Lecture 22: Numerical Linear Algebra (UMA021): Integration

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Numerical Quadrature:

Quadrature formulas:

1. The quadrature formula is called Newton-Cotes formula if all points are equally spaced.
2. All the Newton-Cotes formulas use values of the function at equally-spaced points
3. It can significantly decrease the accuracy of the approximation.

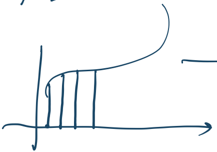
$x_0, x_1, x_2 \; - - -$

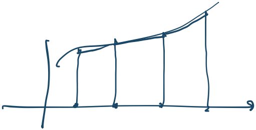
$\int \quad \approx$

$x_0, x_1,$

x_0, x_1, x_2

$h = 0.5$





Numerical Quadrature:

Gaussian Quadrature:

Gaussian quadrature chooses the best points for evaluation rather than equally spaced. So, Gaussian quadrature is more accurate.

multipliers nodes

In the numerical integration method $\int_a^b f(x)dx \approx \sum_{i=1}^{n} \lambda_i f(x_i)$, if both nodes and multipliers are unknown then method is called Gaussian quadrature.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \lambda_i f(x_i)$$

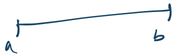
$$= \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) \cdots \lambda_n f(x_n)$$

Numerical Quadrature:

$[a, b]$

Gaussian Quadrature:

The coefficients $\lambda_1, \lambda_2, \cdots, \lambda_n$ in the approximation formula are arbitrary, and the nodes $x_1, x_2, \cdots, x_n$ are restricted only by the fact that they must lie in $[a, b]$. This gives us $\underbrace{2n \text{ unknowns}}$ to choose.



Numerical Quadrature:

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We can obtain these unknowns by making the method exact for the class of polynomials of degree at most $2n - 1$ which gives $2n$ equations in these $2n$ unknowns.

for e.g. $\int_0^2 f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$ → These are 4 unknowns

for $f(x) = 1$

for $f(x) = x$

$\int_0^2 x\,dx = \lambda_0 x_0 + \lambda_1 x_1$

$$\int_{-a}^{a} f(x)\,dx$$

$$\int_0^2 1\,dx = \lambda_0 + \lambda_1 \quad -(1)$$

$$2 = \lambda_0 x_0 + \lambda_1 x_1 \quad -(2)$$

for $f(x) = x^2$

$$\int_0^2 x^2\,dx = \lambda_0 x_0^2 + \lambda_1 x_1^2$$

$$\frac{8}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 \quad -(3)$$

for $f(x) = x^3$

$$\int_0^2 x^3\,dx = \lambda_0 x_0^3 + \lambda_1 x_1^3$$

$$4 = \lambda_0 x_0^3 + \lambda_1 x_1^3 \quad -(4)$$

Numerical Quadrature:

$\int_{-1}^{1} P(x)\, P_3(x)\, dx = 0$

Legendre polynomials:

$0x^2+2x+1$

We use Legendre polynomials, a collection $\{P_0(x), P_1(x), \cdots, P_n(x), \cdots\}$ with properties:

1. For each n, $P_n(x)$ is a monic polynomial of degree n.
2. $\int_{-1}^{1} P(x)P_n(x)dx = 0$, whenever $P(x)$ is a polynomial of degree less than n.
3. The first few polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - \frac{1}{3}$, $P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.
4. The roots of these polynomials are distinct, lie in the interval $(-1, 1)$ have a symmetry with respect to the origin, and, most importantly, are the correct choice for determining the parameters that give us the nodes and coefficients for our quadrature method.

Numerical Quadrature:

Gauss-Legendre Integration Methods::

The Gaussian quadrature formulas are derived for the interval $[-1, 1]$, and any interval $[a, b]$, can be transformed to $[-1, 1]$, by taking the transformation $x = \frac{b-a}{2}t + \frac{b+a}{2}$.

$$\int_a^b f(x)\,dx$$

$$x = \left(\frac{b-a}{2}\right)t + \frac{b+a}{2}$$

$$\boxed{\int_0^2 f(x)\,dx} \longrightarrow \int_{-1}^{1} g(t)\,dt$$

$$x = \frac{(2-0)}{2}t + \frac{2+0}{2}$$

$$x = t + 1$$

$$dx = dt$$

Gauss-Legendre one-point formula:

For $n = 1$, the formula is given by $\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1)$.

