Summary: Improper integrals

Consider $f, g > 0$. We say that g grows faster than f as x tends towards ∞, and write this as

$$0 < f(x) << g(x),$$

if $\quad f, g \longrightarrow \infty \quad$ and $\quad \dfrac{f(x)}{g(x)} \longrightarrow 0 \quad$ as $\quad x \to \infty$.

Examples

For $p > 0$,
$\ln x << x^p << e^x << e^{x^2}$.

Note that $e^{x^2} = e^{(x^2)}$, not $(e^x)^2 = e^{2x}$.

Rate of Decay (as $x \to \infty$)

Consider $f, g > 0$. We say that f decays faster than g as x tends towards ∞, and write this as

$$g(x) >> f(x) > 0,$$

if $\quad f, g \longrightarrow 0 \quad$ and $\quad \dfrac{f(x)}{g(x)} \longrightarrow 0 \quad$ as $\quad x \to \infty$.

Examples

For $p > 0$,

$$\frac{1}{\ln x} >> \frac{1}{x^p} >> e^{-x} >> e^{-x^2}.$$

Improper integrals definition

An **improper integral** is defined by $\displaystyle\int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx$.

This improper integral **converges** if the limit exists and is finite.
This improper integral **diverges** if the limit does not exist (this includes when the limit is $\pm\infty$).

Conclusion of example: powers of x

$$\int_a^\infty \frac{dx}{x^p} \qquad \begin{cases} \text{diverges} & \text{if } \ p \le 1 \\ \text{converges to } \frac{a^{-p+1}}{p-1} & \text{if } \ p > 1 \end{cases}$$

Notation

We say that $f(x) \sim g(x)$ as $x \to \infty$ if $\dfrac{f(x)}{g(x)} \underset{x\to\infty}{\longrightarrow} 1.$

In words, we say that $f(x)$ and $g(x)$ are **similar** as $x \to \infty$. (The idea is that $f(x)$ and $g(x)$ have the same asymptotic behavior as x tends to infinity.)

Limit comparison

The idea behind limit comparison is that if the asymptotic behavior is the same, then the improper integrals have the same behavior.

If $f(x) \sim g(x)$ as $x \to \infty$,
then the two integrals $\displaystyle\int_a^\infty f(x)\,dx$ and $\displaystyle\int_a^\infty g(x)\,dx$ (for large a) either **both converge** or **both diverge**.

This also works in the case where one function decays faster than the other as x tends towards infinity.

Suppose that $g(x)$ decays faster than $f(x)$ as $x \to \infty$. That is $f(x) >> g(x)$ as $x \to \infty$.

- If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges.
- If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

Comparison

Suppose $f(x) \ge g(x) > 0$ for $x \ge a$.

If $\displaystyle\int_a^\infty f(x)\,dx$ converges, then $\displaystyle\int_a^\infty g(x)\,dx$ converges also.

If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ diverges also.

Definition of singularity

A **singularity** of a function $f(x)$ is a point $x = s$ such that the function $f(x)$ does not exist at $x = s$.

There are three main ways that the function can fail to exist at a point:

- $\lim_{x\to s^+} |f(x)|$ and/or $\lim_{x\to s^-} |f(x)|$ tends to ∞. This is the case of most interest in this section.

- $\lim_{x\to s^\pm} f(x)$ does not exist. In this case the function f may oscillate, or have a jump discontinuity.

- $\lim_{x\to s^\pm} f(x)$ exists and is finite. In this case, the function f has a removable discontinuity.

Definition of improper integrals of the 2nd type

An **improper integral of the 2nd type** is an integral $\int_a^b f(x)\,dx$ such that the function $f(x)$ has a singularity at $x = s$ for some s with $a \le s \le b$.

For example, if $f(x)$ has a singularity at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{C\to b^-} \int_a^C f(x)\,dx.$$

We say

- the integral **converges** if the limit exists and is finite.

