Lecture 16: Numerical Linear Algebra (UMA021): Interpolation

Dr. Meenu Rani

Department of Mathematics
TIET, Patiala
Punjab-India

Lagrange Interpolating polynomials:

$x_0, x_1, \dots x_n$

Result (error term):

Suppose $x_0, x_1, \cdots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \cdots, x_n$, and hence in (a, b), exists with

exact   Approximation   error term

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P_n(x)$ is $n-$th degree Lagrange's interpolating polynomial.

$|f(x) - P_n(x)| = |e(x)|$

error function

$\xi \in [a, b]$

$f(x+h) = f(h) + (x-h) f'(h) + (x-h)^2 f''(h) + \cdots (x-h)^n f^{(n)}(h)$

error $= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$ $2!$ $n!$ $+ - - - -$

Lagrange Interpolating polynomials:

Example:

Use the error formula to find the error bound for the polynomial which is used to approximate $f(x) = \frac{1}{x}$ on $[2, 4]$ with the nodes $x_0 = 2, x_1 = 2.75$, and $x_2 = 4$.

Error is given by $= E(x)$

$n = 2$

$$E(x) = \frac{f^{(3)}(\xi)}{3!} (x - x_0)(x - x_1)(x - x_2)$$

$$\max_{x \in (2,4)} |E(x)| = \max_{x \in (2,4)} \left| \frac{f^{(3)}(\xi)}{3!} (x-2)(x-2.75)(x-4) \right|$$

$$= \underbrace{\max_{\xi \in (2,4)} \left| \frac{f^{(3)}(\xi)}{6} \right|}\ \underbrace{\max_{x \in (2,4)} |(x-2)(x-2.75)(x-4)|}$$

$$= \boxed{M} * \max_{x \in (2,4)} |g(x)|$$

Now

$$M = \max_{\xi \in (2,4)} \left| \frac{f^{(3)}(\xi)}{6} \right|$$

$$= \max_{\xi \in (2,4)} \left| \frac{-6}{\xi^4} * \frac{1}{6} \right| = \frac{1}{2^4}$$

$$\because \begin{cases} f(x) = \frac{1}{x} \\ f'(x) = \frac{-1}{x^2} \\ f''(x) = \frac{2}{x^3} \\ f'''(x) = \frac{-6}{x^4} \end{cases}$$

$$M = \frac{1}{16}$$

&

$$\max_{x \in (2,4)} |g(x)|$$

$$g(x) = (x - 2.75)(x^2 - 6x + 8)$$

$$g'(x) = (x - 2.75)(2x - 6) + (x^2 - 6x + 8)(1) = 0$$

$$2x^2 - 5.5x - 6x + 16.5 + x^2 - 6x + 8 = 0$$

$$= \quad 3x^2 - 17.5x + 24.5 = 0$$

$$x = \frac{7}{3}, \ \frac{7}{2}$$

$$\left|g\left(\frac{7}{3}\right)\right| = \left|\left(\frac{7}{3} - 2.75\right)\left(\left(\frac{7}{3}\right)^2 - 6\left(\frac{7}{3}\right) + 8\right)\right| = \frac{25}{108}$$

$$\left|g\left(\frac{7}{2}\right)\right| = \left|\left(\frac{7}{2} - 2.75\right)\left(\left(\frac{7}{2}\right)^2 - 6\left(\frac{7}{2}\right) + 8\right)\right| = \frac{9}{16}$$

$$\Rightarrow \max_{x \in [2,4)} |g(x)| = \frac{9}{16}$$

$$\Rightarrow \max_{x \in [2,4} E(x) = m * \frac{9}{16} = \frac{1}{16} * \frac{9}{16} = \frac{9}{256} = 0.035$$

Ans

Lagrange Interpolating polynomials:

Example:

Determine the spacing h in a table of equally spaced values of the function $f(x) = e^x$ between 0 and 1, so that interpolation with a linear polynomial will yield an accuracy of 10^{-6}.

