Lecture 25: Numerical Linear Algebra (UMA021): Matrix Algebra

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System of linear equations

LU Factorization:

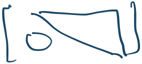
An another method to find the solution of linear system of equations

$$AX = b$$

$$\boxed{A} \to LU$$

lower triangular matrix

Upper triangular.



$$A = \begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_n \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

$$a_{11} \neq 0$$

$$E_2 \to E_2 - \underbrace{\frac{a_{21}}{a_{11}}}_{= l_{21}} E_1$$

$$E_3 \to E_3 - \underbrace{\frac{a_{31}}{a_{11}}}_{= l_{31}} E_1$$

$$- \quad - \quad -$$

$$E_n \to E_n - \underbrace{\frac{a_{n1}}{a_{11}}}_{= l_{n1}} E_1$$

$$\begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_n \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a^*_{22} & \cdots & a^*_{2n} \\ 0 & & & \\ \vdots & & & \\ 0 & a^*_{n2} & \cdots & a^*_{nn} \end{bmatrix}$$

$$E_3 \to E_3 - \frac{a^*_{32}}{a^*_{22}} E_2 \qquad = l_{32}$$

$$E_4 \to E_4 - \frac{a^*_{42}}{a^*_{22}} E_2 \qquad = l_{42}$$

$$\sim \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ 0 & a^*_{22} & a^*_{32} & \cdots & a^*_{n2} \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & a^{**}_{n2} & \cdots & a^{**}_{nn} \end{bmatrix}$$

$$E_n \to E_n - \frac{a^*_{n2}}{a^*_{22}} E_2 \qquad = l_{n2}$$

$$E_j \to E_j - \frac{a_{ji}}{a_{ii}} E_i \qquad = l_{ji}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a^*_{22} & \cdots & a^*_{2n} \\ 0 & 0 & & \\ \vdots & & & \\ 0 & 0 & \cdots & a^*_{nn} \end{bmatrix} = U \text{ (say)}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & 0 & & \cdots & 0 \\ l_{31} & l_{32} & 1 & 0 & & \cdots & 0 \\ \vdots & & & & & & \\ l_{n1} & l_{n2} & 0 & & \cdots & 0 & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}}, \quad l_{ji} = \frac{a_{ji}}{a_{ii}}$$

$$L * U = A$$

$$\boxed{AX = b} = (LU)X = b$$

$$L(UX) = b$$

$$\boxed{LY = b} \quad \because \quad \text{let } \boxed{UX = Y}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & 0 & & \cdots & 0 \\ l_{31} & l_{32} & 1 & 0 & \cdots & 0 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

find Y by using forward sub.

$$y_1 = b_1, \quad l_{21} y_1 + y_2 = b_2$$

$\parallel^{ly}$ find y_n

After getting y vector, take

$$UX = y$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}}_{\text{known}}$$

Use back substitution to find vector X.

System of linear equations

Example:

Determine the LU factorization for matrix A in the linear system $Ax = b$, where $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$. Then use the factorization to solve the system.

Solution

$$A = \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$$

$E_2 \to E_2 - 2E_1$, $E_3 \to E_3 - 3E_1$, $E_4 \to E_4 + E_1$

$$\sim \begin{array}{c} E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{bmatrix}$$

$E_3 \to E_3 - 4E_2$, $E_4 \to E_4 + 3E_2$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & \boxed{0} & -13 \end{bmatrix} = U \text{ (say)}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}$$

System of linear equations

Solution:

$$A x = b$$

$$(L U) x = b$$

$$L(U x) = b$$

Let $U x = y$

$$L y = b$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

Use forward sub.

$$y_1 = 1$$

$$2y_1 + y_2 = 1$$

$$2 + y_2 = 1 \Rightarrow y_2 = -1$$

$$3y_1 + 4y_2 + y_3 = -3$$

$$3 - 4 + y_3 = -3$$

$$y_3 = -3 + 1 = -2$$

$$-y_1 - 3y_2 + y_4 = 4$$

$$-1 - 3(-1) + y_4 = 4$$

$$y_4 = 2 \qquad \Rightarrow Y = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$$

Now $UX = Y$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$$

Use back sub. $x_4 = \frac{-2}{13}$, $3x_3 + 13x_4 = -2$

$$3x_3 - 2 = -2$$

$x_3 = 0$

System of linear equations:

$x_1 + x_2 + 3x_4 = 1$

$x_1 + \frac{23}{13} - \frac{6}{13} = 1$

$x_1 = -4/13$

$-x_2 - x_3 - 5x_4 = -1$

$-x_2 - 0 + \frac{10}{13} = -1$

$X = \left[\frac{-4}{13}, \frac{23}{13}, 0, \frac{-2}{13}\right]^t$ Ans

$-x_2 = -1 - \frac{10}{13}$

$x_2 = \frac{23}{13}$

Exercise:

1. Modify the *LU* Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear system:

$$
\begin{aligned}
2x_1 - x_2 + x_3 &= -1 \\
3x_1 + 3x_2 + 9x_3 &= 0 \\
3x_1 + 3x_2 + 5x_3 &= 4.
\end{aligned}
$$

$\boxed{\text{Norm}}$

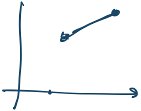
$\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

ℓ_2 - Norm

$\boxed{\ell_\infty \text{ - Norm}}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$



Iterative methods to solve System of linear equations:

Distance between $n-$dimensional vectors

To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between $n-$dimensional column vectors.

