General Program

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$$
\begin{array}{ll}
L_1: & \textbf{do } I_1 = p_1, q_1, \theta_1 \\
L_2: & \quad \textbf{do } I_2 = p_2, q_2, \theta_2 \\
\vdots & \qquad \vdots \\
L_m: & \qquad \textbf{do } I_m = p_m, q_m, \theta_m \\
 & \qquad\quad H(I_1, I_2, \ldots, I_m) \\
 & \qquad \textbf{enddo} \\
 & \qquad \vdots \\
 & \quad \textbf{enddo} \\
 & \textbf{enddo}
\end{array}
$$

Dependence concepts

- Taking single assignment statement S in the program. Let $\mathbf{m_s}$ denote the number of loops containing S and L_s the loop nest determined by S.

- For L_s, let Is denote the index vector, $\hat{I}_s$ the iteration vector, p_{S0} and q_{S0} the initial and final vectors, P_s and Q_s the initial and final matrices, and θ_s the stride matrix.

- The normalized final vector and the normalized final matrix of L_s are denoted by $\hat{q}_{S0}$ and $\hat{Q}_S$,

$$\hat{\mathbf{Q}}_S = \Theta_S \mathbf{P}_S^{-1} \mathbf{Q}_S \Theta_S^{-1}$$
$$\hat{\mathbf{q}}_{S0} = \left(\mathbf{q}_{S0} - \mathbf{p}_{S0} \mathbf{P}_S^{-1} \mathbf{Q}_S\right) \Theta_S^{-1}$$

the loops of the program $L_1, L_2, .. , L_{ms+mt-m}$

$$\mathbf{L} = (L_1, L_2, \ldots, L_m)$$
$$\mathbf{L}_S = (L_1, L_2, \ldots, L_m, L_{m+1}, \ldots, L_{m_S})$$
$$\mathbf{L}_T = (L_1, L_2, \ldots, L_m, L_{m_S+1}, \ldots, L_{m_S+m_T-m})$$

$i = (i_1, i_2, \ldots, i_m, i_{m+1}, \ldots, i_{ms})$ of the index vector I_s determines an instance of statement S that is denoted by S(i), and a value $j = (j_1, j_2, \ldots, j_m, j_{ms}, \ldots, j_{mS+mT-m})$ of the index vector I_T determines an instance of statement T that is denoted by T(j).

The distance from the instance S(i) to the instance T(j) is defined to be the m-vector $(\hat{j}_1 - \hat{i}_1, \hat{j}_2 - \hat{i}_2 \; . \; . \; \hat{j}_m - \hat{i}_m)$, where

$$\hat{\boldsymbol{i}} = (\hat{i}_1, \ldots, \hat{i}_m, \hat{i}_{m+1}, \ldots, \hat{i}_{m_S})$$

$$\hat{\boldsymbol{j}} = (\hat{j}_1, \ldots, \hat{j}_m, \hat{j}_{m_S+1}, \ldots, \hat{j}_{m_S+m_T-m})$$

Example

$$
\begin{array}{lllll}
L_1: & \textbf{do } I_1 = 1, 100, 1 \\
L_2: & \quad \textbf{do } I_2 = 1, I_1, 2 \\
L_3: & \quad\quad \textbf{do } I_3 = I_1, I_1 + I_2, 1 \\
\quad S: & \quad\quad\quad X(2I_1 - 1, 3I_2 + 1, 2I_3) = \cdots \\
 & \quad\quad \textbf{enddo} \\
L_4: & \quad\quad \textbf{do } I_4 = 100, 0, -3 \\
L_5: & \quad\quad\quad \textbf{do } I_5 = I_1, I_4, 1 \\
\quad T: & \quad\quad\quad\quad \cdots = \cdots X(2I_1 + 1, 4I_2 + 6, I_4 + I_5) \cdots \\
 & \quad\quad\quad \textbf{enddo} \\
 & \quad\quad \textbf{enddo} \\
 & \quad \textbf{enddo} \\
 & \textbf{enddo}
\end{array}
$$

