Recurrences

Instructor: Dr. Tarunpreet Bhatia
Assistant Professor, CSED
Thapar Institute of Engineering and Technology

Recursion Tree Method

- Recursion Tree Method is a pictorial representation of an iteration method which is in the form of a tree where at each level nodes are expanded.
- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- Each node represents the cost of a single subproblem somewhere in the set of recursive function invocations.
- A Recursion Tree is best used to generate a good guess, which can be verified by the Substitution Method.

Steps to solve a recurrence relation using recursion tree

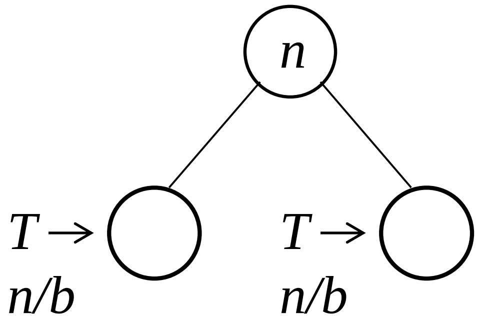
- Draw the **recursion tree** for the given recurrence relation.
- Calculate the **height of the recursion tree** formed.
- Calculate the **cost** (*time required to solve all the subproblems at a level*) at each level.
- Calculate the total number of nodes at each level in the recursion tree.
- Sum up the **cost of all the levels** in the recursion tree.

Recursion Tree Method

Here while solving recurrences, we divide the problem into subproblems.

For e.g., $T(n) = a\,T(n/b) + f(n)$ where $a \geq 1$,$b > 1$ and $f(n)$ is a given function .

F(n) is the cost of splitting or combining the sub problems.



Example 1

$T(n) = 3T(n/4) + cn^2$

Appendix: Geometric series

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

for $x \neq 1$ *and* $x > 1$

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for $x \neq 1$ *and* $x < 1$

$$1 + x + x^2 + \ldots. = \frac{1}{1 - x}$$

for $|x| < 1$

Example 2

$$T(n) = 2T(n/2) + n$$

Example 3

$T(n) = T(n/3) + T(2n/3) + n$

Example 4

Solve $T(n) = T(n/4) + T(n/2) + n^2$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$T(n/4) \qquad\qquad\qquad T(n/2)$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

