Lecture 2: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

Dr. Meenu Rani

Department of Mathematics
TIET, Patiala
Punjab-India

Root-finding problem

Methods for root-finding problem:

To find a solution of an equation $f(x) = 0$, we discuss the following three methods:

1. Bisection method
2. Fixed point Iteration
3. Newton method

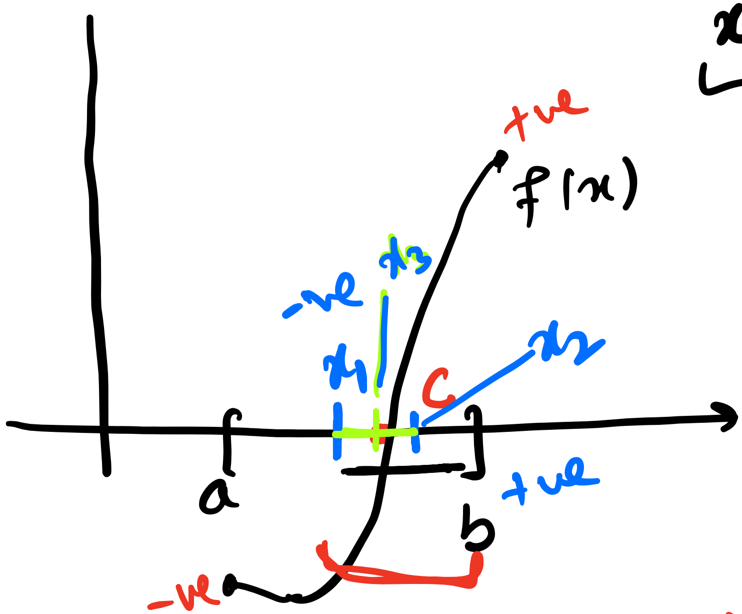
Root-finding problem

eqn $\boxed{f(x)} = 0$

Intermediate Value Theorem (IVT)

Let $f(x)$ be continuous function on $[a,b]$ and $\underbrace{f(a) * f(b) < 0}_{f(c)=0}$, then $\exists$ a number $c \in (a,b)$ such that

$$\underbrace{x^2 - 3x + 1}=0$$



$[a, x_1]$

$[x_1, b]$

$\underbrace{[x_1, x_2]}$ ✓ or $[x_1, b]$ ✗

Root-finding problem

How to find c ?

To find the root of $f(x)=0$

Bisection method: Procedure

Step 1 find the interval in which $f(a) * f(b) < 0$
-ve $\boxed{\text{+ve}}$

Step 2 use bisection method $x_1 = \frac{a+b}{2}$

Step 3 Check the sign of $f(x_1) < 0$

The root lies in $[x_1, b]$ by IVT

Step 4 use bisection method $x_2 = \frac{x_1+b}{2}$

The root lies in either in $[x_1, x_2]$ or $[x_2, b]$
-ve +ve +ve +ve

$|x_2 - x_3| < 10^{-2}$

check the sign of $f(x_2) > 0$ $\Rightarrow$ The root lies in $[x_1, x_2]$

Step 5 use bisection method $x_3 = \frac{x_1+x_2}{2}$.

continue this process untill you get

the desired accuracy

Root-finding problem

Bisection method: Stopping Criteria

$$|x_n - x_{n-1}| < \text{tolerance} \quad (\text{given})$$

$$\text{tolerance} = \quad \text{e.g. } 10^{-1}, \text{ or } 10^{-2} \text{ or } 10^{-3}$$

Root-finding problem

Bisection method: Example

Show that $f(x) = x^3 + 2x^2 - 3x - 1 = 0$ has a root in $[1, 2]$ and use the bisection method to determine an approximation to the root i.e. is accurate to at least within 10^{-4}.

Solution:

$f(x)$ is 3rd polynomial function, so it continuous on $[1, 2]$

and $f(1) = 1 + 2 - 3 - 1 = -1 < 0$

$f(2) = 8 + 8 - 6 - 1 = 9 > 0$

$\Rightarrow$ by IVT, we say that $f(x)$ has a root in $[1, 2]$

Root-finding problem

By using bisection method

$x_1 = \frac{1+2}{2} = 1.5$

Root lies in either $[1, 1.5]$ or $[1.5, 2]$
$-ve$ $+ve$ $+ve$ $+ve$

Check $f(1.5) > 0$

$\Rightarrow$ By IVT, Root lies in $[1, 1.5]$

Using bisection method $x_2 = \frac{1+1.5}{2} = 1.25$

Root lies in either $[1, 1.25]$ or $[1.25, 1.5]$
$-ve$ $+ve$ $+ve$ $+ve$

Check $f(1.25) > 0$

$\Rightarrow$ By IVT, Root lies in $[1, 1.25]$

$x_3 = \frac{1+1.25}{2} = 1.125$

$|x_2 - x_3| = |1.25 - 1.125| \nless 10^{-1}$

Root-finding problem

Check $f(1.125) < 0$

Root lies in $[\overset{-ve}{1.125}, \overset{+ve}{1.25}]$

$$x_4 = \frac{1.125 + 1.25}{2} = 1.1875$$

$$|x_3 - x_4| = |1.125 - 1.1875| < 10^{-1}$$

$\Rightarrow$ $x_4 = 1.1875$ is the app. root of given eqn

Table of Bisection method

n	a (−ve)	b (+ve)	$x_n = \frac{a+b}{2}$	sign of $f(x_n)$	$\lvert x_n - x_{n-1}\rvert$
1	1	2	$x_1 = 1.5$	+ve	$\lvert x_1 - x_2\rvert \nless 10^{-1}$
2	1	1.5	$x_2 = 1.25$	+ve	$\lvert x_2 - x_3\rvert \nless 10^{-1}$
3	1	1.25	$x_3 = 1.125$	−ve	$\lvert x_3 - x_4\rvert < 10^{-1}$
4	1.125	1.25	$x_4 = 1.1875$	−ve	

$\Rightarrow$ $x_4 = 1.1875$ is the root of given eqn

Root-finding problem

Exercise:

1. Use intermediate value theorem to get the first positive root of $x - 2^{-x} = 0$ and hence apply bisection method to find the root accurate to within 10^{-1}.
2. Using the bisection method, determine the point of intersection of the curves given by $y = 3x$ and $y = e^x$ in the interval $[0, 1]$ with an accuracy 0.1.