Method of Bounds

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Contents

- Introduction to method of bounds
- Perfect Nest, One-Dimensional Array
- Rectangular Loops, One-Dimensional Array
- Rectangular Loops, Multi-Dimensional Array
- General program

Introduction

- The *method of bounds,* consists of two major tests;

1) the generalized gcd test to check for an integer solution to the equations, and
2) the bounds test to check for a real solution to the equations and the inequalities.

“The *bounds test* works by testing if a certain real number lies between the extreme values of a certain real-valued function in a certain set”

- A linear dependence problem involving an n-dimensional array X, there are n scalar dependence equations that can be written in the form Eq. (1)

$$f_k(\mathbf{x}) = c_k \qquad (1 \le k \le n)$$

  - f_l, f_2, ..., f_n, are real-valued linear functions on some Euclidean space $R^{\mathbf{N}}$,
  - c_1, c_2, ..., c_n are integers
- a single dependence equation, (2) $f(\mathbf{x}) = c$
- there is a real solution to Equation is represented as

$$\min_{\mathbf{x} \in P} f(\mathbf{x}) \le c \le \max_{\mathbf{x} \in P} f(\mathbf{x})$$

  - The inequalities of the problem define a subset *P* of $R^{\mathbf{N}}$

"The bounds test for a one-dimensional array consists of testing this inequality"

- When n > 1, the situation gets more complicated. The system of equations (1) has a real solution in P **iff** i.e. Equation (3)

$$\max_{\boldsymbol{\lambda}\in\mathbf{R}^{n-1}} \min_{\mathbf{x}\in P}\left[f_1(\mathbf{x})+\sum_{k=2}^{n}\lambda_k(f_k(\mathbf{x})-c_k)\right]\le c_1$$

$$\le \min_{\boldsymbol{\lambda}\in\mathbf{R}^{n-1}} \max_{\mathbf{x}\in P}\left[f_1(\mathbf{x})+\sum_{k=2}^{n}\lambda_k(f_k(\mathbf{x})-c_k)\right]$$

- The general inequality for the bounds test; it includes (1) as a special case (n = 1). If (1) or (3) fails to hold, then there is no dependence.

Lemma 6.1 *Consider a bounded real-valued function $f : D \rightarrow \mathbf{R}$ defined on a nonempty set D. Let μ denote a lower and ν an upper bound for f. If the equation $f(x) = c$ has a solution in D, then $\mu \leq c \leq \nu$.*

- In case of a multi-dimensional array, instead of seeking a *simultaneous* real solution to the system of n equations (1) in *P*, we may simply test for *n* *individual* real solutions to the n individual equations in *P*. Then, we would test the following sequence of inequalities:

$$\min_{\mathbf{x} \in P} f_k(\mathbf{x}) \leq c_k \leq \max_{\mathbf{x} \in P} f_k(\mathbf{x}) \qquad (1 \leq k \leq n).$$

- If at least one of these inequalities fails to hold, then there is no dependence.

Perfect Nest, One-Dimensional Array

- Example

$$
\begin{array}{ll}
L_1: & \textbf{do } I_1 = p_1, q_1, 1 \\
L_2: & \quad \textbf{do } I_2 = p_2, q_2, 1 \\
\vdots & \qquad \vdots \\
L_m: & \qquad \textbf{do } I_m = p_m, q_m, 1 \\
 & \qquad\quad H(I_1, I_2, \ldots, I_m) \\
 & \qquad \textbf{enddo} \\
 & \qquad \vdots \\
 & \quad \textbf{enddo} \\
 & \textbf{enddo}
\end{array}
$$

Theorem 6.2 *If in the loop nest* **L**, *a variable* $X\,(a_0 + \sum_{r=1}^{m} a_r I_r)$ *of* S *and a variable* $X\,(b_0 + \sum_{r=1}^{m} b_r I_r)$ *of* T *cause a dependence of* T *on* S *at a level* ℓ, *then the following two conditions hold:*

