Lecture 20: Numerical Linear Algebra (UMA021): Integration

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Example:

Compare the Trapezoidal rule and Simpson's rule approximations to $\int_0^1 x\, e^x \, dx$. Find the absolute error and maximum bound for the errors.

Solution:

$$\int_0^1 \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{e^x}\, dx$$

Exact $\quad x\, e^x - \int_0^1 1\, e^x\, dx = \left(x\, e^x - e^x\right)_0^1$

$$= e^1 - e^1 - 0 + e^0 = 1$$

By trap. rule $\quad a = 0, \quad b = 1, \quad f(x) = x e^x, \quad h = b - a = 1 - 0$

$$\int_0^1 x e^x\, dx = \frac{1}{2}\left[f(0) + f(1)\right] = \frac{1}{2}\left[0 + e^1\right]$$

$$= 1.359$$

By Simpson's

$a = 0, \quad b = 1, \quad h = \frac{b-a}{2} = \frac{1-0}{2} = 0.5$

$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1$

$$\int_0^1 x e^x \, dx = \frac{0.5}{3}\left[f(0) + 4f(0.5) + f(1) \right]$$

$$= \frac{0.5}{3}\left[0 + 4 * \frac{1}{2} e^{1/2} + e^1 \right]$$

$$= \frac{0.5}{3}\left[2e^{1/2} + e \right] = 1.0026$$

A.E. $\quad |1 - 1.359| = \quad\quad |1.359 - 1.0026| =$

$|1 - 1.0026| =$

Max error bounds in trap. Rule

$$\max_{c \in (0,1)} \left| -\frac{h^3}{12} f''(c) \right| \quad\quad h = 1-0$$

$$\left\{\begin{array}{l} f(x) = x e^x \\ f'(x) = x e^x + e^x \\ f''(x) = x e^x + 2e^x \\ \qquad\quad (x+2)e^x \end{array}\right\}$$

$$\max_{c \in [0,1]} \frac{1}{12} \left| (c+2) e^c \right|$$

$$= \frac{1}{12} (1+2) e^1 = \frac{e}{4} = (0.679)$$

<u>max error bound in Simpson's Rule</u>:

$$\left\{ f^{iv}(x) = (x+4) e^x \right\}$$

$$\max_{c \in [0,1]} \left| -\frac{h^5}{90} f^{iv}(c) \right| \qquad h = \frac{1-0}{2}$$

$$\max_{c \in [0,1]} \left| \left(\frac{1}{2}\right)^5 \frac{1}{90} (c+4) e^c \right|$$

$$= \frac{1}{32 * 90} (5e) = \frac{e}{32 * 18} =$$

$$0.0047$$

Numerical Quadrature:

Exercise:

1. Approximate the integral $I = \int\limits_{-0.25}^{0.25} (\cos x)^2 dx$ using the trapezoidal and compare with exact value.
2. Approximate the integral $I = \int\limits_{0}^{2} \frac{1}{x+1}\, dx$ using the trapezoidal and Simpson's formulas and compare with exact values. Also, find the maximum bound for the errors.
3. The Trapezoidal rule applied to $\int\limits_{0}^{2} f(x)dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?

$\#$

$$\int_0^{10} f(x)\,dx = \frac{10-0}{2}\left(f(0)+f(10)\right) = 5\left(f(0)+f(10)\right)$$

$h = 10-0 = 10$

$$\left| -\frac{(10)^3}{12} f''(c) \right|$$

$[0,10]$

$$\int_0^{10} f(x)\,dx = \underbrace{\int_0^1 f(x)\,dx}_{h=1} + \underbrace{\int_1^2} + \underbrace{\int_2^3} - - - \underbrace{\int_9^{10}}$$

Numerical Quadrature:

Composite Trapezoidal Rule:

We divide the interval $[a, b]$, into n subintervals with step size $h = \frac{b-a}{n}$, and taking nodal points $a = x_0 < x_1 < \cdots < x_n = b$, where $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots, n$.

$\int_a^b f(x)\,dx \qquad [a, b] \to \frac{b-a}{n} = h$

$a = x_0, x_1, x_2, \; - - - \; x_i \; - - - \; x_n = b$

$x_i = x_0 + ih$

$$\int_{a=x_0}^{b=x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + - - - \int_{x_{n-1}}^{x_n} f(x)\,dx$$

using trapezoidal in each integration.

