Lecture 13: Numerical Linear Algebra (UMA021): Interpolation

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$f(x) = 0$

Polynomial approximation:

Approximation of function:

$\boxed{f(x)} \to g(x)$

$f(x)$

$f(x) \approx g(x)$

Weierstrass Approximation Theorem.

Let $f \in C[a,b]$ then we can approximate a function by a polynomial in $[a,b]$

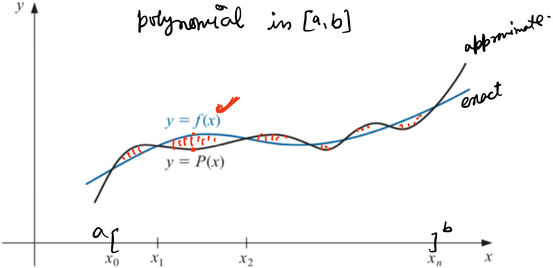
$\left| e^x - \left(1 + x + \frac{x^2}{2!}\right) \right|$

$f(3)$

$= \sin x$

$f(x) \approx P(x)$

max error $= \max e(x) = \max\limits_{x \in [a,b]} \left| f(x) - P(x) \right|$



Using Taylor's series about $x = a$

$a = 0$

Polynomial approximation:

$$\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots}$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$h \to x$

$a \to 0$

Taylor's polynomial:

For example: Let $f(x) = e^x$

$f(x) = e^x$

Taylor's series about $x = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

for approximation

$$e^x \approx 1 = P_0(x)$$

$$e^x \approx 1 + x = P_1(x)$$

$$e^x \approx 1 + x + \frac{x^2}{2!} = P_2(x)$$

$\lim_{n \to \infty} P_n(x) = e^x$

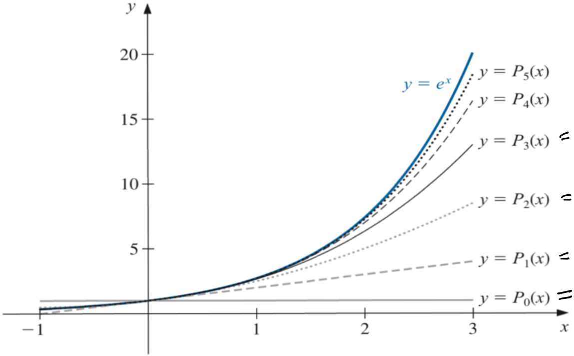
$$e^x \approx 1 + x + x^2 + \cdots \frac{x^n}{} = P_n(x)$$

$2!$ $n!$ $n \to +\infty$

Polynomial approximation:

Approximation of function:

From the following graph of the polynomial approximation we can see that for the higher-degree polynomials, the error becomes less and it approaches to the function $f(x) = e^x$.



$y = P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$y = P_2(x) = 1 + x + \frac{x^2}{2!}$

$y = P_1(x) = 1 + x$

$y = P_0(x) = 1$

Polynomial Approximation

Limitations of approximation by Taylor's Polynomial:

1. By Taylor's polynomials approximation, we can get better approximation only for higher order differentiable functions.
2. This approximation does not give better for all functions.

for

e.g. $f(x) = \frac{1}{x}$, $\Rightarrow f'(x) = \frac{-1}{x^2}$ $f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-6}{x^4}$

Taylor's series about $x=1$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) - - -$$

$$= 1 + (x-1)(-1) + \frac{(x-1)^2}{\cancel{2!}}(\cancel{2}) + \frac{(x-1)^3}{\cancel{3!}}(-6) + - - \cdot$$

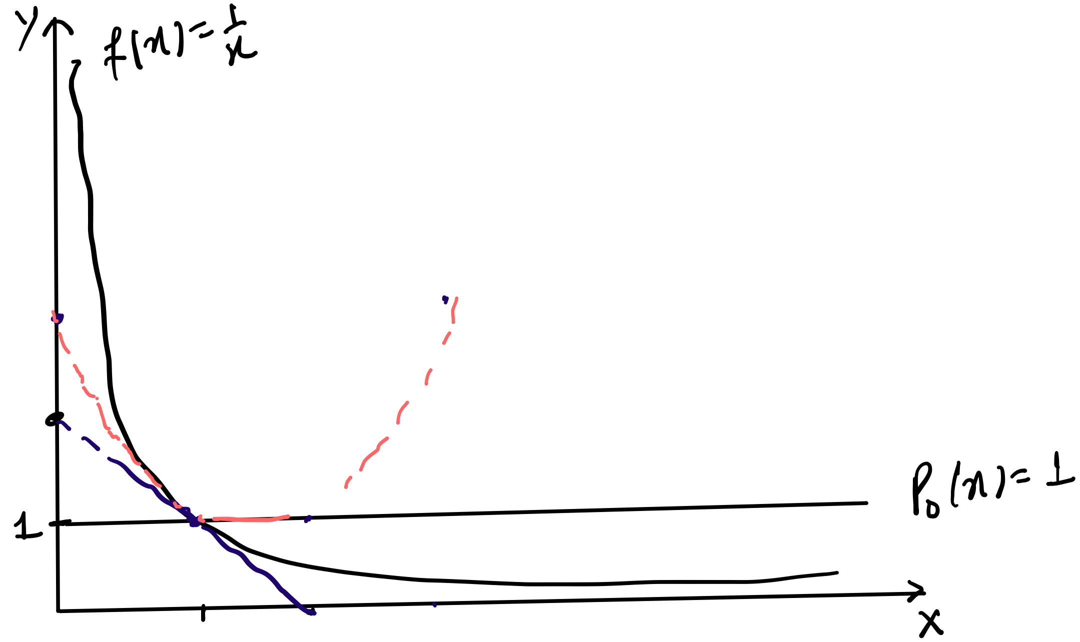
$$f(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + - - -$$

$$f(x) \approx 1 = P_0(x)$$

$$f(x) \approx 1 - x + 1 = 2 - x = P_1(x)$$

$$f(x) \approx 1 - (x-1) + (x^2+1-2x) = x^2 - 3x + 3$$

$y = 2 - x$



not getting better approximation by Taylor's polynomial

Polynomial interpolation:

x_0, x_1

Interpolation:

Given $(n+1)$ points $(x_i, y_i), i = 0, 1, 2, \cdots, n$ at which function $f(x)$ passes, then approximate in form of a polynomial $P(x)$ of degree n such that is

$f(x_0) = P(x_0)$

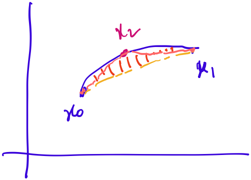
$f(x_1) = P(x_1)$

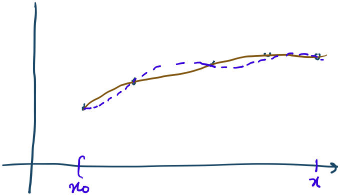
$$f(x_i) = P(x_i), i = 0, 1, 2, \cdots, n.$$

Using this polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.

$f(x_0) =$

$f(x_1)$





Polynomial interpolation:

Result:

Existence and Uniqueness: Given a real valued function $f(x)$ and $(n+1)$ distinct points $x_0, x_1, \cdots, x_n$ there exists a unique polynomial $P_n(x)$ of degree $\leq n$ which interpolates the unknown $f(x)$ at the points $x_0, x_1, \cdots, x_n$

$x_0, x_1, \cdots x_n$ $\qquad$ $P_n(x) \approx f(x)$

Polynomial interpolation:

Methods to interpolate:

1. Lagrange Interpolation
2. Newton's Interpolation