EKF Localization

(자율주행 핵심기술 SLAM 단기강좌)

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Kalman Filter

- Algorithm

Algorithm Kalman filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)

$$\left.\begin{aligned}\bar{\mu}_t &= A_t\mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t &= A_t\Sigma_{t-1}A_t^T + R_t\end{aligned}\right\} \text{Predicted belief } \overline{bel}(x_t)$$

Kalman Gain

$$\boxed{K_t} = \bar{\Sigma}_t C_t^T (C_t\bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

$$\left.\begin{aligned}\mu_t &= \bar{\mu}_t + K_t\boxed{(z_t - C_t\bar{\mu}_t)} \\ \Sigma_t &= (I - K_t C_t)\bar{\Sigma}_t\end{aligned}\right\} \text{Posterior belief } bel(x_t)$$

Return μ_t, Σ_t

innovation

Innovation : output prediction error, measurement residual

Extended Kalman Filter (EKF)

- Extension of KF to non-linear systems

<u>Non-linear - Gaussian system:</u>

$$\begin{cases} x_t = g(x_{t-1}, u_t + \boxed{\varepsilon_t}) \\ z_t = h(x_t) + \boxed{\delta_t} \end{cases}$$

Process noise

$$\varepsilon_t \sim p_\varepsilon(\cdot) = \mathbb{N}(0, R_t)$$

Measurement noise

$$\delta_t \sim p_\delta(\cdot) = \mathbb{N}(0, Q_t)$$

The key idea of the EKF approximation is linearization.

EKF utilizes Taylor expansion for linearization.

Extended Kalman Filter (EKF)

- Linearization
  - Prediction

$$g(x_{t-1}, u_t) \approx g(\mu_{t-1}, u_t) + \underbrace{\frac{\partial g(\mu_{t-1}, u_t)}{\partial x_{t-1}}}_{=: G_t}(x_{t-1} - \mu_{t-1}) = g(\mu_{t-1}, u_t) + G_t(x_{t-1} - \mu_{t-1})$$

$G_t \rightarrow$ Jacobian

  - Correction

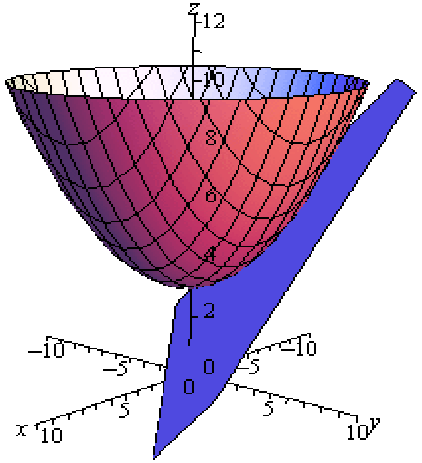
$$h(x_t) \approx h(\bar{\mu}_t) + \underbrace{\frac{\partial h(\bar{\mu}_t)}{\partial x_t}}_{=: H_t}(x_t - \bar{\mu}_t) = h(\bar{\mu}_t) + H_t(x_t - \bar{\mu}_t)$$

$H_t \rightarrow$ Jacobian

Extended Kalman Filter (EKF)

- Jacobian matrix
  - Typically $m \times n$ non-square matrix
  - Given a vector-valued function

$$g(x) = \begin{bmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{bmatrix}$$



  - Jacobian matrix is defined as

$$G_t = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}$$

  - The orientation of the tangent plane to the vector-valued function at a given point
  - Generalizes the gradient of a scalar valued function

Extended Kalman Filter (EKF)

$$\begin{cases} x_t = g(x_{t-1}, u_t + \varepsilon_t) \\ z_t = h(x_t) + \delta_t \end{cases}$$

KF

Prediction Step

$$\bar{\mu}_t = A_t\mu_{t-1} + B_t u_t$$
$$\bar{\Sigma}_t = A_t\Sigma_{t-1}A_t^T + R_t$$

Correction Step

$$K_t = \bar{\Sigma}_t C_t^T (C_t\bar{\Sigma}_t C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t)$$
$$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$$

