Numerical Analysis

Solution of Exercises : Chapter 3[1]
Direct Methods for Solving Linear Systems

1. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = -1,\ x_2 = 1,\ x_3 = 3.$)

(a)

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11. \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1 \\ x_1 + 4x_2 + 2x_3 &= 9. \end{aligned}$$

Sol.

(a) We use the two-digit rounding arithmetic and apply Gauss elimination. Firstly we write the augmented matrix:

$$\begin{bmatrix} -1 & 4 & 1 & 8 \\ 1.7 & 0.67 & 0.67 & 1 \\ 2 & 1 & 4 & 11 \end{bmatrix}.$$

The multipliers are $m_{21} = \dfrac{1.7}{-1} = -1.7, m_{31} = \dfrac{2}{-1} = -2$. We perform $E_2 \to E_2 + 1.7E_1$, $E_3 \to E_3 + 2E_1$, and obtain

$$\begin{bmatrix} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 9 & 6 & 27 \end{bmatrix}.$$

Multiplier is $m_{32} = \dfrac{9}{7.5} = 1.2$ and perform $E_3 \to E_3 - 1.2E_2$.

$$\begin{bmatrix} -1 & 4 & 1 & 8 \\ 0 & 7.5 & 2.4 & 15 \\ 0 & 0 & 3.1 & 9 \end{bmatrix}.$$

Using back substitution, the solution is

$$\begin{aligned} &3.1x_3 = 9 \quad \Longrightarrow \quad x_3 = 2.9 \\ &7.5x_2 + (2.4)(2.9) = 15 \quad \Longrightarrow \quad x_2 = 1.1 \\ &-x_1 + (4)(1.1) + 2.9 = 8 \quad \Longrightarrow \quad x_1 = -0.7. \end{aligned}$$

(b) We use the two-digit rounding arithmetic and apply Gauss elimination. The augmented matrix is:

$$\begin{bmatrix} 4 & 2 & -1 & -5 \\ 0.11 & 0.11 & -0.33 & -1 \\ 1 & 4 & 2 & 9 \end{bmatrix}.$$

The multipliers are $m_{21} = \dfrac{0.11}{4} = 0.028, m_{31} = \dfrac{1}{4} = 0.25$. We perform $E_2 \to E_2 - 0.028E_1$, $E_3 \to E_3 - 0.25E_1$, and obtain

$$\begin{bmatrix} 4 & 2 & -1 & -5 \\ 0 & 0.054 & -0.30 & -0.86 \\ 0 & 3.5 & 2.3 & 10 \end{bmatrix}.$$

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Multiplier is $m_{32} = \dfrac{3.5}{0.054} = 65$ and perform $E_3 \to E_3 - 65E_2$.

$$\begin{bmatrix} 4 & 2 & -1 & -5 \\ 0 & 0.054 & -0.30 & -0.86 \\ 0 & 0 & 22 & 56 \end{bmatrix}.$$

Using back substitution, the solution is

$$\begin{aligned} & 22x_3 = 56 \quad \Longrightarrow \quad x_3 = 2.5 \\ & 0.054x_2 + (-0.30)(2.5) = -0.86 \quad \Longrightarrow \quad 0.054x_2 - 0.75 = -0.86 \quad \Longrightarrow \quad x_2 = -2 \\ & 4x_1 + (2)(-2) - 1(2.5) = -5 \quad \Longrightarrow \quad 4x_1 - 6.5 = -5 \quad \Longrightarrow \quad x_1 = 0.38. \end{aligned}$$

In this case, solution is wrong due to rounding errors and limited digits.

2. Using the four-digit arithmetic solve the following system of equations by Gaussian elimination with partial pivoting

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000. \end{aligned}$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!

