Lecture 14: Numerical Linear Algebra (UMA021): Interpolation

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e next

$$\boxed{f(x)} \approx \overset{\text{App.}}{g(x)}$$

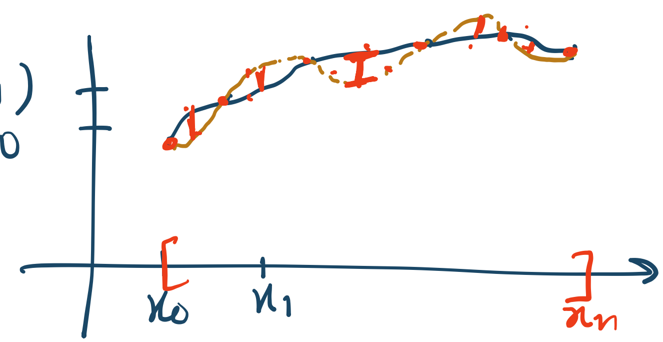
$$f(1.25) \approx g(1.25)$$

$$\boxed{f(x) \approx P(x)}$$

$\boxed{\text{Taylor's poly.}}$

$\boxed{\text{Interpolation}}$

$$\boxed{f(x)} \approx p(x)$$



$$f(x_0) = P(x_0)$$

$$f(x_1) = P(x_1)$$

$$f(x_2) = P(x_2)$$

$$\text{---} \quad \text{---}$$

$$f(x_n) = P(x_n)$$

$$\text{error} = |f(x) - P(x)|$$

$$= \max_{x \in [x_0, x_n]} |e(x)|$$

Lagrange Interpolation & Newton D.D.

Polynomial interpolation:

Lagrange Interpolating polynomials:

Linear Interpolation: The linear Lagrange's interpolating polynomial passes through $(x_0, f(x_0)), (x_1, f(x_1))$ at which function $f(x)$ passes is

$L_{1,0}$ $L_{1,1}$

$P_n \to n^{th}$ degree

$$\boxed{P_1(x)} = L_0(x)f(x_0) + L_1(x)f(x_1),$$

1^{st} degree

$f(x_0) = P_1(x_0)$

$f(x_1) = P_1(x_1)$

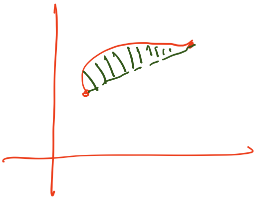
where $L_0(x) = \frac{x-x_1}{x_0-x_1}$ and $L_1(x) = \frac{x-x_0}{x_1-x_0}$.

$f(x)$ x_0, x_1

$(x_0, f(x_0))$, $(x_1, f(x_1))$

$$\boxed{P_1(x_0)} = l_0(x_0)\, f(x_0) + l_1(x_0)\, f(x_1)$$

$$= (1)\, f(x_0) + (0) = f(x_0)$$



$$P_1(x_1) = l_0(x_1)\, f(x_0) + l_1(x_1)\, f(x_1)$$

$$= 0 + 1\, f(x_1) = f(x_1)$$

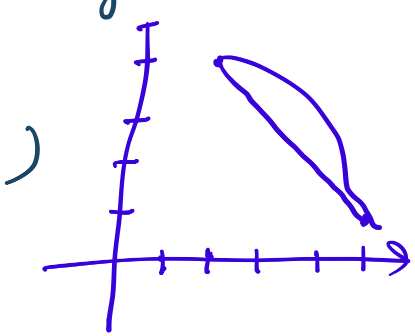
If x_0, x_1 are two points. then $l_0(x) = \dfrac{x - x_1}{x_0 - x_1}$, $\quad l_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

Example

Determine the linear langrange Interpolating polynomial that passes through the points $(2,4)$ and $(5,1)$

Sol^n

Linear Langrange int. poly. is given by



$x_0 = 2$

$x_1 = 5$

$f(x_0) = 4$

$f(x_1) = 1$

$$P_1(x) = \frac{(x - x_1)}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$P_1(x) = \frac{x - 5}{2 - 5}\, 4 + \frac{x - 2}{5 - 2}\, (1)$$

$$= \frac{1}{3}\left[-4x + 20 + x - 2\right] = \frac{1}{3}\left[18 - 3x\right] = 6 - x$$

Lagrange Interpolating polynomials:

Quadratic Lagrange Interpolating polynomial:

Let function $f(x)$ passes through 3 points

$$(\underline{x_0}, f(x_0)), (\underline{x_1}, f(x_1)), (\underline{x_2}, f(x_2)).$$

Consider the construction of a polynomial of degree at most 2 that passes through these 3 points.

