Lecture 37: Numerical Linear Algebra (UMA021): Matrix Computations

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Constructing U in the factorization $A = U\,S\,V^t$:

The non-zero eigenvalues of $A^t\,A$ and those of $A\,A^t$ are the same. In addition, the corresponding eigenvectors of the symmetric matrices $A^t\,A$ and $A\,A^t$ form complete orthonormal subsets of R^n and R^m, respectively.
So the orthonormal set of n eigenvectors for $A^t\,A$ form the columns of V, as stated above, and the orthonormal set of m eigenvectors for $A\,A^t$ form the columns of U in the same way.

Singular Value Decomposition: Example

Determine the singular value decomposition of the 5×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

from prev. e.g.

$$S = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

To find U, we have found the e.vectors of AA^t corresp. to each non-zero e.values

5, 2, 1

ie $X_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$\boxed{A^tA}$ $\boxed{AA^t}$

3x3 5x5

5, 2, 1 5, 2, 1, 0, 0

The other e.values of AA^t are 0, 0

To find e.vector X_4 corresp. to $\lambda = 0$

ie $(AA^t - 0I) X_4 = 0$

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_3 + x_5 = 0, \quad -(1)$

$x_2 + x_3 + x_4 + x_5 = 0, \quad -(2)$

$x_1 + x_2 + 2x_3 + x_4 + x_5 = 0 \quad -(3)$

$x_1 + x_2 + x_3 + x_4 + 2x_5 = 0 \quad -(4)$

from (2) & (3)

$x_1 + x_3 = 0$

$x_3 = -x_1$

from (3) & (4)

$x_3 - x_5 = 0$

$x_5 = x_3$

$x_5 = -x_1$

from (2)

$x_2 - x_1 + x_4 - x_1 = 0$

$x_2 + x_4 - 2x_1 = 0$

$x_4 = 2x_1 - x_2$

from (4)

$\cancel{x_1} + x_2 - \cancel{x_1} + x_4 - 2x_1 = 0$

$x_4 + x_2 - 2x_1 = 0$

$$X_4 = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 \\ 2x_1 - x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \\ 2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ -x_2 \\ 0 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The E.vectors corresp to $\lambda = 0$, are $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

$= X_4 \qquad = X_5$

$\Rightarrow$ All the eigenvectors are

$$X_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} \ \& \ X_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$\{X_1, X_2, X_3, X_4\}$ and $\{X_1, X_2, X_3, X_5\}$ are orthogonal set

X_4 & X_5 are not orthogonal

because they are the eigenvectors associated with same eigenvalue ie $\lambda = 0$.

So, we keep x_4 as one of eigenvectors used to form U and determine x_5 which will give orthogonal set.

for this, we use Gram-Schmidth process on x_1, x_2, x_3, x_4 & x_5

As x_1, x_2, x_3, x_4 are already orthogonal

So apply G.S. on x_4 & x_5, we have

$$v_5 = x_5 - \frac{x_4 \cdot x_5}{\|x_4\|^2} x_4$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{-2}{7}\right) \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 1 \\ -2/7 \\ -3/7 \\ -2/7 \end{bmatrix}$$

$$\Rightarrow v_5 = \begin{bmatrix} 2/7 \\ 1 \\ -2/7 \\ -3/7 \\ -2/7 \end{bmatrix}$$ which is orthogonal to x_1, x_2, x_3, x_4

$$\|v_5\| = \frac{\sqrt{70}}{7}$$

$$x_5^* = \frac{v_5}{\|v_5\|} = \begin{bmatrix} 2/\sqrt{70} \\ \sqrt{70}/7 \\ -2/\sqrt{70} \\ -3/\sqrt{70} \\ -2/\sqrt{70} \end{bmatrix}$$

$$\Rightarrow \quad U = \begin{bmatrix} \frac{2}{\sqrt{30}} & \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{70}} \\ \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{\sqrt{70}}{7} \\ \frac{3}{\sqrt{30}} & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{7}} & \frac{-2}{\sqrt{70}} \\ \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{7}} & \frac{-3}{\sqrt{70}} \\ \frac{3}{\sqrt{30}} & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{7}} & \frac{-2}{\sqrt{70}} \end{bmatrix}$$

$\Rightarrow \quad A = USV^t$ verify it !

Singular Value Decomposition:

Singular Value Decomposition: Example

Determine the singular value decomposition of the 2×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times 3}$$

To find the E·values of A^tA

$$A^tA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

let λ be the E·value of A^tA

$$|A^tA - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)^2\right] - 1\left[(1-\lambda)\right] = 0$$

$$(1-\lambda)\left[1+\lambda^2-2\lambda-1\right] = 0$$

$\lambda = 1, \quad \lambda^2 - 2\lambda = 0$

$\lambda = 1, \; 2, 0$

$\lambda = 2, 1, 0$ only singular values are $\sqrt{2}, \; 1$

$$S = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Now to find V :- find the E.vector corresp. each $\lambda = 2, 1, 0$

To find Evector X_1 corresp to $\lambda = 2$

$$(A^tA - 2I)X_1 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0, \quad x_1 - x_2 = 0, \quad x_3 = 0$

$x_1 = x_2$

$$X_1 = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

To find E.vector X_2 corresp to $\lambda = 1$

$$(A^tA - I)\, x_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0, \quad x_1 = 0$$

$$X_2 = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To find Evector X_3 corresp to $\lambda = 0$

$$(A^tA - 0I)\, X_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 = 0, \quad x_3 = 0$$

$$x_2 = -x_1 \Rightarrow X_3 = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \checkmark$$

To find $U_{2\times 2}$ The Non-zero E.values of AA^t & A^tA are same, so the E.values of

$$AA^t = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$ are $2, 1$

To find the E.vector y_1 corresp to $\lambda = 2$

$$(AA^t - 2I)\,Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0 \quad \Rightarrow \quad Y_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To find the E.vector y_2 corresp. to $\lambda = 1$

$$(AA^t - 1I)\,Y_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0$$

$$Y_2 = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = USV^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Singular Value Decomposition: Exercise

1. Determine the singular value decomposition of the 2×2 matrix

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Ans:

$$A = USV^t = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Singular Value Decomposition:

Singular Value Decomposition: Exercise

2. Determine the singular value decomposition of the 2×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Ans: $A = USV^t =$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}.$$