Lecture 23: Numerical Linear Algebra (UMA021): Matrix Algebra

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System of linear equations

$2x + 3y = 5$

$-3x + 7y = 9$

$x = ?\ y = ?$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots a_{1n}x_n = b_1 & - (1) \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots a_{2n}x_n = b_2 & - (2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots a_{nn}x_n = b_n. & -- (n) \end{cases} \tag{1.1}$$

This is a linear system of n equation in n unknowns $x_1, x_2 \dots\dots x_n$. This system can simply be written in the matrix equation form

$$\text{Ax=b}$$

$$\underset{n \times n}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}} \times \underset{n \times 1}{\overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{n \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}} = b \tag{1.2}$$

$x = ?$

Direct Methods

1. Gauss Elimination
2. LU Decomposition(Factorization)

$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}\begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \\ -1 & 2 & 1 \end{bmatrix}$$

1 2 3

3 4 9

$E_2 \to E_2 - \frac{3}{2} E_1$

5 − 4

7 − 9

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & -2 \end{bmatrix}$$

$E_3 \to E_3 - \frac{-1}{2} E_1$

System of linear equations

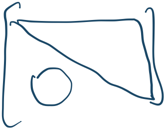
Gauss Elimination

To solve larger system of linear equation, we consider a following $n \times n$ system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \longrightarrow (E_1)\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \rightarrow (E_2)\\
&\cdots\cdots\cdots\\
a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n &= b_i \rightarrow (E_i)\\
&\cdots\cdots\cdots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3} + \cdots + a_{nn}x_n &= b_n \rightarrow (E_n).
\end{aligned}
$$

Here E_i denote the i-th row of the coefficients matrix A, $i = 1, 2, \cdots, n$.

Step (i) write Augumented Matrix $[A : b]$ or $[A | b]$



$$
\begin{array}{c}
E_1 \\ E_2 \\ \\ E_n
\end{array}
\underbrace{\left[\begin{array}{cccccc:c}
a_{11} & a_{12} & a_{13} & - - - & a_{1n} & & b_1 \\
a_{21} & a_{22} & a_{23} & - - - & a_{2n} & & b_2 \\
- & - & - & - - - & - & & \\
a_{n1} & a_{n2} & a_{n3} & - - - & a_{nn} & & b_n
\end{array}\right]}
$$

Elementary operation

$E_i \leftrightarrow E_j$

$E_2 \rightarrow E_2 - 2E_1$

Step 2 Reduce Matrix A into upper triangular matrix by using elementary operations.

Check $a_{11} \neq 0$ but if $a_{11} = 0$ then $E_1 \leftrightarrow E_j$ in which $a_{j1} \neq 0$

$E_2 \to E_2 - \frac{a_{21}}{a_{11}} E_1$, $E_3 \to E_3 - \frac{a_{31}}{a_{11}} E_1$, $E_4 \to E_4 - \frac{a_{41}}{a_{11}} E_1$ —— $E_n \to E_n - \frac{a_{n1}}{a_{11}} E_1$ $\longrightarrow$ $E_j \to E_j - \frac{a_{j1}}{a_{11}} E_1$

$j = 2, 3, \cdots n$

Let $a_{11} \neq 0$ and eliminate x_1 from $E_2, E_3, \cdots, E_n$.
Define multipliers $m_{j1} = \dfrac{a_{j1}}{a_{11}}$, for each $i = 2, 3, \cdots, n$.
We write each entry in E_j as $E_j - m_{j1}E_1$ and b_j as $b_j - m_{j1}b_1$; $j = 1, 2, \cdots, n$.

$$\left[\begin{array}{cccccc:c}
a_{11} & a_{12} & a_{13} & \cdots & & a_{1n} & b_1 \\
0 & a^*_{22} & a^*_{23} & \cdots & & a^*_{2n} & b^*_2 \\
0 & a^*_{32} & - & - & - & - & \\
0 & - & - & - & - & & \\
0 & a^*_{n2} & - & - & - & a^*_{nn} & b^*_n
\end{array}\right]$$

$E_3 \to E_3 - \frac{a^*_{32}}{a^*_{22}} E_2$, $E_4 \to E_4 - \frac{a^*_{42}}{a^*_{22}} E_2$ ---- $E_n \to E_n - \frac{a^*_{n2}}{a^*_{22}} E_2$