EKF

Prediction Step

$$\bar{\mu}_t = g(\mu_{t-1}, u_t)$$
$$\bar{\Sigma}_t = G_t\Sigma_{t-1}G_t^T + G_t R_t G_t^T$$

Correction Step

$$K_t = \bar{\Sigma}_t H_t^T (H_t\bar{\Sigma}_t H_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$
$$\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$$

Jacobian

Extended Kalman Filter (EKF)

- Kalman filter vs. EKF

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)

$$\bar{\mu}_t = A_t\mu_{t-1} + B_t u_t$$
$$\bar{\Sigma}_t = A_t\Sigma_{t-1}A_t^T + R_t$$
$$K_t = \bar{\Sigma}_t C_t^T (C_t\bar{\Sigma}_t C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t)$$
$$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$$

return μ_t, Σ_t

Algorithm EKF($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)

$$\bar{\mu}_t = g(\mu_{t-1}, u_t)$$
$$\bar{\Sigma}_t = G_t\Sigma_{t-1}G_t^T + G_t R_t G_t^T$$
$$K_t = \bar{\Sigma}_t H_t^T (H_t\bar{\Sigma}_t H_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$$
$$\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$$

return μ_t, Σ_t

EKF Localization

- EKF localization with known correspondences
  - Input to the algorithm

    Mean μ_{t-1} and covariance Σ_{t-1}

    A control u_t, a map m

    A set of features $z_t = \{z_t^1, z_t^2, \cdots\}$ measured at time t

    The correspondence variables $c_t = \{c_t^1, c_t^2, \cdots\}$
  - Consideration in the algorithm

    A map is a collection of features.

    Velocity motion model

    Point landmarks

EKF Localization

- Prediction step
  - State vector

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} -\frac{\hat{v}_t}{\hat{w}_t}\sin\theta + \frac{\hat{v}_t}{\hat{w}_t}\sin(\theta + \hat{w}_t\Delta t) \\ \frac{\hat{v}_t}{\hat{w}_t}\cos\theta - \frac{\hat{v}_t}{\hat{w}_t}\cos(\theta + \hat{w}_t\Delta t) \\ \hat{w}_t\Delta t \end{bmatrix}$$

$$\begin{bmatrix} \hat{v}_t \\ \hat{w}_t \end{bmatrix} = \begin{bmatrix} v_t \\ w_t \end{bmatrix} + \begin{bmatrix} \epsilon_{\alpha_1|v|+\alpha_2|w|} \\ \epsilon_{\alpha_3|v|+\alpha_4|w|} \end{bmatrix} = \begin{bmatrix} v_t \\ w_t \end{bmatrix} + \mathbb{N}(0, M_t)$$

$$\underbrace{\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix}}_{x_t} = \underbrace{\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} -\frac{v_t}{w_t}\sin\theta + \frac{v_t}{w_t}\sin(\theta + w_t\Delta t) \\ \frac{v_t}{w_t}\cos\theta - \frac{v_t}{w_t}\cos(\theta + w_t\Delta t) \\ w_t\Delta t \end{bmatrix}}_{g(x_{t-1}, u_t)} + \mathbb{N}(0, R_t)$$

EKF Localization

- Prediction step
  - Jacobian

$$g(x_{t-1}, u_t) \approx g(\mu_{t-1}, u_t) + G_t(x_{t-1} - \mu_{t-1})$$

$$G_t = \begin{bmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{bmatrix}$$

$$G_t = \frac{\partial g(\mu_{t-1}, u_t)}{\partial x_{t-1}} = \begin{bmatrix} 1 & 0 & -\frac{v_t}{w_t}\cos\theta + \frac{v_t}{w_t}\cos(\theta + w_t \Delta t) \\ 0 & 1 & -\frac{v_t}{w_t}\sin\theta + \frac{v_t}{w_t}\sin(\theta + w_t \Delta t) \\ 0 & 0 & 1 \end{bmatrix}$$

$$V_t = \begin{bmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial w_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial w_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial w_t} \end{bmatrix}$$

