School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

UMA011 : Numerical Analysis
Assignment 2
Roots of Non-linear Equations-A

1. Use the bisection method to find solution accurate to within 10^{-3} for $x - 2^{-x} = 0$ for $0 \leq x \leq 1$.

2. Using the bisection method, determine the point of intersection of the curves given by $y = 3x$ and $y = e^x$ in the interval $[0, 1]$ with an accuracy 0.1.

3. Find a bound for the number of iterations needed to achieve an approximation of $\sqrt[3]{25}$ by the bisection method with an accuracy 10^{-2}. Hence find the approximation with given accuracy.

4. The function defined by $f(x) = \sin(\pi x)$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the bisection method converges to

(a) 0, if $a + b < 2$
(b) 2, if $a + b > 2$
(c) 1, if $a + b = 2$.

5. Show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-2}.

6. For $x = \dfrac{5}{x^2} + 2$ determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-2}, and perform the calculations.

7. Use the fixed-point iteration method to find smallest and second smallest positive roots of the equation $\tan x = 4x$, correct to 4 decimal places.

8. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-2}.

9. The iterates $x_{n+1} = 2 - (1 + c)x_n + cx_n^3$ will converge to $\alpha = 1$ for some values of constant c (provided that x_0 is sufficiently close to α). Find the values of c for which convergence occurs? For what values of c, if any, convergence is quadratic.

10. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

(a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $\alpha = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.
(b) Find an interval about $1/A$ for which fixed-point iteration converges, provided x_0 is in that interval.

11. Consider the root-finding problem $f(x) = 0$ with root α, with $f'(x) \neq 0$. Convert it to the fixed-point problem

$$x = x + cf(x) = g(x)$$

with c a nonzero constant. How should c be chosen to ensure rapid convergence of

$$x_{n+1} = x_n + cf(x_n)$$

to α (provided that x_0 is chosen sufficiently close to α)? Apply your way of choosing c to the root-finding problem $x^3 - 5 = 0$.

12. Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$. What happens if $x_0 < 0$?

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13. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.