Lecture 33: Numerical Linear Algebra (UMA021): Matrix Computations

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Orthogonal and Orthonormal Matrices:

A matrix Q is said to be an orthogonal if its columns form an orthonormal set.

Note: A matrix Q is $n \times n$ matrix, then Q is invertible with $Q^{-1} = Q^t, \quad Q^t Q = I.$

$Q^{-1} = Q^t$

$QQ^{-1} = QQ^t$

$\boxed{I = QQ^t}$

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

$q_1 \quad q_2 \quad \cdots \quad q_n \rightarrow$ orthonormal

$\|q_1\| = 1, \quad \|q_2\| = 1 \quad \cdots \quad \|q_n\| = 1$

$q_i \cdot q_j = 0 \quad \forall\, i \neq j$

$q_i\, q_j = 1 \quad \forall\, i = j$

Orthogonal and Orthonormal Matrices:

Orthogonal and Orthonormal Matrices: Example

Show that the matrix $Q = \begin{bmatrix} 0 & \frac{-\sqrt{30}}{6} & \frac{\sqrt{6}}{6} \\ \frac{2\sqrt{5}}{5} & \frac{-\sqrt{30}}{30} & \frac{-\sqrt{6}}{6} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{30}}{15} & \frac{\sqrt{6}}{3} \end{bmatrix}$ is an orthogonal matrix.

Sol^n

$$Q^t Q = \begin{bmatrix} 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ -\frac{\sqrt{30}}{6} & -\frac{\sqrt{30}}{30} & \frac{\sqrt{30}}{15} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{30}}{6} & \frac{\sqrt{6}}{6} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{30}}{30} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{30}}{15} & \frac{\sqrt{6}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \overset{1}{\frac{20}{25} + \frac{5}{25}} & \overset{0}{\frac{\cancel{2}\sqrt{5}}{5} * -\frac{\sqrt{30}}{\cancel{30}\,15} + \frac{\sqrt{5}}{5} * \frac{\sqrt{30}}{15}} & \overset{0}{-\frac{\cancel{2}\sqrt{30}}{\cancel{30}\,15} + \frac{\sqrt{30}}{15}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ Q is orthogonal matrix.

Orthogonal and Orthonormal Matrices:

Orthogonal and Orthonormal Matrices: Example

Two matrices A and B are said to be similar if a non-singular matrix S exists with $A = S^{-1}BS$.
Note: Similar matrices have the same eigenvalues.

A B

$A = S^{-1}BS$

S

$\Downarrow$

A & B have same eigen values

$A = QR$ → triangular matrix, Q → orthogonal

QR decomposition:

The QR factorization of a matrix factorize in to an orthogonal matrix and triangular matrix. The decomposition gives

$A = [a_1 \ a_2 \ --- \ a_n]$

$$A = \boxed{QR},$$

where matrix A be an $\underbrace{m \times n}$ matrix with linearly independent columns.

Apply Gram-Schmidth orthogonalization process to the columns of A produces an $m \times n$ matrix Q whose columns are orthonormal. → to find Q

$Q = [e_1 \ e_2 \ - \ — \ e_n]$

and R is an $n \times n$ upper triangular matrix with positive entries on the diagonal.

$$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & -- & a_n \cdot e_1 \\ 0 & a_2 \cdot e_2 & — & a_n \cdot e_2 \\ 0 & -- & — & - \\ 0 & — & — & a_n \cdot e_n \end{bmatrix}$$

If A is non-singular then this factorization is unique.

QR decomposition: Example

Apply Gram-Schmidth orthogonalization process to find QR factorization of matrix $\begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$.

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

let $a_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Apply Gram-Schmidth process to columns of A

$$q_1 = a_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$q_2 = a_2 - \frac{a_2 \cdot q_1}{\|q_1\|^2} * q_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \left(\frac{3(-1) + 5(1)}{(\sqrt{1+1})^2}\right) * \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$q_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $q_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \rightarrow$ orthogonal

To make it orthonormal, we have

$e_1 = \dfrac{q_1}{\|q_1\|}$ ✓, $e_2 = \dfrac{q_2}{\|q_2\|}$ ✓

$$= \frac{1}{\sqrt{1+1}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{16+16}}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad e_2 = \frac{1}{4\sqrt{2}}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These are orthonormal vectors

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 \\ 0 & a_2 \cdot e_2 \end{bmatrix} = \begin{bmatrix} -1\left(-1/\sqrt{2}\right) + 1\left(1/\sqrt{2}\right) & 3\left(\frac{-1}{\sqrt{2}}\right) + 5\left(\frac{1}{\sqrt{2}}\right) \\ 0 & 3\left(\frac{1}{\sqrt{2}}\right) + 5\left(\frac{1}{\sqrt{2}}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{bmatrix}$$

$$A = QR = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \sqrt{2} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$$

QR decomposition: Another method to find R

$A = QR$

$Q^t Q = I$

$Q^t A = Q^t Q R$

$Q^t A = I R = R$

$\boxed{R = Q^t A}$ ✓

QR decomposition:

QR decomposition: Exercise

Apply Gram-Schmidth orthogonalization process to find QR factorization of matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

QR Algorithm: → To find all Eigenvalues of A.

$Q_1^t Q_1 = I$

$\boxed{A = QR}$

$A_1 = Q_1 R_1$

$\boxed{A_2} = R_1 Q_1 = Q_2 R_2$

$A_1 = Q_1 R_1$

$A_2 = I R_1 Q_1$

$A_2 = Q_1^t Q_1 (R_1 Q_1)$

$= Q_1^t (Q_1 R_1) Q_1$

$A_2 = Q_1^{-1} A_1 Q_1$

$A = QR$

$A_1 = Q_1 R_1$ Reverse

$A_2 = R_1 Q_1 = Q_2 R_2$

$A_3 = R_2 Q_2 = Q_3 R_3$

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$A_k = Q_k R_k$

$A_{k+1} = R_k Q_k$

$\Rightarrow$ A_1 & A_2 are similar

$\Rightarrow$ They have same e. value

||ly $A_3 = Q_2^{-1} A_2 Q_2$ $\Rightarrow$ A_2 & A_3 have same e.value

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At last we got diagonal matrix as A_n

$\Rightarrow$ diagonal entries of A_n are the e.values of A.

A_2 and A_1 are symmetric

QR Algorithm:

QR Algorithm:

Write the *QR* factorization of matrix *A* and perform two iterations of *QR* algorithm to find all the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$