Lecture 27: Numerical Linear Algebra (UMA021): Matrix Algebra

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Jacobi Method

$$a_{11} x_1 + a_{12} x_2 \ldots a_{1n} x_n = b_1 \quad -(1)$$

$$a_{21} x_1 + a_{22} x_2 \ldots a_{2n} x_n = b_2 \quad -(2)$$

$$\ldots$$

$$a_{n1} x_1 + a_{n2} x_2 \ldots a_{nn} x_n = b_n \quad -(n)$$

from (1)

$$x_1^{(1)} = \frac{1}{a_{11}} \left[b_1 - \left(a_{12} x_2^{(0)} + a_{13} x_3^{(0)} \ldots a_{1n} x_n^{(0)} \right) \right]$$

$$= \frac{1}{a_{11}} \left[b_1 - \sum_{j=2}^{n} a_{1j} x_j^{(0)} \right]$$

from (2)

$$x_2^{(1)} = \frac{1}{a_{22}} \left[b_2 - \left(a_{21} x_1^{(0)} + a_{23} x_3^{(0)} \ldots a_{2n} x_n^{(0)} \right) \right]$$

$$= \frac{1}{a_{22}} \left[b_2 - \sum_{\substack{j=1 \\ j \neq 2}}^{n} a_{2j} x_j^{(0)} \right]$$

$$\ldots$$

from (n)

$$x_n^{(1)} = \frac{1}{a_{nn}} \left[b_n - \left(a_{n1} x_1^{(0)} \ldots a_{n,n-1} x_{n-1}^{(0)} \right) \right]$$

$$= \frac{1}{a_{nn}} \left[b_n - \sum_{j=1}^{n-1} a_{nj} x_j^{(0)} \right]$$

Use initial guess $x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}$ find $x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}$

$$x_i^{(1)} = \frac{1}{a_{ii}}\left[b_n - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(0)}\right]$$

$\boxed{\text{Stopping Criterion}}$

$\|X^{(2)} - X^{(1)}\|_\infty <$ tol.

$$x_i^{(2)} = \frac{1}{a_{ii}}\left[b_n - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(1)}\right]$$

$$x_i^{(3)} = \frac{1}{a_{ii}}\left[b_n - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(2)}\right]$$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_n - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(k)}\right]}$$

$k = 0, 1, 2 \ldots n$

Jacobi Method.

Gauss Seidel Method

Consider the system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad -(1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad -(2)\\
\vdots \qquad\quad \vdots \qquad\quad \vdots \quad &= \vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad -(n)
\end{aligned}
$$

from (1)

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(0)} + a_{13}x_3^{(0)} - - - a_{1n}x_n^{(0)}\right)\right]$$

$$= \frac{1}{a_{11}}\left[b_1 - \sum_{j=2}^{n} a_{1j}x_j^{(0)}\right]$$

Use initial guess

$$x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \end{bmatrix}$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(1)} + a_{23}x_3^{(0)} - - - a_{2n}x_n^{(0)}\right)\right]$$

$$= \frac{1}{a_{22}}\left[b_2 - \sum_{j=1}^{1} a_{2j}x_j^{(1)} - \sum_{j=3}^{n} a_{2j}x_j^{(0)}\right]$$

$$\left[\, x_n^{(0)} \,\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31} x_1^{(1)} + a_{32} x_2^{(1)} + a_{34} x_4^{(0)} - a_{3n} x_n^{(0)} \right) \right]$$

$$X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}$$

$$= \frac{1}{a_{33}}\left[b_3 - \sum_{j=1}^{2} a_{3j} x_j^{(1)} - \sum_{j=4}^{n} a_{3j} x_j^{(0)} \right]$$

— — —

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1} x_1^{(1)} + a_{n2} x_2^{(1)} - - - a_{n,n-1} x_{n-1}^{(1)} \right) \right]$$

$$= \frac{1}{a_{nn}}\left[b_n - \sum_{j=1}^{n-1} a_{nj} x_j^{(1)} \right]$$

$$x_i^{(1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(0)} \right]$$

$$x_i^{(2)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(2)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(1)} \right]$$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]}$$

Gauss - Seidel method

Iterative methods to solve System of linear equations

Result:

If A is strictly diagonally dominant, then for any choice of $x^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{x^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $Ax = b$.

$\boxed{A}x = b$

If A is S.D.D.

$\Downarrow$

$x^{(k)} \longrightarrow x$ → solution of $Ax=b$.

by J.M.
G.S.M.

Example:

Use Jacobi and Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned}
10x_1 - x_2 + 2x_3 &= 6 \\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\
2x_1 - x_2 + 10x_3 - x_4 &= -11 \\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned}$$

starting with $x = (0, 0, 0, 0)^t$ and iterating until $\|x^{(k)} - x^{(k-1)}\|_\infty < 10^{-3}$.

Solution

$$A = \begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix}$$

Check whether A is S.D.D.