Norms

Vector norms

Let $\mathbb{R}^n$ denote the set of all $n-$dimensional column vectors with real-number components.

To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Norms

Vector norms

We will need only two specific norms on $\mathbb{R}^n$,

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|.$$

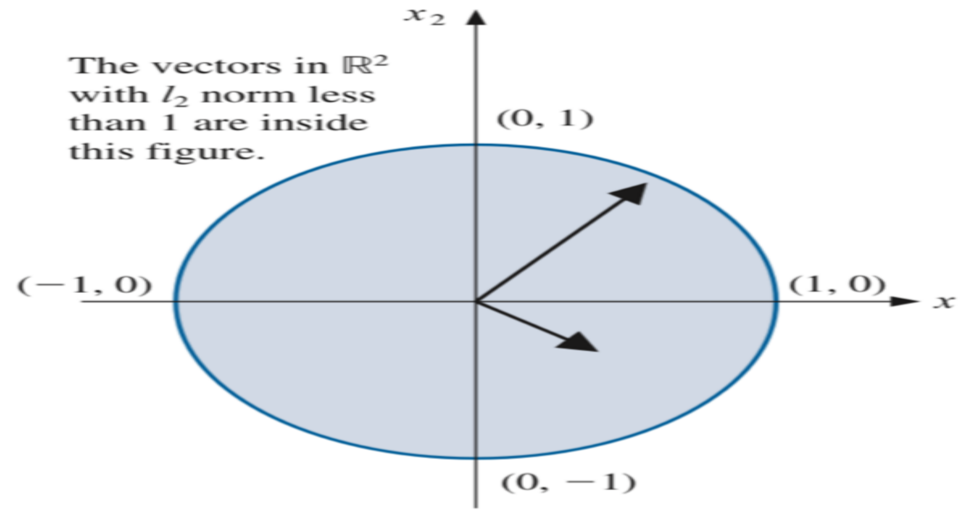
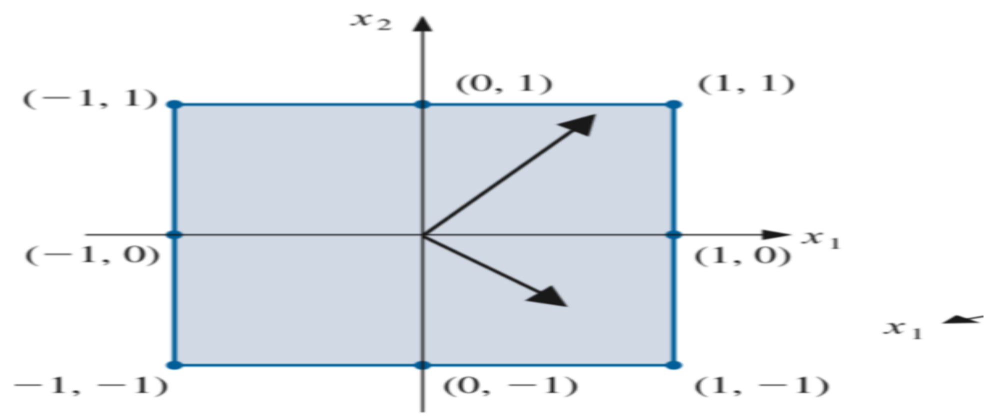


Figure 7.2



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.

Norms

Example:

Determine the l_2 norm and the l_∞ norm of the vector $x = (-1, 1, -2)^t$.

Norms

Distance between Vectors in $\mathbb{R}^n$:

If $x = (x_1, x_2, \cdots, x_n)^t$ and $y = (y_1, y_2, \cdots, y_n)^t$ are vectors in $\mathbb{R}^n$, then l_2 and l_∞ distances between x and y are defined by

$$\|x - y\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|x - y\|_\infty = \max_{1 \le i \le n} |x_i - y_i|.$$

Norms

Convergence of a sequence in $\mathbb{R}^n$:

The sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \cdots, n$.

Convergence of a sequence in $\mathbb{R}^n$

Example:

Show that
$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t = \left(1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin(k)\right)^t$
converges to $x = (1, 2, 0, 0)^t$ with respect to l_∞ norm.

System of linear equations:

Exercise:

1. Find l_∞ and l_2 norms of the vectors.
   a) $x = (3, -4, 0, \frac{3}{2})^t$.
   b) $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.
2. Find the limit of the sequence $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^t = \left(\frac{4}{k+1}, \frac{2}{k^2}, k^2 e^{-k}\right)^t$ with respect to l_∞ norm.