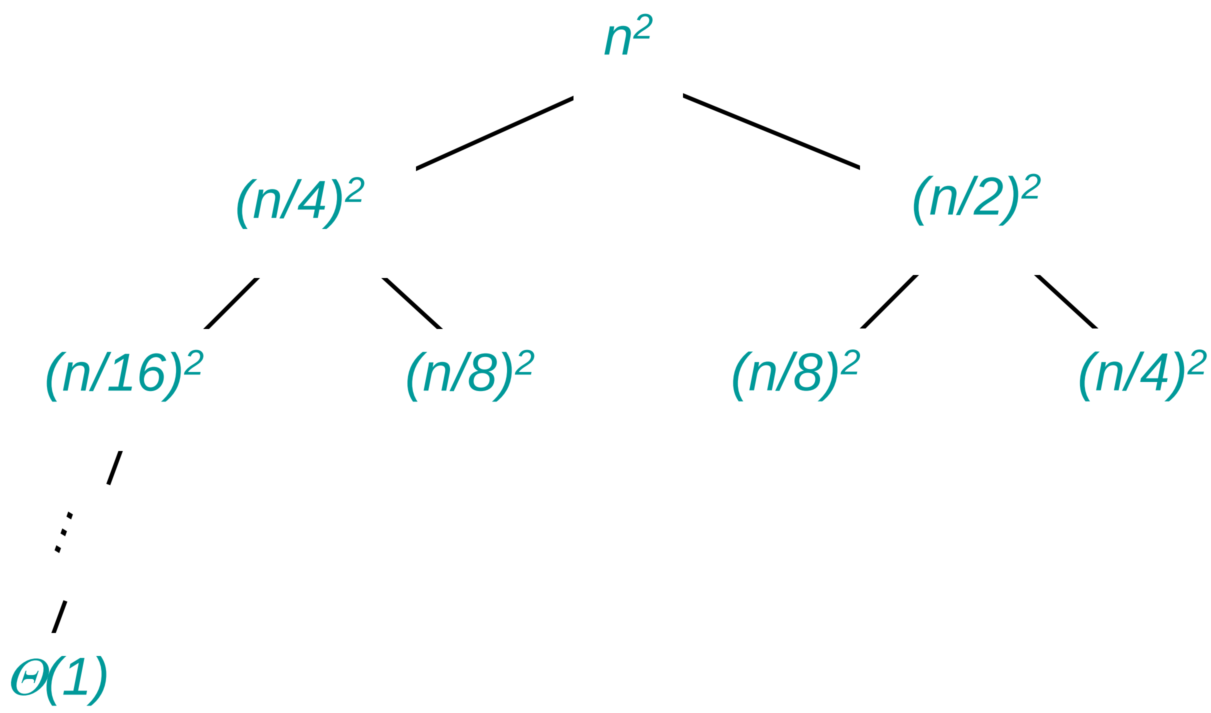
$$n^2$$

$$(n/4)^2 \qquad\qquad (n/2)^2$$

$$T(n/16) \qquad T(n/8) \qquad T(n/8) \qquad T(n/4)$$

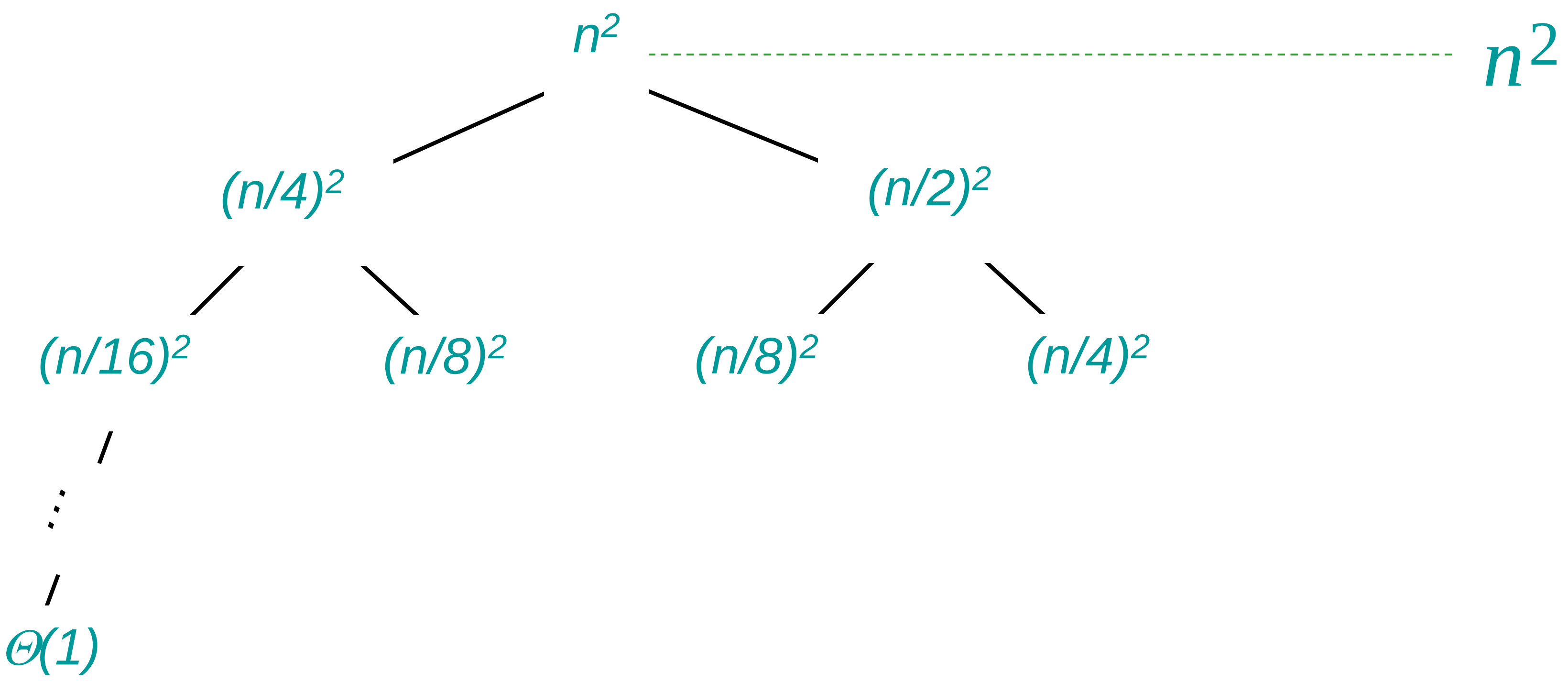
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