$|10| > |-1| + |2| + |0|$ ✓

$|11| > |-1| + |-1| + |3|$ ✓

$|10| > |2| + |-1| + |-1|$ ✓

Using Jacobi method

$$x_1^{(k+1)} = \frac{1}{10}\left(6 + x_2^{(k)} - 2x_3^{(k)}\right) \qquad |8| > |0| + |3| + |-1| \checkmark$$

$$x_2^{(k+1)} = \frac{1}{11}\left(25 + x_1^{(k)} + x_3^{(k)} - 3x_4^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{10}\left(-11 - 2x_1^{(k)} + x_2^{(k)} + x_4^{(k)}\right)$$

$$x_4^{(k+1)} = \frac{1}{8}\left(15 - 3x_2^{(k)} + x_3^{(k)}\right)$$

Let $k = 0$, $X^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1^{(1)} = \frac{1}{10}(6 + 0) = 0.6$$

$$x_2^{(1)} = \frac{1}{11}(25) = 2.2727$$

$$x_3^{(1)} = \frac{1}{10}(-11) = -1.1$$

$$x_4^{(1)} = \frac{15}{8} = 1.875$$

$$X^{(1)} = \begin{pmatrix} 0.6 \\ 2.2727 \\ -1.1 \\ 1.875 \end{pmatrix}$$

for $k=1$, $\quad X^{(1)} = \begin{bmatrix} 0.6 \\ 2.2727 \\ -1.1 \\ 1.875 \end{bmatrix} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{bmatrix}$

$$x_1^{(2)} = \frac{1}{10}\left[6 + 2.2727 - 2(-1.1)\right] = 1.0472$$

$$x_2^{(2)} = \frac{1}{11}\left[25 + 0.6 - 1.1 - 3(1.875)\right] = 1.7159$$

$$x_3^{(2)} = \frac{1}{10}\left[-11 - 2(0.6) + 2.2727 + 1.875\right] = -0.8052$$

$$x_4^{(2)} = \frac{1}{8}\left[15 - 3(2.2727) - 1.1\right] = 0.8852$$

$$\|X^{(2)} - X^{(1)}\|_\infty$$

$$= \max\left\{ |x_1^{(1)} - x_1^{(2)}|, |x_2^{(1)} - x_2^{(2)}|, |x_3^{(1)} - x_3^{(2)}|, |x_4^{(1)} - x_4^{(2)}| \right\}$$

$$\not< 10^{-3}$$

for $k=3$, $\quad X^{(2)} = \begin{bmatrix} 1.0472 \\ 1.7159 \\ -0.8052 \\ 0.8852 \end{bmatrix}$

$$x_1^{(3)} = \frac{1}{10}\left[6 + 1.7159 + 2(0.8052)\right] = 0.9326$$

$$x_2^{(3)} = \frac{1}{11}\left[25 + 1.0472 - 0.8052 - 3(0.8852)\right] = 2.053$$

$$x_3^{(3)} = \frac{1}{10}\left[-11 - 2(1.0472) + 1.7159 + 0.8852\right] = -1.0493$$

$$x_4^{(3)} = \frac{1}{8}\left[15 - 3(1.7159) - 0.8052\right] = 1.1309$$

$$X^{(3)} = \begin{bmatrix} 0.9326 \\ 2.053 \\ -1.0493 \\ 1.1309 \end{bmatrix} \checkmark$$

Using Gauss - Seidel Method

$$x_1^{(k+1)} = \frac{1}{10}\left(6 + x_2^{(k)} - 2x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{11}\left(25 + x_1^{(k+1)} + x_3^{(k)} - 3x_4^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{10}\left(-11 - 2x_1^{(k+1)} + x_2^{(k+1)} + x_4^{(k)}\right)$$

$$x_4^{(k+1)} = \frac{1}{8}\left(15 - 3x_2^{(k+1)} + x_3^{(k+1)}\right)$$

Use $X^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $k=0$

$$x_1^{(1)} = \frac{1}{10}(6) = 0.6$$

$$x_2^{(1)} = \frac{1}{11}(25 + 0.6 + 0) = 2.327$$

$$x_3^{(1)} = \frac{1}{10}\left(-11 - 2(0.6) + 2.327) + 0\right) = -0.9873$$

$$x_4^{(1)} = \frac{1}{8}\left(15 - 3(2.327) - 0.9873\right) = 0.8789$$

$$X^{(1)} = \begin{bmatrix} 0.6 \\ 2.327 \\ -0.9873 \\ 0.8789 \end{bmatrix} \text{ and } k=1$$

$||^{ly}$ find $X^{(2)}$, $X^{(3)}$ - - - & Check the tolerance to get 10^{-3} accuracy

Jacobi Method

k	0	1	2	3	4	5	6	7	8	9	10	
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001	$\to 1$
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998	$\to 2$
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998	$\to -1$
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998	$\to 1$

Gauss-Seidel Method

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

System of linear equations:

Exercise:

1. The linear system

$$\begin{aligned} x_1 - x_3 &= 0.2 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425 \\ x_1 - \frac{1}{2}x_2 + x_3 &= 2 \end{aligned}$$

has the solution $(0.9, -0.8, 0.7)^T$.

   a. Is the coefficient matrix strictly diagonally dominant?
   b. Perform four iterations of the Jacobi and Gauss-Seidel iterative method to approximate the solution. Take $x^{(0)} = 0$.