(a) *The gcd of*

$$a_1 - b_1, a_2 - b_2, \ldots, a_{\ell-1} - b_{\ell-1}, a_\ell, \ldots, a_m, b_\ell, \ldots, b_m$$

divides $(b_0 - a_0)$; *and*

(b) $\overline{\mu} \le b_0 - a_0 \le \overline{\nu}$,

where

$$\overline{\mu} = \sum_{r=1}^{\ell-1} \mu(a_r - b_r, 0, p_r, q_r, 0) + \mu(a_\ell, -b_\ell, p_\ell, q_\ell, 1) + \sum_{r=\ell+1}^{m} [\mu(a_r, 0, p_r, q_r, 0) + \mu(-b_r, 0, p_r, q_r, 0)]$$

and

$$\overline{\nu} = \sum_{r=1}^{\ell-1} \nu(a_r - b_r, 0, p_r, q_r, 0) + \nu(a_\ell, -b_\ell, p_\ell, q_\ell, 1) + \sum_{r=\ell+1}^{m} [\nu(a_r, 0, p_r, q_r, 0) + \nu(-b_r, 0, p_r, q_r, 0)].$$

Theorem 6.3 *Let* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ *denote a vector whose elements are members of the set* $\{0, 1, -1, *\}$. *If in the loop nest* $\mathbf{L}$, *the variables* $X\left(a_0 + \sum_{r=1}^{m} a_r I_r\right)$ *of* S *and* $X\left(b_0 + \sum_{r=1}^{m} b_r I_r\right)$ *of* T *cause a dependence of* T *on* S *with the direction vector* $\boldsymbol{\sigma}$, *then the following two conditions hold:*

(a) *The gcd of all integers in the three lists:*

$$\{(a_r - b_r) : \sigma_r = 0\}, \{a_r : \sigma_r \neq 0\}, \{b_r : \sigma_r \neq 0\}$$

divides $(b_0 - a_0)$; *and*

(b) $\overline{\mu} \leq b_0 - a_0 \leq \overline{\nu}$,

Corollary 1 *If statement T depends on statement S in the loop nest* **L**, *then the conditions of Theorem 6.2 hold for some dependence level ℓ such that $1 \le \ell \le m+1$ (or $1 \le \ell \le m$, if $T \le S$).*

Corollary 1 *If statement T depends on statement S in the loop nest* **L**, *then the conditions of Theorem 6.3 hold for some direction vector $\boldsymbol{\sigma} \succeq \mathbf{0}$ (or $\boldsymbol{\sigma} \succ \mathbf{0}$, if $T \le S$).*

Example

$$
\begin{array}{lll}
L_1: & \textbf{do } I_1 = 10, 100, 1 & \\
L_2: & \quad \textbf{do } I_2 = 2, 50, 1 & \\
\quad S: & \qquad X(I_1 + I_2 - 10) = \cdots & \\
\quad T: & \qquad \cdots = \cdots X(2I_1 + I_2 + 31) \cdots & \\
 & \quad \textbf{enddo} & \\
 & \textbf{enddo} &
\end{array}
$$

We check for dependence of T on S with the direction vector (1, - 1).

The dependence equation is

$$(i_1 - 2j_1) + (i_2 - j_2) = 41$$

- The gcd of the coefficients is 1, so that this equation has integer solutions. Thus, the gcd test is not effective in this case.
- The dependence constraints are

$$10 \le i_1 \le 100, \qquad 2 \le i_2 \le 50,$$
$$10 \le j_1 \le 100, \qquad 2 \le j_2 \le 50.$$

- The direction vector (1, - 1) imposes the additional conditions:

$$i_1 \le j_1 - 1 \quad \text{and} \quad j_2 \le i_2 - 1$$

- All these inequalities together define a polytope P in R^4. In terms of real variables, the dependence equation can be written as

$$f(x) = 41$$

where, $x = (x_1, x_2, y_1, y_2)$ and $f : \mathbf{R}^4 \rightarrow \mathbf{R}$ is a linear function defined by

$$f(\mathbf{x}) = (x_1 - 2y_1) + (x_2 - y_2)$$

The bounds test says that dependence cannot exist unless

$$\min_{\mathbf{x} \in P} f(\mathbf{x}) \leq 41 \leq \max_{\mathbf{x} \in P} f(\mathbf{x})$$