Sol. We use the four-digit arithmetic and apply Gauss elimination with partial pivoting. Firstly we write the augmented matrix in given format as:

$$\begin{bmatrix} 0.7290 & 0.8100 & 0.9000 & 0.6867 \\ 1.000 & 1.000 & 1.000 & 0.8338 \\ 1.331 & 1.210 & 1.100 & 1.000 \end{bmatrix}.$$

We bring the largest element as the pivot. So we interchange first and third rows.

$$\begin{bmatrix} 1.331 & 1.210 & 1.100 & 1.000 \\ 1.000 & 1.000 & 1.000 & 0.8338 \\ 0.7290 & 0.8100 & 0.9000 & 0.6867 \end{bmatrix}.$$

The multipliers are $m_{21} = 0.7513, m_{31} = 0.5477$. We perform $E_2 \to E_2 - 0.7513E_1$, $E_3 \to E_3 - 0.5477E_1$, and obtain

$$\begin{bmatrix} 1.331 & 1.210 & 1.100 & 1.000 \\ 0 & 0.09090 & 0.1736 & 0.08250 \\ 0 & 0.1473 & 0.2975 & 0.1390 \end{bmatrix}.$$

Now $|a_{32}| > |a_{22}|$, we interchange second and third rows.

$$\begin{bmatrix} 1.331 & 1.210 & 1.100 & 1.000 \\ 0 & 0.1473 & 0.2975 & 0.1390 \\ 0 & 0.09090 & 0.1736 & 0.08250 \end{bmatrix}.$$

Multiplier is $m_{32} = 0.6171$ and perform $E_3 \to E_3 - 0.6171E_2$.

$$\begin{bmatrix} 1.331 & 1.210 & 1.100 & 1.000 \\ 0 & 0.1473 & 0.2975 & 0.1390 \\ 0 & 0 & -0.01000 & -0.003280 \end{bmatrix}.$$

The solution is (using back substitution)

$$\begin{aligned} & -0.01000x_3 = -0.003280 \quad \Longrightarrow \quad x_3 = 0.3280 \\ & 70.1473x_2 + (0.2975)(0.3820) = 0.1390 \quad \Longrightarrow \quad x_2 = 0.2812 \\ & 1.331x_1 + (1.210)(0.2812) + (1.100)(0.3820) = 1.000 \quad \Longrightarrow \quad x_1 = 0.2246. \end{aligned}$$

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3. Use the Gaussian elimination algorithm to solve the following linear systems, if possible, and determine whether row interchanges are necessary:

(a)

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 &= 3. \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3. \end{aligned}$$

Sol. Do it yourself.

4. Use Gaussian elimination with scaled pivoting and three-digit chopping arithmetic to solve the following linear system and compare the approximations to the actual solution $[0, 10, 1/7]^T$.

$$\begin{aligned} 3.03x_1 - 12.1x_2 + 14x_3 &= -119 \\ -3.03x_1 + 12.1x_2 - 7x_3 &= 120 \\ 6.11x_1 - 14.2x_2 + 21x_3 &= -139. \end{aligned}$$

Sol. We will perform each calculation with three-digit chopping. The augmented matrix is

$$\begin{bmatrix} 3.03 & -12.1 & 14 & -119 \\ -3.03 & 12.1 & -7 & 120 \\ 6.11 & -14.2 & 21 & -139. \end{bmatrix}$$

The scale factors are $s_1 = 14, s_2 = 12.1, \&\ s_3 = 14.2$. We need to pick the largest $(3.03/14 = 0.216, 3.03/12.1 = 0.250, 6.11/14.2 = 0.430)$, which is the third entry, and thus we interchange first and third rows and also interchange s_1 and s_3 to get

$$\begin{bmatrix} 6.11 & -14.2 & 21 & -139 \\ -3.03 & 12.1 & -7 & 120 \\ 3.03 & -12.1 & 14 & -119. \end{bmatrix}$$

Multipliers are $m_{21} = -0.495$ and $m_{31} = 0.495$. Performing $E_2 \to E_2 + 0.495E_1$, $E_3 \to E_3 - 0.495E_1$, we obtain

$$\begin{bmatrix} 6.11 & -14.2 & 21 & -139 \\ 0 & 5.08 & 3.3 & 51.2 \\ 0 & -5.08 & 3.7 & -50.2. \end{bmatrix}$$

Now comparing $(5.08/12.1, 5.08/14)$, the first ratio is largest so no need to interchange rows. Multiplier is $m_{32} = -1$. Thus we perform $E_3 \to E_3 + E_2$ to obtain

$$\begin{bmatrix} 6.11 & -14.2 & 21 & -139 \\ 0 & 5.08 & 3.3 & 51.2 \\ 0 & 0 & 7 & 1. \end{bmatrix}$$

Backward substitution gives

$$x_3 = 1/7 = 0.142$$

$$5.08x_2 + (3.3)(0.142) = 51.2 \implies x_2 = \frac{50.7}{5.08} = 9.98$$

$$6.11x_1 - (14.2)(9.98) + (21)(0.142) = -139 \implies 6.11x_1 - 138 = -139 \implies x_1 = \frac{-1}{6.11} = -0.163.$$

Comparing with the exact values, we observe that, due to chopping, we got wrong answer.

