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1 Basic

1.1 vimrc

```
se is nu ru et tgc sc hls cin cino+=j1 sw=2 sts=2 bs=2
mouse=a "encoding=utf-8 ls=2
syn on | colo desert | filetype indent on
inoremap {<CR> {<CR>}<ESC>O
map <F8> <ESC>:w<CR>:!g++ "%<" -o "%<" -g -std=gnu++20 -
    DCKISEKI -Wall -Wextra -Wshadow -Wfatal-errors -
    Wconversion -fsanitize=address,undefined,float-
    divide-by-zero,float-cast-overflow && echo success<
    CR>
map <F9> <ESC>:w<CR>:!g++ "%<" -o "%<" -O2 -g -std=gnu
    ++20 && echo success<CR>
map <F10> <ESC>:!. / "%<"<CR>
ca Hash w !cpp -dD -P -fpreprocessed \ | tr -d '[:space
    :]' \ | md5sum \ | cut -c-6
let c_no_curly_error=1
" setxkbmap -option caps:ctrl_modifier
```

1.2 Debug Macro [851d50]

```
#define all(x) begin(x), end(x)
#ifndef CKISEKI
#include <experimental/iterator>
#define safe cerr<<__PRETTY_FUNCTION__<<" line "<<
    __LINE__<<" safe\n"
#define debug(a...) debug_(#a, a)
#define orange(a...) orange_(#a, a)
void debug_(const char *s, auto ...a) {
    cerr << "[1;32m" << s << " ] = (" ;
    int f = 0;
    (... , (cerr << (f++ ? ", " : "") << a));
    cerr << " ]\e[0m\n";
}
void orange_(const char *s, auto L, auto R) {
    cerr << "[1;33m" << s << " ] = [ " ;
    using namespace experimental;
    copy(L, R, make_ostream_joiner(cerr, ", "));
    cerr << " ]\e[0m\n";
}
#else
#define safe ((void)0)
#define debug(...) safe
#define orange(...) safe
#endif
```

1.3 Increase Stack [b6856c]

```
const int size = 256 << 20;
register long rsp asm("rsp");
char *p = (char*)malloc(size)+size, *bak = (char*)rsp;
__asm__("movq %0, %%rsp\n::"r"(p));
// main
__asm__("movq %0, %%rsp\n::"r"(bak));
```

1.4 Pragma Optimization [6006f6]

```
#pragma GCC optimize("Ofast,no-stack-protector")
#pragma GCC optimize("no-math-errno,unroll-loops")
#pragma GCC target("sse,sse2,sse3,ssse3,sse4")
#pragma GCC target("popcnt,abm,mmx,avx,arch=skylake")
__builtin_ia32_ldmxcsr(__builtin_ia32_stmxcsr()|0x8040)
```

1.5 IO Optimization [c9494b]

```
static inline int gc() {
    constexpr int B = 1<<20; static char buf[B], *p, *q;
    if (p == q) q = (p = buf) + fread(buf, 1, B, stdin);
    return q == buf ? EOF : *p++;
}
```

1.6 SVG Writer [57436c]

```
class SVG {
    void p(string_view s) { o << s; }
    void p(string_view s, auto v, auto... vs) {
        auto i = s.find('$');
        o << s.substr(0, i) << v, p(s.substr(i + 1), vs...);
    }
    ofstream o; string c = "red";
public:
    SVG(auto f, auto x1, auto y1, auto x2, auto y2) : o(f) {
        p("<svg xmlns='http://www.w3.org/2000/svg' "
            "viewBox='$ $ $ $'>\n"
            "<style>{*stroke-width:0.5%;}</style>\n",
            x1, -y2, x2 - x1, y2 - y1); }
```

```

~SVG() { p("</svg>\n"); }
SVG &color(string nc) { return c = nc, *this; }
void line(auto x1, auto y1, auto x2, auto y2) {
    p("<line x1='$' y1='$' x2='$' y2='$' stroke='$' />\n",
      x1, -y1, x2, -y2, c); }
void circle(auto x, auto y, auto r) {
    p("<circle cx='$' cy='$' r='$' stroke='$' "
      "fill='none' />\n", x, -y, r, c); }
void text(auto x, auto y, string s, int w = 12) {
    p("<text x='$' y='$' font-size='$px'>$</text>\n",
      x, -y, w, s); }
};

```

2 Data Structure

2.1 Dark Magic [095f25]

```

#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/priority_queue.hpp>
using namespace __gnu_pbds;
// heap tags: pairing/binary/binomial/rc_binomial/thin
template<typename T>
using pbds_heap = __gnu_pbds::prioity_queue<T, less<T>, \
    pairing_heap_tag>;
// pbds_heap::point_iterator
// x = pq.push(10); pq.modify(x, 87); a.join(b);
// tree tags: rb_tree_tag/ov_tree_tag/splay_tree_tag
template<typename T>
using ordered_set = tree<T, null_type, less<T>,
    rb_tree_tag, tree_order_statistics_node_update>;
// find_by_order, order_of_key
// hash tables: cc_hash_table/gp_hash_table

```

2.2 Link-Cut Tree [60627f]

```

template<typename Val, typename SVal> class LCT {
    struct node {
        int pa, ch[2];
        bool rev;
        Val v, prod, rprod;
        SVal sv, sub, vir;
        node() : pa{0}, ch{0, 0}, rev{false}, v{}, prod{},
            rprod{}, sv{}, sub{}, vir{} {};
    };
    #define cur o[u]
    #define lc cur.ch[0]
    #define rc cur.ch[1]
    vector<node> o;
    bool is_root(int u) const {
        return o[cur.pa].ch[0] != u && o[cur.pa].ch[1] != u;
    }
    bool is_rch(int u) const {
        return o[cur.pa].ch[1] == u && !is_root(u);
    }
    void down(int u) {
        if (not cur.rev) return;
        if (lc) set_rev(lc);
        if (rc) set_rev(rc);
        cur.rev = false;
    }
    void up(int u) {
        cur.prod = o[lc].prod * cur.v * o[rc].prod;
        cur.rprod = o[rc].rprod * cur.v * o[lc].rprod;
        cur.sub = cur.vir + o[lc].sub + o[rc].sub + cur.sv;
    }
    void set_rev(int u) {
        swap(lc, rc), swap(cur.prod, cur.rprod);
        cur.rev ^= 1;
    }
    void rotate(int u) {
        int f = cur.pa, g = o[f].pa, l = is_rch(u);
        if (cur.ch[l ^ 1]) o[cur.ch[l ^ 1]].pa = f;
        if (not is_root(f)) o[g].ch[is_rch(f)] = u;
        o[f].ch[l] = cur.ch[l ^ 1];
        cur.ch[l ^ 1] = f;
        cur.pa = g, o[f].pa = u;
        up(f);
    }
    void splay(int u) {
        vector<int> stk = {u};
        while (not is_root(stk.back()))
            stk.push_back(o[stk.back()].pa);
        while (not stk.empty()) {
            down(stk.back());
            stk.pop_back();
        }
    }
};

```

```

for (int f = cur.pa; not is_root(u); f = cur.pa) {
    if (!is_root(f))
        rotate(is_rch(u) == is_rch(f) ? f : u);
    rotate(u);
}
up(u);
}
void access(int x) {
    for (int u = x, last = 0; u; u = cur.pa) {
        splay(u);
        cur.vir = cur.vir + o[rc].sub - o[last].sub;
        rc = last; up(last = u);
    }
    splay(x);
}
int find_root(int u) {
    int la = 0;
    for (access(u); u; u = lc) down(la = u);
    return la;
}
void split(int x, int y) { chroot(x); access(y); }
void chroot(int u) { access(u); set_rev(u); }
public:
    LCT(int n = 0) : o(n + 1) {}
    int add(const Val &v = {}) {
        o.push_back(v);
        return int(o.size()) - 2;
    }
    void set_val(int u, const Val &v) {
        splay(++u); cur.v = v; up(u);
    }
    void set_sval(int u, const SVal &v) {
        access(++u); cur.sv = v; up(u);
    }
    Val query(int x, int y) {
        split(++x, ++y); return o[y].prod;
    }
    SVal subtree(int p, int u) {
        chroot(++p); access(++u);
        return cur.vir + cur.sv;
    }
    bool connected(int u, int v) {
        return find_root(++u) == find_root(++v);
    }
    void link(int x, int y) {
        chroot(++x); access(++y);
        o[y].vir = o[y].vir + o[x].sub;
        up(o[x].pa = y);
    }
    void cut(int x, int y) {
        split(++x, ++y);
        o[y].ch[0] = o[x].pa = 0; up(y);
    }
    #undef cur
    #undef lc
    #undef rc
};

```

2.3 LiChao Segment Tree [b9c827]

```

struct L {
    int m, k, id;
    L() : id(-1) {}
    L(int a, int b, int c) : m(a), k(b), id(c) {}
    int at(int x) { return m * x + k; }
};
class LiChao {
private:
    int n; vector<L> nodes;
    static int lc(int x) { return 2 * x + 1; }
    static int rc(int x) { return 2 * x + 2; }
    void insert(int l, int r, int id, L ln) {
        int m = (l + r) >> 1;
        if (nodes[id].id == -1)
            return nodes[id] = ln, void();
        bool atLeft = nodes[id].at(l) < ln.at(l);
        if (nodes[id].at(m) < ln.at(m))
            atLeft ^= 1, swap(nodes[id], ln);
        if (r - l == 1) return;
        if (atLeft) insert(l, m, lc(id), ln);
        else insert(m, r, rc(id), ln);
    }
    int query(int l, int r, int id, int x) {
        int m = (l + r) >> 1, ret = 0;
    }
};

```

```

    if (nodes[id].id != -1) ret = nodes[id].at(x);
    if (r - l == 1) return ret;
    if (x < m) return max(ret, query(l, m, lc(id), x));
    return max(ret, query(m, r, rc(id), x));
}
public:
    LiChao(int n_) : n(n_), nodes(n * 4) {}
    void insert(L ln) { insert(0, n, 0, ln); }
    int query(int x) { return query(0, n, 0, x); }
};

```

2.4 Treap [ae576c]

```

__gnu_cxx::sfmt19937 rnd(7122); // <ext/random>
namespace Treap {
    struct node {
        int size, pri; node *lc, *rc, *pa;
        node() : size(1), pri(rnd()), lc(0), rc(0), pa(0) {}
        void pull() {
            size = 1; pa = 0;
            if (lc) { size += lc->size; lc->pa = this; }
            if (rc) { size += rc->size; rc->pa = this; }
        }
    };
    int SZ(node *x) { return x ? x->size : 0; }
    node *merge(node *L, node *R) {
        if (not L or not R) return L ? L : R;
        if (L->pri > R->pri)
            return L->rc = merge(L->rc, R), L->pull(), L;
        else
            return R->lc = merge(L, R->lc), R->pull(), R;
    }
    void splitBySize(node *o, int k, node *&L, node *&R) {
        if (not o) L = R = 0;
        else if (int s = SZ(o->lc) + 1; s <= k)
            L = o, splitBySize(o->rc, k - s, L->rc, R), L->pull();
        else
            R = o, splitBySize(o->lc, k, L, R->lc), R->pull();
    }
    // SZ(L) == k
    int getRank(node *o) { // 1-base
        int r = SZ(o->lc) + 1;
        for (; o->pa; o = o->pa)
            if (o->pa->rc == o) r += SZ(o->pa->lc) + 1;
        return r;
    }
} // namespace Treap

```

2.5 Linear Basis [138d5d]

```

template <int BITS, typename S = int> struct Basis {
    static constexpr S MIN = numeric_limits<S>::min();
    array<pair<llu, S>, BITS> b;
    Basis() { b.fill({0, MIN}); }
    void add(llu x, S p) {
        for (int i = BITS - 1; i >= 0; i--) if (x >> i & 1) {
            if (b[i].first == 0) return b[i] = {x, p}, void();
            if (b[i].second < p)
                swap(b[i].first, x), swap(b[i].second, p);
            x ^= b[i].first;
        }
    }
    optional<llu> query_kth(llu v, llu k) {
        vector<pair<llu, int>> o;
        for (int i = 0; i < BITS; i++)
            if (b[i].first) o.emplace_back(b[i].first, i);
        if (k >= (1ULL << o.size())) return {};
        for (int i = int(o.size()) - 1; i >= 0; i--)
            if ((k >> i & 1) ^ (v >> o[i].second & 1))
                v ^= o[i].first;
        return v;
    }
    Basis filter(S l) {
        Basis res = *this;
        for (int i = 0; i < BITS; i++)
            if (res.b[i].second < l) res.b[i] = {0, MIN};
        return res;
    }
};

```

2.6 Binary Search On Segtree [6c61c0]

```

// find_first = l -> minimal x s.t. check([l, x])
// find_last = r -> maximal x s.t. check([x, r])
int find_first(int l, auto &&check) {
    if (l >= n) return n + 1;

```

```

    l += sz; push(l); Monoid sum; // identity
    do {
        while ((l & 1) == 0) l >>= 1;
        if (auto s = sum + nd[l]; check(s)) {
            while (l < sz) {
                prop(l); l = (l << 1);
                if (auto nxt = sum + nd[l]; not check(nxt))
                    sum = nxt, l++;
            }
            return l + 1 - sz;
        } else sum = s, l++;
    } while (lowbit(l) != l);
    return n + 1;
}
int find_last(int r, auto &&check) {
    if (r <= 0) return -1;
    r += sz; push(r - 1); Monoid sum; // identity
    do {
        r--;
        while (r > 1 and (r & 1)) r >>= 1;
        if (auto s = nd[r] + sum; check(s)) {
            while (r < sz) {
                prop(r); r = (r << 1) | 1;
                if (auto nxt = nd[r] + sum; not check(nxt))
                    sum = nxt, r--;
            }
            return r - sz;
        } else sum = s;
    } while (lowbit(r) != r);
    return -1;
}

```

3 Graph

3.1 2-SAT (SCC) [09167a]

```

class TwoSat { // test @ CSES Giant Pizza
private:
    int n; vector<vector<int>> G, rG, sccs;
    vector<int> ord, idx, vis, res;
    void dfs(int u) {
        vis[u] = true;
        for (int v : G[u]) if (!vis[v]) dfs(v);
        ord.push_back(u);
    }
    void rdfs(int u) {
        vis[u] = false; idx[u] = sccs.size() - 1;
        sccs.back().push_back(u);
        for (int v : rG[u]) if (vis[v]) rdfs(v);
    }
public:
    TwoSat(int n_) : n(n_), G(n), rG(n), idx(n), vis(n),
        res(n) {}
    void add_edge(int u, int v) {
        G[u].push_back(v); rG[v].push_back(u);
    }
    void orr(int x, int y) {
        if ((x ^ y) == 1) return;
        add_edge(x ^ 1, y); add_edge(y ^ 1, x);
    }
    bool solve() {
        for (int i = 0; i < n; ++i) if (not vis[i]) dfs(i);
        for (int u : ord | views::reverse)
            if (vis[u]) sccs.emplace_back(), rdfs(u);
        for (int i = 0; i < n; i += 2)
            if (idx[i] == idx[i + 1]) return false;
        vector<bool> c(sccs.size());
        for (size_t i = 0; i < sccs.size(); ++i)
            for (int z : sccs[i])
                res[z] = c[i], c[idx[z ^ 1]] = !c[i];
        return true;
    }
    bool get(int x) { return res[x]; }
    int get_id(int x) { return idx[x]; }
    int count() { return sccs.size(); }
};

```

3.2 BCC [6ac6db]

```

class BCC {
    int n, ecnt, bcnt;
    vector<vector<pair<int, int>>> g;
    vector<int> dfn, low, bcc, stk;
    vector<bool> ap, bridge;
    void dfs(int u, int f) {

```

```

dfn[u] = low[u] = dfn[f] + 1;
int ch = 0;
for (auto [v, t] : g[u]) if (bcc[t] == -1) {
    bcc[t] = 0; stk.push_back(t);
    if (dfn[v]) {
        low[u] = min(low[u], dfn[v]);
        continue;
    }
    ++ch, dfs(v, u);
    low[u] = min(low[u], low[v]);
    if (low[v] > dfn[u]) bridge[t] = true;
    if (low[v] < dfn[u]) continue;
    ap[u] = true;
    while (not stk.empty()) {
        int o = stk.back(); stk.pop_back();
        bcc[o] = bcnt;
        if (o == t) break;
    }
    bcnt += 1;
}
ap[u] = ap[u] and (ch != 1 or u != f);
}
public:
BCC(int n_) : n(n_), ecnt(0), bcnt(0), g(n), dfn(n),
    low(n), stk(), ap(n) {}
void add_edge(int u, int v) {
    g[u].emplace_back(v, ecnt);
    g[v].emplace_back(u, ecnt++);
}
void solve() {
    bridge.assign(ecnt, false); bcc.assign(ecnt, -1);
    for (int i = 0; i < n; ++i) if (!dfn[i]) dfs(i, i);
}
int bcc_id(int x) const { return bcc[x]; }
bool is_ap(int x) const { return ap[x]; }
bool is_bridge(int x) const { return bridge[x]; }
};

```

3.3 Round Square Tree [cf6d74]

```

struct RST { // be careful about isolate point
    int n; vector<vector<int>> T;
    RST(auto &G) : n(G.size()), T(n) {
        vector<int> stk, vis(n), low(n);
        auto dfs = [&](auto self, int u, int d) -> void {
            low[u] = vis[u] = d; stk.push_back(u);
            for (int v : G[u]) if (!vis[v]) {
                self(self, v, d + 1);
                if (low[v] == vis[u]) {
                    int cnt = int(T.size()); T.emplace_back();
                    for (int x = -1; x != v; stk.pop_back())
                        T[cnt].push_back(x = stk.back());
                    T[u].push_back(cnt); // T is rooted
                } else low[u] = min(low[u], low[v]);
            } else low[u] = min(low[u], vis[v]);
        };
        for (int u = 0; u < n; u++)
            if (!vis[u]) dfs(dfs, u, 1);
    } // T may be forest; after dfs, stk are the roots
}; // test @ 2020 Shanghai K

```

3.4 Edge TCC [5a2668]

```

vector<vector<int>> ETCC(auto &adj) {
    const int n = static_cast<int>(adj.size());
    vector<int> up(n), low(n), in, out, nx, id;
    in = out = nx = id = vector<int>(n, -1);
    int dfc = 0, cnt = 0; Dsu dsu(n);
    auto merge = [&](int u, int v) {
        dsu.join(u, v); up[u] += up[v]; };
    auto dfs = [&](auto self, int u, int p) -> void {
        in[u] = low[u] = dfc++;
        for (int v : adj[u]) if (v != u) {
            if (v == p) { p = -1; continue; }
            if (in[v] == -1) {
                self(self, v, u);
                if (nx[v] == -1 && up[v] <= 1) {
                    up[u] += up[v]; low[u] = min(low[u], low[v]);
                    continue;
                }
            }
            if (up[v] == 0) v = nx[v];
            if (low[u] > low[v])
                low[u] = low[v], swap(nx[u], v);
            for (; v != -1; v = nx[v]) merge(u, v);
        }
    };
}

```

```

} else if (in[v] < in[u]) {
    low[u] = min(low[u], in[v]); up[u]++;
} else {
    for (int &x = nx[u]; x != -1 &&
        in[x] <= in[v] && in[v] < out[x]; x = nx[x])
        merge(u, x);
    up[u]--;
}
}
out[u] = dfc;
};
for (int i = 0; i < n; i++)
    if (in[i] == -1) dfs(dfs, i, -1);
for (int i = 0; i < n; i++)
    if (dsu.anc(i) == i) id[i] = cnt++;
vector<vector<int>> comps(cnt);
for (int i = 0; i < n; i++)
    comps[id[dsu.anc(i)]].push_back(i);
return comps;
} // test @ yosupo judge

```

3.5 DMST [f4317e]

```

using lld = int64_t;
struct E { int s, t; lld w; }; // 0-base
struct PQ {
    struct P {
        lld v; int i;
        bool operator>(const P &b) const { return v > b.v; }
    };
    min_heap<P> pq; lld tag;
    void push(P p) { p.v -= tag; pq.emplace(p); }
    P top() { P p = pq.top(); p.v += tag; return p; }
    void join(PQ &b) {
        if (pq.size() < b.pq.size())
            swap(pq, b.pq), swap(tag, b.tag);
        while (!b.pq.empty()) push(b.top()), b.pq.pop();
    }
};
vector<int> dmst(const vector<E> &e, int n, int root) {
    vector<PQ> h(n * 2);
    for (int i = 0; i < int(e.size()); ++i)
        h[e[i].t].push({e[i].w, i});
    vector<int> a(n * 2); iota(all(a), 0);
    vector<int> v(n * 2, -1), pa(n * 2, -1), r(n * 2);
    auto o = [&](auto Y, int x) -> int {
        return x == a[x] ? x : a[x] = Y(Y, a[x]); };
    auto S = [&](int i) { return o(o, e[i].s); };
    int pc = v[root] = n;
    for (int i = 0; i < n; ++i) if (v[i] == -1)
        for (int p = i; v[p] < 0 || v[p] == i; p = S(r[p])) {
            if (v[p] == i)
                for (int q = pc++; p != q; p = S(r[p])) {
                    h[p].tag -= h[p].top().v; h[q].join(h[p]);
                    pa[p] = a[p] = q;
                }
            while (S(h[p].top().i) == p) h[p].pq.pop();
            v[p] = i; r[p] = h[p].top().i;
        }
    vector<int> ans;
    for (int i = pc - 1; i >= 0; i--) if (v[i] != n) {
        for (int f = e[r[i]].t; f != -1 && v[f] != n; f = pa[f])
            v[f] = n;
        ans.push_back(r[i]);
    }
    return ans; // default minimize, returns edgeid array
}

