Lecture 3: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Bisection method

$f(x) = 0$ $\quad [1, 2] \quad x_1 = 1.5,\ x_2 =$

$\langle x_n \rangle \longrightarrow x$

Maximum error bound

Suppose that $f \in C[a, b]$ and $f(a) * f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a ~~zero~~ p of f with $|p_n - p| \le \boxed{\frac{b-a}{2^n}}$ when $n \ge 1$. to root of $f(x)=0$

set of continuous function on $[a, b]$

Proof:

$p_n \to$ nth approximate root

$\boxed{p} \to$ exact root.

$f(x)=0$

$[a,b]$

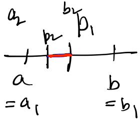
$p_1 = \frac{a+b}{2}$

$[a, p_1]$, $\boxed{[p_1, b]}$ $\quad p_2 = \frac{p_1+b}{2}$

$p_1, p_2, p_3 \cdots \boxed{p_n}$

Bisection method

Proof



$f(x) = 0$ be an equation. By IVT suppose that it has a root in $[a, b] = [a_1, b_1]$ (say)

Use bisection method $p_1 = \frac{a+b}{2}$

length of interval $[a_1, b_1] = |b_1 - a_1|$

Now Root of $f(x) = 0$ lies in either $[a_1, p_1]$ or $[p_1, b]$

Again Using IVT on these interval, we get

Root lies in $[a_2, b_2]$ (say)

length of interval of $[a_2, b_2]$ is $|b_2 - a_2| = \frac{|b_1 - a_1|}{2}$

Again using bisection method we get $p_2 = \frac{a_2 + b_2}{2}$

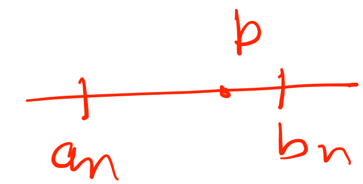
Now Root of $f(x) = 0$ lies in either $[a_2, p_2]$ or $[p_2, b_2]$

By Using IVT, we get root lies in $[a_3, b_3]$ (say)

length of interval of $[a_3, b_3]$ is $|b_3 - a_3| = \frac{|b_1 - a_1|}{2^2}$

Continue this process with n iterations, then we get root lies in $[a_n, b_n]$

and $p_n = \dfrac{a_n+b_n}{2}$



$p \to$ exact root

$p \in (a_n, b_n)$

$\boxed{a_n < p} < b_n$

$a_n < p$

$-a_n > -p$

$\boxed{-p < -a_n}$

Now

$$|p_n - p| = \left|\frac{a_n+b_n}{2} - p\right|$$

$$< \left|\frac{a_n+b_n}{2} - a_n\right|$$

$$|p_n - p| < \left|\frac{a_n+b_n-2a_n}{2}\right|$$

$$|p_n - p| < \frac{|b_n - a_n|}{2} = \frac{|b_1 - a_1|}{2 \times 2^{n-1}} = \frac{|b-a|}{2^n}$$

Bisection method

Example

Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$.

Solution:

$a_1 = 1, \quad b_1 = 2$

$a \quad\quad b$

$|p_n - p| \leq \left|\frac{b-a}{2^n}\right| \leq 10^{-3}$

$\frac{|2-1|}{2^n} \leq 10^{-3}$

$\frac{1}{2^n} \leq 10^{-3}$

$2^{-n} \leq 10^{-3}$

$2^n \geq 10^3$

$n \log 2 \geq 3 \log 10 \quad \Rightarrow \quad n \geq 3 \frac{\log 10}{\log 2}$

$\Rightarrow \quad n \geq 9.96$

$\boxed{n = 10}$

$f(x) = 0$

Exercise:

1. Find a bound for the number of iterations needed to achieve an approximation of $(25)^{1/3}$ by the bisection method with an accuracy 10^{-2}. Hence find the approximation with given accuracy.

make a non-linear eqn

$\boxed{x = (25)^{1/3}}$

$x^3 = 25$

$x^3 - 25 = 0$

not linear eqn

$x - (25)^{1/3} = 0$

$\boxed{x^3 - 25 = 0}$

$f(0) = -ve \qquad f(3) = +ve$

$f(1) = -ve$

$f(2) = -ve$

$[2, 3]$