Partial Differential Equations Linear and Homogeneous in Partial Derivatives with Constant Coefficients

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + k_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

is linear and homogeneous in partial derivatives of order n and has constant coefficients.

If we write $\frac{\partial^r}{\partial x^r} \equiv D^r$ and $\frac{\partial^r}{\partial y^r} = D'^r$ the equation can be written as

$$f(D,D')z = F(x,y) \quad \text{——— (1)}$$

where $f(D,D') = D^n + k_1 D^{n-1}D' + k_2 D^{n-2}D'^2 + \cdots + k_n D'^n$

The complete solution of (1) is given by

z = complementary function (C.F.) + particular integral (P.I.)

Rules to write complementary function (C.F.)

C.F. is the complete solution of

$$f(D,D')z = 0$$

(1) Replacing D by m and D' by 1, auxiliary equation is

$$m^n + k_1 m^{n-1} + k_2 m^{n-2} + \cdots + k_n = 0 \quad \text{——— (2)}$$

Let the roots of this equation be $m_1, m_2, \ldots, m_n$.

Case I — When all roots are different

$$C.F. = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx)$$

Case II — If two roots of equation (2) are equal say $m_1 = m_2 = m$ and all others are different

$$C.F. = \phi_1(y+mx) + x\phi_2(y+mx) + \phi_3(y+m_3x) + \cdots + \phi_n(y+m_nx)$$

Similarly if $m_1 = m_2 = m_3 = m$ and others are different then

$$C.F. = \phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx) + \phi_4(y+m_4x) + \cdots + \phi_n(y+m_nx)$$

Que Solve the following partial differential equations

(i) $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0$

(ii) $(D^3 - D^2D' - DD'^2 + D'^3)z = 0$ or $(D_x^3 - D_x^2 D_y - D_x D_y^2 + D_y^3)z = 0$

Sol (i) The given p.d.e. can be written symbolically as

$$(D^4 - D'^4)z = 0$$

A.E. is $m^4 - 1 = 0 \Rightarrow (m^2-1)(m^2+1) = 0$

$\Rightarrow m = \pm 1, \pm i$

$\therefore$ General solution is $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(ii) p.d.e. is $(D^3 - D^2D' - DD'^2 + D'^3)z = 0$

A.E. is $m^3 - m^2 - m + 1 = 0$

$\Rightarrow m^2(m-1) - 1(m-1) = 0$

$\Rightarrow (m-1)(m^2-1) = 0 \Rightarrow m = 1, 1, -1$

$\therefore$ General solution is

$$z = \phi_1(y+x) + x\,\phi_2(y+x) + \phi_3(y-x)$$

Particular Integral (P.I.)

If $f(D,D')z = F(x,y)$ then

$$P.I. = \frac{1}{f(D,D')} F(x,y)$$

Formulae regarding P.I.

(1) If $F(x,y) = e^{ax+by}$, then

$$P.I. = \frac{1}{f(D,D')} e^{ax+by} = \begin{cases} \dfrac{1}{f(a,b)} e^{ax+by} & \text{if } f(a,b) \neq 0 \\[2ex] \dfrac{x^k}{\left[\dfrac{d^k}{dD^k} f(D,D')\right]_{D=a,\,D'=b}} & \text{if } f(a,b) = 0 \end{cases}$$

where k is the least +ive integer so that it reduces to above form when $\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,D'=b} \neq 0$

(2) If $F(x,y) = \sin(ax+by+c)$ or $\cos(ax+by+c)$, then

$$P.I. = \frac{1}{f(D,D')} \underset{\cos}{\sin}(ax+by+c)$$

$$= \frac{1}{[f(D,D')]_{D^2=-a^2,\ D'^2=-b^2,\ DD'=-ab}} \underset{\cos}{\sin}(ax+by+c)$$

provided den. $\neq 0$

(3) If $F(x,y) =$ polynomial in x & y

$$P.I. = [f(D,D')]^{-1} \cdot F(x,y)$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and operate on $F(x,y)$.

(4) If $F(x,y) = e^{ax+by} V(x,y)$, then

$$P.I. = \frac{1}{f(D,D')} e^{ax+by} V(x,y) = e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x,y)$$

(5) If $F(x,y) = \phi(ax+by)$ and $f(D,D')$ is a homogeneous function of degree n (say) then

$$P.I. = \frac{1}{f(D,D')} \phi(ax+by) = \frac{1}{f(a,b)} G(ax+by) \text{ if } f(a,b) \neq 0$$

where $G(ax+by)$ is obtained after integrating $\phi(z)$ w.r.t. z, n times and then taking $z = ax+by$.

and $$P.I. = \frac{x^k}{\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,\ D'=b}} \quad \text{if } f(a,b) = 0$$

where k is the least positive integer so that it reduces to above form when $\left[\frac{d^k}{dD^k} f(D,D')\right]_{D=a,\ D'=b} \neq 0$

Now apply the above formula.

Remark (1) If $F(x,y) = \sinh(ax+by+c)$ or $\cosh(ax+by+c)$ then convert it in exponential forms and use case 1.

(2) Sometimes it is easy to use the formula

General Formula $\dfrac{1}{(D-mD')} F(x,y) = \int F(x, a-mx)\,dx$

where $y = a - mx$

After integrating it we put $a = y + mx$.

Que Solve the following p.d.e's

(1) $(D^2+4DD'-5D'^2)z = \sin(2x+3y) + 3e^{2x+y}$

(2) $(D^3-7DD'^2-6D'^3)z = \sin(x+2y)$

(3) $(2D^2-5DD'+2D'^2)z = 5\sin(2x+y)$

(4) $4r+12s+9t = e^{3x-2y}$

(5) $r-2s+t = 2x\cos y$

(6) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y\cos x$

(7) $r-s-2t = (y-1)e^x$

Sol (1) $(D^2+4DD'-5D'^2)z = \sin(2x+3y) + 3e^{2x+y}$

A.E. is $m^2+4m-5=0 \Rightarrow (m+5)(m-1)=0$

$\Rightarrow m = 1, -5$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-5x)$

$$P.I. = \frac{1}{(D^2+4DD'-5D'^2)}\left[\sin(2x+3y) + 3e^{2x+y}\right]$$

$$= \frac{1}{(D^2+4DD'-5D'^2)}\sin(2x+3y) + 3 \cdot \frac{1}{(D^2+4DD'-5D'^2)} e^{2x+y}$$

$$= \frac{1}{-4-24+45}\sin(2x+3y) + 3.\frac{1}{(4+8-5)}e^{2x+y}$$

$$= \frac{1}{17}\sin(2x+3y) + \frac{3}{7}e^{2x+y}$$

Hence the complete solution is

$$\boxed{z = \phi_1(y+x) + \phi_2(y-5x) + \frac{1}{17}\sin(2x+3y) + \frac{3}{7}e^{2x+y}}$$ A

(2) $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y)$

A.E. is $m^3 - 7m - 6 = 0$

$m = -1$ satisfies it.

