Lecture 29: Numerical Linear Algebra (UMA021): Matrix Algebra

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Matrix Representation of Jacobi & Gauss-seidel methods

Jacobi Method

$$x^{(k)} = -D^{-1}(L+U)\,x^{(k-1)} + D^{-1}b$$

Gauss-seidel method

$$x^{(k)} = -(D+L)^{-1}U\,x^{(k-1)} + (D+L)^{-1}b$$

$$x^{(k)} = T\,x^{(k-1)} + C$$

$\boxed{Ax = b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ 0 & \cdots & a_{33} & 0 \\ \cdots & \cdots & 0 & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & 0 & \\ a_{31} & a_{32} & & \\ \vdots & \cdots & a_{nn} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & & \cdots & a_{2n} \\ & 0 & & \vdots \\ & & & 0 \end{bmatrix}$$

System of linear equations: Matrix representation of iterative methods

Result:(Stronger condition for the convergence of iterative methods):

For any $X^{(0)} \in \mathbb{R}^n$, the sequence $\{X^{(k)}\}_{k=0}^{\infty}$ defined by $X^{(k)} = TX^{(k-1)} + C$, for each $k \geq 1$ converges to unique solution $X = TX + C$ iff $\rho(T) < 1$.

↳ spectral radius of T < 1

max |E. values of T| < 1

System of linear equations:

Example:

Check whether you can apply Gauss-Seidel iterative techniques to solve the following linear system of equations.

$$
\begin{aligned}
2x_1 - x_2 + x_3 &= -1 \\
2x_1 + 2x_2 + 2x_3 &= 4 \\
-x_1 - x_2 + 2x_3 &= -5.
\end{aligned}
$$

Check for S.D.D.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|2| \ngtr |-1| + |1|$$

$$|2| \ngtr |2| + |2|$$

for Gauss-seidel method, The iterative scheme is

$$x^{(k)} = \underbrace{-(D+L)^{-1}U}_{= T_g} x^{(k-1)} + c$$

for convergence condition check

$$\rho\left(-(D+L)^{-1}U\right) < 1$$

$$\rho(T_g) < 1 \quad \text{or not}$$

$$T_g = -(D+L)^{-1}U = -\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & -1/4 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

To find E.values of T_g, we take $|T_g - \lambda I| = 0$

$$|T_g - \lambda I| = \begin{vmatrix} -\lambda & \frac{1}{2} & -1/2 \\ 0 & \frac{-1}{2}-\lambda & -1/2 \\ 0 & 0 & \frac{-1}{2}-\lambda \end{vmatrix} = 0$$

$$-\lambda \left[\left(\frac{-1}{2} - \lambda\right)\left(\frac{-1}{2} - \lambda\right) \right] = 0$$

$\lambda = 0, \frac{-1}{2}, \frac{-1}{2}$ $\rightarrow$ E. values of T_g

$\max |\text{E. value of } T_g| = \frac{1}{2} < 1$

$\Rightarrow \rho(T_g) < 1$

$\Rightarrow$ Yes, we can apply Gauss-Seidel method for which we get the convergence for any initial guess.

System of linear equations:

Exercise:

Check whether you can apply Jacobi and Gauss-Seidel iterative techniques to solve the following linear systems.

1.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= -1 \\ 3x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 1 \end{aligned}$$

2.

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7 \\ x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

Iterative methods to find the solution of linear system:

Question:

Use the Gauss-Seidel method to approximate the solution of the following system:

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 3 \\ 2x_1 + 7x_2 + x_3 &= 19 \\ x_1 - 3x_2 + 12x_3 &= 31 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Iterative methods to find the solution of linear system:

Question:

The linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 2x_1 + 2x_2 + 2x_3 &= 4 \\ -x_1 - x_2 + 2x_3 &= -5 \end{aligned}$$

has the solution $(1, 2, -1)^t$. Then show that $\rho(T_g) = \frac{1}{2}$.

Condition Number:

The condition number of the non-singular matrix A relative to maximum norm $\|.\|$ is

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty.$$

$\|I\|_\infty =$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\|I\|_\infty = \max\{|1|+|0|+|0|,\ |0|+|1|+|0|,\ |0|+|0|+|1|\}$$

$$= 1$$

$A\,x = b$ $\quad A \to A+\delta = B$

$B\,x = b$

Iterative methods to find the solution of linear system:

$\|AB\| \le \|A\|\,\|B\|$

Condition Number:

For any non-singular matrix A and infinity norm $\|A\|$

$$1 = \|I\|_\infty = \|A.A^{-1}\| \le \|A\|\,\|A^{-1}\| = K(A).$$

$\boxed{K(A) \geq 1}$

Note: A matrix A is well conditioned if $K(A)$ is close to 1, and is ill conditioned when $K(A)$ is significantly greater than 1.

$$1 = \|I\|_\infty = \|AA^{-1}\|_\infty \le \|A\|\,\|A^{-1}\| = K(A)$$

$$K(A) \geq 1$$

Example:

Determine the condition number for the matrix:

let $A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}.$ $|A| = 2 - 2.0002$
$= -0.0002$

$$A^{-1} = \frac{-1}{0.0002} \begin{bmatrix} 2 & -2 \\ -1.0001 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10000 & 10000 \\ 5000.5 & -5000 \end{bmatrix}$$

$$K(A) = \|A\|_\infty \, \|A^{-1}\|_\infty = \max\{|1| + |2|, |-1.0001| + |2|\}$$

$$* \max\{|-10000| + |10000|, |5000.5| + |-5000|\}$$

$= 3.0001 * 20000 = 60002 >> 1$

Iterative methods to find the solution of linear system:

Exercise:

Determine the condition number for the following matrices and check whether these matrices are ill-conditioned or well conditioned

1.)

$$\begin{bmatrix} 5 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

2.)

$$\begin{bmatrix} 1 & 2 \\ 0.9999 & 2 \end{bmatrix}.$$