SHANGHAI JIAOTONG UNIVERSITY
X071571: OPTIMIZATION METHODS

Problem Set 2

Problem 1. Let $f : C \to \mathbb{R}$ be a convex function defined on an open convex set $C \subset \mathbb{R}^n$. Show that for every $x_0 \in C$ there is an affine function g such that $g(x_0) = f(x_0)$ and $g \le f$. We say that a convex function can be *minorized* by an affine function at every point.

Problem 2. Prove the following:

- Let A be a symmetric matrix. Prove that the function $f_A(x) = x^T Ax$ is convex if and only if A is positive semi-definite.
- Prove the strict convexity of the function $f(x) = \log\left(\frac{1}{1-\|x\|^2}\right)$ on the set $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : \|x\| < 1\}$.

Problem 3. Let f be the Kullbak-Leibler divergence between $u, v \in (\mathbb{R}_{>0})^n$ given by

$$D_{KL}(u,v) = \sum_{i=1}^{n}(u_i \log \frac{u_i}{v_i} - u_i + v_i)$$

- Prove that $D_{KL}(u,v)$ is convex on $(\mathbb{R}_{>0})^n \times (\mathbb{R}_{>0})^n$.
- Prove that $D_{KL}(u,v) \ge 0$ for all $u, v \in (\mathbb{R}_{>0})^n$, called the *information inequality.*
- Prove that $D_{KL}(u,v) = 0$ if and only if $u = v$.

Problem 4. For $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ let $f_{x,v} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$f_{x,v}(t) = f(x + tv), \qquad x, v \in \mathbb{R}^n.$$

Prove that f is convex if and only if $f_{x,v}$ is convex for any $x, v \in \mathbb{R}^n$.

Problem 5. Prove that the functions

- $f(X) = \log\det(X)$, $\mathrm{dom}(f) = \mathbb{S}^n_{++} = \{\text{symmetric positive definite matrices}\}$. *[Hint: Use Problem 4]*
- $g(x) = (x_1 \cdots x_n)^{1/n}$, $\mathrm{dom}(g) = (\mathbb{R}_{>0})^n$

are concave

Problem 6. Prove the following inequalities:

- $\sqrt{ab} \le \frac{a+b}{2}$, for $a, b \ge 0$.
- The Holder inequality:

$$\sum_{i=1}^{n} x_i y_i \le (|x_i|^p)^{1/p} (|y_i|^q)^{1/q},$$

  where $1/p + 1/q = 1$.

[Hint: Use the Jensen inequality]

Problem 7. Using Fenchel's inequality, show that for any $c > 0$ and any $s, x \in \mathbb{R}^n$, one has

$$c\|x\|^2 + \frac{1}{c}\|s\|^2 \geq 2\langle s, x\rangle.$$

Problem 8. Compute the conjugate of the following functions:

- The standard norm on $\mathbb{R}^n$: $f(x) = \|x\|$.
- Maximum function: $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbb{R}^n$.
- Piecewise-linear function on $\mathbb{R}$: $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbb{R}$. You can assume that the a_i's are sorted in increasing order, i.e., $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, i.e. for each i there is at least one x with $f(x) = a_i x + b_i$.

Problem 9. Support function calculus. The *support function* of a set $C \subset \mathbb{R}^n$ is defined as $S_C(y) = \sup\{\langle y, x\rangle | x \in C\}$.

(1) Show that S_C is a convex function.
(2) Show that $S_{A+B} = S_A + S_B$.
(3) Show that $S_{A\cup B} = \max\{S_A, S_B\}$.
(4) Let B be closed and convex. Show that $A \subset B$ if and only if $S_A(y) \leq S_B(y)$ for all y.

Problem 10. *Properties of conjugate functions.*

- Conjugate of convex plus affine function. Define $g(x) = f(x) + c^T x + d$, where f is convex. Express g^* in terms of f^* (and c, d).
- Conjugate and minimization. Let $f(x, z)$ be convex in (x, z) and define $g(x) = \inf_z f(x, z)$. Express the conjugate g^* in terms of f^*. As an application, express the conjugate of $g(x) = \inf_z\{h(z) | Az + b = x\}$, where h is convex, in terms of h^*, A and b.