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Huber loss

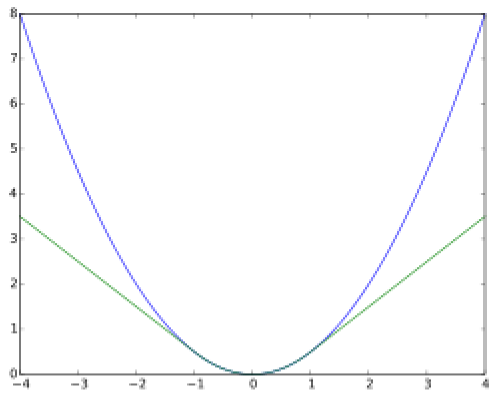
In statistics, the **Huber loss** is a loss function used in robust regression, that is less sensitive to outliers in data than the squared error loss. A variant for classification is also sometimes used.

Contents

Definition

The Huber loss function describes the penalty incurred by an estimation procedure f. Huber (1964) defines the loss function piecewise by[1]

$$L_\delta(a) = \begin{cases} \frac{1}{2}a^2 & \text{for } |a| \le \delta, \\ \delta(|a| - \frac{1}{2}\delta), & \text{otherwise.} \end{cases}$$



Huber loss (green, $\delta = 1$) and squared error loss (blue) as a function of $y - f(x)$

This function is quadratic for small values of a, and linear for large values, with equal values and slopes of the different sections at the two points where $|a| = \delta$. The variable a often refers to the residuals, that is to the difference between the observed and predicted values $a = y - f(x)$, so the former can be expanded to[2]

$$L_\delta(y, f(x)) = \begin{cases} \frac{1}{2}(y - f(x))^2 & \text{for} |y - f(x)| \le \delta, \\ \delta\, |y - f(x)| - \frac{1}{2}\delta^2 & \text{otherwise.} \end{cases}$$

Motivation

Two very commonly used loss functions are the squared loss, $L(a) = a^2$, and the absolute loss, $L(a) = |a|$. The squared loss function results in an arithmetic mean-unbiased estimator, and the absolute-value loss function results in a median-unbiased estimator (in the one-dimensional case, and a geometric median-unbiased estimator for the multi-dimensional case). The squared loss has the disadvantage that it has the tendency to be dominated by outliers—when summing over a set of a

's (as in $\sum_{i=1}^{n} L(a_i)$), the sample mean is influenced too much by a few particularly large a-values when the distribution is heavy tailed: in terms of estimation theory, the asymptotic relative efficiency of the mean is poor for heavy-tailed distributions.

As defined above, the Huber loss function is strongly convex in a uniform neighborhood of its minimum $a = 0$; at the boundary of this uniform neighborhood, the Huber loss function has a differentiable extension to an affine function at points $a = -\delta$ and $a = \delta$. These properties allow it to combine much of the sensitivity of the mean-unbiased, minimum-variance estimator of the mean (using the quadratic loss function) and the robustness of the median-unbiased estimator (using the absolute value function).

Pseudo-Huber loss function

The **Pseudo-Huber loss function** can be used as a smooth approximation of the Huber loss function. It combines the best properties of **L2** squared loss and **L1** absolute loss by being strongly convex when close to the target/minimum and less steep for extreme values. This steepness can be controlled by the δ value. The **Pseudo-Huber loss function** ensures that derivatives are continuous for all degrees. It is defined as[3][4]

$$L_\delta(a) = \delta^2 \left(\sqrt{1 + (a/\delta)^2} - 1 \right).$$

As such, this function approximates $a^2/2$ for small values of a, and approximates a straight line with slope δ for large values of a.

While the above is the most common form, other smooth approximations of the Huber loss function also exist.[5]

Variant for classification

For classification purposes, a variant of the Huber loss called *modified Huber* is sometimes used. Given a prediction $f(x)$ (a real-valued classifier score) and a true binary class label $y \in \{+1, -1\}$, the modified Huber loss is defined as[6]

$$L(y, f(x)) = \begin{cases} \max(0, 1 - y\, f(x))^2 & \text{for } y\, f(x) \geq -1, \\ -4y\, f(x) & \text{otherwise.} \end{cases}$$

The term $\max(0, 1 - y\, f(x))$ is the hinge loss used by support vector machines; the quadratically smoothed hinge loss is a generalization of L.[6]

Applications

The Huber loss function is used in robust statistics, M-estimation and additive modelling.[7]

See also

- Winsorizing
- Robust regression
- M-estimator
- Visual comparison of different M-estimators

References

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