5. Suppose that

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 6x_2 + 8x_3 &= 5 \\ 6x_1 + \alpha x_2 + 10x_3 &= 5, \end{aligned}$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

(a) $\alpha = 6$.

(b) $\alpha = 9$.

(c) $\alpha = -3$.

Sol. The augmented matrix is:

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 6 & 8 & 5 \\ 6 & \alpha & 10 & 5 \end{bmatrix}.$$

The scale factors are $s_1 = 3,\ s_2 = 8,\ s_3 = 10$ as $|\alpha| < 10$. So no row interchange is required.
The multipliers are $m_{21} = 2, m_{31} = 3$. We perform $E_2 \to E_2 - 2E_1$, $E_3 \to E_3 - 3E_1$, and obtain

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & \alpha - 3 & 1 & 2 \end{bmatrix}.$$

As no row interchange is required, thus we have

$$\begin{aligned} \left|\frac{4}{s_2}\right| &> \left|\frac{\alpha - 3}{s_3}\right| \\ \implies |\alpha - 3| &< 5 \\ \implies -2 &< \alpha < 8. \end{aligned}$$

Clearly $\alpha = 6$ lies in the given range. So with $\alpha = 6$, we have

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{bmatrix}.$$

Now multiplier is $m_{32} = 3/4$ and perform $E_3 \to E_3 - 3/4E_2$.

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & -1/2 & 1/4 \end{bmatrix}.$$

Using back substitution, the solution is

$$\begin{aligned} x_3 &= 1/2 \\ x_2 &= 1/2 \\ x_1 &= -1/2. \end{aligned}$$

6. Use the LU factorization to solve the following linear system:

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

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Sol. We first apply the Gaussian elimination on the matrix A and collect the multipliers m_{21}, m_{31}, and m_{32}. We have

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}.$$

Multipliers are $m_{21} = 3/2 = 1.5$, $m_{31} = 3/2 = 1.5$.
$E_2 \to E_2 - 1.5E_1$ and $E_3 \to E_3 - 1.5E_1$.

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 4.5 & 3.5 \end{bmatrix}.$$

Multiplier is $m_{32} = 4.5/4.5 = 1$ and we perform $E_3 \to E_3 - E_2$.

$$\sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix} = LU.$$

Therefore,

$$Ax = b \implies LUx = b.$$

Assuming $Ux = y$, we obtain,

$$Ly = b$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1.5 & 1 & 0 \\ 1.5 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

Using forward substitution, we obtain $y_1 = -1$, $y_2 = 1.5$, and $y_3 = 4$. Now

$$Ux = y \implies \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4.5 & 7.5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.5 \\ 4 \end{bmatrix}.$$

Now, using the backward substitution process, we obtain the final solution as $x_3 = -1,\ x_2 = 2$, and $x_1 = 1$.

7. We require to solve the following system of linear equations using LU decomposition.

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ x_1 + 2x_2 - 2x_3 &= 2 \\ -2x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Find the matrices L and U using Gauss elimination. Using those values of L and U, solve the given system of equations.

Sol. From the given system of linear equations

$$\text{A=} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

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To determine L and U we apply gauss elimination to matrix A.
Here multipliers at first step are $m_{21} = 1$ and $m_{31} = -2$. Operating $E_2 \to E_2 - E_1$ and $E_3 \to E_3 + 2E_1$.
We get

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

At second step $m_{31} = 3$ and operating $E_3 \to E_3 - 3E_2$ we get

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus we have

$$\begin{aligned} Ax &= b \\ LUx &= b. \end{aligned}$$

Let $Ux = y$ then first we solve $Ly = b$ for y. We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Using forward elimination, we get $y_1 = 3, y_2 = -1$ and $y_3 = 10$.
Now we consider $Ux = y$:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 10 \end{bmatrix}$$

Using backward elimination, we get $x_1 = 4, x_2 = 4$ and $x_3 = 5$.