Lecture 9: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Order of Convergence of a sequence $\langle p_n \rangle$

$p_n \to p$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda \neq 0$$

$p_n \to p$ with order α

for $\alpha = 1$, $\lambda_i < 1$ $0 \leq i \leq n$

$$\begin{cases} \alpha = 2 \\ \alpha = 3 \\ \alpha = 4 \\ \cdots \end{cases}$$

$\lambda_0, \lambda_1, \lambda_2, \lambda_3 \; - - -$ decreasing seq^n

Order of convergence:

Order of convergence of fixed point iteration method:

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, $\forall x \in [a,b]$. Suppose, in addition, that g' is continuous on $[a,b]$ and a positive constant $k < 1$ exists with $|g'(x)| \le k < 1$, for all $x \in (a,b)$

$p_n = g(p_{n-1})$
$\Rightarrow p_n \to p$

(i) If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, then the sequence $p_n = g(p_{n-1})$, $n \geq 1$ converges ~~only~~ linearly to the unique fixed point p in $[a,b]$.

$p \to$ exact fixed pt   atleast

Using Taylor's poly of $g(x)$ at a pt. p $[p_n, p]$

$$g(p_n) \approx g(p) + g'(c_n)(p_n - p)$$

$p_n \le c_n \le p$, $p_n \to p$, $c_n \to p$

$a_n \le b_n \le c_n$, $a_n \to p$, $b_n \to p$, $c_n \to p$

$$p_{n+1} = p + g'(c_n)(p_n - p)$$

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{|p_n - p|^1} \right| = \lim_{n\to\infty} |g'(c_n)| = |g'(p)| < 1 \quad \because p \in (a,b)$$

$\nexists$ also if g' is cont. $+$ $c_n \to p$

$\Rightarrow \quad p_n \to p$ linearly. $\qquad g(c_n) = g'(p)$

Order of convergence:

Order of convergence of fixed point iteration method:

(ii) If $g'(p) = 0$ and $g''(x)$ is continuous function with $|g''(x)| < M$ on an open neighbourhood of p, the there exists a $\delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$ the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$\boxed{|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.}$$

Using Taylor's poly. of $g(x)$ at a pt. p up to 2nd order

$$g(p_n) = g(p) + (p_n - p)\, g'(p) + \frac{(p_n-p)^2}{2!}\, g''(c_n)$$

$p_n < c_n < p$

$$p_{n+1} = p + 0 + \frac{(p_n-p)^2}{2}\, g''(c_n)$$

$\downarrow \quad \Downarrow \quad \downarrow$

$p \qquad\qquad p$

p $\qquad$ p

g'' is cont

$\Rightarrow g''(c_n) \to g''(p)$

$$p_{n+1} - p = \frac{(p_n - p)^2}{2} g''(c_n)$$

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{(p_n - p)^2} \right| = \lim_{n\to\infty} \frac{|g''(c_n)|}{2} = \frac{|g''(p)|}{2} < M$$

$\Rightarrow p_n \to p$ quadratically.

$$|p_{n+1} - p| < \frac{|p_n - p|^2}{2} M$$

$\because \left(|g''(c_n)| < M \right.$ (given) $\left. \right)$

Order of convergence:

Order of convergence of fixed point iteration method:

In general, if $g'(p) = 0$, $g''(p) = 0, \cdots, g^{m-1}(p) = 0$, then the sequence defined by $p_n = g(p_{n-1})$, when $n \geq 1$, converges at least of order m to p.

$g^m(p) \neq 0$

$p_n \to p$ with order $\alpha = m$ (exact)

Order of convergence:

p is the exact root of $f(x)=0$

ie $f(p)=0$

$f'(p)\neq 0$

Order of convergence of Newton's method:

$$p_{n+1} = \underbrace{p_n - \frac{f(p_n)}{f'(p_n)}}_{} = g(p_n)$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g(p) = p - \frac{f(p)}{\text{non-zero}} = p - \frac{0}{\text{non-zero}} = p$$

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}$$

$$g'(x) = \cancel{1} - \cancel{1} + \frac{f(x)f''(x)}{(f'(x))^2}$$

Put $x=p$

$$g'(p) = \frac{\overset{=0}{f(p)}\, f''(p)}{\underset{\neq 0}{(f'(p))^2}} = 0$$

$p_n \to p$ atleast quadratically.

Order of convergence:

Example:

Given that the iterates $x_{n+1} = \frac{2}{3}x_n + \frac{a}{3x_n^2}$, $a \in \mathbb{R}$ converges to $p = a^{1/3}$. Find the order of convergence of the iteration scheme.

$$p_{n+1} = \frac{2}{3}p_n + \frac{a}{3p_n^2} \quad \to \quad p = a^{1/3}$$

$$= g(p_n)$$

$$\Rightarrow g(x) = \frac{2}{3}x + \frac{a}{3x^2}$$

$$g(a^{1/3}) = \frac{2}{3}a^{1/3} + \frac{a}{3a^{2/3}}$$

$$= \frac{2}{3}a^{1/3} + \frac{a^{1/3}}{3}$$

$$= a^{1/3}$$

$$g'(x) = \frac{2}{3} - \frac{2}{3}\frac{a}{x^3}$$

$x = a^{1/3}$

$$g'(a^{1/3}) = \frac{2}{3} - \frac{2}{3}\frac{\cancel{a}}{\cancel{a}} = 0$$

Order of convergence:

$g''(x) = 0 + \frac{2}{3}\frac{3a}{x^4}$

$g''(a^{1/3}) \neq 0 \implies x_{n+1} \to x = a^{1/3}$
with order 2

Exercise:

1. What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a} = g(x_n), \; a \in \mathbb{R}$$

as it converges to the fixed point $p = \sqrt{a}$?

2. The iterates $x_{n+1} = 2 - (1 + c)x_n + cx_n^3$ converges to $p = 1$ for some values of constant c (provided that x_0 is sufficiently close to p). For what values of c, if any, convergence is quadratic.

$f(x) = x^2 - 2x + 1 = 0$

$x = 1, 1$

Repeated multiple roots.

$f(x) = (x^2 - 2x + 1)(x-3)(x-4)$

$x = 1, 1$

$f(x) = (x-1)^2 \, q(x) \qquad q(1) \neq 0$

$f(x) = 0$ has a root at $x = p$ which is repeating 2 times

$f(x) = (x-p)^2 \, q(x) \qquad q(p) \neq 0$

If repition is m

$f(x) = (x-p)^m \, q(x), \qquad q(p) \neq 0$

Multiple roots (Repeated Roots)

Definition:

An equation $f(x) = 0$ has a root p with multiplicity m if for $x \neq p$, we can write $f(x) = (x - p)^m q(x)$, $q(p) \neq 0$.

If $m = 1$, then equation $f(x) = 0$ has a simple root at p.

Multiple roots

Result:

The function $f \in C^1[a, b]$ has a simple zero at p in $[a, b]$ iff $f(p) = 0$ but $f'(p) \neq 0$.

Generalized result:

The function $f \in C^m[a, b]$ has a zero of multiplicity m at p in $[a, b]$ iff $f(p) = 0$, $f'(p) = 0, \cdots, f^{m-1}(p) = 0$, but $f^m(p) \neq 0$.

Multiple roots

Remarks:

(i) The Newton's method (in which $f'(p) \neq 0$ is required) works for those functions which has a simple zero not for multiplicity.

(ii) Newton's method gives quadratically convergent sequence but quadratic convergence might not occur if the zero is not simple.