Lecture 30: Numerical Linear Algebra (UMA021): Matrix Algebra

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Least Square Approximation Method:

x_0 $f(x_0)$

x_1 $f(x_1)$ $y(x_i)$

x_2

x_n $f(x_n)$

Least Square Approximation Method:

Suppose that the data points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, where x_i are the independent variable and y_i are the dependent variable.

Let $e_i = y_i - f(x_i)$ be the error at each data points.

According to the method of least squares, the best fitting curve has the property that $\sum_{i=0}^{n} e_i^2 = \boxed{\sum_{i=0}^{n} (y_i - f(x_i))^2}$ is minimum. $= E$

$\boxed{y_i} \rightarrow$ exact $\quad f(x_i) \rightarrow$ app $\quad (e_i)^2 = \sum_{i=0}^{n} (y_i - f(x_i))^2$

points Exact

x_0	$\boxed{y_0}$	$f(x_0)$
x_1	y_1	—
x_2	y_2	—
x_3	y_3	⋮

$$e_i = \underset{\text{exact}}{\underset{\downarrow}{y_i}} - \underset{\text{approximation}}{\underset{\downarrow}{f(x_i)}}$$

$$\sum_{i=0}^{n} y_i - f(x_i)$$

$\boxed{E}$

$$\frac{dE}{d(\,)}$$

∂

$E(x, y)$

$$\frac{\partial E}{\partial x}, \quad \frac{\partial E}{\partial y}$$

Least Square Approximation Method:

Condition of maximum & minima in two variables $f(x,y)$ $\quad f_x=0, \; f_y=0$ $\quad$ Let $D = f_{xx} f_{yy} - (f_{xy})^2$

If $D>0$ & $f_{xx}>0 \rightarrow$ min.

$D<0$ $\quad f_{xx}<0 \rightarrow$ max.

Least Square fit of a straight line:

Suppose that the data points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Let $f(x) = \underbrace{a + bx}$, where a, b are the constants to be determined to the given data.

Now residuals is given by

$$\underset{\text{error}}{e_i} = \underbrace{y_i - f(x_i)} = y_i - (a + bx_i) \Rightarrow E = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} (y_i - (a + bx_i))^2.$$

We need to find a and b such that error E is minimum.

The necessary condition for minimum is $\dfrac{\partial E}{\partial a} = 0$ and $\dfrac{\partial E}{\partial b} = 0$

$$e_i^2 = \left(y_i - (a+bx_i)\right)^2 = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} \left(y_i - (a+bx_i)\right)^2$$

$$E = \sum_{i=0}^{n} \left(y_i - (a+bx_i)\right)^2$$

$(2x+3)^4$

$E(a, b)$

$$E(a, b) = \sum_{i=0}^{n} \left(\underbrace{y_i - (a + b x_i)}\right)^2$$

$$\frac{\partial E}{\partial a} = \sum_{i=0}^{n} 2 \left(y_i - (a + b x_i) \right)^1 (-1) = 0$$

$$= -2 \sum_{i=0}^{n} y_i - (a + b x_i) = 0$$

$$\Rightarrow \sum_{i=0}^{n} (y_i - (a + b x_i)) = 0$$

$$\sum_{i=0}^{n} y_i - a \sum_{i=0}^{n} 1 - b \sum_{i=0}^{n} x_i = 0 \quad \text{--- (1)}$$

& $\frac{\partial E}{\partial b} = 0$

$$\sum_{i=0}^{n} 2 (y_i - (a + b x_i)) (-x_i) = 0$$

$$= \sum_{i=0}^{n} (y_i x_i - a x_i - b x_i^2) = 0$$

$$\Rightarrow \quad \sum_{i=0}^{n} x_i y_i - a \sum_{i=0}^{n} x_i - b \sum_{i=0}^{n} x_i^2 = 0 \qquad \text{--- (2)}$$

Least Square Approximation Method:

Example:

Obtain the least square straight line fit to the following data:

x	5	10	15	20
$f(x)=y$	16	19	23	26

Solution:

i	x_i	y_i	$x_i y_i$	x_i^2
0	5	16	80	25
1	10	19	190	100
2	15	23	345	225
3	20	26	520	400
	50	84	1135	750

let the straight line be $ax+b$

So we take the error function $E(a,b)$

$$= \sum_{i=0}^{3} (y_i - f(x_i))^2$$

$$E(a,b) = \sum_{i=0}^{3}\left(y_i - (ax_i+b)\right)^2$$

$$\frac{\partial E}{\partial a} = 0 \Rightarrow \sum_{i=0}^{3} y_i - a\sum_{i=0}^{3} x_i - b\sum_{i=0}^{3} 1 = 0$$

$$84 - a(50) - b(4) = 0$$

$$50a + 4b = 84$$

$$\boxed{25a + 2b = 42} \quad - (1)$$

$$\& \quad \frac{\partial E}{\partial b} = 0 \Rightarrow \sum_{i=0}^{3} x_i y_i - a\sum_{i=0}^{3} x_i^2 - b\sum_{i=0}^{3} x_i = 0$$

$$1135 - a(750) - b(50) = 0$$

$$750a + 50b = 1135$$

$$\boxed{150a + 10b = 227} \quad - (2)$$

On solving (1) & (2) $a = 0.68$, $b = 12.5$ $\Rightarrow$ Str. line is $0.65x + 12.5$

Least Square Approximation Method:

Exercise:

1. Use the method of least squares to fit the linear polynomial to the following data:

x	-2	-1	0	1	2
$f(x)$	15	1	1	3	19

Power method:

Power method:

It is an iterative method which is used to determine the dominant eigenvalue i.e the eigenvalue with largest magnitude.

Power method:

Procedure of Power method:

$\boxed{A}_{n\times n}$ $\qquad X^{(0)}_{n\times 1} \rightarrow$ initial guess.

for e.g.

$$y^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1/4 \\ 2/4 \\ 3/4 \\ 1 \end{bmatrix} = K_1 X^{(1)}$$

$$A X^{(0)} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n} \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{n\times 1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{n\times 1} = y^{(1)}$$

Take largest element in magnitude common from the elements of $y^{(1)}$ ie K_1

then we get $\quad y^{(1)} = K_1 X^{(1)}$

$$A X^{(1)} = y^{(2)} = K_2 X^{(2)}$$

$$A X^{(2)} = y^{(3)} = K_3 X^{(3)}$$

$|K_n - K_{n-1}| < tol$

and $\| X^{(n)} - X^{(n-1)} \|_\infty < tol.$

Power method:

Example:

Determine the largest eigenvalue and the corresponding eigenvector of the following matrix correct to one decimal using the power method with $x^{(0)} = (1, 0, 0)^t$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

We will do this question again in the next lecture.