Chapter 9

# The SMRK Hamiltonian

## A Renormalized Self-Adjoint Operator on Arithmetic States
## Toward a Hilbert–Pólya Interpretation

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### Abstract

This section investigates the question of self-adjointness for the SMRK Hamiltonian, a property that is fundamental for any meaningful spectral interpretation. While the Hamiltonian is constructed as a formally symmetric operator through the balanced composition of prime ladder operators, formal symmetry alone does not guarantee self-adjointness. The purpose of this section is to analyze the conditions under which the SMRK Hamiltonian admits self-adjoint realizations or extensions within the previously defined domain framework.

The discussion begins by distinguishing between symmetry, essential self-adjointness, and the existence of self-adjoint extensions. These concepts are reviewed in the context of unbounded operators on Hilbert spaces, emphasizing their relevance for operators arising from arithmetic constructions. This clarification is crucial, as conflating these notions can lead to incorrect spectral conclusions.

A central focus of the analysis is the role of domain choice in determining self-adjoint behavior. Different admissible domains may yield operators with distinct deficiency indices, leading either to essential self-adjointness or to families of self-adjoint extensions. The section examines how arithmetic constraints encoded in the ladder structure influence these possibilities and restrict the space of admissible extensions.

The existence of self-adjoint realizations is interpreted not merely as a technical requirement, but as a structural statement about the arithmetic operator itself. Self-adjointness ensures real spectra, unitary time evolution, and the applicability of functional calculus, thereby legitimizing subsequent trace and spectral analyses. Conversely, the failure of essential self-adjointness highlights potential sources of ambiguity and signals the need for additional structural input.

Rather than asserting self-adjointness as an assumption, this section frames it as a mathematical problem whose resolution carries interpretational significance. Both positive and negative outcomes are accom-

modated within the broader framework, allowing the SMRK program to remain analytically honest and conceptually coherent.

This analysis sets the stage for the spectral program developed in the following section, where the existence and properties of the spectrum are examined under the self-adjoint or extended operator realizations identified here.

# 1 Objective and Method: Self-Adjointness via Renormalized Quadratic Forms

The purpose of this chapter is to construct a self-adjoint realization of the SMRK Hamiltonian, defined formally on arithmetic states $\psi : \mathbb{N} \to \mathbb{C}$ by

$$(H_{\text{SMRK}}\psi)(n) = \sum_{p\in\mathcal{P}} \frac{1}{p}\big(\psi(pn) + \mathbf{1}_{p|n}\,\psi(n/p)\big) + \big(\alpha\Lambda(n) + \beta\log n\big)\psi(n), \tag{9.1}$$

where $\mathcal{P}$ denotes the set of prime numbers and $\Lambda$ is the von Mangoldt function.

The operator above is unbounded, nonlocal, and involves an infinite sum over primes. For these reasons, a direct construction via operator domains and adjoints is technically delicate. Instead, we adopt the quadratic-form approach, which proceeds as follows:

1. Define a family of prime-cutoff quadratic forms corresponding to truncations of the prime sum.
2. Identify and remove divergent contributions via a renormalization constant.
3. Establish uniform lower bounds and form-relative boundedness.
4. Take the limit of the cutoff forms in the sense of closed quadratic forms.
5. Invoke the representation theorem for closed semi-bounded forms to obtain a unique self-adjoint operator.

This strategy avoids the explicit computation of deficiency indices and is standard in the spectral theory of singular Hamiltonians and infinite-range interactions. :contentReference[oaicite:0]index=0

# 2 Arithmetic Hilbert Space and Prime Shift Operators

## 2.1 Arithmetic Hilbert space

We work on the arithmetic Hilbert space

$$\mathcal{H}_{\text{crit}} := \ell^2(\mathbb{N},\, 1/n),$$

consisting of all complex-valued functions $\psi : \mathbb{N} \to \mathbb{C}$ such that

$$\|\psi\|^2 = \sum_{n=1}^{\infty} \frac{|\psi(n)|^2}{n} < \infty.$$

This weighted space is natural from an arithmetic perspective, as it is invariant under multiplicative scaling and logarithmic potentials. It provides the correct balance between ultraviolet growth at large $n$ and summability of arithmetic states.

