Optimizing Dart Throwing Strategies for the Elderly Based on Markov Decision Process

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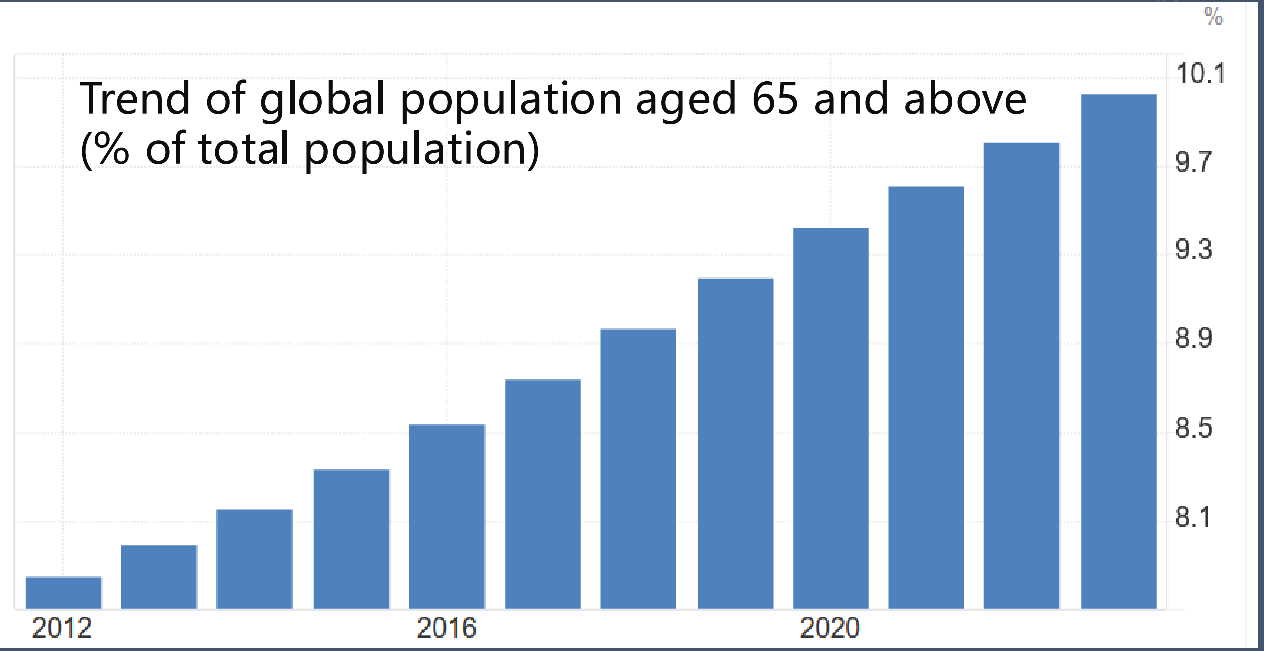
Table of Contents

1. Introduction (1)

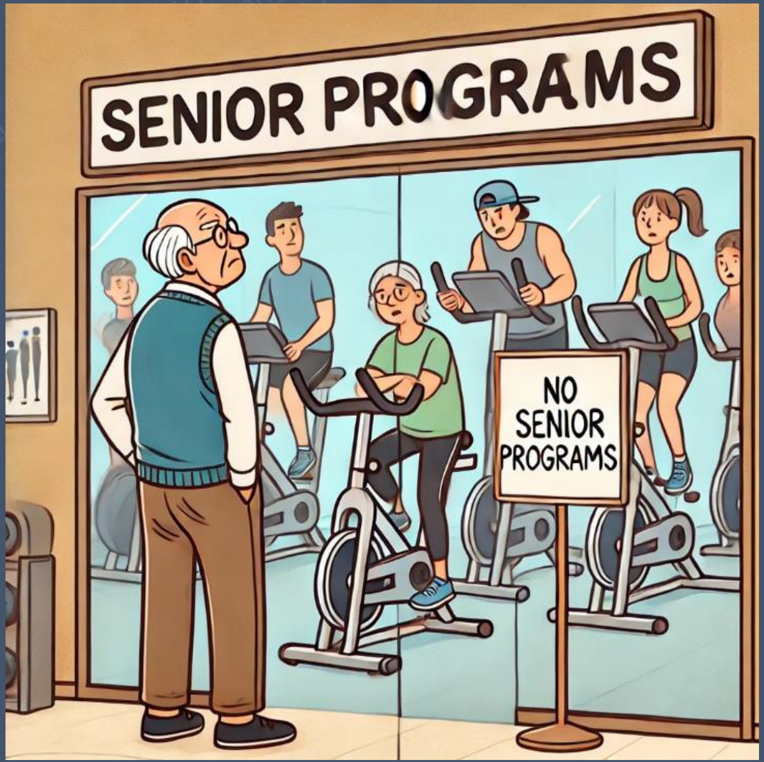
- The World Bank's data confirms a rapidly accelerating trend: the world's population is aging at an unprecedented rate.

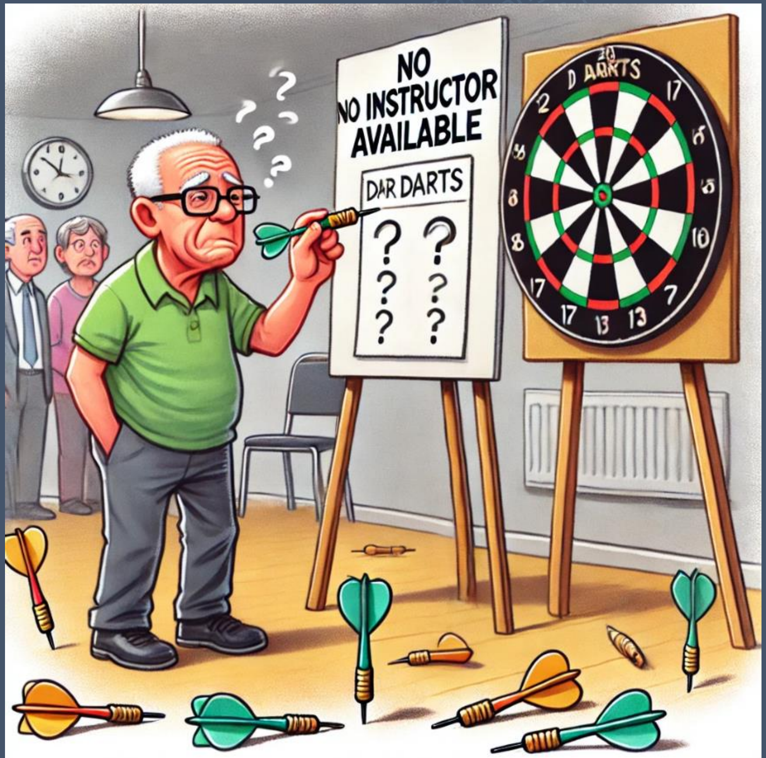
- In 2023, the global population aged 65 and above accounted for 10.03% of the total population.



- WHO delivered a work on the UN Decade of Healthy Aging (2021–2030).
- National Health Commission of China issued the "14th Five-Year Plan for Healthy Aging“.
- Encouraging physical activity and enhancing exercise experiences for older adults are valuable strategies for improving their quality of life.

1. Introduction (2)





Fitness challenges facing by the elderly:

- Limited activity options
- Risk of injury
- Insufficient guidance

Darts are excellent for the elderly:

- Hand-eye coordination
- Calculation and memory
- Low risk of injury
- Reducing loneliness
- Low barriers to entry and minimal requirements

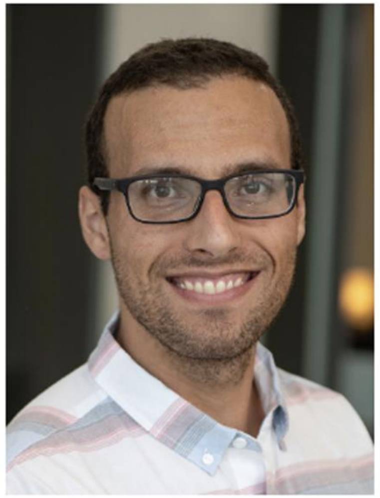
The elderly require guidance:

- Understanding rules
- Setting appropriate fitness goals
- Building confidence
- Developing a suitable plan
- Successfully transitioning through the beginner phase

2. Related Works (1)

Ryan Tibshirani

Research Software Teaching Group Other



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Amazon Scholar in AWS AI Labs

Short bio

I am a Professor in the Department of Statistics at UC Berkeley. I am also a Principal Investigator in the Delphi group, and an Amazon Scholar in AWS AI Labs. From 2011-2022, I was a faculty member in Statistics and Machine Learning at Carnegie Mellon University. From 2007-2011, I did my Ph.D. in Statistics at Stanford University, with Jonathan Taylor as my thesis advisor. From 2003-2007, I did my B.S. in Mathematics at Stanford.

Journal of the Royal Statistical Society Series A (2011)

Ryan T. et al. investigated dart throwing strategies using statistical methods. They employed bivariate normal and skew-normal distributions to establish throwing models, utilizing the standard deviation of the distribution to characterize a player's skill level. The Expectation-Maximization (EM) algorithm was applied to estimate the standard deviations of dart throws in both the X and Y directions. Their findings indicated that this method can accurately estimate the true skill level of a dart player, which can assist players in achieving higher scores in a single round.

