NATIONAL UNIVERSITY OF SINGAPORE

MA4230 - MATRIX COMPUTATION

(Semester 1 : AY2021/2022)

Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. Use A4 size paper and pen (blue or black ink) to write your answers.
2. Write down your student number clearly on the top left of every page of the answers. Do not write your name.
3. Write on one side of the paper only. Write the question number and page number on the top right corner of each page (e.g. Q1P1, Q1P2, . . . , Q2P1, . . .).
4. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** pages. Answer **ALL** questions from Section A and **ONE** question from Section B.
5. The total mark for this paper is **ONE HUNDRED (100)**.
6. This is an **OPEN BOOK** examination.
7. You may use any calculator. However, you should lay out systematically the various steps in the calculations.
8. Join the Zoom conference using your mobile phone and turn on the video camera at all times during the exam. Adjust your camera such that your face and upper body (including your hands), as well as your computer screen are captured on Zoom.
9. You may go for a short toilet break (not more than 5 minutes) during the exam.
10. At the end of the exam,
    - Scan or take pictures of your work (make sure the images can be read clearly);
    - Merge all your images into one pdf file (arrange them in the order from Q1 to Q6 in their page sequence);
    - Name the pdf file by Matric No_Moule Code (e.g. A123456R_MA4230);
    - Upload your pdf into the LumiNUS folder "Exam Submission".
      - The Exam Submission folder will close at 19:45. After the folder is closed, exam answers that are not submitted will not be accepted, unless there is a valid reason.

SECTION A (answer ALL questions)

Question 1 [17 marks]
Let $n \in \mathbb{N}$. Let $M, P \in \mathbb{R}^{n\times n}$ be symmetric matrices and $P^{\mathrm{T}}P = I_n$. Consider the matrix

$$A := \left(\begin{array}{c|c} M & PM \\ \hline MP & PMP \end{array}\right) \in \mathbb{R}^{2n\times 2n}.$$

Let $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ be the vectors satisfying $(e_1|e_2|\cdots|e_n) = I_n$.

(i) Show that $A^{\mathrm{T}} = A$.

(ii) Show that $\|A\|_F = 2\|M\|_F$ and $\|A\|_2 \le 2\|M\|_2$.
(Hint: For $w = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}$ with $x, y \in \mathbb{R}^n$ and $\|w\|_2^2 = \|x\|_2^2 + \|y\|_2^2 = 1$, show that $\|Aw\|_2^2 \le 4\|M\|_2^2$.)

(iii) Set $n := 4$, $M := \mathrm{diag}_{4\times 4}(-2, 1, 0, 0)$, $P := (e_4|e_3|e_2|e_1)$. Show that $\|A\|_p = 2$ for all $p \in [1, \infty)$.

Question 2 [20 marks]
We consider the matrix $A \in \mathbb{R}^{4\times 2}$ and the vector $b \in \mathbb{R}^4$ defined by

$$A := \begin{pmatrix} 1 & -2 \\ 2 & 2 \\ -1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad b := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We denote the Moore–Penrose inverse of A by $A^\dagger \in \mathbb{R}^{2\times 4}$. For $k \in \mathbb{N}_0$, we define $\gamma_k \in \mathbb{R}$ by

$$\gamma_k := \inf\{\, \|A^\dagger - M\|_2 \mid M \in \mathbb{R}^{2\times 4},\ \mathrm{rk}(M) \le k\}.$$

(i) Compute $A^\dagger$ via a reduced SVD of A and find the orthogonal projector onto $\mathcal{R}(A)$.

(ii) Find all vectors $x \in \mathbb{R}^2$ satisfying $\|Ax - b\|_2 = \inf_{v\in\mathbb{R}^2} \|Av - b\|_2$.

(iii) For each $k \in \mathbb{N}_0$, compute γ_k and find a matrix $A_k \in \mathbb{R}^{2\times 4}$ with $\mathrm{rk}(A_k) \le k$ and $\|A^\dagger - A_k\|_2 = \gamma_k$.

Question 3 [20 marks]
Let $n \in \mathbb{N}$. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$. We define

$$\mathcal{S}_-^n := \{ M \in \mathbb{R}^{n\times n} \mid M \text{ is symmetric negative definite}\}.$$

(i) Suppose that n is odd and $X \in \mathcal{S}_-^n$ is a matrix with distinct singular values. For each $k \in \mathbb{N}_0$, find the number of SVDs $X = U\Sigma V^{\mathrm{T}}$ of X with the property that $|\det(U) + \det(V)| = k$.