- Statement S determines the loop nest Ls = *(L_1,L_2,L_3) with ms = 3* loops. For this nest, we have:

$$p_1 = p_{10} = 1,$$
$$q_1 = q_{10} = 100,$$
$$p_2 = p_{20} + p_{21}I_1 = 1 + 0 \times I_1,$$
$$q_2 = q_{20} + q_{21}I_1 = 0 + 1 \times I_1,$$
$$p_3 = p_{30} + p_{31}I_1 + p_{32}I_2 = 0 + 1 \times I_1 + 0 \times I_2,$$
$$q_3 = q_{30} + q_{31}I_1 + q_{32}I_2 = 0 + 1 \times I_1 + 1 \times I_2,$$
$$\theta_1 = 1, \theta_2 = 2, \theta_3 = 1.$$

- the initial and final vectors are

$$\mathbf{p}_{S0} = (p_{10}, p_{20}, p_{30}) = (1, 1, 0),$$
$$\mathbf{q}_{S0} = (q_{10}, q_{20}, q_{30}) = (100, 0, 0)$$

- the initial, final, and stride matrices are

$$\mathbf{P}_S = \begin{pmatrix} 1 & -p_{21} & -p_{31} \\ 0 & 1 & -p_{32} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{Q}_S = \begin{pmatrix} 1 & -q_{21} & -q_{31} \\ 0 & 1 & -q_{32} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{\Theta}_S = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- We compute the normalized final vector and matrix:

$$\hat{\mathbf{q}}_{S0} = \left(\mathbf{q}_{S0} - \mathbf{p}_{S0}\mathbf{P}_S^{-1}\mathbf{Q}_S\right)\mathbf{\Theta}_S^{-1} = (99, 0, 1),$$

$$\hat{\mathbf{Q}}_S = \mathbf{\Theta}_S\mathbf{P}_S^{-1}\mathbf{Q}_S\mathbf{\Theta}_S^{-1} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

- The iteration vector $\hat{I}_s = (\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3)$ satisfies the constraints

$$\mathbf{0} \le \hat{\mathbf{I}}_S$$
$$\hat{\mathbf{I}}_S \hat{\mathbf{Q}}_S \le \hat{\mathbf{q}}_{S0}$$

- Thus, the iteration space of the loop nest Ls is the set of all integer vectors $(\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3) <= (0, 0, 0)$ such that

$$\begin{aligned} \hat{\imath}_1 \qquad\qquad & \le 99 \\ -\hat{\imath}_1/2 + \hat{\imath}_2 \qquad & \le 0 \\ -2\hat{\imath}_2 + \hat{\imath}_3 & \le 1 \end{aligned}$$

- that is, the set of all integer 3-vectors $(\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3)$ such that

$$\begin{aligned} 0 \le \hat{\imath}_1 &\le 99 \\ 0 \le \hat{\imath}_2 &\le \hat{\imath}_1/2 \\ 0 \le \hat{\imath}_3 &\le 2\hat{\imath}_2 + 1 \end{aligned}$$

- Given an index point (i_1, i_2, i_3) of L_s, the corresponding iteration point ($\hat{i}_1$, $\hat{i}_2$, $\hat{i}_3$) is computed using the equation

$$\hat{\mathbf{I}} = (\mathbf{IP} - \mathbf{p}_0)\Theta^{-1}$$

$$\begin{aligned}(\hat{i}_1, \hat{i}_2, \hat{i}_3) &= [(i_1, i_2, i_3)\mathbf{P}_S - \mathbf{p}_{S0}]\,\Theta_S^{-1} \\ &= \left[(i_1, i_2, i_3)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (1,1,0)\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (i_1 - 1, (i_2 - 1)/2, i_3 - i_1)\,.\end{aligned}$$

Statement T determines the loop nest $L_T=(L_1,L_2,L_4,L_5)$ with $m_T = 4$ loops. For this nest, the initial and final vectors are

$$\mathbf{p}_{T0} = (1, 1, 100, 0), \qquad \mathbf{q}_{T0} = (100, 0, 0, 0)$$

the initial, final, and stride matrices are

$$\mathbf{P}_T = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{Q}_T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{\Theta}_T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We compute the normalized final vector and matrix:

$$\hat{\mathbf{q}}_{T0} = \left(\mathbf{q}_{T0} - \mathbf{p}_{T0}\mathbf{P}_T^{-1}\mathbf{Q}_T\right)\Theta_T^{-1} = (99, 0, 100/3, 99)$$

