Lecture 35: Numerical Linear Algebra (UMA021): Matrix Computations

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Singular Value Decomposition:

Constructing V in the factorization $A = U\,S\,V^t$:

The $n \times n$ matrix $A^t\,A$ is symmetric, we have a factorization $A^t\,A$, where D is a diagonal matrix whose diagonal entries are the eigenvalues of $A^t\,A$ and V is an orthogonal matrix whose *ith* column is an eigenvector with l_2 norm corresponding to the eigenvalue on the *ith* diagonal entry of $D = S^2$.

Singular Value Decomposition:

5, 2, 1 → 3x3

Constructing U in the factorization $A = U\,S\,V^t$:

The non-zero eigenvalues of $A^t\,A$ and those of $A\,A^t$ are the same. In addition, the corresponding eigenvectors of the symmetric matrices $A^t\,A$ and $A\,A^t$ form complete orthonormal subsets of R^n and R^m, respectively.

So the orthonormal set of n eigenvectors for $A^t\,A$ form the columns of V, as stated above, and the orthonormal set of m eigenvectors for $A\,A^t$ form the columns of U in the same way.

5x5

5, 2, 1, 0, 0

$(AA^t - \lambda I)x = 0$

Singular Value Decomposition:

Singular Value Decomposition: Example

Determine the singular value decomposition of the 5×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{5\times 3}$$

Sol^n

from prev. e.g. $S = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

e.values of $A^t A$ are $5, 2, 1$

To find V, we find the e.vectors corresp. to each e.values.

To find e-vector X corresp. to $\lambda = 5$

$$(A^t A - \lambda I) X_1 = 0$$

$$\begin{bmatrix} 2-5 & 1 & 1 \\ 1 & 4-5 & 1 \\ 1 & 1 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -3x_1 + x_2 + x_3 = 0, \quad x_1 - x_2 + x_3 = 0, \quad x_1 + x_2 - 3x_3 = 0$

$$\begin{array}{r} x_1 - x_2 + x_3 = 0 \\ \hline 2x_1 - 2x_3 = 0 \end{array}$$

$x_1 = x_3$

$-3x_1 + x_2 + x_1 = 0$

$+2x_1 = x_2$

$$\Rightarrow \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ +2x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ corresp to $\lambda = 5$

To find X_2 corresp to $\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 + x_3 = 0$, $\quad x_1 + 2x_2 + x_3 = 0$, $\quad x_1 + x_2 = 0$

$x_3 = -x_2 = x_1 \qquad x_2 = -x_1$

$$X_2 = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

To find X_3 Corresp to $\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0, \qquad x_1 + 3x_2 + x_3 = 0 \qquad x_1 + x_2 + x_3 = 0$$

$$\begin{array}{r} x_1 + 3x_2 + x_3 = 0 \\ \underline{-\ \ -\ \ -} \\ \hline -2x_2 = 0 \end{array} \qquad\qquad x_1 = -x_3$$

$$x_2 = 0$$

$$X_3 = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ X_1 & X_2 & X_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \rightarrow \text{orthogonal matrix.}$$

To find U

We write

$$AA^t = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

The non-zero E.values of AA^t are A^tA are same

So Evalues of AA^t are $5, 2, 1, 0, 0$ ✓

To find E.vector corresp to $\lambda = 5$

$$(AA^t - 5I)X_1 = 0 \Rightarrow \begin{bmatrix} -3 & 0 & 1 & 0 & 1 \\ 0 & -4 & 1 & 1 & 1 \\ 1 & 1 & -3 & 1 & 1 \\ 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + x_3 + x_5 = 0, \quad -(1)$

$-4x_2 + x_3 + x_4 + x_5 = 0, \quad -(2)$

$x_1 + x_2 - 3x_3 + x_4 + x_5 = 0 \quad -(3)$

$x_2 + x_3 - 4x_4 + x_5 = 0, \quad -(4)$

$x_1 + x_2 + x_3 + x_4 - 3x_5 = 0 \quad -(5)$

from (3) & (5)

$-4x_3 + 4x_5 = 0$

$x_3 = x_5$

from eqn (1)

$-3x_1 + x_3 + x_3 = 0$

$-3x_1 = -2x_3$

$x_3 = +\frac{3}{2}x_1$

$x_5 = +\frac{3}{2}x_1$

from eqn (2) & (4)

$-5x_2 + 5x_4 = 0$

$x_2 = x_4$

from eqn (4)

$x_2 + \frac{3}{2}x_1 - 4x_2 + \frac{3}{2}x_1 = 0$

$-3x_2 = -3x_1$

$x_2 = +x_1 \quad \Rightarrow x_4 = +x_1$

$$X = \begin{bmatrix} x_1 \\ +x_1 \\ +\frac{3}{2}x_1 \\ +x_1 \\ +\frac{3}{2}x_1 \end{bmatrix} = +x_1 \begin{bmatrix} +1 \\ 1 \\ \frac{3}{2} \\ 1 \\ 3/2 \end{bmatrix}$$

$$+2x_4 \begin{bmatrix} +2 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow X_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} \checkmark \qquad \sqrt{4+4+9+4+9}$$