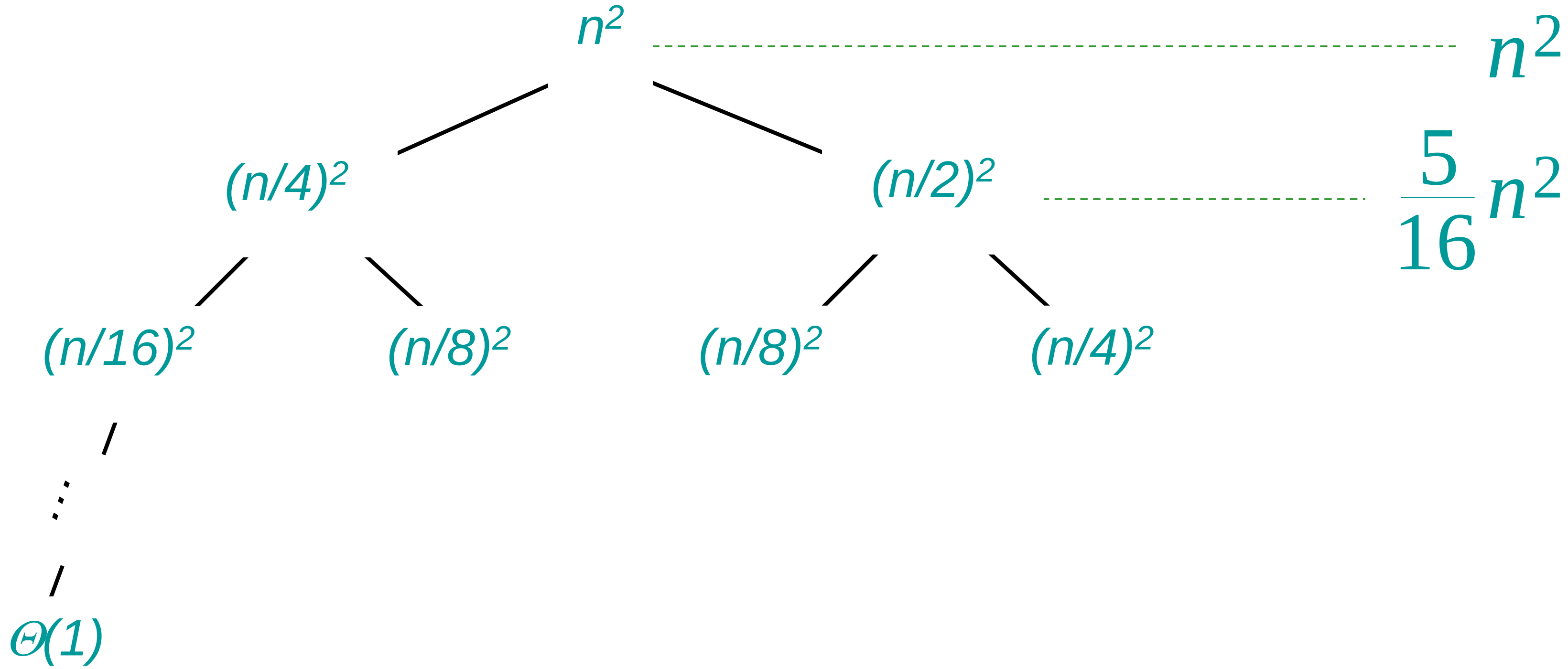
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



