Internal Assessment

An Exploration of the Golden Ratio in the Aperiodic Order of Penrose Tilings

1.0 Introduction and Rationale

Humanity has a deep and ancient fascination with patterns. From the repeating hexagons of a honeycomb to the intricate symmetry of a snowflake, we are drawn to systems governed by discernible rules. My mathematical interests have always been rooted in this search for order, particularly in how simple rules can generate structures of profound complexity. My initial exposure was to periodic patterns — those that repeat at regular intervals, like a tiled floor or wallpaper. The beauty of these patterns lies in their predictability; once you understand the unit cell, you understand the entire infinite structure.

My curiosity, however, was truly ignited when I encountered the concept of aperiodic tilings. During research for a design project, I came across images of the stunning geometric mosaics adorning the Darb-e Imam shrine in Isfahan, built in the 15th century. Historians have argued that these patterns exhibit features of aperiodic tiling, centuries before their formal discovery in the West by mathematician and physicist Roger Penrose. This discovery was a revelation to me: here was a pattern that could fill an entire plane without ever repeating itself. It was a perfect manifestation of "ordered, yet non-repeating" structure.

This led me directly to the study of Penrose tilings, specifically the P3 tiling composed of two simple shapes: a "thick" and a "thin" rhombus. The construction rules were surprisingly simple, yet they produced a structure of infinite variety and complexity. This presented a fascinating paradox: if the pattern is not periodic, what prevents it from descending into complete chaos? Is there a hidden, quantitative law governing its composition? This question formed the catalyst for my investigation. I wanted to move beyond a purely aesthetic appreciation and use mathematics to probe the deep structure of this aperiodic order. My aim became to discover if a stable, predictable relationship exists between the quantities of the two rhombuses as the tiling expands towards infinity.

1.1 Aim of the Exploration

The aim of this exploration is to investigate the quantitative relationship between the two constituent tiles of the P3 Penrose tiling. Specifically, this investigation will determine the limit of the ratio of the number of thick rhombuses to thin rhombuses as the tiling populates an infinite plane. Furthermore, this exploration will seek to formally prove that this limiting ratio is precisely the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$.

1.2 Scope and Approach

To achieve this aim, this investigation will proceed in two distinct, complementary stages:

1. **Method 1: Iterative Calculation and Algebraic Derivation.** The first approach will be direct and computational. Starting from a single rhombus, I will apply the tiling's "inflation rules" iteratively to generate successive generations of the pattern. By calculating the number of thick and thin rhombuses at each stage, I will track the evolution of their ratio, observe its convergence, and form a hypothesis. This hypothesis will then be verified through a rigorous algebraic derivation of the limit from the system's recurrence relations.
2. **Method 2: Linear Algebra and Eigenvector Analysis.** To gain a more profound and structural understanding of why this ratio is so stable, the second approach will employ the tools of linear algebra. The recurrence relations will be formulated into a matrix equation. By finding the eigenvalues and eigenvectors of this "substitution matrix," I will analyze the long-term behavior of the system. This method is intended to reveal the inherent properties of the transformation itself, showing that the golden ratio emerges not as a numerical coincidence, but as a fundamental property of the tiling's geometric growth.

By approaching the problem from these two distinct mathematical perspectives, this exploration aims to provide a robust and comprehensive verification of the profound connection between the aperiodic order of Penrose tiling and the fundamental constant of the golden ratio.

2.0 Method 1: Iterative Calculation and Algebraic Derivation

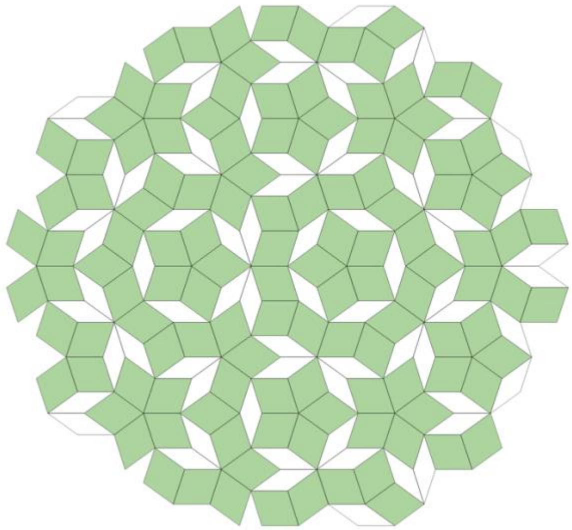
This first method employs a direct computational approach to observe the emerging pattern in the ratio of rhombuses. The process begins by defining the geometric rules of the tiling's growth, using these rules to generate numerical data across several generations, and then forming a hypothesis based on this data. Finally, the hypothesis is proven through a formal algebraic derivation of the limit.