- Now, we have

$$\min_{\mathbf{x}\in P} f(\mathbf{x}) = \min_{\substack{10\le x_1\le 100\\ 10\le y_1\le 100\\ x_1\le y_1-1}} (x_1 - 2y_1) + \min_{\substack{2\le y_2\le 50\\ 2\le x_2\le 50\\ y_2\le x_2-1}} (-y_2 + x_2)$$

$$= \mu(1, -2, 10, 100, 1) + \mu(-1, 1, 2, 50, 1)$$

$$= -189$$

$$\max_{\mathbf{x}\in P} f(\mathbf{x}) = \max_{\substack{10\le x_1\le 100\\ 10\le y_1\le 100\\ x_1\le y_1-1}} (x_1 - 2y_1) + \max_{\substack{2\le y_2\le 50\\ 2\le x_2\le 50\\ y_2\le x_2-1}} (-y_2 + x_2)$$

$$= \nu(1, -2, 10, 100, 1) + \nu(-1, 1, 2, 50, 1)$$

$$= 36$$

Since the condition **-189 < 41 < 36** does not hold, there is no dependence of T on S with the direction vector (1, - 1).

Rectangular Loops, One-Dimensional Array

- Let S and T denote any two statements in the program. Let m_s *denote the number of loops in the nest* L_s *determined* by S, m_T the number of loops in the nest L_T determined by T, and m the number of loops in the nest determined by both.

$$
\begin{aligned}
\mathbf{L} \ &= (L_1, L_2, \ldots, L_m) \\
\mathbf{L}_S &= (L_1, L_2, \ldots, L_m, L_{m+1}, \ldots, L_{m_S}) \\
\mathbf{L}_T &= (L_1, L_2, \ldots, L_m, L_{m_S+1}, \ldots, L_{m_S+m_T-m})
\end{aligned}
$$

Theorem 6.4 *If in the model program of this section, two variables*

$$X\left(a_0 + \sum_{r=1}^{m_S} a_r I_r\right) \text{ and } X\left(b_0 + \sum_{r=1}^{m} b_r I_r + \sum_{r=m+1}^{m_T} b_r I_{m_S+r-m}\right)$$

of statements S and T, respectively, cause a dependence of T on S at a level ℓ, then the following two conditions hold:

(a) *The gcd of*

$$a_1 - b_1, a_2 - b_2, \ldots, a_{\ell-1} - b_{\ell-1}, a_\ell, \ldots, a_{m_S}, b_\ell, \ldots, b_{m_T}$$

divides $(b_0 - a_0)$; and

(b) $\overline{\mu} \leq b_0 - a_0 \leq \overline{\nu}$,

Corollary 1 *If statement T depends on statement S in the model program, then the conditions of Theorem 6.4 hold for some dependence level ℓ such that $1 \leq \ell \leq m + 1$ (or $1 \leq \ell \leq m$, if $T \leq S$).*

Theorem 6.5 *Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ denote a vector whose elements are members of the set $\{0, 1, -1, *\}$. If in the model program of this section, two variables*

$$X\left(a_0 + \sum_{r=1}^{m_S} a_r I_r\right) \text{ and } X\left(b_0 + \sum_{r=1}^{m} b_r I_r + \sum_{r=m+1}^{m_T} b_r I_{m_S+r-m}\right)$$

of statements S and T, respectively, cause a dependence of T on S with the direction vector $\boldsymbol{\sigma}$, then the following two conditions hold:

(a) *The gcd of all integers in the three lists:*

$$\{a_r - b_r : 1 \le r \le m \text{ and } \sigma_r = 0\},$$
$$\{a_r : 1 \le r \le m \text{ and } \sigma_r \ne 0, \text{ or } m+1 \le r \le m_S\},$$
$$\{b_r : 1 \le r \le m \text{ and } \sigma_r \ne 0, \text{ or } m+1 \le r \le m_T\}$$

divides $(b_0 - a_0)$; and

(b) $\overline{\mu} \le b_0 - a_0 \le \overline{\nu}$,

Corollary 1 *If statement T depends on statement S in the model program, then the conditions of Theorem 6.5 hold for some direction vector $\boldsymbol{\sigma} \succeq \mathbf{0}$ (or $\boldsymbol{\sigma} \succ \mathbf{0}$, if $T \le S$).*

