Lecture 26: Numerical Linear Algebra (UMA021): Matrix Algebra

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$\boxed{\text{Norm}}$

x, y

$|a-b|$

Iterative methods to solve System of linear equations:

Distance between $n-$dimensional vectors

To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between $n-$dimensional column vectors.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ $\qquad \|X - Y\|$

in $\mathbb{R}$ $\qquad \| \; \| \rightarrow | \; |$

Norms

Vector norms

Let $\mathbb{R}^n$ denote the set of all $n-$dimensional column vectors with real-number components.
To define a distance in $\mathbb{R}^n$ we use the notion of a $\overset{\|\cdot\|}{\text{norm}}$, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$

A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

$\alpha \in \mathbb{R}$ $\quad X \in \mathbb{R}^n$

αX

$\alpha = -2 \quad X = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$|-3 * 2| = |-3| * |2|$

$\alpha X = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$

$\|X\|_2 = \left(\sum_{i=1}^{n} (x_i - 0)^2 \right)^{1/2}$

$\|\alpha X\| = |\alpha|\,\|X\|$

$\|X - Y\|_2 \to$

$\|X - 0\|_2$

$\|X\|_\infty = \max_{1 \leq i \leq n} \{ |x_i - 0| \}$

$\|X - Y\|_\infty$

$\|X - Y\|_p = \left(\sum_{i=1}^{n} (x_i)^p \right)^{1/p}$

Norms

Vector norms

We will need only two specific norms on $\mathbb{R}^n$,

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\underbrace{\|\mathbf{x}\|_2} = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \underbrace{\|\mathbf{x}\|_\infty} = \max_{1\le i\le n} |x_i|.$$

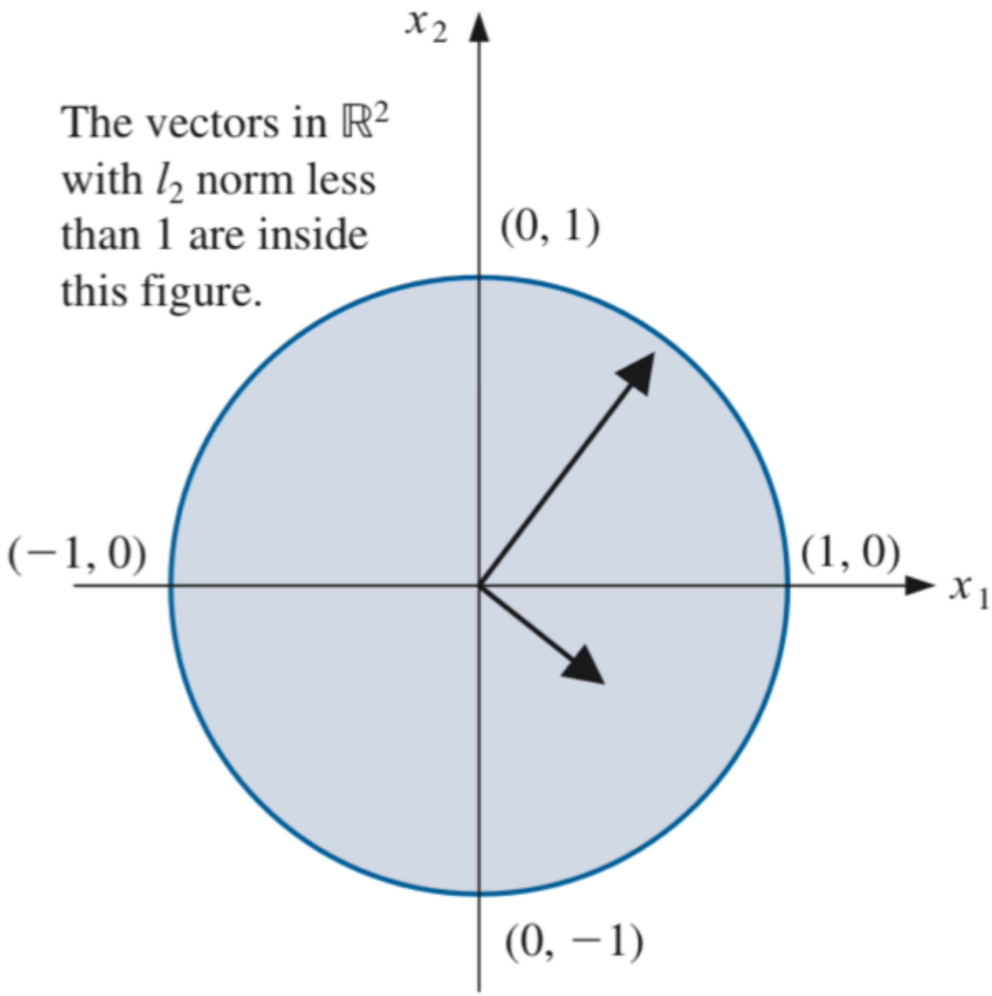
$X = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