- the integral **diverges** if the limit does not exist (which includes the case that the limit is $\pm\infty$.)

Overview of improper integrals

The improper integral $\displaystyle\int_a^\infty \frac{dx}{x^p}$ $\begin{cases} \text{diverges} & \text{if } p \le 1 \\ \text{converges to } \frac{a^{1-p}}{p-1} & \text{if } p > 1 \end{cases}$.

The improper integral $\displaystyle\int_0^a \frac{dx}{x^p}$ $\begin{cases} \text{diverges} & \text{if } p \ge 1 \\ \text{converges to } \frac{a^{1-p}}{1-p} & \text{if } p < 1 \end{cases}$

Comparison tests for improper integrals of 2nd type

Suppose that $f(x)$ and $g(x)$ both have a singularity at $x = s$.

Suppose $f(x) > g(x) \ge 0$ for all $a \le x \le b$ except at $x = s$.

If $\int_a^b f(x)\,dx$ converges, then $\int_a^b g(x)\,dx$ converges also.

If $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges also.

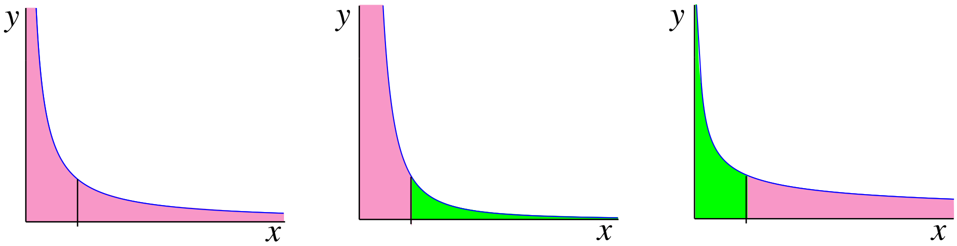

Figure 1: From left to right, we see the areas for $0 \le x \le 1$ and $1 \le x < \infty$ under the graphs of $\frac{1}{x}$, $\frac{1}{x^2}$, and $\frac{1}{\sqrt{x}}$. The areas shaded in pink are infinite. The areas shaded in green are finite.

More notation

Suppose that $f(x)$ and $g(x)$ have a singularity at $x = s$. ($f,\ g \longrightarrow \pm\infty$ as $x \to s^+$ and/or as $x \to s^-$.)

We say that $f(x)$ is **similar** to $g(x)$, and write $f(x) \sim g(x)$ as $x \to s^+$ or $x \to s^-$ if

$$\frac{f(x)}{g(x)} \longrightarrow \qquad 1 \qquad \text{as} \qquad x \longrightarrow s^{\pm}. \tag{1}$$

We say that $f(x)$ grows faster than $g(x)$ as x tends towards s, and write $f(x) >> g(x)$ as $x \to s^{\pm}$, if $\begin{cases} f(x) \longrightarrow \infty \\ g(x) \longrightarrow \infty \\ \frac{g(x)}{f(x)} \longrightarrow 0 \end{cases}$ as $x \longrightarrow s^{\pm}$.

Limit comparison tests for improper integrals of 2nd type

Suppose that $f(x)$ and $g(x)$ both have a singularity at $x = s$.

Suppose $f(x), g(x) \ge 0$ for all $a \le x \le b$ except at $x = s$.

1. If $f(x) \sim g(x)$ as $x \to s^{\pm}$, then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either **both converge** or **both diverge**.

2. Suppose that $f(x)$ grows faster than $g(x)$ as x tends towards $s^{\pm}$. In other words, $f(x) >> g(x)$ as $x \to s^{\pm}$.

   - If $\int_a^b f(x)\,dx$ converges, then $\int_a^b g(x)\,dx$ converges.

- If $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.

Note that this notation is exactly the same notation that we had before. The only difference is that instead of having $x \to \infty$, we have $x \to s^+$ or $x \to s^-$, where s is a finite number that is a singularity of the function of interest.