```

3.6 Dominator Tree [ea5b7c]

```

struct Dominator {
    vector<vector<int>> g, r, rdom; int tk;
    vector<int> dfn, rev, fa, sdom, dom, val, rp;
    Dominator(int n) : g(n), r(n), rdom(n), tk(0) {
        dfn = rev = fa = sdom = dom =
            val = rp = vector<int>(n, -1); }
    void add_edge(int x, int y) { g[x].push_back(y); }
    void dfs(int x) {
        rev[dfn[x] = tk] = x;
        fa[tk] = sdom[tk] = val[tk] = tk; tk++;
        for (int u : g[x]) {
            if (dfn[u] == -1) dfs(u), rp[dfn[u]] = dfn[x];
            r[dfn[u]].push_back(dfn[x]);
        }
    }
}

```

```

}
void merge(int x, int y) { fa[x] = y; }
int find(int x, int c = 0) {
    if (fa[x] == x) return c ? -1 : x;
    if (int p = find(fa[x], 1); p != -1) {
        if (sdom[val[x]] > sdom[val[fa[x]]])
            val[x] = val[fa[x]];
        fa[x] = p;
        return c ? p : val[x];
    } else return c ? fa[x] : val[x];
}
vector<int> build(int s, int n) {
    // return the father of each node in dominator tree
    dfs(s); // p[i] = -2 if i is unreachable from s
    for (int i = tk - 1; i >= 0; --i) {
        for (int u : r[i])
            sdom[i] = min(sdom[i], sdom[find(u)]);
        if (i) rdom[sdom[i]].push_back(i);
        for (int u : rdom[i]) {
            int p = find(u);
            dom[u] = (sdom[p] == i ? i : p);
        }
        if (i) merge(i, rp[i]);
    }
    vector<int> p(n, -2); p[s] = -1;
    for (int i = 1; i < tk; ++i)
        if (sdom[i] != dom[i]) dom[i] = dom[dom[i]];
    for (int i = 1; i < tk; ++i)
        p[rev[i]] = rev[dom[i]];
    return p;
} // test @ yosupo judge
};

```

3.7 Edge Coloring [029763]

```

// max(d_u) + 1 edge coloring, time: O(NM)
int C[kN][kN], G[kN][kN]; // 1-based, G: ans
void clear(int N) {
    for (int i = 0; i <= N; i++)
        for (int j = 0; j <= N; j++)
            C[i][j] = G[i][j] = 0;
}
void solve(vector<pair<int, int>> &E, int N) {
    int X[kN] = {}, a;
    auto update = [&](int u) {
        for (X[u] = 1; C[u][X[u]]; X[u]++);
    };
    auto color = [&](int u, int v, int c) {
        int p = G[u][v];
        G[u][v] = G[v][u] = c;
        C[u][c] = v, C[v][c] = u;
        C[u][p] = C[v][p] = 0;
        if (p) X[u] = X[v] = p;
        else update(u), update(v);
        return p;
    };
    auto flip = [&](int u, int c1, int c2) {
        int p = C[u][c1];
        swap(C[u][c1], C[u][c2]);
        if (p) G[u][p] = G[p][u] = c2;
        if (!C[u][c1]) X[u] = c1;
        if (!C[u][c2]) X[u] = c2;
        return p;
    };
    for (int i = 1; i <= N; i++) X[i] = 1;
    for (int t = 0; t < E.size(); t++) {
        auto [u, v] = E[t];
        int v0 = v, c = X[u], c0 = c, d;
        vector<pair<int, int>> L; int vst[kN] = {};
        while (!G[u][v0]) {
            L.emplace_back(v, d = X[v]);
            if (!C[v][c]) for (a=L.size()-1; a>=0; a--)
                c = color(u, L[a].first, c);
            else if (!C[u][d]) for (a=L.size()-1; a>=0; a--)
                color(u, L[a].first, L[a].second);
            else if (vst[d]) break;
            else vst[d] = 1, v = C[u][d];
        }
        if (!G[u][v0]) {
            for (; v; v = flip(v, c, d), swap(c, d));
            if (C[u][c0]) { a = int(L.size()) - 1;
                while (--a >= 0 && L[a].second != c);
                for (; a>=0; a--) color(u, L[a].first, L[a].second);
            }
        }
    }
}

```

```

} else t--;
}
}
}

```

3.8 Centroid Decomposition [670cdd]

```

class Centroid {
    vector<vector<pair<int, int>>> g; // g[u] = {(v, w)}
    vector<int> pa, dep, vis, sz, mx;
    vector<vector<int64_t>> Dist;
    vector<int64_t> Sub, Sub2;
    vector<int> Cnt, Cnt2;
    void DfsSz(vector<int> &tmp, int x) {
        vis[x] = true, sz[x] = 1, mx[x] = 0;
        for (auto [u, w] : g[x]) if (not vis[u]) {
            DfsSz(tmp, u); sz[x] += sz[u];
            mx[x] = max(mx[x], sz[u]);
        }
        tmp.push_back(x);
    }
    void DfsDist(int x, int64_t D = 0) {
        Dist[x].push_back(D); vis[x] = true;
        for (auto [u, w] : g[x])
            if (not vis[u]) DfsDist(u, D + w);
    }
    void DfsCen(int x, int D, int p) {
        vector<int> tmp; DfsSz(tmp, x);
        int M = int(tmp.size()), C = -1;
        for (int u : tmp)
            if (max(M - sz[u], mx[u]) * 2 <= M) C = u;
        for (int u : tmp) vis[u] = false;
        DfsDist(C);
        for (int u : tmp) vis[u] = false;
        pa[C] = p, vis[C] = true, dep[C] = D;
        for (auto [u, w] : g[C])
            if (not vis[u]) DfsCen(u, D + 1, C);
    }
public:
    Centroid(int N) : g(N), pa(N), dep(N),
        vis(N), sz(N), mx(N), Dist(N),
        Sub(N), Sub2(N), Cnt(N), Cnt2(N) {}
    void AddEdge(int u, int v, int w) {
        g[u].emplace_back(v, w);
        g[v].emplace_back(u, w);
    }
    void Build() { DfsCen(0, 0, -1); }
    void Mark(int v) {
        int x = v, z = -1;
        for (int i = dep[v]; i >= 0; --i) {
            Sub[x] += Dist[v][i], Cnt[x]++;
            if (z != -1)
                Sub2[z] += Dist[v][i], Cnt2[z]++;
            x = pa[z = x];
        }
    }
    int64_t Query(int v) {
        int64_t res = 0;
        int x = v, z = -1;
        for (int i = dep[v]; i >= 0; --i) {
            res += Sub[x] + 1LL * Cnt[x] * Dist[v][i];
            if (z != -1)
                res -= Sub2[z] + 1LL * Cnt2[z] * Dist[v][i];
            x = pa[z = x];
        }
        return res;
    }
}; // pa, dep are centroid tree attributes

```

3.9 Lowbit Decomposition [760ac1]

```

class LBD {
    int timer, chains;
    vector<vector<int>> G;
    vector<int> tl, tr, chain, head, dep, pa;
    // chains : number of chain
    // tl, tr[u] : subtree interval in the seq. of u
    // head[i] : head of the chain i
    // chain[u] : chain id of the chain u is on
    void predfs(int u, int f) {
        dep[u] = dep[pa[u] = f] + 1;
        for (int v : G[u]) if (v != f) {
            predfs(v, u);
            if (lowbit(chain[u]) < lowbit(chain[v]))

```



```

    chain[u] = chain[v];
}
if (chain[u] == 0) chain[u] = ++chains;
}
void dfschain(int u, int f) {
    tl[u] = timer++;
    if (head[chain[u]] == -1)
        head[chain[u]] = u;
    for (int v : G[u])
        if (v != f and chain[v] == chain[u])
            dfschain(v, u);
    for (int v : G[u])
        if (v != f and chain[v] != chain[u])
            dfschain(v, u);
    tr[u] = timer;
}
public:
LBD(int n) : timer(0), chains(0), G(n), tl(n), tr(n),
    chain(n), head(n + 1, -1), dep(n), pa(n) {}
void add_edge(int u, int v) {
    G[u].push_back(v); G[v].push_back(u);
}
void decompose() { predfs(0, 0); dfschain(0, 0); }
PII get_subtree(int u) { return {tl[u], tr[u]}; }
vector<PII> get_path(int u, int v) {
    vector<PII> res;
    while (chain[u] != chain[v]) {
        if (dep[head[chain[u]]] < dep[head[chain[v]]])
            swap(u, v);
        int s = head[chain[u]];
        res.emplace_back(tl[s], tl[u] + 1);
        u = pa[s];
    }
    if (dep[u] < dep[v]) swap(u, v);
    res.emplace_back(tl[v], tl[u] + 1);
    return res;
}
};

```

3.10 Virtual Tree [ad5cf5]

```

vector<pair<int, int>> build(vector<int> vs, int r) {
    vector<pair<int, int>> res;
    sort(vs.begin(), vs.end(), [](int i, int j) {
        return dfn[i] < dfn[j]; });
    vector<int> s = {r};
    for (int v : vs) if (v != r) {
        if (int o = lca(v, s.back()); o != s.back()) {
            while (s.size() >= 2) {
                if (dfn[s[s.size() - 2]] < dfn[o]) break;
                res.emplace_back(s[s.size() - 2], s.back());
                s.pop_back();
            }
            if (s.back() != o) {
                res.emplace_back(o, s.back());
                s.back() = o;
            }
        }
        s.push_back(v);
    }
    for (size_t i = 1; i < s.size(); ++i)
        res.emplace_back(s[i - 1], s[i]);
    return res; // (x, y): x->y
}

```

3.11 Tree Hashing [d6a9f9]

```

vector<int> g[maxn]; ll h[maxn];
llu F(llu z) { // xorshift64star from iwiwi
    z ^= z >> 12; z ^= z << 25; z ^= z >> 27;
    return z * 2685821657736338717LL;
}
llu hsah(int u, int f) {
    llu r = 127; // bigger?
    for (int v : g[u]) if (v != f) r += hsah(v, u);
    return h[u] = F(r);
} // test @ UOJ 763 & yosupo library checker

```

3.12 Mo's Algorithm on Tree

```

dfs u:
    push u
    iterate subtree
    push u

```

```

Let P = LCA(u, v) with St(u) <= St(v)
if (P == u) query[St(u), St(v)]
else query[Ed(u), St(v)], query[St(P), St(P)]

```

3.13 Count Cycles [c7e8f2]

```

// ord = sort by deg decreasing, rk[ord[i]] = i
// D[i] = edge point from rk small to rk big
for (int x : ord) { // c3
    for (int y : D[x]) vis[y] = 1;
    for (int y : D[x]) for (int z : D[y]) c3 += vis[z];
    for (int y : D[x]) vis[y] = 0;
}
for (int x : ord) { // c4
    for (int y : D[x]) for (int z : adj[y])
        if (rk[z] > rk[x]) c4 += vis[z]++;
    for (int y : D[x]) for (int z : adj[y])
        if (rk[z] > rk[x]) --vis[z];
} // both are O(M*sqrt(M)), test @ 2022 CCPC guangzhou

```

3.14 MaximalClique [293730]

```

// contain a self loop u to u, than u won't in clique
template <size_t maxn> class MaxClique {
public:
    using bits = bitset<maxn>;
    bits popped, G[maxn], ans;
    size_t deg[maxn], deo[maxn], n;
    void sort_by_degree() {
        popped.reset();
        for (size_t i = 0; i < n; ++i)
            deg[i] = G[i].count();
        for (size_t i = 0; i < n; ++i) {
            size_t mi = maxn, id = 0;
            for (size_t j = 0; j < n; ++j)
                if (not popped[j] and deg[j] < mi)
                    mi = deg[id = j];
            popped[deo[i] = id] = 1;
            for (size_t u = G[i]._Find_first(); u < n;
                u = G[i]._Find_next(u))
                --deg[u];
        }
    }
    void BK(bits R, bits P, bits X) {
        if (R.count() + P.count() <= ans.count()) return;
        if (not P.count() and not X.count()) {
            if (R.count() > ans.count()) ans = R;
            return;
        }
        /* greedily choose max degree as pivot
        bits cur = P | X; size_t pivot = 0, sz = 0;
        for (size_t u = cur._Find_first();
            u < n; u = cur._Find_next(u))
            if (deg[u] > sz) sz = deg[pivot = u];
        cur = P & (~G[pivot]);
        */ // or simply choose first
        bits cur = P & (~G[(P | X)._Find_first()]);
        for (size_t u = cur._Find_first(); u < n;
            u = cur._Find_next(u)) {
            if (R[u]) continue;
            R[u] = 1;
            BK(R, P & G[u], X & G[u]);
            R[u] = P[u] = 0, X[u] = 1;
        }
    }
public:
    void init(size_t n_) {
        n = n_;
        for (size_t i = 0; i < n; ++i) G[i].reset();
        ans.reset();
    }
    void add_edges(int u, bits S) { G[u] = S; }
    void add_edge(int u, int v) { G[u][v] = G[v][u] = 1; }
    int solve() {
        sort_by_degree(); // or simply iota(deo... )
        for (size_t i = 0; i < n; ++i)
            deg[i] = G[i].count();
        bits pob, nob = 0; pob.set();
        for (size_t i = n; i < maxn; ++i) pob[i] = 0;
        for (size_t i = 0; i < n; ++i) {
            size_t v = deo[i];
            bits tmp;
            tmp[v] = 1;
            BK(tmp, pob & G[v], nob & G[v]);
        }
    }
};

```

```

    pob[v] = 0, nob[v] = 1;
}
return static_cast<int>(ans.count());
}
};

```

3.15 MaximumClique [aee5d8]

```

constexpr size_t kN = 150; using bits = bitset<kN>;
struct MaxClique {
    bits G[kN], cs[kN];
    int ans, sol[kN], q, cur[kN], d[kN], n;
    void init(int _n) {
        n = _n;
        for (int i = 0; i < n; ++i) G[i].reset();
    }
    void add_edge(int u, int v) { G[u][v] = G[v][u] = 1; }
    void pre_dfs(vector<int> &v, int i, bits mask) {
        if (i < 4) {
            for (int x : v) d[x] = (int)(G[x] & mask).count();
            sort(all(v), [&](int x, int y) {
                return d[x] > d[y]; });
        }
        vector<int> c(v.size());
        cs[1].reset(), cs[2].reset();
        int l = max(ans - q + 1, 1), r = 2, tp = 0, k;
        for (int p : v) {
            for (k = 1; (cs[k] & G[p]).any(); ++k);
            if (k >= r) cs[+r].reset();
            cs[k][p] = 1;
            if (k < l) v[tp++] = p;
        }
        for (k = l; k < r; ++k)
            for (auto p = cs[k]._Find_first();
                 p < kN; p = cs[k]._Find_next(p))
                v[tp] = (int)p, c[tp] = k, ++tp;
        dfs(v, c, i + 1, mask);
    }
    void dfs(vector<int> &v, vector<int> &c,
             int i, bits mask) {
        while (!v.empty()) {
            int p = v.back(); v.pop_back(); mask[p] = 0;
            if (q + c.back() <= ans) return;
            cur[q++] = p;
            vector<int> nr;
            for (int x : v) if (G[p][x]) nr.push_back(x);
            if (!nr.empty()) pre_dfs(nr, i, mask & G[p]);
            else if (q > ans) ans = q, copy_n(cur, q, sol);
            c.pop_back(); --q;
        }
    }
    int solve() {
        vector<int> v(n); iota(all(v), 0);
        ans = q = 0; pre_dfs(v, 0, bits(string(n, '1')));
        return ans; // sol[0 ~ ans-1]
    }
} cliq; // test @ yosupo judge

```

3.16 Minimum Mean Cycle [e23bc0]

```

// WARNING: TYPE matters
struct Edge { int s, t; llf c; };
llf solve(vector<Edge> &e, int n) {
    // O(VE), returns inf if no cycle, mmc otherwise
    vector<VI> prv(n + 1, VI(n)), prve = prv;
    vector<vector<llf>> d(n + 1, vector<llf>(n, inf));
    d[0] = vector<llf>(n, 0);
    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < (int)e.size(); ++j) {
            auto [s, t, c] = e[j];
            if (d[i][s] < inf && d[i + 1][t] > d[i][s] + c) {
                d[i + 1][t] = d[i][s] + c;
                prv[i + 1][t] = s; prve[i + 1][t] = j;
            }
        }
    }
    llf mmc = inf; int st = -1;
    for (int i = 0; i < n; ++i) {
        llf avg = -inf;
        for (int k = 0; k < n; ++k) {
            if (d[n][i] < inf - eps)
                avg = max(avg, (d[n][i] - d[k][i]) / (n - k));
            else avg = inf;
        }
    }
}

```

```

    if (avg < mmc) tie(mmc, st) = tie(avg, i);
}
if (st == -1) return inf;
vector<int> vst(n), eid, cycle, rho;
for (int i = n; !vst[st]; st = prv[i--][st]) {
    vst[st]++; eid.emplace_back(prve[i][st]);
    rho.emplace_back(st);
}
while (vst[st] != 2) {
    int v = rho.back(); rho.pop_back();
    cycle.emplace_back(v); vst[v]++;
}
reverse(all(eid)); eid.resize(cycle.size());
return mmc;
}

```

4 Flow & Matching

4.1 HopcroftKarp [930040]

```

struct HK {
    vector<int> l, r, a, p; int ans;
    HK(int n, int m, auto &g) : l(n, -1), r(m, -1), ans(0) {
        for (bool match = true; match;) {
            match = false; a.assign(n, -1); p.assign(n, -1);
            queue<int> q;
            for (int i = 0; i < n; ++i)
                if (l[i] == -1) q.push(a[i] = p[i] = i);
            // bitset<maxn> nvis, t; nvis.set();
            while (!q.empty()) {
                int z, x = q.front(); q.pop();
                if (l[a[x]] != -1) continue;
                for (int y : g[x]) { // or iterate t = g[x]&nvis
                    // nvis.reset(y);
                    if (r[y] == -1) {
                        for (z = y; z != -1; )
                            r[z] = x, swap(l[x], z), x = p[x];
                        match = true; ++ans; break;
                    } else if (p[r[y]] == -1)
                        q.push(z = r[y]), p[z] = x, a[z] = a[x];
                }
            }
        }
    }
};

```

4.2 Kuhn Munkres [2c09ed]

```

struct KM { // maximize, test @ UOJ 80
    int n, l, r; llf ans; // fl and fr are the match
    vector<llf> hl, hr; vector<int> fl, fr, pre, q;
    void bfs(const auto &w, int s) {
        vector<int> vl(n), vr(n); vector<llf> slk(n, INF);
        l = r = 0; vr[q[r++] = s] = true;
        const auto check = [&](int x) -> bool {
            if (vl[x] || slk[x] > 0) return true;
            vl[x] = true; slk[x] = INF;
            if (fl[x] != -1) return vr[q[r++] = fl[x]] = true;
            while (x != -1) swap(x, fr[fl[x] = pre[x]]);
            return false;
        };
        while (true) {
            while (l < r)
                for (int x = 0, y = q[l++]; x < n; ++x) if (!vl[x])
                    if (chmin(slk[x], hl[x] + hr[y] - w[x][y]))
                        if (pre[x] = y, !check(x)) return;
                llf d = ranges::min(slk);
                for (int x = 0; x < n; ++x)
                    vl[x] ? hl[x] += d : slk[x] -= d;
                for (int x = 0; x < n; ++x) if (vr[x]) hr[x] -= d;
                for (int x = 0; x < n; ++x) if (!check(x)) return;
            }
        }
        KM(int n_, const auto &w) : n(n_), ans(0),
            hl(n), hr(n), fl(n, -1), fr(n), pre(n), q(n) {
            for (int i = 0; i < n; ++i) hl[i] = ranges::max(w[i]);
            for (int i = 0; i < n; ++i) bfs(w, i);
            for (int i = 0; i < n; ++i) ans += w[i][fl[i]];
        }
};

```

4.3 Flow Models

- Maximum/Minimum flow with lower bound / Circulation problem
 - Construct super source S and sink T .
 - For each edge (x, y, l, u) , connect $x \rightarrow y$ with capacity $u - l$.