$\therefore\ m^3 - 7m - 6 = m^2(m+1) - m(m+1) - 6(m+1) = 0$

$\Rightarrow (m+1)(m^2 - m - 6) = 0$

$\Rightarrow (m+1)(m-3)(m+2) = 0$

$\Rightarrow m = -2, -1, 3$

$\therefore$ C.F. $= \phi_1(y-2x) + \phi_2(y-x) + \phi_3(y+3x)$

$$P.I. = \frac{1}{(D^3 - 7DD'^2 - 6D'^3)}\sin(x+2y)$$

$$= \frac{1}{1^3 - 7(1)(2)^2 - 6(2)^3}\cos(x+2y)$$

(after integrating $\sin z$ w.r.t. z three times and putting $z = x+2y$, $D = 1$, $D' = 2$)

$$= -\frac{1}{75}\cos(x+2y)$$

Hence the complete solution is

$$\boxed{z = \phi_1(y-2x) + \phi_2(y-x) + \phi_3(y+3x)}$$

Other Method to find P.I.

$$\frac{1}{(D^3 - 7DD'^2 - 6D'^3)}\sin(x+2y)$$

$$= \frac{1}{(-D + 14D' + 24D')}\sin(x+2y)$$

$$= \frac{(-D - 38D')}{(-D + 38D')(-D - 38D')}\sin(x+2y)$$

$$= \frac{(-D - 38D')}{D^2 - (38)^2 D'^2}\sin(x+2y)$$

$$= \frac{(-D - 38D')}{-1 + 4\times 38\times 38}\sin(x+2y)$$

$$= \frac{-\cos(x+2y) - 76\cos(x+2y)}{5775}$$

$$= -\frac{1}{75}\cos(x+2y)$$

(3) $(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)$

A.E. is $2m^2 - 5m + 2 = 0$

$\Rightarrow m = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4} = \frac{1}{2}, 2$

$\therefore$ C.F. $= \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x) = \psi_1(2y+x) + \psi_2(y+2x)$

P.I. $= \frac{1}{(2D^2 - 5DD' + 2D'^2)} \cdot 5\sin(2x+y)$

$= \frac{5x}{(4D - 5D')} \sin(2x+y) = 5x \frac{1}{4(2) - 5(1)} [-\cos(2x+y)]$

$= \frac{-5x}{3} \cos(2x+y)$

Hence the solution is

$$\boxed{z = \psi_1(2y+x) + \psi_2(y+2x) - \frac{5x}{3}\cos(2x+y)}$$

(4) $4r + 12s + 9t = e^{3x-2y}$

The given equation can be written as

$(4D^2 + 12DD' + 9D'^2)z = e^{3x-2y}$

A.E. is $4m^2 + 12m + 9 = 0 \Rightarrow (2m+3)^2 = 0 \Rightarrow m = -\frac{3}{2}, -\frac{3}{2}$

$\therefore$ C.F. $= \phi_1\left(y - \frac{3}{2}x\right) + x\phi_2\left(y - \frac{3}{2}x\right) = \psi_1(2y-3x) + x\psi_2(2y-3x)$

P.I. $= \frac{1}{(4D^2 + 12DD' + 9D'^2)} e^{3x-2y}$

$= \frac{1}{(2D+3D')^2} e^{3x-2y}$

$= \frac{x^2}{\frac{d^2}{dD^2}(2D+3D')^2} e^{3x-2y} \quad [\because 2.3 + 3.(-2) = 0]$

$= \frac{x^2}{8} e^{3x-2y}$

$\therefore$ Complete sol. is $\boxed{z = \psi_1(2y-3x) + x\psi_2(2y-3x) + \frac{x^2}{8}e^{3x-2y}}$

(5)

$$r-2s+t = 2x\cos y$$

The given equ. can be written as

$$(D^2-2DD'+D'^2)z = 2x\cos y$$

A.E. is $m^2-2m+1=0$ or $(m-1)^2=0 \Rightarrow m=1,1$

$\therefore$ C.F. $= \phi_1(y+x) + x\phi_2(y+x)$

$$\text{P.I.} = \frac{1}{(D-D')^2} 2x\cos y$$

$$= 2 \text{ Re part of } \frac{1}{(D-D')^2} x e^{iy}$$

$$= 2 \text{ Re part of } e^{iy} \frac{1}{(D-D'-i)^2} x$$

$$= 2 \text{ Re part of } e^{iy}(-1)\left[1+i(D-D')\right]^{-2}. x$$

$$= 2 \text{ Re part of } (-e^{iy})\left[1-2iD+\cdots\right]x$$

$$= 2 \text{ Re part of } (-1)(\cos y + i\sin y)(x-2i)$$

$$= -2(x\cos y + 2\sin y)$$

$\therefore$ The complete solution is

$$\boxed{y = \phi_1(y+x) + x\phi_2(y+x) - 2(x\cos y + 2\sin y)}$$

(6) $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

The given equ. can be written as

$$(D^2+DD'-6D'^2)z = y\cos x$$

A.E. is $m^2+m-6=0 \Rightarrow (m-2)(m+3)=0 \Rightarrow m=2,-3$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-3x)$

$$\text{P.I.} = \frac{1}{(D^2+DD'-6D'^2)} y\cos x = \frac{1}{(D+3D')(D-2D')} y\cos x$$

$$= \frac{1}{(D+3D')} \int (a-2x)\cos x\, dx \qquad \text{where } y = a-2x$$

$$= \frac{1}{(D+3D')} \left[(a-2x)\sin x - (-2)(-\cos x) \right]$$

$$= \frac{1}{(D+3D')} \left[y\sin x - 2\cos x \right]$$

$$= \int \left[(b+3x)\sin x - 2\cos x \right] dx \qquad \text{where } y = b+3x$$

$$= \left[(b+3x)(-\cos x) - (3)(-\sin x) - 2\sin x \right]$$

$$= -y\cos x + 3\sin x - 2\sin x = -y\cos x + \sin x$$

$\therefore$ The complete sol. is

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x}$$

(7) $r - s - 2t = (y-1)e^x$

The given eqn. can be written as

$$(D^2 - DD' - 2D'^2) z = (y-1)e^x$$

A.E. is $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$

$$\Rightarrow m = 2, -1$$

$$\therefore \text{ C.F.} = \phi_1(y+2x) + \phi_2(y-x)$$

$$\text{P.I.} = \frac{1}{(D^2 - DD' - 2D'^2)} (y-1)e^x$$

$$= \frac{e^x}{(D+1)^2 - (D+1)D' - 2D'^2} (y-1)$$

$$= \frac{e^x}{D^2 - DD' - 2D'^2 + 2D - D' + 1} (y-1)$$

$$= e^x \left[1 + (2D - D' - 2D'^2 - DD' + D^2) \right]^{-1} (y-1)$$

$$= e^x \left[1 - 2D + D' + \cdots \right] (y-1) = e^x (y-1+1) = ye^x$$

$\therefore$ The complete sol. is

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-x) + ye^x}$$

Linear p.d.e's with constant coefficients, non-homogeneous in partial derivatives

$$f(D, D')z = F(x, y) \quad \text{———} \quad (1)$$

where $f(D, D')$ is not homogeneous i.e., sum of powers of D and D' in terms may not be equal.

Here we also have complete solution = C.F. + P.I.

Rules for finding C.F.

Case I When $f(D, D')$ can be factorized into linear factors of the type $(D - mD' - c)$ where m, c be any constant (may be zero) and factors are not repeated.

$\therefore$ C.F. (corresponding to this factor) $= e^{cx}\phi(y+mx)$

The solution corresponding to various factors added up, give the C.F. of (1).

Case II When factors are repeated. Suppose $(D - mD' - c)$ is repeated twice. Then C.F. (corresponding to these two factors)

$$= e^{cx}\left[\phi_1(y+mx) + x\,\phi_2(y+mx)\right]$$

Similarly C.F. $= e^{cx}\left[\phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx)\right]$

when factor $(D - mD' - c)$ is repeated thrice.

Case III If $f(D, D')$ cannot be factorized into linear factors. Then C.F. (corresponding to non-linear factor)

$$= \sum_{i=1}^{\infty} c_i e^{h_i x + k_i y} \quad \text{where } f(h_i, k_i) = 0$$

Rules for finding P.I. Formulae (1) to (4) of homogeneous form can be used in the same way for finding P.I. for non-homogenous form.

