Algorithm Foundations of Data Science and Engineering

Lecture 8: Community Detection

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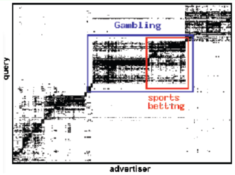
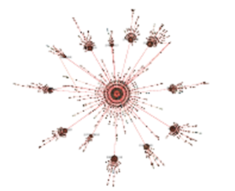
Outline

Network and communities

We often think of networks being organized into modules, clusters, communities.





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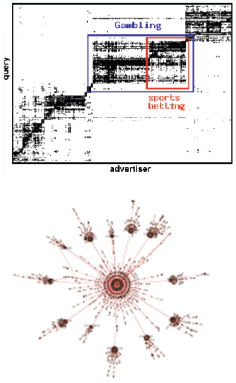

- Network visualization.

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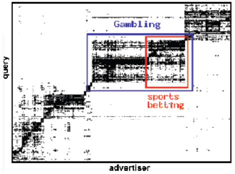

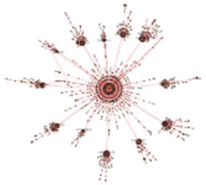

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- Find densely linked clusters.

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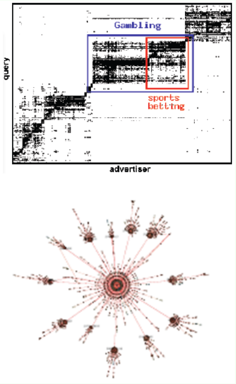

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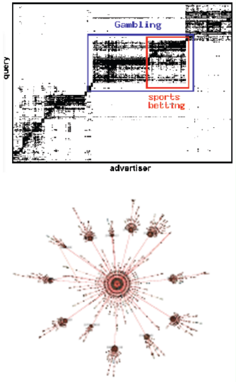
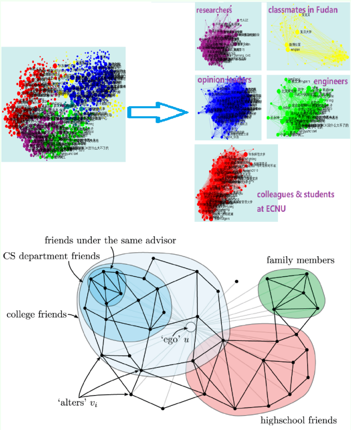


- Network visualization.
- Find densely linked clusters.
- Find micro-markets by partitioning the query VS. advertiser graph.
- Spammer detection (water army).

Network and communities Cont'd

Discovering social circles, circles of trust:



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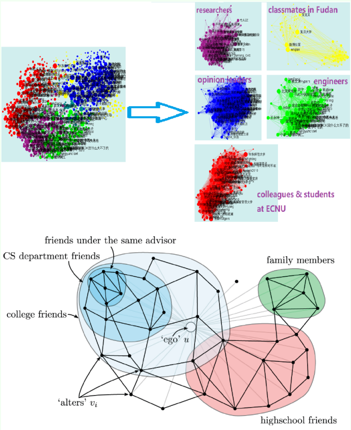

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Network and communities Cont'd

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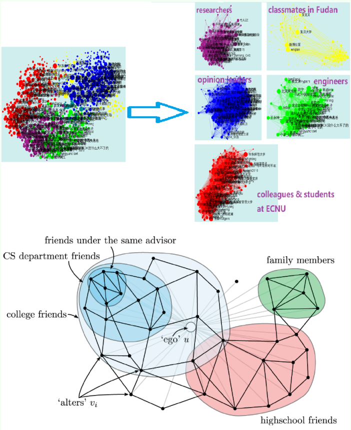

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Partition nodes of a network into disjoint sets.

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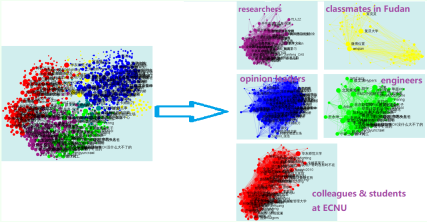

Global community structures

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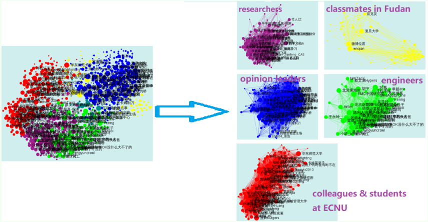

- Clustering based on vertex similarity

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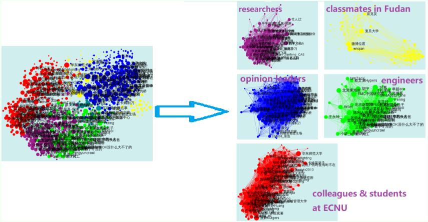

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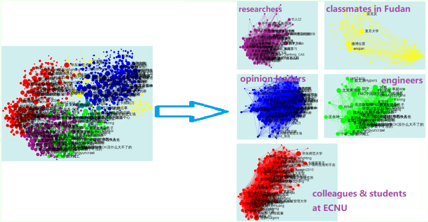