2.1 The Inflation Rule and Recurrence Relations

The P3 Penrose tiling is constructed from two fundamental shapes: a thick rhombus with interior angles of 72° and 108°, and a thin rhombus with angles of 36° and 144°. The tiling's ability to grow infinitely while maintaining its aperiodic structure is governed by a set of "inflation" (or substitution) rules. These rules dictate how each existing tile is replaced by a new configuration of tiles in the next generation.

The inflation rules are as follows:
1. **Thick Rhombus Inflation:** Each thick rhombus is replaced by two new thick rhombuses and one new thin rhombus.
2. **Thin Rhombus Inflation:** Each thin rhombus is replaced by one new thick rhombus and one new thin rhombus.

To analyze the growth of the tiling quantitatively, these rules can be translated into a system of linear recurrence relations. Let T_n represent the total count of thick rhombuses at generation n, and let N_n represent the total count of thin rhombuses at generation n.



Based on the rules above:

$$T_{n+1} = 2T_n + N_n$$

$$N_{n+1} = T_n + N_n$$

This pair of equtions forms the mathematical basis for tracking the systems's evolution.

2.2 Generational Growth and Ratio Calculation

To initiate the process, I will begin with the simplest possible non-trivial configuration: a single thick rhombus at generation zero ($n = 0$). This gives the initial conditions: $T_0 = 1$ and $N_0 = 0$. Applying the recurrence relations iteratively yields the data shown in the table below.

Generation (n)	Thick Rhombuses (T_n)	Thin Rhombuses (N_n)	Total Rhombuses (T_n+N_n)	Ratio ($R_n = T_n/N_n$)	Decimal Approx. of R_n
0	1	0	1	Undefined	-
1	2	1	3	2/1	2.000000
2	5	3	8	5/3	1.666667
3	13	8	21	13/8	1.625000
4	34	21	55	34/21	1.619048
5	89	55	144	89/55	1.618182
6	233	144	377	233/144	1.618056
7	610	377	987	610/377	1.618037

Table 2.1: Generational Growth of Thick and Thin Rhombuses

An interesting secondary observation is that the counts for T_n and N_n are interleaved terms of the Fibonacci sequence, where $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots$. Specifically, $T_n = F_{2n+2}$ and $N_n = F_{2n+1}$.

2.3 Observation and Hypothesis

The data in Table 2.1 clearly indicates that the ratio R_n is not random; it appears to be converging rapidly towards a fixed value. After only a few generations, the decimal approximation stabilizes around 1.618. This value is characteristic of the golden ratio, denoted by the Greek letter phi (ϕ).

Based on this strong computational evidence, I hypothesize that the limit of the ratio of thick to thin rhombuses, as the number of generations approaches infinity, is the golden ratio.

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{T_n}{N_n} = \phi$$

2.4 Algebraic Derivation of the Limit

While the numerical data provides compelling evidence, a formal mathematical proof is required to confirm the hypothesis. To do this, I will solve for the limit of the ratio algebraically.

Let us assume that this limit exists and is a positive, finite value, which we will call R.

$$R = \lim_{n\to\infty} \frac{T_n}{N_n}$$

If this limit exists, then for a sufficiently large n, the ratio at generation n will be virtually identical to the ratio at generation $n + 1$. Therefore, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} R_{n+1} = R$