Journal of the Royal Statistical Society
SERIES A Statistics in Society

J. R. Statist. Soc. A (2011)
174, Part 1, pp. 213–226

A statistician plays darts

Ryan J. Tibshirani, Andrew Price and Jonathan Taylor
Stanford University, USA

[Received June 2009. Revised April 2010]

Summary. Darts is enjoyed both as a pub game and as a professional competitive activity. Yet most players aim for the highest scoring region of the board, regardless of their level of skill. By modelling a dart throw as a two-dimensional Gaussian random variable, we show that this is not always the optimal strategy. We develop a method, using the EM algorithm, for a player to obtain a personalized heat map, where the bright regions correspond to the aiming locations with high (expected) pay-offs. This method does not depend in any way on our Gaussian assumption, and we discuss alternative models as well.

Keywords: EM algorithm; Importance sampling; Monte Carlo methods; Statistics of games

1. Introduction

Familiar to most, the game of darts is played by throwing small metal missiles (darts) at a circular target (dartboard). Fig. 1 shows a standard dartboard. A player receives a different score for landing a dart in different sections of the board. In most common dart games, the board's small concentric circle, called the 'double bulls-eye' (DB) or just 'bulls-eye', is worth 50 points. The surrounding ring, called the 'single bulls-eye' (SB), is worth 25. The rest of the board is divided into 20 pie-sliced sections, each having a different point value from 1 to 20. There is a 'doubles' ring and a 'triples' ring spanning these pie slices, which multiply the score by a factor of 2 or 3 respectively.

Not being expert dart players, but statisticians, we were curious whether there is some way to optimize our score. In Section 2, under a simple Gaussian model for dart throws, we describe an efficient method to try to optimize your score by choosing an optimal location at which to aim. If you can throw relatively accurately (as measured by the variance in a Gaussian model), there are some surprising places that you might consider aiming the dart.

The optimal aiming spot changes depending on the variance. Hence we describe an algorithm by which you can estimate your variance based on the scores of as few as 50 throws aimed at the DB. The algorithm is a straightforward implementation of the EM algorithm (Dempster *et al.*, 1977), and the simple model that we consider allows a closed form solution. In Sections 3 and 4 we consider more realistic models, Gaussian with general covariance and skew Gaussian, and we turn to importance sampling (Liu, 2008) to approximate the expectations in the E-steps. The M-steps, however, remain analogous to the maximum likelihood calculations; therefore we feel that these provide nice teaching examples to introduce the EM algorithm in conjunction with Monte Carlo methods.

Not surprisingly, we are not the first to consider optimal scoring for darts: Stern (1997) compared aiming at the triple 19 and triple 20 for players with an advanced level of accuracy, and

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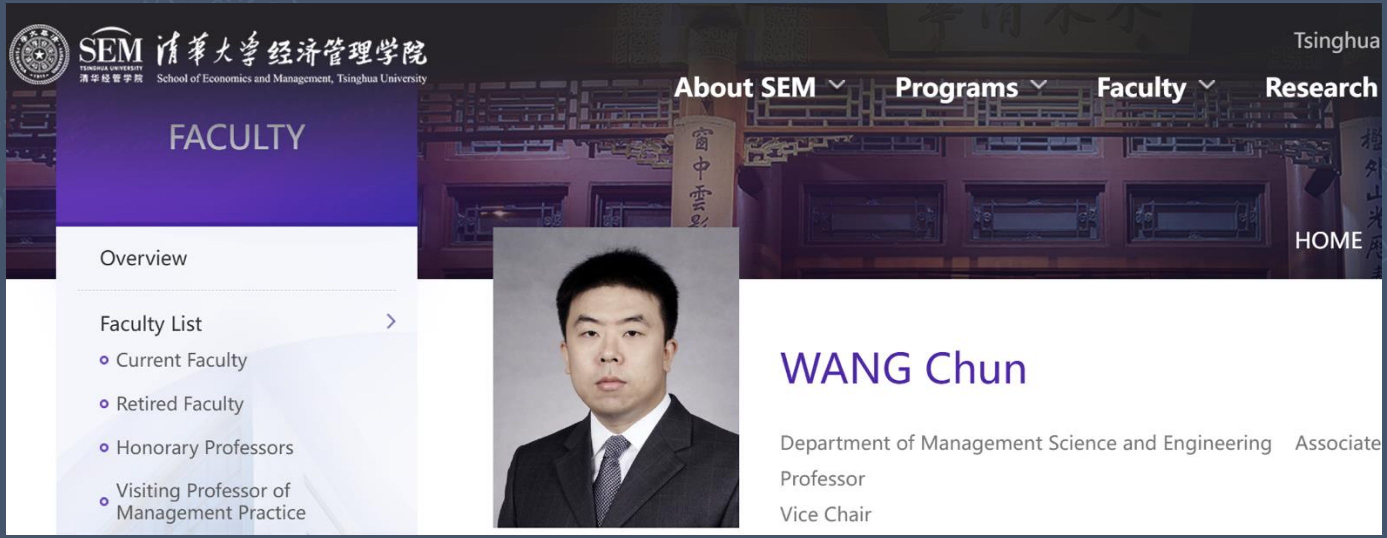
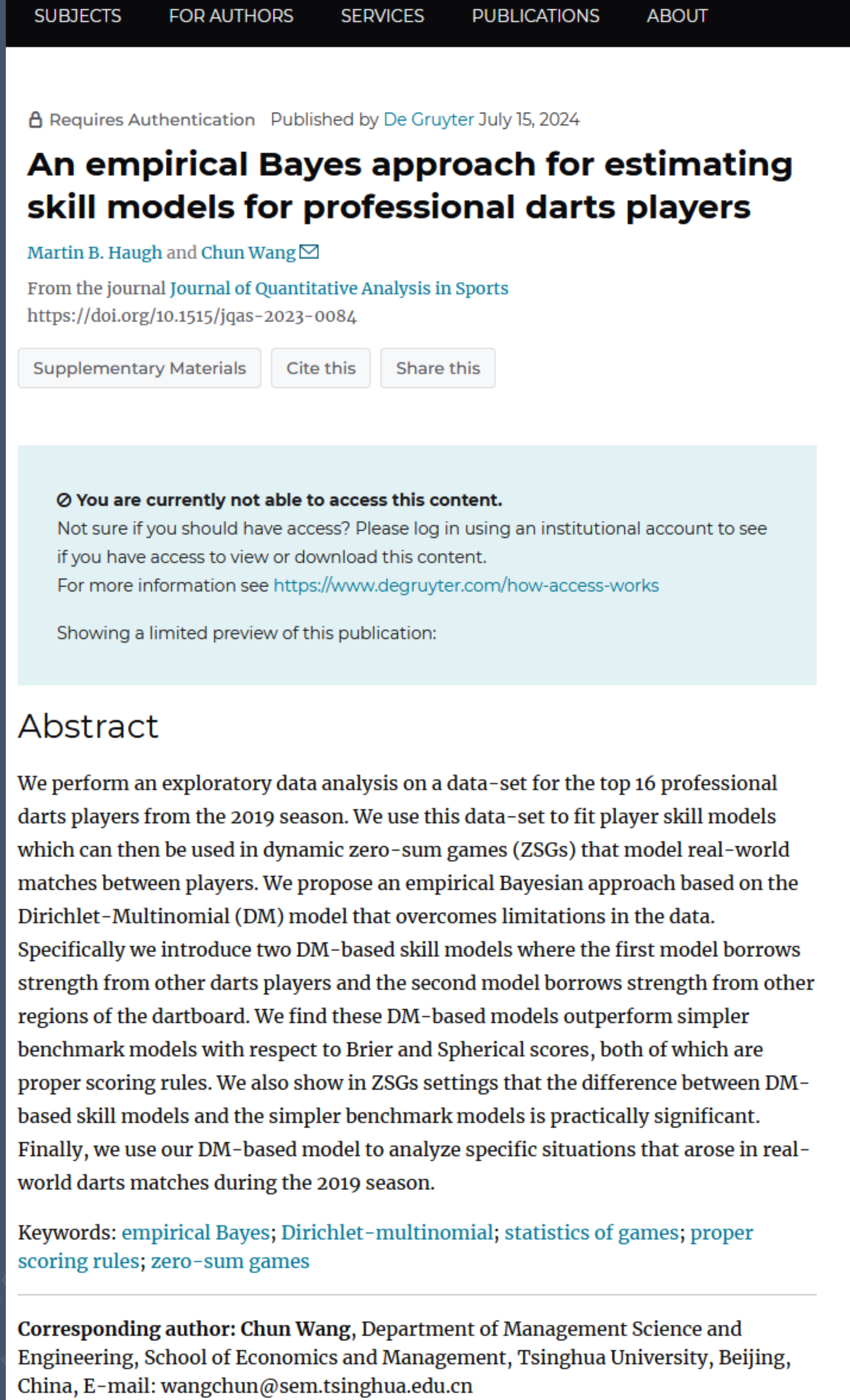
2. Related Works (2)



Journal of Quantitative Analysis in Sports (2024)

Haugh and Wang conducted an analysis on a dataset of 16 top professional dart players from the 2019 season. This study utilized the dataset to construct models that fit player skills, subsequently employing these models to simulate the dynamic zero-sum game (ZSG) process of actual matches between dart players. Their experiments demonstrated that employing a game strategy based on this ZSG model, which involves strategic play considering both the player's own skill level and the opponent's score, can increase the probability of winning by 2% to 3%.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An empirical Bayes approach for estimating skill models for professional darts players

Martin B. Haugh and Chun Wang

From the journal Journal of Quantitative Analysis in Sports
https://doi.org/10.1515/jqas-2023-0084

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Abstract

We perform an exploratory data analysis on a data-set for the top 16 professional darts players from the 2019 season. We use this data-set to fit player skill models which can then be used in dynamic zero-sum games (ZSGs) that model real-world matches between players. We propose an empirical Bayesian approach based on the Dirichlet-Multinomial (DM) model that overcomes limitations in the data. Specifically we introduce two DM-based skill models where the first model borrows strength from other darts players and the second model borrows strength from other regions of the dartboard. We find these DM-based models outperform simpler benchmark models with respect to Brier and Spherical scores, both of which are proper scoring rules. We also show in ZSGs settings that the difference between DM-based skill models and the simpler benchmark models is practically significant. Finally, we use our DM-based model to analyze specific situations that arose in real-world darts matches during the 2019 season.