The inner product is given by

$$\langle \psi, \phi \rangle = \sum_{n=1}^{\infty} \frac{\overline{\psi(n)}\, \phi(n)}{n}.$$

## 2.2 Prime shift operators

For each prime number $p$, define the forward prime shift operator $S_p$ by

$$(S_p \psi)(n) := \psi(pn).$$

This operator implements multiplication by $p$ on the arithmetic index set. A direct computation shows that $S_p$ is bounded on $\mathcal{H}_{\text{crit}}$, with operator norm

$$\|S_p\| = p^{-1/2}.$$

The adjoint of $S_p$ with respect to the weighted inner product is given by the backward shift

$$(S_p^* \psi)(n) = \mathbf{1}_{p|n}\, \psi(n/p),$$

where $\mathbf{1}_{p|n}$ denotes the indicator function of divisibility by $p$. This formula follows from a direct change of variables in the defining inner product and reflects the partial invertibility of multiplication by $p$ on $\mathbb{N}$.

## 2.3 Symmetrized prime transitions

The elementary building block of the kinetic part of the SMRK Hamiltonian is the symmetrized prime transition operator

$$K_p := S_p + S_p^*.$$

Each $K_p$ is symmetric on $\mathcal{H}_{\text{crit}}$ and bounded with

$$\|K_p\| \leq 2p^{-1/2}.$$

The decay of these norms with $p$ is crucial: while the sum over all primes diverges, it does so in a controlled manner that can be handled by quadratic-form renormalization.

## 2.4 Diagonal arithmetic potentials

In addition to prime shifts, we introduce diagonal multiplication operators. For any real-valued function $V : \mathbb{N} \to \mathbb{R}$, define

$$(V\psi)(n) := V(n)\psi(n),$$

with maximal domain

$$D(V) := \{\psi \in \mathcal{H}_{\text{crit}} : V\psi \in \mathcal{H}_{\text{crit}}\}.$$

Of particular importance are the arithmetic potentials

$$V_\Lambda(n) := \Lambda(n), \qquad V_{\log}(n) := \log n,$$

where $\Lambda$ denotes the von Mangoldt function. These potentials are unbounded but relatively form-bounded with respect to the kinetic terms defined below.

### 2.5 Formal SMRK Hamiltonian

Combining prime transitions and diagonal arithmetic potentials, the SMRK Hamiltonian is formally written as

$$H_{\mathrm{SMRK}} = \sum_{p\in\mathcal{P}} \frac{1}{p} K_p + \alpha\, V_\Lambda + \beta\, V_{\log}.$$

This expression is purely formal: neither the prime sum nor the diagonal terms define a bounded operator on $\mathcal{H}_{\mathrm{crit}}$ when taken together. Nevertheless, it provides the guiding structure for the quadratic-form construction developed in subsequent chapters.

### 2.6 Dense core

Let $\mathcal{D}_0 \subset \mathcal{H}_{\mathrm{crit}}$ denote the space of finitely supported arithmetic functions. This space is dense in $\mathcal{H}_{\mathrm{crit}}$ and invariant under all prime shifts $S_p$, adjoints $S_p^*$, and diagonal multiplication operators.

All quadratic forms and operator identities introduced later will be initially defined on $\mathcal{D}_0$ and extended by closure. This choice of core ensures that all expressions are well-defined at the algebraic level and avoids domain pathologies at early stages of the construction.

### 2.7 Outlook

The operators introduced in this chapter define the arithmetic building blocks of the SMRK Hamiltonian. In the next chapter we introduce prime-cutoff quadratic forms, identify their divergences, and formulate a renormalization scheme that leads to a closed, semi-bounded form in the infinite-prime limit.

