Summary: Manipulating Taylor Series

Multiplying two power series

We multiply two powers series using the same rule as when we multiply two polynomials.

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, which converges for $|x| < A$, and the power series $\sum_{n=0}^{\infty} b_n x^n$, which converges for $|x| < B$.

- The product of the two power series converges for $|x| < \min(A, B)$, but it *could* have a larger radius of convergence.

Dividing two power series

If

$$F(x) = \frac{G(x)}{H(x)}$$

where $F(x)$, $G(x)$, $H(x)$ are all power series, then we can find $F(x)$ by solving the following equation of power series degree by degree:

$$F(x)H(x) = G(x).$$

The radius of convergence of $F(x)$ is more difficult to track, since $F(x)$ diverges whenever $H(x) = 0$ and $G(x) \neq 0$.

Substitution and Taylor series

We can find the composition of two power series $f(g(x))$ by using similar rules as composition of polynomials. In this course, we will only substitute a polynomial $p(x)$ into a power series $f(x)$.

If the radius of convergence of the power series $f(x)$ is R, then the power series $f(p(x))$ converges whenever $|p(x)| < R$.

In particular, we can find a Taylor series for a function $f(x)$ at $x = 0$ and then substitute in polynomial of the form $p(x) = ax^n$ for x since $p(0) = 0$.

Error function

Recall the **error function** is defined by the following integral formula:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and cannot be expressed in terms of functions that we already know with algebraic operations such that addition and multiplication.

To obtain the Taylor series of the error function, we replace the integrand e^{-t^2} with its Taylor series:

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \\
&= \frac{2}{\sqrt{\pi}} \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) dt \\
&= \frac{2}{\sqrt{\pi}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \right).
\end{aligned}$$

Taylor polynomials

If the Taylor series of $f(x)$ is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (\text{for } |x| < R),$$

then the polynomial

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_n x^n$$

is called the **Taylor polynomial** of degree n of $f(x)$. In other words, a Taylor polynomial is the polynomial obtained by truncating the Taylor series to degree n.

The degree n Taylor polynomial $P_n(x)$ is the best fit degree n polynomial of $f(x)$ at $x = 0$, in the sense that

$$\frac{d^k}{dx^k} P_n(x) = \frac{d^k}{dx^k} f(x)\bigg|_{x=0} \qquad \text{for } 0 < k \leq n.$$

Hence, the degree 1 and degree 2 Taylor polynomials for $f(x)$, are the linear and quadratic approximations of $f(x)$ respectively.

Taylor polynomials are especially useful as for approximating functions, functions that cannot be expressed algebraically in terms of the elementary functions that we know, such as the error function. Often, numerical tools, such as the graphing tool on your calculator or computer, use Taylor polynomials to approximate these functions.

We can use Taylor polynomials to approximate a function with arbitrarily high accuracy inside the radius of convergence of the Taylor series. But how do we know the degree of the Taylor polynomial needed to achieve a certain accuracy? The answer is in the **Taylor remainder theorem** below.

Taylor remainder theorem

Suppose the Taylor series of $f(x)$ is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad \text{for } |x| < R,$$

and let $P_n(x)$ be the degree n Taylor polynomial of $f(x)$:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Then, for any $|x| < R$,

- if $f(x)$, is $n + 1$ times differentiable on the open interval $(0, x)$, that is, $f^{(n+1)}(x)$ and all lower derivatives f exist on $(0, x)$, and
- if $f^{(n)}$ is continuous on the closed interval $[0, x]$,

then there is a number c in $(0, x)$ such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

This is the **Taylor remainder theorem**.

Note that the $n = 0$ case is the Mean Value Theorem (MVT). As in the MVT, we do not know exactly where c is.

Nevertheless, we can use the Taylor remainder theorem to find upper bounds on the error caused by approximating a function by a Taylor polynomial.

Taylor series centered at $x = a$

Let $g(t) = f(t+a)$. That is, g is the translation of f to the left by a.

Recall the Taylor series of $g(t)$ at $t = 0$ is

$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n \qquad (|t| < R).$$

Then

$$\begin{aligned} f(t+a) = g(t) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n \qquad (|t| < R) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} t^n \qquad (|t| < R). \end{aligned}$$

Now let $x = t + a$. In terms of x, the above formula becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad (|x-a| < R).$$

This is the Taylor series of $f(x)$ at $x = a$.

Note that the radius of convergence of the Taylor series of $f(x)$ at $x = a$ is the number R such that $f(x)$ converges when $|x - a| < R$, and diverges when $|x - a| > R$.

If $a = 0$, we get the formula for the Taylor series that we started with in this section. This special case of Taylor's formula gives us a power series often referred to as the **Maclaurin series**.