Remark: We cannot use the formula (5) of homogeneous form as $f(D, D')$ is not homogeneous here.

General Formula

$$\frac{1}{(D-mD'-c)} F(x,y) = e^{cx}\int e^{-cx} F(x, a-mx)\,dx$$

where $y = a - mx$

After integration, we substitute $a = y + mx$

Que Find the solutions of the following p.d.e's

(1) $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y) + e^{x+2y}$

(2) $(D^2-D')z = xe^{x+y}$

(3) $(D^2-D'^2+3D'-3D)z = xy$

(4) $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

Sol (1) $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y) + e^{x+2y}$

$\Rightarrow \quad (D+D')(D+D'-2)z = \sin(x+2y) + e^{x+2y}$

$\therefore \quad C.F. = \phi_1(y-x) + e^{2x}\phi_2(y-x)$

$$P.I. = \frac{1}{(D^2+2DD'+D'^2-2D-2D')}\sin(x+2y) + \frac{1}{(D+D')(D+D'-2)}e^{x+2y}$$

$$= \frac{1}{-1+2(-2)+(-4)-2D-2D'}\sin(x+2y) + \frac{1}{(1+2)(1+2-2)}e^{x+2y}$$

$$= \frac{-1}{(2D+2D'+9)}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= -\frac{(2D+2D'-9)}{(2D+2D'+9)(2D+2D'-9)}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= -\frac{(2D+2D'-9)}{(4D^2+4D'^2+8DD')-81}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= -\frac{(2D+2D'-9)}{-4-16-16-81}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= \frac{(2D+2D'-9)}{117}\sin(x+2y) + \frac{1}{3}e^{x+2y}$$

$$= \frac{1}{117}\left[2\cos(x+2y)+4\cos(x+2y)-9\sin(x+2y)\right]+\frac{1}{3}e^{x+2y}$$

$$= \frac{1}{39}\left[2\cos(x+2y)-3\sin(x+2y)+\frac{1}{3}e^{x+2y}\right.$$

$\therefore$ The complete solution is

$$z = \phi_1(y-x)+e^{2x}\phi_2(y-x)+\frac{1}{39}\left[2\cos(x+2y)-3\sin(x+2y)\right] + \frac{1}{3}e^{x+2y}$$

(2) $(D^2-D')z = xe^{x+y}$

Here D^2-D' cannot be factorized into linear factors in D and D'.

$\therefore$ C.F. $= \sum_{i=1}^{\infty} c_i e^{h_i x + k_i y}$ where $h_i^2 - k_i = 0$ i.e., $k_i = h_i^2$

$$= \sum_{i=1}^{\infty} c_i e^{h_i x + h_i^2 y}$$

$$\text{P.I.} = \frac{1}{(D^2-D')} x e^{x+y}$$

$$= \frac{e^{x+y}}{(D+1)^2-(D'+1)} x$$

$$= \frac{e^{x+y}}{(D^2+2D-D')} x$$

$$= e^{x+y}\frac{1}{(-D')}\left[1-\left(\frac{2D}{D'}+\frac{D^2}{D'}\right)\right]^{-1}.x$$

$$= e^{x+y}\frac{1}{(-D')}\left[1+\frac{2D}{D'}+\frac{D^2}{D'}+\cdots\right].x$$

$$= e^{x+y}\frac{1}{(-D')}(x+2y) = e^{x+y}(-xy-y^2)$$

$\therefore$ The complete sol. is

$$z = \sum_{i=1}^{\infty} c_i e^{h_i x + h_i^2 y} - e^{x+y}(xy+y^2)$$

where c_i and h_i are arbitrary constants.

(3) $(D^2 - D'^2 + 3D' - 3D)z = xy$

or $(D-D')(D+D'-3)z = xy$

$\therefore$ C.F. $= \phi_1(y+x) + e^{3x}\phi_2(y-x)$

$$P.I. = \frac{1}{(D^2-D'^2+3D'-3D)}xy$$

$$= \frac{1}{(D-D')(D+D'-3)}xy$$

$$= -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left[1-\left(\frac{D}{3}+\frac{D'}{3}\right)\right]^{-1} \cdot xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left(1+\frac{D}{3}+\frac{D'}{3}+\frac{2}{9}DD'+\cdots\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{D}{3}+\frac{2D'}{3}+\frac{D'}{D}+\frac{2}{9}DD'+\cdots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{y}{3}+\frac{2x}{3}+\frac{1}{D}x+\frac{2}{9}\right) = -\frac{1}{3}\left(\frac{1}{2}x^2y+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right)$$

$\therefore$ The complete solution is

$$\boxed{z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{54}(9x^2y+6xy+6x^2+3x^3+4x)}$$

(4) $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

C.F. $= e^{2x}\phi_1(y+3x) + xe^{2x}\phi_2(y+3x)$

$$P.I. = \frac{1}{(D-3D'-2)^2}2e^{2x}\tan(y+3x)$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$$= 2e^{2x}\frac{1}{(D-3D')}\int \tan(a-3x+3x)\,dx \quad \text{where } a-3x=y$$

$$= 2e^{2x}\frac{1}{(D-3D')}x\tan a = 2e^{2x}\frac{1}{(D-3D')}x\tan(y+3x)$$

$$= 2e^{2x}\int x\tan a\,dx \quad \text{where } a-3x=y$$

$$= 2e^{2x}\cdot\frac{x^2}{2}\tan a = x^2e^{2x}\tan(y+3x)$$

$\therefore$ The complete sol. is

$$\boxed{z = e^{2x}\phi_1(y+3x) + xe^{2x}\phi_2(y+3x) + x^2e^{2x}\tan(y+3x)}$$

Classification of linear second order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \quad \text{——— (1)}$$

where A, B, C, D, E and F are functions of x, y or are real constants. The p.d.e. is said to be a

parabolic equ. if $B^2 - 4AC = 0$
hyperbolic equ. if $B^2 - 4AC > 0$
elliptic equ. if $B^2 - 4AC < 0$

Some simple examples of the above equ's are the following

(i) One dimensional heat equ. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is parabolic.

(ii) One dimensional wave equ. $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ is hyperbolic.

(iii) Two dimensional Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is elliptic.

Principle of superposition If $c_1u_1, c_2u_2, c_3u_3, ----$ are solutions of equ. (1) then the complete sol. of (1) is

$$u(x,y) = \sum_{n=1}^{\infty} c_n u_n$$

Method of separation of variables

Qu Using the method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$ [Ist term Feb. 15, 2.5 marks]

where $u(x,0) = 6e^{-3x}$, $x > 0$, $t > 0$

Sol Let $u(x,t) = X(x).T(t)$

$\therefore \frac{\partial u}{\partial x} = X'T$, $\frac{\partial u}{\partial t} = XT'$

where dashes denote derivatives w.r.t. their variables.

$\therefore$ The given equ. becomes

$$X'T = 2XT' + XT$$

$$\Rightarrow \frac{X'}{X} = \frac{2T'+T}{T} = \text{constant } (\lambda) \text{ say}$$

$$\Rightarrow X' - \lambda X = 0 \text{ and } T' - \frac{(\lambda-1)}{2}T = 0$$

$$\Rightarrow X = Ae^{\lambda x}, \quad T = Be^{\frac{(\lambda-1)}{2}t}$$

$$\therefore u(x,t) = Ce^{\lambda x} \cdot e^{\frac{(\lambda-1)}{2}t} \quad \text{where } AB = C$$

Given $u(x,0) = 6e^{-3x}$

$$\therefore \quad Ce^{\lambda x} = 6e^{-3x} \Rightarrow C = 6, \ \lambda = -3$$

Hence $\boxed{u(x,t) = 6e^{-(3x+2t)}}$ A

<u>Que</u> (for practice) Use the method of separation of variables to solve the p.d.e. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$

[A $u(x,y) = 4e^{-x}e^{3y/2}$]

<u>Que</u> Solve the equation $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ given $u = 3e^{-y} - e^{-5y}$ when $x = 0$.