The formula has two unknowns λ_1 and x_1. Make the method exact for $f(x) = 1, x$, we obtain

For $f(x) = 1$, we have $\int_{-1}^{1} 1\, dx = \lambda_1 .1 \Rightarrow \lambda_1 = 2$

For $f(x) = x$, we have $\int_{-1}^{1} x\, dx = \lambda_1 .x_1 \Rightarrow 2x_1 = 0.$ $\Rightarrow x_1 = 0$

Therefore, one point formula is given by $\boxed{\int_{-1}^{1} f(x)dx = 2\, f(0).}$

for e.g. Use Gaussian legendre one-pt. formula to integrate

$$\int_{-1}^{1} \frac{1}{x+4}\,dx = 2\,f(0) = 2\,\frac{1}{4}$$

$$= \frac{1}{2} = 0.5$$

but if int. is

$$\int_{0}^{2} \underbrace{\frac{1}{x+4}}_{=f(x)}\,dx$$

$$x = \frac{2-0}{2}\,t + \left(\frac{2+0}{2}\right)$$

$$x = t+1$$

$$dx = dt$$

$$\int_{-1}^{1} \boxed{\frac{1}{t+5}}_{=f(t)}\,dt = 2\,f(0) = \frac{2}{5} = 0.4$$

$$f(t) = \frac{1}{t+5}$$

Exact $\displaystyle\int_{0}^{2} \frac{1}{x+4}\,dx = \Big[\ln|x+4|\Big]_{0}^{2} = \ln(6) - \ln(4)$

$$= \ln\left(3/2\right) = 0.405$$

Numerical Quadrature:

$f(0)\ (\lambda_1+\lambda_2)$

Gauss-Legendre two-point formula:

For $n = 2$, the formula is given by

$\int_{-1}^{1} f(x)dx = \lambda_1 f(x_1) + \lambda_2 f(x_2)$.

$(\lambda_1+\lambda_2)\ f(x_1)$

The formula has four unknowns $\lambda_1, \lambda_2, x_1$ and x_2. Make the method exact for $f(x) = 1, x, x^2, x^3$, we obtain

$$\text{For } f(x) = 1, \quad \int_{-1}^{1} 1dx = \lambda_1 .1 + \lambda_2 .1 \Rightarrow \lambda_1 + \lambda_2 = 2. \tag{1}$$

$$\text{For } f(x) = x, \quad \int_{-1}^{1} xdx = \lambda_1 .x_1 + \lambda_2 .x_2 \Rightarrow \lambda_1 x_1 + \underline{\lambda_2 x_2} = 0. \tag{2}$$

$\cancel{x}\ \lambda_1 x_1 = 0$

$x_1 = 0$

$$\text{For } f(x) = x^2, \quad \int_{-1}^{1} x^2dx = \lambda_1 .x_1^2 + \lambda_2 .x_2^2 \Rightarrow \lambda_1 x_1^2 + \lambda_2 x_2^2 = \frac{2}{3}. \tag{3}$$

Numerical Quadrature:

For $f(x) = x^3$, $\quad \int_{-1}^{1} x^3 dx = \lambda_1 . x_1^3 + \lambda_2 . x_2^3 \Rightarrow \lambda_1 x_1^3 + \lambda_2 x_2^3 = 0.$ (4)

Multiply eqⁿ (2) by x_1^2 & sub. it from eqⁿ (4)

$$\cancel{\lambda_1 x_1^3} + \lambda_2 x_2^3 = 0 \qquad - (4)$$

$$\underline{-\cancel{\lambda_1 x_1^3} \underset{-}{+} \lambda_2 x_1^2 x_2 = 0}$$

$$\lambda_2 x_2 (x_2^2 - x_1^2) = 0$$

$ab = 0$

$$\lambda_2 x_2 (x_2 - x_1)(x_2 + x_1) = 0$$

X $\lambda_2 \neq 0$ ∵ it is two pt. formula

X $x_2 \neq 0$ ∵ $\lambda_1 x_1 = 0$ (from (1))
$\lambda_1 \neq 0 \Rightarrow x_1 = 0$
both nodes can't be zero ∵ it reduces to one pt. formula

X $x_2 - x_1 = 0$
$x_2 = x_1$
⇓
reduces to 1-pt formula.

$x_2 + x_1 = 0$
$x_2 = -x_1$

Numerical Quadrature:

from eqⁿ ② $\lambda_1 x_1 - \lambda_2 x_1 = 0$

$x_1(\lambda_1 - \lambda_2) = 0$

$x_1 \neq 0, \quad \lambda_1 = \lambda_2$

from eqⁿ ① $\lambda_1 + \lambda_1 = 2$

$2\lambda_1 = 2$

$\lambda_1 = 1 \Rightarrow \lambda_2 = 1$

from eqⁿ ③ $x_1^2 + x_1^2 = \frac{2}{3} \Rightarrow 2x_1^2 = \frac{2}{3} \Rightarrow x_1 = \sqrt{\frac{1}{3}}$

$x_2 = \frac{-1}{\sqrt{3}}$.

Therefore, two point formula is given by

$$\int_{-1}^{1} f(x)dx = 1f\left(\frac{1}{\sqrt{3}}\right) + 1f\left(\frac{-1}{\sqrt{3}}\right).$$

Numerical Quadrature:

Example:

Approximate the integral $\int_0^{\pi/4} (\cos x)^2 \, dx$ using Gauss-Legendre $1, 2$ and 3 point formula. Also compare with the exact value.

Solution:

Numerical Quadrature:

Exercise:

1. Evaluate the integral

$$\int_{-1}^{1} e^{-x^2} \cos x \; dx$$

by using the Gauss-Legendre one, two and three point formulas.

2. Evaluate

$$I = \int_{0}^{1} \frac{\sin x \; dx}{2 + x}$$

by subdividing the interval $[0, 1]$ into two equal parts and then by using Gauss-Legendre two point formula.