Lecture 31: Numerical Linear Algebra (UMA021): Matrix Computations

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Power method:

Power method:

It is an iterative method which is used to determine the dominant eigenvalue i.e the eigenvalue with largest magnitude.

Power method:

Procedure of Power method:

Power method:

Example:

Determine the largest eigenvalue and the corresponding eigenvector of the following matrix correct to one decimal using the power method with $x^{(0)} = (1, 0, 0)^t$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

$$A x^{(0)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix} = K_1 x^{(1)}$$

$$A x^{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 5 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} -0.3 \\ 1 \\ -0.2 \end{bmatrix} = K_2 x^{(2)}$$

$$A X^{(2)} = \begin{bmatrix} -1.3 \\ 5 \\ -1.4 \end{bmatrix} = 5 \begin{bmatrix} -0.26 \\ 1 \\ -0.28 \end{bmatrix} = K_3 X^{(3)}$$

$|K_3 - K_4| < 0.1$

$$A X^{(3)} = \begin{bmatrix} -1.26 \\ 5.08 \\ -1.56 \end{bmatrix} = 5.08 \begin{bmatrix} -0.248 \\ 1 \\ -0.307 \end{bmatrix} = K_4 X^{(4)}$$

$\|X^{(3)} - X^{(4)}\|_\infty < 0.1$

$\Rightarrow$ $K_4 = 5.08$ is the E.value

& $X^{(4)} = \begin{bmatrix} -0.248 \\ 1 \\ -0.307 \end{bmatrix}$ is E.vector

corresp. to 5.08

Ans

Error bounds in solutions of system of linear equations:

Exercise:

1. Use power method to approximate the most dominant eigenvalue of the matrix until a tolerance of 10^{-1} is achieved with $x^{(0)} = (1, 1, 1)^t$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Power method:

Inverse power method:

If λ is an eigenvalue of a matrix A. & X is E.vector corresp. to λ then

$$AX = \lambda X$$

If A is non-singular matrix then premultiplying by A^{-1}

$$A^{-1}AX = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$$IX = \lambda(A^{-1}X)$$

$$\Rightarrow \quad A^{-1}X = \frac{1}{\lambda}X$$

$\Rightarrow \quad \frac{1}{\lambda}$ is an E.value of A^{-1}

$A \to$ matrix $2, 4, 5$ are the E.values of A
then $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}$ - - - - - - A^{-1}

Inverse power method:

Apply P.M on A^{-1} to find the largest e.value of A^{-1} which will give smallest E.value of A.

Example:

Perform first four iterations of inverse power method to approximate the smallest eigenvalue of the matrix by using an initial vector $x^{(0)} = (1, -1, 2)^t$

$$\text{let } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

To find the smallest E.value of A, apply P.M on

$$\underset{(\text{say})}{B} = A^{-1} = \begin{bmatrix} 0.75 & -0.25 & -0.25 \\ -0.25 & 0.75 & -0.25 \\ -0.25 & -0.25 & 0.75 \end{bmatrix}$$

$$B X^{(0)} = \begin{bmatrix} 0.5 \\ -1.5 \\ 1.5 \end{bmatrix} = -1.5 \begin{bmatrix} -0.3333 \\ 1 \\ -1 \end{bmatrix} = k_1 X^{(1)}$$

$$Bx^{(1)} = \begin{bmatrix} -0.2500 \\ 1.0833 \\ -0.9167 \end{bmatrix} = 1.0833 \begin{bmatrix} -0.2308 \\ 1 \\ -0.8462 \end{bmatrix} = k_2 x^{(2)}$$

$$Bx^{(2)} = \begin{bmatrix} -0.21155 \\ 1.01925 \\ -0.8270 \end{bmatrix} = 1.01925 \begin{bmatrix} -0.2075 \\ 1 \\ -0.8114 \end{bmatrix} = k_3 x^{(3)}$$

$$Bx^{(3)} = \begin{bmatrix} -0.2028 \\ 1.0047 \\ -0.8067 \end{bmatrix} = 1.0047 \begin{bmatrix} -0.2019 \\ 1 \\ -0.8029 \end{bmatrix} = k_4 x^{(4)}$$

$k_4 = 1.0047$ is largest e.value of $B = A^{-1}$

$\lambda = \dfrac{1}{1.0047} = 0.9953$ is the smallest E.value of A

$X = \begin{bmatrix} -0.2019 \\ 1 \\ -0.8029 \end{bmatrix}$ is the E.vector corresp. to λ.

Exercise:

1. Use inverse power method to approximate the smallest eigenvalues of the matrix until a tolerance of 10^{-1} is achieved with $x^{(0)} = (1, 1, 1)^t$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Linear Independent or dependent vectors:

Linear Independent or dependent vectors:

The set of vectors $v_1, v_2, \cdots v_n$ is called linearly independent if the linear combination $c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_n v_n = 0$
$\implies c_1 = c_2 = c_3 = \cdots = c_n = 0.$
Otherwise, the set of vectors is linear dependent.

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + 3c_2 - c_3 = 0, \quad 2c_1 + 4c_2 + 2c_3 = 0, \quad 3c_1 + 5c_2 + 3c_3 = 0$

Linear Independent or dependent vectors:

Linear Independent or dependent vectors: Example

1. Check whether the vectors $v_1 = (1, 2)^t, v_2 = (3, 6)^t$ are L.I. or not.
2. Check whether the vectors $v_1 = (1, 0, 0)^t, v_2 = (-1, 1, 1)^t, v_3 = (0, 4, 2)^t$ are L.I. or not.

(1) Let c_1, c_2 are scalars

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$c_1 + 3c_2 = 0, \quad 2c_1 + 6c_2 = 0$

$c_1 + 3c_2 = 0 \quad c_1 + 3c_2 = 0$

If $c_2 = 1$

$c_1 = -3$

$\Rightarrow v_1 \,\&\, v_2$ are L.D.

(2) Let c_1, c_2 & c_3 are three scalars, then

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$c_1 - c_2 = 0$, $\cancel{c_2} + 4c_3 = 0$, $c_2 + 2c_3 = 0$

$c_1 = c_2$

$$\begin{array}{r} c_2 + 4c_3 = 0 \\ -\ c_2 + 2c_3 = 0 \\ \hline 2c_3 = 0 \end{array}$$

$c_3 = 0$ $\Rightarrow c_2 = 0$

$c_1 = 0$

$\Rightarrow v_1, v_2, v_3$ are L.I.