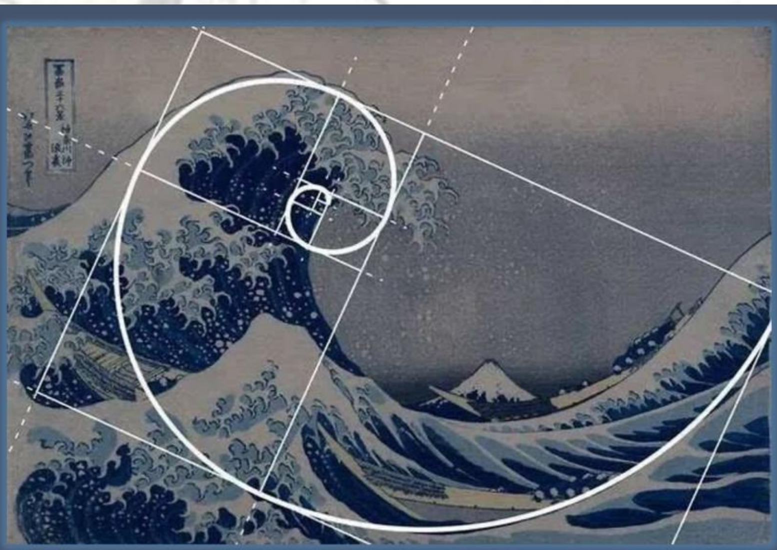


Math + Art

Lecturer: Vicky Gong
Instructor: Yushan Lai
Final Assessment: Completion of an original final project
Time: Tuesdays, 2:30 PM - 4:20 PM

Course Information

Course Title: The Art of Logic: Math + Art

Lecture: Vicky Gong, Class 18, Grade 12 (15611662317)

Special Classroom Requirements: None

Class Size: 15 ~30 students

Instructor: Yushan Lai

Final Assessment: Completion of an original final project with art and math

Course Description

This course aims to cross disciplinary boundaries to explore how the logic, structure, and patterns of mathematics blossom into astonishing beauty in artistic creation. From the harmonious classicism of the Renaissance to the visual magic of modern logic and the infinite mysteries of modern science, we will unveil the mathematical codes hidden behind great works of art. Ultimately, we will apply the knowledge gained to create our own "mathematical art" pieces.

Syllabus

- **Lesson 1: Course Introduction: All is Number**
    - **Content:** Introduce course goals and structure. Through patterns in nature like the nautilus shell and sunflowers, we will begin to explore the universality of mathematical patterns, raising the core question: How does rational mathematics create the ultimate in emotional beauty?
- **Lesson 2: The Order of the Cosmos and the Poetry of the Human Body**

- **Theme:** The Charm of the Golden Ratio.
    - **Content:** In-depth analysis of Leonardo da Vinci's "Vitruvian Man," studying how he combined ancient Roman theory with the Golden Ratio to express the "ideal human form" mathematically.
    - **Activity:** Discovering the "Golden Code" in art and life.
- **Lesson 3: Nature's Design Language: The Fibonacci Sequence**
    - **Theme:** The relationship between the Fibonacci sequence and the Golden Ratio.
    - **Content:** Learn the definition of the Fibonacci sequence and explore its mathematical connection to the Golden Ratio. Analyze Fibonacci patterns in natural phenomena like plant growth and petal arrangements.
- **Lesson 4: The Geometric Mystery of the Mona Lisa's Smile**
    - **Theme:** Golden Rectangle and Compositional Analysis.
    - **Content:** Learn to draw Golden Rectangles and Golden Triangles. Focus on the "Mona Lisa" to analyze the hidden Golden Rectangles and spirals in the painting.
- **Lesson 5: From Temples to Logos: Applications of the Golden Ratio**
    - **Theme:** Extended Applications of the Golden Ratio.
    - **Content:** Analyze classical architecture like the Parthenon and modern designs like the Apple logo to understand the timeless appeal of the Golden Ratio.
    - **Activity:** Try re-composing a favorite photo using the Golden Ratio.
- **Lesson 6: Creating a 3D Illusion on a 2D Plane**
    - **Theme:** The Geometry of Linear Perspective.
    - **Content:** Introduce Brunelleschi's experiments with linear perspective. Learn core geometric principles like the horizon line, vanishing points, and orthogonal lines.
    - **Activity:** Practice drawing a cube in single-point perspective.
- **Lesson 7: The Extension of Space: Two- and Three-Point Perspective**
    - **Theme:** Principles of Multi-Point Perspective.
    - **Content:** Introduce two-point and three-point perspective, understanding how they represent more complex spatial angles. Appreciate and analyze the use of space in Raphael's "The School of Athens."
- **Lesson 8: A Master's Practice: Comprehensive Analysis of <The Last Supper>**

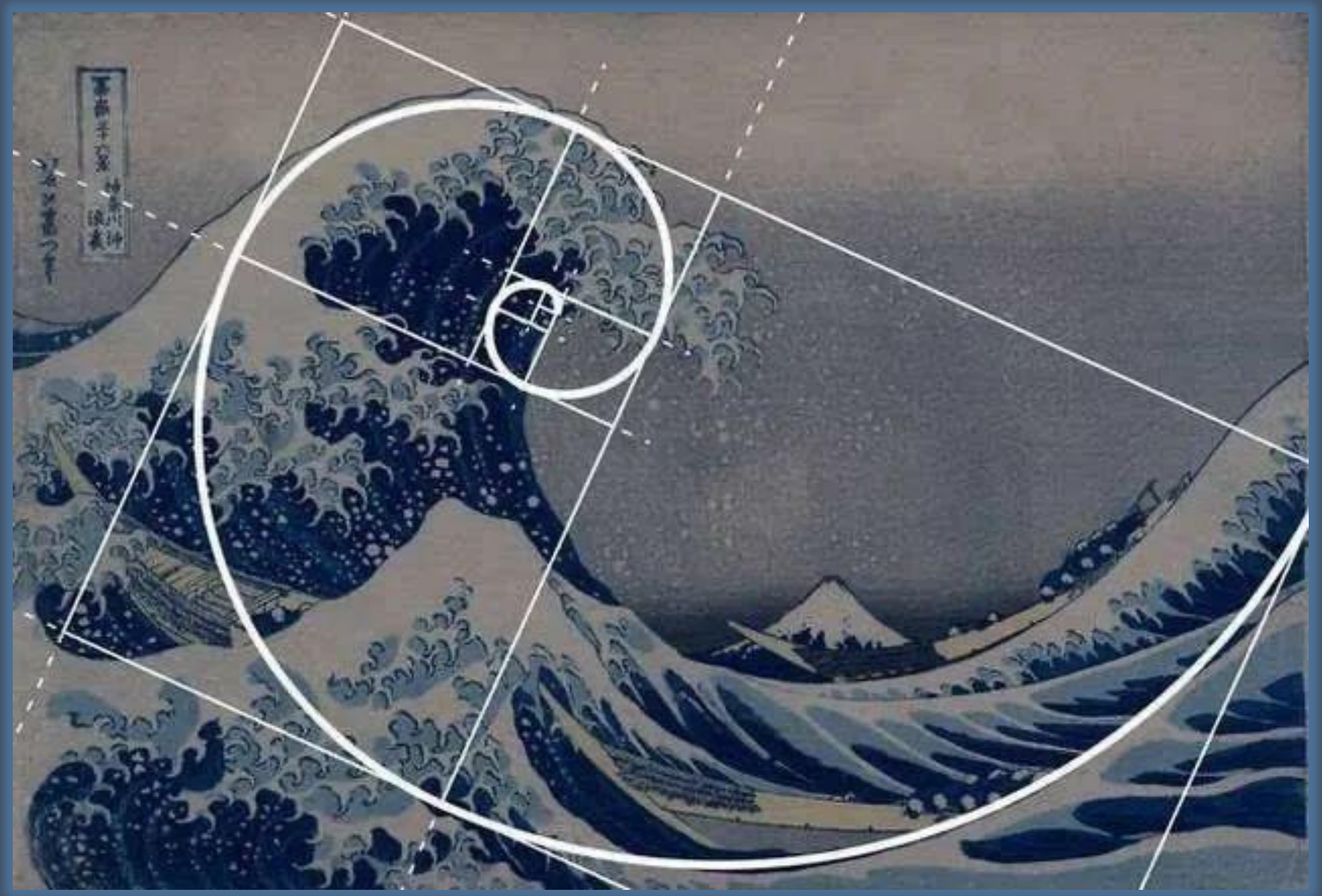
    - **Theme:** Integrated Application of Perspective.
    - **Content:** A comprehensive analysis of single-point perspective, atmospheric perspective, and Golden Ratio composition in Leonardo da Vinci's "The Last Supper."
    - **Activity:** Kick off the "My Ideal Room" perspective drawing project.
- **Lesson 9: Part One Creative Workshop & Summary**
    - **Activity:** Complete the "My Ideal Room" perspective drawing. Summarize the first part of the course.
- **Lesson 10: The Infinite Chessboard: Welcome to the World of Tessellation**
    - **Theme:** What is a Tessellation?
    - **Content:** Appreciate M.C. Escher's works like "Sky and Water" and "Metamorphosis." Learn the mathematical definition of tessellation and explore why only equilateral triangles, squares, and regular hexagons can form regular tessellations.
    - **Activity:** Practice regular polygon tessellation with paper cutting or drawing.
- **Lesson 11: The Language of Symmetry: Mathematical Laws Behind Patterns**
    - **Theme:** Translation, Rotation, Reflection, and Symmetry Groups.
    - **Content:** Learn the three basic symmetry transformations for creating tessellations: translation, rotation, and reflection. Casually introduce the idea of classifying symmetry.
- **Lesson 12: From Rigid to Vivid: Escher's Metamorphosis**
    - **Theme:** Design Principles of "Escher-style" Tessellations.
    - **Content:** Learn how Escher transformed simple geometric shapes into vivid fish, birds, or lizards through "cutting and pasting" modifications.
    - **Activity:** Design a simple tessellation unit.
- **Lesson 13: Tessellation Design Workshop (I)**
    - **Theme:** Refining the Tessellation Unit.
    - **Content:** Building on the previous lesson, refine your own tessellation unit design.
- **Lesson 14: Tessellation Design Workshop (II)**
    - **Theme:** Repetition and Creation.
    - **Content:** Learn to repeat a single design unit to fill the paper, then add color and details.

- **Lesson 15: Inception: Impossible Architecture**
    - **Theme:** Impossible Spaces and Visual Paradoxes.
    - **Content:** Appreciate Escher's "Waterfall" and "Ascending and Descending." Analyze the construction principles of "impossible figures" like the Penrose Stairs and Penrose Triangle.
- **Lesson 16: Deceiving Your Eyes: The Art of Optical Illusions**
    - **Theme:** From Escher to Contemporary Optical Illusion Art.
    - **Content:** Explore how artists use principles of perspective and psychology to "trick" the brain. Introduce other types of optical illusion artworks.
- **Lesson 17: Modern Echoes: From <Monument Valley> to <Inception>**
    - **Theme:** The application of Escher's concepts in modern culture.
    - **Content:** Discuss how modern works like the game "Monument Valley" and the film "Inception" have borrowed from Escher's ideas.
- **Lesson 18: Part Two Creative Sharing & Summary**
    - **Activity:** Share your tessellation projects; summarize the key concepts from the second part of the course.
- **Lesson 19: The Pattern That Never Repeats: An Intro to Penrose Tiling**
    - **Theme:** An Introduction to Penrose Tiling.
    - **Content:** Introduce the story of Sir Roger Penrose's discovery. Learn about the "darts" and "kites" that form Penrose tilings and their unique matching rules.
- **Lesson 20: Penrose Tiling Workshop**
    - **Activity:** Use pre-made "dart" and "kite" paper cutouts to experience the tiling process.
- **Lesson 21: From a Math Game to a Nobel Prize: Quasicrystals**
    - **Theme:** Penrose Tiling and Quasicrystals.
    - **Content:** Tell the story of how Dan Shechtman discovered quasicrystals and their consistency with Penrose tilings. Discuss the "useless usefulness" of mathematics.
- **Lesson 22: Entering the World of Fractals: Coastlines, Snowflakes, and Broccoli**
    - **Theme:** Fractal Art and Self-Similarity.
    - **Content:** Introduce the fractal characteristics of "self-similarity" and "infinite detail" through

examples in nature like coastlines, lightning, and broccoli.