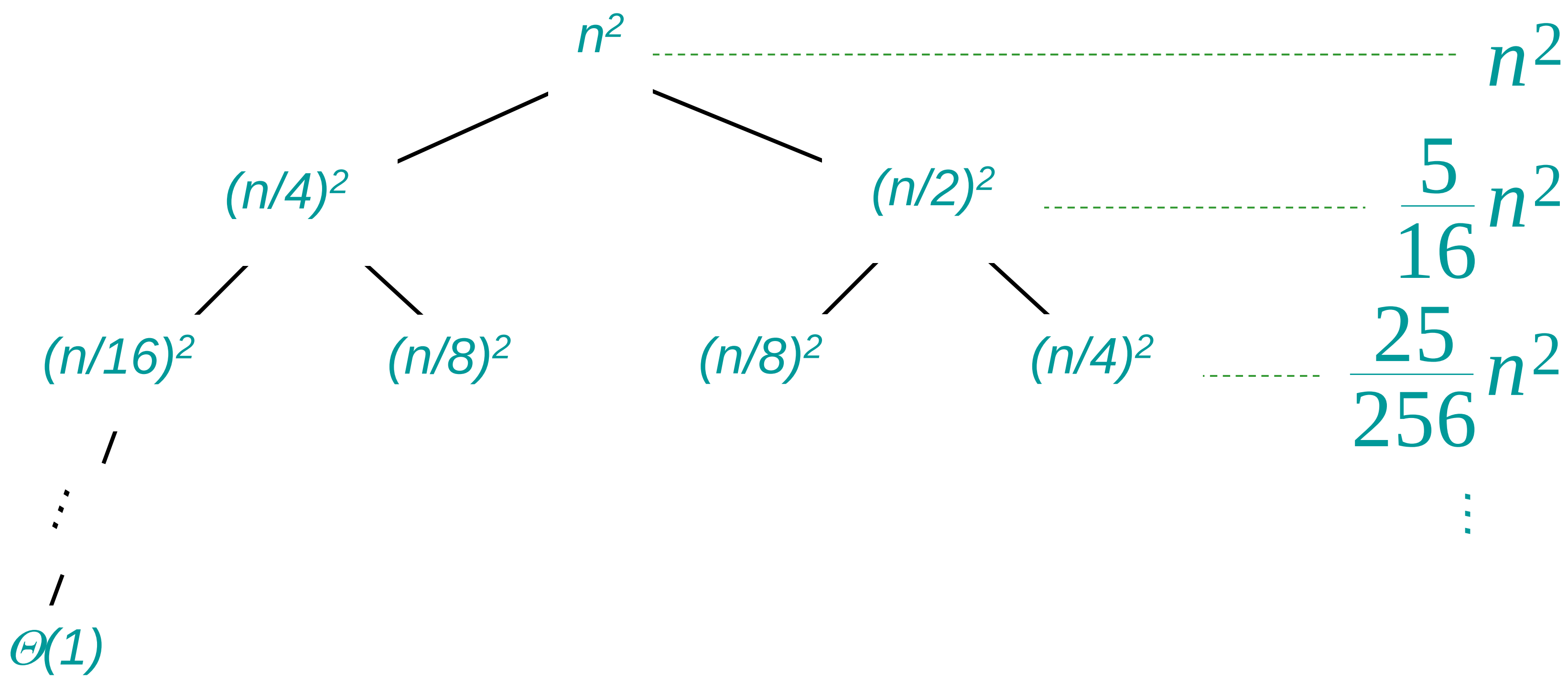
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