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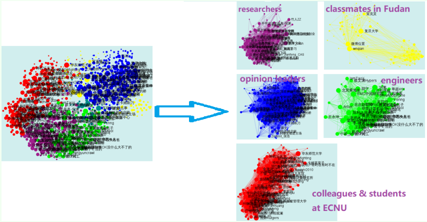

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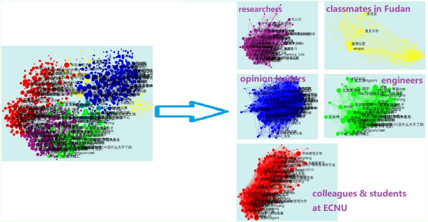

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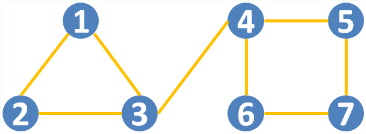
In the study of complex networks, a network is said to have community structure if the nodes of the network can be easily grouped into (potentially overlapping) sets of nodes such that each set of nodes is densely connected internally.

What makes a good community?



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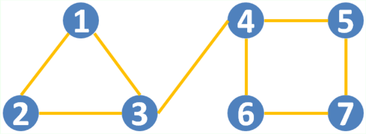

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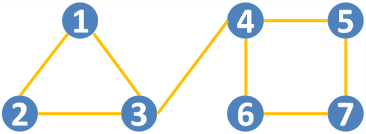

Input an undirected graph $G = (V, E)$:

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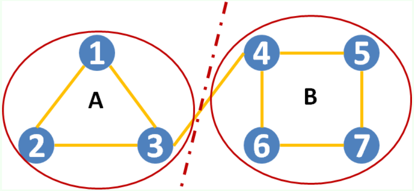
- How can we define a "good" community in G?



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Express community quality as a function of the "edge cut" of the community, where cut is the set of edges (edge weights) with only one node in the community, and can be defined as

$$cut(A) = \sum_{i \in A, j \notin A} w_{ij}.$$

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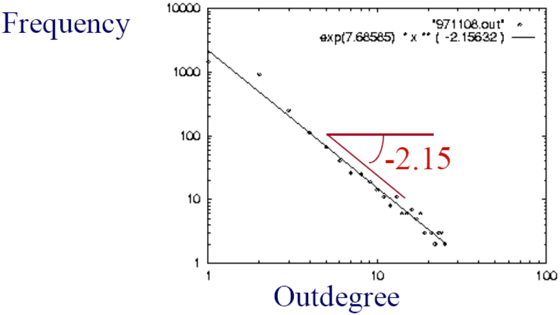
A good community makes a minimal cut. A cut is minimum if the size or weight of the cut is not larger than the size of any other cut. There are polynomial-time methods to solve the min-cut problem, notably the EdmondsCKarp algorithm, which complexity is $O(|V||E|^2)$.

Outline

Power-law I



Internet topology [SIGCOMM 99]

- Out-degree distribution is plotted in log-log scale.

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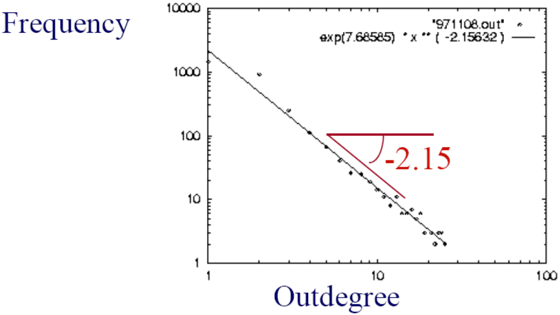

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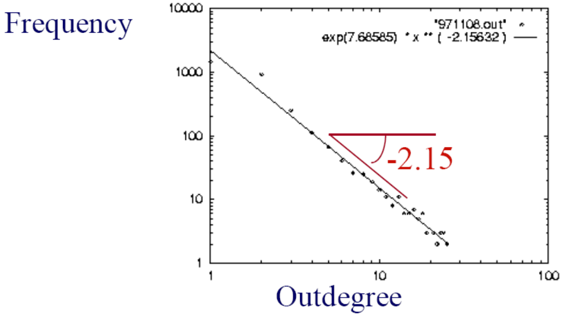