- **Lesson 23: The Koch Snowflake and the Sierpinski Triangle**
    - **Theme:** Iterative Generation of Classic Fractals.
    - **Content:** Learn the mathematical construction process for classic fractals like the Koch snowflake and the Sierpinski triangle, understanding the concept of "iteration."
- **Lesson 24: The Computer-Drawn Demon: The Mandelbrot Set**
    - **Theme:** Exploring the Most Famous Fractal.
    - **Content:** Introduce the Mandelbrot set, appreciating its infinitely complex and beautiful details. Briefly touch upon the concept of the complex plane behind it.
- **Lesson 25: Part Three Summary & Project Announcement**
    - **Activity:** Review the concepts of aperiodic tiling and fractals, and introduce the requirements for the final project in Part Four.
- **Lesson 26: Project Kick-off: Choose Your Creative Track**
    - **Theme:** Final Project Introduction and Brainstorming.
    - **Content:** Introduce creative tracks like fractal art, 3D mathematical models, anamorphic art, and generative art code. Brainstorm and determine initial project directions.
- **Lesson 27: Skills Lab (I): Static Visual Creation**
    - **Content:** Group instruction. One group learns to use an online fractal generator; the other learns advanced techniques for drawing Anamorphic Art.
- **Lesson 28: Skills Lab (II): 3D and Dynamic Creation**
    - **Content:** Group instruction. One group learns to create 3D mathematical models like polyhedra and Möbius strips; the other gets a simple introduction to coding with Processing/P5.js to generate geometric patterns.
- **Lesson 29: Project Planning and Mid-Point Check-in**
    - **Activity:** Refine project proposals and discuss them with the class.
- **Lessons 30-32: Immersive Creation Workshop**
    - **Theme:** Project Implementation and Guidance.
    - **Activity:** Hands-on creation based on project plans.

- **Lesson 33: Final Touches and Artist's Statement**
  - **Theme:** Project Completion and Elaboration.
  - **Content:** Add the final touches to, frame, or debug the projects.
- **Lesson 34: Setting Up the Exhibition**
  - **Theme:** Exhibition Preparation.
  - **Activity:** Transform the classroom into a "Math and Art" exhibition space.
- **Lesson 35: Course Exhibition and Project Showcase**
  - **Theme:** Presentation and Exchange.
- **Lesson 36: Course Summary and Reflection**
  - **Theme:** What Have We Gained?



Math + Art

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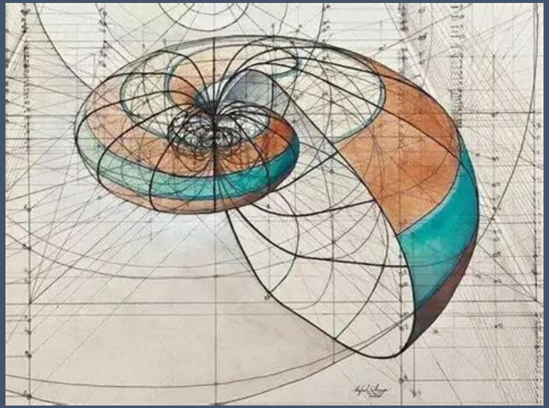
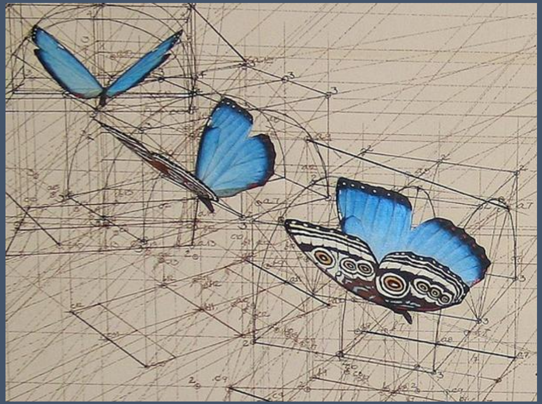
About This Course



1. Break down disciplinary barriers: Learn to appreciate the artistic beauty in mathematical structures and understand the logical essence behind works of art.
2. Inspire interdisciplinary thinking: Cultivate a unique perspective that can transcend the boundaries between liberal arts and sciences and integrate analysis and creativity.
3. Provide a creative practice platform: Encourage students not only to stop at appreciation, but also to take practical actions to transform mathematical concepts into original works of art.

The Aesthetics of Rationality: The Interplay Between Mathematics and Art

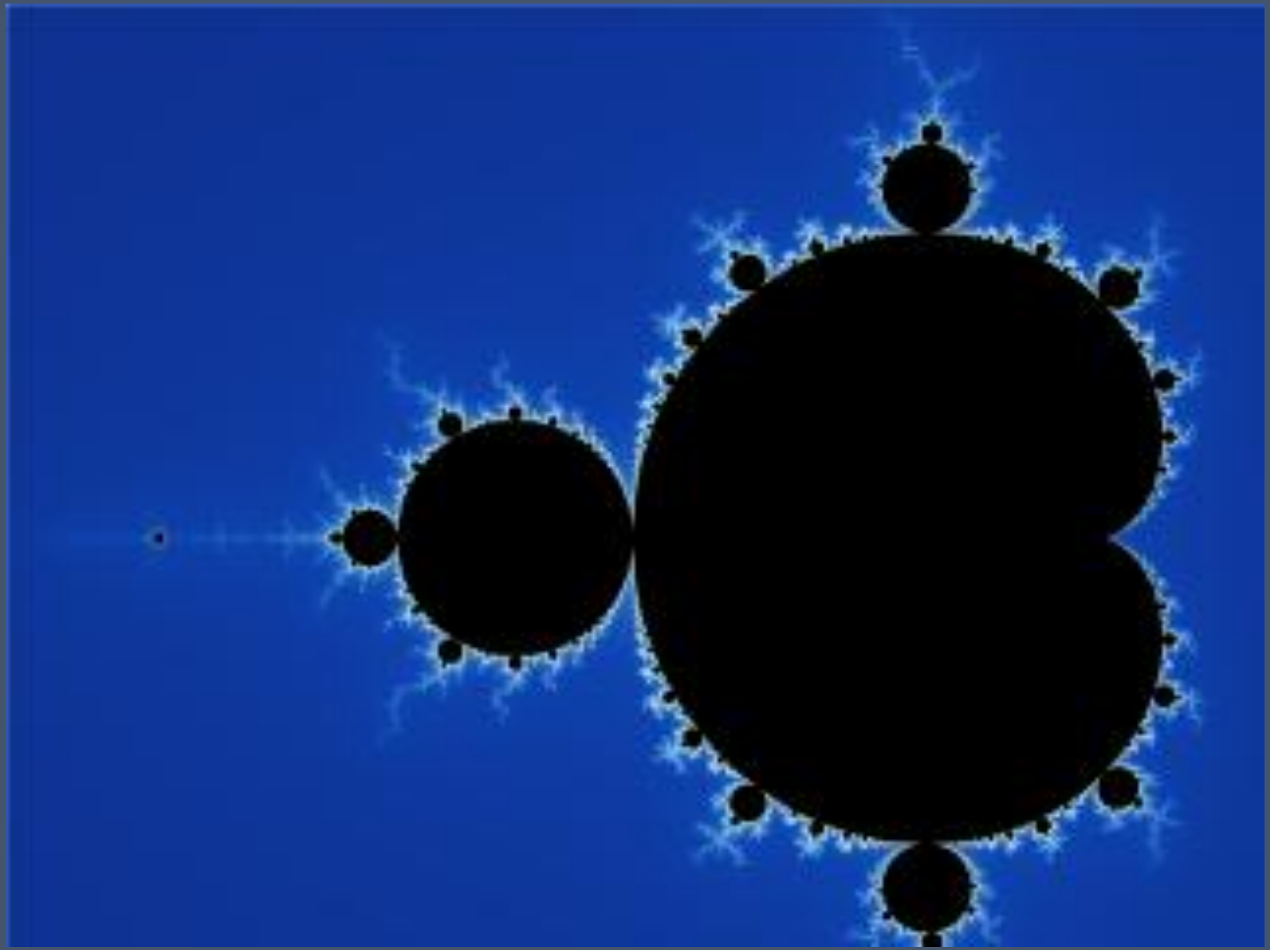


Content of This Session: Famous Examples of Fractals in Mathematics:

Koch Snowflake

Decorative Floor Tiles

Mandelbrot Set



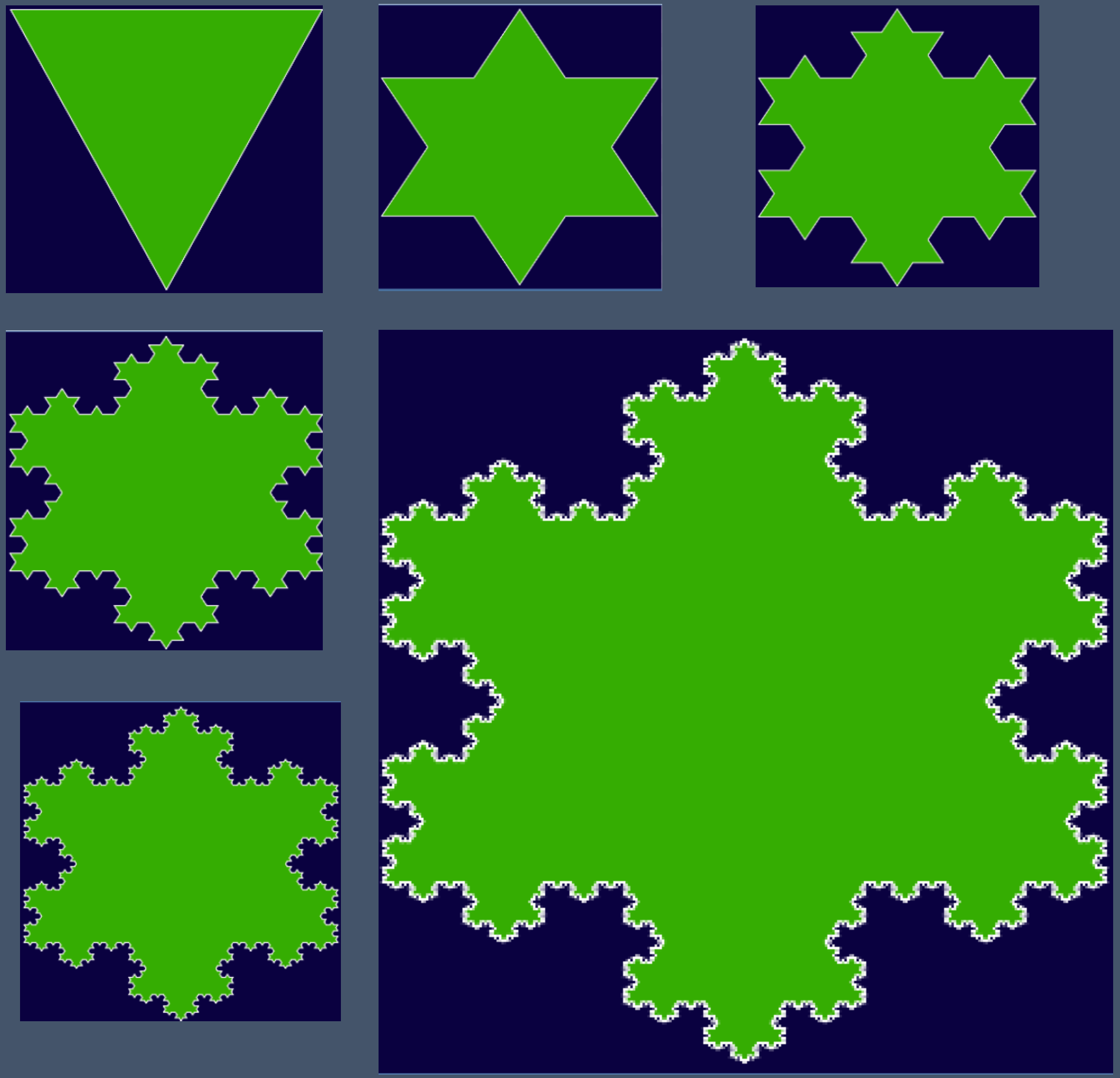
Fantasy scenes in games and films (for atmosphere enhancement)