Keywords: empirical Bayes; Dirichlet-multinomial; statistics of games; proper scoring rules; zero-sum games

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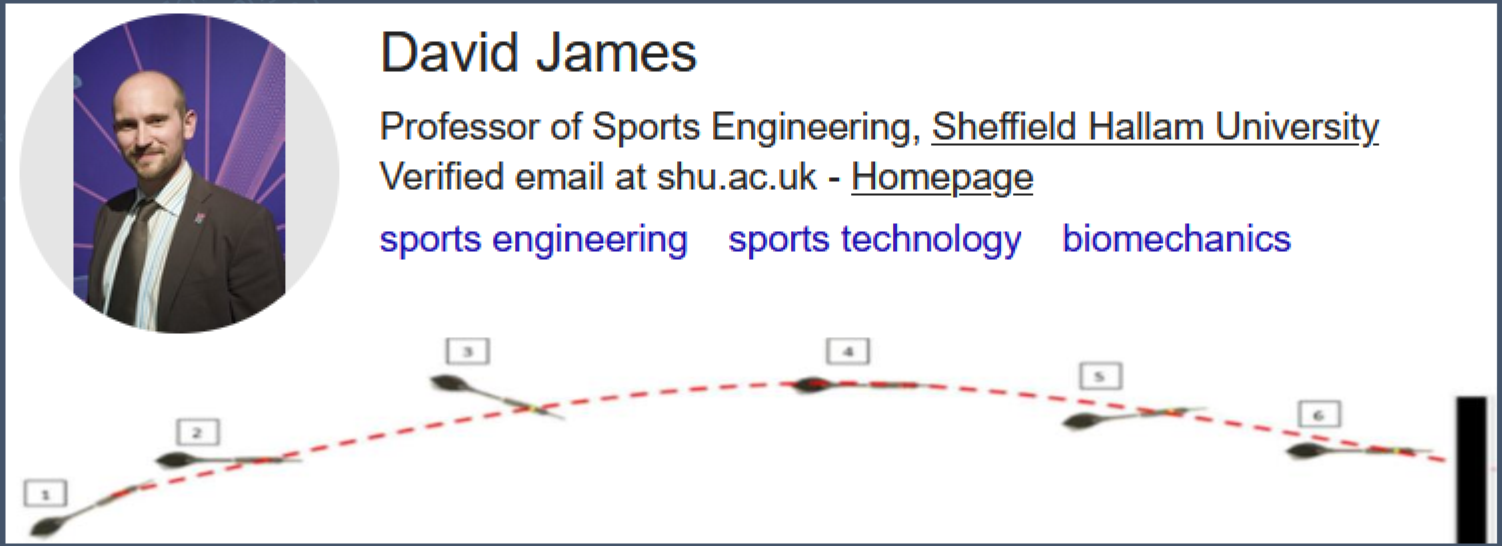
2. Related Works (3)



Journal: Sports Engineering (2018)

James et al. recorded the trajectories of 225 dart throws by 19 amateur players utilizing high-speed video capture. Their analysis revealed that the pitch angle of the dart during flight oscillates in a manner akin to damped harmonic motion. Furthermore, the study found a strong correlation between this oscillation frequency and launch speed, while the characteristic wavelength and damping ratio were independent of launch speed. Notably, the measured oscillation wavelength (2.16 m) closely approximated the regulated throwing distance (2.37 m). The authors suggest "tuning" the dart throwing distance to the wavelength, allowing the dart to undergo one complete oscillation before striking the dartboard.

Sports Engineering
https://doi.org/10.1007/s12283-018-0279-9

ORIGINAL ARTICLE

CrossMark

Experimental validation of dynamic stability analysis applied to dart flight

David James[1] · Jonathan Potts[1]

Abstract

The game of darts attracts a large international following and can be fiercely competitive. Despite its popularity, and the large equipment market, no previous peer-reviewed studies have examined the trajectory of a dart in flight. This study used high-speed video techniques to measure the trajectories of 225 dart throws from 19 amateur players. The dart's pitch and angle of attack were found to oscillate during flight in a manner that is analogous to damped harmonic motion. It was also found that the dart's oscillation frequency was strongly correlated to launch speed, whilst its characteristic wavelength and damping ratio were independent of launch speed. The measured wavelength of oscillation (2.16 m) was found to be similar to the regulation throwing distance (2.37 m). It is proposed that the dart is 'tuned' to the throwing distance such that it undergoes one full oscillation before striking the board. The dart flight was modelled using a classical dynamic stability analysis and good agreement was found between the experimental observations and the theoretical predictions. The success of the model confirms that the approach can be used to explore the dynamics of different dart designs through parametric sensitivity analyses.

Keywords Aerodynamics · Darts · Oscillation · Stability analysis · Trajectory

List of symbols

$\bar{x}_{ac}$	Aerodynamic centre
AR	Aspect ratio
B	Wing span length
$\bar{x}_{dart}$	Centre of gravity
c_{MAC}	Mean aerodynamic chord
$C_{M\alpha}$	Pitching moment gradient coefficient
$c_{N\alpha}$	Normal force gradient coefficient, infinite span
$C_{N\alpha}$	Normal force gradient coefficient, finite span
c_r	Fin root chord length
c_t	Wing tip chord length
F	Planform parameter
f	Function
h	Cone section axial height
i	Section number
I_y	Section moment of inertia
l_b	Barrel length (including needle)
l_s	Stem length
M	Pitching moment
M_α	Pitching moment alpha stability derivative
m_i	Section mass
m_L	Mass of full height (not truncated) cone
M_q	Pitching moment pitch rate stability derivative
m_S	Equivalent mass missing cone
n	Number of sections
N	Aerodynamic normal force
q	Pitch rate
r^2	Coefficient of determination
r_{rot}	Radius of rotation
S	Flight surface area
t	Time
x_{sm}	Static margin
$\bar{y}_{fin}$	Single fin mass centre
α	Angle of attack
Δ	Perturbation from steady flight condition
λ	Wavelength of the angle of attack oscillation
Λ	Sweep angle
λ_c	Fin taper ratio
π	Ratio of circle circumference to diameter
ρ	Density of air
θ	Pitch angle
ζ	Damping ratio

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[1] Department of Engineering and Mathematics, Centre for Sports Engineering Research, Sheffield Hallam University, Sheffield, UK

Published online: 18 June 2018

isea

2. Related Works (4)

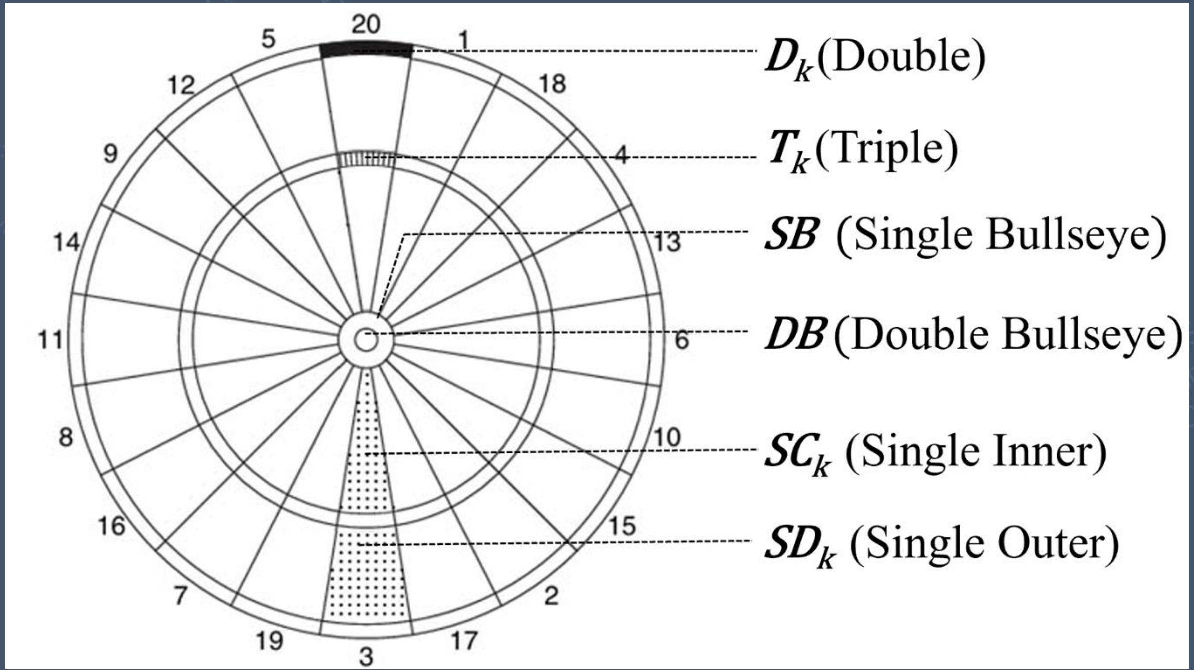
Existing research primarily focuses on the performance of high-level players or professional athletes. When the Darts Society at 101 High School promoted the sport within the local communities, we found that these research findings were not highly applicable to the elderly. Therefore, tailored strategies specifically designed for older adults are necessary.

