Lecture 12: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Newton's method to system of non-linear equations.

Newton's method for non-linear systems:

Consider the non-linear system:

$AX=b$

$X=\begin{bmatrix} x \\ y \end{bmatrix}$

$$f(x,y)=0$$

$$g(x,y)=0$$

$\boxed{\begin{matrix} x^2+y^2-9=0 & -① \\ x^2+e^y=0 & -② \end{matrix}}$

Let $X^{(k)} = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$, $F = \begin{bmatrix} f \\ g \end{bmatrix}$, $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$

$X = (x,y)^t = \begin{bmatrix} x \\ y \end{bmatrix}$

$= ?$

$$F(X^{(k)}) = F(x_k, y_k) = \begin{bmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{bmatrix},$$

$\boxed{F = \begin{bmatrix} x^2+y^2-9 = f \\ x^2+e^y = g \end{bmatrix}_{2\times 1}}$

$$J(x_k,y_k) = \begin{bmatrix} f_x(x_k,y_k) & f_y(x_k,y_k) \\ g_x(x_k,y_k) & g_y(x_k,y_k) \end{bmatrix}$$

$X^{(0)} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $X^{(1)} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$X^{(2)} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$\|X^{(n)} - X^{(n-1)}\| < tol.$

$\rightarrow$ norm

$\max\{|x_n - x_{n-1}|, |y_n - y_{n-1}|\} < tol$

Newton's method to system of non-linear equations.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$X^{(k+1)} = X^{(k)} - \frac{F(X^{(k)})}{F'(X^{(k)})}$$

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{2\times 2} = F' = J$$

The Newton's method for non-linear system is given by

$$X^{(k+1)} = X^{(k)} - J^{-1}F(X^{(k)})$$

$(2\times 2)(2\times 1)$

Newton’s method to system of non-linear equations.

Newton’s method for nonlinear systems: Example

Perform 3 iterations of Newton’s method to the following non-linear system:

$$\begin{aligned} x^2 + xy + y^2 &= 7 \\ x^3 + y^3 &= 9 \end{aligned}$$

with $x^{(0)} = (1.5, 0.5)^t$.

$$F = \begin{bmatrix} x^2 + xy + y^2 - 7 \\ x^3 + y^3 - 9 \end{bmatrix}, \quad J = \begin{array}{cc} a_{11} & a_{12} \\ \left[\begin{matrix} 2x + y \\ 3x^2 \end{matrix}\right. & \left.\begin{matrix} x + 2y \\ 3y^2 \end{matrix}\right] \end{array}$$

$$X^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} \Rightarrow x_0 = 1.5, \quad y_0 = 0.5$$

$$\det(J) = (2x+y)\,3y^2 - (x+2y)\,3x^2$$

$$= 3(y^3 - x^3) + 6xy(x+y)$$

adj(J)

$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$A_{11} = (-1)^{1+1} 3y^2$

$A_{12} = (-1)^{1+2} 3x^2$

$= -3x^2$

$A_{21} = (-1)^{2+1} (x+2y)$

$A_{22} = (-1)^{2+2} (2x+y)$

$= 6xy^2 + 3y^3 - 3x^3 - 6x^2y \neq 0$

$$\text{adj}(J) = \begin{bmatrix} 3y^2 & -3x^2 \\ -(x+2y) & 2x+y \end{bmatrix}^t = \begin{bmatrix} 3y^2 & -x-2y \\ -3x^2 & 2x+y \end{bmatrix}$$

$$J^{-1} = \frac{1}{3(y^3-x^3)+6xy(x+y)} \begin{bmatrix} 3y^2 & -x-2y \\ -3x^2 & 2x+y \end{bmatrix}$$

Apply Newton's method for non-linear system:-

$$X^{(k+1)} = X^k - J^{-1}(X^{(k)}) \, F(X^{(k)})$$

let $k=0$

$$X^{(1)} = X^{(0)} - J^{-1}(X^{(0)}) \, F(X^0)$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left\{ \frac{1}{3(y^3-x^3)+6xy(x+y)} \begin{bmatrix} 3y^2 & -x-2y \\ -3x^2 & 2x+y \end{bmatrix} \right\} \begin{bmatrix} x^2+xy+y^2-7 \\ x^3+y^3-9 \end{bmatrix} \Bigg|_{(x_0,y_0)}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} - \frac{1}{3\left((0.5)^3 - (1.5)^3\right) + 6(0.5)(1.5)(1.5+0.5)}$$

$$\times \begin{bmatrix} 3(0.5)^2 & -1.5-2(0.5) \\ -3(1.5)^2 & 2(1.5)+0.5 \end{bmatrix} \begin{bmatrix} (1.5)^2 + (1.5)(0.5) + (0.5)^2 - 7 \\ (1.5)^3 + (0.5)^3 - 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} - \frac{1}{-14.25} \begin{bmatrix} 0.75 & -0.25 \\ -6.75 & 3.5 \end{bmatrix} \begin{bmatrix} -3.75 \\ -5.50 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2675 \\ 0.9254 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Let $k = 1$

$$J^{-1}(x^{(1)}) \, f(x^{(1)})$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2.2675 \\ 0.9254 \end{bmatrix} - \frac{1}{(\det(J))}$$

at $x' = (2.2675, 0.9254)$

$$= \begin{bmatrix} 2.2675 \\ 0.9254 \end{bmatrix} - \frac{1}{-49.495} \begin{bmatrix} 2.5691 & -4.1183 \\ -15.4247 & 5.4604 \end{bmatrix} \begin{bmatrix} 1.0963 \\ 3.4510 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 2.0373 \\ 0.9654 \end{bmatrix}$$

$$X^{(3)} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2.00013 \\ 0.9987 \end{bmatrix}$$

Ans.

Newton's method to system of non-linear equations.

Newton's method for nonlinear systems: Exercise

Perform 3 iterations of Newton's method to the following non-linear system:

$$\begin{aligned} x^2 - y^2 + 2y &= 0 \\ 2x + y^2 - 6 &= 0 \end{aligned}$$

with $x^{(0)} = (0.1, 2)^t$.