Koch Snowflake

https://onlinetools.com/math/generate-koch-snowflake

Definition of the Koch Snowflake

The Koch Snowflake, also known as the Koch Star or Koch Island, is one of the most famous fractals in mathematics. It is a curve with infinitely complex details generated through simple geometric iteration rules.
Essentially, it is a continuous but nowhere differentiable curve—meaning it is a complete, unbroken line, yet every point on it is filled with infinite sharp corners, making it impossible to draw a unique tangent at any point.



Construction Method

The generation process of the Koch Snowflake is intuitive and follows an infinitely repeatable recursive rule:

1. Initial State (n=0): Start with a standard equilateral triangle.
2. First Iteration (n=1): Divide each side of the triangle into three equal parts.
3. Iteration Rule:
    1. Remove the middle segment of each side.
    2. Using the removed segment as the base, construct a new equilateral triangle outward, with a side length of 1/3 of the original side.
4. Infinite Repetition: Repeat Step 3 for all newly formed segments (there are 12 sides total after the first iteration). This process can continue indefinitely.

As the number of iterations increases, the figure's boundary becomes increasingly complex and delicate, eventually forming the Koch Snowflake.

Koch Snowflake

Brief History of the Koch Snowflake

In the 19th century, mathematicians generally believed that continuous curves must be smooth (i.e., differentiable) at most points. However, in 1872, the German mathematician Karl Weierstrass proposed a surprising counterexample: a function that is continuous everywhere but differentiable nowhere. While valid in analysis, this function was difficult to visualize geometrically. Such counterintuitive curves were dubbed "Pathological Curves" or "Mathematical Monsters" by some mathematicians at the time.

The Swedish mathematician Niels Fabian Helge von Koch was fascinated by these peculiar curves. He sought a more intuitive, understandable geometric construction method than Weierstrass' function.

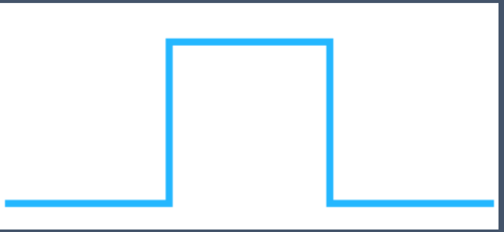
> *1904: Von Koch published his landmark paper Sur une courbe continue sans tangente, obtenue par une construction géométrique élémentaire (On a Continuous Curve Without Tangents, Obtained by an Elementary Geometric Construction).*

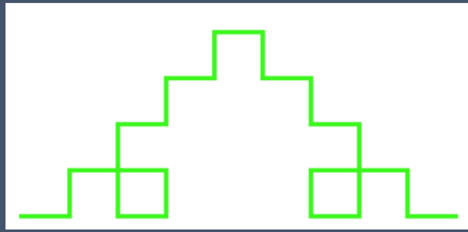
The Koch Snowflake has many variant forms.

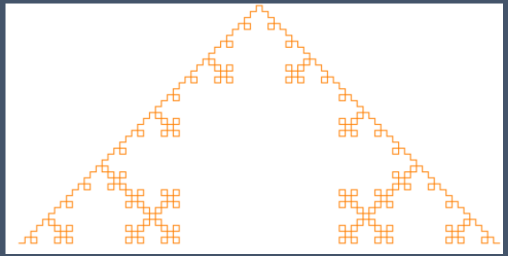
Variant Form: Square Koch Fractal Curves

https://demonstrations.wolfram.com/SquareKochFractalCurves/

The Quadratic Koch Curve—sometimes called the Box Fractal—is a classic variant of the Koch Snowflake. Its construction rules are nearly identical to the standard Koch Snowflake, but the "generator" shape is a square instead of an equilateral triangle.







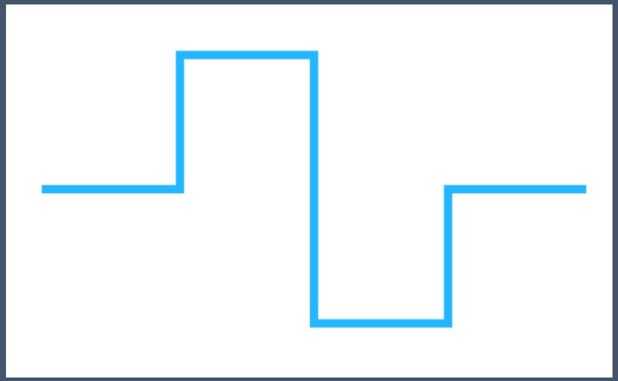
Koch Snowflake

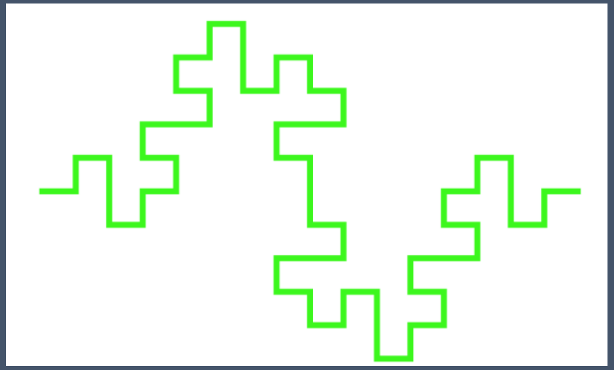
Properties of the Koch Snowflake

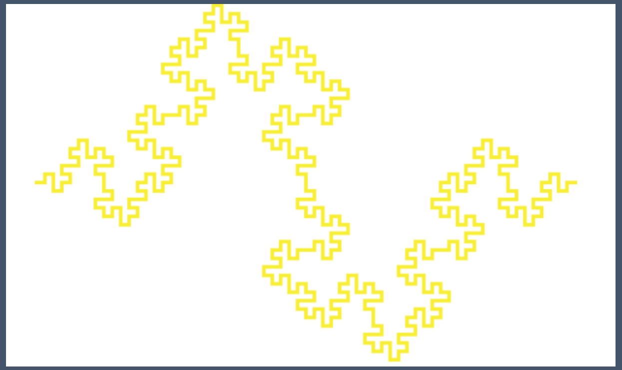
Property	Description
Self-Similarity	Any part of the figure, when magnified, matches the overall structure.
Infinite Perimeter	The boundary length is infinite; each iteration multiplies the length by 4/3.
Finite Area	The enclosed area is finite, eventually converging to 1.6 times the initial area of the equilateral triangle.
Continuous but Nowhere Differentiable	The curve is unbroken, but it is too "sharp" at every point to be differentiated.

Another Type of Quadratic Koch Curve

Please Draw It Yourself (for Students)







Koch Snowflake

https://onlinetools.com/math/generate-koch-snowflake

3D **Koch Snowflake**

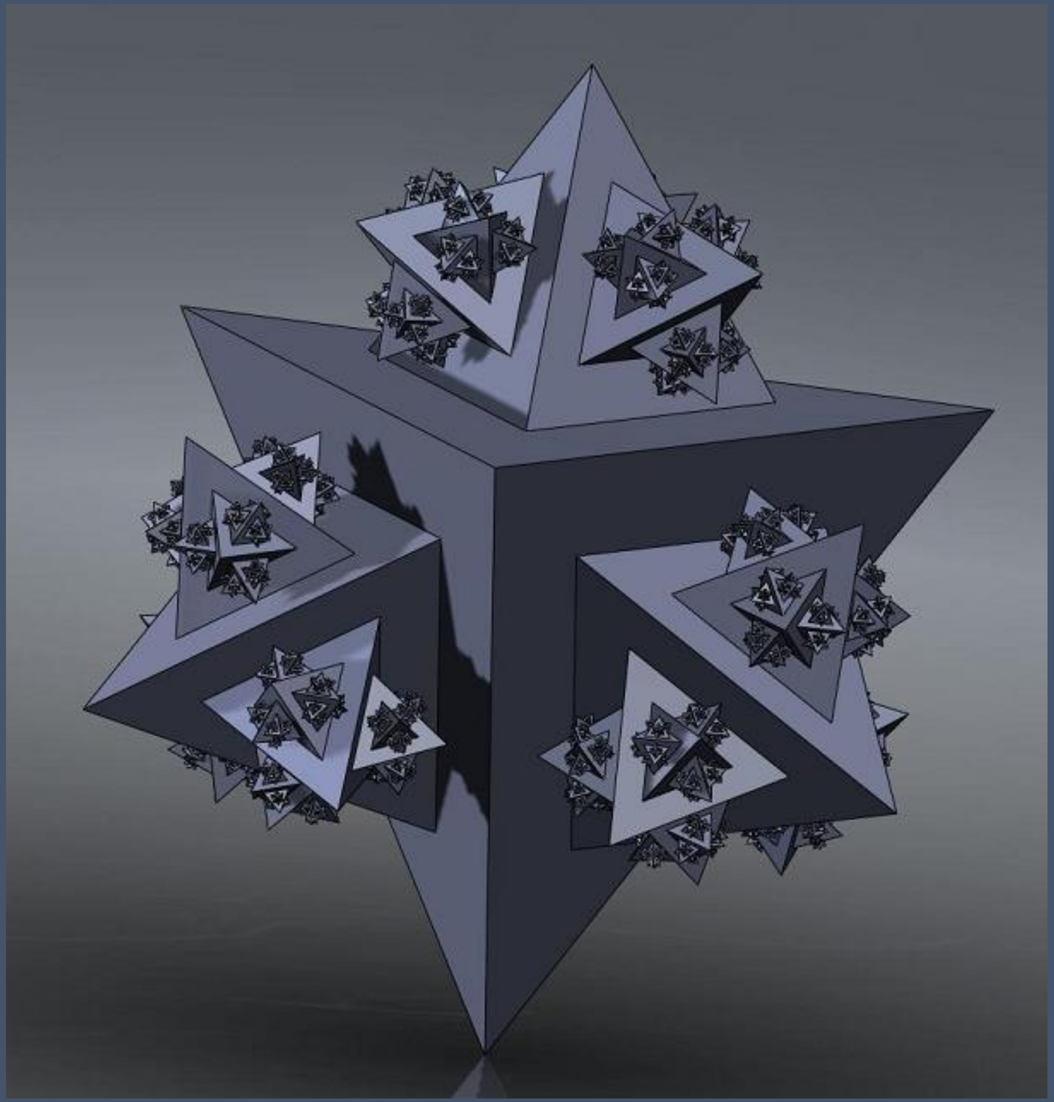
Also often called the Koch Surface, it extends the iterative logic of the 2D Koch Snowflake to 3D space. Its construction and properties are equally remarkable.