3. For each vertex v , denote by $in(v)$ the difference between the sum of incoming lower bounds and the sum of outgoing lower bounds.
 4. If $in(v) > 0$, connect $S \rightarrow v$ with capacity $in(v)$, otherwise, connect $v \rightarrow T$ with capacity $-in(v)$.
 - To maximize, connect $t \rightarrow s$ with capacity ∞ (skip this in circulation problem), and let f be the maximum flow from S to T . If $f \neq \sum_{v \in V, in(v) > 0} in(v)$, there's no solution. Otherwise, the maximum flow from s to t is the answer. Also, f is a mincost valid flow.
 - To minimize, let f be the maximum flow from S to T . Connect $t \rightarrow s$ with capacity ∞ and let the flow from S to T be f' . If $f + f' \neq \sum_{v \in V, in(v) > 0} in(v)$, there's no solution. Otherwise, f' is the answer.
 5. The solution of each edge e is $l_e + f_e$, where f_e corresponds to the flow of edge e on the graph.
- Construct minimum vertex cover from maximum matching M on bipartite graph (X, Y)
 1. Redirect every edge: $y \rightarrow x$ if $(x, y) \in M$, $x \rightarrow y$ otherwise.
 2. DFS from unmatched vertices in X .
 3. $x \in X$ is chosen iff x is unvisited.
 4. $y \in Y$ is chosen iff y is visited.
 - Minimum cost cyclic flow
 1. Construct super source S and sink T
 2. For each edge (x, y, c) , connect $x \rightarrow y$ with $(cost, cap) = (c, 1)$ if $c > 0$, otherwise connect $y \rightarrow x$ with $(cost, cap) = (-c, 1)$
 3. For each edge with $c < 0$, sum these cost as K , then increase $d(y)$ by 1, decrease $d(x)$ by 1
 4. For each vertex v with $d(v) > 0$, connect $S \rightarrow v$ with $(cost, cap) = (0, d(v))$
 5. For each vertex v with $d(v) < 0$, connect $v \rightarrow T$ with $(cost, cap) = (0, -d(v))$
 6. Flow from S to T , the answer is the cost of the flow $C + K$
 - Maximum density induced subgraph
 1. Binary search on answer, suppose we're checking answer T
 2. Construct a max flow model, let K be the sum of all weights
 3. Connect source $s \rightarrow v, v \in G$ with capacity K
 4. For each edge (u, v, w) in G , connect $u \rightarrow v$ and $v \rightarrow u$ with capacity w
 5. For $v \in G$, connect it with sink $v \rightarrow t$ with capacity $K + 2T - (\sum_{e \in E(v)} w(e)) - 2w(v)$
 6. T is a valid answer if the maximum flow $f < K|V|$
 - Minimum weight edge cover
 1. For each $v \in V$ create a copy v' , and connect $u' \rightarrow v'$ with weight $w(u, v)$.
 2. Connect $v \rightarrow v'$ with weight $2\mu(v)$, where $\mu(v)$ is the cost of the cheapest edge incident to v .
 3. Find the minimum weight perfect matching on G' .
 - Submodular functions minimization
 - For a function $f: 2^V \rightarrow \mathbb{R}$, f is a submodular function iff
 - * $\forall S, T \subseteq V, f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$, or
 - * $\forall X \subseteq Y \subseteq V, x \notin Y, f(X \cup \{x\}) - f(X) \geq f(Y \cup \{x\}) - f(Y)$.
 - To minimize $\sum_i \theta_i(x_i) + \sum_{i < j} \phi_{ij}(x_i, x_j) + \sum_{i < j < k} \psi_{ijk}(x_i, x_j, x_k)$
 - If $\theta_i(1) \geq \theta_i(0)$, add edge $(S, i, \theta_i(1) - \theta_i(0))$ and $\theta_i(0)$ to answer; otherwise, $(i, T, \theta_i(0) - \theta_i(1))$ and $\theta_i(1)$.
 - Add edges $(i, j, \phi_{ij}(0, 1) + \phi_{ij}(1, 0) - \phi_{ij}(0, 0) - \phi_{ij}(1, 1))$.
 - Denote x_{ijk} as helper nodes. Let $P = \psi_{ijk}(0, 0, 0) + \psi_{ijk}(0, 1, 1) + \psi_{ijk}(1, 0, 1) - \psi_{ijk}(1, 1, 1) - \psi_{ijk}(0, 1, 0) - \psi_{ijk}(1, 0, 0) - \psi_{ijk}(1, 1, 0)$. Add $-P$ to answer. If $P \geq 0$, add edges $(i, x_{ijk}, P), (j, x_{ijk}, P), (k, x_{ijk}, P), (x_{ijk}, T, P)$; otherwise $(x_{ijk}, i, -P), (x_{ijk}, j, -P), (x_{ijk}, k, -P), (S, x_{ijk}, -P)$.
 - The minimum cut of this graph will be the the minimum value of the function above.

4.4 Dinic [32c53e]

```
template <typename Cap = int64_t> class Dinic {
private:
    struct E { int to; rev; Cap cap; }; int n, st, ed;
    vector<vector<E>> G; vector<size_t> lv, idx;
    bool BFS(int k) {
        lv.assign(n, 0); idx.assign(n, 0);
        queue<int> bfs; bfs.push(st); lv[st] = 1;
        while (!bfs.empty() and not lv[ed]) {
            int u = bfs.front(); bfs.pop();
            for (auto e: G[u]) if (e.cap >> k and !lv[e.to])
                bfs.push(e.to), lv[e.to] = lv[u] + 1;
        }
        return lv[ed];
    }
    Cap DFS(int u, Cap f = numeric_limits<Cap>::max()) {
        if (u == ed) return f;
        Cap ret = 0;
        for (auto &i = idx[u]; i < G[u].size(); ++i) {
            auto &[to, rev, cap] = G[u][i];
```

```
            if (cap <= 0 or lv[to] != lv[u] + 1) continue;
            Cap nf = DFS(to, min(f, cap));
            ret += nf; cap -= nf; f -= nf;
            G[to][rev].cap += nf;
            if (f == 0) return ret;
        }
        if (ret == 0) lv[u] = 0;
        return ret;
    }
public:
    void init(int n_) { G.assign(n = n_, vector<E>()); }
    void add_edge(int u, int v, Cap c) {
        G[u].push_back({v, int(G[v].size()), c});
        G[v].push_back({u, int(G[u].size())-1, 0});
    }
    Cap max_flow(int st_, int ed_) {
        st = st_, ed = ed_; Cap ret = 0;
        for (int i = 63; i >= 0; --i)
            while (BFS(i)) ret += DFS(st);
        return ret;
    }
}; // test @ luogu P3376
```

4.5 HLPP [198e4e]

```
template <typename T> struct HLPP {
    struct Edge { int to, rev; T flow, cap; };
    int n, mx; vector<vector<Edge>> adj; vector<T> excess;
    vector<int> d, cnt, active; vector<vector<int>> B;
    void add_edge(int u, int v, int f) {
        Edge a{v, (int)size(adj[v]), 0, f};
        Edge b{u, (int)size(adj[u]), 0, 0};
        adj[u].push_back(a), adj[v].push_back(b);
    }
    void enqueue(int v) {
        if (!active[v] && excess[v] > 0 && d[v] < n) {
            mx = max(mx, d[v]);
            B[d[v]].push_back(v); active[v] = 1;
        }
    }
    void push(int v, Edge &e) {
        T df = min(excess[v], e.cap - e.flow);
        if (df <= 0 || d[v] != d[e.to] + 1) return;
        e.flow += df, adj[e.to][e.rev].flow -= df;
        excess[e.to] += df, excess[v] -= df;
        enqueue(e.to);
    }
    void gap(int k) {
        for (int v = 0; v < n; v++) if (d[v] >= k)
            cnt[d[v]]--, d[v] = n, cnt[d[v]]++;
    }
    void relabel(int v) {
        cnt[d[v]]--; d[v] = n;
        for (auto e: adj[v])
            if (e.cap > e.flow) d[v] = min(d[v], d[e.to] + 1);
        cnt[d[v]]++; enqueue(v);
    }
    void discharge(int v) {
        for (auto &e: adj[v])
            if (excess[v] > 0) push(v, e);
            else break;
        if (excess[v] <= 0) return;
        if (cnt[d[v]] == 1) gap(d[v]);
        else relabel(v);
    }
    T max_flow(int s, int t) {
        for (auto &e: adj[s]) excess[s] += e.cap;
        cnt[0] = n; enqueue(s); active[t] = 1;
        for (mx = 0; mx >= 0; --mx)
            if (!B[mx].empty()) {
                int v = B[mx].back(); B[mx].pop_back();
                active[v] = 0; discharge(v);
            } else --mx;
        return excess[t];
    }
    HLPP(int n_) : n(n_), adj(n), excess(n),
        d(n), cnt(n + 1), active(n), B(n) {}
};
```

4.6 Global Min-Cut [ae7013]

```
void add_edge(auto &w, int u, int v, int c) {
    w[u][v] += c; w[v][u] += c; }
auto phase(const auto &w, int n, vector<int> id) {
```



```

vector<lld> g(n); int s = -1, t = -1;
while (!id.empty()) {
    int c = -1;
    for (int i : id) if (c == -1 || g[i] > g[c]) c = i;
    s = t; t = c;
    id.erase(ranges::find(id, c));
    for (int i : id) g[i] += w[c][i];
}
return tuple{s, t, g[t]};
}
lld mincut(auto w, int n) {
    lld cut = numeric_limits<lld>::max();
    vector<int> id(n); iota(all(id), 0);
    for (int i = 0; i < n - 1; ++i) {
        auto [s, t, gt] = phase(w, n, id);
        id.erase(ranges::find(id, t));
        cut = min(cut, gt);
        for (int j = 0; j < n; ++j)
            w[s][j] += w[t][j], w[j][s] += w[j][t];
    }
    return cut;
} // O(V^3), can be O(VE + V^2 log V)?

```

4.7 GomoryHu Tree [5ed29]

```

vector<tuple<int, int, int>> GomoryHu(int n) {
    vector<tuple<int, int, int>> rt;
    vector<int> g(n);
    for (int i = 1; i < n; ++i) {
        int t = g[i];
        auto f = flow;
        rt.emplace_back(f.max_flow(i, t), i, t);
        f.walk(i); // bfs points that connected to i (use
                    // edges with .cap > 0)
        for (int j = i + 1; j < n; ++j)
            if (g[j] == t && f.connect(j)) // check if i can reach j
                g[j] = i;
    }
    return rt;
}
/* for our dinic:
 * void walk(int) { BFS(0); }
 * bool connect(int i) { return lv[i]; } */

```

4.8 Minimum Cost Max Flow [6d1b01]

```

template <typename F, typename C> class MCMF {
    static constexpr F INF_F = numeric_limits<F>::max();
    static constexpr C INF_C = numeric_limits<C>::max();
    struct E {
        int to, r;
        F f; C c;
        E() {}
        E(int a, int b, F x, C y)
            : to(a), r(b), f(x), c(y) {}
    };
    vector<vector<E>> g;
    vector<pair<int, int>> f;
    vector<bool> inq;
    vector<F> up; vector<C> d;
    optional<pair<F, C>> step(int S, int T) {
        queue<int> q;
        for (q.push(S), d[S] = 0, up[S] = INF_F;
             !q.empty(); q.pop()) {
            int u = q.front(); inq[u] = false;
            if (up[u] == 0) continue;
            for (int i = 0; i < int(g[u].size()); ++i) {
                auto e = g[u][i]; int v = e.to;
                if (e.f <= 0 || d[v] <= d[u] + e.c)
                    continue;
                d[v] = d[u] + e.c; f[v] = {u, i};
                up[v] = min(up[u], e.f);
                if (not inq[v]) q.push(v);
                inq[v] = true;
            }
        }
        if (d[T] == INF_C) return nullopt;
        for (int i = T; i != S; i = f[i].first) {
            auto &eg = g[f[i].first][f[i].second];
            eg.f -= up[T]; g[eg.to][eg.r].f += up[T];
        }
        return pair{up[T], d[T]};
    }
}

```

```

public:
MCMF(int n) : g(n), f(n), inq(n), up(n), d(n, INF_C) {}
void add_edge(int s, int t, F c, C w) {
    g[s].emplace_back(t, int(g[t].size()), c, w);
    g[t].emplace_back(s, int(g[s].size()) - 1, 0, -w);
}
pair<F, C> solve(int a, int b) {
    F c = 0; C w = 0;
    while (auto r = step(a, b)) {
        c += r->first, w += r->first * r->second;
        fill(inq.begin(), inq.end(), false);
        fill(d.begin(), d.end(), INF_C);
    }
    return {c, w};
}
};

```

4.9 Dijkstra Cost Flow [9675fa]

```

template <typename F, typename C> class MCMF {
    static constexpr F INF_F = numeric_limits<F>::max();
    static constexpr C INF_C = numeric_limits<C>::max();
    struct E {
        int to, r; F f; C c;
        E(int a, int b, F x, C y)
            : to(a), r(b), f(x), c(y) {}
    };
    vector<vector<E>> g; vector<pair<int, int>> f;
    vector<F> up; vector<C> d, h;
    optional<pair<F, C>> step(int S, int T) {
        priority_queue<pair<C, int>> q;
        q.emplace(d[S] = 0, S), up[S] = INF_F;
        while (not q.empty()) {
            auto [l, u] = q.top(); q.pop();
            if (up[u] == 0 || l != -d[u]) continue;
            for (int i = 0; i < int(g[u].size()); ++i) {
                auto e = g[u][i]; int v = e.to;
                auto nd = d[u] + e.c + h[u] - h[v];
                if (e.f <= 0 || d[v] <= nd) continue;
                f[v] = {u, i}; up[v] = min(up[u], e.f);
                q.emplace(-(d[v] = nd), v);
            }
        }
        if (d[T] == INF_C) return nullopt;
        for (size_t i = 0; i < d.size(); ++i) h[i] += d[i];
        for (int i = T; i != S; i = f[i].first) {
            auto &eg = g[f[i].first][f[i].second];
            eg.f -= up[T]; g[eg.to][eg.r].f += up[T];
        }
        return pair{up[T], h[T]};
    }
}
public:
MCMF(int n) : g(n), f(n), up(n), d(n, INF_C) {}
void add_edge(int s, int t, F c, C w) {
    g[s].emplace_back(t, int(g[t].size()), c, w);
    g[t].emplace_back(s, int(g[s].size()) - 1, 0, -w);
}
pair<F, C> solve(int a, int b) {
    h.assign(g.size(), 0);
    F c = 0; C w = 0;
    while (auto r = step(a, b)) {
        c += r->first, w += r->first * r->second;
        fill(d.begin(), d.end(), INF_C);
    }
    return {c, w};
}
};

```

4.10 Minimum Cost Circulation [3f7d84]

```

template <typename F, typename C>
struct MinCostCirculation {
    struct ep { int to; F flow; C cost; };
    int n; vector<int> vis; int visc;
    vector<int> fa, fae; vector<vector<int>> g;
    vector<ep> e; vector<C> pi;
    MinCostCirculation(int n_) : n(n_), vis(n), visc(0), g(n), pi(n) {}
    void add_edge(int u, int v, F fl, C cs) {
        g[u].emplace_back((int)e.size());
        e.emplace_back(v, fl, cs);
        g[v].emplace_back((int)e.size());
        e.emplace_back(u, 0, -cs);
    }
}

```

```

C phi(int x) {
    if (fa[x] == -1) return 0;
    if (vis[x] == visc) return pi[x];
    vis[x] = visc;
    return pi[x] = phi(fa[x]) - e[fae[x]].cost;
}
int lca(int u, int v) {
    for (; u != -1 || v != -1; swap(u, v)) if (u != -1) {
        if (vis[u] == visc) return u;
        vis[u] = visc;
        u = fa[u];
    }
    return -1;
}
void pushflow(int x, C &cost) {
    int v = e[x ^ 1].to, u = e[x].to;
    ++visc;
    if (int w = lca(u, v); w == -1) {
        while (v != -1)
            swap(x ^ 1, fae[v]), swap(u, fa[v]), swap(u, v);
    } else {
        int z = u, dir = 0; F f = e[x].flow;
        vector<int> cyc = {x};
        for (int d : {0, 1})
            for (int i = (d ? u : v); i != w; i = fa[i]) {
                cyc.push_back(fae[i] ^ d);
                if (chmin(f, e[fae[i] ^ d].flow)) z = i, dir = d;
            }
        for (int i : cyc) {
            e[i].flow -= f; e[i ^ 1].flow += f;
            cost += f * e[i].cost;
        }
        if (dir) x ^ 1, swap(u, v);
        while (u != z)
            swap(x ^ 1, fae[v]), swap(u, fa[v]), swap(u, v);
    }
}
void dfs(int u) {
    vis[u] = visc;
    for (int i : g[u])
        if (int v = e[i].to; vis[v] != visc and e[i].flow)
            fa[v] = u, fae[v] = i, dfs(v);
}
C simplex() {
    C cost = 0;
    fa.assign(g.size(), -1); fae.assign(e.size(), -1);
    ++visc; dfs(0);
    for (int fail = 0; fail < ssize(e); )
        for (int i = 0; i < ssize(e); i++)
            if (e[i].flow and e[i].cost < phi(e[i ^ 1].to) -
                phi(e[i].to))
                fail = 0, pushflow(i, cost), ++visc;
            else ++fail;
    return cost;
}
};

```

4.11 General Graph Matching [5f2293]

```

struct Matching {
    queue<int> q; int ans, n;
    vector<int> fa, s, v, pre, match;
    int Find(int u) {
        return u == fa[u] ? u : fa[u] = Find(fa[u]);
    }
    int LCA(int x, int y) {
        static int tk = 0; tk++; x = Find(x); y = Find(y);
        for (; swap(x, y)) if (x != n) {
            if (v[x] == tk) return x;
            v[x] = tk;
            x = Find(pre[match[x]]);
        }
    }
    void Blossom(int x, int y, int l) {
        for (; Find(x) != l; x = pre[y]) {
            pre[x] = y, y = match[x];
            if (s[y] == 1) q.push(y), s[y] = 0;
            for (int z : {x, y}) if (fa[z] == z) fa[z] = l;
        }
    }
    bool Bfs(auto &g, int r) {
        iota(all(fa), 0); ranges::fill(s, -1);
        q = queue<int>(); q.push(r); s[r] = 0;
        for (; !q.empty(); q.pop()) {

```

```

            for (int x = q.front(); int u : g[x])
                if (s[u] == -1) {
                    if (pre[u] = x, s[u] = 1, match[u] == n) {
                        for (int a = u, b = x, last;
                            b != n; a = last, b = pre[a])
                            last = match[b], match[b] = a, match[a] = b;
                        return true;
                    }
                    q.push(match[u]); s[match[u]] = 0;
                } else if (!s[u] && Find(u) != Find(x)) {
                    int l = LCA(u, x);
                    Blossom(x, u, l); Blossom(u, x, l);
                }
            }
        }
        return false;
    }
    Matching(auto &g) : ans(0), n(int(g.size())),
        fa(n+1), s(n+1), v(n+1), pre(n+1, n), match(n+1, n) {
        for (int x = 0; x < n; ++x)
            if (match[x] == n) ans += Bfs(g, x);
    } // match[x] == n means not matched
}; // test @ yosupo judge

```

4.12 Weighted Matching [94ca35]

```

#define pb emplace_back
#define rep(i, l, r) for (int i=(l); i<=(r); ++i)
struct WeightGraph { // 1-based
    static const int inf = INT_MAX;
    struct edge { int u, v, w; }; int n, nx;
    vector<int> lab; vector<vector<edge>> g;
    vector<int> slack, match, st, pa, S, vis;
    vector<vector<int>> flo, flo_from; queue<int> q;
    WeightGraph(int n_) : n(n_), nx(n * 2), lab(nx + 1),
        g(nx + 1, vector<edge>(nx + 1)), slack(nx + 1),
        flo(nx + 1), flo_from(nx + 1, vector(n + 1, 0)) {
        match = st = pa = S = vis = slack;
        rep(u, 1, n) rep(v, 1, n) g[u][v] = {u, v, 0};
    }
    int ED(edge e) {
        return lab[e.u] + lab[e.v] - g[e.u][e.v].w * 2;
    }
    void update_slack(int u, int x, int &s) {
        if (!s || ED(g[u][x]) < ED(g[s][x])) s = u;
    }
    void set_slack(int x) {
        slack[x] = 0;
        for (int u = 1; u <= n; ++u)
            if (g[u][x].w > 0 && st[u] != x && S[st[u]] == 0)
                update_slack(u, x, slack[x]);
    }
    void q_push(int x) {
        if (x <= n) q.push(x);
        else for (int y : flo[x]) q_push(y);
    }
    void set_st(int x, int b) {
        st[x] = b;
        if (x > n) for (int y : flo[x]) set_st(y, b);
    }
    vector<int> split_flo(auto &f, int xr) {
        auto it = find(all(f), xr);
        if (auto pr = it - f.begin(); pr % 2 == 1)
            reverse(1 + all(f), it = f.end() - pr);
        auto res = vector(f.begin(), it);
        return f.erase(f.begin(), it), res;
    }
    void set_match(int u, int v) {
        match[u] = g[u][v].v;
        if (u <= n) return;
        int xr = flo_from[u][g[u][v].u];
        auto &f = flo[u], z = split_flo(f, xr);
        rep(i, 0, int(z.size())-1) set_match(z[i], z[i ^ 1]);
        set_match(xr, v); f.insert(f.end(), all(z));
    }
    void augment(int u, int v) {
        for (;;) {
            int xnv = st[match[u]]; set_match(u, v);
            if (!xnv) return;
            set_match(xnv, st[pa[xnv]]);
            u = st[pa[xnv]], v = xnv;
        }
    }
    int lca(int u, int v) {
        static int t = 0; ++t;
        for (++t; u || v; swap(u, v)) if (u) {