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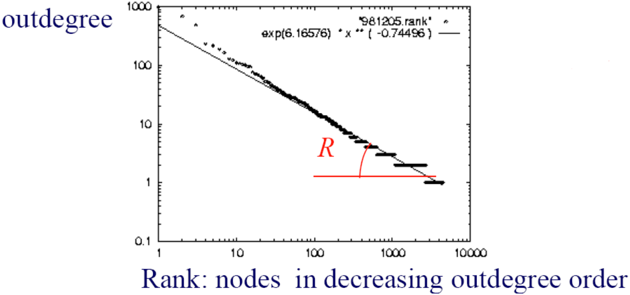
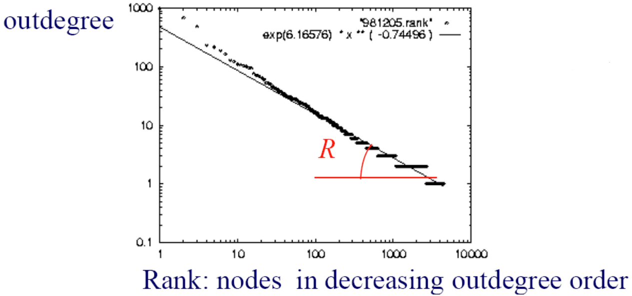
  - 80% of a company's sales are made by 20% of its sales staff

Power-law II



Rank of out-degrees [ICDE 09]

Power-law II



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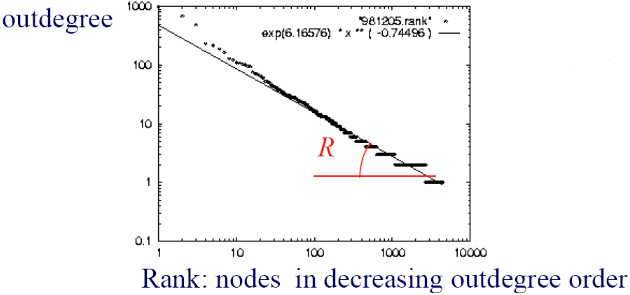

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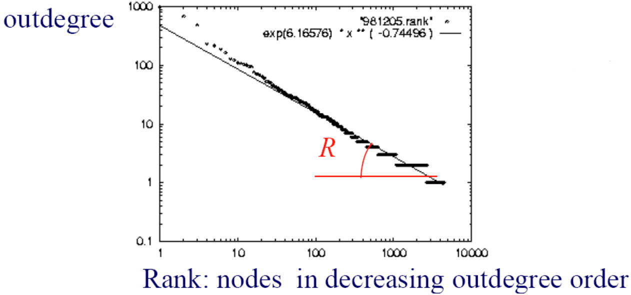

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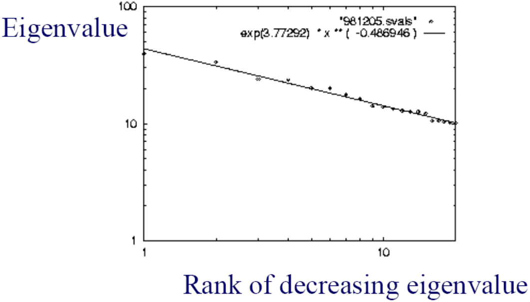
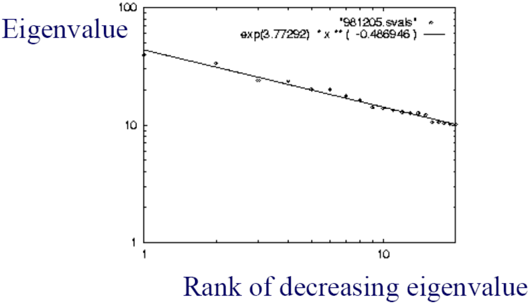


Rank of eigenvalues [ICDE 09]

Power-law III



Rank of eigenvalues [ICDE 09]

- Eigenvalues of adjacency matrix (top 20) are ranked in decreasing order, and plotted in log-log scale.

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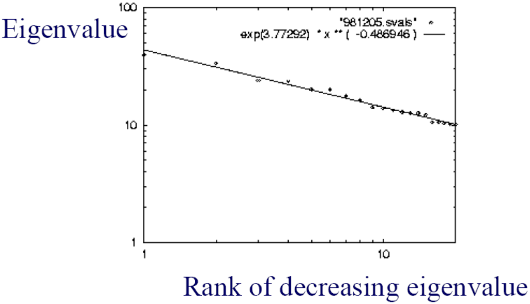

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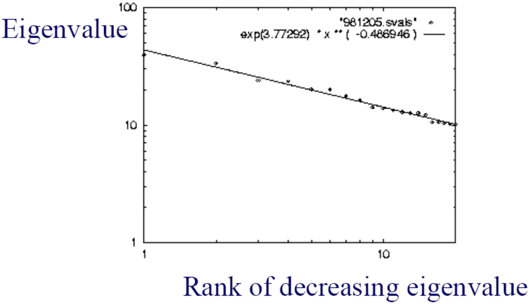

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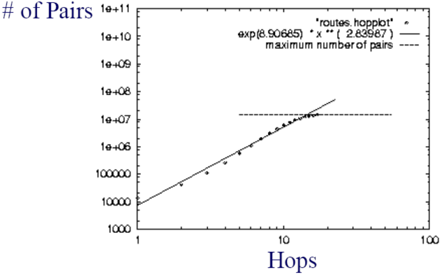
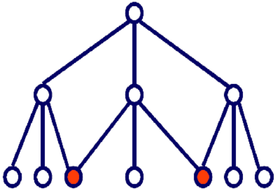
Power-law IV