Construction Rules

The most classic method to construct a 3D Koch Snowflake is as follows:

1. Initial State: Start with a regular tetrahedron (the simplest 3D convex polyhedron, composed of four equilateral triangles).
2. Iteration Rule:
    1. At the center of each triangular face of the initial tetrahedron, "attach" a smaller, proportionally scaled-down regular tetrahedron.
    2. Typically, the side length of the new small tetrahedron is half the side length of the original triangular face.
3. Infinite Repetition: Repeat Step 2 for all newly formed, exposed triangular faces.

As iterations increase, the 3D object’s surface becomes extremely complex and delicate, filled with countless tiny spines, forming a three-dimensional snowflake-like structure.

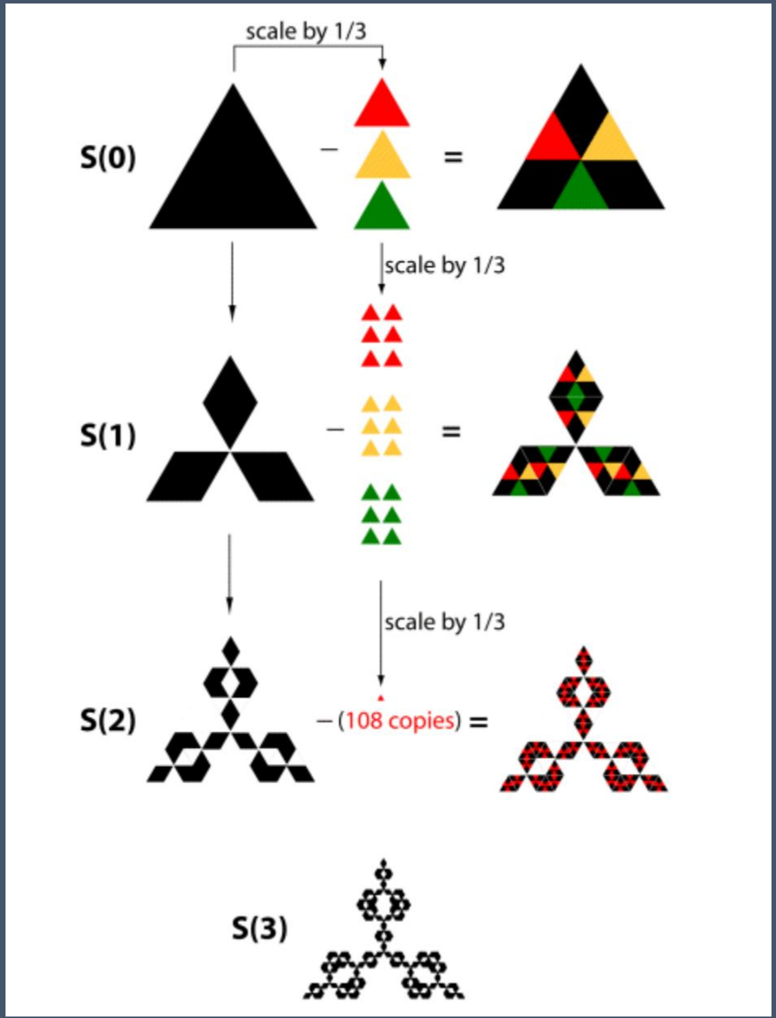


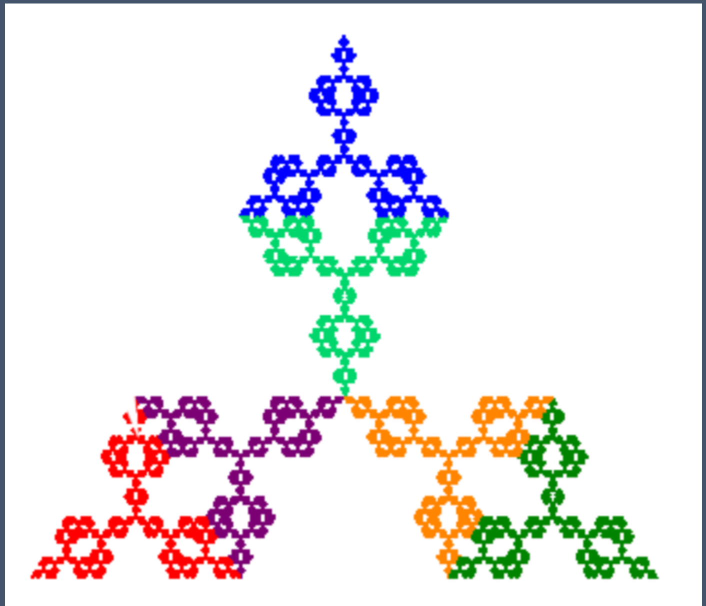
Koch Snowflake

Koch Antisnowflake

https://larryriddle.agnesscott.org/ifs/ksnow/kantisnow.htm

Modification Method: Construct the generator's triangle inward instead of outward.
Characteristics: Instead of growing outward, it continuously "erodes" the initial triangular area. Eventually, it forms a figure with an extremely complex boundary and numerous internal holes. Its perimeter is also infinite, but its area converges to a value smaller than the initial triangle.



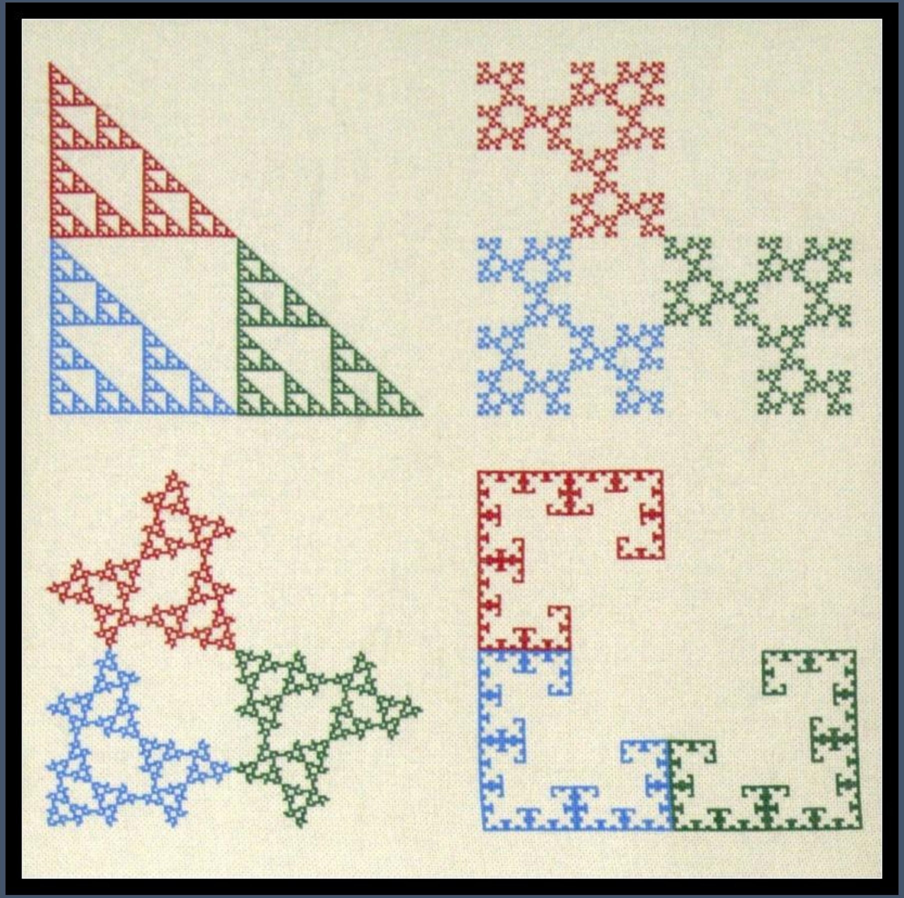


https://larryriddle.agnesscott.org/ifs/ksnow/constAnimationAnti.htm

Relevant Artistic Creations

Cross-Stitch

"The generation of fractals can be an artistic attempt, a mathematical exploration, or simply a soothing pastime."— Kerry Mitchell, The Fractal Art Manifesto

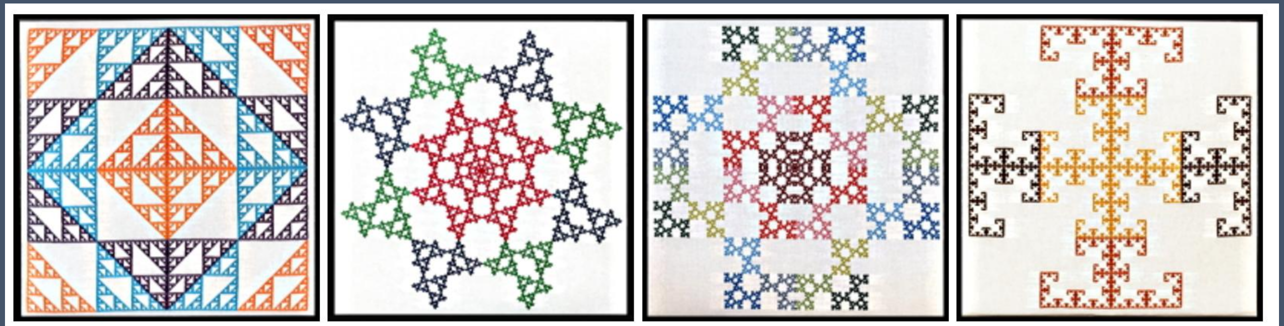


Sierpiński Theme and Variations (Cross-Stitch)

The Sierpiński Triangle is a fractal generated by:

1. Dividing a square into four equal sub-squares;
2. Removing the top-right sub-square;
3. Repeating this process recursively on the remaining three sub-squares.

This is our "theme." The "variations" are created by leveraging the square’s symmetry.



https://larryriddle.agnesscott.org/artwork/artwork.htm?utm_source=chatgpt.com

Relevant Artistic Creations : Carpet

Sierpiński Carpet (Digital Printing and Backstitch)

The Sierpiński Carpet is constructed as follows:

1. Start with a solid square;
2. Subdivide the square into nine congruent sub-squares and remove the central one;
3. Recursively apply this process to the remaining eight sub-squares (the recursion is infinite).

