Understanding Cryptography – A Textbook for Students and Practitioners

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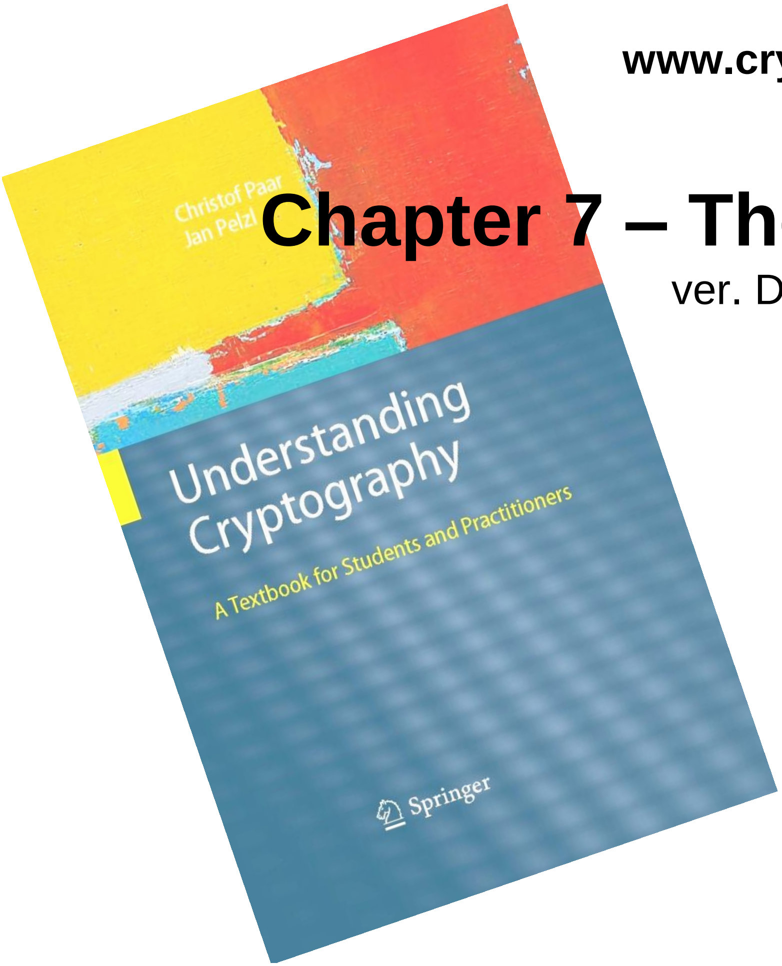
www.crypto-textbook.com



Chapter 7 – The RSA Cryptosystem

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These slides were prepared by Benedikt Driessen, Christof Paar and Jan Pelzl

Some legal stuff (sorry): Terms of use

Content of this Chapter

- The RSA Cryptosystem
- Implementation aspects
- Finding Large Primes
- Attacks and Countermeasures
- Lessons Learned

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The RSA Cryptosystem

- Martin Hellman and Whitfield Diffie published their landmark public-key paper in 1976
- Ronald Rivest, Adi Shamir and Leonard Adleman proposed the asymmetric RSA cryptosystem in1977
- Until now, RSA is the most widely use asymmetric cryptosystem although elliptic curve cryptography (ECC) becomes increasingly popular
- RSA is mainly used for two applications
  - Transport of (i.e., symmetric) keys (cf. Chptr 13 of *Understanding Cryptography*)
  - Digital signatures (cf. Chptr 10 of *Understanding Cryptography*)

Encryption and Decryption

- RSA operations are done over the integer ring Z_n (i.e., arithmetic modulo n), where $n = p * q$, with p, q being large primes
- Encryption and decryption are simply exponentiations in the ring

Definition

Given the public key $(n,e) = k_{pub}$ and the private key $d = k_{pr}$ we write

$$y = e_{k_{pub}}(x) \equiv x^e \bmod n$$

$$x = d_{k_{pr}}(y) \equiv y^d \bmod n$$

where $x, y \in Z_n$.

We call $e_{k_{pub}}()$ the encryption and $d_{k_{pr}}()$ the decryption operation.