Uniform norm

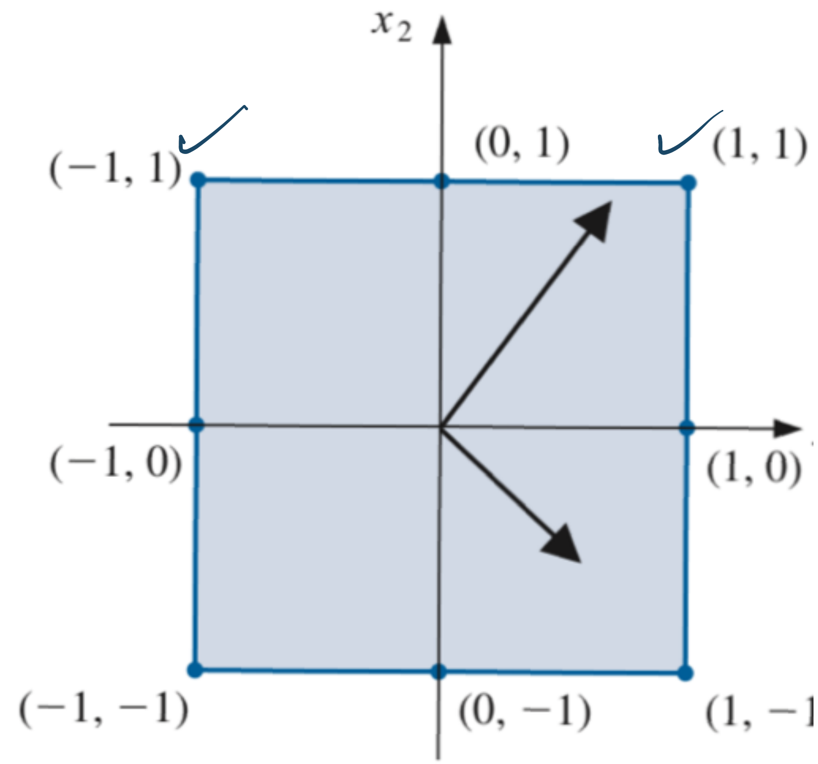
or Infinity norm

or Maximum Norm

$\|X\|_2 = \sqrt{(x_1-0)^2+(y_1-0)^2+(z_1-0)^2}$



The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are inside this figure.



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.

$\boxed{x = (x, y)^t} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\|x\|_2 \leq 1$

$\sqrt{(x-0)^2 + (y-0)^2} \leq 1$

$x^2 + y^2 \leq 1$

$\|x\|_\infty \leq 1$

$\max \{|x-0|, |y-0|\} \leq 1$

$|x| \leq 1, \quad |y| \leq 1$

Example:

Determine the l_2 norm and the l_∞ norm of the vector $x = (-1, 1, -2)^t$.

$$\|X\|_2 = \sqrt{(-1-0)^2 + (1-0)^2 + (-2-0)^2}$$

$$= \sqrt{1+1+4} = \sqrt{6}$$

$$\|X\|_\infty = \max\{|-1|, \ |1|, \ |-2|\} = 2$$

Norms

Distance between Vectors in $\mathbb{R}^n$:

If $x = (x_1, x_2, \cdots, x_n)^t$ and $y = (y_1, y_2, \cdots, y_n)^t$ are vectors in $\mathbb{R}^n$, then l_2 and l_∞ distances between x and y are defined by

$$\|x - y\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|x - y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|.$$

Norms

Convergence of a sequence in $\mathbb{R}^n$:

The sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in $\mathbb{R}^n$ with respect to the l_∞-norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \cdots, n$.