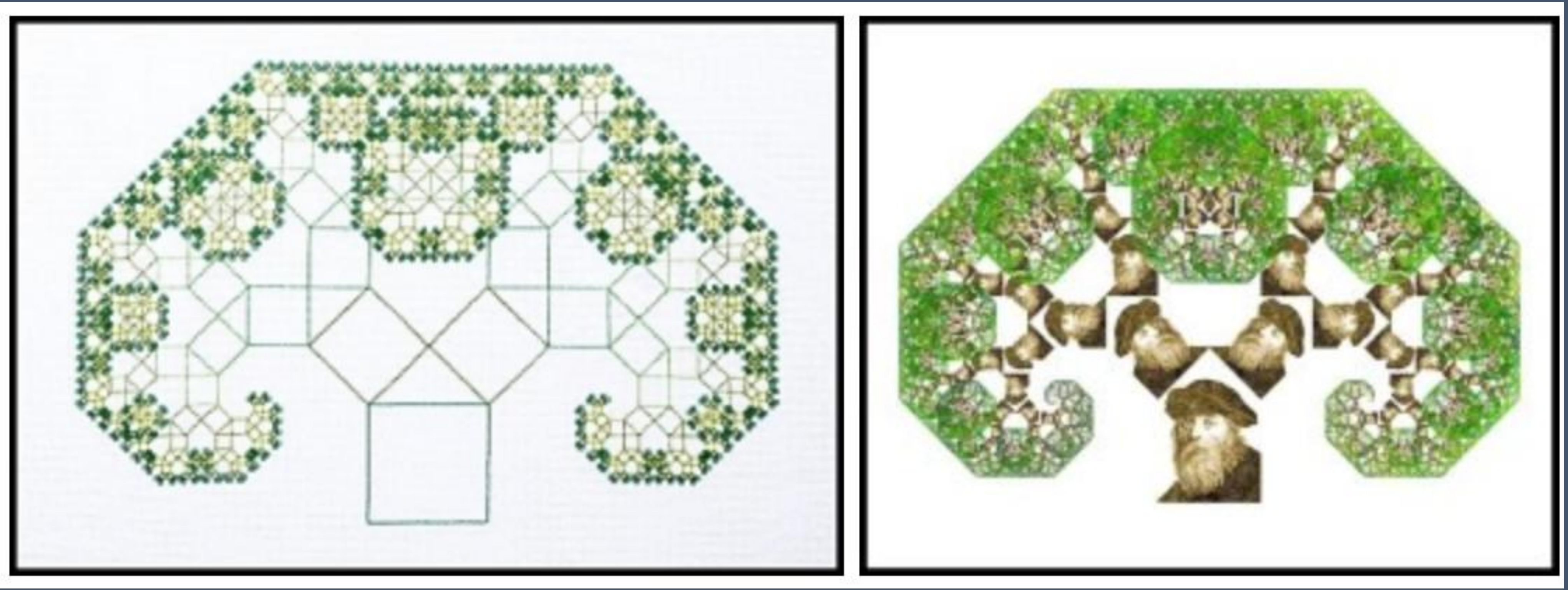
https://larryriddle.agnesscott.org/artwork/artwork.htm?utm_source=chatgpt.com

Relevant Artistic Creations : Pythagorean Tree (Backstitch and Digital Printing)

The traditional Pythagorean Tree fractal is constructed by:

1. Starting with a square;
2. Constructing two smaller squares that form a right triangle with the original square (along the hypotenuse);
3. Recursively applying this construction to the two smaller squares.

- The backstitch pattern on the left shows the first 10 iterations (on an 18-count canvas, 14 inches × 11 inches).
- The stitching uses different shades of green for each new square added in iterations. The smallest set of squares is colored dark green to demonstrate the Lévy Dragon at the iteration limit.



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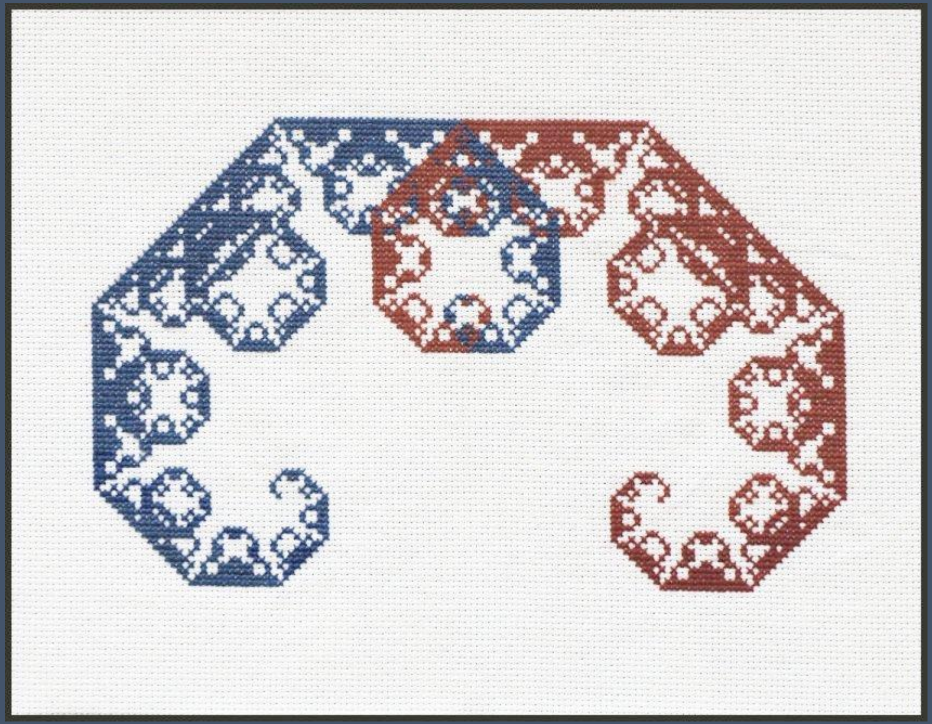
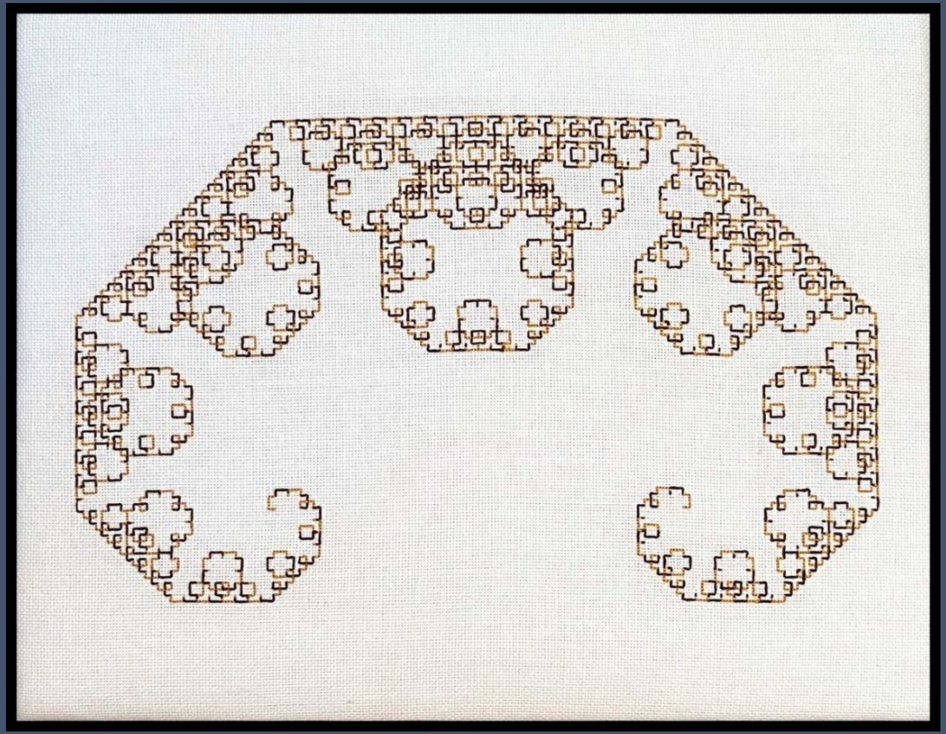
Relevant Artistic Creations : Lévy Dragon (Cross-Stitch and Backstitch)

The fractal now known as the "Lévy Dragon" was first studied by Paul Lévy in 1938—long before computers could generate images or the term "fractal" was invented!

1. The first image is a dragon embroidered with cross-stitch on a 16-count canvas (10 inches × 8 inches). Two colors illustrate how the dragon is composed of two smaller self-similar copies.
2. The second image is the 12th iteration of the dragon curve, embroidered with backstitch on a 22-count canvas (14 inches × 11 inches). The curve consists of identical "staple-like" blocks, alternating between two colors to make it easier to trace the continuous path from the left start to the right end.





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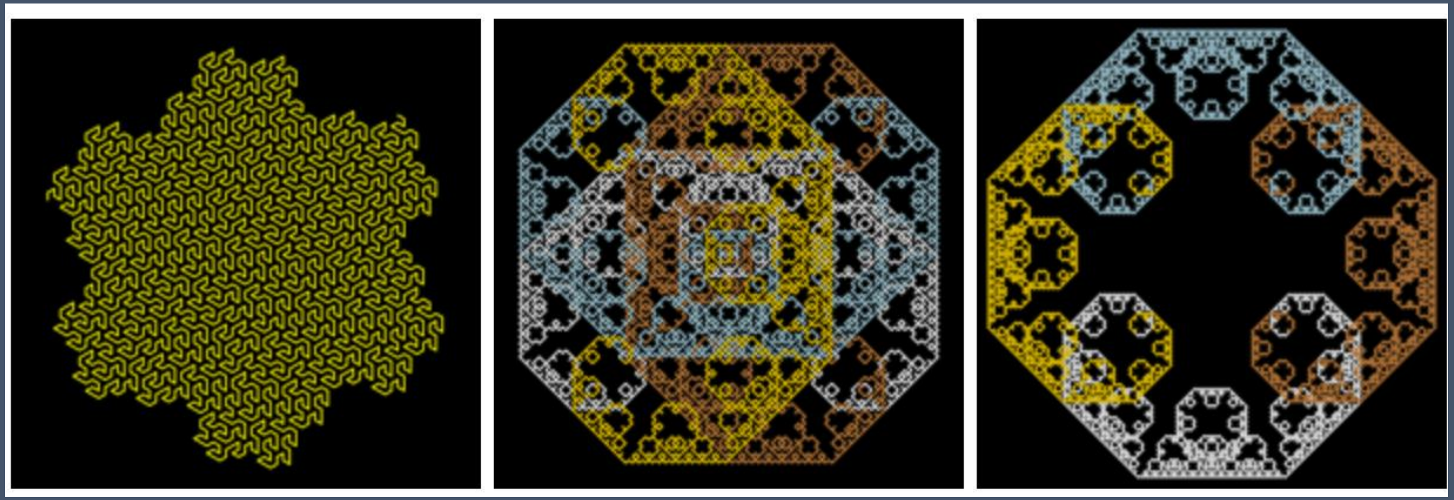
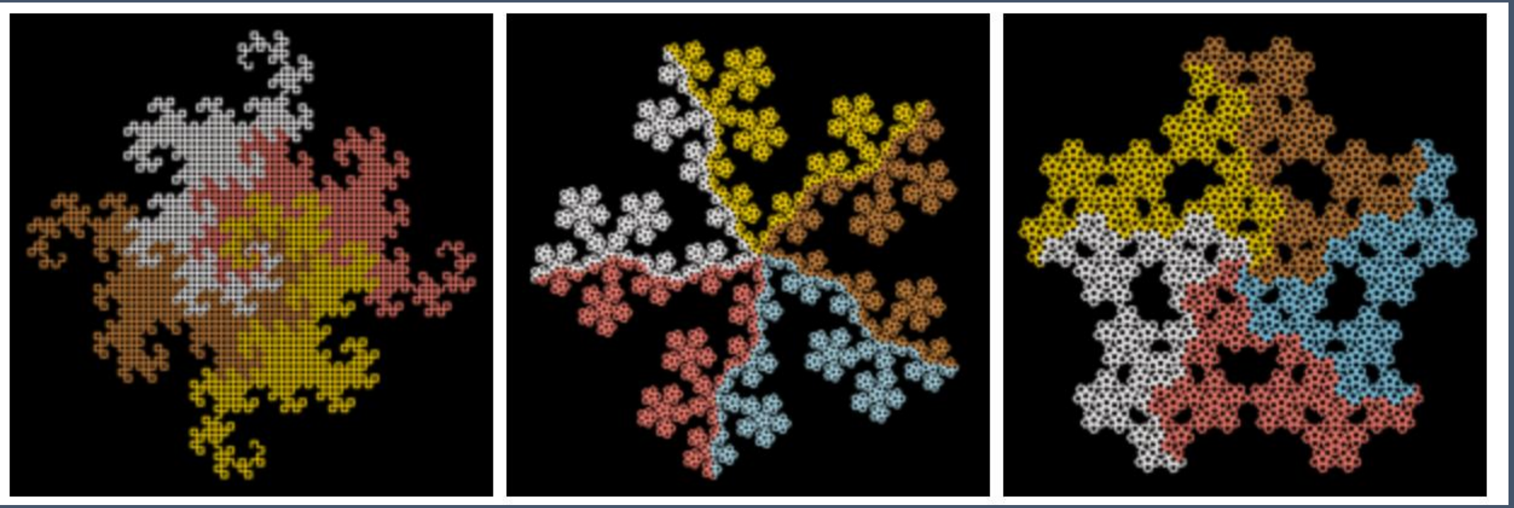
Koch Snowflake: Computer Program Implementation