We can express the ratio R_{n+1} using the recurrence relations:

$$R_{n+1} = \frac{T_{n+1}}{N_{n+1}} = \frac{2T_n + N_n}{T_n + N_n}$$

To analyze the limit, we can divide the numerator and the denominator of the right-hand side by N_n (assuming $N_n \neq 0$, which is true for all $n \geq 1$):

$$R_{n+1} = \frac{2\left(\frac{T_n}{N_n}\right) + 1}{\left(\frac{T_n}{N_n}\right) + 1} = \frac{2R_n + 1}{R_n + 1}$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty} \frac{2R_n + 1}{R_n + 1}$$

$$R = \frac{2R + 1}{R + 1}$$

We can now solve for R. Multiply both sides by $(R + 1)$:

$$R(R+1) = 2R + 1$$

$$R^2 - R - 1 = 0$$

Solving this using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$R = \frac{1 \pm \sqrt{5}}{2}$$

Since R represents a ratio of the number of physical tiles, it must be a positive quantity. Therefore, we discard the negative root.

$$R = \frac{1 + \sqrt{5}}{2}$$

This algebraic derivation rigorously confirms the hypothesis formed from the iterative data. The limit of the ratio of thick to thin rhombuses in the P3 Penrose tiling is indeed the golden ratio, ϕ.

3.0 Method 2: The Matrix Model and Eigenvalue Analysis

Having Established and proven the limit using a direct computational and algebraic approach, this section will now employ the more abstract and powerful tools of linear algebra to analyze the system's long-term behavior. This method will not only reconfirm the result from Method 1 but will also offer a more profound insight into why the system so consistently converges to this specific ratio.

3.1 Rationale for a Deeper Model

Method 1 successfully identified the limit by observing a numerical trend and solving a resulting algebraic equation. However, it does not fully explain the inherent stability of this ratio. Why does this particular system, regardless of the starting configuration of tiles, always tend toward the same proportional relationship?

A matrix model allows us to move beyond a simple step-by-step calculation and instead analyze the properties of the transformation rule itself. By studying the eigenvalues and eigenvectors of the system's transformation matrix, we can uncover the intrinsic, stable properties of the growth process. The eigenvectors represent "stable directions" within the system, and the dominant eigenvalue dictates the long-term behavior, providing a more fundamental explanation for the observed convergence.

3.2 Formulating the Matrix Model

The system of linear recurrence relations derived in section 2.1 provides the foundation for the matrix model:

$$T_{n+1} = 2T_n + N_n$$

$$N_{n+1} = T_n + N_n$$

Let the state of the system at generation n be represented by the column vector v_n:

$$v_n = \begin{pmatrix} T_n \\ N_n \end{pmatrix}$$

The transformation from one generation to the text can then be expressed as a single matrix multiplication, $v_{n+1} = Mv_n$, where M is the substitution matrix:

$$\begin{pmatrix} T_{n+1} \\ N_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} T_n \\ N_n \end{pmatrix}$$

This matrix, $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, encapsulates the complete "genetic code" for the tiling's inflation. The long-term state of the tiling, $\lim_{n\to\infty} v_n$, can be determined by analyzing the properties of this matrix.

3.3 Calculating Eigenvalues and Eigenvectors

The long-term behavior of a system described by $v_{n+1} = Mv_n$ is governed by the eigenvalues (λ) and eigenvectors (x) of the matrix M, which satisfy the equation $Mx = \lambda x$.

To find the eigenvalues, we solve the characteristic equation, $\det(M - \lambda I) = 0$:

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - (1)(1) = 0$$

$$2 - 2\lambda - \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

Using the quadratic formula to solve for λ:

$$\lambda = \frac{3 \pm \sqrt{5}}{2}$$

This yields two distinct real eigenvalues:

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \approx 2.618$$

$$\lambda_2 = \frac{3-\sqrt{5}}{2} \approx 0.382$$

It is noteworthy that these eigenvalues relate directly to the golden ratio: $\lambda_1 = \phi^2$ and $\lambda_2 = (1-\phi)^2$. Since $|\lambda_1| > |\lambda_2|$, $\lambda_1$is the dominant eigenvalue and will govern the system's long term growth rate.

