Lecture 28: Numerical Linear Algebra (UMA021): Matrix Algebra

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$\|x\|_2$ $\|x\|_\infty$ $\|x-y\|_2$ $\|x-y\|_\infty$

System of linear equations: Matrix representation of iterative methods

$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Matrix norm:

A matrix norm on a set of all $n \times n$ matrices is a real-valued function, $\|.\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

- (i) $\|A\| \geq 0$;
- (ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;
- (iii) $\|\alpha A\| = |\alpha| \|A\|$;
- (iv) $\|A + B\| \leq \|A\| + \|B\|$;
- (v) $\|AB\| \leq \|A\|.\|B\|$

The distance between $n \times n$ matrices A and B with respect to this matrix norm is $\underbrace{\|A - B\|}$.

Matrix norm

Matrix norm in l_∞-space

If $A = (a_{ij})$ is $n \times n$ matrix, then the $l_\infty-$norm is given by

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \left\{ |a_{i1}| + |a_{i2}| + |a_{i3}| \cdots |a_{in}| \right\}$$

$$\max \left\{ \begin{array}{l} |a_{11}| + |a_{12}| + |a_{13}| \cdots |a_{1n}|, \\ |a_{21}| + |a_{22}| + |a_{23}| \cdots |a_{2n}|, \end{array} \right.$$

$|a_{n1}| + |a_{n2}| \cdots a_{nn}|\}$

Matrix norm

Example:

Determine $\|A\|_\infty$ norm for the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$.

Solution:

$$\|A\|_\infty = \max_{1 \le i \le 3} \sum_{j=1}^{3} |a_{ij}|$$

$$= \max\{|1| + |2| + |-1|,\ |0| + |3| + |-1|,\ |5| + |-1| + |1|\}$$

$$= \max\{4, 4, 7\} = 7$$

$$\Rightarrow \quad \|A\|_\infty = 7.$$

$\|A\|_2 \rightarrow$ Matrix norm in l_2-space

Matrix norm

Eigenvalues and Eigenvectors:

If A is a square matrix, the characteristic polynomial of A is given by $|A - \lambda I| = p(\lambda)$(say). The zeros of $p(\lambda)$ are the eigenvalues for the matrix A.

If λ is an eigenvalue of A and $X \neq 0$ satisfies $(A - \lambda I)X = 0$, the X is an eigenvector corresponding to eigenvalue λ.

$$\boxed{(A - \lambda I)X = 0}$$

To get $X \neq 0$ we have

$$BX = 0$$

$|B| \neq 0 \Rightarrow$ Unique $\Rightarrow X = 0$

$|B| = 0$: no soln, Infi

X

$$\Rightarrow \underbrace{|A - \lambda I|}_{} = 0$$

Example:

Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}.$$

Solution: Let λ be eigenvalue of the given matrix A

$$|A-\lambda I| = 0 \qquad \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\{(1-\lambda)(4-\lambda) + 2\} = 0$$

$$(2-\lambda)\{4 - 4\lambda - \lambda + \lambda^2 + 2\} = 0$$

$$(2-\lambda)\{\lambda^2 - 5\lambda + 6\} = 0 \quad \Rightarrow \lambda = 2, 2, 3 \rightarrow \text{Eigen}$$

values

for $\lambda = 2$, $(A - 2I)X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + 2x_3 = 0$$

$$x_1 = x_2 - 2x_3$$

$$X = \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

E. vectors corresponding to E. value $\lambda=2$ are

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \& \begin{bmatrix}-2\\0\\1\end{bmatrix}$$

E. vector corresp. to $\lambda=3$

$$\begin{bmatrix}-1 & 0 & 0\\1 & -2 & 2\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1=0$, $x_1 - 2x_2 + 2x_3 = 0$, $x_1 - x_2 + x_3 = 0$

$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

$$\Rightarrow x = \begin{bmatrix}0\\x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}0\\1\\1\end{bmatrix}$$

Matrix norm

Spectral radius:

The spectral radius $\underbrace{\rho(A)}$ of a matrix A is defined by $\rho(A) = \max|\lambda|$, where λ is an eigenvalue of A.