- L-System Fractals Drawn Using ColabTurtlePlus





Attribution: Lawrence H. Riddle, Professor Emeritus of Mathematics, Agnes Scott College

Mandelbrot Set

Definition of the Mandelbrot Set

In simple terms, the Mandelbrot Set is one of the most famous fractals in mathematics. It is an extremely complex set of figures on the complex plane, formed by a simple iterative formula.

Step-by-Step Explanation of the Definition

To determine if a complex number c belongs to the Mandelbrot Set, follow these rules:

1. Define the iterative function: $z_{n+1} = z_n^2 + c$.
2. Set the initial value: Iteration always starts at $z_0 = 0$.
3. Perform iterative calculations: Substitute $z_0 = 0$ and the test complex number c into the formula, and compute repeatedly to generate a sequence:
   - $z_1 = z_0^2 + c = 0^2 + c = c$
   - $z_2 = z_1^2 + c = c^2 + c$
   - $z_3 = z_2^2 + c = (c^2 + c)^2 + c$
   - $z_4 = z_3^2 + c = \dots$

   (Continue infinitely.)
4. Determine membership:
   - If the sequence $z_0, z_1, z_2, \dots$ remains "bounded" (its modulus never exceeds a finite limit), then c belongs to the Mandelbrot Set.
   - If the sequence "diverges" (its modulus tends to infinity), then c does **not** belong to the Mandelbrot Set.

Formally: The Mandelbrot Set M is the set of all complex numbers c such that when starting from $z_0 = 0$ on the complex plane and iterating via the formula $z_{n+1} = z_n^2 + c$, the resulting sequence $\{z_n\}$ does **not** diverge (i.e., its modulus remains bounded, rather than tending to infinity).

In symbols:$M = \{c \in \mathbb{C} : \limsup |z_n| < \infty,\ z_{n+1} = z_n^2 + c,\ z_0 = 0\}$

Understanding in Popular Language

Imagine this process as a game:

1. Map: The entire complex plane (a 2D plane) is the game map.
2. Choose a point: Pick any point on the map—its coordinates are the complex number c.
3. Game Rules: You have a piece starting at the origin (0). Move the piece using the rule $z_{n+1} = z_n^2 + c$: at each step, square the piece's current position z_n, then add the coordinates of your chosen point c to get the next position z_{n+1}.
4. Win-Loss Judgment:
   - If the piece stays within a "safe zone" (centered at the origin, radius = 2) no matter how it moves, your chosen point c belongs to the Mandelbrot Set.
   - If the piece jumps out of this "safe zone" at any step, it will move infinitely far away —and c does **not** belong to the Mandelbrot Set.

The Mandelbrot Set is the collection of all starting points c that keep the piece "in the safe zone forever."

https://mandel.gart.nz/?utm_source=chatgpt.com#/

Mandelbrot Set

Example 1: Test Point $c = -1$

1. Set Parameters:
   - $c = -1$
   - $z_0 = 0$
2. Start Iterative Calculations:
   - Step 1 (n=1): $z_1 = z_0^2 + c = 0^2 + (-1) = -1$; Modulus $|z_1| = |-1| = 1$ (less than 2, continue).
   - Step 2 (n=2): $z_2 = z_1^2 + c = (-1)^2 + (-1) = 1 - 1 = 0$; Modulus $|z_2| = |0| = 0$ (less than 2, continue).
   - Step 3 (n=3): $z_3 = z_2^2 + c = 0^2 + (-1) = -1$; Modulus $|z_3| = |-1| = 1$ (less than 2, continue).
   - Step 4 (n=4): $z_4 = z_3^2 + c = (-1)^2 + (-1) = 0$; Modulus $|z_4| = |0| = 0$ (less than 2, continue).
3. Observation Result: The sequence is $0, -1, 0, -1, 0, -1, \ldots$. It oscillates forever between 0 and -1, with a modulus of 0 or 1 (never exceeding 2).
4. Conclusion: Since the sequence is bounded, $c = -1$ belongs to the Mandelbrot Set. In images, this point is colored black.

https://mandel.gart.nz/?utm_source=chatgpt.com#/

Example 2: Testing the Point $c = 1$

1. Set Parameters:
   - $c = 1$
   - $z_0 = 0$
2. Start Iterative Calculations:
   - Step 1 (n=1):$z_1 = z_0^2 + c = 0^2 + 1 = 1$

     Magnitude $|z_1| = |1| = 1$. *(Less than 2, continue)*
   - Step 2 (n=2):$z_2 = z_1^2 + c = 1^2 + 1 = 2$

     Magnitude $|z_2| = |2| = 2$. *(Equal to 2, on the boundary; continue to observe the next step)*
   - Step 3 (n=3):$z_3 = z_2^2 + c = 2^2 + 1 = 5$

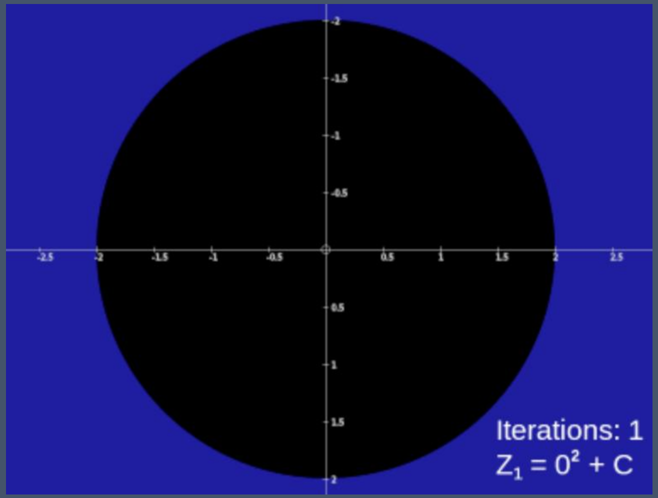
     Magnitude $|z_3| = |5| = 5$. *(Greater than 2, stop!)*
3. Observation Result:

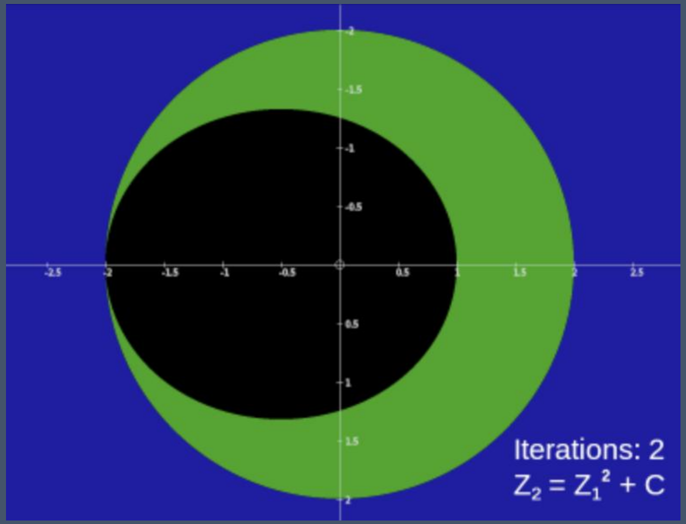
   We obtain the sequence $0, 1, 2, 5, \ldots$. At Step 3, the magnitude of the sequence value has already exceeded 2. We can predict that in the next step, $z_4 = 5^2 + 1 = 26$, and this value will grow rapidly to infinity.
4. Conclusion:

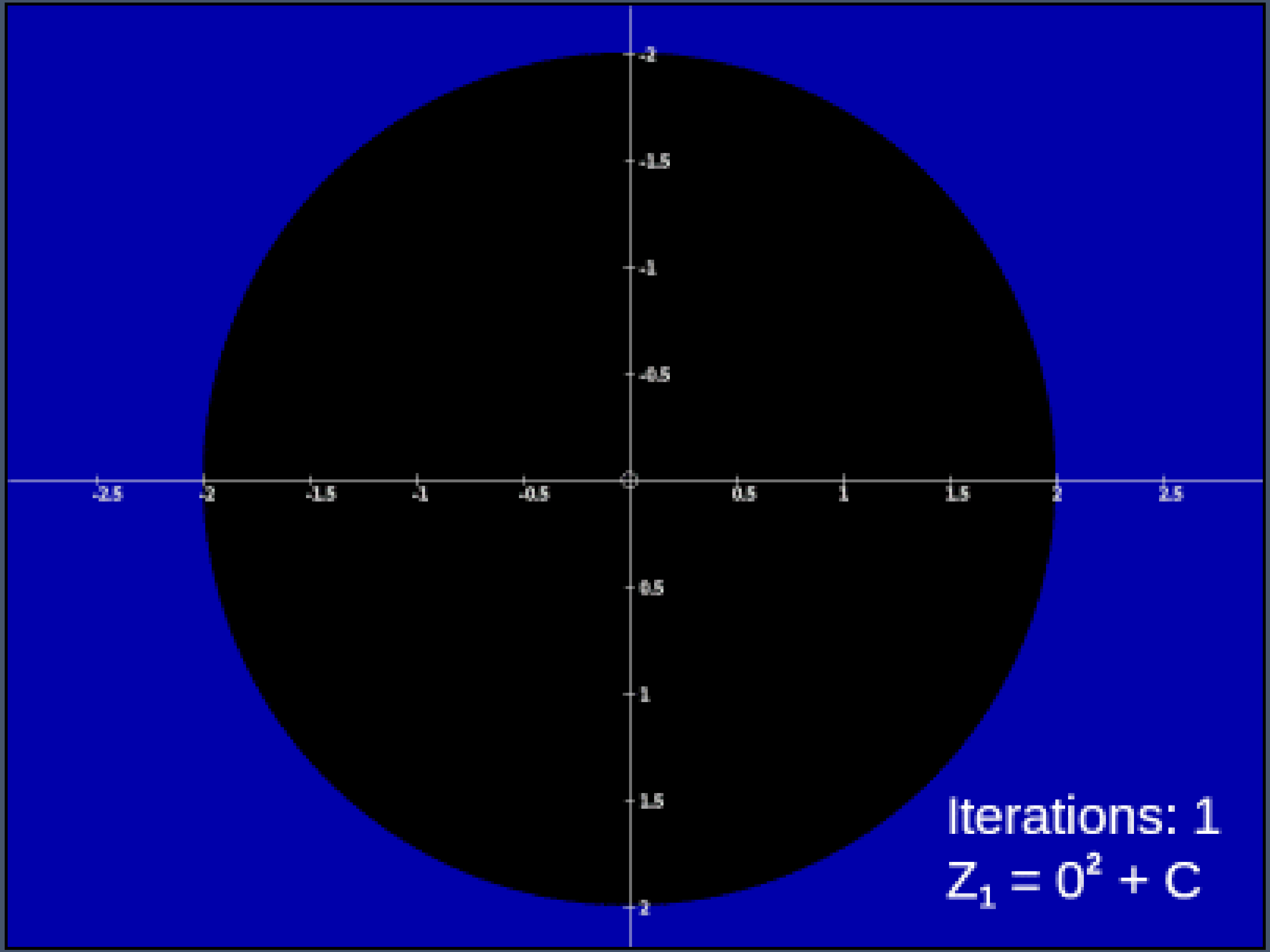
   Since the sequence is divergent (it has "escaped"), $c = 1$ does not belong to the Mandelbrot Set. In visualizations, this point will be assigned a specific color (non-black) based on its "escape speed" (which is 3 steps here).

