Numerical Analysis

Solution of Exercises : Chapter 2[1]
Roots of Non-linear Equations in One Variable

1. Use the bisection method to find solutions accurate to within 10^{-3} for the following problems.

(a) $x - 2^{-x} = 0$ for $0 \leq x \leq 1$.

(b) $e^x - x^2 + 3x - 2 = 0$ for $0 \leq x \leq 1$.

(c) $x + 1 - 2\sin(\pi x) = 0$. for $0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$.

Sol.

(a) Let $f(x) = x - 2^{-x} = 0$, which is continuous. We have $f(0)f(1) < 0$. Thus we take $a = 0$ and $b = 1$. Approximations to the roots (middle points) are given in Table:

n	a	$c = (a+b)/2$	b	sign$(f(a).f(c))$	$\|c_{\text{new}} - c_{\text{old}}\|$
1	0	0.5	1	+	
2	0.5	0.75	1	−	0.25
3	0.5	0.625	0.75	−	0.125
4	0.625	0.6875	0.75	+	0.0625
5	0.625	0.65625	0.6875	−	0.03125
6	0.625	0.640625	0.65625	+	0.015625
7	0.640625	0.648438	0.65625	−	0.007813
8	0.640625	0.644531	0.648438	−	0.003907
9	0.640625	0.642578	0.644531	−	0.001953
10	0.640625	0.641602	0.642578	−	**0.000976 < 0.001**

We accept 0.641602 as root with given accuracy.

(b) Let $f(x) = e^x - x^2 + 3x - 2 = 0$, which is continuous. Root lies in $[0, 1]$ as $f(0)f(1) < 0$. Thus we take $a = 0$ and $b = 1$. Approximations to the roots (middle points) are given in the following Table.
Let initial choices are $a = 0$, $b = 1$, such that $f(0)f(1) < 0$.
Then $c = \dfrac{a+b}{2}$.
Approximations to roots (middle points) are given as follows:

n	a	$c = (a+b)/2$	b	sign$(f(a).f(c))$	$\|c_{\text{new}} - c_{\text{old}}\|$
1	0	0.5	1	−	
2	0	0.25	0.5	+	0.25
3	0.25	0.375	0.5	−	0.125
4	0.25	0.3125	0.375	−	0.0625
5	0.25	0.28125	0.3125	−	0.03125
6	0.25	0.265625	0.28125	−	0.015625
7	0.25	0.257812	0.265625	−	0.007813
8	0.25	0.253906	0.257812	+	0.003906
9	0.253906	0.255859	0.257812	+	0.001953
10	0.255859	0.256836	0.257812	+	**0.000976 < 0.001**

We accept 0.256836 as root with preassigned accuracy.

2. Using the bisection method, determine the point of intersection of the curves given by $y = 3x$ and $y = e^x$ in the interval $[0, 1]$ with an accuracy 0.1.
Sol. We consider $f(x) = 3x - e^x = 0$, which is continuous. Root lies in $[0, 1]$ as $f(0)f(1) < 0$. Thus we take $a = 0, b = 1$. We have:

$$
\begin{aligned}
c_1 &= 0.75 \\
c_2 &= 0.625 \\
c_3 &= 0.5625
\end{aligned}
$$

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As$|c_3 - c_2| = 0.0625 < 0.1$, so x coordinate of the point with given accuracy is $x = 0.625$. Thus $y = 3x = 1.875$. So point of intersection of the curves is $(0.625, 1.875)$.

3. Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-2} using the bisection algorithm.

Sol. Let $f(x) = x^3 - 25 = 0$ which is continuous. Also $f(2)f(3) < 0$ which implies that root lies in interval $(2, 3)$.

Thus we take $a = 2$, $b = 3$, $c = \dfrac{a+b}{2}$ and bracket the root at each stage. Below are the iterations to get an accuracy of 10^{-3}.

n	a	$c = (a+b)/2$	b	sign$(f(a).f(c))$	$\lvert c_{\text{new}} - c_{\text{old}}\rvert$
1	2	2.5	3	+	
2	2.5	2.75	3	+	0.25
3	2.75	2.875	3	+	0.125
4	2.875	2.9375	3	−	0.0625
5	2.875	2.90625	2.9375	+	0.03125
6	2.90625	2.921875	2.9375	+	0.015625
7	2.921875	2.929688	2.9375	−	$\mathbf{0.007813 < 0.01}$

We accept 2.929688 as a root with given accuracy.

4. Find a bound for the number of iterations needed to achieve an approximation by bisection method with accuracy 10^{-2} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

Sol. Let $f(x) = x^3 - x - 1 = 0$ and $f(1)f(2) < 0$. Also f is a continuous function.

Thus we take $a = 1$, $b = 2$, and then we calculate the middle points $c = \dfrac{a+b}{2}$ by bracketing the root.

Number of iterations required are $N \geq \dfrac{\log(2-1) - \log(10^{-2})}{\log 2} \approx 6.6438$.