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \frac{h}{2}\left[f(x_2) + f(x_3)\right] - -$$

$$- - - \quad \frac{h}{2}\left[f(x_n) + f(x_{n-1})\right]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_n) + 2\left(f(x_1) + f(x_2) - - - f(x_{n-1})\right]\right.$$

Comp. Trap Rule

$\|^{ly}$ Error in each integration.

$$-\frac{h^3}{12}f''(c_1) - \frac{h^3}{12}f''(c_2) - - - - \frac{h^3}{12}f''(c_n)$$

$$x_0 < c_1 < x_2, \quad x_1 < c_2 < x_3 \quad - - - , \quad x_{n-1} < c_n < x_n$$

Numerical Quadrature:

Composite Trapezoidal Rule:

$$\int_{a=x_0}^{x_n=b} f(x)dx = \frac{h}{2}(f(x_0)+f(x_n)+2(f(x_1)+f(x_2)+\cdots+f(x_{n-1}))).$$

$h = x_{i+1} - x_i$

Composite Trapezoidal Rule: Error formula (Proof)

$$E(f) = -\frac{h^2(b-a)}{12} f''(c),\ c \in (a,b)$$

Numerical Quadrature:

$N=6$

Composite Simpson's $\frac{1}{3}rd$ Rule:

We divide the interval $[a,b]$, into $\boxed{N \text{ (even)}}$ subintervals with step size $h=\frac{b-a}{N}$, and taking nodal points $a=x_0<x_1<\cdots<x_N=b$, where $x_i=x_0+ih$, $i=0,1,2,\cdots,N$. Take $\underline{N=2n}$.

$6=2*3$

$$\int_{x_0}^{x_{2n}=N} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots \int_{x_{2n-2}=N-2}^{x_{2n}=N} f(x)\,dx$$

Apply simpson's Rule in each int.

$$\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right] + \frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$$

$$+\frac{h}{3}\left[f(x_4)+4f(x_5)+f(x_6)\right] \cdots \frac{h}{3}\left[f(x_{2n-2}^{N-2})+4f(x_{2n-1}^{N-1}) + f(\underbrace{x_{2n}}_{=N})\right]$$

$$= \frac{h}{3}\Big[f(x_0) + f(\underset{N}{x_{2n}}) + 4\big(f(x_1) + f(x_3) + f(x_5) \cdots f(\underset{N-1}{x_{2n-1}}) \big)$$

$$+ 2\big(f(x_2) + f(x_4) \cdots f(\underset{N}{x_{2n}}) \big) \Big]$$

Numerical Quadrature:

Composite Simpson's Rule:

$$\int_{a=x_0}^{x_n=b} f(x)dx$$
$$= \frac{h}{3}[f(x_0) + f(x_{2n}) + 2\{f(x_2) + f(x_4) + \cdots + f(x_{2n-2})\} +$$
$$4\{\{f(x_1) + f(x_3) + \cdots + f(x_{2n-1})\})].$$

Composite Simpson's Rule: Error formula (Proof)

$$E(f) = -\frac{h^4(b-a)}{180} f^{iv}(c), \; c \in (a, b)$$

Example:

Determine the values of subintervals n and step-size h required to approximate $\int_0^2 \frac{1}{x+4}\,dx$ using composite trapezoidal to within 10^{-3} and composite Simpson's rule to within 10^{-5} resp. and hence compute the approximation using both the rules.

Solution: Using comp Trap Rule

max error bound in trap $\leq 10^{-3}$

$a=0$
$b=2$

$$\max_{c\in(0,2)} \left| -\frac{h^2}{12}(b-a) f''(c) \right| \leq 10^{-3}$$

$f(x)=\frac{1}{x+4}$

$$\max_{c\in[0,2]} \left| \frac{h^2}{12}(2) \frac{2}{(c+4)^3} \right| \leq 10^{-3}$$

