hw7_51215903008_陈诺

作业链接

https://www.wolai.com/tbegzKgBnmQbztZBTG5HHP

习题部分

$$\begin{gathered}\text{习题1.求解线性规划}\\ \begin{aligned}\min \quad & e^T x\\ \text{s.t.} \quad & Gx \le h\\ & Ax = b\end{aligned}\\ \text{的对偶函数,给出对偶问题。}P210\end{gathered}$$

$$\begin{aligned}
&L(x,\lambda,v) = e^\top x + \lambda^\top (Gx-h) + v^\top (Ax-b)\\
&\frac{\partial L}{\partial x} = e + G^\top \lambda + A^\top v\\
&\quad\ \ = -h^\top \lambda - b^\top v + \left(e + G^\top \lambda + A^\top v\right)^\top x\\
&g(\lambda,v) = \inf_x L(x,\lambda,v)\\
&= -h^\top \lambda - b^\top v + \inf_x \left(e + G^\top \lambda + A^\top v\right)^\top x\\
&= -h^\top \lambda - b^\top v - \sup_x \left(\left(-G^\top \lambda - A^\top v\right)^\top x - e^\top x\right)\\
&= -h^\top \lambda - b^\top v - f^*\left(-G^\top \lambda - A^\top v\right)\\
&\text{由} f(x) = e^\top x \quad f^*(y) = \sup_x \left\{y^\top x - e^\top x\right\} = \left\{\begin{array}{c} 0, y = e\\ +\infty, others\end{array}\right.\\
&\text{故} g(\lambda,v) = \left\{\begin{array}{c} -h^\top \lambda - b^\top v, -G^\top \lambda - A^\top v = e\\ -\infty, others\end{array}\right.\\
&\text{对偶问题:}\\
&\max_{\lambda,v} g(\lambda,v)\\
&\text{s.t. } \lambda \geqslant 0
\end{aligned}$$

$$\begin{gathered}\text{习题2.求优化问题}\\ \arg\min_{x_1,x_2,x_3} \ x_1x_2x_3\\ \text{当}x_1,x_2,x_3\text{满足}x_1^2+x_2^2+x_3^2=1\text{的解。}\end{gathered}$$

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$f=x_1x_2x_3=\left(e_1^\top x\right)\left(e_2^\top x\right)\left(e_3^\top x\right)$$

$$=\prod_i e_i^\top x$$

$$\text{s.t. } x^\top x=1$$

$$L(x,\lambda)=\prod_i e_i^\top x+\lambda\left(x^\top x\right)$$

$$\frac{\partial L}{\partial x}=\sum_j\left(\prod_{i\neq j}e_i^\top x\right)e_j+2\lambda x=0$$

$$=\begin{pmatrix}x_2x_3\\0\\0\end{pmatrix}+\begin{pmatrix}0\\x_1x_3\\0\end{pmatrix}+\begin{pmatrix}0\\0\\x_1x_2\end{pmatrix}+2\lambda\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

(当然这里也可对$x_1x_2x_3$分别求导)

$$\begin{cases}x_1x_2+2\lambda x_3=0(1)\\x_1x_3+2\lambda x_2=0(2)\\x_2x_3+2\lambda x_1=0(3)\\x_1^2+x_2^2+x_3^2=1(4)\end{cases}$$

$$(4)\text{代入}x_3(1)+x_2(2)+x_1(3):$$

$$3x_1x_2x_3+2\lambda=0$$

$$2\lambda=-3x_1x_2x_3$$

$$(5)\text{代入}(1):$$

$$x_1x_2-3x_1x_2x_3^2=0$$

$$x_1x_2\left(1-3x_3^2\right)=0(6)$$

$$\text{同理}x_1x_3\left(1-3x_2^2\right)=0(7)$$

$$x_1x_3\left(1-3x_2^2\right)=0(8)$$

$$\text{由}(6),\text{若}x_1=0,\text{代入}(7)\text{得}x_2x_3=0$$

$$\text{若}x_2=0\text{由}(6)\text{得}x_3\neq 0\text{矛盾}$$

$$\text{若}x_3=0\text{由}(8)\text{得}x_2\neq 0\text{矛盾}$$

$$\text{故由}(6)(7)(8)\text{得}1-3x_i^2=0,i=1,2,3$$

$$\text{故}x_i=-\frac{1}{\sqrt{3}},i=1,2,3\text{或}x_i=-\frac{1}{\sqrt{3}},x_j=\frac{1}{\sqrt{3}},i=1,2,3,j\neq i$$

习题3.已知矩阵$A\in R^{p\times q},B\in R^{p\times r},\operatorname{rank}(A)=\min(p,q)$,

未知矩阵$X\in R^{q\times r}$,求以下优化问题。

(1)若$p<q$,求$Frobenius$范数最小的矩阵X,使得$AX=B$,即求解代化问题:

$$\min\quad f(X)=\frac{1}{2}\|X\|_F^2$$

$$\text{s.t.}\quad AX=B$$

$$f=\frac{1}{2}\operatorname{Tr}\left(X^{\top} X\right)$$
$$L=\operatorname{Tr}\left(\frac{1}{2} X^{\top} X+V^{\top}(A X-B)\right)$$
$$\frac{\partial L}{\partial X}=X+A^{\top} V=0$$
$$X=-A^{\top} V \ (1)$$
(1) 代入$AX=B$：
$$-AA^{\top} V=B$$
由于$rank(A)=\min(p,q) \overset{p\leq q}{=} P$
故A行满秩，$AA^{\top}$可逆
$$V=-\left(AA^{\top}\right)^{-1} B \ (2)$$
(2) 代回(1)：
$$X=A^{\top}\left(AA^{\top}\right)^{-1} B$$

习题4.梯度下降法是最常用的优化方法之一。考虑优化问题
$$\min f(x)=x_1^2+x_2^2+2x_3^2$$
证明：在点$x_0=(x_1,x_2,x_3)$处沿负梯度方向迭代的最佳步长为
$$\lambda=\frac{x_1^2+x_2^2+4x_3^2}{2x_1^2+2x_2^2+16x_3^2}$$

$$\nabla f(x_0')=\left(\begin{array}{c}2x_1\\2x_2\\4x_3\end{array}\right)$$
$$x_1'=x_0'-\lambda_{x_0'}\nabla f\left(x_0'\right)$$
$$\min_{\lambda} f\left(x_1'\right)$$
$$f\left(x_0'-\lambda\nabla f\right)=\left(x_1-2\lambda x_1\right)^2+\left(x_2-2\lambda x_2\right)^2+2\left(x_3-4\lambda x_3\right)^2$$
$$\frac{\partial f}{\partial\lambda}=-4(1-2\lambda)\left(x_1^2+x_2^2\right)-16(1-4\lambda)x_3^2=0$$
$$\left(8x_1^2+8x_2^2+64x_3^2\right)\lambda=4x_1^2+4x_2^2+16x_3^2$$
$$\lambda=\frac{x_1^2+x_2^2+x_3^2}{2x_1^2+2x_2^2+16x_3^2}$$
(得到x_1'后，$x_2'=x_1'-\lambda_{x_1'}\nabla f\left(x_1'\right)$以此类推)