Thus seven iterations are required to achieve an accuracy of 10^{-2}, which are given below in table. So root

n	a	$c = (a+b)/2$	b	sign$(f(a).f(c))$	$\lvert c_{\text{new}} - c_{\text{old}}\rvert$
1	1	1.5	2	−	
2	1	1.25	1.5	+	0.25
3	1.25	1.375	1.5	−	0.125
4	1.25	1.3125	1.375	+	0.0625
5	1.3125	1.34375	1.375	−	0.03125
6	1.3125	1.328125	1.34375	−	0.015625
7	1.3125	1.320313	1.328125	+	$\mathbf{0.007812 < 0.01}$

with given accuracy is 1.320313.

5. Sketch the graphs of $y = x$ and $y = 2\sin x$. Use the bisection method to find an approximation to within 10^{-2} to the first positive value of x with $x = 2\sin x$.
Sol. The graphs of $y = x$ and $y = 2\sin x$ has shown in the next Figure. They intersect near 2.
More precisely $f(1)f(2) < 0$, thus first positive root lies in $(1, 2)$.
Iterations with desired accuracy are given below. So root with given accuracy is 1.8984375.

6. The function defined by $f(x) = \sin(\pi x)$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the bisection method converges to

(a) 0, if $a + b < 2$

(b) 2, if $a + b > 2$

(c) 1, if $a + b = 2$.

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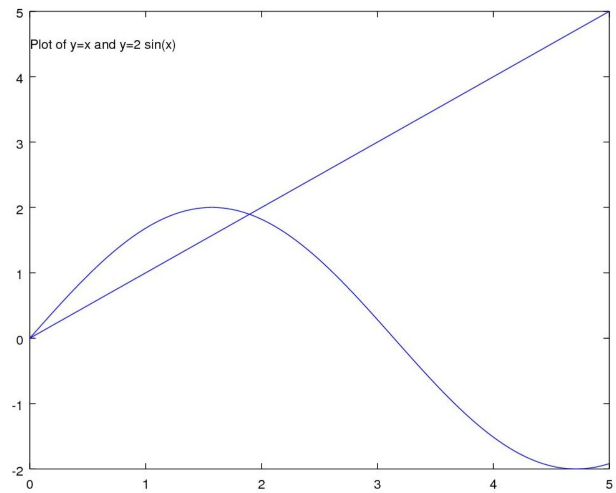

n	a	$c = (a+b)/2$	b	sign$(f(a).f(c))$	$\lvert c_{\text{new}} - c_{\text{old}}\rvert$
1	1	1.5	2	+	
2	1.5	1.75	2	+	0.25
3	1.75	1.875	2	+	0.125
4	1.875	1.9375	2	−	0.0625
5	1.875	1.90625	1.9375	−	0.03125
6	1.875	1.890625	1.90625	+	0.015625
7	1.890625	1.8984375	1.90625	−	$\mathbf{0.0078125 < 0.01}$

Sol.

(a) In this case $a + b < 2$.
The first step is to add the two inequalities given in the problem to get $1 < a + b < 3$ and note that

$$c = a + \frac{b-a}{2} = \frac{a+b}{2}.$$

Since $\sin(\pi a) < 0$ and

$$1 < a + b < 2 \implies 1/2 < c < 1, \implies c \in (1/2, 1) \implies f(c) = \sin(\pi c) > 0.$$

Thus

$$f(a) \cdot f(c) < 0 \implies b = c \in (1/2, 1).$$

The algorithm will now search within the interval $[a, c]$ and 0 is the only root within this new interval.

(b) In this case $a + b > 2$, $f(a) = \sin(\pi a) < 0$ and

$$3 > a + b > 2 \implies 3/2 > c > 1, \quad \implies c \in (1, 3/2) \implies f(c) = \sin(\pi c) < 0.$$

Thus

$$f(a) \cdot f(c) > 0 \implies a = c \in (1, 3/2).$$

The algorithm will now search within the interval $[c, b]$ and 2 is the only root within this new interval.

(c) In this case $a + b = 2$ and $f(a) = \sin(\pi a) < 0$.

$$a + b = 2 \implies c = 1, \; f(c) = \sin(\pi c) = 0.$$

The algorithm will terminate with $c = 1$ as the root.

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7. Show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\dfrac{1}{3}, 1\right]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-2}.

Sol. The function $g(x) = 2^{-x}$ is continuous. Further

$$\begin{aligned}
g(1/3) &= 0.793700526 \\
g(1) &= 0.5 \\
g'(x) &= -2^{-x}\ln 2 \\
|g'(1/3)| &= 0.550151282 \\
|g'(1)| &= 0.34657 \\
g''(x) &= 2^{-x}\ln^2 2 \neq 0.
\end{aligned}$$

Thus $g(x) \in \left[\dfrac{1}{3}, 1\right]$ and $|g'(x)| \leq 0.550151282 < 1$. Hence $g(x)$ has a unique fixed point in $\left[\dfrac{1}{3}, 1\right]$. Now with $x_0 = 0.5$, we have

$$\begin{aligned}
x_1 &= 0.70711 \\
x_2 &= 0.61255 \\
x_3 &= 0.65404 \\
x_4 &= 0.63550 \\
x_5 &= 0.64372 \\
|x_5 - x_4| &= 0.00822 < 0.01.
\end{aligned}$$

So root is 0.64372.