<u>Sol</u>. Let $u(x,y) = X(x)Y(y)$

$\therefore \frac{\partial u}{\partial x} = X'Y, \ \frac{\partial u}{\partial y} = XY'$

where dashes denote derivatives w.r.t. their variables.

$\therefore$ The given equation becomes

$$4X'Y + XY' = 3XY$$

$$\Rightarrow \frac{4X'}{X} = \frac{-Y'+3Y}{Y} = \text{constant } (\lambda) \text{ say}$$

$$\therefore X' - \frac{\lambda}{4}X = 0, \quad Y' - (3-\lambda)Y = 0$$

$$\therefore X = Ae^{\frac{\lambda}{4}x}, \quad Y = Be^{(3-\lambda)y}$$

$$\therefore u(x,y) = Ce^{\frac{\lambda}{4}x}e^{(3-\lambda)y} \quad \text{where } AB = C$$

Now $u(0,y) = 3e^{-y} - e^{-5y}$

$\therefore$ $u(x,y)$ is sum of two solutions as

$$u(x,y) = C_1 e^{\frac{\lambda_1}{4}x} e^{(3-\lambda_1)y} + C_2 e^{\frac{\lambda_2}{4}x} e^{(3-\lambda_2)y}$$

$$u(0,y) = C_1 e^{(3-\lambda_1)y} + C_2 e^{(3-\lambda_2)y} = 3e^{-y} - e^{-5y} \text{ (given)}$$

$\therefore$ Either $C_1 = 3,\ \lambda_1 = 4,\ C_2 = -1,\ \lambda_2 = 8$

or $C_1 = -1,\ \lambda_1 = 8,\ C_2 = 3,\ \lambda_2 = 4$

In both cases, solution is $\boxed{u(x,y) = 3e^{x-y} - e^{2x-5y}}$

Que Use the method of separation of variables to solve the equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$ given that $V = 0$ when $t \to \infty$ as well as $V = 0$ at $x = 0$ and $x = l$.

Sol Let $V = X(x) \cdot T(t)$

$\therefore$ $\frac{\partial^2 V}{\partial x^2} = X''T,\ \frac{\partial V}{\partial t} = XT'$ where dashes denote derivatives w.r.to their variables

$\therefore$ The given equ. becomes $X''T = XT' \Rightarrow \frac{X''}{X} = \frac{T'}{T} = \text{constant } (\lambda)$ say

$\therefore$ $X'' - \lambda X = 0,\ T' - \lambda T = 0$

Now, $T' - \lambda T = 0 \Rightarrow T = Ae^{\lambda t}$

As $V = XT = 0$ when $t \to \infty$ so λ must be negative.

Take $\lambda = -p^2,\ p > 0$

$\therefore$ $T = Ae^{-p^2 t}$

Now, $X'' - \lambda X = 0 \Rightarrow X'' + p^2 = 0$

$\Rightarrow X = B\cos px + C\sin px$

$\therefore$ $V = (D\cos px + E\sin px)e^{-p^2 t}$ where $AB = D$ and $AC = E$

$V(0,t) = 0 \Rightarrow De^{-p^2 t} = 0 \Rightarrow D = 0$

$\therefore$ $V = E\sin px\, e^{-p^2 t}$

Now $V(l,t) = 0 \Rightarrow E\sin pl\, e^{-p^2 t} = 0 \Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l},\ n = 1,2,3,\ldots$ $(\because p > 0)$

$\therefore$ Solutions are $V(x,t) = E_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2}{l^2}t},\ n = 1,2,3,\ldots$

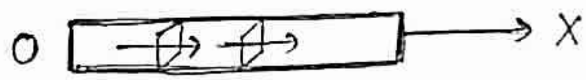
$\therefore$ By principle of superposition, complete solution is

$$\boxed{V(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{l^2}}}$$

One Dimensional Heat Flow Equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Consider the flow of heat by conduction in a uniform bar. We assume the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible.

Take one end of the bar as origin and x-axis along the direction of flow of heat. Let $u(x,t)$ be temperature at a point distance x from origin at time t and the temperature at all points of a cross section is same. (Also, we know that in a body heat flows in a direction of decreasing temperature).



Then $u(x,t)$ is given by the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

where c^2 is diffusivity of bar. (the amount of heat that passes through a given area in unit time).

Let its solution be given by

$$u(x,t) = X(x).T(t) \quad \text{—— (2)}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = X''T \text{ and } \frac{\partial u}{\partial t} = XT'$$

$\therefore$ The p.d.e. (1) becomes,

$$XT' = c^2 X'' T$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \text{constant} = \lambda \text{ (say)} \quad \text{—— (3)}$$

<u>Case I</u> If $\lambda = 0$, then from (3),

$$X'' = 0, \quad T' = 0$$

$$\therefore X = Ax + B, \quad T = C$$

$\therefore$ By (2), $\quad u(x,t) = C(Ax+B)$

$$\therefore \boxed{u(x,t) = Dx + E} \quad \text{where } D = CA,\ E = CB \quad \text{——— (4)}$$

Case II If $\lambda > 0$ i.e., $\lambda = p^2$, $p > 0$, we have from (3)

$$X'' - p^2 X = 0 \text{ and } T' - p^2c^2 T = 0$$

$$\therefore X = (Ae^{px} + Be^{-px}) \text{ and } T = Ce^{p^2c^2t}$$

$\therefore$ By (2), $\quad u(x,t) = (Ae^{px} + Be^{-px})\, C e^{p^2c^2t}$

$$\therefore \boxed{u(x,t) = (De^{px} + Ee^{-px})\, e^{p^2c^2t}} \quad \text{——— (5)}$$

Case III If $\lambda < 0$ i.e. $\lambda = -p^2$, $p > 0$ then from (3)

$$X'' + p^2 X = 0, \quad T' + p^2c^2 T = 0$$

$$\therefore X = A\cos px + B\sin px \text{ and } T = Ce^{-p^2c^2t}$$

$\therefore$ By (2), $\quad u(x,t) = (A\cos px + B\sin px)Ce^{-p^2c^2t}$

$$\therefore \boxed{u(x,t) = (D\cos px + E\sin px)\, e^{-p^2c^2t}} \quad \text{——— (6)}$$

Out of these three solutions, we are to choose solutions satisfying initial and boundary conditions. Since the temperature $u(x,t)$ decreases as the time t increases (as we are dealing with problems on heat conduction, it must be a transient solution), therefore the solution given by (5) can be rejected.

Remark (1) The solution $Dx+E$ is the solution of $\frac{\partial^2 u}{\partial x^2} = 0$ and hence $\frac{\partial u}{\partial t} = 0$ in heat equation (1). Thus, solution $Dx+E$ is steady state solution.

(2) If $u(0,t) = u(l,t) = 0 \ \forall t$, then there is no steady state solution and $u(x,t)$ decreases as time t increases and hence in this case solution is given by equ. (6).

$$\left[\begin{array}{l} \text{by (4), } u(0,t)=0 \Rightarrow E=0 \text{ and } u(l,t)=0 \Rightarrow Dl+E=0 \\ \therefore D=E=0, \text{ So no steady state solution exist.} \end{array}\right]$$

Que! A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to $0°c$, and are kept at that temperature. Find the temperature $u(x,t)$.