Example

$$
\begin{array}{lll}
L_1: & \textbf{do } I_1 = 10, 100, 1 & \\
L_2: & \quad \textbf{do } I_2 = 0, 100, 2 & \\
L_3: & \quad\quad \textbf{do } I_3 = I_1, I_1 + 10, 1 & \\
\quad S: & \quad\quad\quad X(2I_1 + 2I_2 - 2I_3 + 5) = \cdots & \\
 & \quad\quad \textbf{enddo} & \\
L_4: & \quad\quad \textbf{do } I_4 = I_2 + 10, I_2, -1 & \\
\quad T: & \quad\quad\quad \cdots = \cdots X(2I_2 + 2I_4 + 9) \cdots & \\
 & \quad\quad \textbf{enddo} & \\
 & \quad \textbf{enddo} & \\
 & \textbf{enddo} &
\end{array}
$$

- The dependence equation in terms of index value

$$(2i_1 + 2i_2 - 2i_3) - (2j_2 + 2j_4) = 4$$

- the index variables I_2, I_3, and I_4 are related to the corresponding iteration variables $\hat{I}_2$, $\hat{I}_3$ and $\hat{I}_4$ by the following equations:

$$\begin{cases} I_2 = 2\hat{I}_2 \\ I_3 = I_1 + \hat{I}_3 \\ I_4 = (I_2 + 10) - \hat{I}_4 = 2\hat{I}_2 - \hat{I}_4 + 10 \end{cases}$$

- we can rewrite the dependence equation in the form

$$4\hat{\imath}_2 - 8\hat{\jmath}_2 - 2\hat{\imath}_3 + 2\hat{\jmath}_4 = 24$$

- The ranges of I_1, $\hat{I}_2$, $\hat{I}_3$, and ,$\hat{I}_4$ are given by

$$10 \le I_1 \le 100, \qquad 0 \le \hat{I}_3 \le 10$$
$$0 \le \hat{I}_2 \le 50, \qquad 0 \le \hat{I}_4 \le 10$$

- From these inequalities, we get the dependence constraints in terms of i_1 , j_1 , $\hat{i}_2$, $\hat{j}_2$, $\hat{i}_3$, and $\hat{j}_4$:

$$10 \le i_1 \le 100, \qquad 0 \le \hat{i}_2 \le 50, \qquad 0 \le \hat{i}_3 \le 10,$$
$$10 \le j_1 \le 100, \qquad 0 \le \hat{j}_2 \le 50, \qquad 0 \le \hat{j}_4 \le 10$$

- The simplified equation

$$2\hat{\imath}_2 - 4\hat{\jmath}_2 - \hat{\imath}_3 + \hat{\jmath}_4 = 12$$

- Denote a vector in R^5 by x = (x_1, x_2, y_2, x_3, y_4), and a linear function **f:R^5 →R** by

$$f(\mathbf{x}) = 2x_2 - 4y_2 - x_3 + y_4$$

- the dependence equation has the form f(x) = 12

$$\min_{\mathbf{x}\in P} f(\mathbf{x}) \leq 12 \leq \max_{\mathbf{x}\in P} f(\mathbf{x})$$

$$10 \le x_1 \le 100, \qquad 0 \le x_2 \le 50, \qquad 0 \le x_3 \le 10,$$

$$0 \le y_2 \le 50, \qquad 0 \le y_4 \le 10.$$

$$x_2 \le y_2 - 1,$$

$$\min_{\mathbf{x} \in P} f(\mathbf{x}) = \min_{\substack{0 \le x_2 \le 50 \\ 0 \le y_2 \le 50 \\ x_2 \le y_2 - 1}} (2x_2 - 4y_2) + \min_{0 \le x_3 \le 10} (-x_3) + \min_{0 \le y_4 \le 10} y_4$$

$$= \mu(2, -4, 0, 50, 1) + \mu(-1, 0, 0, 10, 0) + \mu(1, 0, 0, 10, 0)$$

$$= -210$$

$$\max_{\mathbf{x} \in P} f(\mathbf{x}) = \max_{\substack{0 \le x_2 \le 50 \\ 0 \le y_2 \le 50 \\ x_2 \le y_2 - 1}} (2x_2 - 4y_2) + \max_{0 \le x_3 \le 10} (-x_3) + \max_{0 \le y_4 \le 10} y_4$$

$$= \nu(2, -4, 0, 50, 1) + \nu(-1, 0, 0, 10, 0) + \nu(1, 0, 0, 10, 0)$$

$$= 6$$

- Since the condition $-210 < 12 < 6$ does not hold, the bounds test rules out the dependence of statement T on statement S at level 2.