8. For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-2}, and perform the calculations.

(a) $x = \dfrac{5}{x^2} + 2$.

(b) $2 + \sin x - x = 0$ in interval $[2, 3]$.

(c) $3x^2 - e^x = 0$.

Sol.

(a) Here $f(x) = \dfrac{5}{x^2} + 2 - x$ and we take $g(x) = \dfrac{5}{x^2} + 2$.
As $f(2)f(3) < 0$, so root lies in $(2, 3)$. We have

$$\begin{aligned}
g'(x) &= -\frac{10}{x^3} \\
|g'(2)| &= 1.25 > 1 \\
|g'(2.5)| &= 0.64 < 1 \\
|g'(3)| &= 0.37037 \\
g''(x) &= -\frac{10}{x^3} \neq 0.
\end{aligned}$$

As root lies in $(2, 3)$, we have to restrict the interval and need to consider $(2.5, 3)$ in order to $|g'(x) < 1|$.
Also $k = \max\limits_{x \in [2.5, 3]} |g'(x)| = 0.64$. Further

$$\begin{aligned}
g(2.5) &= 2.8 \\
g(3) &= 2.5556.
\end{aligned}$$

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Thus $g(x) \in [2.5, 3]$ for $x \in [2.5, 3]$. Thus the choice of g will work. We start with $x_0 = 2.5$ and get $x_1 = g(x_0) = 2.8$. We can estimate the number of iteration from the formula

$$|\alpha - x_n| \le \frac{k^n}{1-k}|x_1 - x_0| < 10^{-2}$$

which gives

$$\begin{aligned} \frac{0.64^n}{0.36}(0.3) &< 10^{-2} \\ n &> 9.9103 \approx 10. \end{aligned}$$

We have

$$\begin{aligned} x_1 &= 2.8 \\ x_2 &= 2.63775510204082 \\ x_3 &= 2.71862291377498 \\ x_4 &= 2.67650663176068 \\ x_5 &= 2.69796453650544 \\ x_6 &= 2.68690634940763 \\ x_7 &= 2.69257202494715 \\ |x_7 - x_6| &< 0.01. \end{aligned}$$

We get the root in 7 iterations with given accuracy although theoretical estimate shows that we need to perform minimum 10 iterations. By starting near root, we can get a more closer estimate.

(b) We consider $f(x) = 2 + \sin x - x$. Root lies in $[2, 3]$ as $f(2)f(3) < 0$. Let us take initial guess $x_0 = 2.5$. We take $g(x) = 2 + \sin x$ and get

$$\begin{aligned} g'(x) &= \cos x \\ |g'(2)| &= 0.4161 \\ |g'(3)| &= 0.9900 \\ g''(x) &= -\sin x \ne 0. \end{aligned}$$

So $|g'(x) < 1|$ for $x \in [2, 3]$. Also $k = \max_{x\in[2,3]} |g'(x)| = 0.99$. Further

$$\begin{aligned} g(2) &= 2.9093 \\ g(3) &= 2.1411. \end{aligned}$$

Thus $g(x) \in [2, 3]$ for $x \in [2, 3]$. Thus the choice of g will work. We start with $x_0 = 2.5$ and get $x_1 = g(x_0) = 2.5985$. We can estimate the number of iteration from the formula

$$|\alpha - x_n| \le \frac{k^n}{1-k}|x_1 - x_0| < 10^{-2}$$

which gives

$$\begin{aligned} \frac{0.99^n}{1-0.99}|2.5985 - 2.5| &< 10^{-2} \\ n &> 685.81 \approx 686. \end{aligned}$$

We have

$$\begin{aligned} &x_1 = 2.5985, x_2 = 2.5168, x_3 = 2.5849, x_4 = 2.5284 \\ &x_5 = 2.5755, x_6 = 2.5363, x_7 = 2.5690, x_8 = 2.5418 \\ &x_9 = 2.5644, x_{10} = 2.5456, x_{11} = 2.5613, x_{12} = 2.5483 \\ &x_{13} = 2.5591, x_{14} = 2.5501. \end{aligned}$$

In 14 iterations we get the solution 2.5501 correct to within 10^{-2}.