Sol The temperature $u(x,t)$ is given by the solution of one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{———} (1)$$

Given $u(x,0) = u_0, \ u(0,t) = 0, \ u(l,t) = 0$

Since $u(0,t) = u(l,t) = 0$, therefore the solution of equ.(1) is given by

$$u(x,t) = (A\cos px + B\sin px)\, e^{-p^2c^2t}; \ p>0 \quad \text{———} (2)$$

Now, $u(0,t) = 0 \Rightarrow Ae^{-p^2c^2t} = 0 \Rightarrow A = 0$

$\therefore$ By (2), $u(x,t) = B\sin px \, e^{-p^2c^2t}; \ p>0$ ——— (3)

Now, $u(l,t) = 0 \Rightarrow B\sin pl \, e^{-p^2c^2t} = 0$

$$\Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}, \ n = 1,2,3,\ldots$$

$\therefore$ By (3), $u(x,t) = B_n \sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2}{l^2}t}; \ n = 1,2,3,\ldots$

By the principle of superposition,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2c^2\pi^2}{l^2}t} \quad \text{———} \quad (4)$$

Now, $u(x,0) = u_0 \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = u_0$

which is half range Fourier sine series of u_0 in $(0,l)$.

$$\therefore \; B_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2u_0}{l}\cdot\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l = \frac{2u_0}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore \left.\begin{array}{l} B_{2n} = 0 \\ B_{2n-1} = \dfrac{4u_0}{(2n-1)\pi} \end{array}\right\} \quad n = 1,2,3,\ldots$$

$\therefore$ By (4),

$$\boxed{u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\frac{(2n-1)\pi x}{l} e^{-\frac{(2n-1)^2c^2\pi^2}{l^2}t}}$$

Que 2 (a) An insulated rod of length l has its ends A and B maintained at $0°$ and $100°c$ respectively until steady state conditions prevail. If B is suddenly reduced to $0°c$ and are maintained at $0°c$, find the temperature at a distance x from A at time t.

(b) Solve the above problem, if the change consists of raising the temperature of A to $20°c$ and reducing that of B to $80°c$.

Sol (a) The temperature $u(x,t)$ is given by the one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{———} \quad (1)$$

Before changing the temperature at the end B,
When $t=0$, the heat flow was in steady state (i.e, independent of time) i.e, $\frac{\partial u}{\partial t}=0$. So by (1), $\frac{\partial^2 u}{\partial x^2}=0$

$$\therefore \ u(x,0)=Ax+B$$

Now $u(0,0)=0$ and $u(l,0)=100$

$$\therefore \ B=0 \text{ and } A=\frac{100}{l}$$

$$\therefore \ u(x,0)=\frac{100x}{l}$$

Now, for the subsequent flow,

$$u(0,t)=0 \ \forall t \quad \text{and} \quad u(l,t)=0 \ \forall t$$

$\therefore$ Solution of (1) is of the form

$$u(x,t)=(A\cos px+B\sin px)e^{-p^2c^2t}; \ p>0 \quad \text{——} (2)$$

[Now, do the same steps upto equ.(4) as in above que.(1)]

$$\therefore \ u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2c^2\pi^2}{l^2}t} \quad \text{——} (4)$$

Now, $u(x,0)=\frac{100x}{l}$

$$\therefore \ \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}=\frac{100x}{l}$$

which is half range Fourier sine series of $\frac{100x}{l}$ in $(0,l)$

$$\therefore \ B_n=\frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{200}{l^2}\left[x\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$=\frac{200}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n\right]=\frac{200}{n\pi}(-1)^{n+1}$$

$$\therefore \text{By (4)}, \quad u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}e^{-\frac{n^2c^2\pi^2}{l^2}t}$$

Ans

(b) Here the Initial condition

$$u(x,0) = \frac{100x}{l}$$

remains the same as in (a) part.

Also, the boundary conditions are

$$u(0,t) = 20 \quad \forall t \quad \text{and} \quad u(l,t) = 80 \quad \forall t$$

Since the boundary values $u(0,t)$ and $u(l,t)$ are given to be non-zero, so we split the temperature function $u(x,t)$ into two parts as

$$u(x,t) = u_s(x) + u_t(x,t) \quad \text{———} \quad (2)$$

where $u_s(x) = Ax + B$ s.t. $u_s(0) = 20$ and $u_s(l) = 80$

$$\therefore B = 20, \; A = \frac{60}{l}$$

$$\therefore u_s(x) = \frac{60x}{l} + 20 \quad \text{———} \quad (3)$$

Put $x = 0$ in (2), we have

$$u_t(0,t) = u(0,t) - u_s(0) = 20 - 20 = 0 \quad \text{———} \quad (4)$$

Put $x = l$ in (2), we have

$$u_t(l,t) = u(l,t) - u_s(l) = 80 - 80 = 0 \quad \text{———} \quad (5)$$

Also, by (2), $u_t(x,0) = u(x,0) - u_s(x)$

$$= \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \quad \left(\because u(x,0) = \frac{100x}{l}\right)$$

$$= \frac{40x}{l} - 20 \quad \text{———} \quad (6)$$

Now $u_t(x,t)$ is given by

$$u_t(x,t) = (A\cos px + B\sin px)\, e^{-p^2c^2t} \; ; \; p > 0$$

where $u_t(0,t)=0$, $u_t(l,t)=0$ and $u_t(x,0)=\frac{40x}{l}-20$

[Now, do the same steps as in que (1) upto equation (4)]

$$\therefore \quad u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{n^2c^2\pi^2}{l^2}t} \quad \text{—————} (7)$$

Now, $u_t(x,0) = \frac{40x}{l} - 20$

$$\therefore \quad \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{40x}{l} - 20$$

which is half range Fourier sine series of $\frac{40x}{l}-20$ in $(0,l)$

$$\therefore \quad B_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l}-20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(\frac{40x}{l}-20\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{40}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left(-\frac{l}{n\pi}\right)\left[20\cos n\pi + 20\right] = \frac{-40}{n\pi}\left[(-1)^n+1\right]$$

$$\therefore \quad B_{2n} = \frac{-80}{2n\pi} \text{ and } B_{2n-1} = 0 \quad \Big\} \; n = 1,2,3,\dots$$

$\therefore$ By (7), $$u_t(x,t) = -\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}e^{-\frac{4n^2c^2\pi^2}{l^2}t}$$

$\therefore$ By (2), (3) and (4),

$$u(x,t) = \frac{60x}{l} + 20 - \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}\, e^{-\frac{4n^2c^2\pi^2}{l^2}t}$$

Que3 A bar of length l with insulated sides is initially at $0^\circ c$ temperature throughout. The end $x=0$ is kept at $0^\circ c$ for all time and heat is suddenly applied such that $\frac{\partial u}{\partial x} = 10$ at $x = l$ for all time. Find the temperature function $u(x,t)$.