$\langle \frac{1}{n} \rangle$

$x_1 \quad x_2 \quad x_3$

$1, \frac{1}{2}, \frac{1}{3}$

$x^{(k)} \to x$

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$x^{(1)}, \quad x^{(2)}, x^{(3)} \cdots \quad x^{(k)} \quad \cdots \quad \to \quad x$

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_n^{(1)} \end{bmatrix} \quad \begin{bmatrix} x_1^{(2)} \\ \\ \end{bmatrix}$$

Convergence of a sequence in $\mathbb{R}^n$

Example:

Show that
$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t = \left(1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin(k)\right)^t$
converges to $x = (1, 2, 0, 0)^t$ with respect to l_∞ norm.

$$\lim_{k\to\infty} \left(1,\ 2+\frac{1}{k},\ \frac{3}{k^2},\ e^{-k}\sin(k)\right)^t$$

$$|a_n| \le |b_n|$$

$$\lim_{k\to\infty} \left|\frac{\sin(k)}{e^k}\right| \le \lim_{k\to\infty} \frac{1}{e^k} = 0$$

by comp test.

$$(1,\ 2,\ 0,\ 0)$$

$$\begin{bmatrix} 1 \\ 2+1 \\ \frac{3}{1} \\ e^{-1}\sin 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2+\frac{1}{2} \\ \frac{3}{4} \\ e^{-2}\sin(2) \end{bmatrix}, \begin{bmatrix} 1 \\ 2+\frac{1}{3} \\ \frac{3}{9} \\ e^{-3}\sin 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2+\frac{1}{4} \\ \frac{3}{16} \\ e^{-4}\sin 4 \end{bmatrix}$$

when $k\to\infty$

$$\to 1$$
$$\to 2$$
$$\to 0$$
$$\to 0$$

System of linear equations:

Exercise:

1. Find l_∞ and l_2 norms of the vectors.
   a) $x = (3, -4, 0, \frac{3}{2})^t$.
   b) $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.
2. Find the limit of the sequence
   $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^t = \left(\frac{4}{k+1}, \frac{2}{k^2}, k^2 e^{-k}\right)^t$ with respect to l_∞ norm.

Iterative methods to solve System of linear equations

Jacobi Method

Consider the system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad — (1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad - (2)\\
\vdots \qquad \vdots \qquad \vdots &= \vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad - (n)
\end{aligned}
$$

Take an initial guess $x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{bmatrix}$

$Ax = b$ To find $x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{bmatrix}$.

1st iteration

from (1) $\quad a_{11}x_1 = b_1 - (a_{12}x_2 + a_{13}x_3 - - - a_{1n}x_n)$

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - (a_{12}x_2^{(0)} + a_{13}x_3^{(0)} - - - a_{1n}x_n^{(0)})\right] = \frac{1}{a_{11}}\left[b_1 - \sum_{j=2}^{n} a_{1j}x_j\right]$$

from (2) $\quad x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - (a_{21}x_1^{(0)} + a_{23}x_3^{(0)} - - - a_{2n}x_n^{(0)})\right]$

n

$$= \frac{1}{a_{22}}\left[b_2 - \sum_{\substack{j=1 \\ j\neq 2}} a_{2j} x_j\right]$$

from (3)

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31} x_1^{(0)} + a_{32} x_2^{(0)} \cdots a_{3n} x_n^{(0)}\right)\right]$$

$$= \frac{1}{a_{33}}\left[b_3 - \sum_{\substack{j=1 \\ j\neq 3}}^{n} a_{3j} x_j\right]$$

$$\cdot \quad \text{—} \quad \text{—} \quad \text{—}$$

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1} x_1^{(0)} \text{———} a_{n-1,n} x_n^{(0)}\right)\right]$$

$$= \frac{1}{a_{nn}}\left[b_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j\right]$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k)}\right] \quad , k = 0, 1, 2 \text{ -- -- } n$$

Strictly diagonally dominant matrix:

A square matrix A is called diagonally dominant if

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}| \ \forall \ i.$$

A is called strictly diagonally dominant if

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}| \ \forall \ i.$$

$$\begin{bmatrix} a_{11} & a_{12} & \text{---} & a_{1n} \\ a_{21} & a_{22} & \text{---} & a_{2n} \\ & & & \\ a_{n1} & \text{---} & \text{---} & a_{nn} \end{bmatrix}$$

$|a_{11}| > |a_{12}| + |a_{13}| + \cdots + |a_{1n}|$

$|a_{22}| > |a_{21}| + |a_{23}| + |a_{24}| - -$

Example:

Check whether the following matrices are strictly diagonal dominant or not: $A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -6 & 3 \\ -2 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

A is strictly D.D.

But B is not S.D.D.