Lecture 8: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Order of convergence:

Iterations ← $\langle p_n \rangle \to p \to$ exact root (sequence)

Bisection method, fixed point Iteration, Newton's method

Definition:

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \boxed{\frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}}} = \lambda,$$

finite

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

$\langle p_n \rangle \to p$

$f(x) = 0$

$[a, b]$

p_0, p_1, p_2

$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$

(i) If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.
(ii) If $\alpha = 2$, ($\lambda \neq 0$) the sequence is quadratically convergent.
In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.

for e.g. $\langle \frac{1}{n} \rangle \to 0$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ----$

$\langle \frac{1}{n^2} \rangle \to 0$ $\quad 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ---$

$\alpha = 2$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda \neq 0$$

$\boxed{p_0}$

$n=0$ $$\frac{|p_1 - p|}{|p_0 - p|^2} = \lambda_1$$

$n=1$ $$\frac{|p_2 - p|}{|p_1 - p|^2} = \lambda_2$$

$n=2$ $$\frac{p_3 - p}{|p_2 - p|^2} = \lambda_3$$

$\boxed{\lambda_1, \lambda_2, \lambda_3 \cdots \cdots \lambda_n}$

decreasing.

If $\alpha = 1$ $\quad p_0 \quad p_1$

$$\frac{|p_1 - p|}{|p_0 - p|^1} = \lambda_1 < 1$$

all $\lambda_i < 1 \;\forall n$

$$\frac{p_2 - p}{|p_1 - p|^1} = \lambda_2 < 1$$

$\alpha = 2, 3, 4, 5 \cdots \cdots$

$$\frac{|p_1 - p|}{|p_0 - p|^{\alpha}} = \lambda_1 \neq 0$$

$$\frac{p_2 - p}{|p_1 - p|^{\alpha}} = \lambda_2 \neq 0$$

$\lambda_3, \lambda_4 \cdots \neq 0$

if $\langle \lambda_n \rangle \to$ decreasing seq.

Order of convergence:

Order of convergence of bisection method:

Let $f(x)=0$ be the equation which has root in $[a,b]$

Let p_n be the nth iteration by bisection method and p be exact root of $f(x)=0$

$$|p_{n+1}-p| \le \frac{|b-a|}{2^n}$$

& $$|p_n - p| \le \frac{|b-a|}{2^{n-1}}$$

Now for $\alpha=1$ $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^1} \le \lim_{n\to\infty} \frac{\frac{|b-a|}{2^n}}{\frac{|b-a|}{2^{n-1}}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$$

$\Rightarrow$ B.M. gives atleast linear

for $\alpha = 2$

$$\frac{|p_{n+1} - p|}{|p_n - p|^2} \leq \frac{\frac{|b-a|}{2^n}}{\left(\frac{|b-a|}{2^{n-1}}\right)^2} = \frac{2^{2n-2}}{2^n \; |b-a|}$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} \leq \lim_{n\to\infty} \frac{2^{n-2}}{|b-a|} = \infty$$

$\Rightarrow$ B.M. generates a sequence of linear order only not of 2nd order.

order cgce

Order of convergence:

$[a,b]$ $\quad c \in (a,b)$

$f(a) = f(b) + (b-a) f'(b) + \frac{(b-a)^2}{2!} f''(b) + \frac{(b-a)^3}{3!} f''(c)$

Order of convergence of fixed point iteration method:

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, $\forall x \in [a,b]$. Suppose, in addition, that g' is continuous on $[a,b]$ and a positive constant $k < 1$ exists with $|g'(x)| \leq k < 1$, for all $x \in (a,b)$

(i) If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, then the sequence $p_n = g(p_{n-1})$, $n \geq 1$ converges only linearly to the unique fixed point p in $[a,b]$.

$p_n \to p$

$g(x)$

$g'(x)$

$g'(p) \neq 0$

$p_{n+1} = g(p_n)$

Using Taylor's poly. on $g(x)$ about a pt. p

$g(p_n) = g(p) + \boxed{g'(p)} (p_n - p) + g''(p) (p_n - p)^2/2! + g'''(p) \frac{(p_n-p)^3}{3!} - - -$

series poly. up to 1st order

$g(p_n) \approx g(p) + g'(c_n) \; (p_n - p)$ $\longrightarrow$ $p_n < c_n < p$

p is exact fixed pt. of g

c_n

$[p_n, p]$

$$p_{n+1} \approx p + g'(c_n)(p_n - p)$$

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{p_n - p} \right| \approx \lim_{n\to\infty} | g'(\underset{\text{cont.}}{\downarrow}(\underset{p}{\downarrow} c_n)) |$$

$$= g'(p) \neq 0$$

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{1}} = |g'(p)| < 1$$

$\Rightarrow$ $p_n \to p$ atleast linearly.

$\because$

① $p_n < c_n < p$

$p_n \to p$, $c_n \to p$, $p \to p$

② If $p_n \to p$

g is cont.

then $g(p_n) \to g(p)$

$p_n \to p$

g

$g(p_n) \nrightarrow g(p)$