Sol The temperature function $u(x,t)$ is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 \text{ is diffusivity of bar.}$$

Since the temperatures at both end points are not given to be $0^\circ c$, therefore we split $u(x,t)$ into two parts as

$$u(x,t) = u_s(x) + u_t(x,t) \quad \text{——— (1)}$$

where $u_s(x) = Ax + B$ s.t. $u_s(0) = 0$ and $\left(\frac{\partial u_s}{\partial x}\right)_{x=l} = 10$

$\therefore \quad B = 0$ and $A = 10$

$\therefore \quad u_s(x) = 10x$

Also, boundary conditions are given as

$u(0,t) = 0 \;\forall t, \; \left(\frac{\partial u}{\partial x}\right)_{x=l} = 10 \;\forall t$

and initial condition as $u(x,0) = 0 \quad \forall x$

By (1), $\quad u(x,t) = 10x + (A\cos px + B\sin px)e^{-p^2c^2t} \; (p>0)$ ——— (2)

($\because$ temperature at $x = l$ is not given)

$\therefore \quad u(0,t) = 0 \;\forall t$

$\Rightarrow \quad Ae^{-p^2c^2t} = 0 \;\forall t \Rightarrow A = 0$

$\therefore$ By (2), $\quad u(x,t) = 10x + B\sin px\, e^{-p^2c^2t} \; (p>0)$ ——— (3)

Now, $\left(\frac{\partial u}{\partial x}\right)_{x=l} = 10$ (given)

$\therefore\ 10 + Bp\cos pl\, e^{-p^2c^2t} = 10$

$\Rightarrow Bp\cos pl\, e^{-p^2c^2t} = 0 \Rightarrow \cos pl = 0$

$\Rightarrow p = \frac{(2n-1)\pi}{2l}\ ;\ n = 1, 2, 3, ---$

$\therefore$ By principle of superposition,

$$u(x,t) = 10x + \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l}\, e^{-\frac{(2n-1)^2\pi^2c^2t}{4l^2}} \quad \text{--- (4)}$$

by (3)

Now, $u(x,0) = 0\ \forall x$

$$\therefore \sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2l} = -10x$$

which is half range Fourier sine series of $-10x$ in $[0,l]$

$$\therefore B_n = \frac{2}{l}\int_0^l (-10x)\sin\frac{(2n-1)\pi x}{2l}\,dx$$

$$= \frac{-20}{l}\left[x\left\{\frac{-2l}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2l}\right\} - (1).\left\{\frac{-4l^2}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi x}{2l}\right\}\right]_0^l$$

$$= \frac{-20}{l}\cdot\frac{4l^2}{(2n-1)^2\pi^2}\sin(2n-1)\frac{\pi}{2}$$

$$= \frac{80l}{(2n-1)^2\pi^2}(-1)^n$$

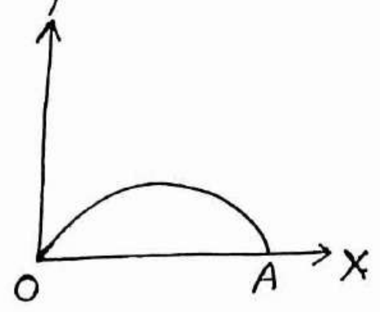
$\left[\because \sin\left(n\pi - \frac{\pi}{2}\right) = -\cos n\pi = -(-1)^n\right]$

$\therefore$ Solution is

$$u(x,t) = 10x + \frac{80l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2l}\, e^{-\frac{(2n-1)^2\pi^2c^2t}{4l^2}}$$

Vibrations of a Streched String (One dimensional wave equation)

Consider a uniform elastic string of length l streched tightly between two points O and A. Taking O as origin, X-axis along OA and Y-axis ⊥ar to OA at O.



We assume each point of the string moves ⊥ar to the equilibrium position OA in XY-place. Then the vertical displacement of points of the string is given by the wave equ.

Solution of Wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{———} \quad (1)$$

where $c^2 = \frac{T}{m}$ is called the diffusivity of string
($T \to$ tension & $m \to$ mass of the string)

To find the solution of wave equ. (1),

let $y(x,t) = X(x).T(t)$ ——— (2)

where $X(x)$ is function of x only and $T(t)$ is function of t only.

$$\therefore \frac{\partial^2 y}{\partial t^2} = XT'', \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

$\therefore$ Equation (1) becomes, $XT'' = c^2 X''T$

$$\text{or} \quad \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \text{constant} = \lambda \text{ (say)} \quad \text{———} \quad (3)$$

Case I If $\lambda = 0$, then from (3),

$$X'' = 0, \quad T'' = 0$$

$$\therefore X = Ax + B, \quad T = Ct + D$$

$$\therefore \boxed{y(x,t) = (Ax+B)(Ct+D)} \quad (\text{by } (2)) \text{ ——— } (4)$$

Case II If $\lambda > 0$ i.e, $\lambda = p^2$; $p > 0$, we have from (3),

$$X'' - p^2 X = 0 \text{ and } T'' - p^2 c^2 T = 0$$

$$\therefore X = (Ae^{px} + Be^{-px}) \text{ and } T = (Ce^{pct} + De^{-pct})$$

$$\therefore \boxed{y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pct} + De^{-pct})} \quad \text{(by (2))} \quad \text{---(5)}$$

Case III If $\lambda < 0$ i.e, $\lambda = -p^2$; $p > 0$, we have from (3),

$$X'' + p^2 X = 0 \text{ and } T'' + p^2 c^2 T = 0$$

$$\therefore X = (A\cos px + B\sin px) \text{ and } T = (C\cos pct + D\sin pct)$$

$$\therefore \boxed{y(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct)} \quad \text{---(6)}$$

The solution of wave equation can be one (4) or (5) or (6) or can be combination of these three solutions.

Remark:
But when the ends O and A are fixed then

$$y(0,t) = y(l,t) = 0 \quad \forall t$$

In this case, solutions given by equ. (4) and (5) are invalid

$\because$ From (4), $y(0,t) = 0 \Rightarrow B(Ct+D) = 0 \Rightarrow B = 0$ or $(Ct+D) = 0$

$y(l,t) = 0 \Rightarrow (Al+B)(Ct+D) = 0 \Rightarrow Al+B = 0$ or $(Ct+D) = 0$

$\therefore$ In this case $y(x,t) = 0$

Similarly from (5), $y(0,t) = 0 \Rightarrow (A+B)(Ce^{pct} + De^{-pct}) = 0$

$\Rightarrow A = -B$ or $(Ce^{pct} + De^{-pct}) = 0$

$y(l,t) = 0 \Rightarrow (Ae^{pl} + Be^{-pl})(Ce^{pct} + De^{-pct}) = 0$

$\Rightarrow Ae^{pl} + Be^{-pl} = 0$ or $Ce^{pct} + De^{-pct} = 0$

$\therefore$ Combining it $\quad A = B = 0$ or $y = 0$

$$\therefore y(x,t) = 0$$

Hence the solution is given by equation (6) only.

D' Alembert's method of solving wave equation

Wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

From D' Alembert method solution is given by

$$y(x,t) = \phi(x+ct) + \psi(x-ct)$$

where ϕ, ψ are arbitrary functions which will be determined from initial conditions.

Remark: If stretched string is infinite then method of solution in article just before D' Alembert's method cannot be applied and solution will be obtained by D' Alembert's method.

Que 1 Solve the p.d.e. $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u = \sin t$ at $x=0$ and $\frac{\partial u}{\partial x} = \sin t$ at $x=0$.

Sol Since boundary conditions at $x=0$ are trigonometric functions, therefore solution of p.d.e. $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ is given by

$$u(x,t) = (A\cos px + B\sin px)(C\cos 2pt + D\sin 2pt) \quad \text{—— (1)}$$

where $p>0$

Now, $u(0,t) = \sin t$ (given)

$\therefore$ By (1) $\quad A(C\cos 2pt + D\sin 2pt) = \sin t$

$\therefore \quad AC = 0,\ AD = 1,\ 2p = 1$

$\Rightarrow \quad C = 0,\ AD = 1,\ p = \frac{1}{2}$

$\therefore$ By (1), $\quad u(x,t) = \cos\frac{x}{2}\sin t + E\sin\frac{x}{2}\sin t$ (where $BD = E$) —— (2)

$\therefore \quad \frac{\partial u}{\partial x} = -\frac{1}{2}\sin\frac{x}{2}\sin t + \frac{1}{2}E\cos\frac{x}{2}\sin t$

Given $\left(\frac{\partial u}{\partial x}\right)_{x=0} = \sin t$

$\therefore \quad \frac{E}{2}\sin t = \sin t \Rightarrow E = 2$

$\therefore$ From (2), $\quad u(x,t) = \cos\frac{x}{2}\sin t + 2\sin\frac{x}{2}\sin t$

i.e., $\boxed{u(x,t) = \cos\left(\frac{x}{2} + 2\sin\frac{x}{2}\right)\sin t}$ Ans

Que2 A thin uniform tightly stretched vibrating string fixed at the points $x=0$ and $x=l$ satisfy the equation

$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$; $y(x,0) = y_0\sin^3\frac{\pi x}{l}$ and released from rest from this position. Find the displacement $y(x,t)$ at any x and any time t.

