. Evaluating the integrals by using Fundamental Theorem of Calculus,

$$\frac{n^7}{7} \leq \sum_{k=1}^n k^6 \leq \frac{(n+1)^7}{7}$$

6. Divide by n^7 :

$$\frac{1}{7} \leq \frac{\sum_{k=1}^n k^6}{n^7} \leq \frac{(n+1)^7}{7n^7}$$

7. Notice that $\lim_{n \rightarrow \infty} \frac{(n+1)^7}{7n^7} = 1/7$. Therefore by the Squeeze Theorem and (6), we get

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n k^6}{n^7} = \frac{1}{7}$$

This proves that $\sum_{k=1}^n k^6 = O(n^7)$ (we can choose $c = 1/7$ and sufficiently large n_0).

8. By (7) we get

$$\lim_{n \rightarrow \infty} \frac{n^7}{\sum_{k=1}^n k^6} = 7$$

So similarly we have $n^7 = O(\sum_{k=1}^n k^6)$. Putting together these two facts we get $\sum_{k=1}^n k^6 = \Theta(n^7)$. \square