$$\hat{\mathbf{Q}}_T = \Theta_T\mathbf{P}_T^{-1}\mathbf{Q}_T\Theta_T^{-1} = \begin{pmatrix} 1 & -1/2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The iteration space of the loop nest L_T is the set of all integer 4-vectors $(\hat{j}_1, \hat{j}_2, \hat{j}_4, \hat{j}_5)$ such that

$$\begin{aligned} &0 \le \hat{j}_1 \le 99 \\ &0 \le \hat{j}_2 \le \hat{j}_1/2 \\ &0 \le \hat{j}_4 \le 33 \\ &0 \le \hat{j}_5 \le 99 - \hat{j}_1 - 3\hat{j}_4 \end{aligned}$$

The iteration value $(\hat{j}_1, \hat{j}_2, \hat{j}_4, \hat{j}_5)$ corresponding to an index value (j_1, j_2, j_4, j_5) of L_T is given by

$$(\hat{j}_1, \hat{j}_2, \hat{j}_4, \hat{j}_5) = [(j_1, j_2, j_4, j_5)\mathbf{P}_T - \mathbf{p}_{T0}]\,\Theta_T^{-1}$$
$$= (j_1 - 1, (j_2 - 1)/2, (100 - j_4)/3, j_5 - j_1)$$

We take three statement instances $S(i_1)$,$S(i_2)$,and $T(j_1)$

Statement Instance	Iteration Point of $\mathbf{L}_S$	Iteration Point of $\mathbf{L}_T$	Iteration Point of $\mathbf{L}$
$S(\mathbf{i}_1) = S(31, 29, 58)$	(30, 14, 27)	—	(30, 14)
$T(\mathbf{j}_1) = T(36, 5, 70, 70)$	—	(35, 2, 10, 34)	(35, 2)
$S(\mathbf{i}_2) = S(41, 25, 61)$	(40, 12, 20)	—	(40, 12)

Where,

$$\mathbf{i}_1 = (31, 29, 58), \quad \mathbf{i}_2 = (41, 25, 61), \quad \mathbf{j}_1 = (36, 5, 70, 70)$$

The distance from $S(i_1)$ to $T(j_1)$ is $(5,-12) = (35, 2) - (30, 14)$, and the distance from $T(j_1)$ to $S(i_2)$ is $(5, 10) = (40, 12) - (35, 2)$.

Since $(5,-12)$ and $(5, 10)$ are both lexicographically positive, $S(i_1)$ is executed before $T(j_1)$, and $T(j_1)$ before $S(i_2)$.

That is, the instances $S(i_1)$, $T(j_1)$, and $S(i_2)$ are executed in this order since

$$(30, 14) \prec (35, 2) \prec (40, 12)$$

Dependence problem

Consider two assignment statements S and T, with variables

$$X(I_S A + a_0) \text{ and } X(I_T B + b_0),$$

Let $X(\hat{\mathbf{I}}_S \hat{\mathbf{A}} + \hat{\mathbf{a}}_0)$ and $X(\hat{\mathbf{I}}_T \hat{\mathbf{B}} + \hat{\mathbf{b}}_0)$ denote the normalized forms of the two variables

$$\begin{aligned}
\hat{\mathbf{A}} &= \Theta_S \mathbf{P}_S^{-1} \mathbf{A} \\
\hat{\mathbf{a}}_0 &= \mathbf{p}_{S0} \mathbf{P}_S^{-1} \mathbf{A} + \mathbf{a}_0 \\
\hat{\mathbf{B}} &= \Theta_T \mathbf{P}_T^{-1} \mathbf{B} \\
\hat{\mathbf{b}}_0 &= \mathbf{p}_{T0} \mathbf{P}_T^{-1} \mathbf{B} + \mathbf{b}_0
\end{aligned}$$

