Lecture 11: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Modified Newton's method

$f(x) = 0$ has a root p

with multiplicity m

$$\boxed{f(x) = (x-p)^m q(x)}, \quad q(p) \neq 0$$

M.N.M. ~~fo~~formula when (m is given)

$$\boxed{p_{n+1} = p_n - m \frac{f(p_n)}{f'(p_n)}}$$

$$p_{n+1} = g(p_n)$$

$$g(x) = x - m \frac{f(x)}{f'(x)}$$

Multiple roots

Order of convergence of modified Newton's method:

$$g(x) = x - m\frac{f(x)}{f'(x)}$$

$$g(x) = x - m\frac{(x-p)^m q(x)}{(x-p)^m q'(x) + q(x)\, m\, (x-p)^{m-1}}$$

$$g(x) = x - m\frac{(x-p)^1\, q(x)}{(x-p)\, q'(x) + m\, q(x)}, \quad x \neq p$$

let $x = p$

$$g(p) = p - m\frac{(0)}{m\, q(p)} = p - \frac{\cancel{m}}{\cancel{m}}\frac{0}{q(p)} = p$$

$$g'(x) = 1 - m\Bigg[\frac{(x-p)\quad q'(x)}{(x-p)q'(x)+mq(x)}$$

$$+ \frac{q(x)\quad \cdot 1}{(x-p)q'(x)+mq(x)} +$$

$$+ (x-p)\, q(x)\, \frac{d}{dx}\, \frac{1}{(x-p)q'(x)+mq(x)}\Bigg]$$

put $x=p$

$$g'(p) = 1 - m\left[\frac{0 \quad q'(p)}{0+mq(p)} + \frac{q(p)}{0+mq(p)} + \frac{(0)}{\text{nonzero}}\right]$$

$$= 1 - m\left[0 + \frac{1}{m} + 0\right] = 1 - \frac{m}{m} = 1-1$$

$$= 0$$

Multiple roots

Modified Newton's method (if multiplicity is not given)

Let $f(x)=0$ be an equation which has a root at p with multiplicity m i.e.

$$f(x) = (x-p)^m q(x), \quad q(p) \neq 0$$

Define

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m q(x)}{(x-p)^m q'(x) + q(x)\, m (x-p)^{m-1}}$$

$$= \frac{(x-p)\, q(x)}{(x-p)\, q'(x) + m\, q(x)}$$

$$= (x-p)\, Q(x)$$

where $Q(x) = \dfrac{q(x)}{(x-p)q'(x) + m q(x)}$

$\mu(x) = (x-p)^1 Q(x)$

$Q(p) \neq 0$

$\Rightarrow$ $\mu(x)$ has a simple root at $x=p$

So, Apply N. M. on $\mu(x)$

$$Q(p) = \frac{g(p)}{0 + m\,g(p)} = \frac{1}{m} \neq 0$$

$$p_{n+1} = p_n - \frac{\mu(p_n)}{\mu'(p_n)}$$

$$= p_n - \frac{\left(\dfrac{f(p_n)}{f'(p_n)}\right)}{\left(\dfrac{f(p_n)}{f'(p_n)}\right)'}$$

$$= p_n - \frac{\dfrac{f(p_n)}{f'(p_n)}}{\dfrac{f'(p_n)\,f'(p_n) - f(p_n)\,f''(p_n)}{(f'(p_n))^2}}$$

$$= \boxed{p_n - \frac{f(p_n)\,f'(p_n)}{(f'(p_n))^2 - f(p_n)\,f''(p_n)}}$$

Multiple roots

Example:

Show that the modification of Newton's method improves the rate of convergence for $f(x) = e^x - x - 1$ at $x = 0$ with $p_0 = 1$.

Apply M.N.M. (when multiplicity is not given)

$$p_{n+1} = p_n - \frac{f(p_n)\, f'(p_n)}{(f'(p_n))^2 - f(p_n)\, f''(p_n)}$$

$$= p_n - \frac{(e^{p_n} - p_n - 1)\,(e^{p_n} - 1)}{(e^{p_n} - 1)^2 - (e^{p_n} - p_n - 1)(e^{p_n})}$$

let $n = 0$, $p_0 = 1$

$$p_1 = 1 - \frac{(e^1 - 2)\,(e^1 - 1)}{\quad}$$

$$(e-1)^2 - (e^1 - 2)(e^1)$$

$$= -0.23421$$

$$p_2 = -0.23421 - \frac{\left(e^{-0.23421} + 0.23421 - 1\right)\left(e^{-0.23421} - 1\right)}{\left(e^{-0.23421} - 1\right)^2 - \left(e^{-0.23421} + 0.23421 - 1\right) * e^{-0.23421}}$$

$$= -0.0084575$$

$$p_3 = - - -$$

To Check order of cgce

$$\frac{|p_{n+1} - p|}{|p_n - p|^2} \quad p=0 \quad \rightarrow \frac{|p_{n+1}|}{|p_n|^2}$$

Newton's method to system of non-linear equations.

Newton's method for non-linear systems:

Consider the non-linear system:

$x^2+2x+3=0$

$$f(x,y)=0$$

$$g(x,y)=0$$

$x^2+y^2=9=0$

$x^2+e^y=3=0$

$X=\begin{bmatrix} x \\ y \end{bmatrix}$ vector

Let $X^{(k)}=\begin{bmatrix} x_k \\ y_k \end{bmatrix}$, $F=\begin{bmatrix} f \\ g \end{bmatrix}$, $J=\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$

$F=\begin{bmatrix} x^2+y^2-9 \\ x^2+e^y-3 \end{bmatrix}$

$$F(X^{(k)})=F(x_k,y_k)=\begin{bmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{bmatrix},$$

$$J(x_k,y_k)=\begin{bmatrix} f_x(x_k,y_k) & f_y(x_k,y_k) \\ g_x(x_k,y_k) & g_y(x_k,y_k) \end{bmatrix}$$

$X^{(0)}=\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \rightarrow$ initial guess

$X^{(1)}, X^{(2)}$ - - -

Newton's method for non-linear eq^n is given by

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

$\boxed{f(x) = 0}$

$\boxed{\begin{matrix} f(x,y)=0 \\ g(x,y)=0 \end{matrix}}$

vector

vector

Matrix

$$X^{(k+1)} = X^{(k)} - \frac{F(X^{(k)})}{F'(X^{(k)})}$$

2x1

Convert Newton's method for system i.e. In Matrix form

$$F = \begin{bmatrix} f \\ g \end{bmatrix}_{2\times 1}$$

2x2

$$\boxed{X^{(k+1)} = X^{(k)} - J^{-1}_{(X^k)} F(X^k)}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}_{2\times 1}$$

$$F' = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{2\times 2} = J$$

Newton's method to system of non-linear equations.

The Newton's method for non-linear system is given by

$$X^{(k+1)} = X^{(k)} - J^{-1}F(X^{(k)})$$