The elderly engaging in darts do not aim to win competitions or earn prize money but focus on fitness, mental stimulation, and improving their quality of life. Their participation is characterized by:

- Lower accuracy;
- Limited physical strength and energy;
- Restricted training time.

Individualized guidance are required for the elderly to **lower the learning curve** and **maintain participation** in this sport.

3.1 Rules of Dart Games



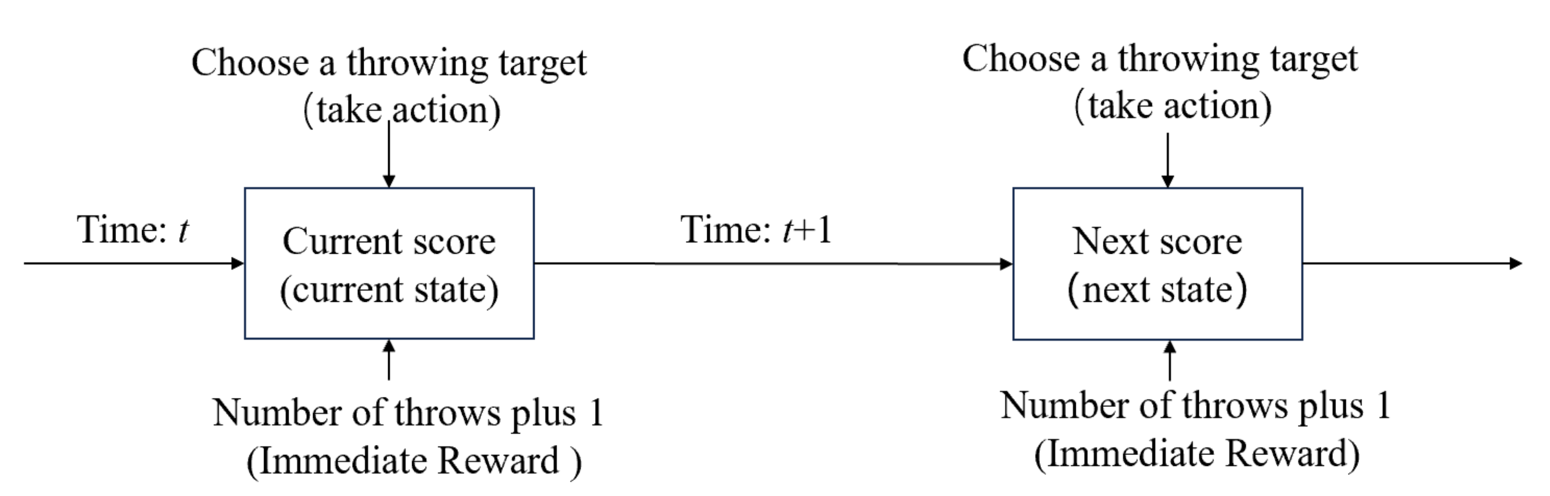
Symbols	description	Score
DB	Double Bull-eyes	50
SB	Single Bull-eyes	25
SC_k	Single ring(inner)	$k \times 1$
T_k	Triple ring	$k \times 3$
SD_k	Single ring (outer)	$k \times 1$
D_k	Double ring	$k \times 2$

Rule 1	Scoring Rule	Each player begins with a score of 501. A player's score decreases with each successful throw. The first player to reach a score of exactly zero wins the game.
Rule 2	Double Out Rule	The final dart thrown must land in the doubles ring (i.e., hit a D_k) to conclude the game.
Rule 3	Bust Rule	If a dart hits out sider of the board, or if the resulting score would be 1, 0 (but does not satisfy the "Double Out" rule), or negative, the player's score remains unchanged from before the throw.
Rule 4	Alternate First Rule	The two players alternate who throws first in each round.

3.2 Model and Mathematical Symbols

The process of completing a "501" game can be regarded as a sequential decision-making process.

The initial state is 501 points.



The remaining score reaches exactly zero.

We model the process using a Markov Decision Process: $Dart\text{-}MDP = \{T, S, A, p(s_{t+1}|s_t, a), r(s_t, a)\}$

Decision Time Point	$\boldsymbol{T} = \{\mathbf{0}, \mathbf{1}, \cdots\}$	The moment a darts player chooses a target for throwing.
State Space	$\boldsymbol{S} = \{\mathbf{501}, \mathbf{500}, \mathbf{499}, \cdots, \mathbf{2}, \mathbf{0}\}$	the player's current remaining score。 $1 \notin \boldsymbol{S}$
Action Set	$\boldsymbol{A} = \{\boldsymbol{DB}, \boldsymbol{SB}, \boldsymbol{SC_k}, \boldsymbol{SD_k}, \boldsymbol{T_k}, \boldsymbol{D_k}\}$	the 82 target areas on the dartboard, $k \in \{1,2,\cdots,20\}$
Transition Probabilities	$p(s_{t+1}\|s_t, a)$	The probability of the system transitioning to a state $s_{t+1} \in S$ after the player, in state $s_t \in S$, takes an action $a \in A$, $\sum_{s_{t+1}\in S} p(s_{t+1}\|s_t, a) = 1$。
Immediate Reward	$r(s_t, a)$	$r(s_t, a) := 1$. When a player makes a throw a, the number of throws increases by one.

3.3 Solving the Transition Probability （1）

The transition probability $p(s_{t+1}|s_t, a)$ represents the probability that, when the player is in state $s_t \in S$ and takes an action $a \in A$ (i.e., aims at a target a and throws), the system transitions to state $s_{t+1} \in S$ at the next decision time point. This can be expressed as the probability of the event that the player, aiming at $a \in A$ and throwing, achieves a score of $(s_t - s_{t+1})$.

$$p(s_{t+1}|s_t, a) = P\big(target = a, score = (s_t - s_{t+1})\big) \tag{1}$$

The scoring rule permits multiple hit locations(h) to achieve the same score,$h \in A \cup \{MISSED\}$.

$$P\big(target = a, score = (s_t - s_{t+1})\big) = \sum_{h \in \{z|Score(z) = (s_t - s_{t+1})\}} P(target = a, hit = h) \tag{2}$$

$P(target = a, hit = h)$ represents the probability that a player aims at the $target = a$ and hits the region $hit = h$. Since darts can miss the board, $h \in A \cup \{MISSED\}$.

For example, the probability of a player transitioning from state s_0 =501 to s_1 =485 after aiming at $a =$ T8 and throwing：

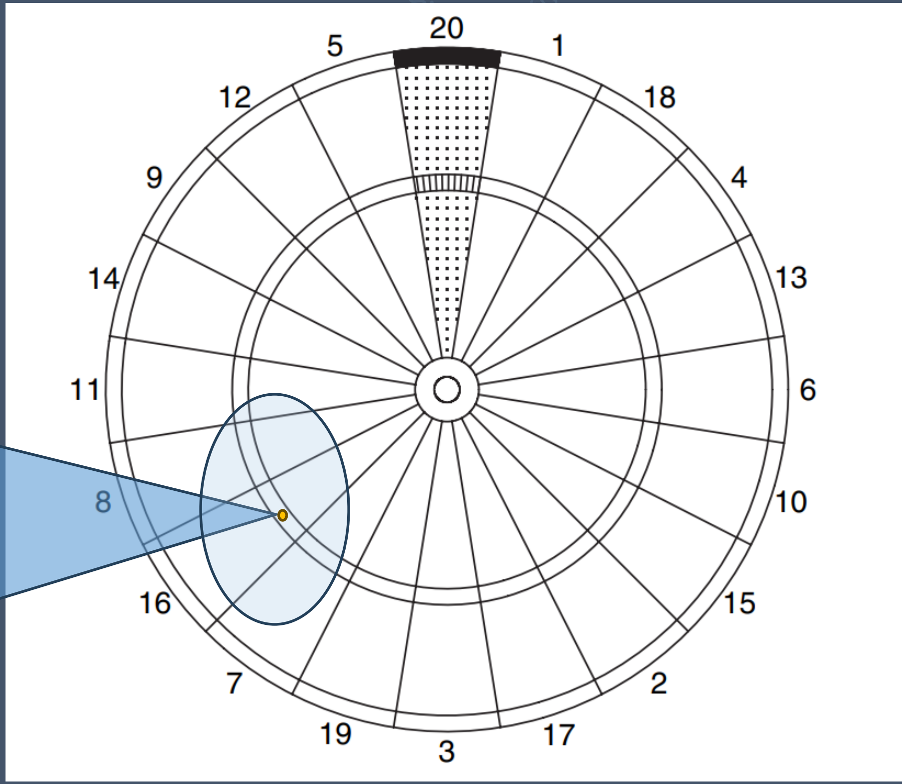
$$p(s_1 = 485|s_0 = 501, a = T8)$$
$$= P(target = T8, hit = SC16) + P(target = T8, hit = SD16) + P(target = T8, hit = D8)$$

3.3 Solving the Transition Probability （2）

Using the method proposed by Ryan J.T. et al. , a player's skill level is characterized by the standard deviations (σ_X, σ_Y) of two independent bivariate normal distributions in the X and Y directions. The probability of hitting a region $hit = h$ after aiming at the $target = a$ can be calculated using the integral formula of a bivariate normal distribution as follows:

$$P(target = a, hit = h) = \iint_h \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left(-\frac{(x-\mu a_X)^2}{2\sigma_X^2} - \frac{(y-\mu a_Y)^2}{2\sigma_Y^2}\right) dxdy \qquad (3)$$

When a player with $(\sigma_Y = 1.2,\ \sigma_Y = 2.0)$ aims at the centroid of T16 ($\mu a_X =$-9.3, $\mu a_Y =$-6.76), the dart may land in multiple areas The coordinates of the landing position follow a normal distribution N(-9.30, 1.2) in X direction and N(-6.76, 2.0) in Y direction.