Rectangular Loops, Multi-Dimensional Array

Theorem 6.6 *Consider the perfect loop nest* $\mathbf{L}$ *of Section 6.2, and two statements* S *and* T *in its body. Let* $X(\mathbf{IA} + \mathbf{a}_0)$ *denote a program variable of* S *and* $X(\mathbf{IB} + \mathbf{b}_0)$ *a variable of* T*, where* X *is an* n*-dimensional array,* $\mathbf{a}_0 = (a_{01}, a_{02}, \ldots, a_{0n})$ *and* $\mathbf{b}_0 = (b_{01}, b_{02}, \ldots, b_{0n})$ *are integer* n*-vectors, and* $\mathbf{A} = (a_{rk})$ *and* $\mathbf{B} = (b_{rk})$ *are* $m \times n$ *integer matrices. If these variables cause a dependence of* T *on* S *at a level* ℓ*, then the following conditions hold:*

(a) *For each k in* $1 \le k \le n$*, the gcd of*

$$a_{1k} - b_{1k}, \ldots, a_{(\ell-1)k} - b_{(\ell-1)k}, a_{\ell k}, \ldots, a_{mk}, b_{\ell k}, \ldots, b_{mk}$$

divides $(b_{0k} - a_{0k})$*; and*

(b) *For each k in* $1 \le k \le n$*, we have*

$$\overline{\mu}_k \le b_{0k} - a_{0k} \le \overline{\nu}_k,$$

Example

$$
\begin{array}{lll}
L_1: & \textbf{do } I_1 = 1, 50, 1 & \\
L_2: & \quad \textbf{do } I_2 = 2, 50, 1 & \\
\quad S: & \quad\quad X(2I_1 + 3I_2 + 50, 3I_1 + I_2 + 49) = \cdots & \\
\quad T: & \quad\quad \cdots = \cdots X(I_1 - I_2 + 51, 2I_1 - I_2 + 48) \cdots & \\
 & \quad \textbf{enddo} & \\
 & \textbf{enddo} &
\end{array}
$$

the dependence of T on S with the direction vector (1, 1). The dependence equations are

$$2i_1 - j_1 + 3i_2 + j_2 = 1$$
$$3i_1 - 2j_1 + i_2 + j_2 = -1$$

- The gcd test in each case is inconclusive. The inequalities for the bounds test applied separately to the two dimensions are

$$\mu(2,-1,1,50,1)+\mu(3,1,2,50,1)\le 1 \le \nu(2,-1,1,50,1)+\nu(3,1,2,50,1),$$

$$\mu(3,-2,1,50,1)+\mu(1,1,2,50,1)\le -1 \le \nu(3,-2,1,50,1)+\nu(1,1,2,50,1),$$

or

$$-39 \le 1 \le 245 \qquad \text{and} \qquad -92 \le -1 \le 146$$

We write,

$$f_1(\mathbf{x}) = 1$$
$$f_2(\mathbf{x}) = -1$$

Where,

$$\mathbf{x} = (x_1, y_1, x_2, y_2)$$
$$f_1(\mathbf{x}) = 2x_1 - y_1 + 3x_2 + y_2$$
$$f_2(\mathbf{x}) = 3x_1 - 2y_1 + x_2 + y_2.$$

Let P denote the polytope in R^4 defined by the inequalities

$$1 \le x_1 \le 50, \qquad 2 \le x_2 \le 50,$$
$$1 \le y_1 \le 50, \qquad 2 \le y_2 \le 50,$$
$$x_1 \le y_1 - 1, \qquad x_2 \le y_2 - 1$$

The real solution in P

$$\max_{\lambda} \min_{\mathbf{x}\in P} [f_1(\mathbf{x}) + \lambda(f_2(\mathbf{x}) + 1)] \le 1$$

$$\le \min_{\lambda} \max_{\mathbf{x}\in P} [f_1(\mathbf{x}) + \lambda(f_2(\mathbf{x}) + 1)]$$

- Since

$$\begin{aligned} f_1(\mathbf{x}) + \lambda(f_2(\mathbf{x}) + 1) \\ &= 2x_1 - y_1 + 3x_2 + y_2 + \lambda(3x_1 - 2y_1 + x_2 + y_2 + 1) \\ &= (2 + 3\lambda)x_1 - (1 + 2\lambda)y_1 + (3 + \lambda)x_2 + (1 + \lambda)y_2 + \lambda \end{aligned}$$

