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Maths Assignment-1

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Q1. The auxiliary equation is: $m^2 - m - 2 = 0$

$$(m-2)(m+1) = 0$$
$$m = 2, -1$$

So, $C.F = \phi_1(y+2x) + \phi_2(y-x)$

$$P.I = \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x = e^x \frac{1}{(D+1)^2 - (D+1)D' - 2D'^2}(y-1)$$

$$= \frac{e^x}{D^2 - DD' - 2D'^2 + 2D - D' + 1}(y-1) = e^x[1 + (2D - 2D' - 2D'^2 - DD' + D^2)]^{-1}(y-1)$$

Using Binomial Theorem:

$$= e^x[1 - 2D + D' + \dots](y+1) = e^x(y-1+1) = ye^x.$$

The complete solution for z is:

$$z = \phi_1(y+2x) + \phi_2(y-x) + ye^x.$$

Q2. A.E. is $m^3 - 7m - 6 = 0$

$m = -1$ satisfies the equation.

$$m^3 - 7m - 6 = (m+1)(m^2 - m - 6) = 0$$
$$= (m+1)(m-3)(m+2) = 0$$
$$= m = -2, 3, -1$$

$\therefore$ $C.F = \phi_1(y-2x) + \phi_2(y+3x) + \phi_3(y-x)$

$$P.I = \frac{1}{D^3 - 7DD'^2 - 6D'^3}[\sin(x+2y) + e^{2x+y}] =$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D^3}\sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{2x+y}$$

$$= \frac{1}{-D + 14D' + 24D'}\sin(x+2y) + \frac{1}{8 - 14 - 6}e^{2x+y}$$

$$= \frac{(-D - 38D')}{(-D + 38D')(-D - 38D')}\sin(x+2y) + \frac{1}{-12}e^{2x+y}$$

$$= \frac{-D-38D'}{-1+4\times 38\times 38}\sin(x+2y) - \frac{1}{12}e^{2x+y} \Rightarrow \frac{-\cos(x+2y)-76\cos(x+2y)}{5775} - \frac{1}{12}e^{2x+y}$$

$$= -\frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

$\therefore$ The complete solution for z is:

$$Z = \phi_1(y-2x) + \phi_2(y+3x) + \phi_3(y-x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

Q3. A.E is $m^2+2m+1-2m-2=0$

$m^2-1=0$

$m=\pm 1$

$(D+D')(D+D'-2)z = \sin(x+2y)$

So, C.F. $= e^{2x}\phi_1(y-x) + \phi_2(y-x)$

$$P.I = \frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(x+2y) = \frac{1}{-1+2(-2)+(-4)-2D-2D'}\sin(x+2y)$$

$$= \frac{-1}{(2D+2D'+9)}\sin(x+2y) = \frac{-(2D+2D'-9)}{(2D+2D'+9)(2D+2D'-9)}\sin(x+2y)$$

$$= -\frac{(2D+2D'-9)}{-4-16-16-81}\sin(x+2y) = \frac{2D+2D'-9}{117}\sin(x+2y)$$

$$= \frac{1}{117}[2\cos(x+2y)+9\cos(x+2y)-9\sin(x+2y)]$$

$$= \frac{1}{39}[2\cos(x+2y)-3\sin(x+2y)]$$

$\therefore$ The complete solution is:

$$Z = e^{2x}\phi_1(y-x) + \phi_2(y-x) + \frac{1}{39}[2\cos(x+2y-3\sin(x+2y)].$$

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Q4. The equation can be written as:

$$(D+D')(D+D'-2) = e^{2x+3y} + xy$$

$$\therefore CF = e^{-2x}\phi_1(y+2x) + \phi_2(y-x)$$

$$P.I = \frac{1}{D^2-DD'-2D'^2+2D+2D'}(e^{2x+3y}+xy)$$

$$= \frac{1}{D^2-DD'-2D'^2+2D+2D'}e^{2x+3y} + \frac{1}{D^2-DD'-2D'^2+2D+2D'}xy.$$

$$= \frac{1}{9-6-18+4+6}e^{2x+3y} + [1+[D^2-DD'-2D'^2+2D+2D'-1]]^{-1}xy.$$

$$= \frac{-e^{2x+3y}}{10} + \underbrace{[1-D^2+DD'+ \cdots]\,xy}_{\text{whole term will be 0 because of partial differentiation terms with } xy}$$

$$P.I = \frac{-e^{2x+3y}}{10}$$

$\therefore$ The solution is

$$z = e^{-2x}\phi_1(y+2x) + \phi_2(y-x) - \frac{1}{10}e^{2x+3y}$$

Q5. Let $u(x,t) = X(x).T(t)$

So, $\frac{\partial u}{\partial x} = X'T$ & $\frac{\partial u}{\partial t} = XT'$

