Lecture 4: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Fixed point iteration method

Root finding problem

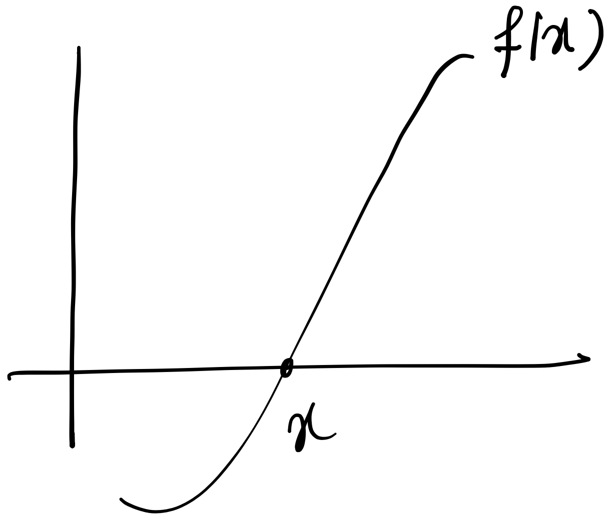
$$f(x) = 0$$

$$f(x) = 0$$

$$\underbrace{f(x) + x}_{g(x)} = x$$

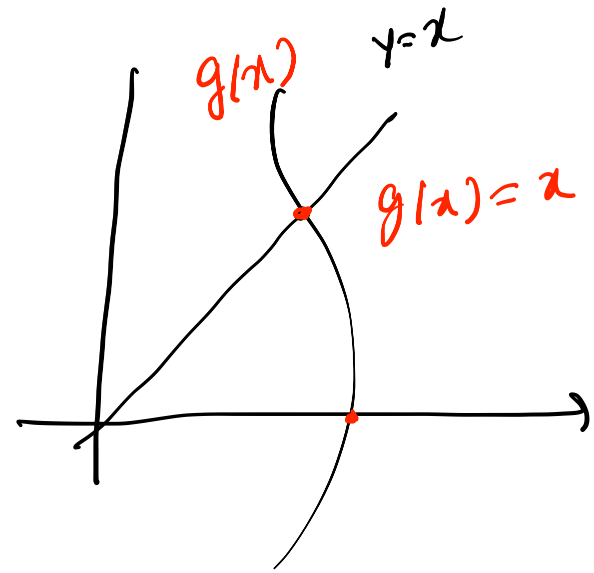
$$g(x) = x$$

$$-f(x) + x = x$$



fixed point problem

$$g(x) = x$$



$$g(x) = x$$

$$\underbrace{g(x) - x}_{f(x)} = 0$$

$$f(x) = 0$$

Fixed point iteration

Fixed Point:

A fixed point for a function $g(x)$ is a number at which the value of function does not change, when function is applied.

$$g(x) = x$$

Fixed point iteration

Example:

Determine any fixed point of the function $g(x) = x^2 - 2$.

$f(x) = 0$ Let p be the fixed pt. of $g(x) = x^2 - 2$

$\Rightarrow g(p) = p$

$p^2 - 2 = p$

$f(x) = p^2 - p - 2 = 0$

$(p-2)(p+1) = 0$

$p = 2, -1$

$\underbrace{x + \frac{1}{2} f(x)} = x$

$f(x) = 0$

$\underbrace{x + f(x)} = x$

$\underbrace{+x - f(x)} = x$

$x + c f(x) = x$

$c \in \mathbb{R}$

Fixed point iteration

$f(x) = 0$

Fixed point forms:

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. write all the possible ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.

1. $x = x^3 + 4x^2 + x - 10$
   $= g_1(x)$

2. $x = x - x^3 - 4x^2 + 10$
   $= g_2(x)$

3. $x = x + c(x^3 + 4x^2 - 10)$
   $= g_3(x)$

4. $x^3 = 10 - 4x^2$
   $x = (10 - 4x^2)^{1/3} = g_4(x)$

5. $4x^2 = 10 - x^3$
   $x = \sqrt{\frac{10 - x^3}{4}} = g_5(x)$

6. $x(x^2 + 4x) = 10$
   $x = \frac{10}{x^2 + 4x} = g_6(x)$

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$$x^2(x+4) = 10$$

$$x = \sqrt{\frac{10}{x+4}}$$

Fixed point iteration

Fixed point forms: Exercise

Write all the possible ways to change the equation $x^3 - 7x + 2 = 0$ to the fixed-point form $x = g(x)$ using simple algebraic manipulation.

Fixed point iteration

$a \le g(x) \le b$

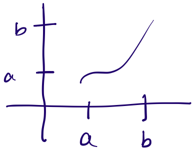
for all

Convergence conditions satisfied by $g(x)$:

(i) (existence) If $g \in C[a,b]$ and $g(x) \in [a,b]$, $\forall x \in [a,b]$, then $g(x)$ has at least one fixed point in $[a,b]$.

g maps into itself.

g is continuous in $[a,b]$



$g(x) = x$

$f(x) = g(x) - x = 0$

Since g is cont. on $[a,b]$, then $\underbrace{f(x) \text{ is cont. on } [a,b]}$

& $f(a) = g(a) - a > 0$

$f(b) = g(b) - b < 0$

By IVT, we say that $\exists$ a no. $c \in [a,b]$ s.t $f(c) = 0$

$\Rightarrow g(c) = c$

Fixed point iteration

If $g(x) \in C[a,b]$ & $g'(x)$ exists in (a,b)
then $|g(a)-g(b)| \leq |g'(c)||a-b|$ $\boxed{L.M.V.T.}$

Convergence conditions satisfied by $g(x)$:

(ii) (uniqueness) If, in addition, $g'(x)$ exists in (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a, b)$, then there is exactly one fixed point in $[a, b]$.

$\boxed{|g'(x)| < 1}$

let p and q be the fixed point for $g(x)$ in $[a,b]$

$\Rightarrow \quad g(p) = p, \quad g(q) = q$

Now $|p-q| = |g(p) - g(q)| \leq \underbrace{|g'(c)|}\,|p-q|$, $c \in (a,b)$, (by L.M.V.T.)

$< 1 \cdot |p-q|$

$\Rightarrow \quad |p-q| < |p-q| \rightarrow$ It is not possible, $p = q$

Fixed point iteration

step 1 → write the fixed pt form $g(x) = x$

step 2 → choose appropriate $g(x)$ in $[a,b]$

Convergence conditions satisfied by $g(x)$:

(iii) (convergence) If conditions of (i) and (ii) are satisfied, then for any number $p_0 \in [a, b]$, the sequence defined by $\boxed{p_n = g(p_{n-1})}$, $n \geq 1$ converges to the unique fixed point p in $[a, b]$.

$$|p_n - p| = |g(p_{n-1}) - g(p)|$$

$$\leq g'(c)\,|p_{n-1} - p|$$

$$\leq k\,|p_{n-1} - p|$$

$$|p_{n-1} - p| \leq k\,|p_{n-2} - p| \Rightarrow |p_n - p| \leq k^2 |p_{n-2} - p|$$

$$\leq k^3\,|p_{n-3} - p| \;\ldots\ldots \leq k^n\,|p_0 - p|$$

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n\,|p_0 - p| = 0 \quad \because (k < 1)$$

step3 Take an initial guess $p_0 \in (a, b)$

$p_1 = g(p_0) \approx p_0$

$p_2 = g(p_1) \approx p_1$

$p_3 = g(p_2) \approx p_2$

$$\boxed{p_n = g(p_{n-1})}$$

$\downarrow$

p