Hop plot [ICDE 09]

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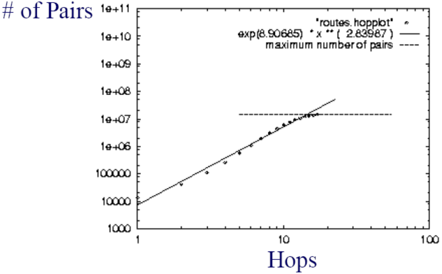

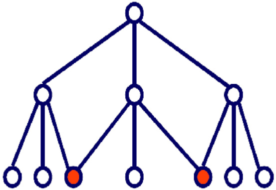

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- How many neighbors within $1, 2, \cdots, h$ hops? ($\sum_{i=1}^{h} avg.^{i}$)

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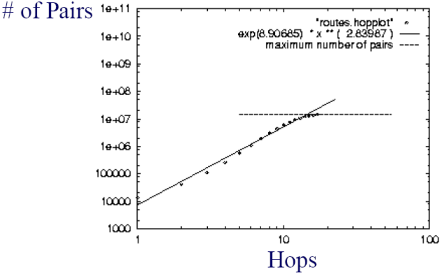

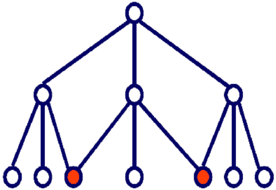

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- Pairs of vertices are plotted in log-log scale. It forms a line with a slope ~ 2.83

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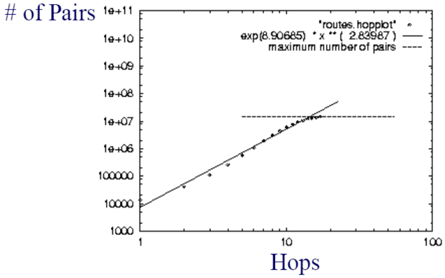

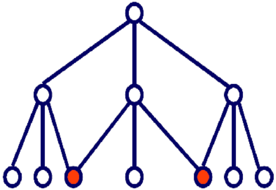

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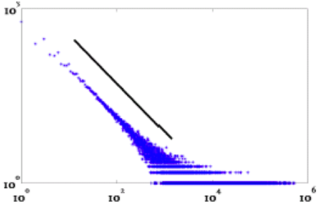
- $pairs. = hop^{2.83}$

Power-law V



Counting of triangle [ICDM 08]

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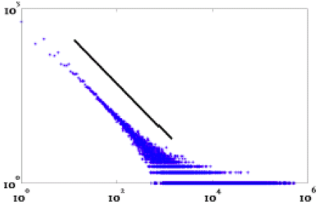

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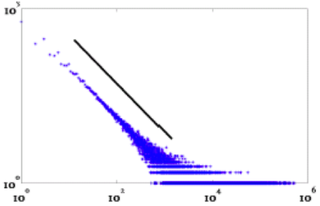
- X-axis: # of triangles a vertex participates in

Power-law V



Counting of triangle [ICDM 08]

- X-axis: # of triangles a vertex participates in
- Y-axis: count of such vertices
- In log-log scale, the plot is almost linear.

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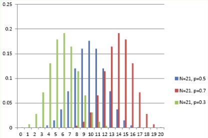

Scale-free network

Preferential attachment model

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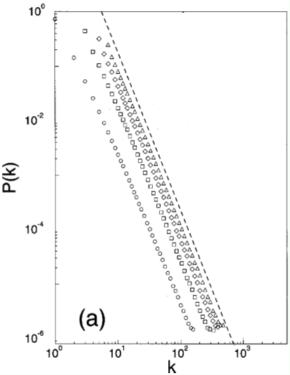
- Power law with exponent $\alpha = 2 + \frac{1}{m}$ [Science 1965]

Barabasi Albert model

Model



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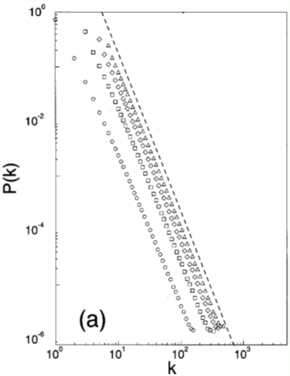

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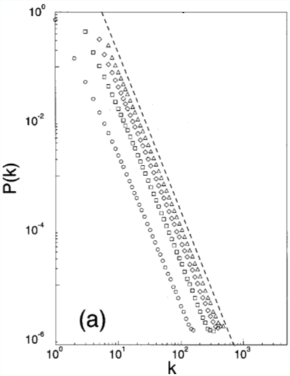
- Start with an initial network of m_0 (≥ 2) nodes, and the degree of each node ≥ 1, otherwise it will always remain isolated.