$\rho(A)$ = max of |eigen values of A|

In prev example

E. values of A are 2, 2, 3

$\rho(A) = 3$

Matrix norm in l_2-space

If $A = (a_{ij})$ is $n \times n$ matrix, then the l_2-norm is given by

$$\|A\|_2 = \sqrt{\rho(A^t A)}\,, \rightarrow \left(\text{spectral radius of } A^t A\right)^{1/2}$$

where A^t is the transpose of A.

Matrix norm

Example:

Determine $\|A\|_2-$norm for the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

Solution:

$$A^t = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

To find the E.values of A^tA, we have $|A^tA - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(6-\lambda)(5-\lambda) - 16\right] - 2\left[10 - 2\lambda + 4\right] - 1\left[8 + 6 - \lambda\right] = 0$$

$$(3-\lambda)\left[\lambda^2 - 11\lambda + 30 - 16\right] - 2\left[14 - 2\lambda\right] - \left[14 - \lambda\right] = 0$$

$$-\lambda^3 + 14\lambda^2 - 42\lambda = 0$$

$$-\lambda\left[\lambda^2 - 14\lambda + 42\right] = 0$$

$$\lambda = 0, \qquad \lambda = \frac{14 \pm \sqrt{196 - 168}}{2}$$

$$\lambda = 0 \qquad \lambda = 7 \pm \frac{\sqrt{28}}{2} = 7 \pm \sqrt{7}$$

$$\max\left\{ |0|, (7+\sqrt{7}), |7-\sqrt{7}| \right\} = 7+\sqrt{7}$$

$$\|A\|_2 = (7+\sqrt{7})^{1/2}.$$

for $AX=b$

Write
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots\ 0 \\ \cdots & \cdots & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots\ a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & \cdots \\ a_{31} & a_{32} & \cdots & \cdots \\ a_{n1} & a_{n2} & & 0 \end{bmatrix}$$

$$= D + U + L$$

System of linear equations: Matrix representation of iterative methods

$Ax = b$

Jacobi method:

The Jacobi method is given by $x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k-1)} \right)$

$$a_{ii}\, x_i^{(k)} = b_i - \sum_{j=1}^{n} a_{ij}\, x_j^{(k-1)}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 \cdots & a_{3n} \\ & & & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix}$$

$$D\, X^{(k)} = b - (L + U)\, X^{(k-1)}$$

$$X^{(k)} = D^{-1}b - D^{-1}(L+U)\,X^{(k-1)}$$

Matrix Rep. of Jacobi method

$$X^{(k)} = \underbrace{-D^{-1}(L+U)}_{T_J}\,X^{(k-1)} + D^{-1}b$$

System of linear equations: Matrix representation of iterative methods

Gauss-Seidel method:

The Gauss-Seidel method is given by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right)$$

$$a_{ii} x^{(k)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}$$

$$D x^{(k)} = b - L x^{(k)} - U x^{(k-1)}$$

$$(D+L) x^{(k)} = b - U x^{(k-1)}$$

$$x^{(k)} = (D+L)^{-1} b - (D+L)^{-1} U x^{(k-1)}$$

$X^{(k)} = -(D+L)^{-1}U \, X^{(k-1)} + (D+L)^{-1}b$

$T_g =$

System of linear equations: Matrix representation of iterative methods

Result:(Stronger condition for the convergence of iterative methods):

For any $X^{(0)} \in \mathbb{R}^n$, the sequence $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} = TX^{(k-1)} + C$, for each $k \geq 1$ converges to unique solution $X = TX + C$ iff $\rho(T) < 1$.

$X^{(k)} = T X^{(k-1)} + c$

$\rho(T) < 1$