- In practice x, y, n and d are very long integer numbers ($\geq$ 1024 bits)
- The security of the scheme relies on the fact that it is hard to derive the „private exponent“ d given the public-key (n, e)

Key Generation

- Like all asymmetric schemes, RSA has set-up phase during which the private and public keys are computed

<u>Algorithm: RSA Key Generation</u>

Output: public key: $k_{pub} = (n, e)$ and private key $k_{pr} = d$

1. Choose two large primes p, q
2. Compute $n = p * q$
3. Compute $\Phi(n) = (p-1) * (q-1)$
4. Select the public exponent $e \in \{1, 2, \ldots, \Phi(n)-1\}$ such that gcd$(e, \Phi(n)) = 1$
5. Compute the private key d such that $d * e \equiv 1 \bmod \Phi(n)$
6. **RETURN** $k_{pub} = (n, e), k_{pr} = d$

Remarks:

- Choosing two large, distinct primes p, q (in Step 1) is non-trivial
- gcd$(e, \Phi(n)) = 1$ ensures that e has an inverse and, thus, that there is always a private key d

Example: RSA with small numbers

ALICE

Message $x = 4$

$K_{pub} = (33,3)$ ←

$y = x^e \equiv 4^3 \equiv 31 \bmod 33$

$y = 31$ →

BOB

1. Choose $p = 3$ and $q = 11$
2. Compute $n = p * q = 33$
3. $\Phi(n) = (3-1) * (11-1) = 20$
4. Choose $e = 3$
5. $d \equiv e^{-1} \equiv 7 \bmod 20$

$y^d = 31^7 \equiv 4 = x \bmod 33$

Content of this Chapter

- The RSA Cryptosystem
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Implementation aspects

- The RSA cryptosystem uses only one arithmetic operation (modular exponentiation) which makes it conceptually a simple asymmetric scheme
- Even though conceptually simple, due to the use of very long numbers, RSA is orders of magnitude slower than symmetric schemes, e.g., DES, AES
- When implementing RSA (esp. on a constrained device such as smartcards or cell phones) close attention has to be paid to the correct choice of arithmetic algorithms
- The square-and-multiply algorithm allows fast exponentiation, even with very long numbers…

Square-and-Multiply

- **Basic principle**: Scan exponent bits from left to right and square/multiply operand accordingly

Algorithm: Square-and-Multiply for x^H *mod n*

Input: Exponent H, base element x, Modulus n

Output: $y = x^H \bmod n$

1. Determine binary representation $H = (h_t, h_{t-1}, ..., h_0)_2$
2. **FOR** $i = t-1$ **TO** 0
3. $\quad\quad y = y^2 \bmod n$
4. $\quad$ **IF** $h_i = 1$ **THEN**
5. $\quad\quad\quad y = y * x \bmod n$
6. **RETURN** y

- Rule: Square in every iteration (Step 3) and multiply current result by x if the exponent bit $h_i = 1$ *(Step 5)*
- Modulo reduction after each step keeps the operand y small

Example: Square-and-Multiply

- Computes x^{26} without modulo reduction
- Binary representation of exponent: $26 = (1,1,0,1,0)_2 = (h_4,h_3,h_2,h_1,h_0)_2$

Step		Binary exponent	Op	Comment
1	$x = x^1$	$(1)_2$		Initial setting, h_4 processed
1a	$(x^1)^2 = x^2$	$(10)_2$	SQ	Processing h_3
1b	$x^2 * x = x^3$	$(11)_2$	MUL	$h_3 = 1$
2a	$(x^3)^2 = x^6$	$(110)_2$	SQ	Processing h_2
2b	-	$(110)_2$	-	$h_0 = 0$
3a	$(x^6)^2 = x^{12}$	$(1100)_2$	SQ	Processing h_1
3b	$x^{12} * x = x^{13}$	$(1101)_2$	MUL	$h_1=1$
4a	$(x^{13})^2 = x^{26}$	$(11010)_2$	SQ	Processing h_0
4b	-	$(11010)_2$	-	$h_0 = 0$

- Observe how the exponent evolves into $x^{26} = x^{11010}$

Complexity of Square-and-Multiply Alg.

- The square-and-multiply algorithm has a logarithmic complexity, i.e., its run time is proportional to the bit length (rather than the absolute value) of the exponent
- Given an exponent with t+1 bits

  $$H = (h_t, h_{t-1}, ..., h_0)_2$$

  with $h_t = 1$, we need the following operations
  - *# Squarings* $= t$
  - Average # multiplications $= 0.5\ t$
  - Total complexity: *#SQ + #MUL* $= 1.5\ t$
- Exponents are often randomly chosen, so *1.5 t* is a good estimate for the average number of operations
- Note that each squaring and each multiplication is an operation with very long numbers, e.g., 2048 bit integers.