(ii) Using a SVD of A, show that there exist $Q \in \mathbb{R}^{n\times n}$, $N \in \mathcal{S}_-^n$ such that $Q^{\mathrm{T}}Q = I_n$ and $A = NQ$.

(iii) Let $Q \in \mathbb{R}^{n\times n}$, $N \in \mathcal{S}_-^n$ with $Q^{\mathrm{T}}Q = I_n$ and $A = NQ$. Show that $\|A + Q\|_2 = \max_{1\le i\le n} |\sigma_i - 1|$.

Question 4 [20 marks]
Consider the matrices $A, \hat{A} \in \mathbb{R}^{3\times 3}$ and the vectors $b, \hat{b} \in \mathbb{R}^3$ defined by

$$A := \begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad b := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{A} := A + \Delta A, \qquad \hat{b} := b + \Delta b,$$

where $\Delta A \in \mathbb{R}^{3\times 3}$ and $\Delta b \in \mathbb{R}^3$ are such that $\|\Delta A\|_\infty \le \frac{1}{20}$ and $\|\Delta b\|_\infty \le \frac{1}{100}$.

(i) Does A have a LU factorization? Justify your answer.

(ii) Perform Gaussian elimination with partial pivoting to obtain a PA=LU factorization and, using your factorization, compute the unique solution $x \in \mathbb{R}^3$ to $Ax = b$.

(iii) Let $x \in \mathbb{R}^3$ denote the unique solution to $Ax = b$. Using that the inverse of A is given by

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 1 \\ 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix},$$

show that $\hat{A}\hat{x} = \hat{b}$ has a unique solution $\hat{x} \in \mathbb{R}^3$ and show that $\|\hat{x} - x\|_\infty < \frac{1}{8}$.

SECTION B (answer **ONE** question)

Question 5 [23 marks]
Consider the matrix $A \in \mathbb{R}^{3\times 4}$ defined by

$$A := \begin{pmatrix} -4 & -4 & 10 & 8 \\ 0 & -2 & 9 & 0 \\ 1 & 0 & 2 & -2 \end{pmatrix},$$

and the full-rank matrix $B := A_{1:2,1:4} \in \mathbb{R}^{2\times 4}$. Let $U \in \mathbb{R}^{4\times 4}$ with $U^{\mathrm{T}}U = I_4$ and define

$$P := I_4 - UB^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}BU^{\mathrm{T}}.$$

(i) Using Householder triangularization, compute matrices $R \in \mathbb{R}^{3\times 4}$ and $Q \in \mathbb{R}^{4\times 4}$ such that the matrix $\tilde{R} := \begin{pmatrix} 0_{1\times 4} \\ R \end{pmatrix} \in \mathbb{R}^{4\times 4}$ is upper-triangular, $Q^{\mathrm{T}}Q = I_4$, and $RQ = A$.

(Hint: Find a factorization $\tilde{A} = \tilde{R}Q$ of a certain matrix $\tilde{A} \in \mathbb{R}^{4\times 4}$.)

(ii) Compute the singular values and the rank of P.

Question 6 [23 marks]
Consider the matrix $A \in \mathbb{R}^{3\times 3}$ given by

$$A := \begin{pmatrix} 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 5 & 1 \\ \sqrt{2} & 1 & 1 \end{pmatrix}$$

and the full-rank matrix $B := A_{1:3,1:2} \in \mathbb{R}^{3\times 2}$. Let $U \in \mathbb{R}^{3\times 3}$ with $U^{\mathrm{T}}U = I_3$ and define

$$P := I_3 - UB(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}U^{\mathrm{T}}.$$

(i) Using only Gerschgorin's theorem and the symmetry of A, show that $\Lambda(A) \subseteq \{x \in \mathbb{R} : |x-3| \le 3+\sqrt{2}\}$.

(ii) Compute matrices $Q, H \in \mathbb{R}^{3\times 3}$ such that $Q^{\mathrm{T}}Q = I_3$, H is an upper-Hessenberg matrix, and $QHQ^{\mathrm{T}} = A$.

(iii) Perform one step of the QR algorithm with Rayleigh quotient shift applied to H, where you may use that

$$\begin{pmatrix} -2 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -2 & 2 & 2 \\ 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iv) Compute the singular values and the rank of P.

End of Paper