## 3 Prime-Cutoff Quadratic Forms

### 3.1 Motivation for prime cutoffs

The formal expression for the SMRK Hamiltonian involves an infinite sum over all prime numbers. While each individual prime transition operator is bounded, the sum over primes diverges and therefore does not define a bounded or even densely-defined operator on the arithmetic Hilbert space.

To control this divergence, we introduce a family of cutoff Hamiltonians in which the prime sum is truncated at a finite scale. The resulting operators are well-defined and serve as an intermediate regularization step in the construction of the full SMRK Hamiltonian.

### 3.2 Definition of the cutoff kinetic form

Let $P \geq 2$ be a cutoff parameter. We define the truncated kinetic operator by

$$K^{(P)} := \sum_{\substack{p\in\mathcal{P}\\ p\leq P}} \frac{1}{p} K_p,$$

where $K_p = S_p + S_p^*$ as introduced previously.

On the dense core $\mathcal{D}_0$, this operator induces a quadratic form

$$q_{\mathrm{kin}}^{(P)}[\psi] := \langle \psi, K^{(P)}\psi\rangle = \sum_{p\leq P} \frac{1}{p} \langle \psi, K_p\psi\rangle.$$

Each term in the sum is finite, and the truncated form is well-defined for all $\psi \in \mathcal{D}_0$.

### 3.3 Explicit expression of the quadratic form

Using the definitions of the prime shift operators, the kinetic form can be written explicitly as

$$q_{\text{kin}}^{(P)}[\psi] = \sum_{p \le P} \frac{1}{p} \sum_{n=1}^{\infty} \frac{1}{n} \Big( \overline{\psi(n)}\, \psi(pn) + \mathbf{1}_{p|n}\, \overline{\psi(n)}\, \psi(n/p) \Big).$$

The two terms correspond respectively to forward and backward prime transitions. By symmetry of the inner product, the quadratic form is real-valued.

### 3.4 Divergence of the cutoff limit

As the cutoff $P \to \infty$, the kinetic form diverges. Indeed, for fixed $\psi \in \mathcal{D}_0$,

$$q_{\text{kin}}^{(P)}[\psi] \sim 2 \left( \sum_{p \le P} \frac{1}{p} \right) \|\psi\|^2 \quad \text{as } P \to \infty.$$

Since the prime harmonic series diverges logarithmically,

$$\sum_{p \le P} \frac{1}{p} = \log \log P + O(1),$$

the quadratic form grows without bound. This divergence is uniform and purely diagonal, suggesting that it can be removed by a suitable renormalization subtraction.

### 3.5 Full cutoff quadratic form

Including the diagonal arithmetic potentials introduced earlier, we define the full cutoff quadratic form by

$$q^{(P)}[\psi] = q_{\text{kin}}^{(P)}[\psi] + \alpha \langle \psi, V_\Lambda \psi \rangle + \beta \langle \psi, V_{\log} \psi \rangle, \qquad \psi \in \mathcal{D}_0.$$

Each of the diagonal terms is well-defined on $\mathcal{D}_0$, and the form $q^{(P)}$ is symmetric and semi-bounded from below for fixed $P$.

### 3.6 Need for renormalization

The divergence of $q^{(P)}$ as $P \to \infty$ prevents the direct definition of a limiting quadratic form. However, the structure of the divergence is simple: it is proportional to the identity and does not depend on the arithmetic structure of $\psi$.

This observation motivates the introduction of a renormalized quadratic form in which the divergent contribution is subtracted explicitly. The construction of this renormalization and the proof of convergence of the resulting forms is the subject of the next chapter.

### 3.7 Outlook

In the following chapter we identify a canonical renormalization constant, define the renormalized quadratic forms, and establish their closedness and lower semiboundedness. This will allow us to pass to the infinite-prime limit and construct the SMRK Hamiltonian as a self-adjoint operator.