Speed-Up Techniques

- Modular exponentiation is computationally intensive
- Even with the square-and-multiply algorithm, RSA can be quite slow on constrained devices such as smart cards
- Some important tricks:
  - Short public exponent e
  - Chinese Remainder Theorem (CRT)
  - Exponentiation with pre-computation *(not covered here)*

Fast encryption with small public exponent

- Choosing a small public exponent *e* does not weaken the security of RSA
- A small public exponent improves the speed of the RSA encryption significantly

Public Key	***e* as binary string**	**#MUL + #SQ**
$2^1+1 = 3$	$(11)_2$	1 + 1 = 2
$2^4+1 = 17$	$(1\ 0001)_2$	4 + 1 = 5
$2^{16} + 1$	$(1\ 0000\ 0000\ 0000\ 0001)_2$	16 + 1 = 17

- This is a commonly used trick (e.g., SSL/TLS, etc.) and makes RSA the fastest asymmetric scheme with regard to encryption!

Fast decryption with CRT

- Choosing a small private key *d* results in security weaknesses!
  - In fact, d must have at least *0.3t* bits, where *t* is the bit length of the modulus *n*
- However, the Chinese Remainder Theorem (CRT) can be used to (somewhat) accelerate exponentiation with the private key *d*
- Based on the CRT we can replace the computation of

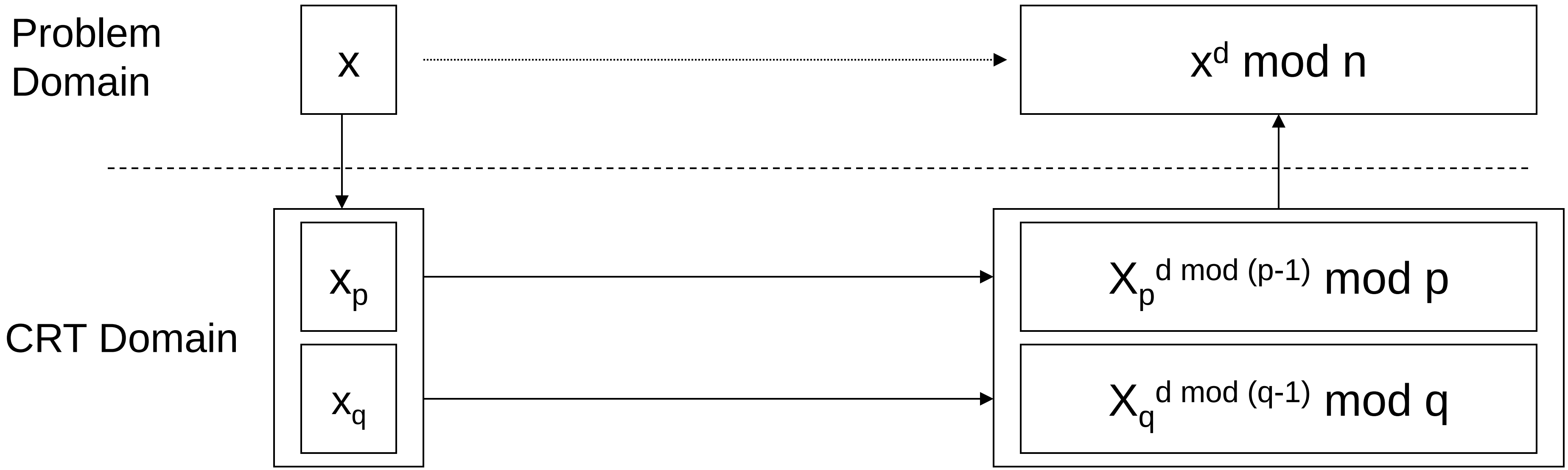
$$x^{d \bmod \Phi(n)} \bmod n$$

by two computations

$$x^{d \bmod (p-1)} \bmod p \quad \text{and} \quad x^{d \bmod (q-1)} \bmod q$$

where *q* and *p* are „small“ compared to *n*

Basic principle of CRT-based exponentiation



- CRT involves three distinct steps

  (1) Transformation of operand into the CRT domain

  (2) Modular exponentiation in the CRT domain

  (3) Inverse transformation into the problem domain

- These steps are equivalent to one modular exponentiation in the problem domain

CRT: Step 1 – Transformation

- Transformation into the CRT domain requires the knowledge of *p* and *q*
- *p* and *q* are only known to the owner of the private key, hence CRT cannot be applied to speed up encryption
- The transformation computes (x_p, x_q) which is the representation of x in the CRT domain. They can be found easily by computing