Proof:

$h = ?$

we take two points

$x_0, x_0 + h$

$n = 1$

$$\max_{x \in [0,1]} |E(x)| = \max_{x \in [0,1]} \left| \frac{f^{(2)}(\xi_x)}{2!} (x - x_0)(x - x_0 - h) \right| \leq 10^{-6}$$

$$= \max_{x \in [0,1]} \left| \frac{f^{(2)}(\xi_x)}{2!} \right| \max_{x \in (0,1)} |(x - x_0)(x - x_0 - h)| \leq 10^{-6}$$

$$= M * \max_{x \in [0,1]} |g(x)| \leq 10^{-6} \quad (*)$$

Now $M = \max_{x \in [0,1]} \left| \frac{f^{(2)}(\xi_y)}{2!} \right| = \max_{x \in [0,1]} \left| \frac{e^{\xi_y}}{2!} \right| = \frac{e^1}{2} = \frac{e}{2}$

& take $g(x) = (x - x_0)(x - x_0 - h)$

$$g'(x) = (x - x_0) + (x - x_0 - h) = 0$$

$$2x - 2x_0 = h$$

$$x = x_0 + \frac{h}{2}$$

$$\max_{x \in [0,1]} |g(x)| = \left| g\left(x_0 + \frac{h}{2}\right) \right|$$

$$= \left| \left(x_0 + \frac{h}{2} - x_0\right)\left(\cancel{x_0} + \frac{h}{2} - \cancel{x_0} - h\right) \right|$$

$$= \left| \frac{h}{2} * -\frac{h}{2} \right| = \frac{h^2}{4}$$

Put M & max value of $|g(x)|$ in $(*)$

$$\Rightarrow \left| \frac{e}{2} * \frac{h^2}{4} \right| \leq 10^{-6}$$

$$|h^2| \leq \frac{10^{-6} * 8}{e}$$

$$|h| \leq \frac{\sqrt{8}}{\sqrt{e}} * 10^{-3}$$

$$= 1.72 * 10^{-3}$$

Lagrange Interpolating polynomials:

Exercise:

1. Determine the spacing h in a table of equally spaced values of the function $f(x) = \sqrt{x}$ between 1 and 2, so that interpolation with a quadratic polynomial will yield an accuracy of 5×10^{-4}.
2. Find a bound for the absolute error on the interval $[x_0, x_n]$.
   - a. $f(x) = \sin x,\ x_0 = 2.0,\ x_1 = 2.4,\ x_2 = 2.6,\ n = 2.$
   - b. $f(x) = e^{2x} \cos 3x,\ x_0 = 0,\ x_1 = 0.3,\ x_2 = 0.6,\ n = 2.$

Newton Divided Difference Interpolation:

Result

Suppose that $f \in C^n[a, b]$ and $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a, b]$. Then a number ξ exists in (a, b) with

nth D.D of f

nth derivative of f

$$f[x_0, x_1, x_2, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Proof: Define a function $g(x) = f(x) - P_n(x)$

$P_n(x)$ is a nth degree Newton-D.D. interpolating poly.

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1](x - x_0)(x - x_1) + \cdots + \underbrace{f[x_0, x_1, \cdots x_n]}(x - x_0) \cdots (x - x_{n-1})$$

Generalized Rolle's thm let $f \in C^n[a,b]$ and $f(x)$ has $(n+1)$ zeros in (a,b) then $\exists$ a no. $c \in (a,b)$ s.t

$$f^{(n)}(c) = 0$$

Rolle's thm.

$f \in C[a,b]$

$f' \in C(a,b)$

then $\exists\, c \in (a,b)$

s.t.

$f'(c) = 0$

Since $g(x) = f(x) - P_n(x)$

Here $f(x) \in C^n[a,b]$, $P_n \in C^\infty[a,b]$

then $g(x) \in C^n[a,b]$

& Put $x = x_0$ in $g(x)$

$$g(x_0) = f(x_0) - P_n(x_0) = 0$$

Put $x = x_1$ in $g(x)$

$$g(x_1) = 0$$

||ly $g(x_n) = 0$

$\|^{ly}$ $g(x_i) = 0 \quad \forall \quad 0 \le i \le n$

$\Rightarrow g(x)$ has $(n+1)$ zeros in (a,b)

So Apply G. R. T. on $g(x)$

then $\exists$ $c \in (a,b)$ s.t.

$$g^{(n)}(c) = 0$$

$$\left(f(x) - p_n(x)\right)^{(n)}_{x=c} = 0$$

$$\left(\frac{d^n}{dx^n} f(x)\right)_{x=c} - f[x_0, x_1 \cdots x_n]\, n! = 0$$

$$\frac{f^{(n)}(c)}{n!} = f[x_0, x_1 \cdots x_n]$$

you can take $c = \xi$