SHANGHAI JIAOTONG UNIVERSITY
X071571: OPTIMIZATION METHODS

Problem Set 1

Problem 1. Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be an affine mapping. Prove:

- if $C \subset \mathbb{R}^m$ is convex, so is $F(C) \subset \mathbb{R}^n$.
- If $D \subset \mathbb{R}^n$ is convex, so is $F^{-1}(D) \subset \mathbb{R}^m$.

Problem 2. Let $C \subset \mathbb{R}^n$ be a convex set. Show that the interior $C^o \subset \mathbb{R}^n$ and the closure $\bar{C} \subset \mathbb{R}^n$ of C are convex.

Problem 3. For two nonempty sets A and B in $\mathbb{R}^n$, show that $\mathrm{co}(A + B) = \mathrm{co}A + \mathrm{co}B$.

Here is a definition needed for the next Problem:

Definition 1. A *cone* is a subset $C \subset \mathbb{R}^n$ satisfying

$$x \in C \Longrightarrow tx \in C \text{ for every } t > 0.$$

A *convex cone* is a cone which is also convex.

Problem 4. A non-empty subset M of $\mathbb{R}^n$ is a convex cone if and only if it possesses the following properties:

(1) it is a cone;
(2) it contains the sums of its elements: $x, y \in M \Longrightarrow x + y \in M$.

Problem 5. Let $\mathbb{S}^n$ be the set of $n \times n$ symmetric matrices and let

$$\mathbb{S}^n_+ = \{\text{positive semi-definite symmetric matrices}\} \subset \mathbb{S}^n.$$

Prove that $\mathbb{S}^n_+$ is a convex cone.

Problem 6. The *normal cone* of a set C at a boundary point x_0 is the set

$$N_C(x_0) = \{y \in \mathbb{R}^n : y^T(x - x_0) \leq 0 \text{ for all } x \in C\}.$$

Show that the normal cone is a convex cone (with no assumption on C). Give a simple description of the normal cone of a polyhedron $\{x : Ax \leq b\}$ at a point in its boundary.

Problem 7. Let $C \subset \mathbb{R}^n$ be a convex and compact set. Show that if $\bar{x} \in C$ is such that $\|\bar{x}\| = \max_{x \in C} \|x\|$, then $\bar{x}$ is an extremal point of C.

Problem 8. Here C and A are two closed sets such that $C \subset A$.

- Show that $p_C \circ p_A = p_C$ if C and A are two linear subspaces.
- Show on an example that the property need not hold under mere convexity of C and A.

Here are two definitions needed for the next Problem:

Definition 2. Let $C \subset \mathbb{R}^n$ be convex. A non-empty convex subset $F \subset C$ is a *face* of C if it satisfies the following property:

$$\left.\begin{array}{c} x_1, x_2 \in C \\ \exists 0 < \alpha < 1 \text{ such that } \alpha x_1 + (1-\alpha) x_2 \in F \end{array}\right\} \Longrightarrow [x_1, x_2] \subset F.$$

Definition 3. A subset $F \subset C$ is called an *exposed face* of C if there is a hyperplane

$$H_{s,r} = \{x \in \mathbb{R}^n : \langle s, x \rangle = r\} \quad (\text{where } s \in \mathbb{R}^n,\ r \in \mathbb{R})$$

such that $\langle y, s \rangle \leq r$ for every $y \in C$, and such that

$$F = C \cap H_{s,r}.$$

Problem 9. Let $C \subset \mathbb{R}^n$ be convex.

- Let $F \subset C$ be a face of C, and let $x \in F$. Show that x is an extremal point of F if and only if it is an extremal point of C.
- Show that an exposed face of C is a face.

Problem 10. Prove Minkowski's Theorem: Let $C \subset \mathbb{R}^n$ be convex and compact. Then $C = \operatorname{co} \operatorname{ext} C$. *[Hint: Proceed by induction on* $\dim(C)$*. Use the the fact that there is a hyperplane supporting* C *at every boundary point and make use of Problem 9.]*