Lecture 19: Numerical Linear Algebra (UMA021): Integration

Dr. Meenu Rani

Department of Mathematics
TIET, Patiala
Punjab-India

Numerical Quadrature:

Numerical Quadrature:

Numerical Quadrature is a basic method for numerically approximating the value of a definite integral $\int_a^b f(x)dx$.

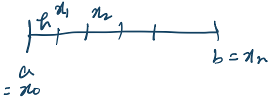
It uses $\sum_{i=0}^{n} a_i f(x_i)$ to approximate $\int_a^b f(x)dx$,

$$i.e. \int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i).$$

$\int_a^b f(x)\,dx$

$[a, b]$

$h = \frac{b-a}{n}$



Numerical Quadrature:

General formula to approximate integral

Divide the interval $[a, b]$ in to a set of $(n+1)$ distinct nodes $\{x_0, x_1, x_2, \cdots, x_n\}$.

Approximate $f(x)$ by Lagrange's interpolating polynomials which is used to approximate $f(x)$.

Thus, we can write

$l_0(x) f(x_0) + l_1(x) f(x_1) - - - l_n(x) f(x_n)$

$$f(x) = \underbrace{P_n(x)}_{n^{th}\text{ degree}} + \underbrace{e_n(x)}_{\text{error}} = \sum_{i=0}^{n} l_i(x) f(x_i) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where $\xi = \xi(x) \in [a, b]$ and $l_i(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$

$$\int_a^b f(x)\,dx = \int_a^b \sum_{i=0}^{n} l_i(x) f(x_i)\,dx + \underbrace{\int_a^b \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0) \cdots (x-x_n)\,dx}$$

$E(f)$

$$\int_a^b f(x)\,dx \approx \int_a^b \sum_{i=0}^{n} l_i(x) \boxed{f(x_i)}\, dx$$

$$= \sum_{i=0}^{n} \underbrace{\left(\int_a^b l_i(x)\,dx\right)} f(x_i)$$

$$= \sum_{i=0}^{n} a_i(x)\; f(x_i)$$

$$a_i(x) = \int_a^b l_i(x)\,dx$$

Numerical Quadrature:

Quadrature formula:

Therefore, $\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i)$, where $a_i = \int_a^b l_i(x)dx$,
for each $i = 0, 1, 2, \cdots, n$
with error given by

$$E(f) = \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x))dx.$$

Numerical Quadrature:

2 points → Trap.
3 points → Simpson's $\frac{1}{3}$rd rule.

Quadrature formula:

We derive formulas by using one and two degree interpolating polynomials with equally spaced nodes. This gives:

1. Trapezoidal Rule
2. Simpson's $\frac{1}{3}$ Rule   resp.

$$\int_a^b f(x)\,dx$$

$a = x_0$ —— $b = x_1$

Numerical Quadrature:

Trapezoidal Rule:

To derive the Trapezoidal rule for approximating $\int_a^b f(x)dx$, let $x_0 = a, x_1 = b, h = b - a$ and use linear Lagrange's interpolating polynomial:

$$
\begin{aligned}
P_1(x) &= \sum_{i=0}^{1} l_i(x) f(x_i) \\
\Rightarrow f(x) &= P_1(x) + e_1(x) \\
\Rightarrow f(x) &= \sum_{i=0}^{1} l_i(x) f(x_i) + \frac{f''(\xi)}{2!}(x - x_0)(x - x_1), \ \xi \in (a, b).
\end{aligned}
$$

$= l_0(x) f(x_0) + l_1(x) f(x_1)$

$= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$

$= \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) +$

Numerical Quadrature:

Trapezoidal Rule:

$$\Rightarrow \int_a^b f(x)dx = \int_{a=x_0}^{b=x_1} \left(\frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) \right) dx$$

$$+ \int_{a=x_0}^{b=x_1} \frac{f''(\xi)}{2!}(x-x_0)(x-x_1)dx = E(f)$$

$$= \sum_{i=0}^{1} a_i(x) f(x_i) + E(f), \text{ (say)}.$$

$$= a_0(x) f(x_0) + a_1(x) f(x_1)$$

$$a_0(x) = \int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} dx \quad , \quad a_1(x) = \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} dx$$