$$V_t = \frac{\partial g(\mu_{t-1}, u_t)}{\partial u_t} = \begin{bmatrix} -\frac{\sin\theta + \sin(\theta + w_t \Delta t)}{w_t} & \frac{v_t(\sin\theta - \sin(\theta + w_t \Delta t))}{w_t^2} + \frac{v_t \cos(\theta + w_t \Delta t)\,\Delta t}{w_t} \\ \frac{\cos\theta - \cos(\theta + w_t \Delta t)}{w_t} & -\frac{v_t(\cos\theta - \cos(\theta + w_t \Delta t))}{w_t^2} + \frac{v_t \sin(\theta + w_t \Delta t)\,\Delta t}{w_t} \\ 0 & \Delta t \end{bmatrix}$$

  - Covariance matrix of the noise in control space

$$M_t = \begin{bmatrix} \alpha_1 v_t^2 + \alpha_2 w_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 w_t^2 \end{bmatrix}$$

$V_t M_t V_t^T$: approximate mapping between the motion noise in control space to the motion noise in state space

EKF Localization

- Correction step

$j = c_t^i$: The identity of the landmark that corresponds to the i th component in the measurement vector.

$$\underbrace{\begin{bmatrix} r_t^i \\ \phi_t^i \end{bmatrix}}_{z_t^i} = \underbrace{\begin{bmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{bmatrix}}_{h(x_t, j, m)} + \mathbb{N}(0, Q_t)$$

$$Q_t = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix}$$

$\begin{bmatrix} m_{j,x} \\ m_{j,y} \end{bmatrix}$: The coordinate of the j th landmark

EKF Localization

- Correction step
  - Linearization of the measurement model

$$h(x_t, j, m) \approx h(\bar{\mu}_t, j, m) + H_t^i(x_t - \bar{\mu}_t)$$

$$H_t = \frac{\partial h(\bar{\mu}_t, j, m)}{\partial x_t} = \begin{bmatrix} \frac{\partial r_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,\theta}} \end{bmatrix}$$

  - Jacobian

$$H_t^i = \begin{bmatrix} -\frac{m_{j,x} - \bar{\mu}_{t,x}}{\sqrt{q}} & -\frac{m_{j,y} - \bar{\mu}_{t,y}}{\sqrt{q}} & 0 \\ \frac{m_{j,y} - \bar{\mu}_{t,y}}{\sqrt{q}} & -\frac{m_{j,x} - \bar{\mu}_{t,x}}{\sqrt{q}} & -1 \end{bmatrix}$$

$$q = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$$

  - Assume all feature measurement probabilities are independent

$$p(z_t \mid x_t, c_t, m) = \prod_i p\left(z_t^i \mid x_t, c_t^i, m\right)$$

EKF Localization

- EKF localization with known correspondences

Algorithm EKF_localization_known_correspondence($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m$)

1: $\theta = \mu_{t-1,\theta}$

2: $$G_t = \begin{bmatrix} 1 & 0 & -\frac{v_t}{w_t}\cos\theta + \frac{v_t}{w_t}\cos(\theta + w_t\Delta t) \\ 0 & 1 & -\frac{v_t}{w_t}\sin\theta + \frac{v_t}{w_t}\sin(\theta + w_t\Delta t) \\ 0 & 0 & 1 \end{bmatrix}$$

3: $$V_t = \begin{bmatrix} -\frac{\sin\theta + \sin(\theta + w_t\Delta t)}{w_t} & \frac{v_t(\sin\theta - \sin(\theta + w_t\Delta t))}{w_t^2} + \frac{v_t\cos(\theta + w_t\Delta t)\,\Delta t}{w_t} \\ \frac{\cos\theta - \cos(\theta + w_t\Delta t)}{w_t} & -\frac{v_t(\cos\theta - \cos(\theta + w_t\Delta t))}{w_t^2} + \frac{v_t\sin(\theta + w_t\Delta t)\,\Delta t}{w_t} \\ 0 & \Delta t \end{bmatrix}$$

4: $$M_t = \begin{bmatrix} \alpha_1 v_t^2 + \alpha_2 w_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 w_t^2 \end{bmatrix}$$

