Lecture 21: Numerical Linear Algebra (UMA021): Integration

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x_0, x_1

$$\int_a^b P_1(x) = \int_a^b l_0(x) f(x_0)\,dx + \int l_1(x) f(x_1)\,dx = f(x_0)\frac{h}{2} + f(x_1)\frac{h}{2}$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

$$\int E(f) = \max \left(f''(c) \quad (x - x_0) \quad (x - x_1) \right)$$

$a<c<b$ $\quad 2!$

$\underbrace{\int_{x_0}^{x_1} f(x)\,dx}$ Error in this int. is $-\frac{h^3}{12} f''(c)$

Trapezoidal Rule: Error formula (Proof)

$$E(f) = -\frac{h^3}{12} f''(c), \;\; c \in (a, b)$$

$h = x_1 - x_0 = b - a$

Proof

$$E(f) = \left.\int_{a=x_0}^{b=x_1} \underbrace{\frac{f''(c)}{2!}}_{f(x)} \underbrace{(x-x_0)(x-x_1)}_{g(x)}\,dx \right] \quad \text{—} \quad (*)$$

Weighted Mean Value Thm (W.M.V.T.)

If $f(x)$ is continuous in $[a,b]$ and $g(x)$ is integrable and does not change its sign over $[a,b]$ then $\exists$ $c \in (a,b)$ s.t

$$\sum f(x)\, g(x) = f(c) \sum g(x)$$

$$\int_a^b f(x)\,g(x)\,dx = f(c_1)\int_a^b g(x)\,dx$$

Use W.M.T. on (*) then $\exists\ c \in (a,b)$

s.t.

$$\frac{f''(c)}{2!}\int_{a=x_0}^{b=x_1} \underset{\mathrm{I}}{(x-x_0)}\ \underset{\mathrm{II}}{(x-x_1)}\,dx$$

use

$$\left.\begin{aligned}&\int (x-x_1)\,dx\\ &= \frac{(x-x_1)^2}{2}\end{aligned}\right\}$$

$$= \frac{f''(c)}{2!}\left[(x-x_0)\frac{(x-x_1)^2}{2} - \int_{x_0}^{x_1}\frac{(x-x_1)^2}{2}\,dx\right]$$

$$= \frac{f''(c)}{2!}\left[(x-x_0)\frac{(x-x_1)^2}{2} - \frac{(x-x_1)^3}{6}\right]_{x_0}^{x_1}$$

$$= \frac{f''(c)}{2!}\left[0 - 0 - 0 + \frac{(x_0-x_1)^3}{6}\right]$$

$$= -\frac{f''(c)}{2!}\left[\frac{(x_1-x_0)^3}{6}\right] = -\frac{f''(c)}{12}h^3 \quad , \quad c\in(a,b)$$

Numerical Quadrature:

Composite Trapezoidal Rule: Error formula (Proof)

$$E(f) = -\frac{h^2(b-a)}{12} f''(c), \quad c \in (a,b)$$

Divide $[a,b]$ into n equal subintervals, $n = \frac{b-a}{h}$

$$\int_{a = x_0}^{b = x_n} f(x)\,dx = \underbrace{\int_{x_0}^{x_1} f(x)\,dx}_{} + \int_{x_1}^{x_2} f(x)\,dx + \cdots \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= -\frac{h^3}{12} f''(c_1) - \frac{h^3}{12} f''(c_2) - \cdots - \frac{h^3}{12} f''(c_n)$$

$x_0 < c_1 < x_1$, $x_1 < c_2 < x_2$, $x_{n-1} < c_n < x_n$

$$= -\frac{h^3}{12} \sum_{i=1}^{n} \underbrace{f''(c_i)}_{f(x)} * \underbrace{1}_{g(x)} \qquad x_{i-1} < c_i < x_i, \; i = 1, 2 \ldots n$$

Use W.M.V.T. for summation $\exists\, c \in (a,b)$

s.t

$$= -\frac{h^3}{12} f''(c) \sum_{i=1}^{n} 1$$

$$= -\frac{h^3}{12} f''(c)\, n = -\frac{h^3}{12} f''(c) \frac{(b-a)}{h}$$

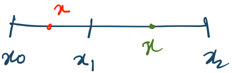
$$= -\frac{h^2}{12}(b-a) f''(c) \quad , \quad c \in (a,b)$$

Numerical Quadrature:

Simpson's Rule: Error formula (Proof)

$$E(f) = -\frac{h^5}{90} f^{iv}(c), \;\; c \in (a,b)$$

$$E(f) = \int_{x_0}^{x_2} \frac{f'''(c)}{3!} \underbrace{\overset{+ve}{(x-x_0)}}_{+ve} \; \underset{+ve}{\overset{-ve}{(x-x_1)}} \; \underset{-ve}{\overset{-ve}{(x-x_2)}} \, dx \qquad \begin{matrix} = +ve \\ = -ve \end{matrix}$$



To apply W.M.V.T on the above int. we add one more pt. x_1

s.t. $$E(f) = \int_{x_0}^{x_2} \frac{f^{iv}(c)}{4!} (x-x_0)(x-x_1)^2 (x-x_2)\, dx$$

Apply W.M.V.T. by taking $f(x) = \dfrac{f^{iv}(c)}{4!}$

$g(x) = (x-x_0)(x-x_1)^2(x-x_2)$

$$E(f) = \frac{f^{iv}(c)}{4!} \int_{x_0}^{x_2} (x-x_0)(x-x_1)^2(x-x_2)\,dx$$

$$= \boxed{-f^{iv}(c)\,\frac{h^5}{90}}$$

Numerical Quadrature:

Composite Simpson's Rule: Error formula (Proof)

$$E(f) = -\frac{h^4(b-a)}{180} f^{iv}(c), \; c \in (a,b)$$

$h = \frac{b-a}{2n = N}$

$$\int_{a=x_0}^{b=x_{2n}=N} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx \;——\; \int_{x_{2n-2}}^{x_{2n}} f(x)\,dx$$

Error in these int.

$$= -\frac{h^5}{90} f^{iv}(c_1) \quad - \frac{h^5}{90} f^{iv}(c_2) \quad , \quad -\frac{h^5}{90} f^{iv}(c_n)$$

$x_0 < c_1 < x_2$, $\quad x_2 < c_2 < x_4$ $\quad x_{2n-2} < c_n < x_{2n}$

$$= -\frac{h^5}{90} \sum_{i=1}^{n} f^{iv}(c_i), \quad x_{2i-2} < c_i < x_{2i}, \quad i = 1, 2 \ldots n$$

Use W.M.V.T for summations

$$-\frac{h^5}{90} f^{iv}(c) \sum_{i=1}^{n} 1$$

$c \in (a,b)$

$$= -\frac{h^5}{90} f^{iv}(c) \, (n)$$

$$\boxed{2n = \frac{b-a}{h}}$$

$$= -\frac{h^5}{90} f^{iv}(c) \frac{(b-a)}{2h}$$

$$= -\frac{h^4}{180} (b-a) f^{iv}(c) \, , \quad c \in (a,b)$$

Simpson's Rule

Exact Trap.

$$\boxed{x^0, x^1}$$

$$\boxed{1, x, x^2, x^3}$$

Numerical Quadrature:Measuring Precision:

Degree of Precision:

The degree of precision of a quadrature formula is the largest positive integer n such that the formula is exact for x^k, for $k = 0, 1, 2, \cdots, n$.

for trap $\quad x^k, \quad k=0,1 \quad$ degree of precision is 1

for simp $\frac{1}{3}$rd $\quad x^k, \quad k=0,1,2,3 \quad$ de. of P. is 3

Numerical Quadrature:

Degree of Precision: Example

The quadrature formula $\int_0^2 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1 and c_2.

0.5

for $f(x) = 1$, the given formula is exact

$$\int_0^2 1\,dx = c_0(1) + c_1(1) + c_2(1)$$

$$(x)_0^2 = c_0 + c_1 + c_2 \Rightarrow \boxed{c_0 + c_1 + c_2 = 2} \quad -(1)$$

for $f(x) = x$, the given formula is exact

$$\boxed{\int_0^2 x\,dx} = c_0(0) + c_1(1) + c_2(2)$$

$$\boxed{2 = c_1 + 2c_2} \quad -(2)$$

for $f(x) = x^2$, the given formula is exact

$$\int_0^2 x^2\,dx = c_0(0) + c_1(1) + c_2(4)$$

$$\left(\frac{x^3}{3}\right)_0^2 = c_1 + 4c_2$$

$$\Rightarrow \boxed{\frac{8}{3} = c_1 + 4c_2} \quad - (3)$$

eq^n (3) $-$ eq^n (2)

$$2c_2 = \frac{8}{3} - 2 = \frac{2}{3}$$

$$\boxed{c_2 = \frac{1}{3}} \Rightarrow$$ from eq^n (2) $$\boxed{c_1 = \frac{4}{3}}$$

from eq^n (1) $c_0 = \frac{1}{3}$

$$\Rightarrow \int_0^2 f(x)\,dx = \frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2)$$

Numerical Quadrature:

Degree of Precision: Example

Find the constants c_0, c_1, and x_1 so that the quadrature formula $\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$ has the highest possible degree of precision.

$-(*)$

for $f(x) = 1$, Make the formula $(*)$ exact

$$\int_0^1 1\,dx = c_0 + c_1 \Rightarrow c_0 + c_1 = 1 \quad -(1)$$

for $f(x) = x$

$$\int_0^1 x\,dx = c_0(0) + c_1 x_1 \Rightarrow c_1 x_1 = 1/2 \quad -(2)$$

for $f(x) = x^2$

$$\int_0^1 x^2 dx = c_0(0) + c_1 x_1^2 = c_1 x_1^2 = 1/3 \quad -(3)$$

eq^n ③ $\div$ eq^n ②

$$x_1 = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$$\frac{2}{3} c_1 = \frac{1}{2}$$

$$c_1 = 3/4$$

from eq^n ① $\quad c_0 = 1/4$

$$\Rightarrow \int_0^1 f(x)\,dx = \frac{1}{4} f(0) + \frac{3}{4} f(2/3)$$

Check for $f(x) = x^3$ [because exact degree is not given. we check the exactness for higher degree]

$$L.H.S. = \int_0^1 x^3\,dx = \left(\frac{x^4}{4}\right)_0^1 = \frac{1}{4}$$

$$R.H.S = \frac{1}{4}(0) + \frac{3}{4}\left(\frac{8}{27}\right) = \frac{2}{9}$$

$\Rightarrow$ $L.H.S \neq R.H.S.$

$\Rightarrow$ degree of precision for given formula is 2

Numerical Quadrature:

Degree of Precision: Exercise:

1. Determine constants a, b, c, and d that will produce a quadrature formula $\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$ that has degree of precision 3.