Lecture 34: Numerical Linear Algebra (UMA021): Matrix Computations

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$A = QR$

Q: orthogonal matrix

R: upper triangular Matrix.

$\boxed{A_1}$

A_2

$\downarrow$

diagonal matrix

or

upper triangular matrix.

$$\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}$$

$A_1 = Q_1 R_1$

Reverse

$A_2 = R_1 Q_1$

Decompose A_2 into QR

$A_2 = Q_2 R_2$

Reverse

$A_3 = R_2 Q_2$

Decompose A_3 into QR

$A_3 = Q_3 R_3$

Reverse

$A_4 = R_3 Q_3$

$A_k = Q_k R_k$

$A_{k+1} = R_k Q_k$

$A_{k+1} = I R_k Q_k$

Since Q_k is orthogonal matrix.

ie $Q_k^t Q_k = I$

$A_{k+1} = Q_k^t (Q_k R_k) Q_k$

$A_{k+1} = Q_k^t A_k Q_k$

$A_{k+1} = Q_k^{-1} A_k Q_k$

Similarity Transformation:

Similar Matrices:

Two matrices A and B are said to be similar if a non-singular matrix S exists with $A = S^{-1}BS$.
Note: Similar matrices have the same eigenvalues.

$\lambda \to$ E-value of A

$X \to$ E-vector corresp. to λ

$A = S^{-1} B S$

$A = S^{-1} B S$

λ is also E-value of B

SX is the Evector of λ

Similarity Transformation:

Similar Matrices:

If A and B are two similar matrices with $A = S^{-1}BS$ and λ is an eigenvalue of A with associated eigenvector X, then λ is an eigenvalue of B with associated eigenvector SX
Note: Similar matrices have the same eigenvalues.

$A \sim D$

Similarity Transformation:

Similar to diagonal matrix:

A square matrix A is similar to diagonal matrix D if there exists an invertible matrix S and diagonal matrix D such that $D = S^{-1}AS$.

$A \sim D$

$D = S^{-1}AS$

Given matrix A

$A_{3\times 3}$

find E.values of A = $\lambda_1, \lambda_2, \lambda_3$ & find E.vectors Corresp. to each E.value x_1, x_2, x_3

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad S = \begin{bmatrix} \underset{3\times 1}{X_1} & \underset{3\times 1}{X_2} & \underset{3\times 1}{X_3} \end{bmatrix}$$

$$\boxed{S^{-1}AS = D}$$

Similarity Transformation:

Similar to triangular matrix

A square matrix A is similar to upper-triangular matrix T if there exists an invertible matrix S and upper-triangular matrix T such that $T = S^{-1}AS$.

QR Algorithm:

QR Algorithm:

QR Algorithm:

Write the QR factorization of matrix A and perform two iterations of QR algorithm to find all the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Decompose matrix A in QR

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Use Gram- Schmidth process to columns of A

ie $a_1 = \underline{(2,1)}^t$, $a_2 = (1,2)$

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{\|v_1\|_2^2} v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1(2) + 2(1)}{(\sqrt{4+1})^2} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 6/5 \end{bmatrix}$$

To make v_1 & v_2 orthonormal, we have

$$e_1 = \frac{v_1}{\|v_1\|}, \quad e_2 = \frac{v_2}{\|v_2\|}$$

$$= \frac{1}{\sqrt{4+1}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{\frac{9}{25} + \frac{36}{25}}} \begin{bmatrix} -3/5 \\ 6/5 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \quad \frac{\sqrt{5}}{3} \begin{bmatrix} -3/5 \\ 6/5 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

✓

$$a_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

✓

$$Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = Q_1 \text{ (say)}, \quad R = \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 \\ 0 & a_2 \cdot e_2 \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \\ 0 & -\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & \frac{4}{\sqrt{5}} \\ 0 & 3/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 4 \\ 0 & 3 \end{bmatrix} = R_1 \text{ (say)}$$

$$A_1 = Q_1 R_1$$

$$A_2 = R_1 Q_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 4 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 14 & 3 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8 & 0.6 \\ 0.6 & 1.2 \end{bmatrix}$$

Again we decompose the matrix A_2 in QR

Apply Gram - Schmidth process to columns of A_2

ie $b_1 = (2.8, 0.6)^t$

$b_2 = (0.6,\ 1.2)^t$

$$u_1 = b_1 = \begin{bmatrix} 2.8 \\ 0.6 \end{bmatrix} \qquad e_1 = \frac{1}{\sqrt{(2.8)^2 + (0.6)^2}} \begin{bmatrix} 2.8 \\ 0.6 \end{bmatrix}$$

$$u_2 = b_2 - \frac{b_2 \cdot u_1}{\|u_1\|_2^2} u_1 \qquad = \frac{1}{\sqrt{8.2}} \begin{bmatrix} 2.8 \\ 0.6 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 0.9778 \\ 0.2095 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 \\ 1.2 \end{bmatrix} - \frac{0.6(2.8) + 1.2(0.6)}{\left(\sqrt{(2.8)^2 + (0.6)^2}\right)^2} \begin{bmatrix} 2.8 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 \\ 1.2 \end{bmatrix} - \frac{2.4}{8.2} \begin{bmatrix} 2.8 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2195 \\ 1.0244 \end{bmatrix} \Rightarrow e_2 = \frac{1}{\sqrt{(-0.2195)^2 + (1.0244)^2}} \begin{bmatrix} -0.2195 \\ 1.0244 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} -0.2095 \\ 0.9778 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.9778 & -0.2095 \\ 0.2095 & 0.9778 \end{bmatrix} = Q_2 \text{ (say)}$$

$$R = \begin{bmatrix} b_1 \cdot e_1 & b_2 \cdot e_1 \\ 0 & b_2 \cdot e_2 \end{bmatrix} = \begin{bmatrix} 2.8635 & 0.8381 \\ 0 & 1.0477 \end{bmatrix} = R_2 \text{ (say)}$$

$$A_3 = R_2 Q_2 = \begin{bmatrix} 2.9756 & 0.2196 \\ 0.2195 & 1.0244 \end{bmatrix} \Rightarrow$$

Approximate E. values are ✓ 2.9756, 1.0244

If we find more iterations

$$A_4 = \begin{bmatrix} 2.973 & 0.0740 \\ 0.0740 & 1.0027 \end{bmatrix}$$

$$A_{10} = \begin{bmatrix} 3.000 & 0.0001 \\ 0.00001 & 1.0000 \end{bmatrix} \rightarrow$$

3, 1 are the E.values of A.