The instance of the variable $X(\hat{I}_S\hat{A} + \hat{a}_0)$ for an iteration point

$$\hat{\imath} = (\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_m, \hat{\imath}_{m+1}, \ldots, \hat{\imath}_{m_S})$$

of the loop nest Ls is $X(\hat{I}_S\hat{A} + \hat{a}_0)$. The instance of for an iteration point $X\left(\hat{\mathbf{I}}_T\hat{\mathbf{B}} + \hat{\mathbf{b}}_0\right)$

$$\hat{\jmath} = (\hat{\jmath}_1, \hat{\jmath}_2, \ldots, \hat{\jmath}_m, \hat{\jmath}_{m_S+1}, \ldots, \hat{\jmath}_{m_S+m_T-m})$$

of the loop nest L_T is $X\left(\hat{\mathbf{I}}_T\hat{\mathbf{B}} + \hat{\mathbf{b}}_0\right)$

These two instances represent the same memory location iff, $\hat{\imath}\hat{\mathbf{A}} + \hat{\mathbf{a}}_0 = \hat{\jmath}\hat{\mathbf{B}} + \hat{\mathbf{b}}_0$ that is, iff

$$\hat{\imath}\hat{\mathbf{A}} - \hat{\jmath}\hat{\mathbf{B}} = \hat{\mathbf{b}}_0 - \hat{\mathbf{a}}_0$$

The matrix equation is the *dependence equation for the program* variables $\mathbf{X}(\hat{\mathbf{I}}_S\hat{\mathbf{A}} + \hat{\mathbf{a}}_0)$ and $X\left(\hat{\mathbf{I}}_T\hat{\mathbf{B}} + \hat{\mathbf{b}}_0\right)$.

It consists of n scalar equations, and there are *$(m_S+ m_T)$ integer variables: the m_S elements of $\hat{\imath}$* and the m_T elements of $\hat{\jmath}$.

Parameter	Type	Size
$\hat{\mathbf{q}}_{S0}$	integer vector	m_S
$\hat{\mathbf{Q}}_S$	integer matrix	$m_S \times m_S$
$\hat{\mathbf{q}}_{T0}$	integer vector	m_T
$\hat{\mathbf{Q}}_T$	integer matrix	$m_T \times m_T$
$\hat{\mathbf{a}}_0$	integer vector	n
$\hat{\mathbf{A}}$	integer matrix	$m_S \times n$
$\hat{\mathbf{b}}_0$	integer vector	n
$\hat{\mathbf{B}}$	integer matrix	$m_T \times n$

- Since Î is a value of $\hat{I}_S$, it must satisfy the inequalities for L_S:

$$\mathbf{0} \le \hat{\boldsymbol{\imath}}$$
$$\hat{\boldsymbol{\imath}}\hat{\mathbf{Q}}_S \le \hat{\mathbf{q}}_{S0}$$

- Since ĵ is a value of $\hat{I}_T$, it must satisfy the inequalities for L_T:

$$\mathbf{0} \le \hat{\boldsymbol{\jmath}}$$
$$\hat{\boldsymbol{\jmath}}\hat{\mathbf{Q}}_T \le \hat{\mathbf{q}}_{T0}$$

- **Corollary 1** If statement T depends on statement S with a direction vector $\sigma = (\sigma_1, \sigma_2 \, . \, . \, . \, . \, . \, \sigma_m)$, then Equation has an integer solution $(\hat{\imath}, \hat{\jmath})$ and the additional condition

$$sig(\hat{\jmath}_r - \hat{\imath}_r) = \sigma \, (1 <= r <= m)$$

- **Corollary 2** If statement T depends on statement S at a level $\boldsymbol{l}$, then Equation has an integer solution and the additional conditions:

$$\{ \hat{\imath}_1 = \hat{\jmath}_1 \, , \hat{\imath}_2 = \hat{\jmath}_2 \, . \, . \, . \, . \, . \, \hat{\imath}_{l-1} = \hat{\jmath}_{l-1}, \hat{\imath}_l <= \hat{\jmath}_{l-1}$$

Example

$$
\begin{array}{lllll}
L_1: & \textbf{do } I_1 = 1, 100, 1 & & & \\
L_2: & \quad \textbf{do } I_2 = 1, I_1, 2 & & & \\
L_3: & \quad\quad \textbf{do } I_3 = I_1, I_1 + I_2, 1 & & & \\
\quad S: & \quad\quad\quad X(2I_1 - 1, 3I_2 + 1, 2I_3) = \cdots & & & \\
 & \quad\quad \textbf{enddo} & & & \\
L_4: & \quad\quad \textbf{do } I_4 = 100, 0, -3 & & & \\
L_5: & \quad\quad\quad \textbf{do } I_5 = I_1, I_4, 1 & & & \\
\quad T: & \quad\quad\quad\quad \cdots = \cdots X(2I_1 + 1, 4I_2 + 6, I_4 + I_5) \cdots & & & \\
 & \quad\quad\quad \textbf{enddo} & & & \\
 & \quad\quad \textbf{enddo} & & & \\
 & \quad \textbf{enddo} & & & \\
 & \textbf{enddo} & & &
\end{array}
$$