- By theorem definition

$$\min_{\mathbf{x}\in P}[f_1(\mathbf{x}) + \lambda(f_2(\mathbf{x}) + 1)]$$

$$= \min_{\substack{1\le x_1\le 50\\ 1\le y_1\le 50\\ x_1\le y_1-1}}[(2+3\lambda)x_1 - (1+2\lambda)y_1] + \min_{\substack{2\le x_2\le 50\\ 2\le y_2\le 50\\ x_2\le y_2-1}}[(3+\lambda)x_2 + (1+\lambda)y_2] + \lambda$$

$$= \mu(2+3\lambda, -1-2\lambda, 1, 50, 1) + \mu(3+\lambda, 1+\lambda, 2, 50, 1) + \lambda$$

$$= [(1+\lambda) - (1+2\lambda) + 48\min(-1-2\lambda, 1+\lambda, 0)] + [(4+2\lambda)2 + (1+\lambda) + 47\min(1+\lambda, 4+2\lambda, 0)] + \lambda$$

$$= 5\lambda + 9 + 48\min(-1-2\lambda, 1+\lambda, 0) + 47\min(1+\lambda, 4+2\lambda, 0)$$

Since

$$\min(-1-2\lambda, 1+\lambda, 0) = \begin{cases} 1+\lambda & \text{if } \lambda \le -1 \\ 0 & \text{if } -1 \le \lambda \le -1/2 \\ -1-2\lambda & \text{if } -1/2 \le \lambda, \end{cases}$$

and

$$\min(1+\lambda, 4+2\lambda, 0) = \begin{cases} 4+2\lambda & \text{if } \lambda \le -3 \\ 1+\lambda & \text{if } -3 \le \lambda \le -1 \\ 0 & \text{if } -1 \le \lambda, \end{cases}$$

- We have

$$\min_{\mathbf{x}\in P}[f_1(\mathbf{x}) + \lambda(f_2(\mathbf{x}) + 1)] = \begin{cases} 8\lambda + 14 & \text{if } \lambda \le -3 \\ 7\lambda + 11 & \text{if } -3 \le \lambda \le -1 \\ 5\lambda + 9 & \text{if } -1 \le \lambda \le -1/2 \\ 3\lambda + 8 & \text{if } -1/2 \le \lambda. \end{cases}$$

- Thus, this minimum is a piecewise-linear function of A. In the interval -∞<= A λ <= -3, its maximum value is equal to

$$\max\left(\lim_{\lambda\to-\infty}(8\lambda + 14), 8(-3) + 14\right) = -10$$

- There are similar results for the other three intervals. Hence, we have

$$\max_{\lambda} \min_{\mathbf{x}\in P} [f_1(\mathbf{x}) + \lambda(f_2(\mathbf{x}) + 1)] = \max(-10, 4, 13/2, \infty)$$
$$= \infty.$$

- T does not depend on S with the direction vector (1, 1)

General program

$$
\begin{array}{lll}
L_1: & \textbf{do } I_1 = 0, 100, 1 & \\
L_2: & \quad \textbf{do } I_2 = 0, I_1, 1 & \\
\quad S: & \qquad X(2I_1 + 3I_2) = \cdots & \\
\quad T: & \qquad \cdots = \cdots X(4I_1 - I_2 + 5) \cdots & \\
 & \quad \textbf{enddo} & \\
 & \textbf{enddo} &
\end{array}
$$

$$2i_1 - 4j_1 + 3i_2 + j_2 = 5$$

$$0 \le i_1 \le 100, \qquad 0 \le i_2 \le i_1,$$
$$0 \le j_1 \le 100, \qquad 0 \le j_2 \le j_1.$$

$$i_1 \le j_1 - 1$$
$$i_2 \le j_2 - 1$$

$$0 \le x_1 \le 100, \qquad 0 \le x_2 \le 100,$$
$$0 \le y_1 \le 100, \qquad 0 \le y_2 \le 100,$$
$$x_1 \le y_1 - 1, \qquad x_2 \le y_2 - 1$$

$$\mathbf{x} = (x_1, x_2, y_1, y_2)$$
$$f(\mathbf{x}) = 2x_1 - 4y_1 + 3x_2 + y_2$$

$$\min_{\mathbf{x} \in P} f(\mathbf{x}) \leq 5 \leq \max_{\mathbf{x} \in P} f(\mathbf{x})$$