Target	Hitted	P(a,h)	Score	$P \times Score$
T16	T16	0.341	48	16.3459
T16	SD16	0.264	16	4.2214
T16	SC16	0.091	16	1.4501
T16	SD7	0.088	7	0.6163
T16	T8	0.087	24	2.0873
T16	SC8	0.055	8	0.4425
T16	T7	0.044	21	0.915
T16	SD8	0.02	8	0.1629
T16	SC7	0.006	7	0.0434
T16	SC11	0.003	11	0.0294
T16	T11	0.001	33	0.0482

3.3 Solving the Transition Probability （3）

Combining equations (1), (2), and (3), we can derive： $p(s_{t+1} \,|s_t, a)$

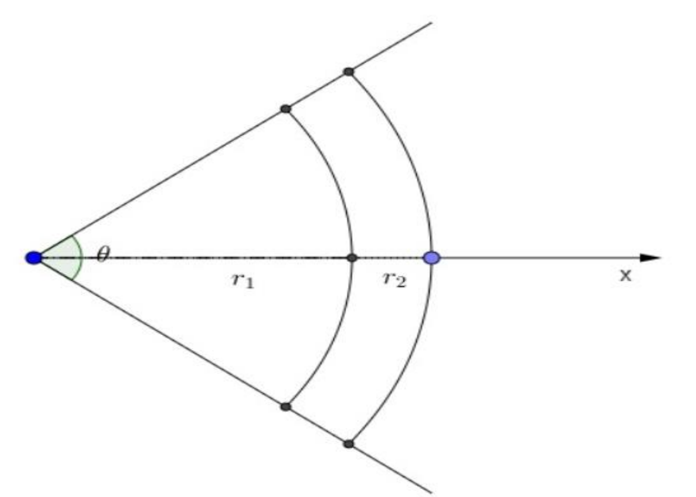
$$=\sum_{h\in\left\{z|score(z)\,=\,(s_t\,-\,s_{t+1})\right\}}\iint_h \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left(-\frac{(x-\mu a_X)^2}{2\sigma_X^2}-\frac{(y-\mu a_Y)^2}{2\sigma_Y^2}\right)dxdy \qquad （4）$$

Converting the double integral into polar coordinates can significantly simplify the computation.

$$=\sum_{h\in\left\{z|Score(z)\,=\,(s_t\,-\,s_{t+1})\right\}}\iint_h \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left(-\frac{(\rho\cos\theta-\mu a_X)^2}{2\sigma_X^2}-\frac{(\rho\sin\theta-\mu a_Y)^2}{2\sigma_Y^2}\right)\rho\, d\rho\, d\theta \qquad （5）$$

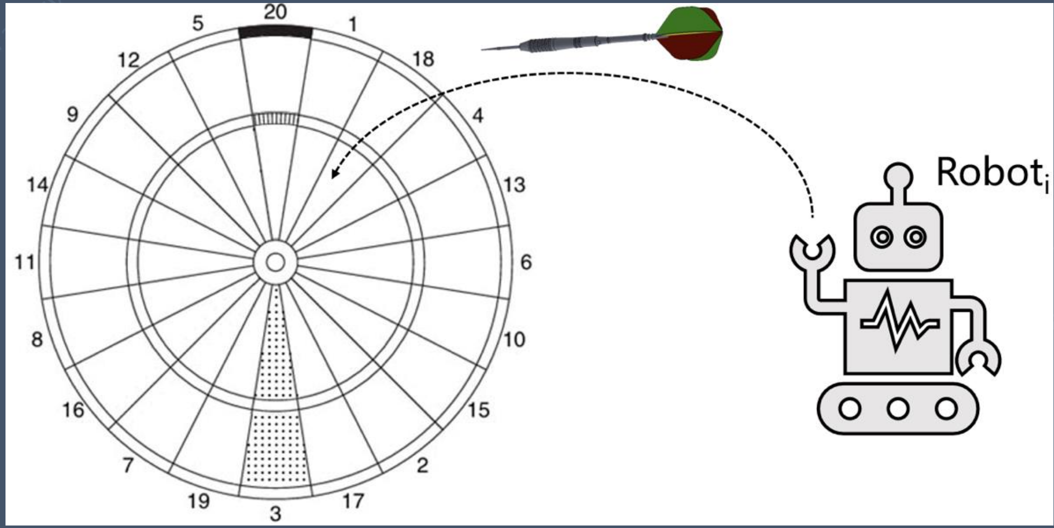
The coordinates (μa_x，μa_Y) of the aiming point (centroid) for each target area (annular sector) can be calculated using the centroid formula for an annular sector. The centroid lies on the bisector of the sector angle, and its distance from the center of the dartboard is as follows：

$$l=\frac{4\sin(\frac{\theta}{2})(r_2^3-r_1^3)}{3\theta(r_2^2-r_1^2)}$$



3.4 Estimating Dart Throwing Accuracy （1）

The standard deviations (σ_X, σ_Y) are critical determinants of the transition probability. This study employs a nearest-neighbor estimation method to estimate.



In the experiment, we construct N virtual robots for dart throwing. the (σ_X, σ_Y) of each robots are different, and the standard deviation in both directions begin at 0.1 centimeters (cm) and increase in increments of 0.1 cm up to 3.0 cm. Beyond 3.0 cm, the increment size changes to 0.2 cm, continuing up to 10 centimeters. In this manner, a total of $N = 4225$ robots with varying skill levels are constructed.

3.4 Estimating Dart Throwing Accuracy （2）

Creating a table to store the $P(target = a, hit = h)$ for each robots.

Robot ID	σ_X	σ_Y	targe	hit	P	Score	Score × P
U034	1.8	2.8	SC20	SC20	0.352	20	7.0477
U034	1.8	2.8	SC20	SC5	0.177	5	0.8868
U034	1.8	2.8	SC20	SC12	0.034	12	0.4069
U034	1.8	2.8	SC20	SC9	0.006	9	0.0538
U034	1.8	2.8	SC20	SC14	0.001	14	0.0202
U034	1.8	2.8	SC20	SC13	0.001	13	0.0188
U034	1.8	2.8	SC20	SC4	0.006	4	0.0239
U034	1.8	2.8	SC20	SC18	0.034	18	0.6103
U034	1.8	2.8	SC20	SC1	0.177	1	0.1774
U034	1.8	2.8	SC20	T20	0.085	60	5.0975
U034	1.8	2.8	SC20	T5	0.025	15	0.3802
U034	1.8	2.8	SC20	T1	0.025	3	0.076
U034	1.8	2.8	SC20	SD20	0.039	20	0.777
U034	1.8	2.8	SC20	SD5	0.008	5	0.0416
U034	1.8	2.8	SC20	SD1	0.008	1	0.0083
U034	1.8	2.8	SC20	DB	0.002	50	0.1166
U034	1.8	2.8	SC20	SB	0.013	25	0.3235

28.4 millions (82 x 82 x 4225) double integrals are required to be computed to fill in the table. Filtering out entries with $P < 0.001$ resulted in ~8.5M records.

Calculate $P(target = a, hit = h)$ in 3 ways.

Method1：

FAILED

Using the integral2() function from the pracma library in R language.

Method2：

Done

Using the dblquad（） function from scipy.integrate library in Python.

Method3：

Done

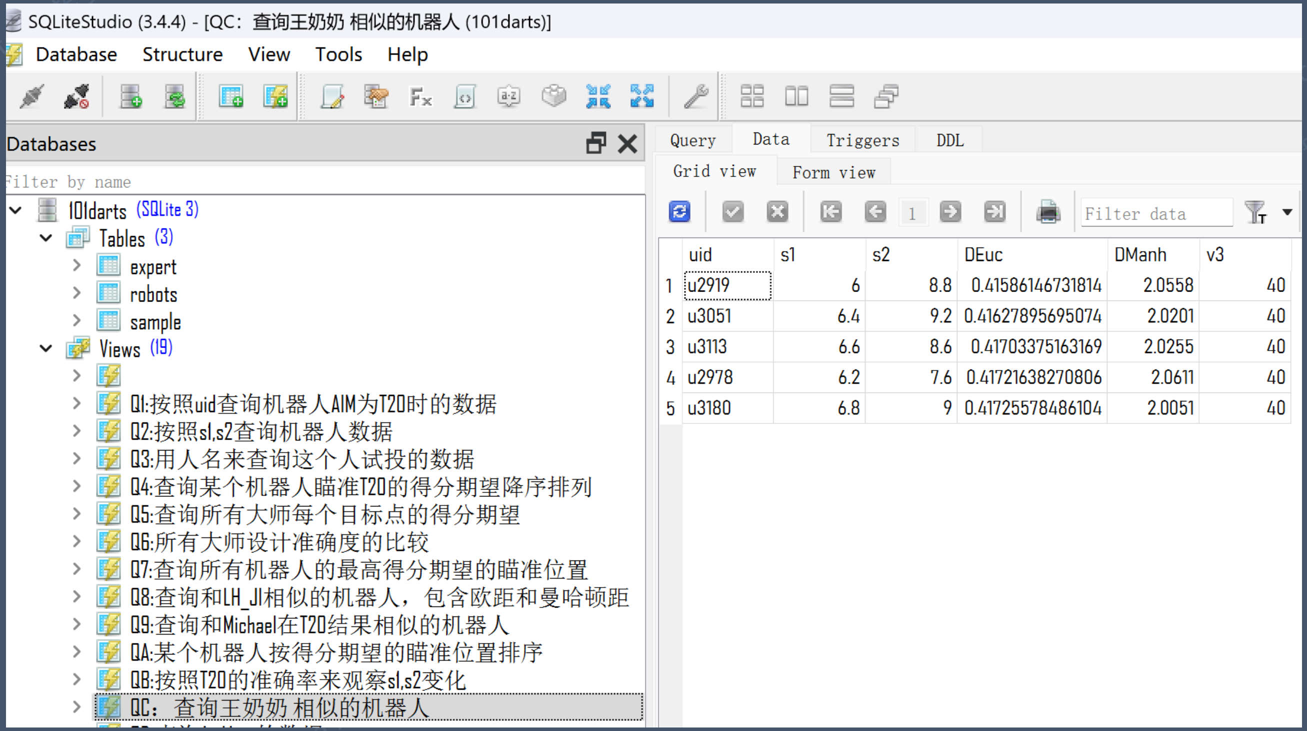
Using the random（） function from numpy library to generate a set of random numbers following a two-dimensional normal distribution, simulating the landing positions of the robot after throwing the dart.