EKF Localization

- EKF localization with known correspondences

$$\bar{\mu}_t = \mu_{t-1} + \begin{bmatrix} -\frac{v_t}{w_t}\sin\theta + \frac{v_t}{w_t}\sin(\theta + w_t\Delta t) \\ \frac{v_t}{w_t}\cos\theta - \frac{v_t}{w_t}\cos(\theta + w_t\Delta t) \\ w_t\Delta t \end{bmatrix}$$

$$\bar{\Sigma}_t = G_t\Sigma_{t-1}G_t^T + V_t M_t V_t^T$$

$$Q_t = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix}$$

for all observed features $z_t^i = (r_t^i, \phi_t^i)^T$ do

$$j = c_t^i$$

$$q = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$$

$$\hat{z}_t^i = \begin{bmatrix} \sqrt{q} \\ \mathrm{atan2}(m_{j,y} - \bar{\mu}_{t,y}, m_{j,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{bmatrix}$$

EKF Localization

- EKF localization with known correspondences

$$H_t^i = \begin{bmatrix} -\dfrac{m_{j,x}-\bar{\mu}_{t,x}}{\sqrt{q}} & -\dfrac{m_{j,y}-\bar{\mu}_{t,y}}{\sqrt{q}} & 0 \\ \dfrac{m_{j,y}-\bar{\mu}_{t,y}}{\sqrt{q}} & -\dfrac{m_{j,x}-\bar{\mu}_{t,x}}{\sqrt{q}} & -1 \end{bmatrix}$$

$$K_t^i = \bar{\Sigma}_t {H_t^i}^T (H_t^i \bar{\Sigma}_t {H_t^i}^T + Q_t)^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^i)$$

$$\Sigma_t = (I - K_t^i H_t^i)\bar{\Sigma}_t$$

end for

$\mu_t = \bar{\mu}_t$

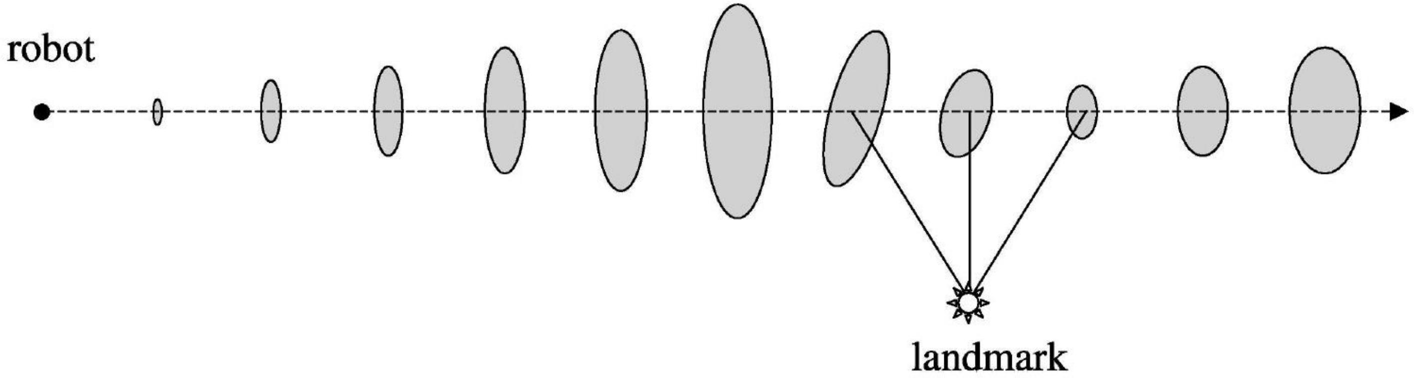
$\Sigma_t = \bar{\Sigma}_t$

$$p_{z_t} = \prod_i \det(2\pi S_t^i)^{-1/2} \exp\{-\frac{1}{2}\left(z_t^i - \hat{z}_t^i\right)^T {S_t^i}^{-1} (z_t^i - \hat{z}_t^i)\}$$

