Lecture 6: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Fixed point iteration

Example of FPI:

Find the root of an equation $\underbrace{x^3 + 4x^2 - 10}_{f(x)} = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.

Solution: By IVT, we first find the interval in which root of $f(x)=0$ lies

Step 1

$f(0) = -10 < 0$

$f(1) = 1+4-10 < 0$

$f(2) = 8+16-10 > 0$

$\Rightarrow$ By IVT the root of $f(x)=0$ lies in $[1,2]$

Step 2 (i) $x = x^3 + 4x^2 - 10 + x$

$= g_1(x)$

(a) $g_1 \in C[1,2]$ or g_1 is cont on $[1,2]$

✗ (b) $g_1(1) = 1+4-10+1 = -4 \notin [1,2]$

(ii.) $$x^3 = 10-4x^2$$
$$x = (10-4x^2)^{1/3} = g_2(x)$$

(a) $g_2 \in C[1,2]$

✗ (b) $g_2(1) = (10-4)^{1/3} = (6)^{1/3} = 1.8 \in [1,2]$

$g_2(2) = (10-16)^{1/3} = (-6)^{1/3} \notin [1,2]$

(iii.) $$4x^2 = 10-x^3$$
$$x = \frac{(10-x^3)^{1/2}}{2} = g_3(x)$$

(a) $g_3 \in C[1,2]$

(b) $g_3(1) = \frac{\sqrt{9}}{2} = \frac{3}{2} \in [1,2]$

✗ $g_3(2) = \frac{\sqrt{10-8}}{2} = \frac{1}{\sqrt{2}} = 0.707 \notin [1,2]$

(iv) $$x^2(x+4) = 10$$

$$x = \sqrt{\frac{10}{x+4}} = g_4(x)$$

(a) $g_4 \in C[1,2]$

(b) $g_4(1) = \sqrt{\frac{10}{5}} = \sqrt{2} \in [1,2]$

$$g_4(2) = \frac{\sqrt{10}}{\sqrt{6}} = \sqrt{\frac{5}{3}} \in [1,2]$$

$$g_4'(x) = \sqrt{10}\ \frac{-1}{2}\ (x+4)^{-3/2} < 0 \quad \forall\ x \in [1,2]$$

$\Rightarrow g_4(x)$ is decreasing function

$\Rightarrow$ $g_4(1) =$ max value $= \sqrt{2} \in [1,2]$

$g_4(2) =$ min value $= \sqrt{\frac{5}{3}} \in [1,2]$

$\Rightarrow$ g maps into itself (or) $g(x) \in [1,2]$ $\forall x \in [1,2]$

(iii) $$|g_4'(x)| = \left|-\frac{\sqrt{10}}{2\ (x+4)^{3/2}}\right|$$

$$g_4''(x) = \frac{\sqrt{10}}{2} \cdot \frac{3}{2} \cdot \frac{1}{(x+4)^{5/2}} > 0 \quad \forall x \in [1,2]$$

$\Rightarrow$ $g_4'(x)$ is increasing on $x \in [1,2]$

$$\Rightarrow \quad |g_4'(1)| = |\text{min value of } g_4'(x) \text{ on } [1,2]| = \left| \frac{-\sqrt{10}}{2(5)^{3/2}} \right| < 1$$

$$\& \quad |g_4'(2)| = |\text{max value } \text{"} \ \text{"} \ \text{"}| = \left| \frac{-\sqrt{10}}{2(6)^{3/2}} \right| < 1$$

$$\Rightarrow \quad |g_4'(x)| < 1 \quad \forall x \in [1,2]$$

$$\Rightarrow \quad g(x) = \sqrt{\frac{10}{x+4}}$$

Step 3 Take an initial guess on $[1,2]$

$$p_0 = 1.5$$

$$p_{n+1} = g(p_n) = \sqrt{\frac{10}{p_n + 4}}, \quad n = 0$$

$$p_1 = \sqrt{\frac{10}{5.5}} = 1.3484$$

$$p_2 = \sqrt{\frac{10}{1.3484+4}} = 1.36737$$

$|p_1 - p_2| \nless 10^{-2}$

$$p_3 = \sqrt{\frac{10}{1.36737+4}} = 1.36500 \qquad |p_2 - p_3| < 10^{-2}$$

$p_3 = 1.365$ is the fixed pt. for $g(x)$ and root of given eq^n $f(x) = 0$

Fixed point iteration

Example of FPI:

Find the root of an equation $\underbrace{x^3 - 7x + 2}_{f(x)} = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.

Solution: step1 By IVT, root of $f(x)=0$ lies in $[0,1]$

step2 (i) $\frac{x^3+2}{7} = x$

$x = g(x)$

(a) $g \in C[0,1]$

(b) $g(0) = \frac{2}{7} \in [0,1]$

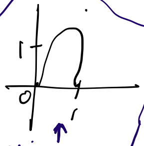
$g(1) = \frac{3}{7} \in [0,1]$

$g'(x) = \frac{3x^2}{7} > 0 \quad \forall x \in [0,1]$

$g(x)$ is increasing.

$\Rightarrow$ $\left.\begin{array}{l} g(0) = \text{min value} \\ g(1) = \text{max value} \end{array}\right\} \in [0,1]$

$\Rightarrow g(x) \in [0,1] \quad \forall x \in [0,1]$



It is necessary to find max. min. otherwise.

if you check only at end pts. without knowing max. & min. then max./min can occur outside the interval.

(c) $|g'(x)| = \left|\frac{3x^2}{7}\right| < 1 \quad \forall x \in [0,1]$

$\Rightarrow$ $g(x)$ satisfies all the conditions of convergence

$\Rightarrow$ $g(x) = \frac{x^3+2}{7}$

Step 3 Take an initial guess on $[0,1]$

$p_0 = 0.5$

$$\boxed{p_{n+1} = g(p_n)} = \frac{p_n^3+2}{7}, \quad n=0$$

$$p_1 = \frac{(0.5)^3+2}{7} = 0.30357$$

$$p_2 = \frac{(0.30357)^3+2}{7} = 0.2897$$

$|p_1 - p_2| \nless 10^{-2}$

$$p_3 = \frac{(0.2897)^3+2}{7} = 0.2891$$

$|p_2 - p_3| < 10^{-2}$

Ans

Fixed point iteration

Example:

The iterates $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ converge to $p = 1$ for some constant c. Find the value or bound for c for which convergence occurs.

Solution:

$p_{n+1} = 2 - (1+c)p_n + cp_n^3 = g(p_n)$

$p_{n+1} = g(p_n) \to 1$

$\Rightarrow$ $g(x) = 2 - (1+c)x + cx^3$ satisfies all the cgce conditions on nbd of 1 or $(1-\delta, 1+\delta)$, $\delta > 0$

$|g'(x)| = |-(1+c) + 3cx^2| < 1 \quad \forall x \in (1-\delta, 1+\delta)$ or $\forall$ nbd of 1.

$\Rightarrow |-1-c+3c| < 1$ at $x = 1$ also

Fixed point iteration

$|-1+2c|<1$

$-1<-1+2c<1$

$0<2c<2$

$\boxed{0<c<1}$ Ans.

Exercise:

1. Find the root of an equation $x^3 - 2x^2 - 5 = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.
2. Let A be a given positive constant and g(x) = $2x - Ax^2$:
   (a) Show that $1/A$ is a fixed point for $g(x)$.
   (b) Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

for $\boxed{x^3-7x+2=0}$

you can choose approximate as $\boxed{g(x) = x - \frac{x^3-7x+2}{3x^2-7} \neq 0}$