- The variable of statement S can be written as X $((I_1,I_2,I_3)A + a_0)$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_0 = (-1, 1, 0)$$

- Its normalized form is X $((\hat{I}_1, \hat{I}_2, \hat{I}_3)\ \hat{A} + \hat{a}_0)$, where

$$\hat{\mathbf{A}} = \mathbf{\Theta}_S \mathbf{P}_S^{-1} \mathbf{A} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\hat{\mathbf{a}}_0 = \mathbf{p}_{S0} \mathbf{P}_S^{-1} \mathbf{A} + \mathbf{a}_0 = (1, 4, 2)$$

The variable of T can be written as X $((I_1, I_2, I_4, I_5)B + b_0)$, where

$$\mathbf{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{b}_0 = (1, 6, 0)$$

Its normalized form is $X\left((\hat{I}_1, \hat{I}_2, \hat{I}_4, \hat{I}_5)\hat{\mathbf{B}} + \hat{\mathbf{b}}_0\right)$, where,

$$\hat{\mathbf{B}} = \mathbf{\Theta}_T \mathbf{P}_T^{-1} \mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 8 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{\mathbf{b}}_0 = \mathbf{p}_{T0} \mathbf{P}_T^{-1} \mathbf{B} + \mathbf{b}_0 = (3, 10, 101)$$

The dependence equation

$$(\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3)\begin{pmatrix} 2 & 0 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} - (\hat{\jmath}_1, \hat{\jmath}_2, \hat{\jmath}_4, \hat{\jmath}_5)\begin{pmatrix} 2 & 0 & 1 \\ 0 & 8 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

It consists of three scalar equations in 7 integer variables:

$$\begin{aligned} 2\hat{\imath}_1 - 2\hat{\jmath}_1 &= 2 \\ 6\hat{\imath}_2 - 8\hat{\jmath}_2 &= 6 \\ 2\hat{\imath}_1 + 2\hat{\imath}_3 - \hat{\jmath}_1 + 3\hat{\jmath}_4 - \hat{\jmath}_5 &= 99 \end{aligned}$$

we found the this problem dependence constraints

$$
\begin{aligned}
&0 \le \hat{\imath}_1 \le 99 \\
&0 \le \hat{\imath}_2 \le \hat{\imath}_1/2 \\
&0 \le \hat{\imath}_3 \le 2\hat{\imath}_2 + 1
\end{aligned}
$$

$$
\begin{aligned}
&0 \le \hat{\jmath}_1 \le 99 \\
&0 \le \hat{\jmath}_2 \le \hat{\jmath}_1/2 \\
&0 \le \hat{\jmath}_4 \le 33 \\
&0 \le \hat{\jmath}_5 \le 99 - \hat{\jmath}_1 - 3\hat{\jmath}_4
\end{aligned}
$$

If we want to test if there is a dependence of T on S with a particular direction vector, say (1, – 1), then we will get two additional conditions: $i_1 <= j_1 - 1$ and $j_2 < i_2 - 1$. Similarly, for dependence at a given level, say 2, the additional conditions are $i_1 = j_1$ and $i_2 <= j_2 - 1$.

Generalized GCD test

A single equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = c$$

with integer coefficients, has an integer solution in x_1, x_2, ..., x_m, iff gcd(a_1, a_2 , . . . , a_m) evenly divides c .