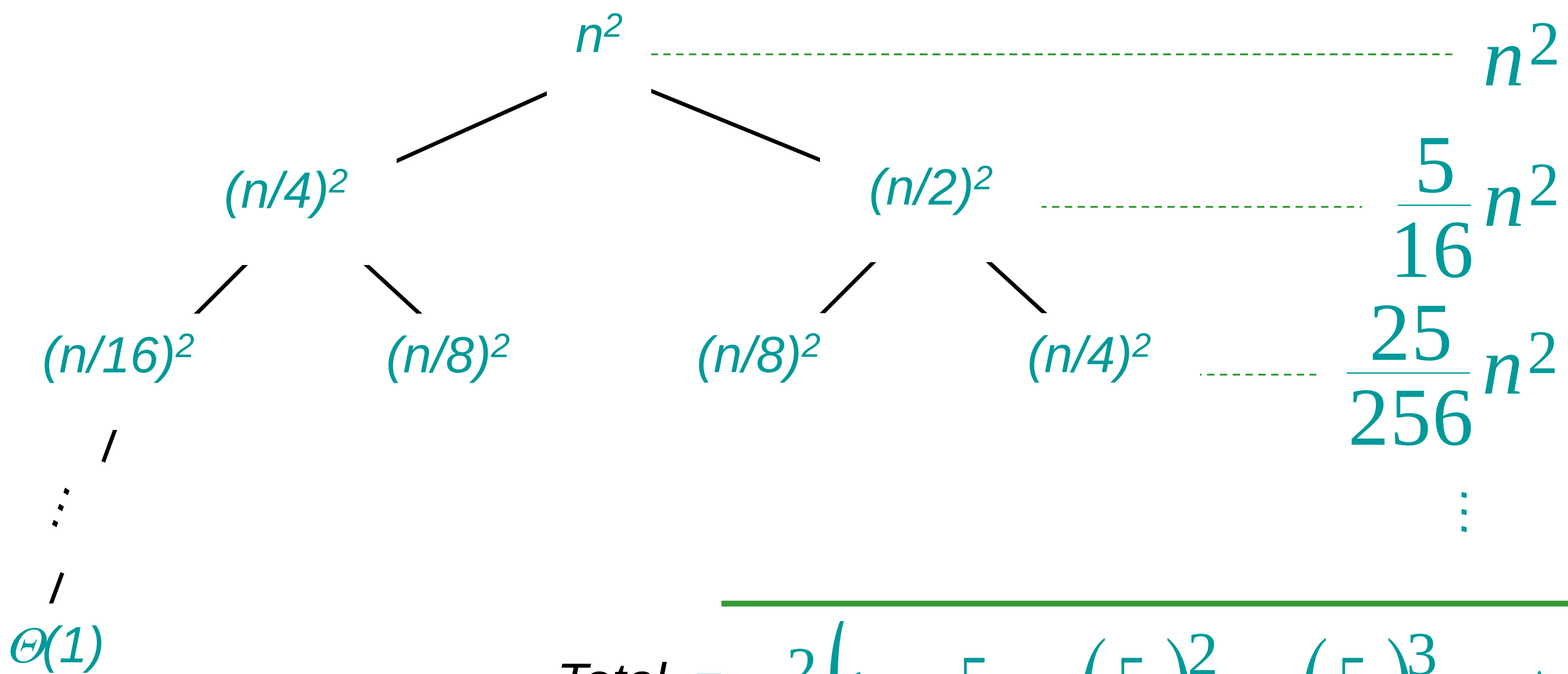
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