Barabasi Albert model



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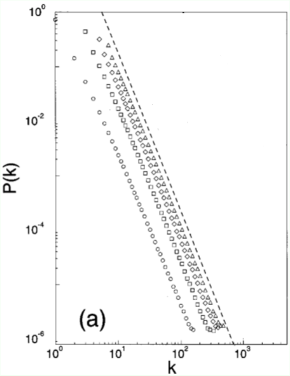

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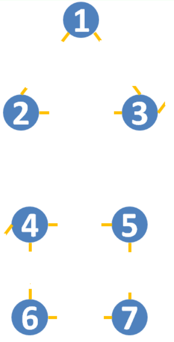
- Results in a single connected component with power-law degree distribution with $\alpha = 3$ [Reviews of Modern Physics 2003].

Null model



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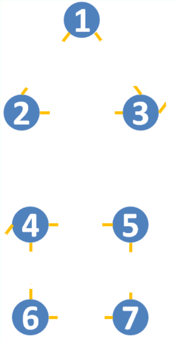

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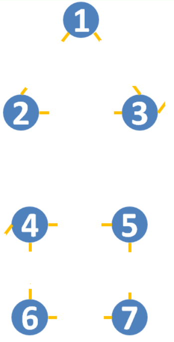

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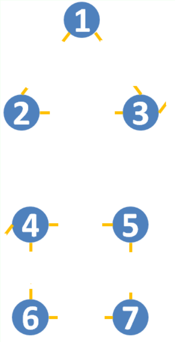

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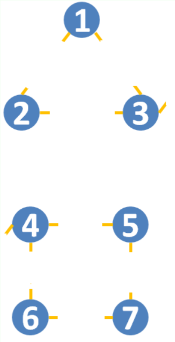

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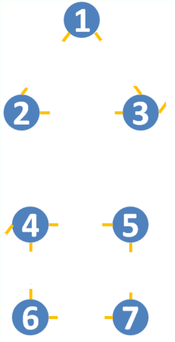

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We have:

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Null model Cont'd

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$$Z_{ij} = \begin{cases} 1, & e_{ij} \in E \\ 0, & \text{otherwise.} \end{cases}$$

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- Thus, the expectation number of edge between vertices v_i and v_j is

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Outline

Modularity

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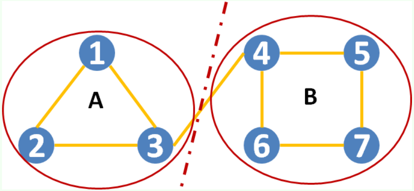
Modularity [Newman 2006]:

$$Q = \frac{1}{2m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{2m}) \delta(C_i, C_j),$$

where m and C_i denote $\#$ edges and the $i-$th community in the graph, k_i is the degree of vertex v_i, and

$$\delta(C_i, C_j) = \begin{cases} 1, & \text{if } C_i = C_j; \\ 0, & \text{otherwise.} \end{cases}$$

Example of modularity



$m = 8, k_1 = 2, k_2 = 2, k_3 = 3,$
$k_4 = 3, k_5 = 2, k_6 = 2, k_7 = 2$

Thus, the modularity of this partition is

$$Q = \frac{1}{2m}\sum_{i,j}(A_{ij} - \frac{k_i k_j}{16})\delta(C_i, C_j) = \frac{1}{16}\Big[(0 - \frac{k_1 k_1}{16}) + 2(1 - \frac{k_1 k_2}{16}) + 2(1 - \frac{k_1 k_3}{16})$$
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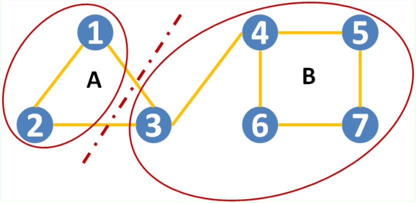
Example of modularity Cont'd



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Intuition of modularity

A measure of how well a network is partitioned into communities.

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- Given a partitioning of the network into groups $c \in C$:

$$Q \propto \sum_{i,j}(A_{ij} - \frac{k_i k_j}{2m})\delta(C_i, C_j) = \big[\sum_{i,j} A_{ij} - \sum_{i,j} \frac{k_i k_j}{2m}\big]\delta(C_i, C_j)$$

$$= \sum_{c \in C} \big[(\#\text{ edges within group } c)$$

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- Given real G on n vertices and m edges, construct a random network G';
  - Same degree distribution but random connections.
  - Consider G' as a multigraph.

Modularity: community VS. Null model

$$Q \propto \sum_{c \in C} \big[(\#\text{ edges within group } c) \\ - (\text{expected } \#\text{ edges within group } c)\big]$$

Modularity: community VS. Null model

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Modularity can be viewed as a score that computes the difference between the observed structure of the network from an expected structure under a random "null" network.