Method 3 is significantly faster, taking 7 seconds vs. 13 seconds for Method 2 to calculate one robot's data (82 x 82 integrations).

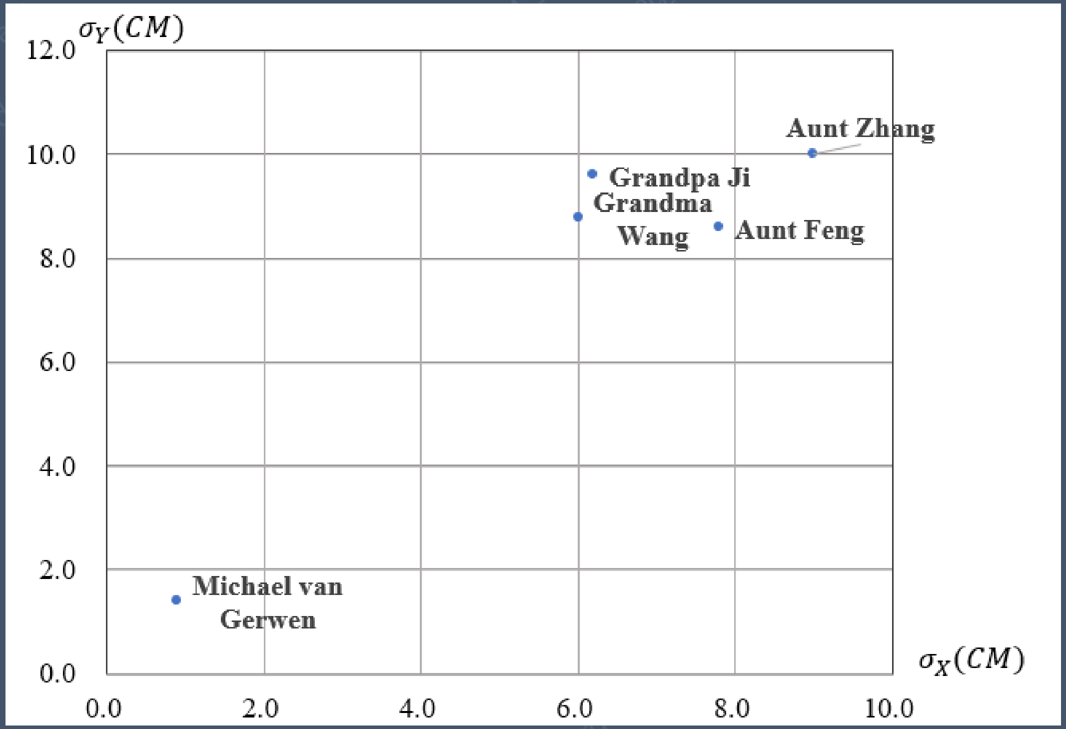
Method 1 terminates abnormally.

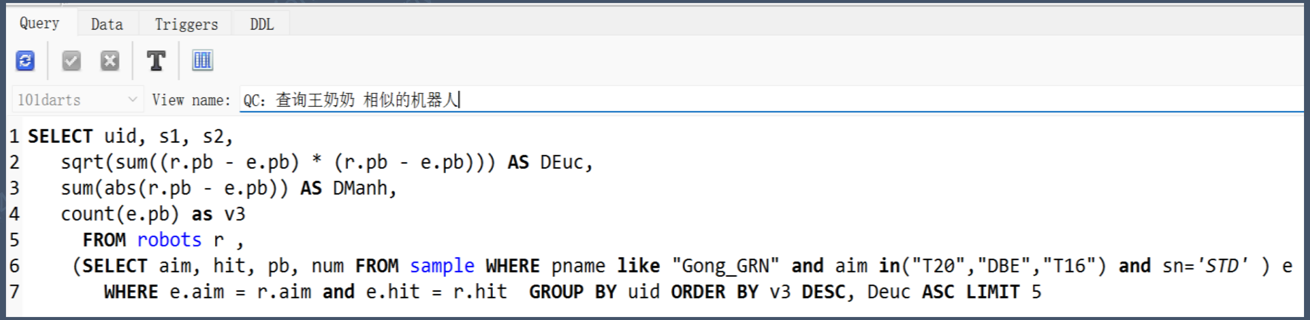
3.4 Estimating Dart Throwing Accuracy (3)

Nearest neighbor calculations are convenient to implement by importing player and robot's data into SQLite.



Accuracy Estimation for the four elderly players and the World Champion





Using SQL language to calculate the nearest neighbor with Euclidean distance, a query can find the dart-throwing robot closest to Grandma Wang.

4.1 Throwing Strategy Optimization（1）

In the“501”Game,each player aims to complete the game with as few throws as possible: $min(\sum r(s_t, a))$.

- $E[s_t, a]$：denote the expected minimum number of throws required for a player to complete the game when in state s_t and taking action a.
- $E\_min[s_t] := \min_{\alpha \in A}(E[s_t, a])$ represents the smallest expected number of throws across all actions $a \in A$ when the player is in state s_t.
- If an action a satisfies $E[s_t, a] = E\_min[s_t]$, it is called the optimal decision for state s_t. The set of all optimal decisions across the entire state space S constitutes the player's optimal strategy.

Dynamic Programming can be utilized to derive the optimal strategy. The first step is to illustrate the state transition diagram when the player is in state $s_t = i$, and aims at target a。

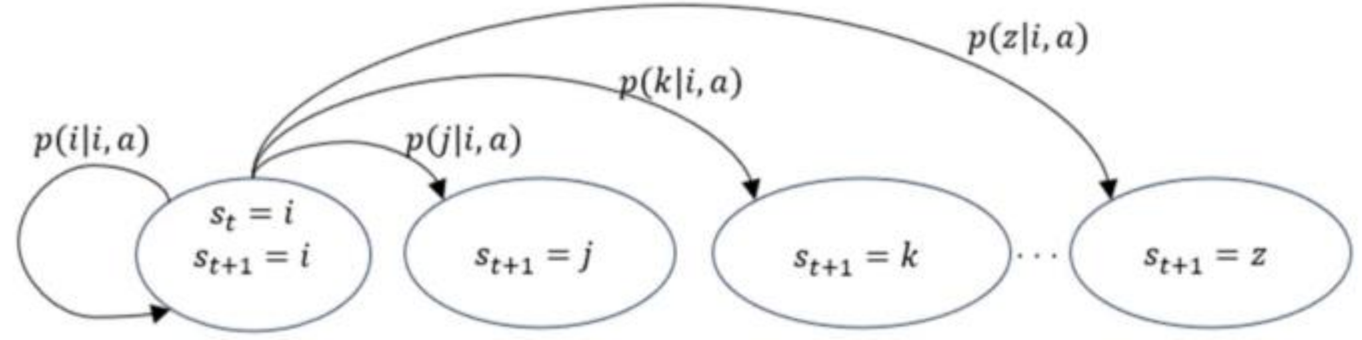


Figure 5: State Transition Diagram for a Player Aiming at Target a in State $s_t = i$

Applying the definition of mathematical expectation,：

$$E[s_t = i, a] = p(s_{t+1} = i | s_t = i, a)(1 + E[s_t = i, a]) + \sum_{j \neq i, j \in S} p(s_{t+1} = j | s_t = i, a) \times E\_min[s_{t+1} = j] \quad (6)$$

4.1 Throwing Strategy Optimization（2）

By reformulating Equation (6), the expectation of the minimum number of throws required for a player to complete the game when in state s_t and taking action a is:

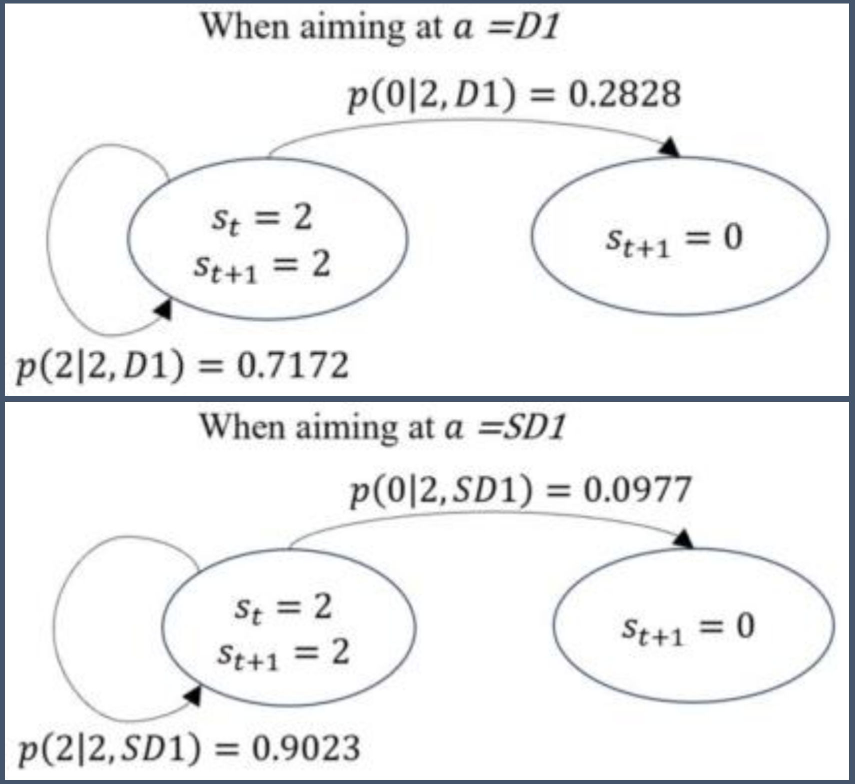
$$E[s_t = i, a] = \frac{p(s_{t+1} = i|s_t = i, a) + \sum_{j \neq \boldsymbol{i}, \boldsymbol{j} \in S}(p(s_{t+1} = j|s_t = i, a) \times E\_min[s_{t+1} = j])}{1 - p(s_{t+1} = i|s_t = i, a)} \qquad (7)$$

written as：

$$E[i, a] = \left(p(i|i, a) + \sum_{j \neq \boldsymbol{i}, \boldsymbol{j} \in S} p(j|i, a) \times E\_min[j]\right) / \left(1 - p(i|i, a)\right) \qquad (8)$$

We firstly calculate $E\_min[2]$.

This Figure right depicts the state transition diagram when this player with $(\sigma_X = 1.8,\ \sigma_Y = 2.2)$ is in state $s_t = 2$ and takes different actions a to transition to the next state $s_{t+1} = 0$.

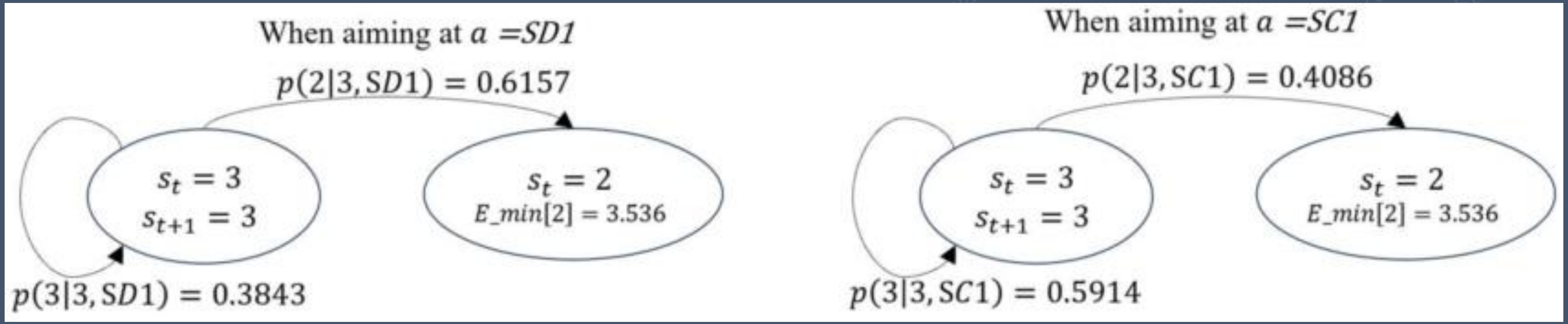


when aiming at *D1*, $E(2, D1) = 3.536$，
when aiming at *SD1*, $E(2, SD1) = 10.235$
Action $a = D1$ is the optimal decision in state $s_t = 2$ for the player, and $E_{min(2)} = E(2, D1) = 3.536$ among the 82 targets.

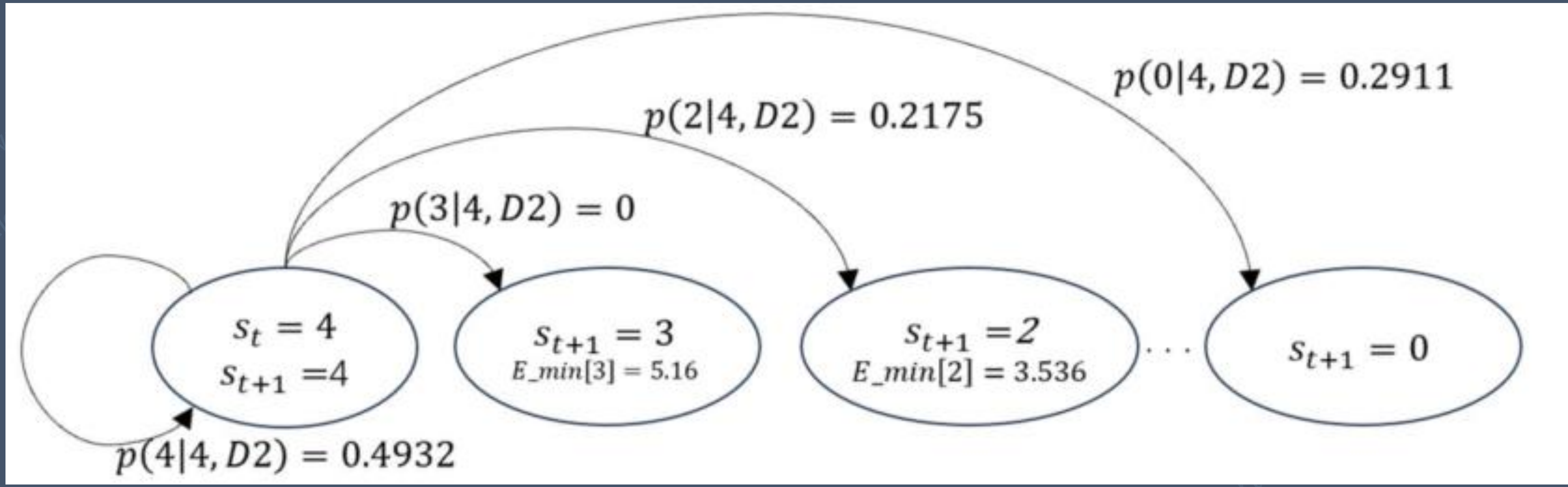
4.1 Throwing Strategy Optimization (3)

Subsequently, by iterating backwards, we can compute the optimal decision for all states,

The player's optimal decision at $s_t = 3$ is $SD1$, with $E\_min[3] = 5.16$.



The player's optimal decision at $s_t = 4$ is $D2$, with $E\_min[4] = 3.478$.



4.1 Throwing Strategy Optimization （4）

A trial calculation using Michael's data yields an expected value of 14.5 throws for optimal game completion, closely approximating his actual performance in 2019 tournament.

Applying the same method, Grandma Wang shows an expected value of 71 throws for game completion, even with consistent concentration and stamina.

The players‘ optimal strategies differ significantly, with only their final-stage decisions (even number scoring target) converging.

	Michael V. Gerwen		Grandma Wang	
S	optimal decision	*E_min*	optimal decision	*E_min*
501	T20	14.5142	SC16	71.0457
500~204	T20	…	SC16	…
500~123	T19~T20	…	SC16	…
122~82	T17~T20	…	SC16	…
82~60	T14~T20	…	SC16,SC11	…
59	SD19	3.0692	SC19	40.9379
58	SD18	3.045	SC13	40.9237
57	SD17	3.0747	SC19	40.9379
56	SD20	2.9762	T8	40.774
55	SD19	3.0247	SC17	40.7332
54	T14	2.9863	T11	40.5589
53	SD17	3.0109	SC2	40.5509
52	SD20	2.8067	SC14	40.3622
51	SD19	2.8563	SC15	40.5869
50	SD18	2.8348	SC11	40.0891
49	SD17	2.8547	SC15	40.4432
48	SD20	2.7801	SC14	40.1702
47	SD19	2.8207	SC15	40.3254
46	SD18	2.8062	T14	40.0954
45	SD17	2.8275	SC16	40.2549
44	SD16	2.7749	T14	39.9323
43	SD15	2.8259	SC19	40.0193
42	SD10	2.8213	T6	39.9147
41	SD13	2.8368	DB	39.9786
40	D20	2.019	D20	38.4819
39	SC19	2.8037	SC19	39.9811
38	D19	2.2728	D19	38.725
37	SD9	2.8208	SC16	39.9076
36	D18	1.9579	D18	37.8867
35	SC19	2.7263	SC7	39.8345
34	D17	2.2717	D17	38.592

