Lecture 17: Numerical Linear Algebra (UMA021): Interpolation

Dr. Meenu Rani

Department of Mathematics
TIET, Patiala
Punjab-India

$h = \frac{b-a}{n} = \frac{2\pi - 0}{8} = \frac{\pi}{4}$

Newton Divided Difference Interpolation:

Result

Suppose that $f \in C^n[a,b]$ and $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a,b]$. Then a number ξ exists in (a,b) with

nth D.D of f   nth derivative of f

$$\underbrace{f[x_0, x_1, x_2, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}}. = \left(\frac{d^n}{dx^n} f(x)\right)_{x=\xi}$$

Proof:

done in previous lecture

G.R.T   If $\underbrace{f \in C^n[a,b]}$ and f has $(n+1)$ zeros in $[a,b]$ then $\exists$ a no. $c \in (a,b)$ s.t.

$f^{(n)}(c) = 0$

$g = f - P_n$

Example:

Let $f(x) = e^x$, show that $f[x_0, x_1, \ldots, x_m] > 0$ for all values of m and all distinct equally spaced nodes $\{x_0 < x_1 < \cdots < x_m\}$.

Solution: Since $f \in C^m[x_0, x_m]$ and $x_0, x_1, \ldots x_m$ are distinct nos. in $[x_0, x_m]$, then a number ξ in (x_0, x_m) with

$$f[x_0, x_1, \ldots x_m] = \frac{f^{(m)}(\xi)}{m!} = \frac{e^{\xi}}{m!} > 0$$

$\forall\, m$ & $\xi \in (x_0, x_m)$

$$\left(\frac{d^m}{dx^m}(e^x)\right)_{x=\xi}$$

$$\Rightarrow \quad f[x_0, x_1, x_2 \ldots x_m] > 0 \quad \forall\ m \text{ \& all values in } (x_0, x_m)$$

$$\frac{d^{m-1}}{dx^{m-1}} e^x$$

Newton Divided Difference Interpolation:

Example:

Let $f(x) = x^3$, compute $f[x_0, x_1, x_2, x_3]$ for the distinct nodes x_0, x_1, x_2, x_3. 3rd D.D.

Solution: Since $f \in C^3[x_0, x_3]$ and x_0, x_1, x_2, x_3 are distinct nodes then by prev. thm., $\exists$ a no. $\xi_3 \in (x_0, x_3)$ s.t.

$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$$f[x_0, x_1, x_2, x_3] = \frac{f^{(3)}(\xi_3)}{3!} = \frac{6}{3!} = 1$$

Newton Divided Difference Interpolation:

Exercise:

1. Let $f(x) = x^3$, compute $f[x_0, x_1, x_2, x_3, x_4]$ for the distinct nodes x_i, $1 \le i \le 4$.

4th D.D.

$$f[x_0, \dots x_4] = \frac{f^{(4)}(\xi)}{4!} = 0$$

Newton Divided Difference Interpolation:

Newton Divided Difference Interpolation:

Newton's divided difference formula can be expressed as

$$\begin{aligned} P_n(x) &= f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &+ \cdots + \underbrace{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}f[x_0, x_1, \cdots, x_n]. \end{aligned}$$

It can be expressed in a simplified form when the nodes are arranged consecutively with equal spacing.

$x_0, x_1, x_2, \ldots x_n$ equally spaced with spacing h.

x_0

$x_1 = x_0 + h$

$\boxed{x_2 = x_0 + 2h}$

$x_3 = x_0 + 3h$

$x_n = x_0 + nh$

$x_i = x_0 + ih$

$x_s =$

x_0	1
x_1	2
x_2	3
x_3	4
x_4	5

$\underbrace{x_s = 1.25}$

$x_s = x_0 + sh$

$$P_n(x_s) = f[x_0] + (x_s - x_0)\, f[x_0, x_1] + (x_s - x_0)(x_s - x_1)\, f[x_0, x_1, x_2]$$

$$+ \cdots (x_s - x_0) \cdots \underbrace{(x_s - x_{n-1})}\, f[x_0 \cdots x_n]$$

$$x_s = x_0 + sh \qquad x_1 = x_0 + h, \quad x_2 = x_0 + 2h$$

$$P_n(x_s) = f[x_0] + (sh)\, f[x_0, x_1] + (sh)(\cancel{x_0} + sh - \cancel{x_0} - h)\, f[x_0, x_1, x_2]$$