return μ_t, Σ_t, p_{z_t}

EKF Localization

- Example



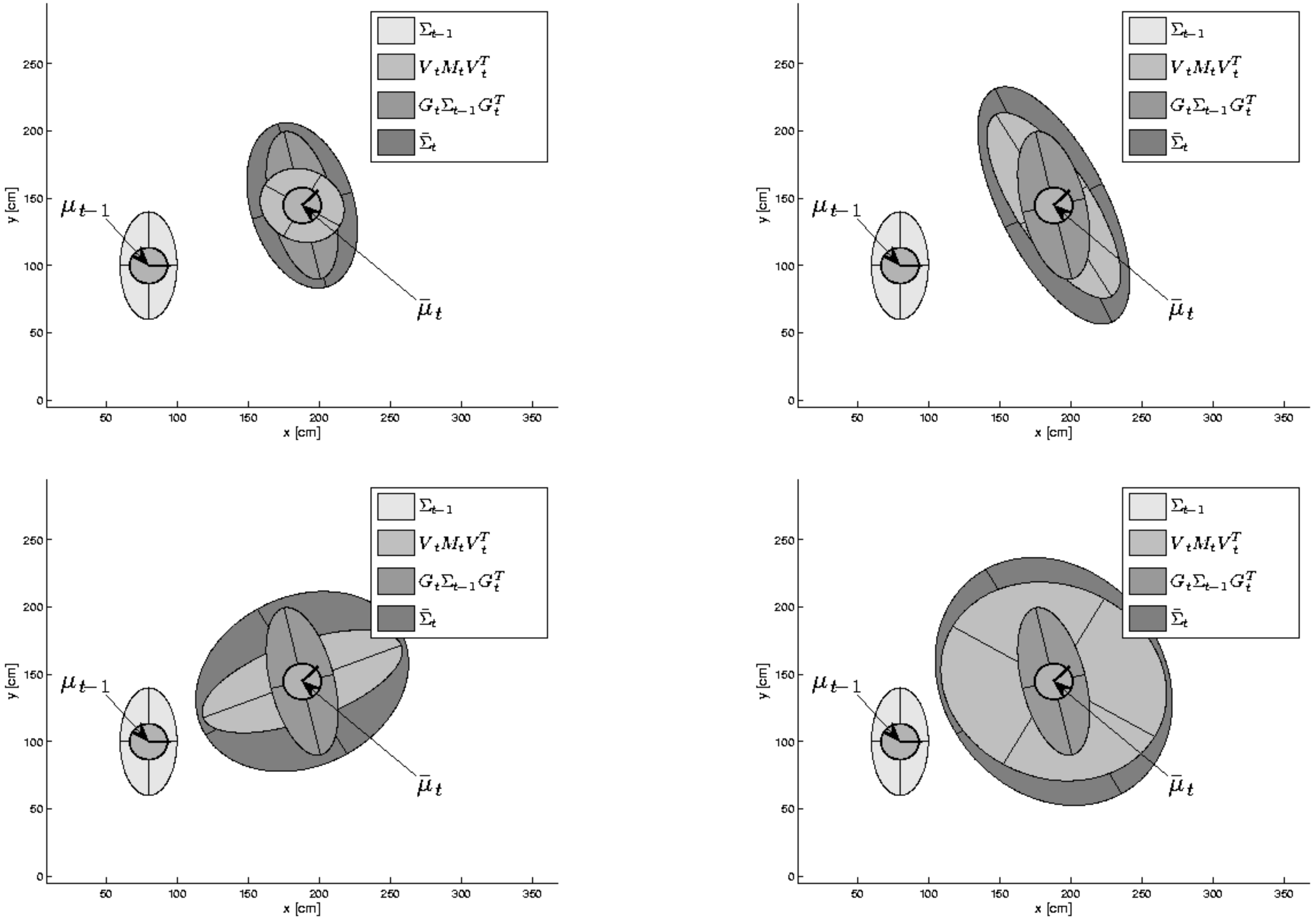
- The robot moves on a straight line.

- As it progresses, its uncertainty increases gradually.