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(c) The equation $3x^2 - e^x = 0$ have two positive roots, one in $[0,1]$ and other in $[3,4]$. There are numerous possibilities to write $g(x)$ and for $g(x) = \sqrt{\frac{1}{3}e^x}$, on $[0,1]$ with $x_0 = 1$, we have

$$\begin{aligned} g'(x) &= \frac{1}{2\sqrt{3}}e^{x/2} \\ |g'(0)| &= 0.28868 \\ |g'(1)| &= 0.47594 \\ g''(x) &= \frac{1}{4\sqrt{3}}e^{x/2} \neq 0. \end{aligned}$$

Thus $|g'(x) < 1|$. Also $k = \max_{x\in[0,1]} |g'(x)| = 0.47594$. Further

$$\begin{aligned} g(0) &= 0.57735 \\ g(1) &= 0.95189. \end{aligned}$$

Thus $g(x) \in [0,1]$ for $x \in [0,1]$. Thus the choice of g will work. We start with $x_0 = 0.5$ and get $x_1 = g(x_0) = 0.74133$. We can estimate the number of iteration from the formula

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0| < 10^{-2}$$

which gives

$$\begin{aligned} \frac{0.47594^n}{1-0.47594}|0.74133 - 0.5| &< 10^{-2} \\ n &> 5.1581 \approx 6. \end{aligned}$$

We have

$$\begin{aligned} x_1 &= 0.74133 \\ x_2 &= 0.83641 \\ x_3 &= 0.87713 \\ x_4 &= 0.89517 \\ x_5 &= 0.90328 \\ |x_5 - x_4| &= 0.00811 < 0.01. \end{aligned}$$

Root is 0.90328.
For another root in $[3,4]$, we take $g(x) = \ln 3x^2$ with $x_0 = 3.5$,

$$\begin{aligned} g'(x) &= \frac{2}{x} \\ |g'(3)| &= 0.66667 \\ |g'(4)| &= 0.5 \\ g''(x) &= -\frac{2}{x^2} \neq 0. \end{aligned}$$

Thus $|g'(x) < 1|$. Also $k = \max_{x\in[0,1]} |g'(x)| = 0.66667$. Further

$$\begin{aligned} g(3) &= 3.2958 \\ g(4) &= 3.8712. \end{aligned}$$

Thus $g(x) \in [3,4]$ for $x \in [3,4]$. We start with $x_0 = 3.5$ and get $x_1 = g(x_0) = 3.6041$. We can estimate the number of iteration from the formula

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0| < 10^{-2}$$

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which gives

$$\begin{aligned}\frac{0.66667^n}{1-0.66667}|3.6041-3.5| &< 10^{-2}\\ n &> 8.4876 \approx 9.\end{aligned}$$

We have

$$\begin{aligned}x_1 &= 3.6041\\ x_2 &= 3.6628\\ x_3 &= 3.6951\\ x_4 &= 3.7126\\ x_5 &= 3.7221\\ |x_5-x_4| &< 0.01.\end{aligned}$$

Even we get the root in five iterations and root is 3.7221.

9. Show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Also estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

Sol. Since $g'(x) = \frac{1}{4}\cos(x/2)$, g is continuous and g' exists on $[0, 2\pi]$.
Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi$ and $g(\pi) = \pi + 1/2$ and thus $g(x) \in [0, 2\pi]$.
$|g'(x)| \le 1/4$, for $0 \le x \le 2\pi$.
Thus a unique fixed point exists in $[0, 2\pi]$. With $k = 1/4$ and $x_0 = \pi$, we have $x_1 = \pi + 1/2$. Now

$$|\alpha - x_n| \le \frac{k^n}{1-k}|x_1 - x_0| < 10^{-2}.$$

For the bound to be less than 0.01, we need $n \ge 4$.
Iterations are given by

$$\begin{aligned}x_1 &= 3.6416\\ x_2 &= 3.6260\\ x_3 &= 3.6270\\ |x_3-x_2| &< 0.01.\end{aligned}$$

Thus $x_3 = 3.6270$ is accurate to within 0.01.

10. Use the fixed-point iteration method to find smallest and second smallest positive roots of the equation $\tan x = 4x$, correct to 4 decimal places.

Sol. Let $f(x) = \tan x - 4x = 0$. In this case we write

$$x = \tan^{-1}(4x) + n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$$

For first positive root, we take $n = 0$ and thus $g(x) = \tan^{-1}(4x)$. Clearly $g(x)$ is continuous and root lies before $\pi/2$. So we take $x_0 = 1.5$ and we have

$$\begin{aligned}x_1 &= 1.4056476\\ x_2 &= 1.3947829\\ x_3 &= 1.3934402\\ x_4 &= 1.3932729\\ x_5 &= 1.3932520.\end{aligned}$$

So smallest positive root correct to four decimals is 1.3932.
For another root which lies before $3\pi/2$, we start with $x_0 = 4.5$ and we take $g(x) = \tan^{-1}(4x) + \pi$.

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Iterations are given by

$$
\begin{aligned}
x_1 &= 4.6568904 \\
x_2 &= 4.6587566 \\
x_3 &= 4.6587780
\end{aligned}
$$

So second smallest positive root correct to four decimals is 4.6588.

11. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-2}.