$$\cdots + (x_0 + sh - x_0)(x_0 + sh - x_0 - h) \cdots (\cancel{x_0} + sh - \cancel{x_0} - (n-1)h)\, f[x_0 \cdots x_n]$$

$$= f[x_0] + sh\, f[x_0, x_1] + (sh)(s-1)h\, f[x_0, x_1, x_2]$$

$$+ (sh)(s-1)h\,(s-2)h\, f[x_0, x_1, x_2, x_3]$$

$$+ \cdots (sh)(s-1)h \cdots (s-(n-1))h\, f[x_0 \cdots x_n]$$

Newton Divided Difference Interpolation:

We take $h = x_{i+1} - x_i$, for $i = 0, 1, \cdots, n-1$ and let $x_s = x_0 + sh$, then polynomial becomes

$$P_n(x) = f[x_0] + sh\, f[x_0, x_1] + s(s-1)h^2\, f[x_0, x_1, x_2] + \cdots + s(s-1)\cdots(s-n+1)h^n\, f[x_0, x_1, \cdots, x_n]$$

$$\binom{s}{1} = \frac{s!}{(s-1)!\,1!} = \frac{s\cancel{(s-1)!}}{\cancel{(s-1)!}}$$

$$= f[x_0] + \binom{s}{1} h\, f[x_0, x_1] + \binom{s}{2} 2!\, h^2 f[x_0, x_1, x_2] + \cdots \binom{s}{n} n!\, h^n f[x_0, x_1 \cdots x_n]$$

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k f[x_0, x_1, \cdots, x_k]$$

N.D.D. when nodes are equally

$$\binom{s}{2} = \frac{s!}{(s-2)!\,2!} = \frac{s(s-1)\cancel{(s-2)!}}{\cancel{(s-2)!}\,2!} = \frac{s(s-1)}{2!}$$

Speed.

Forward Differences

Notations:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{h} = \frac{\triangle f(x_0)}{h},$$

$f(x_1) - f(x_0) = \Delta f(x_0)$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\triangle f(x_1) - \triangle f(x_0)}{h}\right) = \frac{\triangle^2 f(x_0)}{2h^2}$$

and, in general $f[x_0, x_1, \cdots, x_k] = \dfrac{\triangle^k f(x_0)}{k!\, h^k}$.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\Delta f(x_1)}{h} - \frac{\Delta f(x_0)}{h}}{2h}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{\overset{g}{\Delta f}(x_1) - \overset{g}{\Delta f}(x_0)}{2h^2}$$

$$\frac{\Delta g(x_0)}{2h^2} = \frac{\Delta^2 f(x_0)}{2h^2}$$

forward-diff poly. becomes $P_n(x_s) = f(x_0) + \sum \binom{s}{k} \cancel{k!}\, \cancel{h^k}\, \Delta^k f(x_0)$

$k=1$

$\cancel{k!\,h^k}$

Newton Forward Difference Formula

Newton Forward Difference Formula:

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \triangle^k f(x_0), \text{ where } x = x_0 + sh.$$

$$= f(x_0) + \binom{s}{1} \Delta f(x_0) + \binom{s}{2} \Delta^2 f(x_0) + \binom{s}{3} \Delta^3 f(x_0)$$

$$+ \cdots \binom{s}{n} \Delta^n f(x_0)$$

Newton Forward Difference Interpolation:

Newton Forward Difference Table:

i	x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$	$\Delta^4 f(x_i)$
0	x_0	$f(x_0)$	$f(x_1)-f(x_0) = \Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	$\Delta^4 f(x_0)$
1	x_1	$f(x_1)$	$f(x_2)-f(x_1)$	$\Delta^2 f(x_1)$	$\Delta^3 f(x_1)$	
2	x_2	$f(x_2)$	$f(x_3)-f(x_2)$	$\Delta^2 f(x_2)$		
3	x_3	$f(x_3)$	$f(x_4)-f(x_3)$			
4	x_4	$f(x_4)$				

$$P_4(x_s) = f(x_0) + \sum_{k=1}^{4} \binom{s}{k} \Delta^k f(x_0)$$

$$\underbrace{x_s}_{\text{given}} = \underbrace{x_0}_{\text{given}} + s\underbrace{h}_{\text{spacing (given)}}$$

x_0

x_1

$x_2 \longrightarrow x_s$

x_3

$\vdots$

x_{98}

$x_{99} \longrightarrow x_s$

x_{100}

Newton Backward Difference Formula

Newton Backward Difference Formula:

If the interpolating nodes are reordered from last to first as $x_n, x_{n-1}, \cdots, x_0$, we can write the interpolating polynomial as $P_n(x)$

x_n

$x_{n-1} = x_n - h$

$$\begin{aligned} &= f[x_n] + (x - x_n) f[x_n, x_{n-1}] + (x - x_n)(x - x_{n-1}) f[x_n, x_{n-1}, x_{n-2}] \\ &\quad + \cdots + (x - x_n)(x - x_{n-1}) \cdots (x - x_1) f[x_n, x_{n-1}, \cdots, x_0]. \end{aligned}$$

If nodes are equally spaced with $x_s = x_n + sh$, then

$$\begin{aligned} P_n(x_s) &= f[x_n] + s\,h\,f[x_n, x_{n-1}] + s(s+1)\,h^2\,f[x_n, x_{n-1}, x_{n-2}] \\ &\quad + \cdots + s(s+1)\cdots(s+n-1)\,h^n\,f[x_n, x_{n-1}, \cdots, x_0]. \end{aligned}$$

Backward Differences

Notations:

$$f[x_n, x_{n-1}] = \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}} = \frac{f(x_n)-f(x_{n-1})}{h} = \frac{\nabla f(x_n)}{h},$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{f[x_n,x_{n-1}]-f[x_{n-1},x_{n-2}]}{x_n-x_{n-2}} = \frac{1}{2h}\left(\frac{\nabla f(x_n)-\nabla f(x_{n-1})}{h}\right)$$

$$= \frac{\nabla f(x_n) - \nabla f(x_{n-1})}{2\,h^2} = \frac{\nabla^2 f(x_n)}{2h^2}$$

and, in general $f[x_n, x_{n-1}, \cdots, x_0] = \dfrac{\nabla^k f(x_n)}{k!\,h^k}.$

$\Delta f(x_0) = f(x_1) - f(x_0) = \nabla f(x_1)$

Newton Backward Difference Formula:

$$P_n(x_s) = f(x_n) + \sum_{k=1}^{n}(-1)^k \binom{-s}{k} \nabla^k f(x_n), \text{ where } x_s = x_n + sh.$$

x_0

x_1

x_2

x_{n-1}

$x_s = x_n + sh$, $s \to -ve$

$$f(x_4) + \sum_{k=1}^{4} (-1)^k \binom{-s}{k} \boxed{\nabla^k f(x_4)}$$

x_n

$x_{n-1} = x_n - h$

$x_{n-2} = x_n - 2h$

$- - -$

x_n

Newton Backward Difference Interpolation:

Newton Backward Difference Table:

i	x_i	$f(x_i)$	$\Delta f(x_i) = \nabla f(x_{i+1})$	$\nabla^2 f(x_{i+1})$	$\nabla^3 f(x_{i+1})$	
0	x_0	$f(x_0)$				
			$\Delta f(x_0) = \nabla f(x_1)$			
1	x_1	$f(x_1)$		$\nabla^2 f(x_2)$		
			$\Delta f(x_1) = \nabla f(x_2)$		$\nabla^3 f(x_3)$	
2	x_2	$f(x_2)$		$\nabla^2 f(x_3)$		
			$\Delta f(x_2) = \nabla f(x_3)$		$\nabla^3 f(x_4)$	
3	x_3	$f(x_3)$		$\nabla^2 f(x_4)$		
			$\Delta f(x_3) = \nabla f(x_4)$			$\nabla^4 f(x_4)$
4	x_4	$f(x_4)$				

i	x_i	$f(x_i)$	$\nabla f(x_i)$
4	x_4	$f(x_4)$	
3	x_3	$f(x_3)$	
2	x_2		
1	x_1		
0	x_0		