Thapar Institute of Engineering & Technology, Patiala
Department of Mathematics

UMA021: Numerical Linear Algebra
Assignment 3
Matrix Algebra

1. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems (do not reorder the equations). The exact solution to each system is $x_1 = -1$, $x_2 = 1$, $x_3 = 3$.

(a)

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 8 \\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 11 \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5 \\ \frac{1}{9}x_1 + \frac{1}{9}x_2 - \frac{1}{3}x_3 &= -1 \\ x_1 + 4x_2 + 2x_3 &= 9 \end{aligned}$$

2. Using four-digit arithmetic, solve the following system of equations by Gaussian elimination with partial pivoting:

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867 \\ x_1 + x_2 + x_3 &= 0.8338 \\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000 \end{aligned}$$

3. Use Gaussian elimination to solve the following systems and determine if row interchanges are necessary:

(a)

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 6 \\ x_2 - x_3 + x_4 &= 5 \\ x_4 &= 5 \\ x_3 - x_4 &= 3 \end{aligned}$$

4. Use LU factorization to solve the following systems:

(a)

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ x_1 + 2x_2 - 2x_3 &= 2 \\ -2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

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5. Find l_∞ and l_2 norms of the vectors.

(a) $x = \big(3, -4, 0, 32\big)^t$.

(b) $x = \big(\sin k, \cos k, 2^k\big)^t$ for a fixed positive integer k.

6. Find the l_∞ and l_2 norm of the matrix:

$$A = \begin{pmatrix} 4 & -1 & 7 \\ -1 & 4 & 0 \\ -7 & 0 & 4 \end{pmatrix}$$

7. The following linear system $Ax = b$ has x as the actual solution and $\tilde{x}$ as an approximate solution. Compute $\|x - \tilde{x}\|_\infty$ and $\|A\tilde{x} - b\|_\infty$. Also compute $\|A\|_\infty$.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 3x_2 + 4x_3 &= -1 \\ 3x_1 + 4x_2 + 6x_3 &= 2 \end{aligned}$$

Here, actual solution: $x = (0, -7, 5)^t$, approximate solution: $\tilde{x} = (-0.2, -7.5, 5.4)^t$.

8. Find the first two iterations of Jacobi and Gauss-Seidel using $\vec{x}^{(0)} = (0, 0, 0)^t$:

$$\begin{aligned} 4.63x_1 - 1.21x_2 + 3.22x_3 &= 2.22, \\ -3.07x_1 + 5.48x_2 + 2.11x_3 &= -3.17, \\ 1.26x_1 + 3.11x_2 + 4.57x_3 &= 5.11. \end{aligned}$$

9. The linear system:

$$\begin{aligned} x_1 - x_3 &= 0.2 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425 \\ x_1 - \frac{1}{2}x_2 + x_3 &= 2 \end{aligned}$$

has the solution $(0.9, -0.8, 0.7)^t$.

(a) Is the coefficient matrix strictly diagonally dominant?

(b) Compute the spectral radius of the Gauss-Seidel iteration matrix.

(c) Perform four iterations of the Gauss-Seidel iterative method to approximate the solution.

(d) What happens in part (c) when the first equation in the system is changed to $x_1 - 2x_3 = 0.2$?

10. Comment whether the Gauss-Seidel method can be applied for the following system of equations. If applicable, perform two iterations, assuming the initial guess as $\vec{x}^{(0)} = \big(0, 0, 0\big)^t$.

$$\begin{aligned} 12x_1 + 3x_2 - 5x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76 \end{aligned}$$

11. Compute the condition numbers of the following matrices relative to $\|\cdot\|_\infty$:

(a)

$$\begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix}$$

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(b)

$$\begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}$$

Answers: $1(a)$ $\vec{x} = (-0.7, 1.1, 2.9)^t$; $1(b)$ $\vec{x} = (0.38, -2, 2.5)^t$; (2) $\vec{x} = (0.2246, 0.2812, 0.3280)^t$; (3) do it your self; $4(a)$ $\vec{x} = (1, 2, -1)^t$; $4(b)$ $\vec{x} = (4, 4, 5)^t$; $5(a)$ 4, 5.2202; $5(b)$ 2^k, $\sqrt{1 + 4k^2}$; (6) 12, 8.2394; (7) 0.5, 0.3, 13; (8) Jacobi: $\vec{x}^{(1)} = (0.47948, -0.57847, 1.11816)^t$, $\vec{x}^{(2)} = (-0.44934, -0.74039, 1.37962)^t$, Gauss-Seidel: $\vec{x}^{(1)} = (0.47948, -0.30985, 1.19683)^t$, $\vec{x}^{(2)} = (-0.43384, -1 : 28234, 2.11044)^t$; 9. (a) No (b) $\rho(T_g) = 0.625$ (c) $\vec{x}^{(4)} = (1.07090, -0.67183, 0.59319)^t$ (d) Gauss-Seidel method will not converge($\rho(T_g) = 1.375 > 1$) (10) method will work ($\rho(T_g) = 0.18605 < 1$), $\vec{x}^{(2)} = (-0.13729, 3.93515, 3.75891)^t$ (11) (a) 12.24, (b) 198.17.