Mandelbrot Set

https://mandel.gart.nz/







Mandelbrot Set

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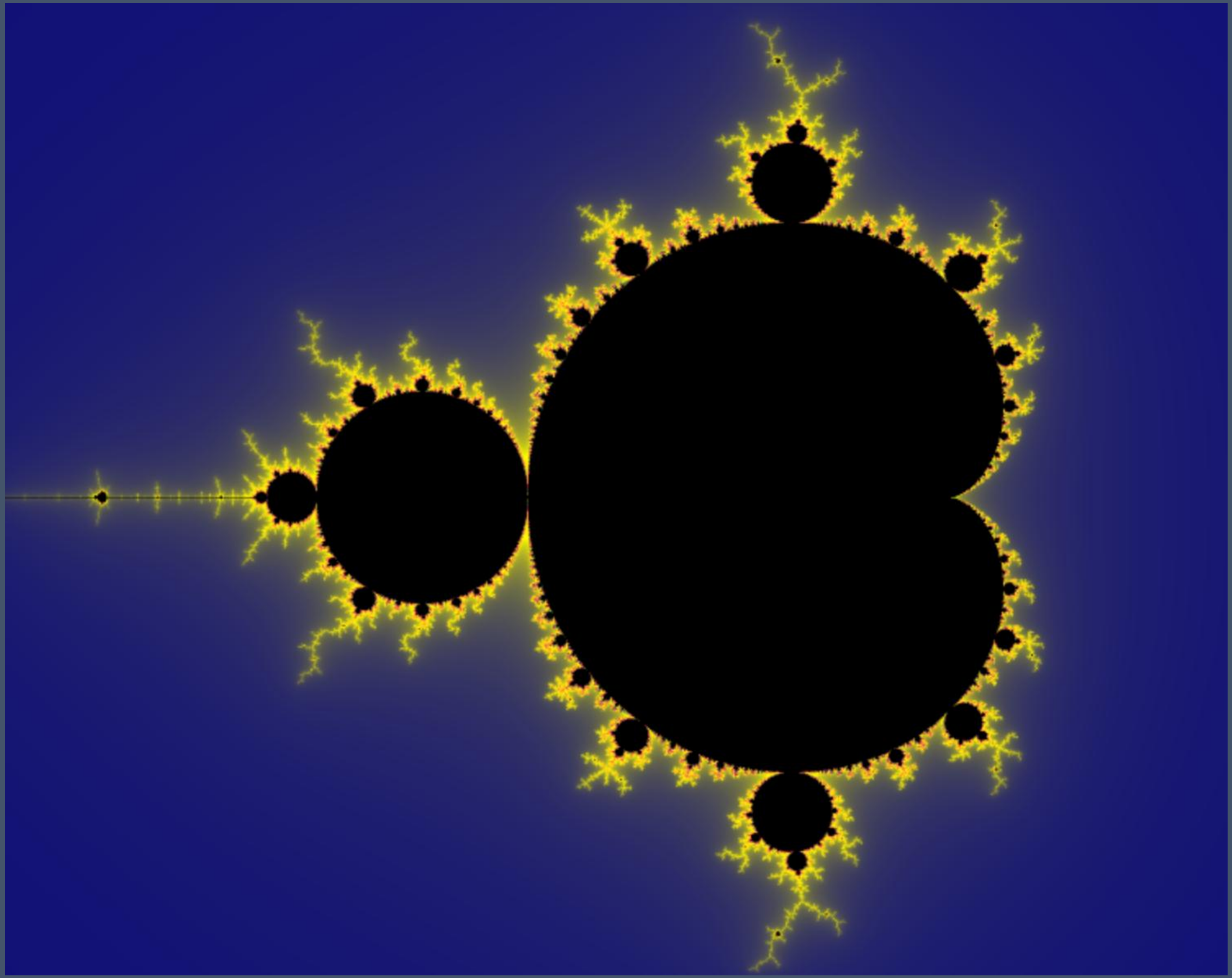

Key Properties of the Mandelbrot Set

Connectivity: While some parts of the set appear isolated, they are all connected—even if only by a thin line. Mathematically, it has been proven that every point in the Mandelbrot Set is connected to all other points, making it a single continuous shape.

Area: The total area of the Mandelbrot Set has been calculated as approximately 1.507 using various methods, but its exact value remains an unsolved mathematical problem.

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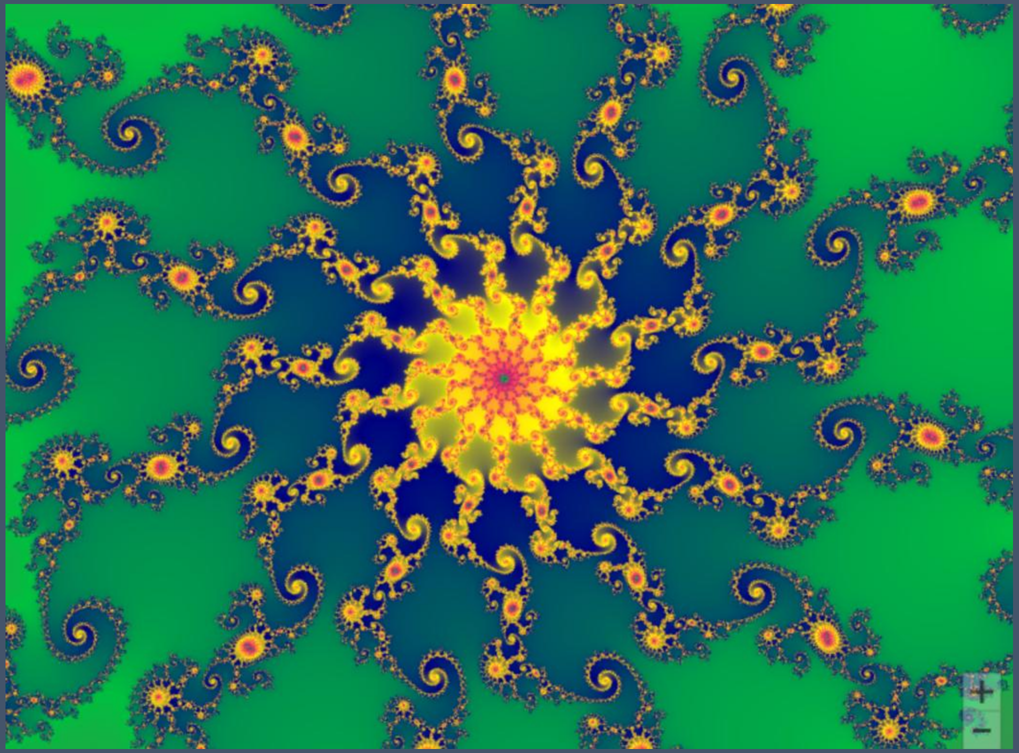
Mandelbrot Set

https://mandel.gart.nz/?utm_source=chatgpt.com#/

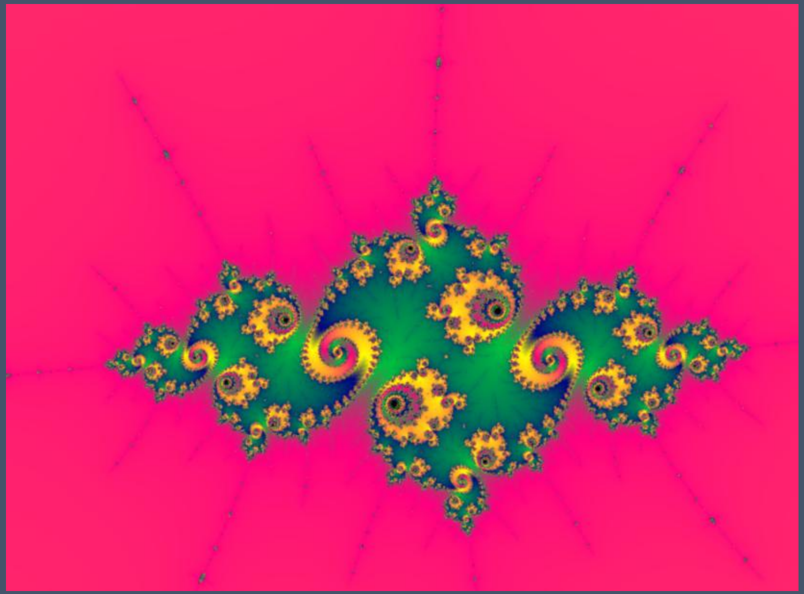
Main Forms of the Mandelbrot Set

Though the Mandelbrot Set is a single connected whole, its interior and boundary consist of several distinct forms with unique mathematical meanings. Their distribution and patterns are fixed, together creating the complex pattern we observe. The main forms include:

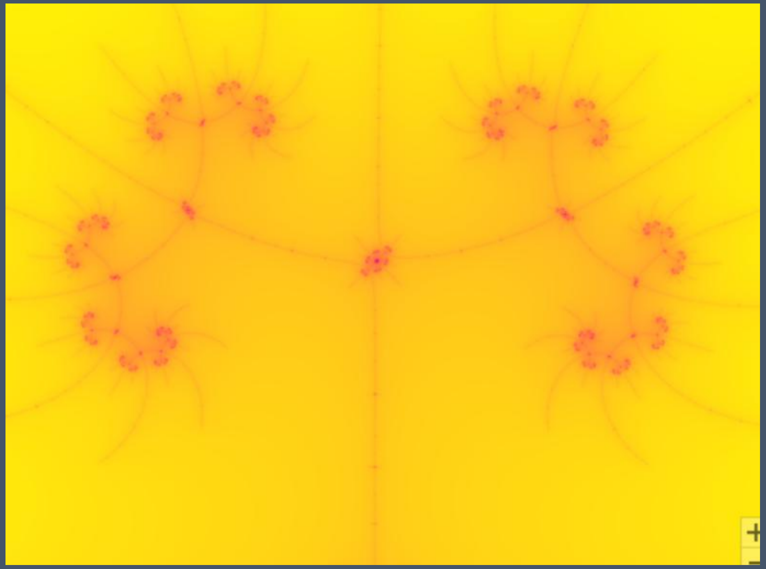


SUN

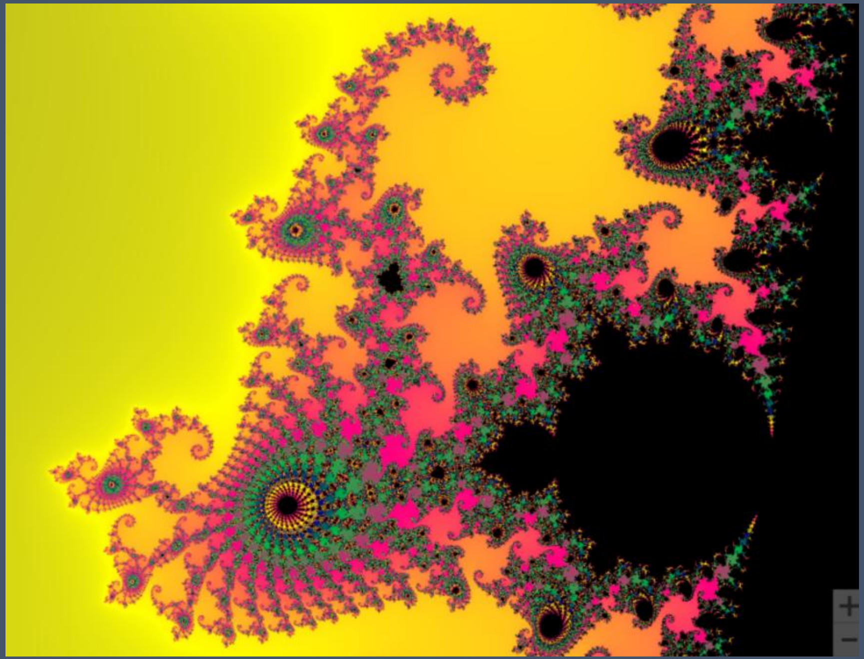
SPIRALS



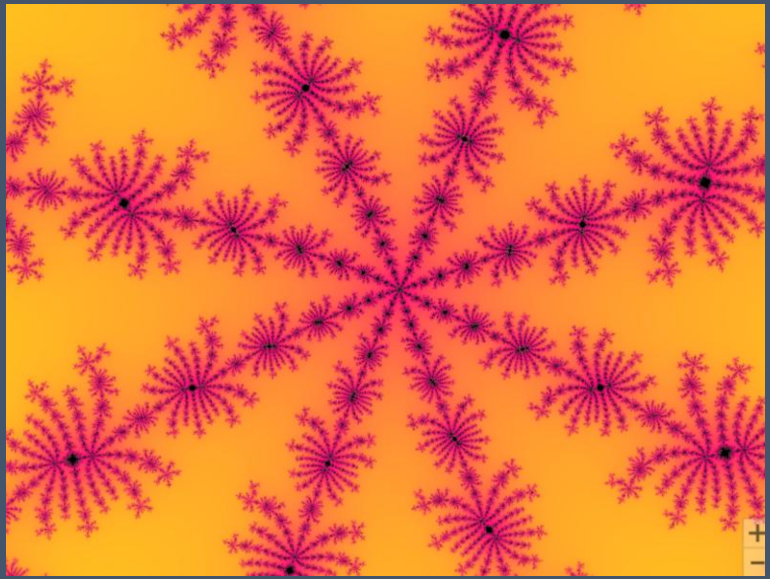
Julia Island



TREE



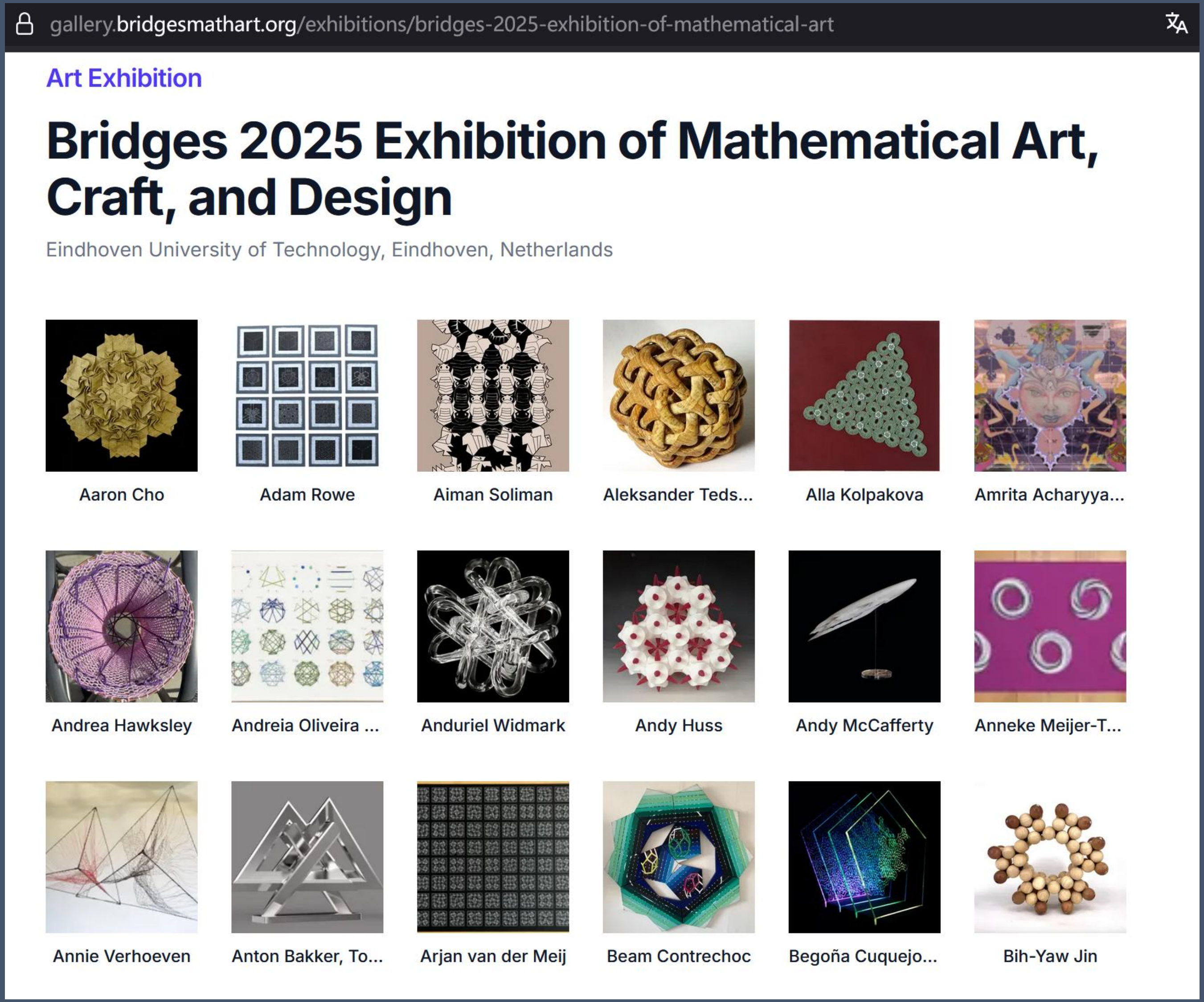
Seahorse Valley



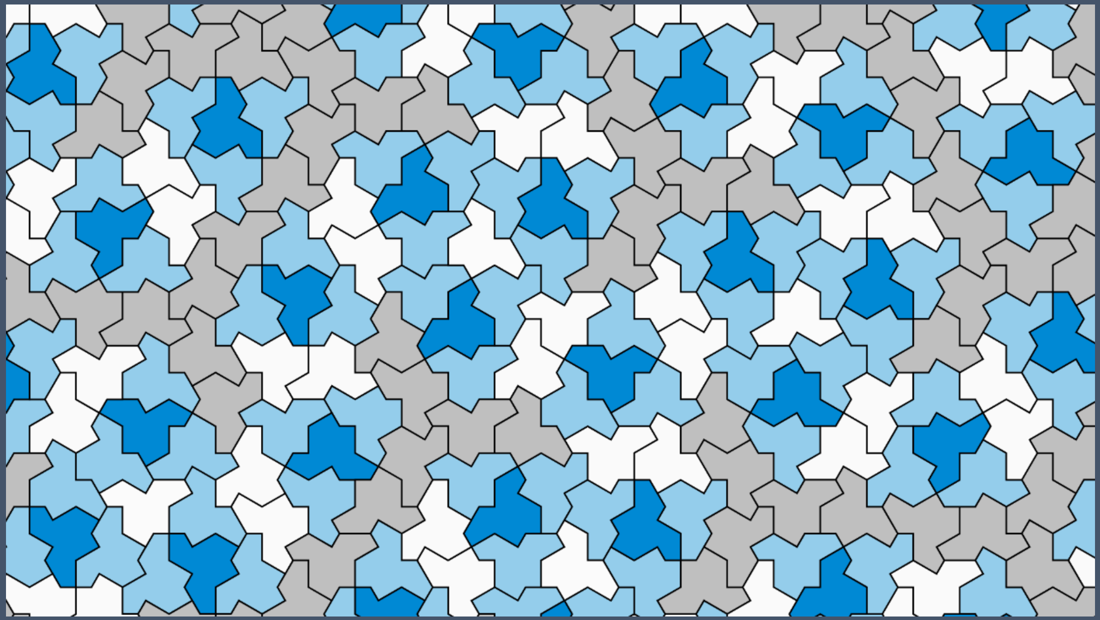
STARFISH

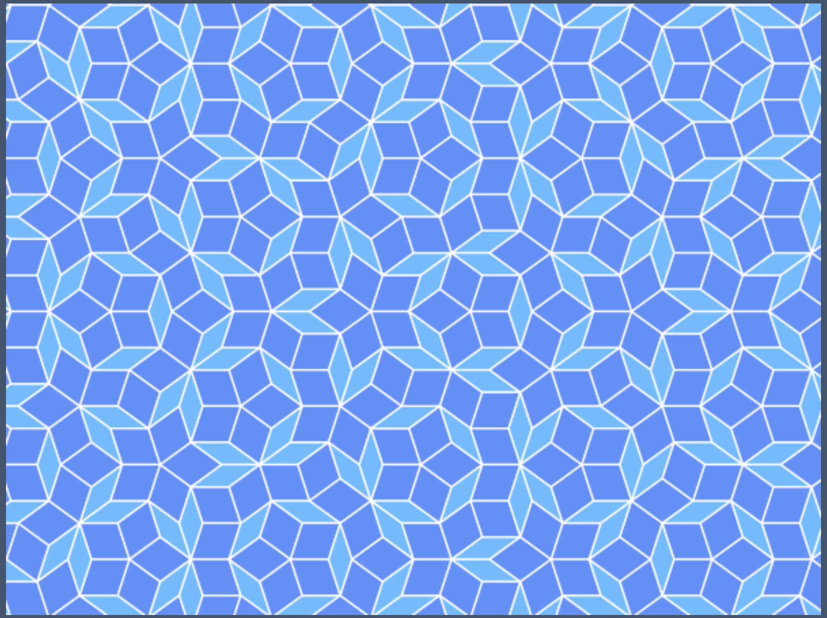
Extracurricular Resources

https://gallery.bridgesmathart.org/exhibitions/bridges-2025-exhibition-of-mathematical-art



Next Topic: Penrose Tiling!





Thanks!