- When it observes a landmark with known position, the uncertainty is reduced.

EKF Localization

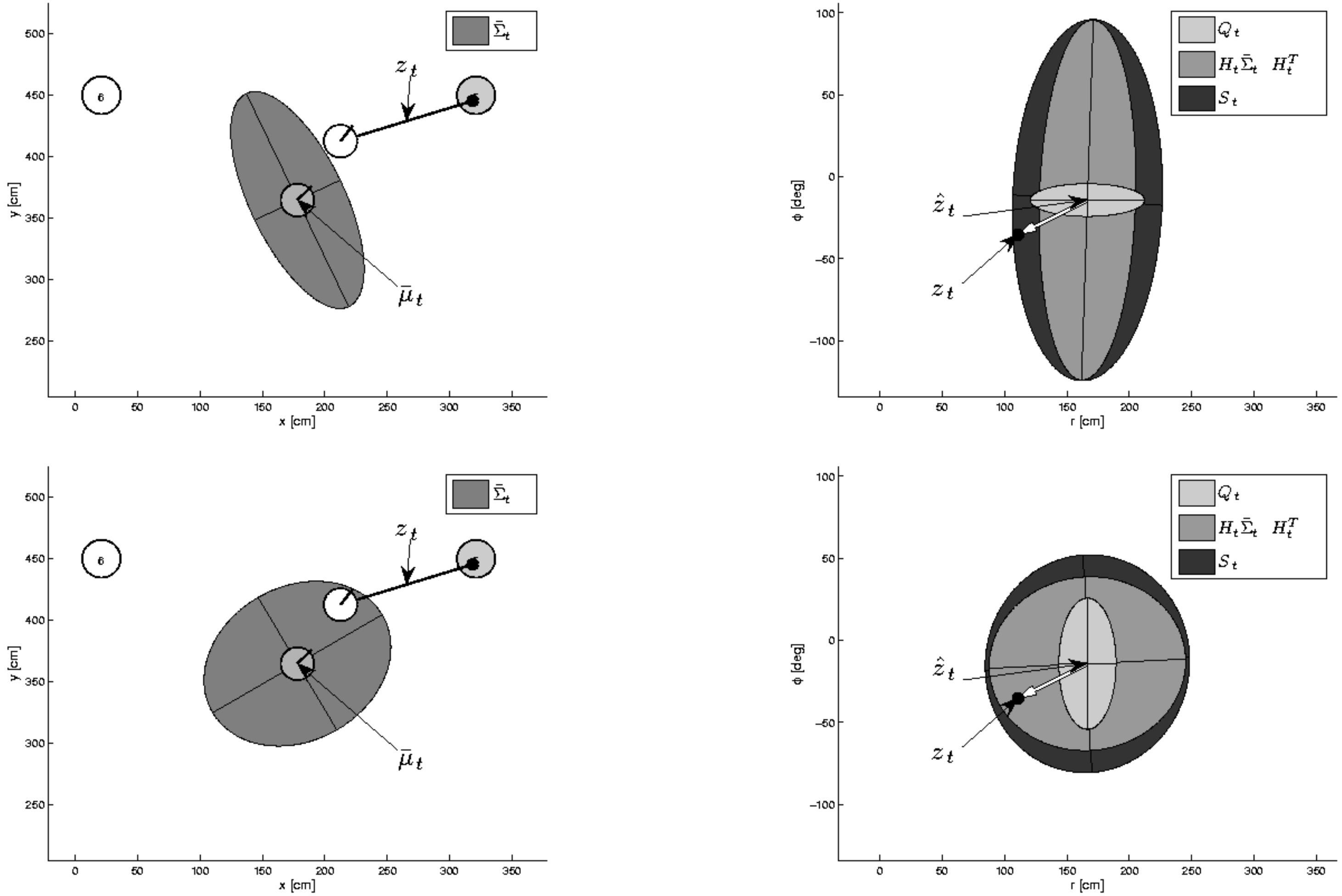
- Prediction step



From Thrun Burgard Fox, Probabilistic Robotics, MIT Press 2006

EKF Localization

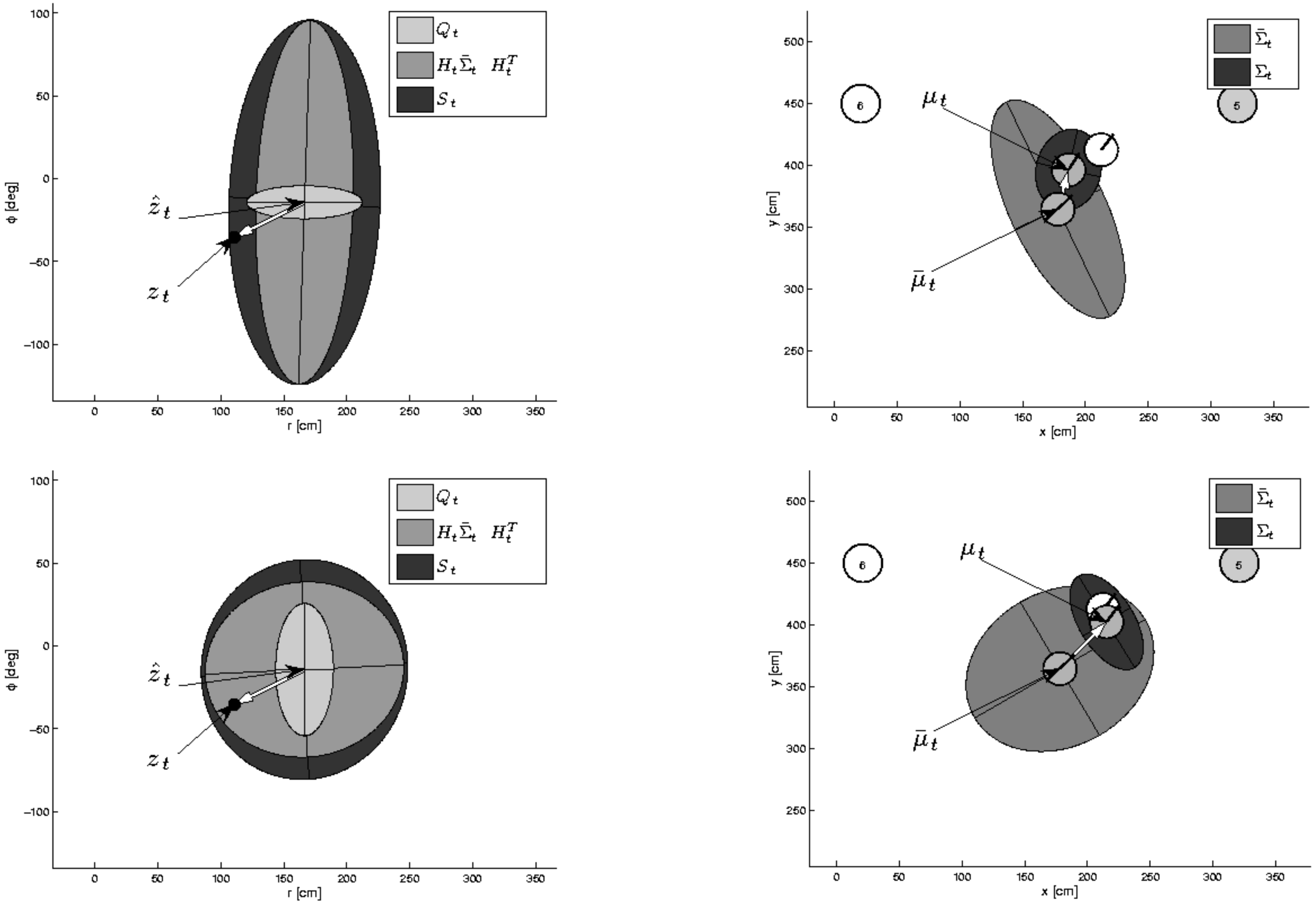
- Measurement prediction



From Thrun Burgard Fox, Probabilistic Robotics, MIT Press 2006

EKF Localization

- Correction step



From Thrun Burgard Fox, Probabilistic Robotics, MIT Press 2006

EKF Localization

- EKF localization with unknown correspondence

Algorithm EKF_localization($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$)