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Outline

Modularity for weighted networks

Definition

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where m and C_i denote total edge weights and the $i-$th community in the graph, w_i is the degree of vertex v_i, and

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- The modularity can be rewritten as

$$\begin{aligned} Q &= \frac{1}{4m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{2m})(s_i s_j + 1) \\ &= \frac{1}{4m} \sum_{i,j} (A_{ij} - \frac{k_i k_j}{2m}) s_i s_j = \frac{1}{4m} \mathbf{s}^T B \mathbf{s}, \end{aligned}$$

where $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$ is called modularity matrix.

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- This observation is reminiscent of the best-known methods of graph partitioning, spectral partitioning.
- We proceed by writing $\mathbf{s}$ as a linear combination of the normalized eigenvectors $\mathbf{u}_i$ of B s.t., $\mathbf{s} = \sum_{i=1}^{n} a_i \mathbf{u}_i$ with $a_i = \mathbf{u}_i^T \cdot \mathbf{s}$. Then

$$Q = \frac{1}{4m} \sum_i a_i \mathbf{u}_i^T B \sum_j a_j \mathbf{u}_j = \frac{1}{4m} \sum_i^n (\mathbf{u}_i^T \cdot \mathbf{s})^2 \beta_i,$$

where β_i is the eigenvalue of B corresponding to eigenvector $\mathbf{u}_i$.

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- If there were no other constraints on **s**, we would simply chose s proportional to the eigenvector $\mathbf{u}_1$ since the eigenvectors are orthogonal.
- Unfortunately, there is another constraint on the problem imposed by the restriction of the elements of s to the values ± 1, which means s cannot normally be chosen parallel to $\mathbf{u}_1$.

Partitioning Cont'd

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  i.e., all vertices whose corresponding elements are positive go in one group and all of the rest in the other;
- As a result, we compute the leading eigenvector of the modularity matrix and divide the vertices into two groups according to the signs of the elements in this vector.

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- In doing this it is crucial to note that it is not correct.
  - The modularity will change if edges are deleted;
  - Any subsequent maximization of modularity would thus maximize the wrong quantity.

Extension Cont'd

Instead, the correct approach is to write the additional contribution $\triangle Q$ to the modularity upon further dividing a group g of size n_g in two as

$$\begin{aligned}
Q &= \frac{1}{2m}\Big[\frac{1}{2}\sum_{i,j\in g} B_{ij}(s_i s_j + 1) - \sum_{i,j\in g} B_{ij}\Big] \\
&= \frac{1}{4m}\Big[\sum_{i,j\in g} B_{ij} s_i s_j - \sum_{i,j\in g} B_{ij}\Big] \\
&= \frac{1}{4m}\sum_{i,j\in g}\Big[B_{ij} - \delta_{ij}\sum_{k\in g} B_{ik}\Big] s_i s_j \\
&= \frac{1}{4m}\sum_{i,j\in g}\Big[B_{ij} - \delta_{ij}\sum_{k\in g} B_{ik}\Big] s_i s_j = \frac{1}{4m}\mathbf{s}^T B^{(g)}\mathbf{s}
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where $B_{ij}^{(g)} = B_{ij} - \delta_{ij}\sum_{k\in g} B_{ik}$ is the new modularity matrix.

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Greedy algorithm for community detection

- Supports directed and weighted graphs;

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Modularity rewriting

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M &= \frac{1}{2m}\sum_{i,j}(A_{ij} - \frac{k_i k_j}{2m})\delta(C_i, C_j) \\
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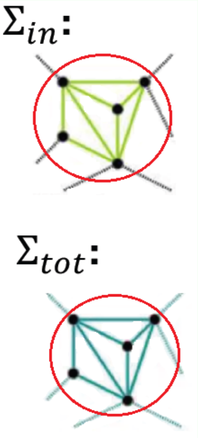
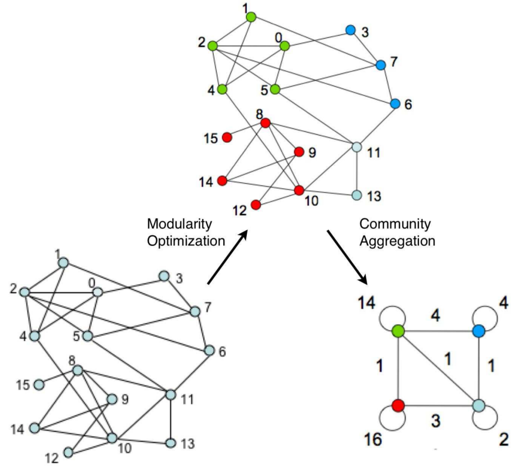


Outline

Louvain method Cont'd

Each iteration



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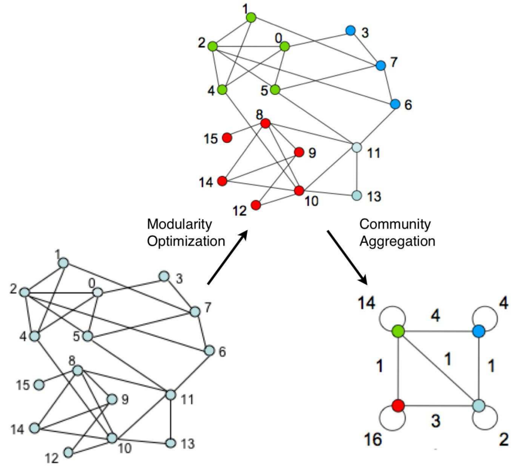