So, $X'T = 2XT' + XT$

$$\frac{X'}{X} = \frac{2T'+T}{T} = \lambda$$

$\Rightarrow X' - \lambda X = 0$ and $T' - \left(\frac{\lambda-1}{2}\right)T = 0$

$X = Ae^{\lambda x}$; $T = Be^{(\lambda-1/2)t}$

$\therefore u(x,t) = Ce^{\lambda x}.e^{\left(\frac{\lambda-1}{2}\right)t}$; $AB = C$

Given, $u(x,0) = 6e^{-3x}$ $\Rightarrow Ce^{\lambda x} = 6e^{-3x}$ $\Rightarrow C = 6$ & $\lambda = -3$

Hence, $u(x,t) = 6e^{-(3x+2t)}$

Q6. Let $u(x,y) = X(x) \cdot Y(y)$

$\Rightarrow \frac{\partial u}{\partial x} = X'Y$, $\frac{\partial u}{\partial y} = XY'$

$\Rightarrow 4X'Y + XY' = 3XY$

$\frac{4X'}{X} = \frac{-Y'+3Y}{Y} = \lambda$

$\therefore X' - \frac{\lambda X}{4} = 0$, $Y' - (3-\lambda)Y = 0$

$X = Ae^{\lambda/4 x}$, $Y = Be^{(3-\lambda)y}$

So, $u(x,y) = Ce^{\frac{\lambda}{4}x} e^{(3-\lambda)y}$

Now, $u(0,y) = 3e^{-y} - e^{-5y}$

$\therefore$ $u(x,y)$ is a sum of two solutions as:

$u(x,y) = c_1 e^{\frac{\lambda_1}{4}x} e^{(3-\lambda_1)y} + c_2 e^{\frac{\lambda_2}{4}x} e^{(3-\lambda_2)y}$

$u(0,y) = c_1 e^{(3-\lambda_1)y} + c_2 e^{(3-\lambda_2)y} = 3e^{-y} - e^{-5y}$

$\therefore$ Either $c_1 = 3$, $\lambda_1 = 4$, $c_2 = -1$, $\lambda_2 = 8$

or $c_1 = -1$, $\lambda_1 = 8$, $c_2 = 8$, $\lambda_2 = 4$

In both cases, solution is $u(x,y) = 3e^{x-y} - e^{2x-5y}$

Q7. Since the ends are fixed:

$y(0,t) = y(l,t) = 0$

$\therefore$ Solution of wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ is given by y

$y(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$ — (1)

Now, $y(0,t) = 0 \Rightarrow A(C\cos cpt + D\sin cpt) = 0 \;\forall t$

$\Rightarrow A = 0$

$\therefore$ By ①, $y(x,t) = (E\cos cpt + F\sin cpt)\sin px$ – ②, $BC = E$ and $BD = F$

Now, $y(l,t) = 0 \Rightarrow (E\cos cpt + F\sin cpt)\sin pl = 0 \quad \forall t$

$\Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}$, $n \in N$

$\therefore$ Solutions are:

$$y(x,t) = \left[E_n\cos\left(\frac{n\pi ct}{l}\right) + F_n\sin\left(\frac{n\pi ct}{l}\right)\right]\sin\frac{n\pi x}{l} \quad ; n = 1, 2, \ldots$$

By principle of superposition,

Solution is given by,

$$y(x,t) = \sum_{n=1}^{\infty}\left[E_n\cos\frac{n\pi ct}{l} + F_n\sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \quad -③$$

Now, $y(x,0) = y_0\sin^3\frac{\pi x}{l}$

$$\therefore y(x,0) = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$\therefore$ From eq ③;

$$\sum_{n=1}^{\infty} E_n\sin\frac{n\pi x}{l} = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

Comparing both sides, we get:

$E_1 = \frac{3y_0}{4}$, $E_2 = 0$, $E_3 = -\frac{y_0}{4}$, $E_n = 0 \ \forall \ n \geq 4$

$$\therefore \text{By ③, } y(x,t) = \frac{y_0}{4}\left[3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l} + \sum_{n=1}^{\infty}F_n\sin\left(\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}\right] \quad -④$$

Now, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ (given)

$\therefore$ From ④: $\sum_{n=1}^{\infty} F_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = 0 \Rightarrow F_n = 0 \ \forall n.$

$\therefore$ Solution is

$$y(x,t) = \frac{y_0}{4}\left(3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right]$$