When this well known result is used in dependence analysis, it is called the *gcd test.*

the dependence equation can be written ;

$$(\hat{\imath}; \hat{\jmath}) \begin{pmatrix} \hat{\mathbf{A}} \\ -\hat{\mathbf{B}} \end{pmatrix} = \hat{\mathbf{b}}_0 - \hat{\mathbf{a}}_0$$

where $(\hat{\imath}\,;\,\hat{\jmath})$ is the vector of size $(m_S + m_T)$ obtained by concatenating the elements of $\hat{\imath}$ and $\hat{\jmath}$, and the coefficient matrix of the equation is the $(m_S + m_T)$ X n matrix obtained by concatenating the rows of matrices $\hat{\mathbf{A}}$, and $\hat{\mathbf{B}}$.

Generalized gcd test: *Find an $(m_S + m_T)$ X $(m_S + m_T)$ unimodular matrix U and an $(m_S + m_T)$ x n echelon matrix S, such that*

$$\mathbf{U} \cdot \begin{pmatrix} \hat{\mathbf{A}} \\ -\hat{\mathbf{B}} \end{pmatrix} = \mathbf{S}$$

If the variables $X(\hat{\mathbf{I}}_S\hat{\mathbf{A}} + \hat{\mathbf{a}}_0)$ of statement S and $X\left(\hat{\mathbf{I}}_T\hat{\mathbf{B}} + \hat{\mathbf{b}}_0\right)$ of statement T cause a dependence between S and T, then there exists an integer vector t of size $(ms + m_T)$ that satisfies the equation

$$\mathbf{tS} = \hat{\mathbf{b}}_0 - \hat{\mathbf{a}}_0$$

Example

$$
\begin{array}{lll}
L_1: & \textbf{do } I_1 = 1, 100, 1 \\
L_2: & \quad \textbf{do } I_2 = 1, I_1, 2 \\
L_3: & \quad\quad \textbf{do } I_3 = I_1, I_1 + I_2, 1 \\
\quad S: & \quad\quad\quad X(2I_1 - 1, 3I_2 + 1, 2I_3) = \cdots \\
 & \quad\quad \textbf{enddo} \\
L_4: & \quad\quad \textbf{do } I_4 = 100, 0, -3 \\
L_5: & \quad\quad\quad \textbf{do } I_5 = I_1, I_4, 1 \\
\quad T: & \quad\quad\quad\quad \cdots = \cdots X(2I_1 + 1, 4I_2 + 6, I_4 + I_5) \cdots \\
 & \quad\quad\quad \textbf{enddo} \\
 & \quad\quad \textbf{enddo} \\
 & \quad \textbf{enddo} \\
 & \textbf{enddo}
\end{array}
$$

The dependence equation

$$(\hat{\imath}_1, \hat{\imath}_2, \hat{\imath}_3, \hat{\jmath}_1, \hat{\jmath}_2, \hat{\jmath}_4, \hat{\jmath}_5) \begin{pmatrix} 2 & 0 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \\ -2 & 0 & -1 \\ 0 & -8 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{pmatrix} = (2, 6, 99)$$

We apply Algorithm 1.2.1 to the coefficient matrix of this system to get the decomposition

$$\begin{pmatrix} 0&0&0&1&0&0&0\\ 0&1&0&0&1&0&0\\ 0&0&0&0&0&0&1\\ 1&0&0&1&0&0&1\\ 0&4&0&0&3&0&0\\ 0&0&1&0&0&0&2\\ 0&0&0&0&0&1&3 \end{pmatrix} \begin{pmatrix} 2&0&2\\ 0&6&0\\ 0&0&2\\ -2&0&-1\\ 0&-8&0\\ 0&0&3\\ 0&0&-1 \end{pmatrix} = \begin{pmatrix} -2&0&-1\\ 0&-2&0\\ 0&0&-1\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$$

where the 7 x 7 matrix on the left (the U matrix) is unimodular and the 7 x 3 matrix on the right (the S matrix) is echelon.

$$(t_1, t_2, t_3, t_4, t_5, t_6, t_7) \cdot \mathbf{S} = (2, 6, 99)$$

The equation is equivalent to the equation

$$(-2t_1, -2t_2, -t_1 - t_3) = (2, 6, 99)$$

that clearly has integer solutions. In fact, any 7-vector of the form $(-1, -3, -98, t_4, t_5, t_6, t_7)$ is an integer solution, where $t_4, \ldots, t_7$ are arbitrary integers.

Thus, the generalized gcd test passes. This means for the current dependence problem, this test does not rule out dependence, nor does it confirm it.