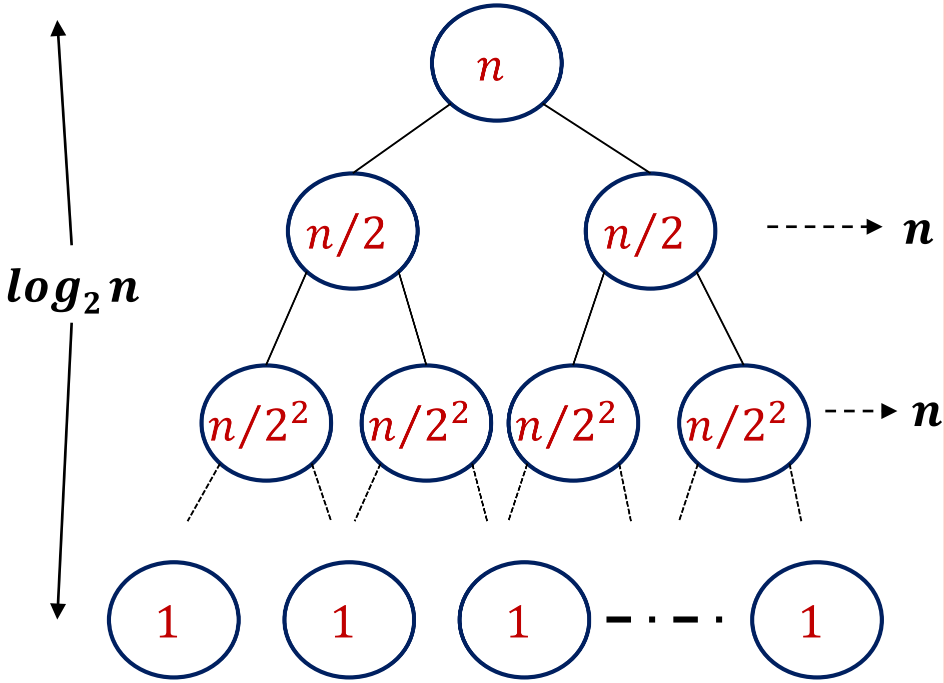


$$Total = n^2\left(1+\frac{5}{16}+\left(\frac{5}{16}\right)^2+\left(\frac{5}{16}\right)^3+\Lambda\right)$$

$= \Theta(n^2)$ *geometric series*

Recurrence Tree Method

The recursion tree for this recurrence is



Example 1: $\boldsymbol{T(n) = 2T(n/2) + n}$

- *When we add the values across the levels of the recursion tree, we get a value of n for every level.*
- *The bottom level has $2^{\log n}$ nodes, each contributing the cost $T(1)$.*
- *We have* $n + n + n + \ldots\ldots$ $\log n$ $times$

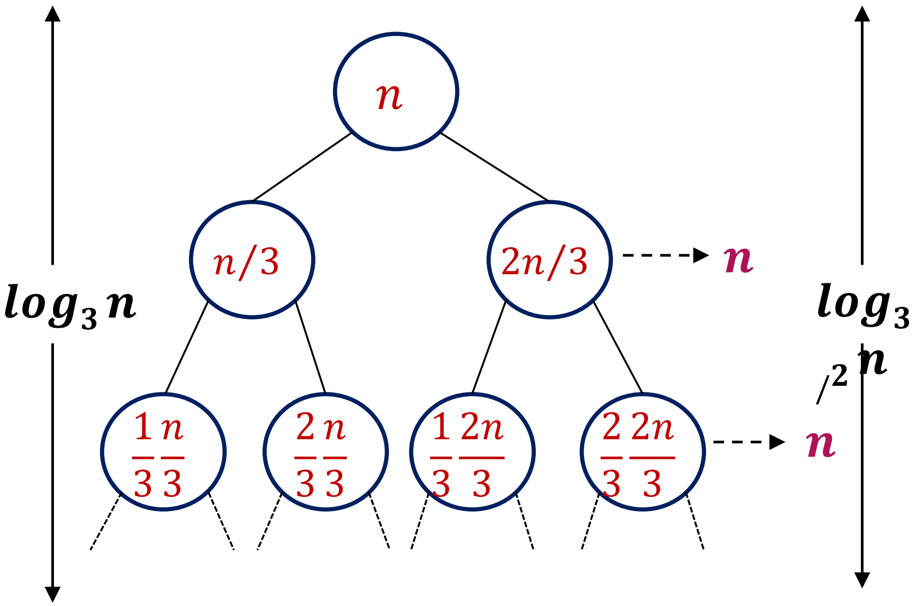
$$\boldsymbol{T(n) = \sum_{i=0}^{log_2 n - 1} n + 2^{log\, n} T(1)}$$