```

```

    if (vis[u] == t) return u;
    vis[u] = t; u = st[match[u]];
    if (u) u = st[pa[u]];
}
return 0;
}

void add_blossom(int u, int o, int v) {
    int b = int(find(n + 1 + all(st), 0) - begin(st));
    lab[b] = 0, S[b] = 0; match[b] = match[o];
    vector<int> f = {o};
    for (int x = u, y; x != o; x = st[pa[y]])
        f.pb(x), f.pb(y = st[match[x]]), q_push(y);
    reverse(1 + all(f));
    for (int x = v, y; x != o; x = st[pa[y]])
        f.pb(x), f.pb(y = st[match[x]]), q_push(y);
    flo[b] = f; set_st(b, b);
    for (int x = 1; x <= nx; ++x)
        g[b][x].w = g[x][b].w = 0;
    for (int x = 1; x <= n; ++x) flo_from[b][x] = 0;
    for (int xs : flo[b]) {
        for (int x = 1; x <= nx; ++x)
            if (g[b][x].w == 0 || ED(g[xs][x]) < ED(g[b][x]))
                g[b][x] = g[xs][x], g[x][b] = g[x][xs];
        for (int x = 1; x <= n; ++x)
            if (flo_from[xs][x]) flo_from[b][x] = xs;
    }
    set_slack(b);
}

void expand_blossom(int b) {
    for (int x : flo[b]) set_st(x, x);
    int xr = flo_from[b][g[b][pa[b]].u], xs = -1;
    for (int x : split_flo(flo[b], xr)) {
        if (xs == -1) { xs = x; continue; }
        pa[xs] = g[x][xs].u; S[xs] = 1, S[x] = 0;
        slack[xs] = 0; set_slack(x); q_push(x); xs = -1;
    }
    for (int x : flo[b])
        if (x == xr) S[x] = 1, pa[x] = pa[b];
        else S[x] = -1, set_slack(x);
    st[b] = 0;
}

bool on_found_edge(const edge &e) {
    if (int u = st[e.u], v = st[e.v]; S[v] == -1) {
        int nu = st[match[v]]; pa[v] = e.u; S[v] = 1;
        slack[v] = slack[nu] = 0; S[nu] = 0; q_push(nu);
    } else if (S[v] == 0) {
        if (int o = lca(u, v)) add_blossom(u, o, v);
        else return augment(u, v), augment(v, u), true;
    }
    return false;
}

bool matching() {
    ranges::fill(S, -1); ranges::fill(slack, 0);
    q = queue<int>();
    for (int x = 1; x <= nx; ++x)
        if (st[x] == x && !match[x])
            pa[x] = 0, S[x] = 0, q_push(x);
    if (q.empty()) return false;
    for (;;) {
        while (q.size()) {
            int u = q.front(); q.pop();
            if (S[st[u]] == 1) continue;
            for (int v = 1; v <= n; ++v)
                if (g[u][v].w > 0 && st[u] != st[v]) {
                    if (ED(g[u][v]) != 0)
                        update_slack(u, st[v], slack[st[v]]);
                    else if (on_found_edge(g[u][v])) return true;
                }
        }
        int d = inf;
        for (int b = n + 1; b <= nx; ++b)
            if (st[b] == b && S[b] == 1)
                d = min(d, lab[b] / 2);
        for (int x = 1; x <= nx; ++x)
            if (int s = slack[x]; st[x] == x && s && S[x] <= 0)
                d = min(d, ED(g[s][x]) / (S[x] + 2));
        for (int u = 1; u <= n; ++u)
            if (S[st[u]] == 1) lab[u] += d;
            else if (S[st[u]] == 0) {
                if (lab[u] <= d) return false;
                lab[u] -= d;
            }
    }
}

```

```

rep(b, n + 1, nx) if (st[b] == b && S[b] >= 0)
    lab[b] += d * (2 - 4 * S[b]);
for (int x = 1; x <= nx; ++x)
    if (int s = slack[x]; st[x] == x &&
        s && st[s] != x && ED(g[s][x]) == 0)
        if (on_found_edge(g[s][x])) return true;
for (int b = n + 1; b <= nx; ++b)
    if (st[b] == b && S[b] == 1 && lab[b] == 0)
        expand_blossom(b);
}
return false;
}

pair<lld, int> solve() {
    ranges::fill(match, 0);
    rep(u, 0, n) st[u] = u, flo[u].clear();
    int w_max = 0;
    rep(u, 1, n) rep(v, 1, n) {
        flo_from[u][v] = (u == v ? u : 0);
        w_max = max(w_max, g[u][v].w);
    }
    for (int u = 1; u <= n; ++u) lab[u] = w_max;
    int n_matches = 0; lld tot_weight = 0;
    while (matching()) ++n_matches;
    rep(u, 1, n) if (match[u] && match[u] < u)
        tot_weight += g[u][match[u]].w;
    return make_pair(tot_weight, n_matches);
}

void set_edge(int u, int v, int w) {
    g[u][v].w = g[v][u].w = w;
}
};

```

5 Math

5.1 Common Bounds

$$p(0) = 1, p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k - 1)/2) \approx 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	$\sim 2e5$	$\sim 2e8$

n	100	1e3	1e6	1e9	1e12	1e15	1e18
$\max_{i \leq n} (d(i))$	12	32	240	1344	6720	26880	103680

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\binom{2n}{n}$	2	6	20	70	252	924	3432	12870	48620	184756	7e5	2e6	1e7	4e7	1.5e8

5.2 Stirling Number

First Kind

$S_1(n, k)$ counts the number of permutations of n elements with k disjoint cycles.

$$S_1(n, k) = (n - 1) \cdot S_1(n - 1, k) + S_1(n - 1, k - 1)$$

$$x(x + 1) \dots (x + n - 1) = \sum_{k=0}^n S_1(n, k) x^k$$

$$g(x) = x(x + 1) \dots (x + n - 1) = \sum_{k=0}^n a_k x^k$$

$$\Rightarrow g(x + n) = \sum_{k=0}^n \frac{b_k}{(n - k)!} x^{n-k},$$

$$b_k = \sum_{i=0}^k ((n - i)! a_{n-i}) \cdot \left(\frac{n^{k-i}}{(k - i)!} \right)$$

Second Kind

$S_2(n, k)$ counts the number of ways to partition a set of n elements into k nonempty sets.

$$S_2(n, k) = S_2(n - 1, k - 1) + k \cdot S_2(n - 1, k)$$

$$S_2(n, k) = \sum_{i=0}^k \binom{k}{i} i^n (-1)^{k-i} = \sum_{i=0}^k \frac{(-1)^i}{i!} \cdot \frac{(k - i)^n}{(k - i)!}$$

5.3 ax+by=gcd [d0cbdd]

```

// ax+ny = 1, ax+ny == ax == 1 (mod n)
void exgcd(lld x, lld y, lld &g, lld &a, lld &b) {
    if (y == 0) g = x, a = 1, b = 0;
    else exgcd(y, x % y, g, b, a), b -= (x / y) * a;
}

```

5.4 Chinese Remainder [d69e74]

```

// please ensure r_i \in [0, m_i)
bool crt(lld &m1, lld &r1, lld m2, lld r2) {
    if (m2 > m1) swap(m1, m2), swap(r1, r2);
    lld g, a, b; exgcd(m1, m2, g, a, b);
    if ((r2 - r1) % g != 0) return false;
    m2 /= g; lld D = (r2 - r1) / g % m2 * a % m2;
    r1 += (D < 0 ? D + m2 : D) * m1; m1 *= m2;
    assert(r1 >= 0 && r1 < m1);
    return true;
}

```

5.5 DiscreteLog [86e463]

```
template<typename Int>
Int BSGS(Int x, Int y, Int M) {
    // x^? \equiv y (mod M)
    Int t = 1, c = 0, g = 1;
    for (Int M_ = M; M_ > 0; M_ >>= 1) g = g * x % M;
    for (g = gcd(g, M); t % g != 0; ++c) {
        if (t == y) return c;
        t = t * x % M;
    }
    if (y % g != 0) return -1;
    t /= g, y /= g, M /= g;
    Int h = 0, gs = 1;
    for (; h * h < M; ++h) gs = gs * x % M;
    unordered_map<Int, Int> bs;
    for (Int s = 0; s < h; bs[y] = ++s) y = y * x % M;
    for (Int s = 0; s < M; s += h) {
        t = t * gs % M;
        if (bs.count(t)) return c + s + h - bs[t];
    }
    return -1;
}
```

5.6 Quadratic Residue [1eabad]

```
int get_root(int n, int P) { // ensure 0 <= n < p
    if (P == 2 or n == 0) return n;
    auto check = [&](int x) {
        return modpow(x, (P - 1) / 2, P);
    };
    if (check(n) != 1) return -1;
    mt19937 rnd(7122); lld z = 1, w;
    while (check(w = (z * z - n + P) % P) != P - 1)
        z = rnd() % P;
    const auto M = [P, w](auto &u, auto &v) {
        auto [a, b] = u; auto [c, d] = v;
        return make_pair((a * c + b * d % P * w) % P,
            (a * d + b * c) % P);
    };
    pair<lld, lld> r(1, 0), e(z, 1);
    for (int w = (P + 1) / 2; w; w >>= 1, e = M(e, e))
        if (w & 1) r = M(r, e);
    return r.first; // sqrt(n) mod P where P is prime
}
```

5.7 Extended Euler

$$a^b \equiv \begin{cases} a^{(b \bmod \varphi(m)) + \varphi(m)} & \text{if } (a, m) \neq 1 \wedge b \geq \varphi(m) \\ a^{b \bmod \varphi(m)} & \text{otherwise} \end{cases} \pmod{m}$$

5.8 Extended FloorSum

$$g(a, b, c, n) = \sum_{i=0}^n i \lfloor \frac{ai+b}{c} \rfloor$$

$$= \begin{cases} \lfloor \frac{a}{c} \rfloor \cdot \frac{n(n+1)(2n+1)}{6} + \lfloor \frac{b}{c} \rfloor \cdot \frac{n(n+1)}{2} \\ + g(a \bmod c, b \bmod c, c, n), & a \geq c \vee b \geq c \\ 0, & n < 0 \vee a = 0 \\ \frac{1}{2} \cdot (n(n+1)m - f(c, c-b-1, a, m-1) \\ - h(c, c-b-1, a, m-1)), & \text{otherwise} \end{cases}$$

$$h(a, b, c, n) = \sum_{i=0}^n \lfloor \frac{ai+b}{c} \rfloor^2$$

$$= \begin{cases} \lfloor \frac{a}{c} \rfloor^2 \cdot \frac{n(n+1)(2n+1)}{6} + \lfloor \frac{b}{c} \rfloor^2 \cdot (n+1) \\ + \lfloor \frac{a}{c} \rfloor \cdot \lfloor \frac{b}{c} \rfloor \cdot n(n+1) \\ + h(a \bmod c, b \bmod c, c, n) \\ + 2 \lfloor \frac{a}{c} \rfloor \cdot g(a \bmod c, b \bmod c, c, n) \\ + 2 \lfloor \frac{b}{c} \rfloor \cdot f(a \bmod c, b \bmod c, c, n), & a \geq c \vee b \geq c \\ 0, & n < 0 \vee a = 0 \\ nm(m+1) - 2g(c, c-b-1, a, m-1) \\ - 2f(c, c-b-1, a, m-1) - f(a, b, c, n), & \text{otherwise} \end{cases}$$

5.9 Extended Euclidean [e09892]

```
template <typename T>
auto euclid(lld a, lld b, lld c, lld n, T U, T R) {
    b %= c;
    if (a >= c)
        return euclid(a % c, b, c, n, U, mpow(U, a / c) * R);
    lld m = (i128(a) * n + b) / c;
    if (!m) return mpow(R, n);
    return mpow(R, (c - b - 1) / a) * U *
        euclid(c, c - b - 1, a, m - 1, R, U) *
        mpow(R, n - (i128(c) * m - b - 1) / a);
}
// time complexity is O(log max(a, c))
```

```
// 給定二維座標系上的一次函數 $y = (ax + b) / c$
// 維護一個矩陣 $A = I$, 考慮 $x \in [0, n)$
// 每次向右穿過網格的垂直線時，乘上一個矩陣 $R$
// 每次向上穿過網格的水平線時，乘上一個矩陣 $U$
// 若剛好經過一個整點，那麼先乘 $U$ 再乘 $R$
```

5.10 FloorSum [fb5917]

```
// @param n `n < 2^32`
// @param m `1 <= m < 2^32`
// @return sum_{i=0}^{n-1} floor((ai + b)/m) mod 2^64
llu floor_sum_unsigned(llu n, llu m, llu a, llu b) {
    llu ans = 0;
    while (true) {
        if (a >= m) ans += n * (n - 1) / 2 * (a / m), a %= m;
        if (b >= m) ans += n * (b / m), b %= m;
        if (llu y_max = a * n + b; y_max >= m) {
            n = (llu)(y_max / m), b = (llu)(y_max % m);
            swap(m, a);
        } else break;
    }
    return ans;
}
lld floor_sum(lld n, lld m, lld a, lld b) {
    llu ans = 0;
    if (a < 0) {
        llu a2 = (a % m + m), d = (a2 - a) / m;
        ans -= 1ULL * n * (n - 1) / 2 * d; a = a2;
    }
    if (b < 0) {
        llu b2 = (b % m + m), d = (b2 - b) / m;
        ans -= 1ULL * n * d; b = b2;
    }
    return ans + floor_sum_unsigned(n, m, a, b);
}
```

5.11 ModMin [253e4d]

```
// min{k | l <= ((ak) mod m) <= r}
optional<llu> mod_min(u32 a, u32 m, u32 l, u32 r) {
    if (a == 0) return l ? nullopt : 0;
    if (auto k = llu(l + a - 1) / a; k * a <= r)
        return k;
    auto b = m / a, c = m % a;
    if (auto y = mod_min(c, a, a - r % a, a - l % a))
        return (l + *y * c + a - 1) / a + *y * b;
    return nullopt;
}
```

5.12 FWT [f82550]

```
/* or convolution:
 * x = (x0, x0+x1), inv = (x0, x1-x0) w/o final div
 * and convolution:
 * x = (x0+x1, x1), inv = (x0-x1, x1) w/o final div */
void fwt(int x[], int N, bool inv = false) {
    for (int d = 1; d < N; d <= 1)
        for (int s = 0; s < N; s += d * 2)
            for (int i = s; i < s + d; i++) {
                int j = i + d, ta = x[i], tb = x[j];
                x[i] = add(ta, tb);
                x[j] = sub(ta, tb);
            }
    if (inv) {
        const int invn = modinv(N);
        for (int i = 0; i < N; i++)
            x[i] = mul(x[i], invn);
    }
}
```

5.13 Packed FFT [7c64ba]

```
int round2k(size_t n) {
    int sz = 1; while (sz < int(n)) sz *= 2; return sz;
}
VL convolution(const VI &a, const VI &b) {
    const int sz = round2k(a.size() + b.size() - 1);
    // Should be able to handle N <= 10^5, C <= 10^4
    vector<P> v(sz);
    for (size_t i = 0; i < a.size(); ++i) v[i].RE(a[i]);
    for (size_t i = 0; i < b.size(); ++i) v[i].IM(b[i]);
    fft(v.data(), sz, /*inv=*/false);
    auto rev = v; reverse(1 + all(rev));
    for (int i = 0; i < sz; ++i) {
        P A = (v[i] + conj(rev[i])) / P(2, 0);
        P B = (v[i] - conj(rev[i])) / P(0, 2);
        v[i] = A * B;
    }
}
```



```

}
VL c(sz); fft(v.data(), sz, /*inv=*/true);
for (int i = 0; i < sz; ++i) c[i] = roundl(RE(v[i]));
return c;
}
VI convolution_mod(const VI &a, const VI &b) {
    const int sz = round2k(a.size() + b.size() - 1);
    vector<P> fa(sz), fb(sz);
    for (size_t i = 0; i < a.size(); ++i)
        fa[i] = P(a[i] & ((1 << 15) - 1), a[i] >> 15);
    for (size_t i = 0; i < b.size(); ++i)
        fb[i] = P(b[i] & ((1 << 15) - 1), b[i] >> 15);
    fft(fa.data(), sz); fft(fb.data(), sz);
    auto rfa = fa; reverse(1 + all(rfa));
    for (int i = 0; i < sz; ++i) fa[i] *= fb[i];
    for (int i = 0; i < sz; ++i) fb[i] *= conj(rfa[i]);
    fft(fa.data(), sz, true); fft(fb.data(), sz, true);
    vector<int> res(sz);
    for (int i = 0; i < sz; ++i) {
        lld A = (lld)roundl(RE((fa[i] + fb[i]) / P(2, 0)));
        lld C = (lld)roundl(IM((fa[i] - fb[i]) / P(0, 2)));
        lld B = (lld)roundl(IM(fa[i])); B %= p; C %= p;
        res[i] = (A + (B << 15) + (C << 30)) % p;
    }
    return res;
} // test @ yosupo judge with long double

```

5.14 CRT for arbitrary mod [e4dde7]

```

const int mod = 1000000007;
const int M1 = 985661441; // G = 3 for M1, M2, M3
const int M2 = 998244353;
const int M3 = 1004535809;
int superBigCRT(lld A, lld B, lld C) {
    static_assert(M1 < M2 && M2 < M3);
    constexpr lld r12 = modpow(M1, M2-2, M2);
    constexpr lld r13 = modpow(M1, M3-2, M3);
    constexpr lld r23 = modpow(M2, M3-2, M3);
    constexpr lld M1M2 = 1LL * M1 * M2 % mod;
    B = (B - A + M2) * r12 % M2;
    C = (C - A + M3) * r13 % M3;
    C = (C - B + M3) * r23 % M3;
    return (A + B * M1 + C * M1M2) % mod;
}

```

5.15 NTT / FFT [41cf2]

```

template <int mod, int G, int maxn> struct NTT {
    static_assert(maxn == (maxn & -maxn));
    int roots[maxn];
    NTT () {
        int r = modpow(G, (mod - 1) / maxn);
        for (int i = maxn >> 1; i; i >>= 1) {
            roots[i] = 1;
            for (int j = 1; j < i; j++)
                roots[i + j] = mul(roots[i + j - 1], r);
            r = mul(r, r);
            // for (int j = 0; j < i; j++) // FFT (tested)
            // roots[i+j] = polar<llf>(1, PI * j / i);
        }
    }
    // n must be 2^k, and 0 <= F[i] < mod
    template <typename T>
    void operator()(int F[], T n, bool inv = false) {
        for (T i = 0, j = 0; i < n; i++) {
            if (i < j) swap(F[i], F[j]);
            for (T k = n >> 1; (j ^= k) < k; k >>= 1);
        }
        for (T s = 1; s < n; s *= 2) {
            for (T i = 0; i < n; i += s * 2) {
                for (T j = 0; j < s; j++) {
                    int a = F[i+j], b = mul(F[i+j+s], roots[s+j]);
                    F[i+j] = add(a, b); // a + b
                    F[i+j+s] = sub(a, b); // a - b
                }
            }
        }
        if (inv) {
            int iv = modinv(int(n));
            for (T i = 0; i < n; i++) F[i] = mul(F[i], iv);
            reverse(F + 1, F + n);
        }
    }
};

```

5.16 Formal Power Series [c6b99a]

```

#define fi(l, r) for (size_t i = (l); i < (r); i++)
using S = vector<int>;
auto Mul(auto a, auto b, size_t sz) {
    a.resize(sz), b.resize(sz);
    ntt(a.data(), sz); ntt(b.data(), sz);
    fi(0, sz) a[i] = mul(a[i], b[i]);
    return ntt(a.data(), sz, true), a;
}
S Newton(const S &v, int init, auto &&iter) {
    S Q = { init };
    for (int sz = 2; Q.size() < v.size(); sz *= 2) {
        S A{begin(v), begin(v) + min(sz, int(v.size()))};
        A.resize(sz * 2), Q.resize(sz * 2);
        iter(Q, A, sz * 2); Q.resize(sz);
    }
    return Q.resize(v.size()), Q;
}
S Inv(const S &v) { // v[0] != 0
    return Newton(v, modinv(v[0]),
        [](S &X, S &A, int sz) {
            ntt(X.data(), sz); ntt(A.data(), sz);
            for (int i = 0; i < sz; i++)
                X[i] = mul(X[i], sub(2, mul(X[i], A[i])));
            ntt(X.data(), sz, true);
        });
}
S Dx(S A) {
    fi(1, A.size()) A[i - 1] = mul(i, A[i]);
    return A.empty() ? A : (A.pop_back(), A);
}
S Sx(S A) {
    A.insert(A.begin(), 0);
    fi(1, A.size()) A[i] = mul(modinv(int(i)), A[i]);
    return A;
}
S Ln(const S &A) { // coef[0] == 1; res[0] == 0
    auto B = Sx(Mul(Dx(A), Inv(A), bit_ceil(A.size()*2)));
    return B.resize(A.size()), B;
}
S Exp(const S &v) { // coef[0] == 0; res[0] == 1
    return Newton(v, 1,
        [](S &X, S &A, int sz) {
            auto Y = X; Y.resize(sz / 2); Y = Ln(Y);
            fi(0, Y.size()) Y[i] = sub(A[i], Y[i]);
            Y[0] = add(Y[0], 1); X = Mul(X, Y, sz);
        });
}
S Pow(S a, lld M) { // period mod*(mod-1)
    assert(!a.empty() && a[0] != 0);
    const auto imul = [&a](int s) {
        for (int &x: a) x = mul(x, s);
        int c = a[0];
        imul(modinv(c)); a = Ln(a); imul(int(M % mod));
        a = Exp(a); imul(modpow(c, int(M % (mod - 1))));
        return a; // mod x^N where N=a.size()
    };
    S Sqrt(const S &v) { // need: QuadraticResidue
        assert(!v.empty() && v[0] != 0);
        const int r = get_root(v[0]); assert(r != -1);
        return Newton(v, r,
            [](S &X, S &A, int sz) {
                auto Y = X; Y.resize(sz / 2);
                auto B = Mul(A, Inv(Y), sz);
                for (int i = 0, inv2 = mod / 2 + 1; i < sz; i++)
                    X[i] = mul(inv2, add(X[i], B[i]));
            });
}
S Mul(auto &&a, auto &&b) {
    const auto n = a.size() + b.size() - 1;
    auto R = Mul(a, b, bit_ceil(n));
    return R.resize(n), R;
}
S MulT(S a, S b, size_t k) {
    assert(b.size()); reverse(all(b)); auto R = Mul(a, b);
    R = vector(R.begin() + b.size() - 1, R.end());
    return R.resize(k), R;
}
S Eval(const S &f, const S &x) {
    if (f.empty()) return vector(x.size(), 0);
    const int n = int(max(x.size(), f.size()));
    auto q = vector(n * 2, S(2, 1)); S ans(n);
    fi(0, x.size()) q[i + n][1] = sub(0, x[i]);
    for (int i = n - 1; i > 0; i--)
        q[i] = Mul(q[i << 1], q[i << 1 | 1]);
}