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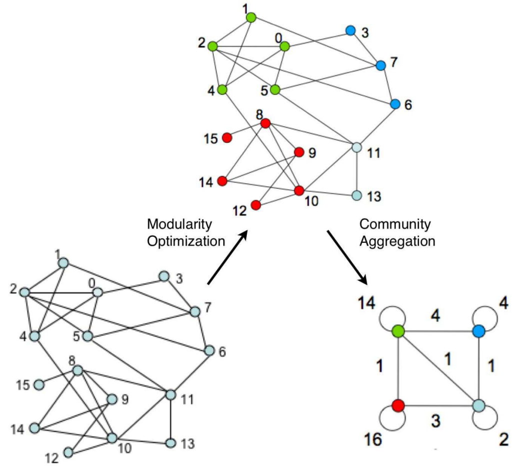

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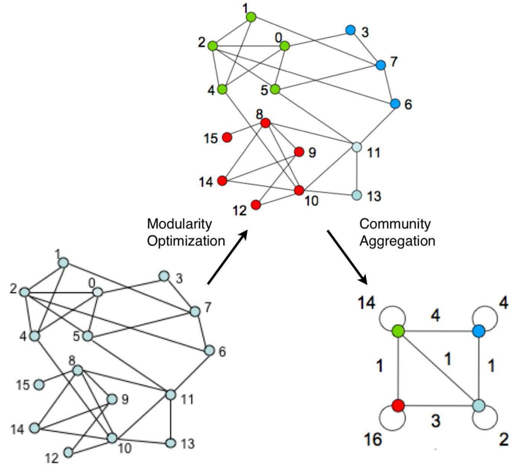

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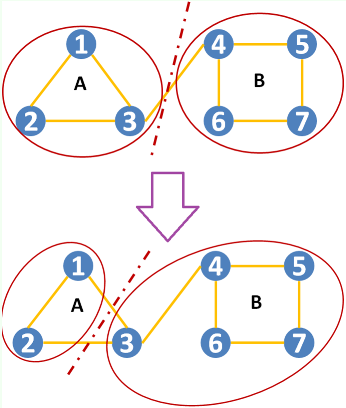

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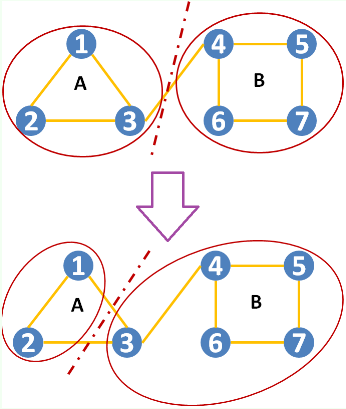

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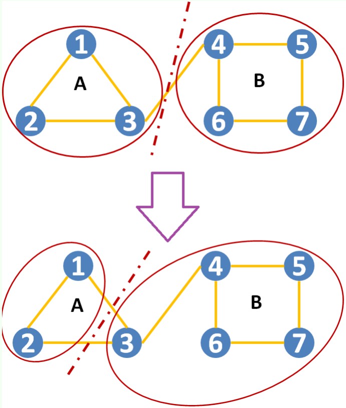

Modularity change is two parts

- Gain: $\triangle Q(v_i \to C)$ means the gain of Q if we move vertex v_i to community C;
- Loss: $\triangle Q(D \to v_i)$ means the loss of Q if we take vertex v_i out of community D.

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- k_i is the sum of all link weights (i.e., degree) of vertex v_i;

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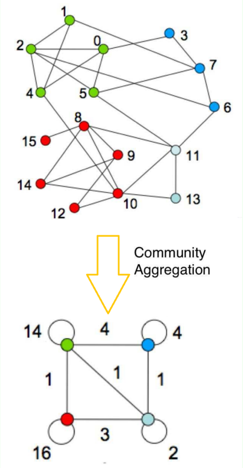
Finally, $\triangle Q = \triangle Q(v_i \rightarrow C) + \triangle Q(D \rightarrow v_i)$.