Next, I will find the eigenvector, $x_1 = \binom{x}{y}$, corresponding to the dominant eigenvalue λ_1. We solve the system

$$\begin{pmatrix} 2-\lambda_1 & 1 \\ 1 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the second row for simplicity:

$$1x + (1-\lambda_1)y = 0$$

$$x = (\lambda_1 - 1)y$$

$$x = \left(\frac{3+\sqrt{5}}{2} - 1\right)y$$

$$x = \left(\frac{1+\sqrt{5}}{2}\right)y$$

$$x = \phi y$$

This shows that the components of the eigenvector are in the ratio of ϕ: 1. We can therefore express the dominant eigenvector as any scalar multiple of:

$$x_1 = \begin{pmatrix} \phi \\ 1 \end{pmatrix}$$

3.4 Interpretation of the Dominant Eigenvector

The power of this method lies in its interpretation. Any initial state vector v_0 can be expressed as a linear combination of the matrix's eigenvectors. After n iterations, the state becomes $v_n = M^n v_0 = c_1 {\lambda_1}^n x_1 + c_2 {\lambda_2}^n x_2$.

As n becomes large, the term associated with the dominant eigenvalue, $c_1 {\lambda_1}^n x_1$, grows exponentially, while the term associated with the smaller eigenvalue, $c_2 {\lambda_2}^n x_2$, diminishes towards zero (since $|\lambda_2| < 1$). Consequently, the state vector $v_n$will proressively align itself with the direction of the dominant eigenvector x_1, regardless of the initial state.

In the context of the tiling, this means the system has an inherent, preferred state of balance. The dominant eigenvector $x_1 = \binom{\phi}{1}$ represents this stable state, and the ratio of its components dictates the fixed, long-term ratio of the tiles.

3.5 Final Derivation from the Eigenvector

As the system evolves through infinite generations, the vector describing the number of tiles, $v_n = \binom{T_n}{N_n}$, becomes proportional to the dominant eigenvector $x_1 = \binom{\phi}{1}$.

Therefore, the limit of the ratio of the components is:

$$\lim_{n\to\infty} \frac{T_n}{N_n} = \frac{\phi}{1} = \phi$$

This analytical result, derived from the fundamental properties of the substitution matrix, provides a more profound comfirmation of the conclusion reached in Method 1. It shows that the golden ratio is not just a numberical outcome but is deeply embedded in the very structure of the tiling's growth rules.

4.0 Comprehensive Analysis and Conclusion

Having derived the ratio limit through two distinct methodologies, this section will now synthesize those findings, compare the approaches, and present a formal conclusion to the investigation.

4.1 Synthesis and Comparison of Methods

Method 1 (Iterative Approach) served as the foundational, empirical part of the inquiry. Its strength lies in its direct and intuitive nature. By generating concrete numerical data (Table 2.1), it provided tangible evidence of a converging trend, allowing for a clear and data-driven hypothesis. Based on iterative calculation and the algebraic solution of the system's recurrence relations, proved that the limit R must satisfy the equation $R^2 - R - 1 = 0$. However, this method is primarily descriptive; it demonstrates *what* happens with each successive generation but offers limited insight into the fundamental reason *why* this specific ratio is the system's inevitable destination.

Method 2 (Matrix Model) provided a more profound and explanatory analysis. By modeling the inflation rules as a matrix transformation, the focus shifted from the sequence of numbers to the properties of the transformation itself. Using linear algebra, showed that the long-term behavior of the system is governed by a dominant eigenvector whose components are in the exact same proportion. The dominant eigenvector is the key in the system's state space, and the system naturally evolves towards this state of equilibrium.

4.2 Formal Conclusion

The aim of this investigation was to determine the limit of the ratio of thick to thin rhombuses in the P3 Penrose tiling as it expands to infinity. This exploration has conclusively demonstrated, through two independent and mutually reinforcing methods, that this limit exists and is precisely the golden ratio.

Thus, it can be formally stated that:

$$\lim_{n\to\infty} \frac{Number\ of\ Thick\ Rhombuses}{Number\ of\ Thin\ Rhombuses} = \phi = \frac{1+\sqrt{5}}{2}$$

This confirms that a profound and predictable mathematical order underpins the beautiful complexity of this aperiodic tiling.

The process of using two different mathematical lenses to view the same problem was particularly illuminating. It highlighted the power of mathematics to not only calculate but also to explain. This exploration has deepened my appreciation for the unifying nature of mathematics, revealing the hidden numerical constants that can govern even the most complex and seemingly irregular structures. It reinforces the idea that within systems that appear chaotic on the surface, there often lies a deep, elegant, and beautiful mathematical order waiting to be discovered.