Sol. A closer look to $f(x) = x^2 + 10\cos x = 0$ or $x^2 = -10\cos x$, shows that this equation has four roots. Two roots are negative which lies in $[-4, -3]$ and $[-2, -1]$.
By symmetry of both x^2 and $-10\cos x$, two positive roots lies in $[1, 2]$ and $[3, 4]$.
There are many possibilities to write $g(x)$. In particular, we take

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 + 10\cos x}{2x - 10\sin x}.$$

Iterations scheme is given by

$$x_{n+1} = g(x_n) = x_n - \frac{x_n^2 + 10\cos x_n}{2x_n - 10\sin x_n}.$$

For root lies in $[-4, -3]$, we take $x_0 = -3$. We have

$$
\begin{aligned}
x_1 &= -3.1961 \\
x_2 &= -3.1630 \\
x_3 &= -3.1620 \\
|x_3 - x_2| &< 0.01.
\end{aligned}
$$

Thus root accurate to within 10^{-2} for the interval $[-4, -3]$ is -3.1620.
Similarly for root lies in $[-2, -1]$, we take initial guess $x_0 = -1.5$. Then iterations are given by

$$
\begin{aligned}
x_1 &= -1.9240 \\
x_2 &= -1.9679 \\
x_3 &= -1.9689 \\
|x_3 - x_2| &= 0.001 < 0.01.
\end{aligned}
$$

Thus root accurate to within 10^{-2} for the interval $[-2, -1]$ is -1.9689.
Just reversing the sign, we get other roots. So positive root in $[1, 2]$ is 1.9689 and second positive root lies in $[3, 4]$ is 3.1620.

12. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

(a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $\alpha = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

(b) Find an interval about $1/A$ for which fixed-point iteration converges, provided x_0 is in that interval.

Sol.

(a) Fixed-point iterations

$$x_{n+1} = g(x_n), \; n = 1, 2, \cdots.$$

If fixed-point converges to the limit α, then

$$\alpha = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (2x_n - Ax_n^2) = 2\alpha - A\alpha^2.$$

Solving for α gives $\alpha = 0, 1/A$. Thus if fixed-point iteration converges to a nonzero limit, then the limit is $\alpha = 1/A$, so the inverse of a number can be found using only multiplications and subtractions from $g(x) = 2x - Ax^2$..

(b) Any subinterval $[a, b]$ of having root $\alpha = \frac{1}{A}$ will work. For example, we take $\left[\frac{3}{4A}, \frac{5}{4A}\right]$ containing $1/A$ suffices. Now $g(x)$ is continuous and $g'(x)$ exists and we have

$$\begin{aligned} g'(x) &= 2 - 2Ax = 0 \quad \implies \quad x = \frac{1}{A} \\ g\left(\frac{1}{A}\right) &= \frac{1}{A} \\ g\left(\frac{3}{4A}\right) &= \frac{15}{16A} \\ g\left(\frac{5}{4A}\right) &= \frac{15}{16A} \\ g'\left(\frac{3}{4A}\right) &= \frac{1}{2} \\ g'\left(\frac{5}{4A}\right) &= -\frac{1}{2}. \end{aligned}$$

Thus if we take $x \in \left[\frac{3}{4A}, \frac{5}{4A}\right]$, then $g(x) \in \left[\frac{3}{4A}, \frac{5}{4A}\right]$ and $|g'(x)| \le \frac{1}{2} < 1$. Thus g satisfy all the condition for fixed point convergence.

13. Consider the root-finding problem $f(x) = 0$ with root α, with $f'(x) \neq 0$. Convert it to the fixed-point problem

$$x = x + cf(x) = g(x)$$

with c a nonzero constant. How should c be chosen to ensure rapid convergence of

$$x_{n+1} = x_n + cf(x_n)$$

to α (provided that x_0 is chosen sufficiently close to α)? Apply your way of choosing c to the root-finding problem $x^3 - 5 = 0$.

Sol. In order to ensure convergence of the iterations

$$|g'(x)| < 1,$$

for x in some interval $[a, b]$ containing root. Now

$$|g'(x)| = |1 + cf'(x)| < 1.$$

Let us start with some initial guess x_0, then

$$\frac{-2}{f'(x_0)} < c < 0.$$

Any c satisfying the above condition will work and $c = -1/f'(x_0)$ will provide rapid convergence (which is actually Newton's method). Now we apply the way to given problem.
We have $f(x) = x^3 - 5 = 0$ and $f(1)f(2) < 0$. So we can start with any point, say $x_0 = 1.5$. Then

$$g(x) = x - \frac{x^3 - 5}{3x^2}.$$

Iterations are given by

$$\begin{aligned} x_1 &= 1.740741 \\ x_2 &= 1.710516 \\ x_3 &= 1.709976. \end{aligned}$$

14. Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$. What happens if $x_0 < 0$?

Sol. Do it yourself.

15. Use secant method to find root accurate to within 10^{-3} for $-x^3 - \cos x =$ with initial guesses -1 and 0.

Sol. The secant iterations are defined as

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \; n = 1, 2 \cdots$$

Start with $x_0 = -1$ and $x_1 = 0$, we obtain

$$\begin{aligned} x_2 &= 0 - \frac{0+1}{f(0) - f(-1)} f(0) = -0.68507 \\ x_3 &= -1.2521 \\ x_4 &= -0.80720 \\ x_5 &= -0.84778 \\ x_6 &= -0.86653 \\ x_7 &= -0.86546 \\ x_8 &= -0.86547 \\ |x_8 - x_7| &= 0.00001 < 0.001. \end{aligned}$$

16. Use secant method to find root of $x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$ with initial guesses 0.02 and 0.05. Use the stopping criterion that the relative error is less than 0.5%.

