Lecture 32: Numerical Linear Algebra (UMA021): Matrix Computations

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Dot product:

$\|v\|$ $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $2v$ $(V, +, \cdot)$

Dot product of two vectors:

The dot product of two vectors $v_1 = (x_1, x_2, \cdots x_n)^t$ and $v_2 = (y_1, y_2, \cdots y_n)^t$ is defined as
$v_1.v_2 = v_1^t v_2 = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots x_n y_n.$

$$v_1 \cdot v_2 = v_1^t v_2 = x_1 y_1 + x_2 y_2 + x_3 y_3 - - - x_n y_n$$

$$[\ ]_{n\times 1} \cdot [\ ]_{n\times 1} = [\ ]_{1\times n} [\ ]_{n\times 1} = [\ ]_{1\times 1}$$

$\langle v_1, v_2 \rangle = v_1 \cdot v_2$ Inner product of two vectors.

$(V, \|v\|)$ Normed space $(V, \langle v_1, v_2 \rangle)$ Inner product space

Orthogonal vectors:

Orthogonal of two vectors:

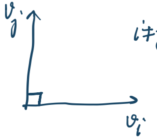
A set of vectors $v_1, v_2, v_3, \cdots v_n$ are said to be orthogonal if their dot product pairwise is zero i.e. $v_i . v_j = 0$, for $i \neq j$.
Note: An orthogonal set of non-zero vectors is linearly independent.

$v_i \cdot v_j = 0 \quad \forall\, i \neq j$

$v . v = |v|$

$v_i \cdot v_i = ||v_i||^2$

$0 \cdot v_j = 0$



$i \neq j$

Orthogonal and Orthonormal vectors:

The length of a vector:

The length of a vector $v = (x_1, x_2. \cdots, x_n)^t$ is defined as
$\|v\|_2 = v.v = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$.

$\|v\|_2 = \sqrt{x_1^2 + x_2^2 - - x_n^2}$

Orthonormal vectors:

Orthonormal set of two vectors:

A set of vectors $v_1, v_2, v_3, \cdots v_n$ are said to be orthonormal if they are orthogonal and every vector has length one.

$$v_i \cdot v_j = \begin{cases} 0 & \forall\ i \neq j \\ 1 & \forall\ i = j \end{cases} \quad \begin{matrix} \Rightarrow \text{orthogonal} \\ \Rightarrow \text{length } 1 \end{matrix}$$

v_1 & v_2 are orthonormal if

$$v_1 \cdot v_2 = 0$$

$$\|v_1\|_2 = 1, \qquad \|v_2\|_2 = 1$$

Orthogonal and Orthonormal vectors:

$\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$

Orthonormal: Example

Show that the vectors $v_1 = (0, 4, 2)^t, v_2 = (-5, -1, 2)^t, v_3 = (1, -1, 2)^t$ form an orthogonal set and use these to determine a set of orthonormal vectors.

$v_1 \cdot v_2 = 0(-5) + 4(-1) + 2(2) = -4 + 4 = 0$

$v_2 \cdot v_3 = -5(1) - 1(-1) + 2(2) = -5 + 1 + 4 = 0$

$v_1 \cdot v_3 = 0(1) + 4(-1) + 2(2) = -4 + 4 = 0$

$\Rightarrow$ v_1, v_2, v_3 form an orthogonal set.

To make it orthonormal, we take

$e_1 = \frac{v_1}{\|v_1\|}$, $e_2 = \frac{v_2}{\|v_2\|}$, $e_3 = \frac{v_3}{\|v_3\|}$

$$e_1 = \frac{(0,4,2)^t}{\sqrt{0+16+4}} = \frac{1}{\sqrt{20}}\begin{bmatrix}0\\4\\2\end{bmatrix}, \qquad \|e_1\| = \frac{\sqrt{20}}{\sqrt{20}} = 1$$

$$e_2 = \frac{(-5,-1,2)^t}{\sqrt{25+1+4}} = \frac{1}{\sqrt{30}}\begin{bmatrix}-5\\-1\\2\end{bmatrix} \Rightarrow \|e_2\| = 1$$

$$e_3 = \frac{(1,-1,2)^t}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix} \qquad \|e_3\| = 1$$

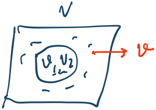
Orthogonal and Orthonormal vectors:

$\langle v_1, v_2 \, -- \, v_n \rangle \to$ L.I
Span(V)

Orthogonal and Orthonormal basis:

A basis that is an orthogonal set is called an orthogonal basis.
A basis that is an orthonormal set is called an orthonormal basis.

$\mathbb{R}^3$ $(1,0,0), (0,1,0)$ $(0,0,1)$



$v = c_1 v_1 \, --- \, c_n v_n$

To check L.I

If $v_1, v_2 \, --- \, v_n$ are L.I

then $c_1 v_1 + c_2 v_2 \, --- \, c_n v_n = 0$

$\Rightarrow c_1 = 0, \; c_2 = 0, \; -- \; c_n = 0$

Gram-Schmidt Orthogonalization Process:

The Gram-Schmidt process or Gram-Schmidt algorithm is a way of finding a set of two or more vectors that are perpendicular to each other.

Gram-Schmidt Orthogonalization Process:

Let $v_1, v_2, v_3, \cdots, v_n$ be n vectors but they are not necessarily orthogonal.

Gram-Schmidt Process takes these vectors and forms a new orthogonal.

original vectors: $v_1, v_2, v_3, \cdots, v_n \to\to$

orthogonal vectors: $u_1, u_2, u_3, \cdots, u_n$

$u_1 = v_1;$

$u_2 = v_2 - \frac{v_2.u_1}{\|u_1\|_2^2} u_1;$ $\qquad v_2 - \left(\frac{v_2 \cdot u_1}{\|u_1\|^2}\right) u_1$

$u_3 = v_3 - \frac{v_3.u_1}{\|u_1\|_2^2} u_1 - \frac{v_3.u_2}{\|u_2\|_2^2} u_2;$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$$u_n = v_n - \sum_{i=1}^{n-1} \frac{v_n.u_i}{\|u_i\|_2^2} u_i.$$

To make it orthonormal we take $e_1 = \frac{u_1}{\|u_1\|}$, $e_2 = \frac{u_2}{\|u_2\|}$, $\cdots e_n = \frac{u_n}{\|u_n\|}$

Gram-Schmidt Orthogonalization Process: Example

Using Gram-Schmidt Orthogonalization Process to find orthogonal basis from basis $v_1 = (1, -1, 1)^t, v_2 = (1, 0, 1)^t, v_3 = (1, 1, 2)^t$.

Let orthogonal basis be u_1, u_2 & u_3

$$u_1 = v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{\|u_1\|^2} u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1(1) - 1(0) + 1(1)}{\left(\sqrt{1+1+1}\right)^2} u_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = u_2$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{\|u_1\|^2} u_1 - \frac{v_3 \cdot u_2}{\|u_2\|^2} u_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1(1) + 1(-1) + 2(1)}{\left(\sqrt{1+1+1}\right)^2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right)}{\left(\sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}\right)^2} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{5/3}{2/3} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{5}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = u_3$$

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$\Rightarrow$ u_1, u_2, u_3 are orthogonal vectors.

Gram-Schmidt Process:

Gram-Schmidt Orthogonalization Process: Exercise

Using Gram-Schmidt Orthogonalization Process to find orthogonal basis from basis
$v_1 = (1, 2, -2)^t, v_2 = (4, 3, 2)^t, v_3 = (1, 2, 1)^t$.