Lecture 15: Numerical Linear Algebra (UMA021): Interpolation

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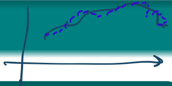
Newton Divided Difference Interpolation:



Derivation:

The divided differences of f with respect to $x_0, x_1, \cdots, x_n$ are used to express $P_n(x)$ in the form

$$\boxed{P_n(x)} = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \qquad (*)$$

for appropriate constants $a_0, a_1, \cdots, a_n$.

$f[x_0] \rightarrow$ zero D.D.

$= f(x_0)$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$

I^{st} D.D. $\boxed{f[x_0, x_1]} = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

2^{nd} D.D. $f[x_0, x_1, x_2] = f[x_1, x_2] - f[x_0, x_1]$

$x_2 - x_0$

$$= \frac{\dfrac{f[x_2]-f[x_1]}{x_2-x_1} - \dfrac{f[x_1]-f[x_0]}{x_1-x_0}}{x_2 - x_0}$$

3^{rd} D.D. $f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$

n^{th} D.D. $f[x_0, x_1 \text{ --- } x_n] = \dfrac{f[x_1, x_2 \text{ -- } x_n] - f[x_0, x_1 \text{ -- } x_{n-1}]}{x_n - x_0}$

Derivation To determine a_0, we put $x = x_0$ in (*)

$P_n(x_0) = a_0 + 0 + 0 \text{ - - -}$

$a_0 = P_n(x_0) = f(x_0)$

$$\boxed{a_0 = f[x_0]}$$

To determine a_1 we put $x = x_1$ in (*)

$$P_n(x_1) = a_0 + a_1(x_1 - x_0) + 0 + 0 \ ----$$

$$f(x_1) = f(x_0) + a_1(x_1 - x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

$$a_1 = f[x_0, x_1]$$

To determine a_2, we put $x = x_2$ in (*)

$$P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{(f(x_1) - f(x_0))}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) - f(x_0) - \frac{(f(x_1) - f(x_0))(x_2 - x_0)}{x_1 - x_0} = a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2(x_2 - x_0)(x_2 - x_1) = \frac{x_1 f(x_2) - x_0 f(x_2) - x_1 f(x_0) + \cancel{x_0 f(x_0)}}{(x_1 - x_0)}$$

$$- x_2 f(x_1) + x_2 f(x_0) + x_0 f(x_1) - \cancel{x_0 f(x_0)}$$

$$a_2 = \frac{x_1 f(x_2) - x_0 f(x_2) - x_1 f(x_0) - x_2 f(x_1) + x_2 f(x_0) + x_0 f(x_1)}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{x_1 f(x_2) - x_0 f(x_2) - x_1 f(x_0) - x_2 f(x_1) + x_2 f(x_0) + x_0 f(x_1) - x_1 f(x_1) + x_1 f(x_1)}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2)(x_1 - x_0) - f(x_1)(x_1 - x_0) - f(x_1)(x_2 - x_1) + f(x_0)(x_2 - x_1)}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{\left(f(x_2) - f(x_1)\right)(x_1 - x_0) - (x_2 - x_1)\left(f(x_1) - f(x_0)\right)}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$$

$$a_2 = f[x_0, x_1, x_2] \quad , a_3 = f[x_0, x_1, x_2, x_3]$$

||ly $\quad a_n = f[x_0, x_1, x_2 \cdots x_n]$

$$\Rightarrow P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+ \cdots \; f[x_0, x_1 \cdots x_n](x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1})$$

Newton Divided Difference Interpolation:

Divided Differences

The zeroth divided difference of the function f with respect to x_i, denoted $f[xi]$, is simply the value of f at x_i : $f[x_i] = f(x_i)$.

The first divided difference of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and defined as: $f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$.

The second divided difference, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as $f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$.

The kth divided difference, $f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k}]$, is defined as
$f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+k}] - f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$.

N·D·D· Table.

$i=0$

i	x_i	$f[x_i] = f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i - - x_{i+3}]$
0	x_0	$f[x_0]$	$\dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$	$\dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
1	x_1	$f[x_1]$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
2	x_2	$f[x_2]$	$f[x_2, x_3]$		
3	x_3	$f[x_3]$	$f[x_3, x_4]$		
4	x_4	$f[x_4]$	$\vdots$		
5	x_5	$f[x_5]$			
6	x_6	$f[x_6]$			

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + - - -$$

Newton Divided Difference Interpolation:

Example:

Complete the divided difference table for the following data:

x	1	1.3	1.6	1.9	2.2
$f(x)$	0.7652	0.6201	0.4554	0.2818	0.1103

and construct the interpolating polynomial that uses all this data and hence find the value of $f(x)$ at $x = 1.5$.

Solution:

x_1, x_2

i	x_i	$f(x_i) = f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i \cdots x_{i+3}]$	
0	1.0	0.7652				
			$\frac{0.6201-0.7652}{1.3-1.0} = -0.4837$			
1	1.3	0.6201		$\frac{-0.5489+0.4837}{1.6-1.0} = -0.1087$		
			$\frac{0.4554-0.6201}{1.6-1.3} = -0.5489$		0.0659	
2	1.6	0.4554		-0.0494		0.001825
			-0.5786		0.0681	
3	1.9	0.2818		0.0118		
			-0.5715			
4	2.2	0.1103				

$$P_4(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3]$$

$$+ (x - x_0)(x - x_1)(x - x_2) + f[x_0 - - - x_4](x - x_0) - - (x - x_3)$$

Put $x = 1.5$

$P_4(1.5) = 0.51182$ $\underline{Ay}$

Newton Divided Difference Interpolation:

Exercise:

1. Using Newton's divided difference interpolation, construct interpolating polynomials of degree one, two, and three for the following data. Approximate $f(0.43)$ using the polynomial.

   $f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.4816.$

2. Show that the Newton polynomial interpolating the following data has degree 3.

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4