Sol. The secant iterations are given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \; n = 1, 2 \cdots$$

Start with $x_0 = 0.02$ and $x_1 = 0.05$, denote the relative error by ε_n, we get

$$\begin{aligned} x_2 = 0.06461, \quad & \varepsilon_1 = \left| \frac{x_1 - x_0}{x_1} \right| \times 100\% = 22.62\% \\ x_3 = 0.06241 \quad & \varepsilon_2 = \left| \frac{x_2 - x_1}{x_2} \right| \times 100\% = 3.525\% \\ x_4 = 0.06238 \quad & \varepsilon_3 = \left| \frac{x_3 - x_2}{x_3} \right| \times 100\% = 0.0595\% < 0.5\%. \end{aligned}$$

17. Use Newton's method to approximate the positive root of $2\cos x = x^4$ correct to six decimal places.

Sol. Let $f(x) = 2\cos x - x^4 = 0$. Clearly f is continuous differentiable function and $f(1)f(2) < 0$.
We have $f'(x) = -2\sin x - 4x^3$.
Starting with $x_0 = 1$, we get

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{2\cos(1) - 1^4}{-2\sin(1) - 4(1)^3} = 1.014183 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.013957 \\ x_3 &= 1.013957. \end{aligned}$$

Thus root correct to six decimal places is 1.013957.

CONTINUED

18. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$.

Sol. Let $(x, y = x^2)$ be a point on the curve $y = x^2$.
Let $s(x)$ be the square of the distance of this point from point $(1, 0)$. Then

$$s(x) = (x-1)^2 + \left(x^2\right)^2 = (x-1)^2 + x^4.$$

Minimizing the square of the distance is equivalent to minimizing the distance.
For finding the minimum, firstly equate first derivate of $s(x)$ to zero, which gives

$$s'(x) = 2(x-1) + 4x^3 = 2[(x-1) + 2x^3] = 0.$$

Write the above expression as root finding problem.

$$f(x) = (x-1) + 2x^3 = 0.$$
$$f'(x) = 1 + 6x^2.$$

Newton iteration's are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n - 1) + 2x_n^3}{1 + 6x_n^2} = \frac{1 + 4x_n^3}{1 + 6x_n^2}.$$

Starting with $x_0 = 1.0$, we have

$$\begin{aligned} x_1 &= 0.6 \\ x_2 &= 0.589873417721519 \\ x_3 &= 0.589754528506440 \\ x_4 &= 0.589754512301459. \end{aligned}$$

Therefore root with desired accuracy 10^{-4} is 0.58975.
Thus the point of minimum distance from $(1, 0)$ is $(0.58975, 0.34781)$.

19. (a) Apply Newton's method to the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$

with the root $\alpha = 0$. What is the behavior of the iterates? Do they converge, and if so, at what rate?

(b) Do the same but with

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & x \geq 0 \\ -\sqrt[3]{x^2}, & x < 0 \end{cases}$$

Sol. Clearly $f(x) = 0$ gives root $x = 0$ in both cases.

(a) For $x > 0$, the Newton's iterations are given by

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{\sqrt{x_n}}{\frac{1}{2\sqrt{x_n}}} \\ &= x_n - 2x_n \\ &= -x_n. \end{aligned}$$

Similarly for $x < 0$, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{-\sqrt{-x_n}}{\frac{-1}{2\sqrt{-x_n}}} \\ &= -x_n. \end{aligned}$$

CONTINUED

Thus in both cases

$$x_{n+1} = -x_n.$$

Hence sequence is oscillatory and provide no convergence.

(b) In this case for $x > 0$, the Newton's iterations are

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^{2/3}}{\frac{2}{3x_n^{1/3}}} \\ &= x_n - \frac{3}{2}x_n \\ &= -\frac{1}{2}x_n. \end{aligned}$$

Similarly for $x < 0$, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{-x_n^{2/3}}{-\frac{2}{3x_n^{1/3}}} \\ &= -\frac{1}{2}x_n. \end{aligned}$$

Thus in both cases

$$x_{n+1} = -\frac{1}{2}x_n.$$

In this case convergence occurs.
Since $\alpha = 0$ is the root, so from iterations scheme, we can write

$$\begin{aligned} |x_{n+1} - 0| &= \frac{1}{2}|x_n - 0| \\ |x_{n+1} - \alpha| &= \frac{1}{2}|x_n - \alpha| \\ e_{n+1} &= \frac{1}{2}e_n, \end{aligned}$$

where error $e_n = |x_n - \alpha|$. Thus by definition of order of convergence, order of convergence is one.

Note that the Newton's method provides second order convergence (minimum) if $f'(\alpha) \neq 0$. In this example, the function is not differentiable at $x = 0$. So method may fail (as in part (a)) or method provide slow convergence (as in part (b)).