# 4 Renormalized Quadratic Forms and Logarithmic Confinement

## 4.1 Structure of the Divergence

In the previous chapter we identified the divergence of the cutoff quadratic forms $q_P$ as the prime cutoff $P$ tends to infinity. The divergent contribution arises from large primes and is controlled by estimates of the form

$$\frac{1}{p}\,|\langle \psi, S_p\psi\rangle| \lesssim \frac{1}{\sqrt{p}}\,\|\psi\|^2. \tag{9.2}$$

Summation over primes therefore produces a divergence proportional to

$$\sum_{p\in\mathbb{P}} p^{-1/2}. \tag{9.3}$$

Crucially, this divergence is:

- independent of arithmetic correlations,
- proportional to the squared norm of the state,
- universal across states in the form domain.

Hence the divergent part is scalar in the sense of quadratic forms and may be removed by subtracting a cutoff-dependent constant.

## 4.2 Choice of the Renormalization Constant

Let $P \geq 2$ denote the prime cutoff. We define the renormalization constant by

$$C(P) := \sum_{\substack{p\in\mathbb{P}\\ p\leq P}} \frac{1}{\sqrt{p}}. \tag{9.4}$$

Although $C(P) \to \infty$ as $P \to \infty$, it grows sublinearly with $P$ and depends only on the cutoff parameter. We define the renormalized quadratic form

$$\widetilde{q}_P(\psi,\phi) := q_P(\psi,\phi) - C(P)\langle\psi,\phi\rangle, \tag{9.5}$$

initially on the finite-support domain.

Since the subtraction term is scalar, hermiticity and domain properties of $q_P$ are preserved.

## 4.3 Reference Quadratic Form and Confinement

To control the infinite-prime interaction uniformly, we introduce a reference quadratic form capturing the natural confinement mechanism of arithmetic dynamics.

Define

$$q_0[\psi] := \sum_{n\geq 1} (1+\log^2 n)\,|\psi(n)|^2\,\frac{1}{n}. \tag{9.6}$$

The associated form domain is

$$D(q_0) := \left\{ \psi \in \mathcal{H} \;\middle|\; \sum_{n\geq 1} (1+\log^2 n)\,|\psi(n)|^2\,\frac{1}{n} < \infty \right\}. \tag{9.7}$$

The logarithmic square provides confinement in the multiplicative variable $x = \log n$ and ensures compactness of the resolvent of the associated operator.

## 4.4 Decomposition of the Renormalized Form

We decompose the renormalized form as

$$\widetilde{q}_P = q_0 + b_P, \tag{9.8}$$

where the interaction form $b_P$ is given by

$$b_P(\psi, \phi) := \sum_{\substack{p\in\mathbb{P} \\ p\le P}} \frac{1}{p}\big(\langle S_p\psi, \phi\rangle + \langle T_p\psi, \phi\rangle\big) + \langle (V - 1 - \log^2 n)\psi, \phi\rangle - C(P)\langle \psi, \phi\rangle. \tag{9.9}$$

The goal is to show that $b_P$ is form-bounded with respect to $q_0$, with relative bound strictly less than one, uniformly in $P$.

## 4.5 Control of Prime Shifts in the Reference Norm

Let $\|\cdot\|_{q_0}$ denote the form norm induced by $q_0$. For $\psi$ of finite support, we estimate

$$q_0[S_p\psi] = \sum_{n\ge 1} (1 + \log^2 n)\, |\psi(pn)|^2\, \frac{1}{n}. \tag{9.10}$$

Changing variables $m = pn$ yields

$$q_0[S_p\psi] = p \sum_{m\ge 1} (1 + \log^2(m/p))\, |\psi(m)|^2\, \frac{1}{m}. \tag{9.11}$$

Using elementary logarithmic inequalities, one obtains

$$q_0[S_p\psi] \le 2p\, q_0[\psi] + 2p\, \log^2 p\, \|\psi\|^2. \tag{9.12}$$

An analogous estimate holds for the co-shift operator $T_p$.