Louvain: Phase 2 (Reconstructing)



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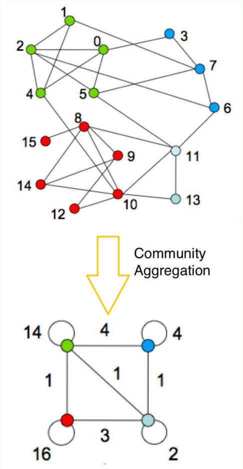

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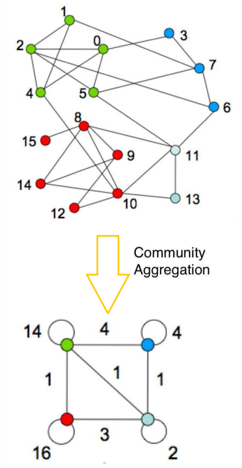

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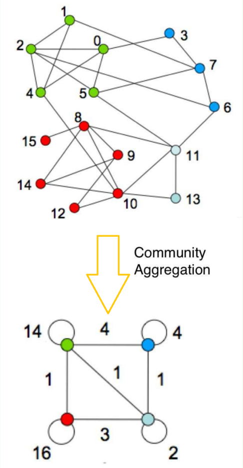

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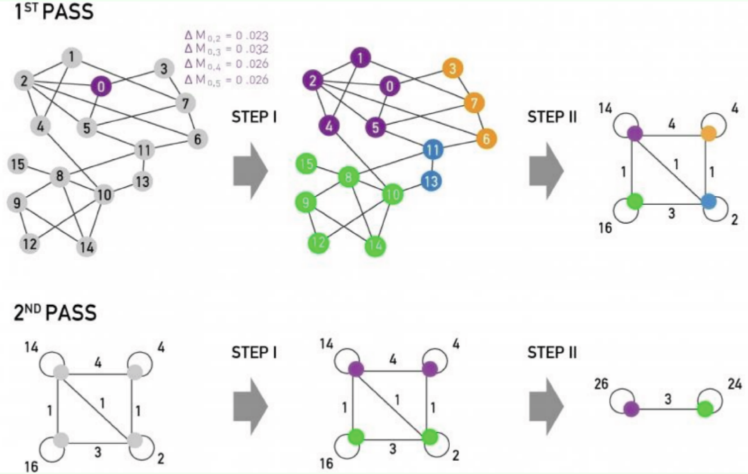
- After aggregation, the graph becomes a weighted graph.

Louvain method



Outline

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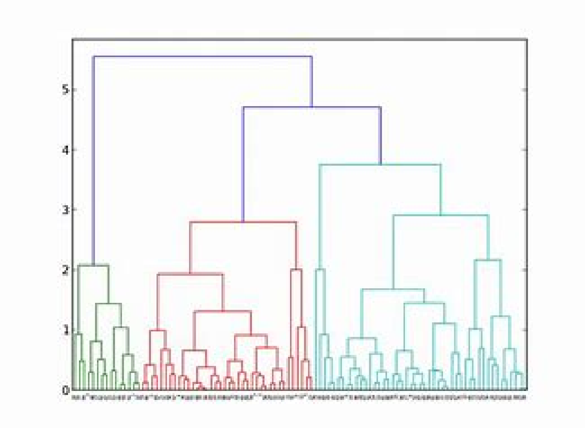
- The time complexity of this algorithm is $O(|E|)$, e.g., the time is less than 1 minutes for finding the community from a graph of 1 million vertices.
- The number of communities is not a hyper-parameter.
- It can be used to evaluate the quality of community structure;

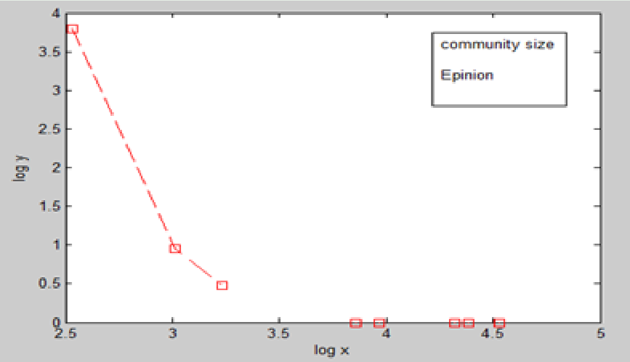
Prons of Louvain method Cont'd

Hierarchical partitions



Cons of Louvain method

The sizes of communities follow the rule of Power-law.



One of the major drawbacks of the Louvain algorithm is resolution limit.

Experimental result

	Karate	Arxiv	Internet	Web nd.edu
Nodes/links	34/77	9k/24k	70k/351k	325k/1M
CNM	.38/0s	.772/3.6s	.692/799s	.927/5034s
PL	.42/0s	.757/3.3s	.729/575s	.895/6666s
WT	.42/0s	.761/0.7s	.667/62s	.898/248s
Our algorithm	.42/0s	.813/0s	.781/1s	.935/3s

	Phone	Web uk-2005	Web WebBase 2001
Nodes/links	2.6M/6.3M	39M/783M	118M/1B
CNM	-/-	-/-	-/-
PL	-/-	-/-	-/-
WT	.56/464s	-/-	-/-
Our algorithm	.769/134s	.979/738s	.984/152mn

Take-home messages

- Motivation
- Modularity
  - Graph Model
  - Definition
  - Variants
- Modularity Matrix
  - Two Communities
  - Multiple Community Partitioning
- Louvain Method
  - Introduction
  - Algorithm
  - Analysis