$$x_p \equiv x \bmod p \qquad \text{and} \qquad x_q \equiv x \bmod q$$

CRT: Step 2 – Exponentiation

- Given d_p and d_q such that

$$d_p \equiv d \bmod (p-1) \qquad \text{and} \qquad d_q \equiv d \bmod (q-1)$$

one exponentiation in the problem domain requires two exponentiations in the CRT domain

$$y_p \equiv x_p^{d_p} \bmod p \qquad \text{and} \qquad y_q \equiv x_q^{d_q} \bmod q$$

- In practice, p and q are chosen to have half the bit length of n, i.e., $|p| \approx |q| \approx |n|/2$

CRT: Step 3 – Inverse Transformation

- Inverse transformation requires modular inversion twice, which is computationally expensive

$$c_p \equiv q^{-1} \bmod p \quad \text{and} \quad c_q \equiv p^{-1} \bmod q$$

- Inverse transformation assembles y_p, y_q to the final result $y \bmod n$ in the problem domain

$$y \equiv [\, q * c_p \,] * y_p + [\, p * c_q \,] * y_q \bmod n$$

- The primes p and q typically change infrequently, therefore the cost of inversion can be neglected because the two expresssions

  $[\, q * c_p \,]$ and $[\, p * c_q \,]$

  can be precomputed and stored

Complexity of CRT

- We ignore the transformation and inverse transformation steps since their costs can be neglected under reasonable assumptions
- Assuming that n has $t+1$ bits, both p and q are about $t/2$ bits long
- The complexity is determined by the two exponentiations in the CRT domain. The operands are only $t/2$ bits long. For the exponentiations we use the square-and-multiply algorithm:
    - # squarings (one exp.): $\#SQ = 0.5\, t$
    - # aver. multiplications (one exp.): #MUL = 0.25t
    - Total complexity: $2 * (\#MUL + \#SQ) = 1.5t$
- This looks the same as regular exponentations, but since the operands have half the bit length compared to regular exponent., each operation (i.e., multipl. and squaring) is 4 times faster!
- Hence CRT is **4 times** faster than straightforward exponentiation

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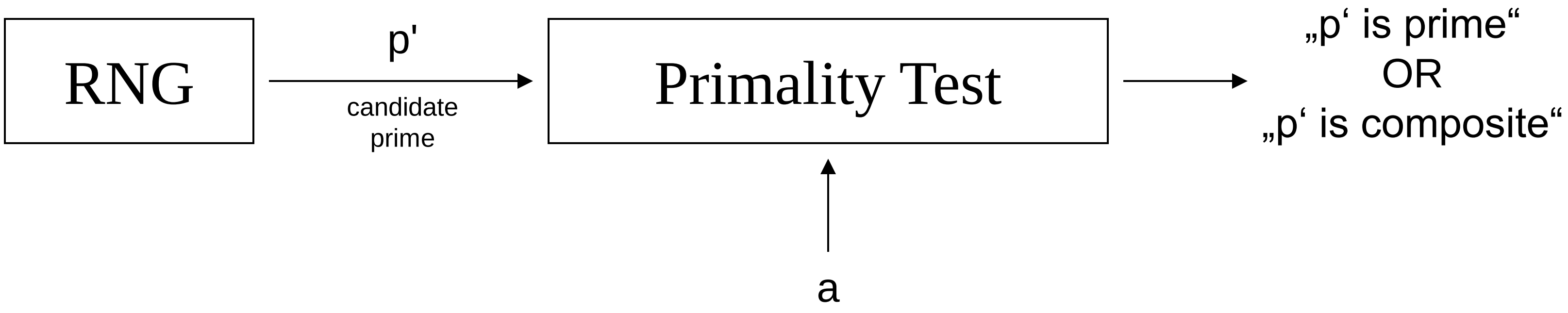
Finding Large Primes

- Generating keys for RSA requires finding two large primes p and q such that $n = p * q$ is sufficiently large
- The size of p and q is typically half the size of the desired size of n
- To find primes, random integers are generated and tested for primality:



- The random number generator (RNG) should be non-predictable otherwise an attacker could guess the factorization of n