To find X_2 corresp to $\lambda = 2$ ie $(AA^t - 2I)X_2 = 0$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 + x_5 = 0$, —(1) $\qquad -x_2 + x_3 + x_4 + x_5 = 0$, —(2) $\qquad x_1 + x_2 + x_4 + x_5 = 0$ —(3)

$x_2 + x_3 - x_4 + x_5 = 0$ —(4) $\qquad x_1 + x_2 + x_3 + x_4 = 0$ —(5)

from (1) & (2)

$-x_2 + x_4 = 0$

$x_4 = x_2$

from eqn (1) & (4)

$x_2 = x_4$

from eqn (3) & (5)

$x_5 - x_3 = 0$

$x_3 = x_5$

$\Downarrow$

from eqⁿ ①

$x_3 = x_5 = 0$

from eqⁿ ⑤

$x_1 + x_2 + x_4 = 0$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$x_2 = -\frac{1}{2}x_1 \Rightarrow x_4 = \frac{-1}{2}x_1$

$$X_2 = \begin{bmatrix} x_1 \\ -\frac{1}{2}x_4 \\ 0 \\ -\frac{1}{2}x_1 \\ 0 \end{bmatrix} = \frac{-1}{2}x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\sqrt{4+1+1}$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

To find X_3 corresp to $\lambda = 1$ ie $(AA^t - I)X_3 = 0$

$$(AA^t - I)X_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 + x_5 = 0$ $-(1)$, $x_3 + x_4 + x_5 = 0$ $-(2)$, $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ $-(3)$

$x_2 + x_3 + x_5 = 0$ $-(4)$

from (1) & (3)

$x_2 + x_4 = 0$

$x_4 = -x_2$

$\Rightarrow x_4 = x_1$ $-(5)$

from (2) & (3)

$x_1 + x_2 = 0$

$x_2 = -x_1$

$\Leftarrow$

from (3) & (4)

$x_1 + x_4 = 0$

$x_4 = -x_1$ $-(6)$

from 5 & 6, we get

$$x_1 = x_4 = 0 \quad \Rightarrow x_2 = 0$$

from 1 $\quad x_5 = -x_3$

$$\Rightarrow X_3 = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ -x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

To find X_4 corresp to $\lambda = 0$, $(AA^t - 0I)X_4 = 0$

$$(AA^t - 0I)X_4 = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_3 + x_5 = 0, \quad -(1)$ $\quad x_2 + x_3 + x_4 + x_5 = 0, \quad -(2)$ $\quad x_1 + x_2 + 2x_3 + x_4 + x_5 = 0 \quad -(3)$

$x_2 + x_3 + x_4 + x_5 = 0 \quad -(4)$ $\quad , \quad x_1 + x_2 + x_3 + x_4 + 2x_5 = 0 \quad -(5)$

from (2) & (3)

$$x_1 + x_3 = 0$$

$$\boxed{x_3 = -x_1} \quad -(6)$$

from (1) & (6)

$$2x_1 - x_1 + x_5 = 0$$

$$\boxed{x_5 = -x_1} \quad -(7)$$

from (2)

$$x_2 - x_1 + x_4 - x_1 = 0$$

$$x_4 = 2x_1 - x_2 \quad -(8)$$

$$\Rightarrow X_4 = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 \\ 2x_1 - x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \\ 2x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ -x_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad X_4 = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ All the Eigenvectors are

$$X_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$\{X_1, X_2, X_3, X_4\}$ and $\{X_1, X_2, X_3, X_5\}$ are orthogonal but X_4 & X_5 are not orthogonal

because they are the eigenvectors associated with same e.value is $\lambda = 0$.

So, we keep x_4 as one of eigenvectors used to form U and determine x_5 which will give orthogonal set.

for this, we use Gram-Schmidth process on x_1, x_2, x_3, x_4, x_5

$$v_5 = x_5 - \frac{x_4 \cdot x_5}{\|x_4\|^2} x_4$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{(-2)}{7} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \frac{2}{7} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 1 \\ -2/7 \\ -3/7 \\ -2/7 \end{bmatrix}$$

$$v_5 = \begin{bmatrix} 2/7 \\ 1 \\ -2/7 \\ -3/7 \\ -2/7 \end{bmatrix}$$ which is orthogonal to x_1, x_2, x_3, x_4

$$\|v_5\| = \frac{\sqrt{70}}{7}$$

$$x_5^* = \frac{v_5}{\|v_5\|} = \begin{bmatrix} 2\sqrt{70} \\ \sqrt{70}/7 \\ -2\sqrt{70} \\ -3\sqrt{70} \\ -2\sqrt{70} \end{bmatrix}$$

$$\Rightarrow \quad U = \begin{bmatrix} \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{7}} & 2\sqrt{70} \\ \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{\sqrt{70}}{7} \\ \frac{3}{\sqrt{30}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{7}} & -2\sqrt{70} \\ \frac{2}{\sqrt{30}} & \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{7}} & -3\sqrt{70} \\ \frac{3}{\sqrt{30}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{7}} & -2\sqrt{70} \end{bmatrix}$$

$$\Rightarrow \quad \underbrace{A = USV^t} \quad \rightarrow \text{ verify it}$$