20. Apply the Newton's method with $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$, and verify that the convergence is only of first-order. Further show that root $\alpha = 1$ has multiplicity 2 and then apply the modified Newton's method with $m = 2$ and verify that the convergence is of second-order.

Sol. Successive iterations in Newton's method are given by

$$x_{n+1} = x_n - \frac{x_n^3 - x_n^2 - x_n + 1}{3x_n^2 - 2x_n - 1}, \quad n = 0, 1, 2, \cdots$$

Starting with $x_0 = 0.8$, we obtain

$$x_1 = 0.905882,\ x_2 = 0.954132,\ x_3 = 0.977338,\ x_4 = 0.988734.$$

Since the exact root is $\alpha = 1$, we have the error in approximations

$$\begin{aligned} e_0 &= |\alpha - x_0| = 0.2 = 0.2 \times 10^0 \\ e_1 &= |\alpha - x_1| = 0.094118 = 0.94 \times 10^{-1} \\ e_2 &= |\alpha - x_2| = 0.045868 = 0.46 \times 10^{-1} \\ e_3 &= |\alpha - x_3| = 0.022662 = 0.22 \times 10^{-1} \\ e_4 &= |\alpha - x_4| = 0.011266 = 0.11 \times 10^{-1}. \end{aligned}$$

CONTINUED

Now we compute the ratios to find the order of convergence.

$$
\begin{aligned}
p &= \frac{\ln(e_2/e_1)}{\ln(e_1/e_0)} = \frac{\ln(0.46/0.94)}{\ln(0.94/0.2)} = 0.946532462,\\
p &= \frac{\ln(e_3/e_2)}{\ln(e_2/e_1)} = \frac{\ln(0.22/0.46)}{\ln(0.46/0.94)} = 1.032107253,\\
p &= \frac{\ln(e_4/e_3)}{\ln(e_3/e_2)} = \frac{\ln(0.11/0.22)}{\ln(0.22/0.46)} = 0.939734509,
\end{aligned}
$$

which shows the linear (almost) convergence .
Iterations in modified Newton's method are given by

$$
x_{n+1} = x_n - 2\frac{x_n^3 - x_n^2 - x_n + 1}{3x_n^2 - 2x_n - 1}, \quad n = 0, 1, 2, \cdots
$$

Starting with $x_0 = 0.8$, we obtain

$$
x_1 = 1.011765,\ x_2 = 1.0000034,\ x_3 = 1.000000.
$$

Now we have the error in approximations

$$
\begin{aligned}
e_0 &= |\alpha - x_0| = 0.2 = 0.2 \times 10^0\\
e_1 &= |\alpha - x_1| = 0.011765 = 0.12 \times 10^{-1}\\
e_2 &= |\alpha - x_2| = 0.000034 = 0.34 \times 10^{-4}\\
p &= \frac{\ln(e_2/e_1)}{\ln(e_1/e_0)} = \frac{\ln(0.34 \times 10^{-4}/0.12 \times 10^{-1}}{\ln(0.12 \times 10^{-1}/0.2)} = 2.085120871,
\end{aligned}
$$

which verifies the second-order convergence.

21. Use Newton's method and the modified Newton's method to find a solution of

$$
\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0, \quad \text{for } -2 \le x \le -1
$$

accurate to within 10^{-3}.

Sol. Newton's iterations are given by

$$
x_{n+1} = x_n - \frac{\cos(x_n + \sqrt{2}) + x_n(x_n/2 + \sqrt{2})}{-\sin(x_n + \sqrt{2}) + x_n + \sqrt{2}}, \quad n = 0, 1, 2, \cdots
$$

Starting with $x_0 = -1.5$, we get the root in 12 iterations with accuracy 10^{-3}.

$$
\begin{aligned}
x_1 &= -1.47855075977922 & x_2 &= -1.46246535074333\\
x_3 &= -1.45040193555133 & x_4 &= -1.44135464479384\\
x_5 &= -1.43456929092347 & x_6 &= -1.42948032375788\\
x_7 &= -1.42566361878740 & x_8 &= -1.42280109901900\\
x_9 &= -1.42065421398883 & x_{10} &= -1.41904405169404\\
x_{11} &= -1.41783644052568 & x_{12} &= -1.41693074432811.
\end{aligned}
$$

If we apply modified Newton's method, then iterations are given by

$$
x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)}, \quad n = 0, 1, 2, \cdots
$$

In this case, we get the answer only in two iterations (with same accuracy)

$$
x_1 = -1.41423460337851, \quad x_2 = -1.41424162427621.
$$

CONTINUED

22. Given the iterative scheme $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n \geq 0$ with $f(\alpha) = f'(\alpha) = 0$ and $f''(\alpha) \neq 0$. Find the order of convergence for this scheme.

