Lecture 10: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Newton's method

$$f(x)=0$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad f'(p_n) \neq 0$$

$$f'(p) \neq 0$$

$$f(x) = (x-2)^2$$

$$= (x-1)^2 \underbrace{(x^2-5x+6)}_{=q(x)} \qquad q(1) \neq 0$$

$$f(x) = (x^2-2x+1) \underbrace{(x-2)(x-3)}_{=(x^2-5x+6)} = 0$$

$x = 1, 1, 2, 3$

$f(1) = 0$ $\qquad f'(1) = 0$

$$\boxed{\begin{aligned} f'(x) &= (x^2-2x+1)(2x-5) \\ &\quad + (x^2-5x+6)(2x-2) \end{aligned}}$$

Multiple roots

Repeated Roots

Definition:

An equation $f(x) = 0$ has a root p with multiplicity m if for $x \neq p$, we can write $f(x) = \underbrace{(x-p)^m q(x)}$, $q(p) \neq 0$.

If $\underbrace{m = 1}$, then equation $f(x) = 0$ has a simple root at p.

$f(x) = (x-p)^m q(x)$

$f(p) = 0$, $q(p) \neq 0$

$f'(x) = (x-p)^m q'(x) + q(x) \, m(x-p)^{m-1}$

$f'(p) = 0 + 0 = 0$

$f''(p) = 0$

$f'''(p) = 0$ —— $f^{m-1}(p) = 0$ $f^m(p) \neq 0$

$f(x) = (x-p) q(x)$

$q(p) \neq 0$

$f(p) = 0$

$f'(x) = (x-p) q'(x) + q(x) \cdot 1$

$f'(p) = 0 + q(p) \neq 0$

Multiple roots

$f, f' \in C[a,b]$

Result:

The function $f \in C^1[a, b]$ has a simple zero at p in $[a, b]$ iff $f(p) = 0$ but $f'(p) \neq 0$.

If $m=1$ $\quad f(x) = (x-p)\,g(x), \quad g(p) \neq 0$

$f(p)=0$

but $f'(p) \neq 0$

$\boxed{f(x)} \to$ zero

$\boxed{f(x)=0}$ Root

$f, f', f'', f''', \cdots f^m \in C[a,b]$

$g(p)=p$

$g'(p) \neq 0$

Generalized result:

The function $f \in C^m[a, b]$ has a zero of multiplicity m at p in $[a, b]$ iff $f(p) = 0,\ f'(p) = 0, \cdots, f^{m-1}(p) = 0$, but $f^m(p) \neq 0$.

$f(x) = (x-2)^2\,(x-3)$

$f(2)=0$

$f'(x) = (x-2)^2\,1 + (x-3)\,2(x-2) \to f''(x) = 2(x-2) + 2(x-3) + 2(x-2)$

$f'(2) = 0 + 0 = 0$

$f''(2) \neq 0$

Multiple roots

Remarks:

(i) The Newton's method (in which $f'(p) \neq 0$ is required) works for those functions which has a simple zero not for multiplicity.

(ii) Newton's method gives quadratically convergent sequence but quadratic convergence might not occur if the zero is not simple.

Multiple roots

Example:

Let $f(x) = e^x - x - 1$

a) Show that f has a zero (root) of multiplicity 2 ($m=2$) at $x = 0$. ($\Rightarrow p=0$)

b) Show that Newton's method with $p_0 = 1$ converges to $x = 0$ but not quadratically.

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$f(x) = e^x - x - 1$

$f(0) = e^0 - 0 - 1 = 0$

$f'(x) = e^x - 1$

$f'(0) = e^0 - 1 = 1 - 1 = 0$

$f''(x) = e^x$

$f''(0) = 1 \neq 0$

$\Rightarrow$ $f = 0$ has root at $x = 0$ with multiplicity 2

(b) Using N.M.

