Lecture 5: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Fixed point iteration

Convergence conditions satisfied by $g(x)$:

(i) (existence) If $g \in C[a,b]$ and $g(x) \in [a,b]$, $\forall x \in [a,b]$, then $g(x)$ has at least one fixed point in $[a,b]$.
(ii) (uniqueness) If, in addition, $g'(x)$ exists in (a,b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a,b)$, then there is exactly one fixed point in $[a,b]$.
(iii) (convergence) If conditions of (i) and (ii) are satisfied, then for any number $p_0 \in [a,b]$, the sequence defined by $p_n = g(p_{n-1})$, $n \geq 1$ converges to the unique fixed point p in $[a,b]$.

Fixed point iteration

Example:

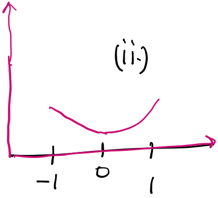
Show that $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on the interval $[-1, 1]$.

existence and uniqueness

Solution: (i) $g(x) = \frac{x^2-1}{3}$ is continuous function on $[-1,1]$

or $g(x) \in C[-1,1]$ ✓

T.P. $g(x) \in [-1,1]$ $\forall x \in [-1,1]$



(ii) $\boxed{g'(x) = \frac{2x}{3}}$ $=0 \Rightarrow x=0$

> 0 when $x \in [0,1]$

< 0 when $x \in [-1,0]$

$\Rightarrow g(x)$ is increasing in $[0,1]$ & decreasing in $[-1,0]$

Fixed point iteration

Solution (continued):

$\Rightarrow$ max. value of $g(x)$ in $[0,1]$ lies at 1

$g(1) = 0 \in [-1,1]$

& min. value of $g(x)$ in $[0,1]$ is

$g(0) = \frac{-1}{3} \in [-1,1]$

Also g has max. value at -1 ie $g(-1) = 0 \in [-1,1]$

min. value at 0 ie $g(0) = \frac{-1}{3} \in [-1,1]$

OR by finding critical pts.

$g''(x) = \frac{2}{3} > 0 \Rightarrow g(x)$ has min. value at $x = 0$

$g(0) = \frac{-1}{3} \in [-1,1]$

$\Rightarrow g(x) \in [-1,1] \quad \forall \; x \in [-1,1]$

(iii) $|g'(x)| = \left|\frac{2x}{3}\right| < 1 \quad \forall \; x \in [-1,1]$

or $g''(x) = \frac{2}{3} > 0 \Rightarrow g'(x)$ is increasing on $[-1,1]$

$|g'(-1)| = \left|-\frac{2}{3}\right| < 1$

$|g'(1)| = \left|\frac{2}{3}\right| < 1$

Fixed point iteration

by finding max: 4 min. if we are not able to guess its value by simply looking into the function

Exercise:

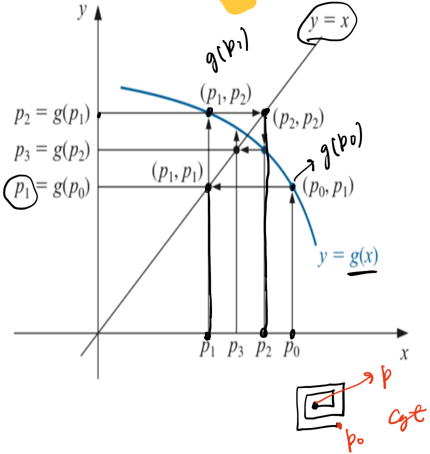
1. Show that $g(x) = 2^{-x}$ has a unique fixed point on the interval $\left[\frac{1}{3}, 1\right]$.

Root finding problem

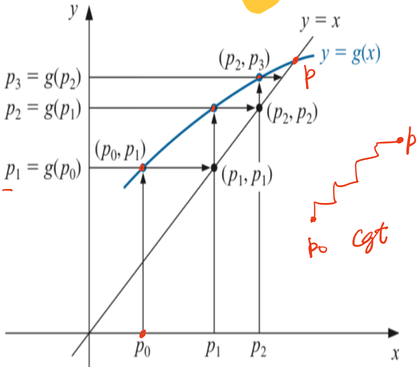
Convergence through graphics:

$|g'(x)| < 1$

$-1 < g'(x) < 0$



$0 < g'(x) < 1$

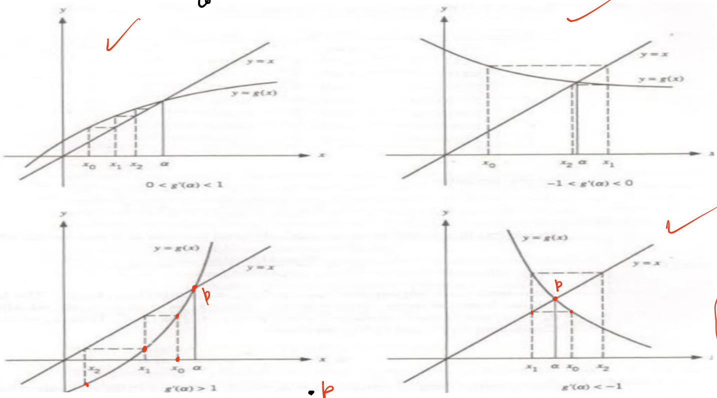


Root finding problem

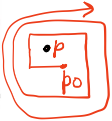
$|g'(x)| < 1$ is required:

$0 < g'(x) < 1$ Convergent

$-1 < g'(x) < 0$ cgt.



divergent



If $g'(x) > 1$ divergent

$g'(x) < -1$

Fixed point iteration

Converse is not true:

If the conditions for the convergence (three conditions on $g(x)$) of a fixed point are satisfied then there is a guarantee for the existence and uniqueness of a fixed point on a given interval but if we have one fixed point in a given interval then condition may or may not be satisfied.

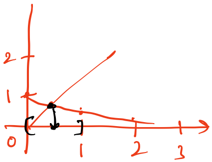
Fixed point iteration

Counter example:

Show that the conditions for the convergence of a fixed point do not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exist.

Solution:

$g(x) = 3^{-x}$ has a fixed pt. in $[0,1]$



(i) $g(x) \in C[0,1]$

(ii) $g'(x) = \boxed{3^{-x} \, (-1) \, \ln(3)} < 0$

$\Rightarrow g(x)$ is decreasing on $[0,1]$

$\Rightarrow$ $g(0) =$ max. value $= 1 \in [0,1]$

$g(1) =$ min. value $= \frac{1}{3} \in [0,1]$

Fixed point iteration

$g \in [0,1) \quad \forall x \in [0,1]$

(iii) $|g'(x)| = |-3^{-x} \ln(3)|$

$g'(0) = |-3^0 \ln(3)| = 1.09 > 1 \Rightarrow |g'(x)| \nless 1$

Exercise:

1. Show that the conditions for the convergence of a fixed point do not ensure a unique fixed point of $g(x) = \frac{x^2-1}{3}$ on the interval $[3, 4]$, even though a unique fixed point on this interval does exist.