Sol. We write the Newton's iterations as fixed point iterations with

$$g(x) = x - \frac{f(x)}{f'(x)}$$
$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

Since $f(\alpha) = f'(\alpha) = 0$, so $g'(0)$ has $\dfrac{(0)}{(0)}$ form. Thus by using the l'Hôpital rule, we have

$$\lim_{x\to 0} g'(0) = \lim_{x\to 0} \frac{f'(x)f''(x) + f(x)f^{(3)}(x)}{2f'(x)f''(x)}$$
$$\lim_{x\to 0} g'(0) = \lim_{x\to 0} \frac{[f''(x)]^2 + 2f'(x)f^{(3)}(x) + f(x)f^{(4)}(x)}{2[f'(x)]^2 + 2f'(x)f^{(3)}(x)}$$
$$g'(\alpha) = \frac{1}{2} \neq 0.$$

Thus the order of convergence for the given scheme is one.

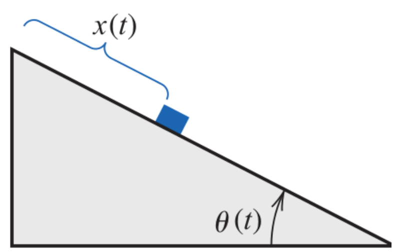
23. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin\omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.



Sol. Substituting the appropriate values, we find ω by finding the root of

$$f(\omega) = -\frac{32.17}{2\omega^2}\left(\frac{e^{\omega} - e^{-\omega}}{2} - \sin\omega\right) - 1.7 = 0.$$

Here we substitute $g = 32.17$ ft/s^2 and $t = 1$.
Clearly f is a continuous function and $f(-1)f(0) < 0$. Solving for ω, by bisection method (we can chose any method), we get root $\omega = -0.317055511474609$.

24. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

CONTINUED

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb, and $k = 0.1$ lb-s/ft. Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.

Sol. When the object hit the ground, the height of the object $s(t) = 0$. This gives

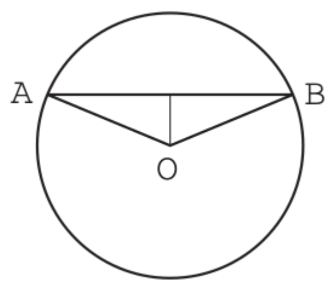
$$\begin{aligned} s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}) &= 0 \\ 300 - 80.425t + 201.0625(1 - e^{-0.4t}) &= 0 \\ t = (501.0625 - 201.0625\, e^{-0.4t})/80.425 &= g(t). \end{aligned}$$

Start with $t_0 = 3$, first three iterations of fixed-point are (within accuracy 0.01 s)

$$t_1 = 5.89184519259234, \quad t_2 = 5.99336160678534, \quad t_3 = 6.00278549575005.$$

So 6.0028 is within 0.01 s of the actual time.

25. The circle below has radius 1, and the longer circular arc joining A and B is twice as long as the chord AB. Find the length of the chord AB, correct to four decimal places. Use Newton's method.



Sol. Draw a perpendicular from O to AB, meeting AB at M. Let $\theta = \angle AOM$. Standard trigonometry shows that the length of AB is $2\sin\theta$. The shorter arc joining A and B has length 2θ, so the longer arc has length $2\pi - 2\theta$. The longer arc is twice the chord, and therefore

$$2\pi - 2\theta = 4\sin\theta.$$

We can use the Newton's Method to solve this equation as it stands by writing $f(\theta) = 2\sin\theta + \theta - \pi$.
Also if we substitute $x = \pi - \theta$ then equation becomes $x = 2\sin x$. This equation can be solved easily with initial guess 1.5 and root is 1.895494267.
The length of the chord is $2\sin\theta$ that is, $2\sin x$, and that is equal to x.

26. It costs a firm $C(q)$ dollars to produce q grams per day of a certain chemical, where

$$C(q) = 1000 + 2q + 3q^{2/3}.$$

The firm can sell any amount of the chemical at \$4 a gram. Find the break-even point of the firm, that is, how much it should produce per day in order to have neither a profit nor a loss. Use the Newton's method and give the answer to the nearest gram.

Sol. If we sell q grams then the revenue is $4q$. The break-even point is when revenue is equal to cost, that is, when

$$4q = 1000 + 2q + 3q^{2/3}.$$

Let $f(q) = 2q - 3q^{2/3} - 1000$. We need to solve the equation $f(q) = 0$.
It is worth asking first whether there is a solution, and whether possibly there might be more than one.
We have $f'(q) = 2 - q^{-1/3}$. The Newton's Method yields

$$q_{n+1} = q_n - \frac{2q_n - 3q_n^{2/3} - 1000}{2 - q_n^{-1/3}} = \frac{q_n + 1000q_n^{1/3}}{2 - q_n^{-1/3}}.$$

CONTINUED

Let us start with $q_0 = 600$. Note that $f(q) < 0$ for small values of q, indeed up to 500 and beyond. Also, $f(1000) > 0$. Since f is continuous, it is equal to 0 somewhere between 500 and 1000.
Further, iterations are given as

$$\begin{aligned} q_1 &= 607.6089386 \\ q_2 &= 607.6067886. \end{aligned}$$

Thus, to the nearest integer, the answer is 608.