```

```

q[1] = Mult(f, Inv(q[1]), n);
for (int i = 1; i < n; i++) {
    auto L = q[i << 1], R = q[i << 1 | 1];
    q[i << 1 | 0] = Mult(q[i], R, L.size());
    q[i << 1 | 1] = Mult(q[i], L, R.size());
}
for (int i = 0; i < n; i++) ans[i] = q[i + n][0];
return ans.resize(x.size()), ans;
}
pair<S, S> DivMod(const S &A, const S &B) {
    assert(!B.empty() && B.back() != 0);
    if (A.size() < B.size()) return {}, A;
    const auto sz = A.size() - B.size() + 1;
    S X = B; reverse(all(X)); X.resize(sz);
    S Y = A; reverse(all(Y)); Y.resize(sz);
    S Q = Mul(Inv(X), Y);
    Q.resize(sz); reverse(all(Q)); X = Mul(Q, B); Y = A;
    fi(0, Y.size()) Y[i] = sub(Y[i], X[i]);
    while (Y.size() && Y.back() == 0) Y.pop_back();
    while (Q.size() && Q.back() == 0) Q.pop_back();
    return {Q, Y};
} // empty means zero polynomial
int LinearRecursionKth(S a, S c, int64_t k) {
    const auto d = a.size(); assert(c.size() == d + 1);
    const auto sz = bit_ceil(2 * d + 1), o = sz / 2;
    S q = c; for (int &x: q) x = sub(0, x); q[0]=1;
    S p = Mul(a, q); p.resize(sz); q.resize(sz);
    for (int r; r = (k & 1), k; k >= 1) {
        fill(d + all(p), 0); fill(d + 1 + all(q), 0);
        ntt(p.data(), sz); ntt(q.data(), sz);
        for (size_t i = 0; i < sz; i++)
            p[i] = mul(p[i], q[(i + o) & (sz - 1)]);
        for (size_t i = 0, j = o; j < sz; i++, j++)
            q[i] = q[j] = mul(q[i], q[j]);
        ntt(p.data(), sz, true); ntt(q.data(), sz, true);
        for (size_t i = 0; i < d; i++) p[i] = p[i << 1 | r];
        for (size_t i = 0; i <= d; i++) q[i] = q[i << 1];
    } // Bostan-Mori
    return mul(p[0], modinv(q[0]));
} // a_n = \sum c_j a_{n-j}, c_0 is not used

```

5.17 Partition Number [9bb845]

```

ans[0] = tmp[0] = 1;
for (int i = 1; i * i <= n; i++) {
    for (int rep = 0; rep < 2; rep++)
        for (int j = i; j <= n - i * i; j++)
            modadd(tmp[j], tmp[j-i]);
    for (int j = i * i; j <= n; j++)
        modadd(ans[j], tmp[j - i * i]);
}

```

5.18 Pi Count [715863]

```

struct S { int rough; lld large; int id; };
lld PrimeCount(lld n) { // n ~ 10^13 => < 1s
    if (n <= 1) return 0;
    const int v = static_cast<int>(sqrtl(n)); int pc = 0;
    vector<int> smalls(v + 1), skip(v + 1); vector<S> z;
    for (int i = 2; i <= v; ++i) smalls[i] = (i + 1) / 2;
    for (int i : views::iota(0, (v + 1) / 2))
        z.emplace_back(2*i+1, (n / (2*i+1) + 1) / 2, i);
    for (int p = 3; p <= v; ++p)
        if (smalls[p] > smalls[p - 1]) {
            const int q = p * p; ++pc;
            if (1LL * q * q > n) break;
            skip[p] = 1;
            for (int i = q; i <= v; i += 2 * p) skip[i] = 1;
            int ns = 0;
            for (auto e : z) if (!skip[e.rough]) {
                lld d = 1LL * e.rough * p;
                e.large += pc - (d <= v ? z[smalls[d] - pc].large :
                    smalls[n / d]);
                e.id = ns; z[ns++] = e;
            }
            z.resize(ns);
            for (int j = v / p; j >= p; --j) {
                int c = smalls[j] - pc, e = min(j * p + p, v + 1);
                for (int i = j * p; i < e; ++i) smalls[i] -= c;
            }
        }
    lld ans = z[0].large; z.erase(z.begin());
    for (auto &[rough, large, k] : z) {
        const lld m = n / rough; --k;

```

```

ans -= large - (pc + k);
for (auto [p, _, l] : z)
    if (l >= k || p * p > m) break;
    else ans += smalls[m / p] - (pc + l);
}
return ans;
} // test @ yosupo library checker w/ n=1e11, 68ms

```

5.19 Miller Rabin [fbd812]

```

bool isprime(llu x) {
    auto withn = [&](llu a, int t) {
        for (llu a2; t--; a = a2) {
            a2 = mmul(a, a, x);
            if (a2 == 1 && a != 1 && a != x - 1) return true;
        }
        return a != 1;
    };
    if (x <= 2 || ~x & 1) return x == 2;
    int t = countr_zero(x-1); llu odd = (x-1) >> t;
    for (llu m: {2, 325, 9375, 28178, 450775, 9780504, 1795265022})
        if (m % x != 0 && withn(mpow(m % x, odd, x), t))
            return false;
    return true;
} // test @ luogu 143 & yosupo judge, ~1700ms for Q=1e5
// if use montgomery, ~250ms for Q=1e5

```

5.20 Pollard Rho [57ad88]

```

// does not work when n is prime or n == 1
// return any non-trivial factor
llu pollard_rho(llu n) {
    static mt19937_64 rnd(120821011);
    if (!(n & 1)) return 2;
    llu y = 2, z = y, c = rnd() % n, p = 1, i = 0, t;
    auto f = [&](llu x) {
        return madd(mmul(x, x, n), c, n);
    };
    do {
        p = mmul(msub(z = f(f(z)), y = f(y), n), p, n);
        if (++i &= 63) if (i == (i & -i)) t = gcd(p, n);
    } while (t == 1);
    return t == n ? pollard_rho(n) : t;
} // test @ yosupo judge, ~270ms for Q=100
// if use montgomery, ~70ms for Q=100

```

5.21 Berlekamp Massey [a94d00]

```

template <typename T>
vector<T> BerlekampMassey(const vector<T> &output) {
    vector<T> d(output.size() + 1), me, he;
    for (size_t f = 0, i = 1; i <= output.size(); ++i) {
        for (size_t j = 0; j < me.size(); ++j)
            d[i] += output[i - j - 2] * me[j];
        if ((d[i] - output[i - 1]) == 0) continue;
        if (me.empty()) {
            me.resize(f = i);
            continue;
        }
        vector<T> o(i - f - 1);
        T k = -d[i] / d[f]; o.push_back(-k);
        for (T x : he) o.push_back(x * k);
        if (o.size() < me.size()) o.resize(me.size());
        for (size_t j = 0; j < me.size(); ++j) o[j] += me[j];
        if (i-f+he.size() > me.size()) he = me, f = i;
        me = o;
    }
    return me;
}

```

5.22 Gauss Elimination [9dea40]

```

void gauss(vector<vector<llf>> A, vector<llf> b) {
    const int n = A.size(), m = A[0].size();
    for (int i = 0; i < m; ++i) {
        int p = -1;
        for (int j = i; j < n; ++j)
            if (abs(A[j][i]) > eps)
                if (p == -1 || abs(A[j][i]) > abs(A[p][i]))
                    p = j;
        if (p == -1) continue;
        swap(A[p], A[i]);
        swap(b[p], b[i]);
        for (int j = 0; j < n; ++j) if (j != i) {
            llf z = A[j][i] / A[i][i];
            for (int k = 0; k < m; ++k)

```

```

    A[j][k] -= z * A[i][k];
    b[j] -= z * b[i];
}
}
}

```

5.23 Characteristic Polynomial [ff2159]

```

#define rep(x, y, z) for (int x=y; x<z; x++)
using VI = vector<int>; using VVI = vector<VI>;
void Hessenberg(VVI &H, int N) {
    for (int i = 0; i < N - 2; ++i) {
        for (int j = i + 1; j < N; ++j) if (H[j][i]) {
            rep(k, i, N) swap(H[i+1][k], H[j][k]);
            rep(k, 0, N) swap(H[k][i+1], H[k][j]);
            break;
        }
        if (!H[i + 1][i]) continue;
        for (int j = i + 2; j < N; ++j) {
            int co = mul(modinv(H[i + 1][i]), H[j][i]);
            rep(k, i, N) subeq(H[j][k], mul(H[i+1][k], co));
            rep(k, 0, N) addeq(H[k][i+1], mul(H[k][j], co));
        }
    }
}
VI CharacteristicPoly(VVI &A) {
    int N = (int)A.size(); Hessenberg(A, N);
    VVI P(N + 1, VI(N + 1)); P[0][0] = 1;
    for (int i = 1; i <= N; ++i) {
        rep(j, 0, i+1) P[i][j] = j ? P[i-1][j-1] : 0;
        for (int j = i - 1, val = 1; j >= 0; --j) {
            int co = mul(val, A[j][i - 1]);
            rep(k, 0, j+1) subeq(P[i][k], mul(P[j][k], co));
            if (j) val = mul(val, A[j][j - 1]);
        }
    }
    if (N & 1) for (int &x: P[N]) x = sub(0, x);
    return P[N]; // test: 2021 PTZ Korea K
}

```

5.24 Simplex [c9c93b]

```

namespace simplex {
    // maximize c^T x under Ax <= B and x >= 0
    // return VD(n, -inf) if the solution doesn't exist
    // return VD(n, +inf) if the solution is unbounded
    using VD = vector<llf>;
    using VVD = vector<vector<llf>>;
    const llf eps = 1e-9, inf = 1e+9;
    int n, m; VVD d; vector<int> p, q;
    void pivot(int r, int s) {
        llf inv = 1.0 / d[r][s];
        for (int i = 0; i < m + 2; ++i)
            for (int j = 0; j < n + 2; ++j)
                if (i != r && j != s)
                    d[i][j] -= d[r][j] * d[i][s] * inv;
        for (int i = 0; i < m + 2; ++i) if (i != r) d[i][s] *= -inv;
        for (int j = 0; j < n + 2; ++j) if (j != s) d[r][j] *= +inv;
        d[r][s] = inv; swap(p[r], q[s]);
    }
    bool phase(int z) {
        int x = m + z;
        while (true) {
            int s = -1;
            for (int i = 0; i <= n; ++i) {
                if (!z && q[i] == -1) continue;
                if (s == -1 || d[x][i] < d[x][s]) s = i;
            }
            if (s == -1 || d[x][s] > -eps) return true;
            int r = -1;
            for (int i = 0; i < m; ++i) {
                if (d[i][s] < eps) continue;
                if (r == -1 ||
                    d[i][n+1]/d[i][s] < d[r][n+1]/d[r][s]) r = i;
            }
            if (r == -1) return false;
            pivot(r, s);
        }
    }
    VD solve(const VVD &a, const VD &b, const VD &c) {
        m = (int)b.size(), n = (int)c.size();
        d = VVD(m + 2, VD(n + 2));
        for (int i = 0; i < m; ++i)
            for (int j = 0; j < n; ++j) d[i][j] = a[i][j];
    }
}

```

```

p.resize(m), q.resize(n + 1);
for (int i = 0; i < m; ++i)
    p[i] = n + i, d[i][n] = -1, d[i][n + 1] = b[i];
for (int i = 0; i < n; ++i) q[i] = i, d[m][i] = -c[i];
q[n] = -1, d[m + 1][n] = 1;
int r = 0;
for (int i = 1; i < m; ++i)
    if (d[i][n + 1] < d[r][n + 1]) r = i;
if (d[r][n + 1] < -eps) {
    pivot(r, n);
    if (!phase(1) || d[m + 1][n + 1] < -eps)
        return VD(n, -inf);
    for (int i = 0; i < m; ++i) if (p[i] == -1) {
        int s = min_element(d[i].begin(), d[i].end() - 1)
            - d[i].begin();
        pivot(i, s);
    }
}
if (!phase(0)) return VD(n, inf);
VD x(n);
for (int i = 0; i < m; ++i)
    if (p[i] < n) x[p[i]] = d[i][n + 1];
return x;
}

```

5.25 Simplex Construction

Standard form: maximize $\sum_{1 \leq i \leq n} c_i x_i$ such that for all $1 \leq j \leq m$, $\sum_{1 \leq i \leq n} A_{ji} x_i \leq b_j$, and $x_i \geq 0$ for all $1 \leq i \leq n$.

1. In case of minimization, let $c'_i = -c_i$
2. $\sum_{1 \leq i \leq n} A_{ji} x_i \geq b_j \rightarrow \sum_{1 \leq i \leq n} -A_{ji} x_i \leq -b_j$
3. $\sum_{1 \leq i \leq n} A_{ji} x_i = b_j$
 - $\sum_{1 \leq i \leq n} A_{ji} x_i \leq b_j$
 - $\sum_{1 \leq i \leq n} A_{ji} x_i \geq b_j$

4. If x_i has no lower bound, replace x_i with $x_i - x'_i$

5.26 Adaptive Simpson [09669e]

```

llf simp(llf l, llf r) {
    llf m = (l + r) / 2;
    return (f(l) + f(r) + 4.0 * f(m)) * (r - l) / 6.0;
}
llf F(llf L, llf R, llf v, llf eps) {
    llf M = (L + R) / 2, vl = simp(L, M), vr = simp(M, R);
    if (abs(vl + vr - v) <= 15 * eps)
        return vl + vr + (vl + vr - v) / 15.0;
    return F(L, M, vl, eps / 2.0) +
        F(M, R, vr, eps / 2.0);
} // call F(l, r, simp(l, r), 1e-6)

```

5.27 Integer Division [cd3c02]

```

int fdiv(int a, int b)
{ return a / b - (a % b && (a < 0) ^ (b < 0)); }
int cdiv(int a, int b)
{ return a / b + (a % b && (a < 0) ^ (b > 0)); }

```

6 Geometry

6.1 Basic Geometry [f50abd]

```

#define IM imag
#define RE real
using lld = int64_t;
using llf = long double;
using PT = std::complex<lld>;
using PF = std::complex<llf>;
using P = PT;
llf abs(P p) { return sqrtl(norm(p)); }
PF toPF(PT p) { return PF(RE(p), IM(p)); }
int sgn(lld x) { return (x > 0) - (x < 0); }
lld dot(P a, P b) { return RE(conj(a) * b); }
lld cross(P a, P b) { return IM(conj(a) * b); }
int ori(P a, P b, P c) {
    return sgn(cross(b - a, c - a));
}
int quad(P p) {
    return (IM(p) == 0) // use sgn for PF
        ? (RE(p) < 0 ? 3 : 1) : (IM(p) < 0 ? 0 : 2);
}
int argCmp(P a, P b) {
    // returns 0/+1, starts from theta = -PI
    int qa = quad(a), qb = quad(b);
    if (qa != qb) return sgn(qa - qb);
    return sgn(cross(b, a));
}

```

```
P rot90(P p) { return P{-IM(p), RE(p)}; }
template <typename V> llf area(const V & pt) {
    lld ret = 0;
    for (int i = 1; i + 1 < (int)pt.size(); i++)
        ret += cross(pt[i] - pt[0], pt[i+1] - pt[0]);
    return ret / 2.0;
}
template <typename V> PF center(const V & pt) {
    P ret = 0; lld A = 0;
    for (int i = 1; i + 1 < (int)pt.size(); i++) {
        lld cur = cross(pt[i] - pt[0], pt[i+1] - pt[0]);
        ret += (pt[i] + pt[i + 1] + pt[0]) * cur; A += cur;
    }
    return toPF(ret) / llf(A * 3);
}
PF project(PF p, PF q) { // p onto q
    return dot(p, q) * q / dot(q, q); // dot<llf>
}
```

6.2 2D Convex Hull [ecba37]

```
// from NaCl, counterclockwise, be careful of n<=2
vector<P> convex_hull(vector<P> v) {
    sort(all(v)); // by X then Y
    if (v[0] == v.back()) return {v[0]};
    int t = 0, s = 1; vector<P> h(v.size() + 1);
    for (int _ = 2; _--; s = t--, reverse(all(v)))
        for (P p : v) {
            while (t > s && ori(p, h[t-1], h[t-2]) >= 0) t--;
            h[t++] = p;
        }
    return h.resize(t), h;
}
```

6.3 2D Farthest Pair [8b5844]

```
// p is CCW convex hull w/o colinear points
int n = (int)p.size(), pos = 1; lld ans = 0;
for (int i = 0; i < n; i++) {
    P e = p[(i + 1) % n] - p[i];
    while (cross(e, p[(pos + 1) % n] - p[i]) >
            cross(e, p[pos] - p[i]))
        pos = (pos + 1) % n;
    for (int j: {i, (i + 1) % n})
        ans = max(ans, norm(p[pos] - p[j]));
} // tested @ AOJ CGL_4_B
```

6.4 MinMax Enclosing Rect [e4470c]

```
// from 8BQube, plz ensure p is strict convex hull
const llf INF = 1e18, qi = acos(-1) / 2 * 3;
pair<llf, llf> solve(const vector<P> &p) {
    llf mx = 0, mn = INF; int n = (int)p.size();
    for (int i = 0, u = 1, r = 1, l = 1; i < n; ++i) {
#define Z(v) (p[(v) % n] - p[i])
        P e = Z(i + 1);
        while (cross(e, Z(u + 1)) > cross(e, Z(u))) ++u;
        while (dot(e, Z(r + 1)) > dot(e, Z(r))) ++r;
        if (!i) l = r + 1;
        while (dot(e, Z(l + 1)) < dot(e, Z(l))) ++l;
        P D = p[r % n] - p[l % n];
        llf H = cross(e, Z(u)) / llf(norm(e));
        mn = min(mn, dot(e, D) * H);
        llf B = sqrt(norm(D)) * sqrt(norm(Z(u)));
        llf deg = (qi - acos(dot(D, Z(u)) / B)) / 2;
        mx = max(mx, B * sin(deg) * sin(deg));
    }
    return {mn, mx};
} // test @ UVA 819
```

6.5 Minkowski Sum [602806]

```
// A, B are strict convex hull rotate to min by (X, Y)
vector<P> Minkowski(vector<P> A, vector<P> B) {
    const int N = (int)A.size(), M = (int)B.size();
    vector<P> sa(N), sb(M), C(N + M + 1);
    for (int i = 0; i < N; i++) sa[i] = A[(i+1)%N]-A[i];
    for (int i = 0; i < M; i++) sb[i] = B[(i+1)%M]-B[i];
    C[0] = A[0] + B[0];
    for (int i = 0, j = 0; i < N || j < M; ) {
        P e = (j >= M || (i < N && cross(sa[i], sb[j]) >= 0))
            ? sa[i++] : sb[j++];
        C[i + j] = e;
    }
    partial_sum(all(C), C.begin()); C.pop_back();
    return convex_hull(C); // just to remove colinear
}
```

6.6 Segment Intersection [60d016]

```
struct Seg { // closed segment
    P st, dir; // represent st + t*dir for 0<=t<=1
    Seg(P s, P e) : st(s), dir(e - s) {}
    static bool valid(llf p, llf q) {
        // is there t s.t. 0 <= t <= 1 && qt == p ?
        if (q < 0) q = -q, p = -p;
        return 0 <= p && p <= q;
    }
    vector<P> ends() const { return { st, st + dir }; }
};
template <typename T> bool isInter(T A, P p) {
    if (A.dir == P(0)) return p == A.st; // BE CAREFUL
    return cross(p - A.st, A.dir) == 0 &&
        T::valid(dot(p - A.st, A.dir), norm(A.dir));
}
template <typename U, typename V>
bool isInter(U A, V B) {
    if (cross(A.dir, B.dir) == 0) { // BE CAREFUL
        bool res = false;
        for (P p: A.ends()) res |= isInter(B, p);
        for (P p: B.ends()) res |= isInter(A, p);
        return res;
    }
    P D = B.st - A.st; lld C = cross(A.dir, B.dir);
    return U::valid(cross(D, B.dir), C) &&
        V::valid(cross(D, A.dir), C);
}
```