	Michael V. Gerwen		Grandma Wang	
S	optimal decision	*E_min*	optimal decision	*E_min*
33	SD17	2.7579	SC19	39.8197
32	D16	1.7776	D16	37.8634
31	SC19	2.7522	SC19	39.7845
30	D15	1.9661	D15	38.3577
29	SD17	2.7687	SC3	39.7002
28	D14	1.742	D14	38.2068
27	SD15	2.7775	SC19	39.6264
26	D13	1.8292	D13	37.71
25	T3	2.7668	T3	39.7535
24	D12	1.9073	D12	37.7735
23	SD7	2.762	T19	39.6309
22	D11	1.7791	D11	37.6871
21	SD5	2.7679	T3	39.34
20	D10	1.7506	D10	38.5093
19	SD3	2.7546	T17	39.4849
18	D9	2.003	D9	38.0932
17	SD1	2.7522	SC13	40.1004
16	D8	1.6798	D8	37.2727
15	SD3	2.7767	SC7	40.182
14	D7	2.2245	T10	38.7079
13	SD1	2.7701	SC9	39.4688
12	D6	1.6912	D6	35.9405
11	SD3	2.9032	T19	40.207
10	D5	2.3478	D5	39.4909
9	SD1	2.9058	T20	40.4486
8	D4	1.8261	D4	34.9834
7	SD3	3.0708	SC5	41.5591
6	D3	2.4361	D3	38.7618
5	SD1	3.055	T1	42.0595
4	D2	1.9856	D2	34.1859
3	SD1	3.1537	T1	43.7396
2	D1	2.0751	D1	34.4826

4.2 Game Rule Optimization

Grandma Wang shows an expected value of 71 throws to complete the game, which is a hard task for the elderly. We tried to explores rule adjustments to help the elderly successfully complete "501" games.

New Rule 1: The last dart no longer needs to be a double; the game ends when the score reaches zero.	The stat space needs to incorporate new states 1: S = {501, 500, 499, ⋯,2, 1, 0} because of the “double-out” rule is removed. Grandma Wang requires 44.17 throws and world champion Michael requires 13.8 throws to complete the game under this new rule.
New Rule 2: The game is completed when the score is reduced to 0 or a negative value. It is not necessary to reduce the score to exactly 0.	The player simply needs to greedily select the target with the highest expected score at each throw, Grandma Wang requires about 35 throws and Michael requires 13 throws.

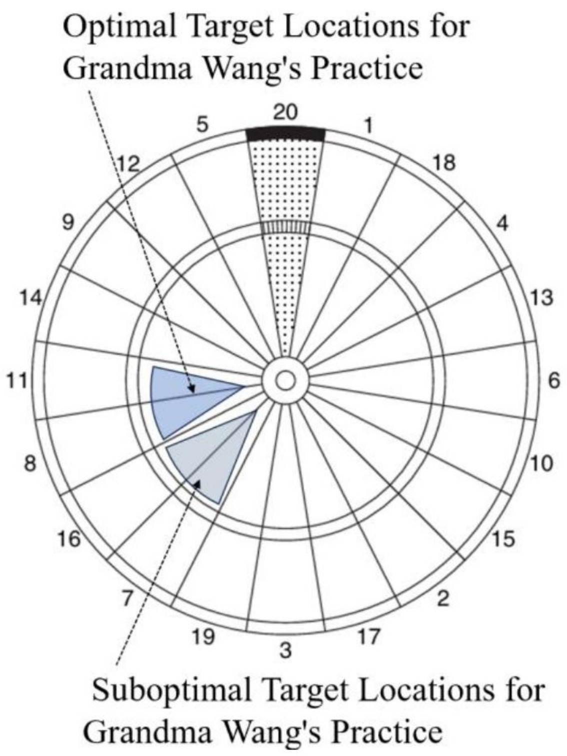
Player	Robot	σ_x	σ_Y	Target	expected score
Michael V. Gerwen	U534	0.9	1.4	$T20$	38
Grandpa Ji	U2998	6.2	9.6	$SC8$	13.71
Aunt Feng	U3503	7.8	8.6	DB	13.64
Aunt Zhang	U3900	9.0	10.0	DB	12.68
Grandma Wang	U2919	6.0	8.8	$SC8$	14.21

4.3 Training Strategy Optimization

It can be observed that Michael's T20 accuracy surpasses that of other high-scoring targets (T19, T18) due to focused practice on this target, which means enhancing accuracy on specific targets can lead to better results.

Assuming Grandma Wang's standard deviations at a specific location a are reduced by 20% ($\sigma_X = 4.8$, $\sigma_Y = 7.0$), after intensive training, with accuracy elsewhere unchanged, applying the methodology for New Rule 1 yields the following result.

Target for practice	Expectation of the Minimum Number of Throws	Decrease in the Number of Throws
$SC8$	41.7	2.4
$SC11$	41.8	2.4
$SC16$	42.2	2.0
$SC7$	42.4	1.8
$SC14$	43.0	1.2
$T11$	43.1	1.1
$SC19$	43.2	1.0
$T19$	43.8	0.4
$T18$	44.0	0.2
$T20$	44.0	0.1
DB	44.1	0.0



- Practicing on all targets does not uniformly contribute to improving Grandma Wang's performance
- Intensive practice on high-scoring targets such as T20, T19, T18, and DB yields negligible improvement.
- Selecting positions like SC8 and SC11 for practice, and leveraging these positions during the game, proves to be more beneficial.

5. Conclusion

This research originated from Beijing No. 101 High School's darts society, addressing the difficulties promoting dart-based fitness for the elderly.

Main Contributions:

① Developed an MDP-based dart throwing strategy model.
② Solving the transition probabilities of the MDP using the polar coordinate form of the bivariate normal distribution.
③ Estimated elderly dart throwing accuracy using a nearest neighbor approach.
④ Proposed three optimization strategies for elderly dart players: target selection, rule adjustment, and personalized training plans.

All above helped the elderly lower the learning curve and maintain participation in this sport.

We also discovered that Darts is accessible even for those with greater care needs (e.g., wheelchair users, bedridden individuals), and these strategies can be applied to them.

One more thing

Calculate $P(target = a, hit = h)$ in 3 ways.

Method1：
Using the integral2() function from the pracma library in R language.
FAILED

Method2： python™
Using the dblquad () function from scipy.integrate library in Python.
Done

Method3： python™
Using the random () function from numpy library to generate a set of random numbers following a two-dimensional normal distribution, simulating the landing positions of the robot after throwing the dart.
Done

Bug report for the integral2 function in the R package pracma

Reporter :Yunshan Gong Email: yunshan.gong@outlook.com Date: 2024-11-20

Problem Description

When using integral2 for double integration over a specific range, the function will halt under certain conditions.

Reproducible Example

```
library(pracma)

fp <- function(d,r){
    rt = 1/(sqrt(2*pi)*e1)*exp(-1*(r*cos(d)-u1)^2/(2*(e1^2)))
    rt = rt*1/(sqrt(2*pi)*e2)*exp(-1*(r*sin(d)-u2)^2/(2*(e2^2)))
    rt = rt*r
    return(rt)
}
u1=0; u2=7.5; e1=1.5; e2=8.6;
d1=2.67; d2=2.985; r1=2.1; r2=10.1;
val=integral2(fp,d1,d2,r1,r2);
```

Output and error message:

```
Error in if (adjerr[1] > localtol) { :
  missing value where TRUE/FALSE needed
Calls: integral2 -> .save2list
Execution halted
```

Session Information

```
> sessionInfo()
R version 4.4.2 (2024-10-31 ucrt)
Platform: x86_64-w64-mingw32/x64
Running under: Windows 11 x64 (build 22631)

Matrix products: default


locale:
[1] LC_COLLATE=Chinese (Simplified)_China.utf8
[2] LC_CTYPE=Chinese (Simplified)_China.utf8
[3] LC_MONETARY=Chinese (Simplified)_China.utf8
[4] LC_NUMERIC=C
[5] LC_TIME=Chinese (Simplified)_China.utf8

time zone: Asia/Shanghai
tzcode source: internal

attached base packages:
[1] stats     graphics  grDevices utils     datasets  methods   base

other attached packages:
[1] pracma_2.4.4

loaded via a namespace (and not attached):
[1] compiler_4.4.2
```

HW Hans W<hwborchers@gmail.com>
收件人: 你
周五 2024/11/22 15:27

Dear Yunshan Gong:

Thank you for your careful bug report.
Unfortunately, there will be no more updates to the 'pracma' package.
Save your corrections for yourself (say, as `integral2c()`) and use this version in the future.

Very best wishes
Hans Werner

答复 转发

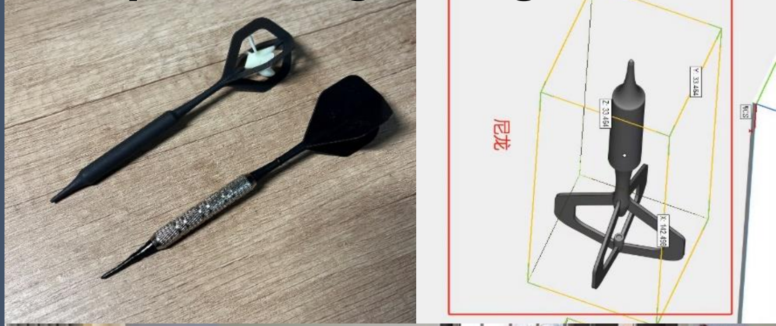
Author:Hans W. Borchers
Former scientist at ABB Research Center

Companionship is always the best strategy.

$$Strategy_{Best} = \underset{strategy}{argmax} \sum_{a \in Strategy} Time\ together\ (a)$$

Thanks！

3D-printed lightweight darts



Icon-assisted aiming