Lecture 7: Numerical Linear Algebra (UMA021): Roots of Non-Linear Equations

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Newton's method:

Importance:

well known and most power full method

$p = 1.2$ $[1, 2]$

Conditions for the convergence:

$p_0 = 1.5$

Suppose $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is small.

IVT

$f \in C[a,b]$

$f \in C^1[a,b] \Rightarrow f, f' \in C[a,b]$

$f \in C^2[a,b] \Rightarrow f, f', f'' \in C[a,b]$

Newton's method:

Derivation:

Let $f(x)=0$ be the equation & p be the exact root of $f(x)=0$ ie $f(p)=0$

let p_0 be an initial guess to the root p

let $f \in C^2[a,b]$ and $|p-p_0|$ is small.

consider the Taylor's polynomial for $f(x)$ about p_0

$$f(p) \approx f(p_0) + (p-p_0) f'(p_0) + \frac{(p-p_0)^2}{2!} f''(p_0).$$

$$0 \approx f(p_0) + (p-p_0) f'(p_0).$$

$$-f(p_0) \approx (p-p_0) f'(p_0)$$

$|p-p_0|$ is small then $|p-p_0|^2$ is very small

$$\frac{-f(p_0)}{f'(p_0)} \approx p - p_0$$

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

(say)

$$\text{IIly} \qquad p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

$$\text{IIly} \qquad p_3 = p_2 - \frac{f(p_2)}{f'(p_2)}$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \quad f'(p_n) \neq 0$$

→ Newton's method

Newton's method:

Graphical representation:

Eq^n of tangent 1

$$y - f(p_0) = f'(p_0)\,(x - p_0)$$

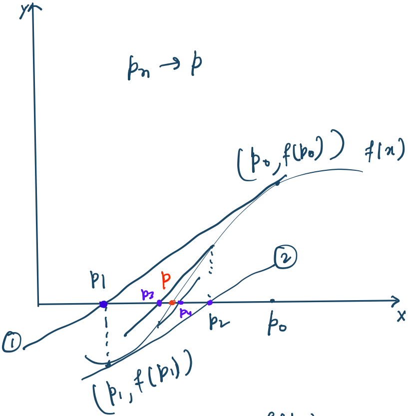
At x-axis $\quad y = 0$

$$-f(p_0) = f'(p_0)\,(x - p_0)$$

$$\frac{-f(p_0)}{f'(p_0)} = x - p_0$$

$$\Rightarrow \quad x = p_0 - \frac{f(p_0)}{f'(p_0)}$$

$(= p_1)$ (say)



Eq^n of tangent 2

$$y - f(p_1) = f'(p_1)\,(x - p_1) \quad \Rightarrow \quad p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

Newton's method:

Example:

Find the root of an equation $f(x) = \cos(x) - x = 0$ with an accuracy of 10^{-2}

$f(x) = \cos x - x = 0 \quad$ — (i)

By IVT, first we find the interval at which root lie so that we can take an initial guess close to the exact root.

$f(0) = \cos 0 - 0 = 1 > 0$

$f(1) = \cos 1 - 1 < 0$

$\Rightarrow$ Root lies in $[0,1]$

Apply Newton's method on (i)

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$= p_n - \frac{\cos(p_n) - p_n}{-\sin p_n - 1}$$

$$= p_n + \frac{\cos p_n - p_n}{\sin p_n + 1}$$

let us take an initial guess $p_0 = 0.5$

$$p_1 = 0.5 + \frac{\cos(0.5) - 0.5}{\sin(0.5) + 1} = 0.7552$$

$$p_2 = 0.7552 + \frac{\cos(0.7552) - 0.7552}{\sin(0.7552) + 1}$$

$$= 0.7391$$

$|p_2 - p_3| < 10^{-2}$

$$p_3 = 0.7391 + \frac{\cos(0.7391) - 0.7391}{\sin(0.7391) + 1}$$

$$= 0.7390$$ Ans.

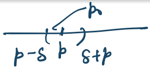
Newton's method:

Convergence result for Newton's method:

Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

nbd of p

ie $|p - p_0|$ is small



$p_n \longrightarrow p$

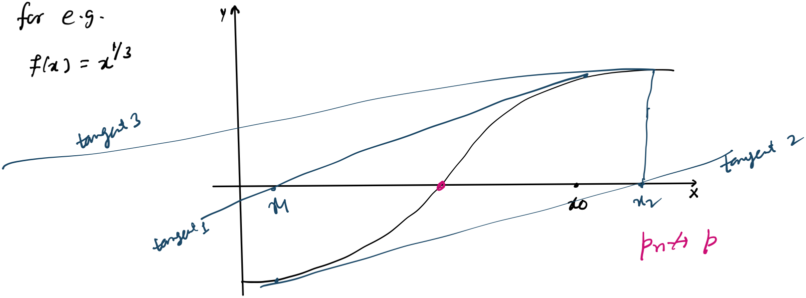
Newton's method:

Case of failure:

(i) When the initial guess is on the inflection of the function i.e. $f''(p_0) = 0$.

for e.g.

$f(x) = x^{1/3}$



Newton's method:

Case of failure:

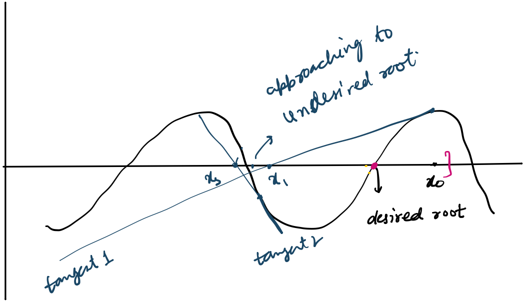
(ii) When there is an another slope near to the initial guess.

for e.g.

$f(x) = \sin x$

or

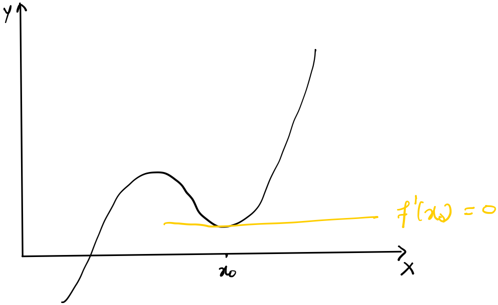
$f(x) = \cos x$



Newton's method:

Case of failure:

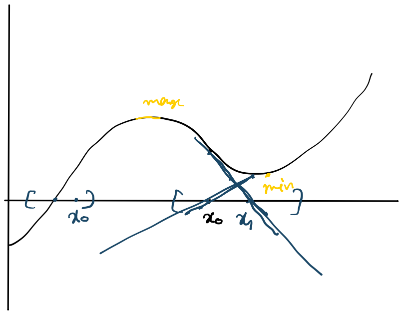
(iii) When the initial guess or any iterative value of function never hits the $x-$axis i.e. $f'(x) = 0$.



Newton's method:

Case of failure:

(iv) When the initial guess is between local maximum or local minimum.



p_n is oscillating
not cging to p

Newton's method:

Exercise:

1. Find the root of an equation $x - e^{-x} = 0$ by using Newton's method with the accuracy of 10^{-2}.
2. The function $f(x) = \sin x$ has a zero on the interval $\boxed{(3, 4)}$ namely, $x = \pi$. Perform three iterations of Newton's method to approximate this zero, using $\boxed{x_0 = 4.}$