$E_j \to E_j - \frac{a^*_{j2}}{a_{22}} E_2$, $j = 3, 4 — n$

We repeat this procedure and follow a sequential procedure for $j = 2, 3, \cdots, n$ and perform the operations

$$\boxed{E_j \longrightarrow E_j - (a_{ji}/a_{ii}E_i),} \quad \text{for each } j = i+1, i+2, \cdots, n.$$

$$b_i - (a_{ij}/a_{jj}b_j), \quad \text{for each } i = j+1, j+2, \cdots, n,$$

provided $\boxed{a_{ii} \neq 0.}$ This eliminates x_i in each row below the i-th values.
Also we replace each b_i with $b(i) = b(i) - (a_{ij}/a_{jj})b(j-1)$. The resulting matrix is triangular and has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{bmatrix} \longrightarrow$$

$0x_1 + 0x_2 + 0x_3 - - - 0x_{n-1}$
$+ a_{nn}^* x_n = b_n$

$x_n = \frac{b_n}{a_{nn}^*}$

Solving the n-th equation for x_n gives

Step3 Use back substitution.

$$x_n = \frac{b_n}{a_{nn}}.$$

Solving the $(n-1)$st equation for x_{n-1} and using the known value for x_n yields (back substitution)

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}.$$

x_1

Continuing this process, we obtain

$$x_i = \frac{b_i - a_{i,i+1}x_{i+1} - a_{i,i+2}x_{i+2} - \cdots - a_{in}x_n}{a_{ii}} = \frac{b_i - \sum\limits_{j=i+1}^{n} a_{ij}x_j}{a_{ii}},$$

for each $i = n-1, n-2, \cdots, 2, 1$.

Example

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear system. (The exact solution to each system is $x_1 = -1,\ x_2 = 1,\ x_3 = 3.$)

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$$-x_1 + 4x_2 + x_3 = 8$$
$$\frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 = 1$$
$$2x_1 + x_2 + 4x_3 = 11.$$

Solution: Step 1 Write Augmented Matrix

$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \begin{bmatrix} -1 & 4 & 1 & : & 8 \\ 1.7 & 0.67 & 0.67 & : & 1 \\ 2 & 1 & 4 & : & 11 \end{bmatrix}$$

Apply

$a_{11} \neq 0$

$E_2 \to E_2 - \frac{1.7}{-1} E_1 \Rightarrow E_2 \to E_2 + 1.7 E_1$

$E_3 \to E_3 + 2E_1$

$$\left[\begin{array}{ccc:c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 9 & 6 & 27 \end{array}\right]$$

$1 + 2 * 4$

$4 -$

$11 + 2 * 8$

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$E_3 \to E_3 - \frac{9}{7.5} E_2$

$\Rightarrow E_3 \to E_3 - 1.2 E_2$

$$\left[\begin{array}{ccc:c} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 0 & 3.1 & 9 \end{array}\right]$$

Use back substitution

$3.1 x_3 = 9 \Rightarrow x_1 = \frac{9}{3.1} = 2.9$

$-x_1 + 4x_2 + x_3 = 8$

$-x_1 + 4 * 1.1 + 2.9 = 8$

$-x_1 = 8 - 7.3 = 0.7$

$x_1 = -0.7$

$7.5 x_2 + 2.4 x_3 = 15$

$7.5 x_2 + (2.4)(2.9) = 15$

$7.5 x_2 = 8 \Rightarrow x_2 = \frac{8}{} = 1.1$

System of linear equations

$$x = \begin{bmatrix} -0.7 \\ 1.1 \\ 2.9 \end{bmatrix}$$ Ans.

Example

Apply Gaussian elimination to the system:

$$
\begin{aligned}
0.003000x_1 + 59.14x_2 &= 59.17 \\
5.291x_1 - 6.130x_2 &= 46.78
\end{aligned}
$$

using four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Solution:

$$
\begin{array}{c} E_1 \\ E_2 \end{array}
\begin{bmatrix} 0.0030000 & 59.14 & : & 59.17 \\ 5.291 & -6.130 & : & 46.78 \end{bmatrix}
$$

$$E_2 \to E_2 - \frac{5.291}{0.0030000} E_1 \Rightarrow E_2 - 1764 E_1$$

$$-6.130 - 1764(59.14)$$

$$-6.130 - 104322.96$$

$\boxed{-6.130 - 1.043 * 10^5}$

$-6.130 - 1.0432296 \times 10^5$

System of linear equations

$$\rightarrow \left[\begin{array}{cc:c} 0.003000 & 59.14 & 59.17 \\ 0 & & \end{array}\right]$$

Complete it.