Sol Since the ends are fixed

$\therefore \; y(0,t) = y(l,t) = 0$

$\therefore$ Solution of wave equation $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ is given

by $\quad y(x,t) = (A\cos px + B\sin px)\,(C\cos cpt + D\sin cpt)$ —— (1)

where $p>0$

Now, $y(0,t) = 0 \Rightarrow A\,(C\cos cpt + D\sin cpt) = 0 \quad \forall t$

$\Rightarrow \quad A = 0$

$\therefore$ By (1), $y(x,t) = (E\cos cpt + F\sin cpt)\sin px$ —————— (2)

where $BC = E$ and $BD = F$

Now, $y(l,t) = 0 \Rightarrow (E\cos cpt + F\sin cpt)\sin pl = 0 \quad \forall t$

$\Rightarrow \quad \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}, \; n\in N$

$\therefore$ Solutions are

$$y(x,t) = \left[E_n\cos\left(\frac{n\pi ct}{l}\right) + F_n\sin\left(\frac{n\pi ct}{l}\right)\right]\sin\frac{n\pi x}{l} \;;\; n=1,2,3,\ldots$$

By principle of superposition,

Solution is given by

$$y(x,t) = \sum_{n=1}^{\infty}\left[E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\frac{n\pi x}{l} \quad \text{--- (3)}$$

Now, $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$ (given)

$$\therefore\ y(x,0) = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$\therefore$ From equ. (3),

$$\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

Comparing both sides, we get

$$E_1 = \frac{3y_0}{4},\quad E_2 = 0,\quad E_3 = -\frac{y_0}{4},\quad E_n = 0 \ \forall\ n \geq 4$$

$\therefore$ By (3),

$$y(x,t) = \frac{y_0}{4}\left[3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right] + \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad \text{--- (4)}$$

Now, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ (given)

$\therefore$ From (4), $\sum_{n=1}^{\infty} F_n \cdot \left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l} = 0 \Rightarrow F_n = 0\ \forall\ n$

$\therefore$ From (4), Solution is

$$y(x,t) = \frac{y_0}{4}\left[3\cos\left(\frac{\pi ct}{l}\right)\sin\left(\frac{\pi x}{l}\right) - \cos\left(\frac{3\pi ct}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\right]$$

Que 3 Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, $0 < x < l$, $0 < t < 4$ with the boundary conditions $u(0,t) = u(l,t) = 0$ and initial conditions $u(x,0) = f(x)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.

Sol As $u(0,t) = u(l,t) = 0$

$\therefore$ Solution of wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $0<x<l$, $0<t<4$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\frac{n\pi x}{l} \quad \text{--- (3)}$$

[Do the same steps as in above question upto equ. (3)]

Given $u(x,0) = f(x)$ $\quad (0<x<l)$

$\therefore$ By (3), $\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} = f(x)$

which is Half range Fourier sine series of $f(x)$ in $(0,l)$

$$\therefore E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

Also, by (3) $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l}\left[-E_n \sin\left(\frac{n\pi ct}{l}\right) + F_n \cos\left(\frac{n\pi ct}{l}\right)\right] \sin\frac{n\pi x}{l}$ --- (4)

Given $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$ $\quad (0<x<l)$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{n\pi c}{l}\right) F_n \sin\frac{n\pi x}{l} = g(x)$$

which is Half range Fourier sine series of $g(x)$ in $(0,l)$

$$\therefore \left(\frac{n\pi c}{l}\right) F_n = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$$

$$\Rightarrow F_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$$

$\therefore$ By (3), Solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\frac{n\pi x}{l}$$

where $E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$, $F_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx$

$0 \le x \le l$; $0 \le t < 4$

Que 4 Using D' Alembert method find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.

Sol The deflection $y(x,t)$ of a vibrating string at any time t is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

By D' Alembert method its solution is given by

$$y(x,t) = \phi(x+ct) + \psi(x-ct) \quad \text{--- (1)}$$

where ϕ and ψ are arbitrary functions.

$\therefore$ Initial deflection $y(x,0) = \phi(x) + \psi(x)$

Given $y(x,0) = f(x) = k(\sin x - \sin 2x)$

$$\therefore \quad \phi(x) + \psi(x) = k(\sin x - \sin 2x) \quad \text{--- (2)}$$

Now, Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ (given)

$\therefore$ By (1), $\quad c\phi'(x) - c\psi'(x) = 0$

i.e, $\quad \phi'(x) - \psi'(x) = 0$

$$\Rightarrow \quad \phi(x) - \psi(x) = d \quad \text{--- (3)}$$

where d is arbitrary constant.

From (2) and (3), $\quad \phi(x) = \frac{k}{2}(\sin x - \sin 2x) + \frac{d}{2}$

$$\psi(x) = \frac{k}{2}(\sin x - \sin 2x) - \frac{d}{2}$$

$\therefore$ By (1), $y(x,t) = \frac{k}{2}\left[\sin(x+ct) - \sin 2(x+ct)\right] + \frac{d}{2}$

$$+ \frac{k}{2}\left[\sin(x-ct) - \sin 2(x-ct)\right] - \frac{d}{2}$$

$$= \frac{k}{2}\left[\{\sin(x+ct) + \sin(x-ct)\} - \{\sin 2(x+ct) + \sin 2(x-ct)\}\right]$$

Hence, $\boxed{y(x,t) = k[\sin x \cos ct - \sin 2x \cos 2ct]}$ Ans

Solution of Two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{——— (1)}$$

Let $u(x,y) = X(x).Y(y)$ ——— (2)

$\therefore \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$

where $X(x)$ is function of x only & $Y(y)$ is function of y only.

$\therefore$ The p.d.e. (1) becomes

$$X''Y + XY'' = 0$$

$$\text{or } \frac{X''}{X} = \frac{-Y''}{Y} = \text{constant} = \lambda \text{ (say)} \qquad \text{——— (3)}$$

Case I: If $\lambda = 0$, then from (3),

$$X'' = 0, \quad Y'' = 0$$

$$\therefore X = Ax + B, \quad Y = Cy + D$$

$\therefore$ By (2), $\boxed{u(x,y) = (Ax+B)(Cy+D)}$ ——— (4)

Case II: If $\lambda > 0$ i.e., $\lambda = p^2; p > 0$, we have from (3),

$$X'' - p^2 X = 0, \quad Y'' + p^2 Y = 0$$

$$\therefore \quad X = Ae^{px} + Be^{-px}, \quad Y = C\cos py + D\sin py$$

$\therefore$ By (2), $\boxed{u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)}$

Case III: If $\lambda < 0$ i.e., $\lambda = -p^2; p > 0$, we have from (3),

$$X'' + p^2 X = 0, \quad Y'' - p^2 Y = 0$$

$$\therefore X = (A\cos px + B\sin px), \quad Y = (Ce^{py} + De^{-py})$$

$\therefore$ By (2), $\boxed{u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})}$

Among these solutions, we are to find those solutions which satisfy initial and boundary conditions consistent with the physical nature.