6.7 Half Plane Intersection [31e216]

```
struct Line {
    P st, ed, dir;
    Line(P s, P e) : st(s), ed(e), dir(e - s) {}
}; using LN = const Line &;
PF intersect(LN A, LN B) {
    llf t = cross(B.st - A.st, B.dir) /
        llf(cross(A.dir, B.dir));
    return toPF(A.st) + toPF(A.dir) * t; // C^3 / C^2
}
bool cov(LN l, LN A, LN B) {
    i128 u = cross(B.st-A.st, B.dir);
    i128 v = cross(A.dir, B.dir);
    // ori(l.st, l.ed, A.st + A.dir*(u/v)) <= 0?
    i128 x = RE(A.dir) * u + RE(A.st - l.st) * v;
    i128 y = IM(A.dir) * u + IM(A.st - l.st) * v;
    return sgn(x*IM(l.dir) - y*RE(l.dir)) * sgn(v) >= 0;
} // x, y are C^3, also sgn<i128> is needed
bool operator<(LN a, LN b) {
    if (int c = argCmp(a.dir, b.dir)) return c == -1;
    return ori(a.st, a.ed, b.st) < 0;
}
// cross(pt-line.st, line.dir)<=0 <-> pt in half plane
// the half plane is the LHS when going from st to ed
llf HPI(vector<Line> &q) {
    sort(q.begin(), q.end());
    int n = (int)q.size(), l = 0, r = -1;
    for (int i = 0; i < n; i++) {
        if (i && !argCmp(q[i].dir, q[i-1].dir)) continue;
        while (l < r && cov(q[i], q[r-1], q[r])) --r;
        while (l < r && cov(q[i], q[l], q[l+1])) ++l;
        q[++r] = q[i];
    }
    while (l < r && cov(q[l], q[r-1], q[r])) --r;
    while (l < r && cov(q[r], q[l], q[l+1])) ++l;
    n = r - l + 1; // q[l .. r] are the lines
    if (n <= 1 || !argCmp(q[l].dir, q[r].dir)) return 0;
    vector<PF> pt(n);
    for (int i = 0; i < n; i++)
        pt[i] = intersect(q[i+l], q[(i+1)%n+l]);
    return area(pt);
} // test @ 2020 Nordic NCPC : BigBrother
```

6.8 SegmentDist (Sausage) [9d8603]

```
// be careful of abs<complex<int>> (replace _abs below)
llf PointSegDist(P A, Seg B) {
    if (B.dir == P(0)) return _abs(A - B.st);
    if (sgn(dot(A - B.st, B.dir)) *
        sgn(dot(A - B.ed, B.dir)) <= 0)
        return abs(cross(A - B.st, B.dir)) / _abs(B.dir);
    return min(_abs(A - B.st), _abs(A - B.ed));
}
llf SegSegDist(const Seg &s1, const Seg &s2) {
```



```

if (isInter(s1, s2)) return 0;
return min({
    PointSegDist(s1.st, s2),
    PointSegDist(s1.ed, s2),
    PointSegDist(s2.st, s1),
    PointSegDist(s2.ed, s1) });
} // test @ QOJ2444 / PTZ19 Summer.D3

```

6.9 Rotating Sweep Line [8aff27]

```

struct Event {
    P d; int u, v;
    bool operator<(const Event &b) const {
        return sgn(cross(d, b.d)) > 0; }
};
P makePositive(P z) { return cmpxy(z, 0) ? -z : z; }
void rotatingSweepLine(const vector<P> &p) {
    const int n = int(p.size());
    vector<Event> e; e.reserve(n * (n - 1) / 2);
    for (int i = 0; i < n; i++)
        for (int j = i + 1; j < n; j++)
            e.emplace_back(makePositive(p[i] - p[j]), i, j);
    sort(all(e));
    vector<int> ord(n), pos(n);
    iota(all(ord), 0);
    sort(all(ord), [&p](int i, int j) {
        return cmpxy(p[i], p[j]); });
    for (int i = 0; i < n; i++) pos[ord[i]] = i;
    const auto makeReverse = [](auto &v) {
        sort(all(v)); v.erase(unique(all(v)), v.end());
        vector<pair<int, int>> segs;
        for (size_t i = 0, j = 0; i < v.size(); i = j) {
            for (; j < v.size() && v[j].v - v[i].v <= j - i; j++);
            segs.emplace_back(v[i].u, v[j].u + 1 + 1);
        }
        return segs;
    };
    for (size_t i = 0, j = 0; i < e.size(); i = j) {
        /* do here */
        vector<size_t> tmp;
        for (; j < e.size() && !(e[i].u < e[j].u); j++)
            tmp.push_back(min(pos[e[j].u], pos[e[j].v]));
        for (auto [l, r] : makeReverse(tmp)) {
            reverse(ord.begin() + l, ord.begin() + r);
            for (int t = l; t < r; t++) pos[ord[t]] = t;
        }
    }
}

```

6.10 Polygon Cut [fdd064]

```

using P = PF;
vector<P> cut(const vector<P> &poly, P s, P e) {
    vector<P> res;
    for (size_t i = 0; i < poly.size(); i++) {
        P cur = poly[i], prv = i ? poly[i-1] : poly.back();
        bool side = ori(s, e, cur) < 0;
        if (side != (ori(s, e, prv) < 0))
            res.push_back(intersect({s, e}, {cur, prv}));
        if (side)
            res.push_back(cur);
    }
    return res;
}

```

6.11 Point In Simple Polygon [037c52]

```

bool PIP(const vector<P> &p, P z, bool strict = true) {
    int cnt = 0, n = (int)p.size();
    for (int i = 0; i < n; i++) {
        P A = p[i], B = p[(i + 1) % n];
        if (isInter(Seg(A, B), z)) return !strict;
        auto zy = IM(z), Ay = IM(A), By = IM(B);
        cnt ^= ((zy < Ay) - (zy < By)) * ori(z, A, B) > 0;
    }
    return cnt;
}

```

6.12 Point In Hull (Fast) [060ba1]

```

bool PIH(const vector<P> &h, P z, bool strict = true) {
    int n = (int)h.size(), a = 1, b = n - 1, r = !strict;
    if (n < 3) return r && isInter(Seg(h[0], h[n-1]), z);
    if (ori(h[0], h[a], h[b]) > 0) swap(a, b);
    if (ori(h[0], h[a], z) >= r || ori(h[0], h[b], z) <= -r)
        return false;
}

```

```

while (abs(a - b) > 1) {
    int c = (a + b) / 2;
    (ori(h[0], h[c], z) > 0 ? b : a) = c;
}
return ori(h[a], h[b], z) < r;
}

```

6.13 Point In Polygon (Fast) [71725b]

```

vector<int> PIPfast(vector<P> p, vector<P> q) {
    const int N = int(p.size()), Q = int(q.size());
    vector<pair<P, int>> evt;
    vector<Seg> edge;
    for (int i = 0; i < N; i++) {
        int a = i, b = (i + 1) % N;
        P A = p[a], B = p[b];
        assert(A < B || B < A); // std::operator<
        if (B < A) swap(A, B);
        evt.emplace_back(A, i);
        evt.emplace_back(B, ~i);
        edge.emplace_back(A, B);
    }
    for (int i = 0; i < Q; i++)
        evt.emplace_back(q[i], i + N);
    sort(all(evt));
    auto vtx = p; sort(all(vtx));
    auto eval = [](const Seg &a, lld x) -> lld {
        if (RE(a.dir) == 0) {
            assert(x == RE(a.st));
            return IM(a.st) + lld(IM(a.dir)) / 2;
        }
        lld t = (x - RE(a.st)) / lld(RE(a.dir));
        return IM(a.st) + IM(a.dir) * t;
    };
    lld cur_x = 0;
    auto cmp = [&](const Seg &a, const Seg &b) -> bool {
        if (int s = sgn(eval(a, cur_x) - eval(b, cur_x)))
            return s == -1;
        int s = sgn(cross(b.dir, a.dir));
        if (cur_x != RE(a.st) && cur_x != RE(b.st)) s *= -1;
        return s == -1;
    };
    namespace pbds = __gnu_pbds;
    using Tree = pbds::tree<Seg, int, decltype(cmp),
        pbds::rb_tree_tag,
        pbds::tree_order_statistics_node_update>;
    Tree st(cmp);
    vector<int> ans(Q);
    for (auto [ep, i] : evt) {
        cur_x = RE(ep);
        if (i < 0) { // remove
            st.erase(edge[~i]);
        } else if (i < N) { // insert
            auto [it, succ] = st.insert({edge[i], i});
            assert(succ);
        } else {
            int qid = i - N;
            if (binary_search(all(vtx), ep)) { // on vertex
                ans[qid] = 1;
                continue;
            }
            Seg H(ep, ep); // ??
            auto it = st.lower_bound(H);
            if (it != st.end() && isInter(it->first, ep)) {
                ans[qid] = 1; // on edge
                continue;
            }
            if (it != st.begin() && isInter(prev(it)->first, ep)) {
                ans[qid] = 1; // on edge
                continue;
            }
            auto rk = st.order_of_key(H);
            if (rk % 2 == 0) ans[qid] = 0; // outside
            else ans[qid] = 2; // inside
        }
    }
    return ans;
} // test @ AOJ CGL_3_C

```

6.14 Tangent of Points To Hull [6d7cd7]

```

pair<int, int> get_tangent(const vector<P> &v, P p) {
    const auto gao = [&, N = int(v.size())](int s) {

```

```

const auto lt = [&](int x, int y) {
    return ori(p, v[x % N], v[y % N]) == s; };
int l = 0, r = N; bool up = lt(0, 1);
while (r - l > 1) {
    int m = (l + r) / 2;
    if (lt(m, 0) ? up : !lt(m, m+1)) r = m;
    else l = m;
}
return (lt(l, r) ? r : l) % N;
}; // test @ codeforces.com/gym/101201/problem/E
return {gao(-1), gao(1)}; // (a,b):ori(p,v[a],v[b])<0
} // plz ensure that point strictly out of hull

```

6.15 Circle Class & Intersection [d5df51]

```

llf FMOD(llf x) {
    if (x < -PI) x += PI * 2;
    if (x > PI) x -= PI * 2;
    return x;
}
struct Cir { PF o; llf r; };
// be carefule when tangent
vector<llf> intersectAngle(Cir a, Cir b) {
    PF dir = b.o - a.o; llf d2 = norm(dir);
    llf (norm(a.r - b.r) >= d2) { // norm(x) := |x|^2
        if (a.r < b.r) return {-PI, PI}; // a in b
        else return {}; // b in a
    } else if (norm(a.r + b.r) <= d2) return {};
    llf dis = abs(dir), theta = arg(dir);
    llf phi = acos((a.r * a.r + d2 - b.r * b.r) /
        (2 * a.r * dis)); // is acos_safe needed?
    llf L = FMOD(theta - phi), R = FMOD(theta + phi);
    return {L, R};
}
vector<PF> intersectPoint(Cir a, Cir b) {
    llf d = abs(a.o - b.o);
    if (d > b.r+a.r || d < abs(b.r-a.r)) return {};
    llf dt = (b.r*b.r - a.r*a.r)/d, d1 = (d+dt)/2;
    PF dir = (a.o - b.o) / d;
    PF u = dir * d1 + b.o;
    PF v = rot90(dir) * sqrt(max(0.0L, b.r*b.r-d1*d1));
    return {u + v, u - v};
} // test @ AOJ CGL probs

```

6.16 Circle Common Tangent [d97f1c]

```

// be careful of tangent / exact same circle
// sign1 = 1 for outer tang, -1 for inter tang
vector<Line> common_tan(const Cir &a, const Cir &b, int
    sign1) {
    if (norm(a.o - b.o) < eps) return {};
    llf d = abs(a.o - b.o), c = (a.r - sign1 * b.r) / d;
    PF v = (b.o - a.o) / d;
    if (c * c > 1) return {};
    if (abs(c * c - 1) < eps) {
        PF p = a.o + c * v * a.r;
        return {Line(p, p + rot90(b.o - a.o))};
    }
    vector<Line> ret; llf h = sqrt(max(0.0L, 1-c*c));
    for (int sign2 : {1, -1}) {
        PF n = c * v + sign2 * h * rot90(v);
        PF p1 = a.o + n * a.r;
        PF p2 = b.o + n * (b.r * sign1);
        ret.emplace_back(p1, p2);
    }
    return ret;
}

```

6.17 Line-Circle Intersection [10786a]

```

vector<PF> LineCircleInter(PF p1, PF p2, PF o, llf r) {
    PF ft = p1 + project(o-p1, p2-p1), vec = p2-p1;
    llf dis = abs(o - ft);
    if (abs(dis - r) < eps) return {ft};
    if (dis > r) return {};
    vec = vec * sqrt(r * r - dis * dis) / abs(vec);
    return {ft + vec, ft - vec}; // sqrt_safe?
}

```

6.18 Poly-Circle Intersection [8e5133]

```

// Divides into multiple triangle, and sum up
// from 8BQube, test by HDU2892 & AOJ CGL_7_H
llf _area(PF pa, PF pb, llf r) {
    if (abs(pa) < abs(pb)) swap(pa, pb);
    if (abs(pb) < eps) return 0;

```

```

    llf S, h, theta;
    llf a = abs(pb), b = abs(pa), c = abs(pb - pa);
    llf cB = dot(pb, pb-pa) / a / c, B = acos_safe(cB);
    llf cC = dot(pa, pb) / a / b, C = acos_safe(cC);
    if (a > r) {
        S = (C / 2) * r * r; h = a * b * sin(C) / c;
        if (h < r && B < PI / 2)
            S -= (acos_safe(h/r)*r*r - h*sqrt_safe(r*r-h*h));
    } else if (b > r) {
        theta = PI - B - asin_safe(sin(B) / r * a);
        S = 0.5 * a*r*sin(theta) + (C-theta)/2 * r * r;
    } else
        S = 0.5 * sin(C) * a * b;
    return S;
}
llf area_poly_circle(const vector<PF> &v, PF O, llf r)
{
    llf S = 0;
    for (size_t i = 0, N = v.size(); i < N; ++i)
        S += _area(v[i] - O, v[(i + 1) % N] - O, r) *
            ori(0, v[i], v[(i + 1) % N]);
    return abs(S);
}

```

6.19 Minimum Covering Circle [92bb15]

```

Cir getCircum(P a, P b, P c) { // P = complex<llf>
    P z1 = a - b, z2 = a - c; llf D = cross(z1, z2) * 2;
    auto c1 = dot(a + b, z1), c2 = dot(a + c, z2);
    P o = rot90(c2 * z1 - c1 * z2) / D;
    return {o, abs(o - a)};
}
Cir minCircleCover(vector<P> p) {
    assert (!p.empty());
    ranges::shuffle(p, mt19937(114514));
    Cir c = {0, 0};
    for (size_t i = 0; i < p.size(); ++i) {
        if (abs(p[i] - c.o) <= c.r) continue;
        c = {p[i], 0};
        for (size_t j = 0; j < i; ++j) {
            if (abs(p[j] - c.o) <= c.r) continue;
            c.o = (p[i] + p[j]) / llf(2);
            c.r = abs(p[i] - c.o);
            for (size_t k = 0; k < j; ++k) {
                if (abs(p[k] - c.o) <= c.r) continue;
                c = getCircum(p[i], p[j], p[k]);
            }
        }
    }
    return c;
} // test @ TIOJ 1093 & luogu P1742

```

6.20 Circle Union [073c1c]

```

#define eb emplace_back
struct Teve { // test@SPOJ N=1000, 0.3~0.5s
    PF p; llf a; int add; // point, ang, add
    Teve(PF x, llf y, int z) : p(x), a(y), add(z) {}
    bool operator<(Teve &b) const { return a < b.a; }
};
// strict: x = 0, otherwise x = -1
bool disjunct(Cir &a, Cir &b, int x)
{ return sgn(abs(a.o - b.o) - a.r - b.r) > x; }
bool contain(Cir &a, Cir &b, int x)
{ return sgn(a.r - b.r - abs(a.o - b.o)) > x; }
vector<llf> CircleUnion(vector<Cir> &c) {
    // area[i] : area covered by at least i circles
    int N = (int)c.size(); vector<llf> area(N + 1);
    vector<vector<int>> overlap(N, vector<int>(N));
    auto g = overlap; // use simple 2darray to speedup
    for (int i = 0; i < N; ++i)
        for (int j = 0; j < N; ++j) {
            /* c[j] is non-strictly in c[i]. */
            overlap[i][j] = i != j &&
                (sgn(c[i].r - c[j].r) > 0 ||
                 (sgn(c[i].r - c[j].r) == 0 && i < j)) &&
                contain(c[i], c[j], -1);
        }
    for (int i = 0; i < N; ++i)
        for (int j = 0; j < N; ++j)
            g[i][j] = i != j && !(overlap[i][j] ||
                overlap[j][i] || disjunct(c[i], c[j], -1));
    for (int i = 0; i < N; ++i) {
        vector<Teve> eve; int cnt = 1;

```

```

for (int j = 0; j < N; ++j) cnt += overlap[j][i];
// if (cnt > 1) continue; (if only need area[1])
for (int j = 0; j < N; ++j) if (g[i][j]) {
    auto IP = intersectPoint(c[i], c[j]);
    PF aa = IP[1], bb = IP[0];
    llf A = arg(aa - c[i].o), B = arg(bb - c[i].o);
    eve.eb(bb, B, 1); eve.eb(aa, A, -1);
    if (B > A) ++cnt;
}
if (eve.empty()) area[cnt] += PI*c[i].r*c[i].r;
else {
    sort(eve.begin(), eve.end());
    eve.eb(eve[0]); eve.back().a += PI * 2;
    for (size_t j = 0; j + 1 < eve.size(); j++) {
        cnt += eve[j].add;
        area[cnt] += cross(eve[j].p, eve[j+1].p) *.5;
        llf t = eve[j + 1].a - eve[j].a;
        area[cnt] += (t-sin(t)) * c[i].r * c[i].r *.5;
    }
}
return area;
}

```

6.21 Polygon Union [2bfff43]

```

llf rat(P a, P b) { return sgn(RE(b)) ? llf(RE(a))/RE(b) : llf(IM(a))/IM(b); }
llf polyUnion(vector<vector<P>>& poly) {
    llf ret = 0; // area of poly[i] must be non-negative
    rep(i, 0, sz(poly)) rep(v, 0, sz(poly[i])) {
        P A = poly[i][v], B = poly[i][(v + 1) % sz(poly[i])];
        vector<pair<llf, int>> segs{{0, 0}, {1, 0}};
        rep(j, 0, sz(poly)) if (i != j) {
            rep(u, 0, sz(poly[j])) {
                P C = poly[j][u], D = poly[j][(u + 1) % sz(poly[j])];
                if (int sc = ori(A, B, C), sd = ori(A, B, D); sc != sd) {
                    llf sa = cross(D-C, A-C), sb = cross(D-C, B-C);
                    if (min(sc, sd) < 0)
                        segs.emplace_back(sa / (sa - sb), sgn(sc - sd));
                    else if (!sc && !sd && j < i && sgn(dot(B-A, D-C)) > 0) {
                        segs.emplace_back(rat(C - A, B - A), 1);
                        segs.emplace_back(rat(D - A, B - A), -1);
                    }
                }
            }
        }
        sort(segs.begin(), segs.end());
        for (auto &s : segs) s.first = clamp<llf>(s.first, 0, 1);
        llf sum = 0;
        int cnt = segs[0].second;
        rep(j, 1, sz(segs)) {
            if (!cnt) sum += segs[j].first - segs[j - 1].first;
            cnt += segs[j].second;
        }
        ret += cross(A, B) * sum;
    }
    return ret / 2;
}

```

6.22 3D Point [46b73b]

```

struct P3 {
    llf x, y, z;
    P3 operator^(const P3 &b) const {
        return {y*b.z-b.y*z, z*b.x-b.z*x, x*b.y-b.x*y};
    }
    //Azimuthal angle (longitude) to x-axis. \in [-pi, pi]
    llf phi() const { return atan2(y, x); }
    //Zenith angle (latitude) to the z-axis. \in [0, pi]
    llf theta() const { return atan2(sqrt(x*x+y*y), z); }
};
P3 ver(P3 a, P3 b, P3 c) { return (b - a) ^ (c - a); }
llf volume(P3 a, P3 b, P3 c, P3 d) {
    return dot(ver(a, b, c), d - a);
}
P3 rotate_around(P3 p, llf angle, P3 axis) {
    llf s = sin(angle), c = cos(angle);
    P3 u = normalize(axis);
    return u*dot(u, p)*(1-c) + p * c + cross(u, p)*s;
}