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad \text{or} \quad p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$p_{n+1} = p_n - \frac{e^{p_n} - p_n - 1}{e^{p_n} - 1}$$

Let $n=0$, $p_0 = 1$

$$p_1 = \boxed{p_0 - \frac{e^{p_0} - p_0 - 1}{e^{p_0} - 1}} \qquad p_0 = 1$$

$$p_1 = 1 - \frac{e^1 - 1 - 1}{e^1 - 1}$$

$$= 1 - \frac{e-2}{e-1} = \frac{\cancel{e} - 1 - \cancel{e} + 2}{e-1} = \frac{1}{e-1} = 0.58198$$

$$p_2 = 0.58198 - \frac{e^{0.58198} - 0.58198 - 1}{e^{0.58198} - 1}$$

$$= 0.31906$$

$$p_3 = 0.31906 - \frac{e^{0.31906} - 0.31906 - 1}{e^{0.31906} - 1}$$

$$= 0.16800$$

$$p_4 = 0.16800 - \frac{e^{0.16800} - 0.16800 - 1}{e^{0.16800} - 1}$$

$$= 0.08635$$

$p_5 = 0.04380$, $p_6 = 0.02206$

To check the order of cgce for $\alpha = 1$

$$\frac{|p_{n+1} - p|}{|p_n - p|^1} = |\lambda_n| < 1 \rightarrow \boxed{\text{we want}}$$ for linear cgce

let $\underbrace{n=0}$, $p = 0 \rightarrow$ exact root (given)

$$\frac{|p_1 - 0|}{|p_0 - 0|} = \frac{|p_1|}{|p_0|} = |0.58198| < 1$$

$$\frac{|p_2 - 0|}{|p_1 - 0|} = \left|\frac{p_2}{p_1}\right| = \frac{0.31906}{0.58198} < 1$$

$$\frac{p_3 - 0}{|p_2 - 0|} = \left|\frac{0.168\omega}{0.31906}\right| < 1$$

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$p_n \to p$ linearly.

To Check for $\alpha = 2$

$$\frac{|p_{n+1} - p|}{|p_n - p|^2}$$

$n = 0$, $\quad p = 0$

$$\frac{|p_1 - 0|}{|p_0 - 0|^2} = 0.58198 = \lambda_0$$

$$\frac{p_2 - 0}{|p_1 - 0|^2} = \frac{|0.31906|}{|0.58198|^2} = \lambda_1 = 0.9304$$

$$\frac{p_3 - 0}{|p_2 - 0|^2} = \frac{0.16800}{|0.31906|^2} = \lambda_2 = 1.65031$$

λ_n should be decreasing for quadratic cgce

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$p_n \to p$ not quadratically.

Multiple roots:

Exercise:

1. Apply the Newton's method with $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$, and verify that the convergence is only of first-order. Further show that root $\alpha = 1$ has multiplicity 2.

Multiple roots

Modified Newton's method (if multiplicity is given)

Modified Newton's method is given by

$$\boxed{p_{n+1} = p_n - m\frac{f(p_n)}{f'(p_n)}} = g(p_n)$$

$$p_{n+1} = g(p_n)$$

Multiple roots

Example:

Show that the modification of Newton's method improves the rate of convergence for $f(x) = e^x - x - 1$ at $x = 0$ with $p_0 = 1$.

M. N. M $\quad p_{n+1} = p_n - \frac{m f(p_n)}{f'(p_n)} \quad m=2$

$$p_{n+1} = p_n - 2 \frac{e^{p_n} - p_n - 1}{e^{p_n} - 1}$$

let $n=0$, $p_0 = 1$

$$p_1 = p_0 - 2 \frac{e^{p_0} - p_0 - 1}{e^{p_0} - 1} = 1 - 2\left(\frac{e^1 - 2}{e - 1}\right)$$

$$= \frac{e - 1 - 2e + 4}{e - 1}$$

$$= 0.163953$$

$$p_2 = 0.163953 - 2\frac{\left(e^{0.163953} - 0.163953 - 1\right)}{e^{0.163953} - 1}$$

$$= 0.0044779$$

$$p_3 = 0.0000033419$$

To check order of conv

$$\frac{|p_{n+1} - p|}{|p_n - p|^2} \qquad n = 0, \quad p = 0$$

$$\frac{p_1 - 0}{|p_0 - 0|^2} = 0.16395$$

$$\frac{|p_2|}{|p_1|^2} = \frac{|0.0044779|}{0.16395} =$$

Multiple roots:

Exercise:

1. Use Newton's method and the modified Newton's method to find a solution of

$$(1-x)\sin(1-x)=0,$$

accurate to within 10^{-2}. Take initial approximation $x_0=0$.

2. Apply modified Newton's method with $m=2$ and $x_0=0.8$ to the equation $f(x)=x^3-x^2-x+1=0$, and verify that the convergence is of second-order.