$\int x \, dx$

$$a_0 = \frac{1}{x_0 - x_1} \int_{x_0}^{x_1} (x - x_1)\, dx \qquad a_1 = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} (x - x_0)\, dx$$

$$\int (ax+b)\, dx = \frac{(ax+b)^2}{2a}$$

$$= \frac{1}{x_0 - x_1} \left(\frac{(x-x_1)^2}{2} \right)_{x_0}^{x_1} \qquad a_1 = \frac{1}{x_1 - x_0} \left(\frac{(x-x_0)^2}{2} \right)_{x_0}^{x_1}$$

$$= \frac{1}{x_0 - x_1} \left[0 - \frac{(x_0 - x_1)^2}{2} \right] \qquad = \frac{(x_1 - x_0)^2}{2(x_1 - x_0)} - 0$$

$$= \frac{-(x_0 - x_1)}{2} = \frac{-(a-b)}{2} \qquad = \frac{(x_1 - x_0)}{2}$$

$$= \frac{b-a}{2} \qquad = \frac{b-a}{2}$$

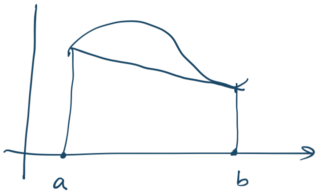
$$\Rightarrow \int_{a = x_0}^{b = x_1} f(x)\, dx \approx \frac{b-a}{2} f(x_0) + \frac{b-a}{2} f(x_1)$$

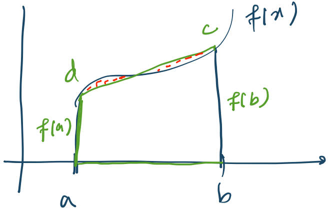
$$= \frac{b-a}{2} \left[f(a) + f(b) \right]$$

Numerical Quadrature:

Trapezoidal Rule:

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)).$$





Area of abcd

$= \frac{1}{2}$ (sum of $\|^{ll}$ sides) * height.

$= \frac{1}{2}\left(f(a)+f(b)\right)(b-a)$

Trapezoidal Rule: Error formula

$$E(f) = -\frac{h^3}{12} f''(c), \ c \in (a, b)$$

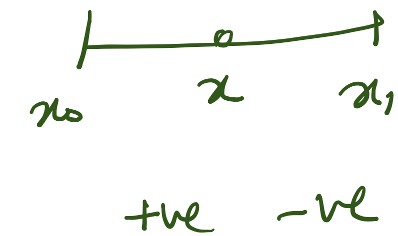
$$E(f) = \int_a^b \frac{f''(\xi_x)}{2!} (x - x_0)(x - x_1) \, dx$$

W.M.V.T weighted mean value Theorem.

Let f & g be two functions in $[a, b]$ s.t. $f(x)$ is continuous in $[a, b)$ and $g(x)$ is integrable function and $g(x)$ does not change its sign in $[a, b]$ then $\exists$ $c \in (a, b)$ s.t.

$$\int_a^b f(x) g(x) \, dx = f(c) \int_a^b g(x) \, dx$$

Take $f = \dfrac{f''(\xi_x)}{2!}$, $g(x) = (x - x_0)\,(x - x_1)$



$\exists \quad c \in (a,b)$ s.t.

$$E(f) = \frac{f''(c)}{2!} \int_{x_0}^{x_1} \underbrace{(x - x_0)}_{\mathrm{I}} \, \underbrace{(x - x_1)}_{\mathrm{II}} \, dx$$

$$= \frac{f''(c)}{2!} \left[(x - x_0) \frac{(x - x_1)^2}{2} - \int \frac{(x - x_1)^2}{2} dx \right]$$

$$= \frac{f''(c)}{2!} \left[(x - x_0) \frac{(x - x_1)^2}{2} - \frac{(x - x_1)^3}{3 * 2} \right]_{x_0}^{x_1}$$

$$= \frac{f''(c)}{2!} \left[0 - 0 - 0 + \frac{(x_0 - x_1)^3}{6} \right]$$

$b - a = h$

$$= \frac{f''(c)}{2} \left[\frac{(a-b)^3}{6} \right] = -\frac{h^3}{12} f''(c), \quad c \in (a,b)$$

Example

$$\int_0^1 2x+3 \, dx$$

Exact value $\left(x^2+3x\right)_0^1 = 1+3 = 4$

By Trap $= \int_0^1 (2x+3)\,dx = \frac{1-0}{2}\left[f(0)+f(1)\right]$

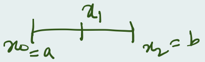
$= 0.5\,(3+5) = \frac{8}{2} = 4$

Numerical Quadrature:

Simpson's Rule:

To derive Simpson's rule for approximating $\int_a^b f(x)dx$, we use second degree Lagrange's interpolating polynomials $P_2(x)$ with equally spaced nodes $x_0 = a, x_2 = b$ and $x_1 = a + h = \frac{a+b}{2}$, where $h = \frac{(b-a)}{2}$,

$h = \frac{b-a}{2}$



$$
\begin{aligned}
P_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) \\
&\quad + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) = \sum_{i=0}^{2} l_i(x)f(x_i) \\
\Rightarrow f(x) &= P_2(x) + e_2(x) \\
\Rightarrow f(x) &= \sum_{i=0}^{2} l_i(x)f(x_i) + \underbrace{\frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)}, \ \xi \in (a,b)
\end{aligned}
$$

Numerical Quadrature:

Simpson's Rule:

$$\Rightarrow \int_a^b f(x)dx$$

$$= \int_{a=x_0}^{b=x_2} \left(\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right) dx$$

$$+ \int_{a=x_0}^{b=x_2} \frac{f'''(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2)dx$$

$$= \sum_{i=0}^{2} a_i(x) f(x_i) + E(f), \text{ (say).}$$

$= a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2)$
$+ E(f)$

$$a_0 = \int_{a=x_0}^{b=x_2} \frac{(x-x_1)(x-x_2)}{\boxed{(x_0-x_1)(x_0-x_2)}}\,dx, \qquad a_1 = \int_{a=x_0}^{b=x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,dx = \frac{4h}{3}, \qquad a_2 = \int_{a=x_0}^{b=x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,dx = h/3$$

$$a_0 = \frac{1}{(x_0-x_1)(x_0-x_2)} \left[(x-x_1)\frac{(x-x_2)^2}{2} - \int \frac{(x-x_2)^2}{2}\,dx \right]_{x_0}^{x_2}$$

$$= \frac{1}{-h\,(-2h)} \left[(x-x_1)\frac{(x-x_2)^2}{2} - \frac{(x-x_2)^3}{6} \right]_{x_0}^{x_2}$$

$$= \frac{1}{2h^2} \left[(x_2-x_1)\,(0) - 0 - (x_0-x_1)\frac{(x_0-x_2)^2}{2} + \frac{(x_0-x_2)^3}{6} \right]$$

$$= \frac{1}{2h^2} \left[0 - (-h)\frac{(-2h)^2}{2} + \frac{(-2h)^3}{6} \right] = \frac{1}{2h^2}\left[\frac{4h^3}{2} - \frac{8h^3}{6} \right]$$

$$= \frac{1}{2h^2}\left[2h^3 - \frac{4}{3}h^3 \right] = \frac{h}{2}\left[2 - \frac{4}{3} \right] = \frac{2h}{2*3} = \frac{h}{3} \checkmark$$

Numerical Quadrature:

Simpson's Rule:

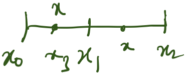
$$\int_{a=x_0}^{b=x_2} f(x)dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)), h = x_i - x_{i-1}, i = 1, 2$$

$h = \frac{b-a}{2}$

Simpson's Rule: Error formula

$$E(f) = -\frac{h^5}{90} f^{(iv)}(c), \;\; c \in (a, b)$$

$$E(f) = \int_{a=x_0}^{b=x_2} \frac{f'''(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2)\, dx$$



We can't apply W.M.V.T $\because$ $g(x) = (x-x_0)(x-x_1)(x-x_2)$ changes its sign

+ve −ve −ve = +ve

+ve +ve −ve = −ve

So, we add one more pt. to the nodes

We take the nodes x_0, x_1, x_1, x_2

$$E^{*}(f) = \int_{x_0}^{x_2} \frac{f^{iv}(\xi_4)}{4!} (x-x_0) (x-x_1)^2 (x-x_2)\, dx$$

Now if we take $f = \frac{f^{iv}(\xi_4)}{4!}$ $\downarrow$ cont on $[x_0, x_2]$

$g = (x-x_0)(x-x_1)^2(x-x_2)$ $\downarrow$ int. f

f does not change its sign on $[x_0, x_2]$

then by W.M.V.T, $\exists$ a $c \in (x_0, x_2)$ s.t.

$$E^{*}(f) = \frac{f^{iv}(c)}{4!} \int_{x_0}^{x_2} (x-x_0)(x-x_1)^2(x-x_2)\, dx$$

$$= -\frac{h^5}{90} f^{iv}(c) \quad , \quad c \in (x_0, x_2) = (a, b)$$