For this, we define $L_{2,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2} \frac{x - x_i}{x_k - x_i}.$

The polynomial is given by
$\underline{P_2(x)} = L_{2,0}(x)f(x_0) + L_{2,1}(x)f(x_1) + L_{2,2}(x)f(x_2).$

$l_0(x)\ f(x_0) + l_1(x)\ f(x_1) + l_2(x)\ f(x_2)$

x_0, x_1, x_2

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \quad l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Lagrange Interpolating polynomials:

Generalization:

If $x_0, x_1, \cdots, x_n$ are $n+1$ distinct points and f is a function whose values are given at these numbers i.e. $f(x_0), f(x_1), \cdots, f(x_n)$, then a unique polynomial $P(x)$ of degree at most n exists with $f(x_k) = P(x_k)$, for each $k = 0, 1, 2, \cdots, n$. The polynomial is given by

$$\begin{aligned} P_n(x) &= L_{n,0}(x)f(x_0) + L_{n,1}(x)f(x_1) + \cdots + L_{n,n}(x)f(x_n) \\ &= \sum_{k=0}^{n} L_{n,k}(x)f(x_k), \end{aligned}$$

$$l_{n,0}(x) = \frac{(x-x_1)\,(x-x_2)\,\cdots\,(x-x_n)}{(x_0-x_1)\,(x_0-x_2)\,\cdots\,(x_0-x_n)}$$

Lagrange Interpolating polynomials:

Generalization (continue):

where for each $k = 0, 1, 2, \cdots, n$

$$
\begin{aligned}
L_{n,k}(x) &= \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \\
&= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.
\end{aligned}
$$

Lagrange Interpolating polynomials:

Example:

1. Use the numbers $x_0 = 2, x_1 = 2.75, x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.
2. Use this polynomial to approximate $f(3) = \frac{1}{3}$.

Solution: (1) $x_0 = 2, \quad x_1 = 2.75, \quad x_2 = 4$

$f(x_0) = \frac{1}{2}, \quad f(x_1) = \frac{1}{2.75}, \quad f(x_2) = \frac{1}{4}$

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)}\left(\frac{1}{2}\right) + \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)}\left(\frac{1}{2.75}\right)$$

$$+ \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)}\left(\frac{1}{4}\right)$$

$$= \frac{(x^2-6.75x+11)}{(-0.75)(-2)(2)} + \frac{x^2-6x+8}{(0.75)(-1.25)(2.75)}$$

$$+ \frac{x^2-4.75+5.5}{(2)(1.25)*4}$$

$$= 0.3333\,(x^2-6.75x+11) - 0.3878\,(x^2-6x+8)$$

$$+ 0.1\,(x^2-4.75x+5.5)$$

$$P_2(x) = 0.0455x^2 - 0.3980x + 1.1139$$

(2) $$P_2(3) = 0.0455(3)^2 - 0.3980(3) + 1.1139 = 0.3294 \approx \frac{1}{3}$$

Lagrange Interpolating polynomials:

Exercise:

1. Find the unique polynomial $P(x)$ of degree 1such that

$$P(1) = 1, P(3) = 27,$$

using Lagrange interpolation. Evaluate $P(1.05)$.

2. For the given functions $f(x)$, let $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.6$. Construct Lagrange interpolation polynomials of degree at most one and at most two to approximate $f(1.4)$, and find the absolute error.
   a. $f(x) = \sin(\pi x)$.
   b. $f(x) = \log_{10}(3x - 1)$.

3. Let $P_3(x)$ be the Lagrange interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$ and $(2, 2)$. Find y if the coefficient of x^3 in $P_3(x)$ is 6.