$f'(x)=\frac{-1}{(x+4)^2}$

$$f''(x) = \frac{2}{(x+4)^3}$$

$$\max_{c \in [0,2]} \left| \frac{h^2}{3} \frac{1}{(c+4)^3} \right| \leq 10^{-3}$$

$$h = \frac{b-a}{n}$$

$$\left| \frac{h^2}{3 (4)^3} \right| \leq 10^{-3}$$

$$|h| \leq \sqrt{10^{-3} * 64 * 3}$$

$$h \leq 0.4381$$

$$\frac{b-a}{n} \leq 0.4381$$

$$\frac{2}{n} \leq 0.4381$$

$$\frac{n}{2} \geq \frac{1}{0.4381}$$

$$n \geq \frac{2}{0.4381} = 4.567$$

$$\boxed{n = 5}$$

$$h = \frac{b-a}{n} = \frac{2-0}{5} = \frac{2}{5}$$

$$x_0 = 0, \quad x_1 = \frac{2}{5}, \quad x_2 = \frac{4}{5}, \quad x_3 = \frac{6}{5}, \quad x_4 = \frac{8}{5}, \quad x_5 = 2$$

Using
<u>Comp Trap. Rule</u> :-

$$\int_0^2 \frac{1}{x+4}\,dx = \frac{2/5}{2}\Big[f(0) + f(2) + 2\big(f(2/5) + f(4/5) + f(6/5) + f(8/5)\big)\Big]$$

$$= \frac{1}{5}\left[\frac{1}{0+4} + \frac{1}{2+4} + 2\left[\left\{\frac{1}{\frac{2}{5}+4}\right\} + \left\{\frac{1}{\frac{4}{5}+4}\right\} + \left\{\frac{1}{\frac{6}{5}+4}\right\} + \left\{\frac{1}{\frac{8}{5}+4}\right\}\right]\right] = ?$$

2nd part

$|$ Max error bounds in Comp Simp $| \le 10^{-5}$

$$f'''(x) = \frac{-6}{(x+4)^4}$$

$$f^{iv} = \frac{24}{(x+4)^5}$$

$$\max_{c\in[0,2]} \left| \frac{-h^4}{180}(b-a)\, f^{iv}(c) \right| \le 10^{-5}$$

$$\max_{c\in[0,2]} \left| \frac{h^4(2-0)}{180} \frac{24}{(c+4)^5} \right| \le 10^{-5}$$

$$h = \frac{b-a}{2n} = \frac{b-a}{N}$$

$$\left| \frac{48}{180} \frac{h^4}{(4)^5} \right| \le 10^{-5}$$

$$|h| \le \left(\frac{10^{-5} * 4^5 * 180}{48} \right)^{1/4}$$

$$|h| \le 0.4426$$

$$\left| \frac{b-a}{N} \right| \le 0.4426$$

$$\left| \frac{2}{N} \right| \le 0.4426$$

$$\frac{N}{2} \ge \frac{1}{0.4426}$$

$$N \ge \frac{2}{0.4426} = 4.5186$$

even

$N = 6$ (next even integer)

$$h = \frac{b-a}{N} = \frac{2}{6} = \frac{1}{3}$$

$x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = \frac{3}{3} = 1, \quad x_4 = \frac{4}{3}, \quad x_5 = \frac{5}{3}, \quad x_6 = 2$

Using comp. simp. Rule

$$\int_0^2 \frac{1}{x+4}\,dx = \frac{\frac{1}{3}}{3}\Big[f(0) + f(2) + 2\big(f(2/3) + f(4/3) \big) + 4\big(f(1/3) + f(1) + f(5/3) \big) \Big]$$

= complete it

Example:

Determine values of h (or n) that will ensure an approximation error of less than 0.01 when approximating $\int_0^2 \sin x \, dx$ and hence compute the approximation using composite Simpson's rule.

Solution:

Do it yourself.

Numerical Quadrature:

Composite integration: Exercise:

1. The area A inside a closed curve $y^2 + x^2 = \cos x$ is given by $A = 4\int_0^{\alpha} (\cos x - x^2)^{1/2} dx$, where α is the positive root of the equation $\cos x = x^2$.
   a. Compute α with three correct decimals by Newton's method.
   b. Use composite trapezoidal rule with 6 subintervals to compute the area A.
2. Determine values of h (or n) that will ensure an approximation error of less than 10^{-2} when approximating $\int_{-1}^{1} \frac{1}{1+x^2}\, dx$ by applying composite trapezoidal rule and composite Simpson's rule.