```

6.23 3D Convex Hull [01652a]

```

struct Face {
    int a, b, c;
    Face(int ta, int tb, int tc) : a(ta), b(tb), c(tc) {}
};
auto preprocess(const vector<P3> &pt) {
    auto G = pt.begin();
    auto a = find_if(all(pt), [&](P3 z) {
        return z != *G; }) - G;
    auto b = find_if(all(pt), [&](P3 z) {
        return ver(*G, pt[a], z) != P3(0, 0, 0); }) - G;
    auto c = find_if(all(pt), [&](P3 z) {
        return volume(*G, pt[a], pt[b], z) != 0; }) - G;
    vector<size_t> id;
    for (size_t i = 0; i < pt.size(); i++)
        if (i != a && i != b && i != c) id.push_back(i);
    return tuple{a, b, c, id};
}
// return the faces with pt indexes
// all points coplanar case will WA
vector<Face> convex_hull_3D(const vector<P3> &pt) {
    const int n = int(pt.size());
    if (n <= 3) return {}; // be careful about edge case
    vector<Face> now;
    vector<vector<int>> z(n, vector<int>(n));
    auto [a, b, c, ord] = preprocess(pt);
    now.emplace_back(a, b, c); now.emplace_back(c, b, a);
    for (auto i : ord) {
        vector<Face> next;
        for (const auto &f : now) {
            llf v = volume(pt[f.a], pt[f.b], pt[f.c], pt[i]);
            if (v <= 0) next.push_back(f);
            z[f.a][f.b] = z[f.b][f.c] = z[f.c][f.a] = sgn(v);
        }
        const auto F = [&](int x, int y) {
            if (z[x][y] > 0 && z[y][x] <= 0)
                next.emplace_back(x, y, i);
        };
        for (const auto &f : now)
            F(f.a, f.b), F(f.b, f.c), F(f.c, f.a);
        now = next;
    }
    return now;
}
// n^2 delaunay: facets with negative z normal of
// convexhull of (x, y, x^2 + y^2), use a pseudo-point
// (0, 0, inf) to avoid degenerate case
// test @ SPOJ CH3D
// llf area = 0, vol = 0; // surface area / volume
// for (auto [a, b, c]: faces)
// area += abs(ver(p[a], p[b], p[c]))/2.0,
// vol += volume(P3(0, 0, 0), p[a], p[b], p[c])/6.0;

```

6.24 3D Projection [68f350]

```

using P3F = valarray<llf>;
P3F toP3F(P3 p) { return {p.x, p.y, p.z}; }
llf dot(P3F a, P3F b) {
    return a[0]*b[0]+a[1]*b[1]+a[2]*b[2];
}
P3F housev(P3 A, P3 B, int s) {
    const llf a = abs(A), b = abs(B);
    return toP3F(A) / a + s * toP3F(B) / b;
}
P project(P3 p, P3 q) {
    P3 o(0, 0, 1);
    P3F u = housev(q, o, q.z > 0 ? 1 : -1);
    auto pf = toP3F(p);
    auto np = pf - 2 * u * dot(u, pf) / dot(u, u);
    return P(np[0], np[1]);
} // project p onto the plane q^Tx = 0

```

6.25 3D Skew Line Nearest Point

- $L_1 : \mathbf{v}_1 = \mathbf{p}_1 + t_1 \mathbf{d}_1, L_2 : \mathbf{v}_2 = \mathbf{p}_2 + t_2 \mathbf{d}_2$
- $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$
- $\mathbf{n}_1 = \mathbf{d}_1 \times \mathbf{n}, \mathbf{n}_2 = \mathbf{d}_2 \times \mathbf{n}$
- $\mathbf{c}_1 = \mathbf{p}_1 + \frac{(\mathbf{p}_2 - \mathbf{p}_1) \cdot \mathbf{n}_2}{\mathbf{d}_1 \cdot \mathbf{n}_2} \mathbf{d}_1, \mathbf{c}_2 = \mathbf{p}_2 + \frac{(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{n}_1}{\mathbf{d}_2 \cdot \mathbf{n}_1} \mathbf{d}_2$

6.26 Delaunay [3a4ff1]

```

/* please ensure input points are unique */
/* A triangulation such that no points will strictly
inside circumcircle of any triangle. C should be big
enough s.t. the initial triangle contains all points */

```

```

#define L(i) ((i)==0 ? 2 : (i)-1)
#define R(i) ((i)==2 ? 0 : (i)+1)
#define F3 for (int i = 0; i < 3; i++)
bool is_inf(P z) { return RE(z) <= -C || RE(z) >= C; }
bool in_cc(const array<P,3> &p, P q) {
    lld inf_det = 0, det = 0, inf_N, N;
    F3 {
        if (is_inf(p[i]) && is_inf(q)) continue;
        else if (is_inf(p[i])) inf_N = 1, N = -norm(q);
        else if (is_inf(q)) inf_N = -1, N = norm(p[i]);
        else inf_N = 0, N = norm(p[i]) - norm(q);
        lld D = cross(p[R(i)] - q, p[L(i)] - q);
        inf_det += inf_N * D; det += N * D;
    }
    return inf_det != 0 ? inf_det > 0 : det > 0;
}
P v[maxn];
struct Tri;
struct E {
    Tri *t; int side;
    E(Tri *t_=0, int side_=0) : t(t_), side(side_) {}
};
struct Tri {
    array<int,3> p; array<Tri*,3> ch; array<E,3> e;
    Tri(int a=0, int b=0, int c=0) : p{a, b, c}, ch{} {}
    bool has_chd() const { return ch[0] != nullptr; }
    bool contains(int q) const {
        F3 if (ori(v[p[i]], v[p[R(i)]], v[q]) < 0)
            return false;
        return true;
    }
    bool check(int q) const {
        return in_cc({v[p[0]], v[p[1]], v[p[2]]}, v[q]);
    }
} pool[maxn * 10], *it, *root;
void link(const E &a, const E &b) {
    if (a.t) a.t->e[a.side] = b;
    if (b.t) b.t->e[b.side] = a;
}
void flip(Tri *A, int a) {
    auto [B, b] = A->e[a]; /* flip edge between A,B */
    if (!B || !A->check(B->p[b])) return;
    Tri *X = new (it++) Tri(A->p[R(a)], B->p[b], A->p[a]);
    Tri *Y = new (it++) Tri(B->p[R(b)], A->p[a], B->p[b]);
    link(E(X, 0), E(Y, 0));
    link(E(X, 1), A->e[L(a)]); link(E(X, 2), B->e[R(b)]);
    link(E(Y, 1), B->e[L(b)]); link(E(Y, 2), A->e[R(a)]);
    A->ch = B->ch = {X, Y, nullptr};
    flip(X, 1); flip(X, 2); flip(Y, 1); flip(Y, 2);
}
void add_point(int p) {
    Tri *r = root;
    while (r->has_chd()) for (Tri *c: r->ch)
        if (c && c->contains(p)) { r = c; break; }
    array<Tri*, 3> t; /* split into 3 triangles */
    F3 t[i] = new (it++) Tri(r->p[i], r->p[R(i)], p);
    F3 link(E(t[i], 0), E(t[R(i)], 1));
    F3 link(E(t[i], 2), r->e[L(i)]);
    r->ch = t;
    F3 flip(t[i], 2);
}
auto build(const vector<P> &p) {
    it = pool; int n = (int)p.size();
    vector<int> ord(n); iota(all(ord), 0);
    shuffle(all(ord), mt19937(114514));
    root = new (it++) Tri(n, n + 1, n + 2);
    copy_n(p.data(), n, v); v[n++] = P(-C, -C);
    v[n++] = P(C * 2, -C); v[n++] = P(-C, C * 2);
    for (int i : ord) add_point(i);
    vector<array<int, 3>> res;
    for (Tri *now = pool; now != it; now++)
        if (!now->has_chd()) res.push_back(now->p);
    return res;
}

```

6.27 Build Voronoi [94f000]

```

void build_voronoi_cells(auto &&p, auto &&res) {
    vector<vector<int>> adj(p.size());
    for (auto f: res) F3 {
        int a = f[i], b = f[R(i)];
        if (a >= p.size() || b >= p.size()) continue;
        adj[a].emplace_back(b);
    }
}

```

```

// use `adj` and `p` and HPI to build cells
for (size_t i = 0; i < p.size(); i++) {
    vector<Line> ls = frame; // the frame
    for (int j : adj[i]) {
        P m = p[i] + p[j], d = rot90(p[j] - p[i]);
        assert (norm(d) != 0);
        ls.emplace_back(m, m + d); // doubled coordinate
    } // HPI(ls)
}
}

```

6.28 kd Tree (Nearest Point) [dbade8]

```

struct KDTree {
    struct Node {
        int x, y, x1, y1, x2, y2, id, f; Node *L, *R;
    } tree[maxn], *root;
    lld dis2(int x1, int y1, int x2, int y2) {
        lld dx = x1 - x2, dy = y1 - y2;
        return dx * dx + dy * dy;
    }
    static bool cmpx(Node& a, Node& b){return a.x<b.x;}
    static bool cmpy(Node& a, Node& b){return a.y<b.y;}
    void init(vector<pair<int,int>> &ip) {
        const int n = ip.size();
        for (int i = 0; i < n; i++) {
            tree[i].id = i;
            tree[i].x = ip[i].first;
            tree[i].y = ip[i].second;
        }
        root = build(0, n-1, 0);
    }
    Node* build(int L, int R, int d) {
        if (L>R) return nullptr; int M = (L+R)/2;
        nth_element(tree+L, tree+M, tree+R+1, d%2?cmpy:cmpx);
        Node &o = tree[M]; o.f = d % 2;
        o.x1 = o.x2 = o.x; o.y1 = o.y2 = o.y;
        o.L = build(L, M-1, d+1); o.R = build(M+1, R, d+1);
        for (Node *s: {o.L, o.R}) if (s) {
            o.x1 = min(o.x1, s->x1); o.x2 = max(o.x2, s->x2);
            o.y1 = min(o.y1, s->y1); o.y2 = max(o.y2, s->y2);
        }
        return tree+M;
    }
    bool touch(int x, int y, lld d2, Node *r){
        lld d = sqrt(d2)+1;
        return x >= r->x1 - d && x <= r->x2 + d &&
            y >= r->y1 - d && y <= r->y2 + d;
    }
    using P = pair<lld, int>;
    void dfs(int x, int y, P &mn, Node *r) {
        if (!r || !touch(x, y, mn.first, r)) return;
        mn = min(mn, P(dis2(r->x, r->y, x, y), r->id));
        if (r->f == 1 ? y < r->y : x < r->x)
            dfs(x, y, mn, r->L), dfs(x, y, mn, r->R);
        else
            dfs(x, y, mn, r->R), dfs(x, y, mn, r->L);
    }
    int query(int x, int y) {
        P mn(INF, -1); dfs(x, y, mn, root);
        return mn.second;
    }
} tree;

```

6.29 kd Closest Pair (3D ver.) [84d9eb]

```

llf solve(vector<P> v) {
    shuffle(v.begin(), v.end(), mt19937());
    unordered_map<lld, unordered_map<lld,
        unordered_map<lld, int>>> m;
    llf d = dis(v[0], v[1]);
    auto Idx = [&d] (llf x) -> lld {
        return round(x * 2 / d) + 0.1; };
    auto rebuild_m = [&m, &v, &Idx](int k) {
        m.clear();
        for (int i = 0; i < k; ++i)
            m[Idx(v[i].x)][Idx(v[i].y)]
                [Idx(v[i].z)] = i;
    }; rebuild_m(2);
    for (size_t i = 2; i < v.size(); ++i) {
        const lld kx = Idx(v[i].x), ky = Idx(v[i].y),
            kz = Idx(v[i].z); bool found = false;
        for (int dx = -2; dx <= 2; ++dx) {
            const lld nx = dx + kx;

```



```

if (m.find(nx) == m.end()) continue;
auto& mm = m[nx];
for (int dy = -2; dy <= 2; ++dy) {
    const lld ny = dy + ky;
    if (mm.find(ny) == mm.end()) continue;
    auto& mmm = mm[ny];
    for (int dz = -2; dz <= 2; ++dz) {
        const lld nz = dz + kz;
        if (mmm.find(nz) == mmm.end()) continue;
        const int p = mmm[nz];
        if (dis(v[p], v[i]) < d) {
            d = dis(v[p], v[i]);
            found = true;
        }
    }
}
if (found) rebuild_m(i + 1);
else m[kx][ky][kz] = i;
}
return d;
}

```

6.30 Simulated Annealing [4e0fe5]

```

llf anneal() {
    mt19937 rnd_engine(seed);
    uniform_real_distribution<llf> rnd(0, 1);
    const llf dT = 0.001;
    // Argument p
    llf S_cur = calc(p), S_best = S_cur;
    for (llf T = 2000; T > EPS; T -= dT) {
        // Modify p to p_prime
        const llf S_prime = calc(p_prime);
        const llf delta_c = S_prime - S_cur;
        llf prob = min((llf)1, exp(-delta_c / T));
        if (rnd(rnd_engine) <= prob)
            S_cur = S_prime, p = p_prime;
        if (S_prime < S_best) // find min
            S_best = S_prime, p_best = p_prime;
    }
    return S_best;
}

```

6.31 Triangle Centers [adb146]

```

O = ... // see min circle cover
G = (A + B + C) / 3;
H = G * 3 - O * 2; // orthogonal center
llf a = abs(B - C), b = abs(A - C), c = abs(A - B);
I = (a * A + b * B + c * C) / (a + b + c);
// FermatPoint: minimizes sum of distance
// if max. angle >= 120 deg then vertex
// otherwise, make eq. triangle AB'C, CA'B, BC'A
// line AA', BB', CC' intersects at P

```

7 Stringology

7.1 Hash [ce7fad]

```

template <int P = 127, int Q = 1051762951>
class Hash {
    vector<int> h, p;
public:
    Hash(const auto &s) : h(s.size()+1), p(s.size()+1) {
        for (size_t i = 0; i < s.size(); ++i)
            h[i + 1] = add(mul(h[i], P), s[i]);
        generate(all(p), [x = 1, y = 1, this]() mutable {
            return y = x, x = mul(x, P), y; });
    }
    int query(int l, int r) const { // 1-base (l, r]
        return sub(h[r], mul(h[l], p[r - l]));
    }
};

```

7.2 Suffix Array [e9e77d]

```

auto sais(const auto &s) {
    const int n = (int)s.size(), z = ranges::max(s) + 1;
    vector<int> c(z); for (int x : s) ++c[x];
    partial_sum(all(c), begin(c));
    vector<int> sa(n); auto I = ranges::iota_view(0, n);
    if (ranges::max(c) <= 1) {
        for (int i : I) sa[--c[s[i]]] = i;
        return sa;
    }
    vector<bool> t(n); t[n - 1] = true;

```

```

for (int i = n - 2; i >= 0; --i)
    t[i] = (s[i]==s[i + 1] ? t[i + 1] : s[i]<s[i + 1]);
auto is_lms = ranges::views::filter([&t](int x) {
    return x && t[x] && !t[x - 1]; });
const auto induce = [&] {
    for (auto x = c; int y : sa)
        if (y-- if (!t[y]) sa[x[s[y] - 1]++] = y;
    for (auto x = c; int y : sa | views::reverse)
        if (y-- if (t[y]) sa[--x[s[y]]] = y;
};
vector<int> lms, q(n); lms.reserve(n);
for (auto x = c; int i : I | is_lms) {
    q[i] = int(lms.size());
    lms.push_back(sa[--x[s[i]]] = i);
}
induce(); vector<int> ns(lms.size());
for (int j = -1, nz = 0; int i : sa | is_lms) {
    if (j >= 0) {
        int len = min({n - i, n - j, lms[q[i] + 1] - i});
        ns[q[i]] = nz += lexicographical_compare(
            begin(s) + j, begin(s) + j + len,
            begin(s) + i, begin(s) + i + len);
    }
    j = i;
}
ranges::fill(sa, 0); auto nsa = sais(ns);
for (auto x = c; int y : nsa | views::reverse)
    y = lms[y], sa[--x[s[y]]] = y;
return induce(), sa;
}
// sa[i]: sa[i]-th suffix is the
// i-th lexicographically smallest suffix.
// hi[i]: LCP of suffix sa[i] and suffix sa[i - 1].
struct Suffix {
    int n; vector<int> sa, hi, rev;
    Suffix(const auto &s) : n(int(s.size())),
        hi(n), rev(n) {
        vector<int> _s(n + 1); _s[n] = 0;
        copy(all(s), begin(_s)); // s shouldn't contain 0
        sa = sais(_s); sa.erase(sa.begin());
        for (int i = 0; i < n; ++i) rev[sa[i]] = i;
        for (int i = 0, h = 0; i < n; ++i) {
            if (!rev[i]) { h = 0; continue; }
            for (int j = sa[rev[i] - 1]; i + h < n && j + h < n
                && s[i + h] == s[j + h];) ++h;
            hi[rev[i]] = h ? h-- : 0;
        }
    }
};

```

7.3 Ex SAM [58374b]

```

struct exSAM {
    int len[maxn * 2], link[maxn * 2]; // maxlen, suflink
    int next[maxn * 2][maxc], tot; // [0, tot), root = 0
    int ord[maxn * 2]; // topo. order (sort by length)
    int cnt[maxn * 2]; // occurence
    int newnode() {
        fill_n(next[tot], maxc, 0);
        return len[tot] = cnt[tot] = link[tot] = 0, tot++;
    }
    void init() { tot = 0, newnode(), link[0] = -1; }
    int insertSAM(int last, int c) {
        int cur = next[last][c];
        len[cur] = len[last] + 1;
        int p = link[last];
        while (p != -1 && next[p][c])
            next[p][c] = cur, p = link[p];
        if (p == -1) return link[cur] = 0, cur;
        int q = next[p][c];
        if (len[p] + 1 == len[q]) return link[cur] = q, cur;
        int clone = newnode();
        for (int i = 0; i < maxc; ++i)
            next[clone][i] = len[next[q][i]] ? next[q][i] : 0;
        len[clone] = len[p] + 1;
        while (p != -1 && next[p][c] == q)
            next[p][c] = clone, p = link[p];
        link[link[cur] = clone] = link[q];
        link[q] = clone;
        return cur;
    }
    void insert(const string &s) {
        int cur = 0;

```

```

for (char ch : s) {
    int &nxt = next[cur][int(ch - 'a')];
    if (!nxt) nxt = newnode();
    cnt[cur = nxt] += 1;
}
}
void build() {
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int i = 0; i < maxc; ++i)
            if (next[cur][i]) q.push(insertSAM(cur, i));
    }
    vector<int> lc(tot);
    for (int i = 1; i < tot; ++i) ++lc[len[i]];
    partial_sum(all(lc), lc.begin());
    for (int i = 1; i < tot; ++i) ord[--lc[len[i]]] = i;
}
void solve() {
    for (int i = tot - 2; i >= 0; --i)
        cnt[link[ord[i]]] += cnt[ord[i]];
}
}

```

7.4 KMP [281185]

```

vector<int> kmp(const auto &s) {
    vector<int> f(s.size());
    for (int i = 1, k = 0; i < (int)s.size(); ++i) {
        while (k > 0 && s[i] != s[k]) k = f[k - 1];
        f[i] = (k += (s[i] == s[k]));
    }
    return f;
}
vector<int> search(const auto &s, const auto &t) {
    // return 0-indexed occurrence of t in s
    vector<int> f = kmp(t), r;
    for (int i = 0, k = 0; i < (int)s.size(); ++i) {
        while (k > 0 && s[i] != t[k]) k = f[k - 1];
        k += (s[i] == t[k]);
        if (k == (int)t.size()) {
            r.push_back(i - t.size() + 1);
            k = f[k - 1];
        }
    }
    return r;
}

```

7.5 Z value [6a7fd0]

```

vector<int> Zalgo(const string &s) {
    vector<int> z(s.size(), s.size());
    for (int i = 1, l = 0, r = 0; i < z[0]; ++i) {
        int j = clamp(r - i, 0, z[i - l]);
        for (; i + j < z[0] && s[i + j] == s[j]; ++j);
        if (i + (z[i] = j) > r) r = i + z[l = i];
    }
    return z;
}

```

7.6 Manacher [c938a9]

```

vector<int> manacher(const string &S) {
    const int n = (int)S.size(), m = n * 2 + 1;
    vector<int> z(m);
    string t = "."; for (char c: S) t += c, t += '.';
    for (int i = 1, l = 0, r = 0; i < m; ++i) {
        z[i] = (r > i ? min(z[2 * l - i], r - i) : 1);
        while (i - z[i] >= 0 && i + z[i] < m) {
            if (t[i - z[i]] == t[i + z[i]]) ++z[i];
            else break;
        }
        if (i + z[i] > r) r = i + z[i], l = i;
    }
    return z; // the palindrome lengths are z[i] - 1
}
/* for (int i = 1; i + 1 < m; ++i) {
    int l = (i - z[i] + 2) / 2, r = (i + z[i]) / 2;
    if (l != r) // [l, r) is maximal palindrome
} */

```

7.7 Lyndon Factorization [d22cc9]

```

// partition s = w[0] + w[1] + ... + w[k-1],
// w[0] >= w[1] >= ... >= w[k-1]
// each w[i] strictly smaller than all its suffix

```

```

void duval(const auto &s, auto &&report) {
    for (int n = (int)s.size(), i = 0, j, k; i < n; ) {
        for (j = i + 1, k = i; j < n && s[k] <= s[j]; j++)
            k = (s[k] < s[j] ? i : k + 1);
        // if (i < n / 2 && j >= n / 2) {
        // for min cyclic shift, call duval(s + s)
        // then here s.substr(i, n / 2) is min cyclic shift
        // }
        for (; i <= k; i += j - k)
            report(i, j - k); // s.substr(l, len)
    }
} // tested @ luogu 6114, 1368 & UVA 719