1: $\theta = \mu_{t-1,\theta}$

2: $$G_t = \begin{bmatrix} 1 & 0 & -\frac{v_t}{w_t}\cos\theta + \frac{v_t}{w_t}\cos(\theta + w_t\Delta t) \\ 0 & 1 & -\frac{v_t}{w_t}\sin\theta + \frac{v_t}{w_t}\sin(\theta + w_t\Delta t) \\ 0 & 0 & 1 \end{bmatrix}$$

3: $$V_t = \begin{bmatrix} -\frac{\sin\theta + \sin(\theta + w_t\Delta t)}{w_t} & \frac{v_t(\sin\theta - \sin(\theta + w_t\Delta t))}{w_t^2} + \frac{v_t\cos(\theta + w_t\Delta t)\,\Delta t}{w_t} \\ \frac{\cos\theta - \cos(\theta + w_t\Delta t)}{w_t} & -\frac{v_t(\cos\theta - \cos(\theta + w_t\Delta t))}{w_t^2} + \frac{v_t\sin(\theta + w_t\Delta t)\,\Delta t}{w_t} \\ 0 & \Delta t \end{bmatrix}$$

4: $$M_t = \begin{bmatrix} \alpha_1 v_t^2 + \alpha_2 w_t^2 & 0 \\ 0 & \alpha_3 v_t^2 + \alpha_4 w_t^2 \end{bmatrix}$$

EKF Localization

- EKF localization

5: $\bar{\mu}_t = \mu_{t-1} + \begin{bmatrix} -\frac{v_t}{w_t}\sin\theta + \frac{v_t}{w_t}\sin(\theta + w_t\Delta t) \\ \frac{v_t}{w_t}\cos\theta - \frac{v_t}{w_t}\cos(\theta + w_t\Delta t) \\ w_t\Delta t \end{bmatrix}$

6: $\bar{\Sigma}_t = G_t\Sigma_{t-1}G_t^T + V_tM_tV_t^T$

7: $Q_t = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\phi^2 \end{bmatrix}$

8: for all observed features $z_t^i = (r_t^i, \phi_t^i)^T$ do

9: for all landmarks k in the map m do

10: $q = (m_{k,x} - \bar{\mu}_{t,x})^2 + (m_{k,y} - \bar{\mu}_{t,y})^2$

11: $\hat{z}_t^k = \begin{bmatrix} \sqrt{q} \\ \mathrm{atan2}(m_{k,y} - \bar{\mu}_{t,y}, m_{k,x} - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{bmatrix}$

EKF Localization

- EKF localization

$$H_t^k = \begin{bmatrix} -\frac{m_{k,x}-\bar{\mu}_{t,x}}{\sqrt{q}} & -\frac{m_{k,y}-\bar{\mu}_{t,y}}{\sqrt{q}} & 0 \\ \frac{m_{k,y}-\bar{\mu}_{t,y}}{\sqrt{q}} & -\frac{m_{k,x}-\bar{\mu}_{t,x}}{\sqrt{q}} & -1 \end{bmatrix}$$

$$S_t^k = H_t^k \bar{\Sigma}_t {H_t^k}^T + Q_t$$

end for

$$j(i) = \underset{k}{\operatorname{argmax}} \det(2\pi S_t^k)^{-1/2} \exp\{-\frac{1}{2}\left(z_t^i - \hat{z}_t^k\right)^T {S_t^k}^{-1} (z_t^i - \hat{z}_t^k)\}$$

$$K_t^i = \bar{\Sigma}_t {H_t^{j(i)}}^T (S_t^{j(i)})^{-1}$$

$$\mu_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^{j(i)})$$

$$\Sigma_t = (I - K_t^i H_t^{j(i)}) \bar{\Sigma}_t$$

end for

$\mu_t = \bar{\mu}_t$

$\Sigma_t = \bar{\Sigma}_t$

return μ_t, Σ_t