$$\boldsymbol{T(n) = n\, log\, n + n}$$

$$\boxed{\boldsymbol{T(n) = O(n \log n)}}$$

Recurrence Tree Method

The recursion tree for this recurrence is



Example 2: $\boldsymbol{T(n) = T(n/3) + T(2n/3) + n}$

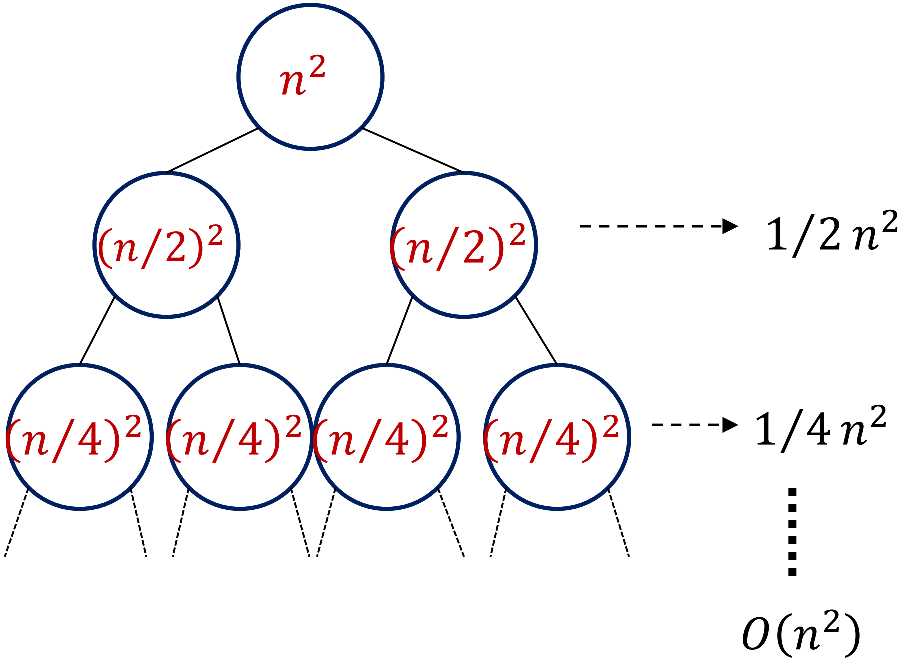
- *When we add the values across the levels of the recursion tree, we get a value of n for every level.*

$$T(n) = \sum_{i=0}^{\log_{3/2} n - 1} n + n^{\log_{3/2} 2} T(1)$$

$$\boxed{\boldsymbol{T(n) \in n \log_{3/2} n}}$$

Recurrence Tree Method

The recursion tree for this recurrence is



Example 3: $\boldsymbol{T(n) = 2T(n/2) + c.n^2}$

- *Sub-problem size at level i is ${}^{n}/_{2^i}$*
- *Cost of problem at level i Is $\left({}^{n}/_{2^i}\right)^2$*
- *Total cost,*

$$\boldsymbol{T(n) \le n^2 \sum_{i=0}^{\log_2 n-1} \left(\frac{1}{2}\right)^i}$$

$$\boldsymbol{T(n) \le n^2 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i}$$

$$\boldsymbol{T(n) \le 2n^2}$$

$$\boxed{\boldsymbol{T(n) = O\left(n^2\right)}}$$

Try this!

$$T(n)=\begin{cases}1 & n=0\\ 2T(n-1)+1 & n>0\end{cases}$$

$$T(n)=\begin{cases}1 & n=1\\ T(n/2)+c & n>1\end{cases}$$

$$T(n)=\begin{cases}1 & n=1\\ 4T(n/2)+n & n>1\end{cases}$$

$$T(n)=\begin{cases}1 & n=1\\ T(n/2)+(3n/2) & n>1\end{cases}$$