Lecture 36: Numerical Linear Algebra (UMA021): Matrix Computations

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$$A_{m\times n} = U_{m\times m} \; S_{m\times n} \; V^t_{n\times n}$$

$$\underset{m\times n}{\left[\quad A \quad\right]} = \underset{m\times m}{\left[\; U \;\right]} \underset{m\times n}{\left[\quad S \quad\right]} \underset{n\times n}{\left[\; V \;\right]}$$

$\boxed{A^tA}$

① To construct S :-

* find the E.values of A^tA $\lambda_1, \lambda_2 - - - \lambda_n$
* Take the square root of the E.values. $\sqrt{\lambda_1}, \sqrt{\lambda_2} - - \sqrt{\lambda_n}$
* list them from largest to smallest
* place them at the diagonal entries of S.

$$S = \begin{bmatrix} \sqrt{\lambda_1} & 0 & - & 0 \\ 0 & \sqrt{\lambda_2} & - & 0 \\ - & - & - & - \\ - & - & - & \sqrt{\lambda_n} \\ 0 & 0 & - & 0 \end{bmatrix}$$

② To construct v :-

* find the eigenvectors x_i of A^tA corresp. to each λ_i

* these vectors are orthogonal then make them orthonormal e_i

$$A = USV^t$$

* place these orthonormal vectors as columns of v.

$$V = \begin{bmatrix} \uparrow & & & \\ e_1 & e_2 & \text{———} & e_n \\ \downarrow & & & \end{bmatrix}$$

③ To construct U :-

Singular Value Decomposition:

Constructing U in the factorization $A = U\,S\,V^t$:

The non-zero eigenvalues of $\boxed{A^t\,A}$ and those of $A\,A^t$ are the same. In addition, the corresponding eigenvectors of the symmetric matrices $A^t\,A$ and $A\,A^t$ form complete orthonormal subsets of R^n and R^m, respectively.
So the orthonormal set of n eigenvectors for $A^t\,A$ form the columns of V, as stated above, and the orthonormal set of m eigenvectors for $A\,A^t$ form the columns of U in the same way.

AA^t

To construct U :- * find the E.vectors of AA^t corresp. to same E.values as of A^tA

* make them orthonormal

* Then place them as columns of U.

Singular Value Decomposition:

Singular Value Decomposition: Example

Determine the singular value decomposition of the 5×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{5\times 3}$$

from prev. E.g. we have already found

S & V

e. values of A^tA are 5, 2, 1

$$S = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{5\times 3}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

To find U :-

Take $AA^t = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$

Since non-zero e. values of AA^t and A^tA are same

$$= \begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}_{5\times 5}$$

$\Rightarrow$ The non-zero e.values of AA^t are 5, 2, 1

To find E.vector corresp. to $\lambda_1 = 5$

i.e. $(AA^t - 5I)X = 0$

$$\begin{bmatrix} -3 & 0 & 1 & 0 & 1 \\ 0 & -4 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 \\ 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + x_3 + x_5 = 0$ — (1)

$-4x_2 + x_3 + x_4 + x_5 = 0$, — (2)

$x_1 + x_2 - 3x_3 + x_4 + x_5 = 0$ — (3)

$x_2 + x_3 - 4x_4 + x_5 = 0$, — (4)

$x_1 + x_2 + x_3 + x_4 - 3x_5 = 0$ — (5)

from eq^n's (2) & (4)

$$\begin{array}{r} -4x_2 + x_3 + x_4 + x_5 = 0 \\ \underline{x_2 + x_3 - 4x_4 + x_5 = 0} \\ -5x_2 + 5x_4 = 0 \end{array}$$

$$\boxed{x_2 = x_4}$$

from eq^n (3) & (5)

$$\begin{array}{r} \cancel{x_1} + \cancel{x_2} - 3x_3 + \cancel{x_4} + x_5 = 0 \\ \underline{\cancel{x_1} + \cancel{x_2} + x_3 + \cancel{x_4} - 3x_5 = 0} \\ -4x_3 + 4x_5 = 0 \end{array}$$

$$x_3 = x_5$$

from eqⁿ 1 $-3x_1 + x_5 + x_5 = 0$

$$-3x_1 + 2x_5 = 0$$

$$x_5 = \frac{3}{2}x_1 \quad \Rightarrow \quad x_3 = \frac{3}{2}x_1$$

from eq 2 $-4x_2 + \frac{3}{2}x_1 + x_2 + \frac{3}{2}x_1 = 0 \Rightarrow -3x_2 = -3x_1$

$$x_2 = x_1$$

$$\Rightarrow x_4 = x_1$$

To find e.vector corresp to 2

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_1 \\ \frac{3}{2}x_1 \\ x_1 \\ 3/2\,x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ \frac{3}{2} \\ 1 \\ 3/2 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 2 \\ 2 \\ \frac{3}{2} \\ 3 \end{bmatrix}$$

$$(AA^t - 2I)\,X = 0$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 + x_5 = 0$, —1 $\quad -x_2 + x_3 + x_4 + x_5 = 0$, —2 $\quad x_1 + x_2 + x_4 + x_5 = 0$ —3

$x_2 + x_3 - x_4 + x_5 = 0$, $\quad x_1 + x_2 + x_3 + x_4 = 0$

from eq^n (2) & (1) $\quad -x_2 + x_4 = 0 \quad$ — (4), $\quad x_2 = x_4$

from eq (3) & (5) $\quad x_5 - x_3 = 0$ — (5)

$x_5 = x_3$

from eq^n (1) & $x_3 = x_5$

$x_3 = x_5 = 0$

from eq^n (3)

$x_1 + x_2 + x_2 + 0 = 0$

$x_1 = -2x_2$

$$X = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

To find the e.vector corresp. to 1

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 + x_5 = 0$, — 1

$x_3 + x_4 + x_5 = 0$, — 2

$x_1 + x_2 + x_3 + x_4 + x_5 = 0$ — 3

$x_2 + x_3 + x_5 = 0$ — 4

from 1 & 2

$x_1 + x_4 = 0$

$x_4 = -x_1$

from eq^n 2 & 3

$x_1 + x_2 = 0$

$x_1 = -x_2$

from eq^n 3 & 4

$x_1 + x_4 = 0$

$x_4 = -x_1$

$x_4 = -x_1$

$x_4 = x_2$

from eq^n 2 & 4

$x_4 - x_2 = 0$

$x_4 = x_2$

from eq^n 1 & 4

$x_1 - x_2 = 0$

$x_1 = x_2$

$\Rightarrow x_1 = x_2 = x_4 = 0$

$\Rightarrow x_3 + x_5 = 0$

$\Rightarrow x_5 = -x_3$

$$X = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ -x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$\Rightarrow$ X_1, X_2 & X_3 are orthogonal

Rest two vectors, we will find in next lecture to complete U matrix.