Primality Tests

- Factoring *p* and *q* to test for primality is typically not feasible
- However, we are not interested in the factorization, we only want to know whether *p* and *q* are composite
- Typical primality tests are probabilistic, i.e., they are not 100% accurate but their output is correct with very high probability
- A probabilistic test has two outputs:
  - „p‘ is composite“ – always true
  - „p‘ is a prime“ – only true with a certain probability
- Among the well-known primality tests are the following
  - Fermat Primality-Test
  - Miller-Rabin Primality-Test

Fermat Primality-Test

- Basic idea: Fermat‘s Little Theorem holds for all primes, i.e., if a number *p‘* is found for which $a^{p'-1} \not\equiv 1 \bmod p'$, it is not a prime

<u>Algorithm: Fermat Primality-Test</u>

Input: Prime candidate *p‘*, security parameter *s*

Output: „*p‘* is composite“ or „*p‘* is likely a prime“

1. **FOR** $i = 1$ **TO** s

2. choose random $a \in \{2,3, \ldots, p'-2\}$

3. **IF** $a^{p'-1} \not\equiv 1 \bmod p'$ **THEN**

4. **RETURN** „*p‘* is composite“

5. **RETURN** „*p‘* is likely a prime“

- For certain numbers („Carchimchael numbers“) this test returns „*p‘* is likely a prime“ often – although these numbers are composite
- Therefore, the Miller-Rabin Test is preferred

Theorem for Miller-Rabin‘s test

- The more powerful Miller-Rabin Test is based on the following theorem

Theorem

Given the decomposition of an odd prime candidate *p‘*

$$p' - 1 = 2^u * r$$

where *r* is odd. If we can find an integer *a* such that

$$a^r \not\equiv 1 \bmod p' \quad \text{and} \quad a^{r2^j} \not\equiv p' - 1 \bmod p'$$

For all $j = \{0,1, ..., u-1\}$, then *p‘* is composite.

Otherwise it is probably a prime.

- This theorem can be turned into an algorithm

Miller-Rabin Primality-Test

Algorithm: Miller-Rabin Primality-Test

Input: Prime candidate p' with $p'-1 = 2^{u} * r$ security parameter s

Output: „p' is composite" or „p' is likely a prime"

1. **FOR** $i = 1$ **TO** s

2. choose random $a \in \{2,3, ..., p'-2\}$

3. $z \equiv a^r \bmod p'$

4. **IF** $z \neq 1$ **AND** $z \neq p'-1$ **THEN**

5. **FOR** $j = 1$ **TO** $u-1$

6. $z \equiv z^2 \bmod p'$

7. **IF** $z = 1$ **THEN**

8. **RETURN** „p' is composite"

9. **IF** $z \neq p'-1$ **THEN**

10. **RETURN** „p' is composite"

11. **RETURN** „p' is likely a prime"

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Attacks and Countermeasures 1/3

- There are two distinct types of attacks on cryptosystems
  - **Analytical attacks** try to break the mathematical structure of the underlying problem of RSA
  - **Implementation attacks** try to attack a real-world implementation by exploiting inherent weaknesses in the way RSA is realized in software or hardware

Attacks and Countermeasures 2/3

RSA is typically exposed to these analytical attack vectors

- **Mathematical attacks**
  - The best known attack is factoring of n in order to obtain $\Phi(n)$
  - Can be prevented using a sufficiently large modulus n
  - The current factoring record is 664 bits. Thus, it is recommended that n should have a bit length between 1024 and 3072 bits
- **Protocol attacks**
  - Exploit the malleability of RSA, i.e., the property that a ciphertext can be transformed into another ciphertext which decrypts to a related plaintext – without knowing the private key
  - Can be prevented by proper padding

Attacks and Countermeasures 3/3

- Implementation attacks can be one of the following
  - **Side-channel analysis**
    - Exploit physical leakage of RSA implementation (e.g., power consumption, EM emanation, etc.)
  - **Fault-injection attacks**
    - Inducing faults in the device while CRT is executed can lead to a complete leakage of the private key

More on all attacks can be found in Section 7.8 of *Understanding Cryptography*

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Lessons Learned

- RSA is the most widely used public-key cryptosystem
- RSA is mainly used for key transport and digital signatures
- The public key e can be a short integer, the private key d needs to have the full length of the modulus n
- RSA relies on the fact that it is hard to factorize n
- Currently 1024-bit cannot be factored, but progress in factorization could bring this into reach within 10-15 years. Hence, RSA with a 2048 or 3076 bit modulus should be used for long-term security
- A naïve implementation of RSA allows several attacks, and in practice RSA should be used together with padding