Q8. Since boundary conditions at $x=0$ are trigonometric functions, therefore sol^n of pde $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$

$$u(x,t) = (A\cos px + B\sin px)(C\cos 2pt + D\sin 2pt) \quad -①$$

Now, $u(0,t) = \sin t$ (given)

$\therefore$ By ① $A(C\cos 2pt + D\sin 2pt) = \sin t$

$\therefore AC=0,\ AD=1,\ 2p=1 \Rightarrow C=0,\ AD=1,\ p=1/2$

$\therefore$ By ①, $u(x,t) = \cos\frac{x}{2}\sin t + E\sin\frac{x}{2}\sin t \quad -②$ (where $BD=E$)

$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2}\sin\frac{x}{2}\sin t + \frac{1}{2}E\cos\frac{x}{2}\sin t.$

Given: $\left(\frac{\partial u}{\partial x}\right)_{x=0} = \sin t$

$\therefore \frac{E}{2}\sin t = \sin t \Rightarrow E = 2$

$\therefore$ from ② $u(x,t) = \cos\left(\frac{x}{2}\sin t\right) + 2\sin\frac{x}{2}\sin t$

$$u(x,t) = \cos\left(\frac{x}{2} + 2\sin\frac{x}{2}\right)\sin t.$$

Q9. The temp. $u(x,t)$ is given by the sol^n of one dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad -①$$

→ Given $u(x,0) = u_0$, $u(0,t)=0$, $u(l,t)=0$

→ Since $u(0,t) = u(l,t) = 0$, therefore the sol^n of eq ① is given by:

$$u(x,t) = (A\cos px + B\sin px)e^{-p^2c^2t} \ ;\ p>0 \quad -②$$

Now, $u(0,t)=0 \Rightarrow Ae^{-p^2c^2t}=0 \Rightarrow A=0$

$\therefore$ By ②, $u(x,t) = B\sin px\, e^{-p^2c^2t}$; $p>0 \quad -③$

→ Now, $u(l,t)=0 \Rightarrow B\sin pl\, e^{-p^2c^2t} = 0$

$\sin pl = 0 \Rightarrow p = \frac{n\pi}{l}$; $n = 1, 2, ---$

$\therefore$ By ③, $u(x,t) = B_n \sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2t}{l^2}}$; $n=1,2,\ldots$

By principle of superposition,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\frac{n^2c^2\pi^2t}{l^2}} \quad -④$$

Now, $u(x,0) = u_0 \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = u_0$

which is half range fourier series sin series of u_0 in $(0,l)$.

$$\therefore B_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx = \frac{2u_0}{l}\times\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

$$\left.\begin{aligned}\therefore\ B_{2n} &= 0\\ B_{2n-1} &= \frac{4u_0}{(2n-1)\pi}\end{aligned}\right\}\ n=1,2,3,4,\ldots$$

$\therefore$ By eq ④

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2(2n-1)^2\pi^2t}{l^2}}$$

Q16 (i) Since $u\to 0$ as $y\to\infty$ for all x

$\therefore$ Solution is of the form,

$u(x,y) = e^{-py}(A\cos px + B\sin px)$; $p>0$ —①

$u(0,y) = 0\ \forall y$ (given)

$\therefore$ By ①, $Ae^{-py} = 0\ \forall y \Rightarrow A = 0$

$\therefore$ By ①, $u(x,y) = Be^{-py}\sin px$; $p>0$ —②

Now $u(l,y) = 0\ \forall y$

$\therefore$ By ②, $Be^{-py}\sin pl = 0\ \forall y$.

$\Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}$, $n=1,2,\ldots$

So, $u(x,y) = B_n e^{-\frac{n\pi y}{l}}\sin\frac{n\pi x}{l}$; $n=1,2,\ldots$

By principal of superposition,

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{\frac{-n\pi y}{l}} \sin \frac{n\pi x}{l} \quad -(3)$$

(iv) Now, $u(x,0) = lx - x^2 \ \forall x \in (0,l)$

By (3), $\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = lx - x^2$

which is half range Fourier sine series in $(0,l)$

$$\therefore B_n = \frac{2}{l}\int_0^l (lx-x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{-l}{n\pi} \cos\frac{n\pi x}{l}\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2} \sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^2}{n^3\pi^3} \cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$B_n = \frac{-4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$\therefore \left.\begin{aligned} B_{2n} &= 0 \\ B_{2n-1} &= \frac{8l^2}{(2n-1)^3\pi^3} \end{aligned}\right\} n = 1, 2, 3, \ldots$$

$\therefore$ Solution is

$$u(x,y) = \frac{8l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cdot e^{-\frac{(2n-1)\pi y}{l}} \cdot \sin\frac{(2n-1)\pi x}{l} \quad //$$

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