Remark

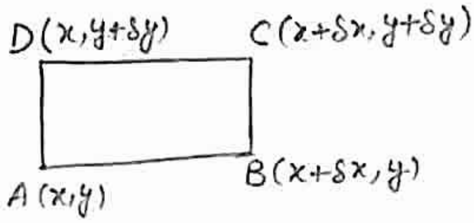
(1) In particular, if $u \to 0$ as $y \to \infty$ for all x, then solution must be

$$\boxed{u(x,y) = e^{-py}\left(E\cos px + F\sin px\right)} \; ; \; p>0$$

and if $u \to 0$ as $x \to \infty$ for all y, then solution must be

$$\boxed{u(x,y) = e^{-px}\left(E\cos py + F\sin py\right)} \; ; \; p>0$$

(2) Two dimensional heat equation is



$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

In steady state it reduces to Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Ques Find the solution of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which satisfies the conditions

(i) $u \to 0$ as $y \to \infty$ for all x (ii) $u = 0$ at $x = 0$ for all y

(iii) $u = 0$ at $x = l$ for all y (iv) $u = lx - x^2$ if $y = 0$ for all $x \in (0, l)$

Solution Since $u \to 0$ as $y \to \infty$ for all x

$\therefore$ Solution is of the form

$$u(x,y) = e^{-py}\left(A\cos px + B\sin px\right) \; ; \; p>0 \qquad \text{——— (1)}$$

$u(0,y) = 0 \;\; \forall \; y$ (given)

$\therefore$ By (1), $\quad A e^{-py} = 0 \;\; \forall \; y \Rightarrow A = 0$

$\therefore$ By (1), $\quad u(x,y) = B e^{-py} \sin px \; ; \; p>0 \qquad \text{——— (2)}$

Now, $u(l,y) = 0 \quad \forall y$

$\therefore$ By (2), $\quad Be^{-py} \sin pl = 0 \quad \forall y$

$\Rightarrow \sin pl = 0 \Rightarrow p = \frac{n\pi}{l}; \; n = 1,2,3,\ldots$

$\therefore$ By (2), $\quad u(x,y) = B_n e^{-\frac{n\pi y}{l}} \sin \frac{n\pi x}{l} \; ; \; n = 1,2,3,\ldots$

$\therefore$ By principle of superposition, solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi y}{l}} \sin \frac{n\pi x}{l} \qquad \text{——— (3)}$$

Now, $u(x,0) = lx - x^2 \quad \forall x \in (0,l) \qquad$ (given)

$\therefore$ By (3), $\quad \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = lx - x^2$

which is half range Fourier sine series in $(0,l)$

$$\therefore B_n = \frac{2}{l}\int_0^l (lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$B_n = \frac{-4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$\left.\begin{aligned} \therefore B_{2n} &= 0 \\ B_{2n-1} &= \frac{8l^2}{(2n-1)^3\pi^3} \end{aligned}\right\} n = 1,2,3,\ldots$$

$\therefore$ By (3), solution is

$$u(x,y) = \frac{8l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-\frac{(2n-1)\pi y}{l}} \sin \frac{(2n-1)\pi x}{l} \qquad \underline{A}$$

Que 2 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions:

$u(0,y) = u(l,y) = u(x,0) = 0$ & $u(x,a) = \sin\frac{n\pi x}{l}$

Sol According to boundary conditions, the solution of given Laplace equation is given by

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \; ; \; p>0 \quad \text{——} (1)$$

Given $u(0,y) = 0$

$\therefore$ By (1) $\quad A(Ce^{py} + De^{-py}) = 0 \Rightarrow A = 0$

$\therefore$ By (1), $\quad u(x,y) = (Ee^{py} + Fe^{-py})\sin px \quad \text{——} (2)$

where $BC = E$, $BD = F$

Now, $u(x,0) = 0$ (given)

$\therefore$ By (2) $\quad (E+F)\sin px = 0 \Rightarrow F = -E$

$\therefore$ By (2), solution is given by

$$u(x,y) = E(e^{py} - e^{-py})\sin px$$

$$\therefore u(x,y) = 2E\sinh py \sin px \quad \text{——} (3)$$

Now, $u(l,y) = 0$ (given)

$\therefore \quad 2E\sinh py \sin pl = 0 \Rightarrow \sin pl = 0 \Rightarrow p = \frac{m\pi}{l}; \; m = 1,2,3,\ldots$

$\therefore$ By (3), solutions are $u(x,y) = 2E_m \sin\frac{m\pi x}{l}\sinh\frac{m\pi y}{l}; \; m = 1,2,\ldots$

$\therefore$ By principle of superposition, solution is

$$u(x,y) = \sum_{m=1}^{\infty} 2E_m \sin\frac{m\pi x}{l}\sinh\frac{m\pi y}{l} \quad \text{——} (4)$$

Now, $u(x,a) = \sin\frac{n\pi x}{l}$ (given)

$$\therefore \sum_{m=1}^{\infty} 2E_m \sin\frac{m\pi x}{l}\sinh\frac{m\pi a}{l} = \sin\frac{n\pi x}{l}$$

$\Rightarrow \quad 2E_n \sinh\frac{n\pi a}{l} = 1 \quad$ and $\quad E_m = 0 \;\; \forall\, m \neq n$

$$\therefore E_n = \frac{1}{2\sinh\frac{n\pi a}{l}}$$

$\therefore$ By (4), Solution is

$$u(x,y) = \frac{\sin\left(\frac{n\pi x}{l}\right)\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \qquad \underline{A}$$

<u>Que 3</u> A long rectangular plate of width π cm with insulated surfaces has its temperature equal to zero on both the long sides and one of the short side so that $u(0,y)=0$, $u(\pi,y)=0$, $u(x,\infty)=0$ and $u(x,0)=kx$. Find the steady state temperature within the plate.

<u>Sol</u> The steady state temperature within the plate is given by the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Since $u\to 0$ as $y\to\infty$ for all x, therefore solution is given by

$$u(x,y)=e^{-py}(A\cos px+B\sin px)\,;\ p>0 \qquad \text{--- (1)}$$

$u(0,y)=0\ \forall\ y$ (given)

$\therefore$ By (1), $\quad Ae^{-py}=0\quad \forall\ y \Rightarrow A=0$

$\therefore$ By (1), $\quad u(x,y)=Be^{-py}\sin px\ ;\ p>0 \qquad \text{--- (2)}$

Now, $u(\pi,y)=0\ \forall\ y$

$\therefore$ By (2), $\quad Be^{-py}\sin p\pi=0\quad\forall\ y$

$\Rightarrow \sin p\pi=0 \Rightarrow p=n\,,\ n=1,2,3,\ldots$

$\therefore$ By (2), $\quad u(x,y)=B_n e^{-ny}\sin nx\ ;\ n=1,2,3,\ldots$

$\therefore$ By principle of superposition, solution is given by

$$u(x,y)=\sum_{n=1}^{\infty} B_n e^{-ny}\sin nx \qquad \text{--- (3)}$$

Now, $u(x,0)=kx$

$\therefore$ By ③, $\sum_{n=1}^{\infty} B_n \sin nx = kx$

It is Fourier half range sine series in $[0, \pi]$

$$\therefore \quad B_n = \frac{2}{\pi}\int_0^{\pi} kx \sin nx \, dx$$

$$= \frac{2k}{\pi}\left[x.\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2k}{\pi}\cdot\frac{\pi}{n}(-1)^{n+1}$$

$$\therefore \quad B_n = \frac{2k}{n}(-1)^{n+1} \quad ; \quad n = 1, 2, 3, \ldots$$

$\therefore$ By ③, solution is

$$\boxed{u(x,y) = 2k \sum_{n=1}^{\infty} (-1)^{n+1} e^{-ny} \frac{\sin nx}{n}}$$