Lecture 18: Numerical Linear Algebra (UMA021): Interpolation

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Newton Forward Difference Formula

Newton Forward Difference Formula:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \triangle^k f(x_0), \text{ where } x_s = x_0 + sh.$$

$s > 0$

$x_0, x_1, x_2 - - x_n$

equally spaced.

$x_i = x_0 + ih$

Newton Backward Difference Formula:

$$P_n(x) = f(x_n) + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n), \text{ where } x_s = x_n + sh.$$

$s < 0$

$x_0, x_1 \; - - \; x_n$

$x_n =$

$x_{n-1} = x_n - h$

$\vdots$

Example:

Given the following data, estimate $f(1.83)$, $f(3.5)$ using Newton forward and backward difference interpolating polynomial: resp.

x	x_0: 1	x_1: 3	x_2: 5	x_3: 7	x_4: 9
$f(x)$	0	1.10	1.61	1.95	2.20

$h = 2$

$x_s = 1.83$

Solution:

$x_s = \underbrace{x_0 + sh} = 1 + s(2)$

$1.83 = 1 + 2s$

$s = 0.415$

$\boxed{n = 4}$

i	x_i	$f(x_i)$	$\Delta f(x_i) = \nabla f(x_{i+1})$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i) = \nabla^3 f(x_{i+3})$
→ 0	1	0	$1.10 - 0 = 1.10$	$0.51 - 1.10 = -0.59$	0.42
1	3	1.10	$1.61 - 1.10 = 0.51$	$0.34 - 0.51 = -0.17 = \Delta^2 f(x_1)$	0.08
2	5	1.61	$1.95 - 1.61 = 0.34$	$0.25 - 0.34 = -0.09$	
3	7	1.95	$2.20 - 1.95 = 0.25$		
→ 4	9	2.20			

-0.34

$$P_4(x_s) = P_n(1.83) = f(x_0) + \sum_{k=1}^{4} \binom{s}{k} \Delta^k f(x_0)$$

$$= \binom{s}{0} f(x_0) + \binom{s}{1} \Delta f(x_0) + \binom{s}{2} \Delta^2 f(x_0) + \binom{s}{3} \Delta^3 f(x_0)$$

$$+ \binom{s}{4} \Delta^4 f(x_0)$$

$$\frac{s!}{(s-1)!\,1!} = \frac{s\cancel{(s-1)!}}{\cancel{(s-1)!}}$$

$$= s$$

$$s = 0.415$$

$$= 0 + s(1.10) + \frac{s(s-1)}{2!}(-0.59)$$

$$+ \frac{s(s-1)(s-2)}{3!}(0.42) + \frac{s(s-1)(s-2)(s-3)}{4!}(-0.34)$$

$$= 0 + (1.1)(0.415) + (0.415)(0.415-1)(-0.59)$$

$$+ \frac{(0.415)(0.415-1)(0.415-2)}{3!}(0.42) + \frac{(0.415)(0.415-1)(0.415-2) * (0.415-3)(-0.34)}{4!}$$

$$= \ ?$$

Using Newton backward Difference to find f(3.5)

$$x_s = 3.5$$

$$3.5 = x_4 + sh$$

$$3.5 = 9 + s(2)$$

$$3.5 - 9 = 2s$$

$$s = \underbrace{-2.75} \rightarrow -ve$$

N.B.D. formula is

$$P_4(x_s) = P_4(3.5) = f(x_4) + \sum_{k=1}^{4} (-1)^k \binom{-s}{k} \nabla^k f(x_4)$$

$$= f(x_4) - \binom{-s}{1} \nabla f(x_4) + \binom{-s}{2} \nabla^2 f(x_4) - \binom{-s}{3} \nabla^3 f(x_4)$$

$$+ \binom{-s}{4} \nabla^4 f(x_4)$$

$$= 2.20 - \binom{2.75}{1}(0.25) + \binom{2.75}{2}(-0.09) - \binom{2.75}{3}(0.08)$$

$$+ \binom{2.75}{4}(-0.34) = ?$$

Example:

For a function f, the forward-divided-differences are given by

$x_0 = 0.0$	$f(x_0) = ?$	$\triangle f(x_0) = ?$ $= 20/7$	$\triangle^2 f(x_0) = \frac{50}{7}$
$x_1 = 0.4$	$-4 = f(x_1) = ?$	$\triangle f(x_1) = 10$	$--$
$x_2 = 0.4$	$f(x_2) = 6$	$--$	$--$

Determine the missing entries in the table.

Solution:

$$\Delta^2 f(x_0) = \Delta f(x_1) - \Delta f(x_0) = \frac{50}{7}$$

$$10 - \Delta f(x_0) = \frac{50}{7}$$

$$\Delta f(x_0) = 10 - \frac{50}{7} = \frac{20}{7}$$

f

$$\Delta f(x_1) = f(x_2) - f(x_1)$$

$$10 = 6 - f(x_1) \Rightarrow f(x_1) = -4$$

4

$$\Delta f(x_0) = f(x_1) - f(x_0)$$

$$\frac{20}{7} = -4 - f(x_0)$$

$$-\left(\frac{20}{7} + 4\right) = f(x_0)$$

$$f(x_0) = -\frac{48}{7}$$

Ans

Newton Forward-Backward Difference

Exercise:

1. Construct the interpolating polynomial that fits the following data using Newton's forward and backward difference interpolation. Hence find the values of $f(x)$ at $x = 0.15$ and 0.45.

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	-1.5	-1.27	-0.98	-0.63	-0.22	0.25

2. The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2	3	4	5	6	7	
$f(x)$	4	9	15	18	-	-	-	-	-

Determine the coefficient of x^3 in $P(x)$ if all fourth-order forward differences are 1.

$\Delta^4 f(x_0) = 1$ -- $\Delta^4 f(x_i) = 1$

$\Delta^4 f(x_1) = 1$ $\quad 0 \le i \le n$

Soln

i	x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$	$\Delta^5 f(x_i)$
0	0	4	5	1	-4	1	0
1	1	9	6	-3	–	1	0
2	2	15	3	–	–	1	0
3	3	18	–	–	–	1	0
4	4	–	–		–		
5	5	–					
6	6	–					
|	|	–					
|	,	–					

$$P(x_s) = f(x_0) + \sum_{k=1}^{4} \binom{s}{k} \Delta^k f(x_0)$$

Exercise:

3. Suppose that $f(x) = \cos x$ to be approximated on $[0, 1]$ by an interpolating polynomial on $n + 1$ equally spaced points. What step size h ensure that linear interpolation gives an absolute error of at most 10^{-6} for all $x \in [0, 1]$.