```

7.8 Main Lorentz [615b8f]

```

vector<pair<int, int>> rep[kN]; // 0-base [l, r]
void main_lorentz(const string &s, int sft = 0) {
    const int n = s.size();
    if (n == 1) return;
    const int nu = n / 2, nv = n - nu;
    const string u = s.substr(0, nu), v = s.substr(nu);
    ru(u.rbegin(), u.rend(), rv(v.rbegin(), v.rend()));
    main_lorentz(u, sft), main_lorentz(v, sft + nu);
    const auto z1 = Zalgo(ru), z2 = Zalgo(v + '#' + u),
        z3 = Zalgo(ru + '#' + rv), z4 = Zalgo(v);
    auto get_z = [](const vector<int> &z, int i) {
        return (0 <= i && i < (int)z.size()) ? z[i] : 0;
    };
    auto add_rep = [&](bool left, int c, int l, int k1,
        int k2) {
        const int L = max(1, l - k2), R = min(l - left, k1);
        if (L > R) return;
        if (left) rep[l].emplace_back(sft + c - R, sft + c - L);
        else rep[l].emplace_back(sft + c - R - l + 1, sft + c - L - l + 1);
    };
    for (int cntr = 0; cntr < n; cntr++) {
        int l, k1, k2;
        if (cntr < nu) {
            l = nu - cntr;
            k1 = get_z(z1, nu - cntr);
            k2 = get_z(z2, nv + 1 + cntr);
        } else {
            l = cntr - nu + 1;
            k1 = get_z(z3, nu + 1 + nv - 1 - (cntr - nu));
            k2 = get_z(z4, (cntr - nu) + 1);
        }
        if (k1 + k2 >= l)
            add_rep(cntr < nu, cntr, l, k1, k2);
    }
}

```

7.9 BWT [5a9b3a]

```

vector<int> v[SIGMA];
void BWT(char *ori, char *res) {
    // make ori -> ori + ori
    // then build suffix array
}
void iBWT(char *ori, char *res) {
    for (int i = 0; i < SIGMA; ++i) v[i].clear();
    const int len = strlen(ori);
    for (int i = 0; i < len; ++i)
        v[ori[i] - 'a'].push_back(i);
    vector<int> a;
    for (int i = 0, ptr = 0; i < SIGMA; ++i)
        for (int j: v[i]) {
            a.push_back(j);
            ori[ptr++] = 'a' + i;
        }
    for (int i = 0, ptr = 0; i < len; ++i) {
        res[i] = ori[a[ptr]];
        ptr = a[ptr];
    }
    res[len] = 0;
}

```

7.10 Palindromic Tree [0673ee]

```

struct PalindromicTree {
    struct node {
        int nxt[26], f, len; // num = depth of fail link
        int cnt, num; // = #pal_suffix of this node
        node(int l = 0) : nxt{}, f(0), len(l), cnt(0), num(0) {}
    };
};

```

```

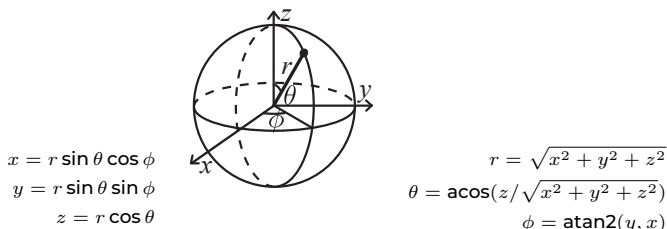
};
vector<node> st; vector<char> s; int last, n;
void init() {
    st.clear(); s.clear();
    last = 1; n = 0;
    st.push_back(0); st.push_back(-1);
    st[0].f = 1; s.push_back(-1);
}
int getFail(int x) {
    while (s[n - st[x].len - 1] != s[n]) x = st[x].f;
    return x;
}
void add(int c) {
    s.push_back(c == 'a'); ++n;
    int cur = getFail(last);
    if (!st[cur].nxt[c]) {
        int now = st.size();
        st.push_back(st[cur].len + 2);
        st[now].f = st[getFail(st[cur].f)].nxt[c];
        st[cur].nxt[c] = now;
        st[now].num = st[st[now].f].num + 1;
    }
    last = st[cur].nxt[c]; ++st[last].cnt;
}
void dpCnt() { // cnt = #occurrence in whole str
    for (int i = st.size() - 1; i >= 0; i--)
        st[st[i].f].cnt += st[i].cnt;
}
int size() { return st.size() - 2; }
} pt;
/* usage
string s; cin >> s; pt.init();
for (int i = 0; i < SZ(s); i++) {
    int prvsz = pt.size(); pt.add(s[i]);
    if (prvsz != pt.size()) {
        int r = i, l = r - pt.st[pt.last].len + 1;
        // pal @ [l,r]: s.substr(l, r-l+1)
    }
}
*/

```

8 Misc

8.1 Theorems

Spherical Coordinate



Sherman-Morrison formula

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}$$

Kirchhoff's Theorem

Denote L be a $n \times n$ matrix as the Laplacian matrix of graph G , where $L_{ii} = d(i)$, $L_{ij} = -c$ where c is the number of edge (i, j) in G .

- The number of undirected spanning in G is $\det(\tilde{L}_{11})$.
- The number of directed spanning tree rooted at r in G is $\det(\tilde{L}_{rr})$.

Random Walk on Graph

Let P be the transition matrix of a strongly connected directed graph, $\sum_j P_{ij} = 1$. Let $F_{i,j}$ be the expected time to reach j from i . Let g_i be the expected time from i to i , $G = \text{diag}(g)$ and J be a matrix all of 1, i.e. $J_{i,j} = 1$. Then, $F = J - G + PF$.

First solve G : let $\pi P = \pi$ be a stationary distribution. Then $\pi_i g_i = 1$. The rank of $I - P$ is $n - 1$, so we first solve a special solution X such that $(I - P)X = J - G$ and adjust X to F by $F_{i,j} = X_{i,j} - X_{j,j}$.

Tutte's Matrix

Let D be a $n \times n$ matrix, where $d_{ij} = x_{ij}$ (x_{ij} is chosen uniform randomly) if $i < j$ and $(i, j) \in E$, otherwise $d_{ij} = -d_{ji}$. $\frac{\text{rank}(D)}{2}$ is the maximum matching on G .

Cayley's Formula

- Given a degree sequence d_1, d_2, \dots, d_n for each labeled vertices, there're $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\dots(d_n-1)!}$ spanning trees.
- Let $T_{n,k}$ be the number of labeled forests on n vertices with k components, such that vertex $1, 2, \dots, k$ belong to different components. Then $T_{n,k} = kn^{n-k-1}$.

Erdős-Gallai theorem

A sequence of non-negative integers $d_1 \geq d_2 \geq \dots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if $d_1 + d_2 + \dots + d_n$ is even and

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k)$$

holds for all $1 \leq k \leq n$.

Havel-Hakimi algorithm

find the vertex who has greatest degree unused, connect it with other greatest vertex.

Gale-Ryser theorem

A pair of sequences of nonnegative integers $a_1 \geq \dots \geq a_n$ and b_1, \dots, b_n is bigraphic if and only if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k \min(b_i, k)$ holds for every $1 \leq k \leq n$.

Fulkerson-Chen-Anstee theorem

A sequence $(a_1, b_1), \dots, (a_n, b_n)$ of nonnegative integer pairs with $a_1 \geq \dots \geq a_n$ is digraphic if and only if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k \min(b_i, k - 1) + \sum_{i=k+1}^n \min(b_i, k)$ holds for every $1 \leq k \leq n$.

Euler's planar graph formula

$$V - E + F = C + 1, E \leq 3V - 6 \text{ (when } V \geq 3)$$

Pick's theorem

For simple polygon, when points are all integer, we have $A = \# \{\text{lattice points in the interior}\} + \frac{\# \{\text{lattice points on the boundary}\}}{2} - 1$

Matroid Intersection

Given matroids $M_1 = (G, I_1)$, $M_2 = (G, I_2)$, find maximum $S \in I_1 \cap I_2$. For each iteration, build the directed graph and find a shortest path from s to t .

- $s \rightarrow x : S \sqcup \{x\} \in I_1$
- $x \rightarrow t : S \sqcup \{x\} \in I_2$
- $y \rightarrow x : S \setminus \{y\} \sqcup \{x\} \in I_1$ (y is in the unique circuit of $S \sqcup \{x\}$)
- $x \rightarrow y : S \setminus \{y\} \sqcup \{x\} \in I_2$ (y is in the unique circuit of $S \sqcup \{x\}$)

Alternate the path, and $|S|$ will increase by 1. Let $R = \min(\text{rank}(I_1), \text{rank}(I_2)), N = |G|$. In each iteration, $|E| = O(RN)$. For weighted case, assign weight $-w(x)$ and $w(x)$ to $x \in S$ and $x \notin S$, resp. Use Bellman-Ford to find the weighted shortest path. The maximum iteration of Bellman-Ford is $2R + 1$.

Dual of LP

Primal	Dual
Maximize $c^T x$ s.t. $Ax \leq b, x \geq 0$	Minimize $b^T y$ s.t. $A^T y \geq c, y \geq 0$
Maximize $c^T x$ s.t. $Ax \leq b$	Minimize $b^T y$ s.t. $A^T y = c, y \geq 0$
Maximize $c^T x$ s.t. $Ax = b, x \geq 0$	Minimize $b^T y$ s.t. $A^T y = c$

Parallel Axis Theorem

The second moment of area $I_z = \int \int x^2 + y^2 dA$ $I_{z'} = I_z + Ad^2$ where d is the distance between two parallel axis z, z' .

8.2 Weight Matroid Intersection [d00ee8]

```

struct Matroid {
    Matroid(bitset<N>); // init from an independent set
    bool can_add(int); // check if break independence
    Matroid remove(int); // removing from the set
};
auto matroid_intersection(const vector<int> &w) {
    const int n = (int)w.size(); bitset<N> S;
    for (int sz = 1; sz <= n; sz++) {
        Matroid M1(S), M2(S); vector<vector<pii>> e(n + 2);
        for (int j = 0; j < n; j++) if (!S[j]) {
            if (M1.can_add(j)) e[n].eb(j, -w[j]);
            if (M2.can_add(j)) e[j].eb(n + 1, 0);
        }
        for (int i = 0; i < n; i++) if (S[i]) {
            Matroid T1 = M1.remove(i), T2 = M2.remove(i);
            for (int j = 0; j < n; j++) if (!S[j]) {
                if (T1.can_add(j)) e[i].eb(j, -w[j]);
                if (T2.can_add(j)) e[j].eb(i, w[i]);
            }
        }
        // maybe implicit build graph for more speed
        vector<pii> d(n + 2, {INF, 0}); d[n] = {0, 0};
        vector<int> prv(n + 2, -1);
        // change to SPFA for more speed, if necessary
        for (int upd = 1; upd--;)
            for (int u = 0; u < n + 2; u++)
                for (auto [v, c] : e[u]) {
                    pii x(d[u].first + c, d[u].second + 1);

```

```

    if (x < d[v]) d[v] = x, prv[v] = u, upd = 1;
}
if (d[n + 1].first >= INF) break;
for (int x = prv[n+1]; x!=n; x = prv[x]) S.flip(x);
// S is the max-weighted independent set w/ size sz
}
return S;
} // from Nac1

```

8.3 Stable Marriage

```

1: Initialize  $m \in M$  and  $w \in W$  to free
2: while  $\exists$  free man  $m$  who has a woman  $w$  to propose to do
3:    $w \leftarrow$  first woman on  $m$ 's list to whom  $m$  has not yet proposed
4:   if  $\exists$  some pair  $(m', w)$  then
5:     if  $w$  prefers  $m$  to  $m'$  then
6:        $m' \leftarrow$  free
7:        $(m, w) \leftarrow$  engaged
8:     end if
9:   else
10:     $(m, w) \leftarrow$  engaged
11:   end if
12: end while

```

8.4 Bitset LCS [4155ab]

```

cin >> n >> m;
for (int i = 1, x; i <= n; ++i)
  cin >> x, p[x].set(i);
for (int i = 1, x; i <= m; ++i) {
  cin >> x, (g = f) |= p[x];
  f.shiftLeftByOne(), f.set(0);
  ((f = g - f) ^= g) &= g;
}
cout << f.count() << '\n';

```

8.5 Prefix Substring LCS [7d8faf]

```

void all_lcs(string S, string T) { // 0-base
  vector<size_t> h(T.size()); iota(all(h), 1);
  for (size_t a = 0; a < S.size(); ++a) {
    for (size_t c = 0, v = 0; c < T.size(); ++c)
      if (S[a] == T[c] || h[c] < v) swap(h[c], v);
    // here, LCS(s[0, a], t[b, c]) =
    // c - b + 1 - sum([h[i] > b] | i <= c)
  }
} // test @ yosupo judge

```

8.6 Convex 1D/1D DP [e5ab4b]

```

struct S { int i, l, r; };
auto solve(int n, int k, auto &w) {
  vector<int64_t> dp(n + 1);
  auto f = [&](int l, int r) -> int64_t {
    if (r - l > k) return -INF;
    return dp[l] + w(l + 1, r);
  };
  dp[0] = 0;
  deque<S> dq; dq.emplace_back(0, 1, n);
  for (int i = 1; i <= n; ++i) {
    dp[i] = f(dq.front().i, i);
    while (!dq.empty() && dq.front().r <= i)
      dq.pop_front();
    dq.front().l = i + 1;
    while (!dq.empty() &&
      f(i, dq.back().l) >= f(dq.back().i, dq.back().l))
      dq.pop_back();
    int p = i + 1;
    if (!dq.empty()) {
      auto [j, l, r] = dq.back();
      for (int s = 1 << 20; s; s >>= 1)
        if (l + s <= n && f(i, l + s) < f(j, l + s))
          l += s;
      dq.back().r = l; p = l + 1;
    }
    if (p <= n) dq.emplace_back(i, p, n);
  }
  return dp;
} // test @ tioj 烏龜疊疊樂

```

8.7 ConvexHull Optimization [b4318e]

```

struct L {
  mutable lld a, b, p;
  bool operator<(const L &r) const {
    return a < r.a; /* here */
  }
  bool operator<(lld x) const { return p < x; }
};
lld Div(lld a, lld b) {
  return a / b - ((a ^ b) < 0 && a % b);
}

```

```

struct DynamicHull : multiset<L, less<>> {
  static const lld kInf = 1e18;
  bool Isect(iterator x, iterator y) {
    if (y == end()) { x->p = kInf; return false; }
    if (x->a == y->a)
      x->p = x->b > y->b ? kInf : -kInf; /* here */
    else x->p = Div(y->b - x->b, x->a - y->a);
    return x->p >= y->p;
  }
  void Insert(lld a, lld b) {
    auto z = insert({a, b, 0}), y = z++, x = y;
    while (Isect(y, z)) z = erase(z);
    if (x!=begin() && Isect(--x, y)) Isect(x, y=erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      Isect(x, erase(y));
  }
  lld Query(lld x) { // default chmax
    auto l = *lower_bound(x); // to chmin:
    return l.a * x + l.b; // modify the 2 "<>"
  }
};

```

8.8 Min Plus Convolution [464dcd]

```

// a is convex a[i+1]-a[i] <= a[i+2]-a[i+1]
vector<int> min_plus_convolution(auto &a, auto &b) {
  const int n = (int)a.size(), m = (int)b.size();
  vector<int> c(n + m - 1, numeric_limits<int>::max());
  auto dc = [&](auto Y, int l, int r, int jl, int jr) {
    if (l > r) return;
    int mid = (l + r) / 2, from = -1, &best = c[mid];
    for (int j = jl; j <= jr; j++)
      if (int i = mid - j; i >= 0 && i < n)
        if (best > a[i] + b[j]) best = a[i] + b[j], from = j;
    Y(Y, l, mid-1, jl, from); Y(Y, mid+1, r, from, jr);
  };
  return dc(dc, 0, n-1+m-1, 0, m-1), c;
}

```

8.9 De-Bruijn [aa7700]

```

vector<int> de_bruijn(int k, int n) {
  // return cyclic string of len k^n s.t. every string
  // of len n using k char appears as a substring.
  vector<int> aux(n + 1), res;
  auto db = [&](auto self, int t, int p) -> void {
    if (t <= n)
      for (int i = aux[t - p]; i < k; ++i, p = t)
        aux[t] = i, self(self, t + 1, p);
    else if (n % p == 0) for (int i = 1; i <= p; ++i)
      res.push_back(aux[i]);
  };
  return db(db, 1, 1), res;
}

```

8.10 Josephus Problem [7f9ceb]

```

lld f(lld n, lld m, lld k) { // n people kill m for
  each turn
  lld s = (m - 1) % (n - k); // O(k)
  for (lld i = n - k + 1; i <= n; i++) s = (s + m) % i;
  return s;
}
lld kth(lld n, lld m, int128 k) { // died at kth
  if (m == 1) return k; // O(m log(n))
  for (k = k*m+m-1; k >= n; k = k-n + (k-n)/(m-1));
  return k;
} // k and result are 0-based, test @ CF 101955

```

8.11 N Queens Problem [31f83e]

```

void solve(VI &ret, int n) { // no sol when n=2,3
  if (n % 6 == 2) {
    for (int i = 2; i <= n; i += 2) ret.push_back(i);
    ret.push_back(3); ret.push_back(1);
    for (int i = 7; i <= n; i += 2) ret.push_back(i);
    ret.push_back(5);
  } else if (n % 6 == 3) {
    for (int i = 4; i <= n; i += 2) ret.push_back(i);
    ret.push_back(2);
    for (int i = 5; i <= n; i += 2) ret.push_back(i);
    ret.push_back(1); ret.push_back(3);
  } else {
    for (int i = 2; i <= n; i += 2) ret.push_back(i);
    for (int i = 1; i <= n; i += 2) ret.push_back(i);
  }
}

```


8.12 Tree Knapsack [f42766]

```
vector<int> G[N]; int dp[N][K]; pair<int,int> obj[N];
void dfs(int u, int mx) {
    for (int s : G[u]) {
        auto [w, v] = obj[s];
        if (mx < w) continue;
        for (int i = 0; i <= mx - w; i++)
            dp[s][i] = dp[u][i];
        dfs(s, mx - w);
        for (int i = w; i <= mx; i++)
            dp[u][i] = max(dp[u][i], dp[s][i - w] + v);
    }
}
```

8.13 Manhattan MST [1008bc]

```
vector<array<int, 3>> manhattanMST(vector<P> ps) {
    vector<int> id(ps.size()); iota(all(id), 0);
    vector<array<int, 3>> edges;
    for (int k = 0; k < 4; k++) {
        sort(all(id), [&](int i, int j) {
            return (ps[i] - ps[j]).x < (ps[j] - ps[i]).y; });
        map<int, int> sweep;
        for (int i : id) {
            for (auto it = sweep.lower_bound(-ps[i].y);
                 it != sweep.end(); sweep.erase(it++)) {
                if (P d = ps[i] - ps[it->second]; d.y > d.x) break;
                else edges.push_back({d.y + d.x, i, it->second});
            }
            sweep[-ps[i].y] = i;
        }
        for (P &p : ps)
            if (k & 1) p.x = -p.x;
            else swap(p.x, p.y);
    }
    return edges; // [{w, i, j}, ...]
} // test @ yosupo judge
```

8.14 Binary Search On Fraction [765c5a]

```
struct Q {
    ll p, q;
    Q go(Q b, ll d) { return {p + b.p*d, q + b.q*d}; }
};
bool pred(Q);
// returns smallest p/q in [lo, hi] such that
// pred(p/q) is true, and 0 <= p,q <= N
Q frac_bs(ll N) {
    Q lo{0, 1}, hi{1, 0};
    if (pred(lo)) return lo;
    assert(pred(hi));
    bool dir = 1, L = 1, H = 1;
    for (; L || H; dir = !dir) {
        ll len = 0, step = 1;
        for (int t = 0; t < 2 && (t ? step/=2 : step*=2);)
            if (Q mid = hi.go(lo, len + step);
                mid.p > N || mid.q > N || dir ^ pred(mid))
                t++;
            else len += step;
        swap(lo, hi = hi.go(lo, len));
        (dir ? L : H) = !!len;
    }
    return dir ? hi : lo;
}
```

8.15 Barrett Reduction [d44617]

```
struct FastMod {
    using Big = __uint128_t; ll b, m;
    FastMod(ll b) : b(b), m(-1ULL / b) {}
    ll reduce(ll a) { // a % b
        ll r = a - (ll)((Big(m) * a) >> 64) * b;
        return r >= b ? r - b : r;
    }
};
```

8.16 Montgomery [47d32c]

```
struct Mont { // Montgomery multiplication
    constexpr static int W = 64;
    ll mod, R1Mod, R2Mod, NPrime;
    void set_mod(llu _mod) {
        mod = _mod; assert(mod & 1);
        ll xinv = 1;
        for (int i = 1; i < W; i++) // Hensel lifting
            if ((xinv * mod) >> i & 1) xinv |= 1ULL << i;
```

```
        assert(xinv * mod == 1);
        const u128 R = (u128(1) << W) % mod;
        R1Mod = static_cast<llu>(R);
        R2Mod = static_cast<llu>(R * R % mod);
        NPrime = -xinv;
    }
    llu redc(llu a, ll b) const {
        auto T = static_cast<u128>(a) * b;
        u128 m = static_cast<llu>(T) * NPrime;
        T += m * mod; T >>= W;
        return static_cast<llu>(T >= mod ? T - mod : T);
    }
    llu from(llu x) const {
        assert(x < mod); return redc(x, R2Mod);
    }
    llu get(llu a) const { return redc(a, 1); }
    llu one() const { return R1Mod; }
} mont;
// a * b % mod == get(redc(from(a), from(b)))
```