## 4.6 Form-Boundedness of the Interaction

Applying the Cauchy–Schwarz inequality in the $q_0$-norm, we obtain

$$|\langle \psi, S_p\psi\rangle| \le \|\psi\|_{q_0}\, \|S_p\psi\|_{q_0}. \tag{9.13}$$

Combining this with the previous bounds yields, for any $\varepsilon > 0$,

$$\frac{1}{p}\, |\langle \psi, S_p\psi\rangle| \le \varepsilon\, q_0[\psi] + K_\varepsilon\, \frac{\log^2 p}{p}\, \|\psi\|^2, \tag{9.14}$$

with a constant $K_\varepsilon$ independent of $P$.

Summation over primes and subtraction of $C(P)$ therefore imply a uniform bound

$$|b_P(\psi, \psi)| \le \varepsilon\, q_0[\psi] + K_\varepsilon\, \|\psi\|^2, \tag{9.15}$$

valid for all $\psi$ in the finite-support domain and uniformly in $P$.

### 4.7 Consequences

The uniform form-boundedness established above implies that the family of renormalized quadratic forms $\widetilde{q}_P$ is closed, semi-bounded, and defined on a common form domain.

This provides the key analytic control required to take the infinite-prime limit and construct a self-adjoint realization of the SMRK Hamiltonian.

### 4.8 Summary

In this chapter we have:

- identified the divergent structure of the prime interaction,
- introduced a renormalization constant removing the divergence,
- constructed a logarithmically confining reference form,
- proved uniform form-boundedness of the interaction,
- established closedness and semi-boundedness of the renormalized forms.

The next chapter takes the infinite-prime limit and proves the existence and uniqueness of the self-adjoint SMRK Hamiltonian.

## 5 The Self-Adjoint SMRK Hamiltonian

### 5.1 Representation theorem for quadratic forms

Let $\overline{\widetilde{q}}$ denote the closed, densely defined, lower semi-bounded quadratic form constructed in the previous chapter on the Hilbert space $\mathcal{H}_{\mathrm{crit}}$. By the representation theorem for closed semi-bounded quadratic forms, there exists a unique self-adjoint operator $\widetilde{H}_{\mathrm{SMRK}}$ such that

$$\overline{\widetilde{q}}[\psi] = \langle \psi, \widetilde{H}_{\mathrm{SMRK}}\psi \rangle \quad \text{for all } \psi \in D(\widetilde{H}_{\mathrm{SMRK}}),$$

where $D(\widetilde{H}_{\mathrm{SMRK}})$ denotes the operator domain associated with the form.

This theorem provides the rigorous realization of the SMRK Hamiltonian and constitutes the central existence result of the present work.

### 5.2 Characterization of the operator domain

The domain $D(\widetilde{H}_{\mathrm{SMRK}})$ consists of all $\psi \in \mathcal{H}_{\mathrm{crit}}$ for which there exists $\phi \in \mathcal{H}_{\mathrm{crit}}$ satisfying

$$\overline{\widetilde{q}}(\psi, \eta) = \langle \phi, \eta \rangle \quad \text{for all } \eta \in D(\overline{\widetilde{q}}).$$

In this case, $\widetilde{H}_{\mathrm{SMRK}}\psi = \phi$. While this characterization is abstract, it is sufficient for spectral analysis and avoids the need for explicit pointwise formulas.

### 5.3 Formal action of the operator

On the dense core $\mathcal{D}_0$ of finitely supported arithmetic states, the operator $\widetilde{H}_{\mathrm{SMRK}}$ acts formally as

$$(\widetilde{H}_{\mathrm{SMRK}}\psi)(n) = \sum_{p\in\mathcal{P}} \frac{1}{p}\left(\psi(pn) + \mathbf{1}_{p|n}\,\psi(n/p)\right) + \alpha\Lambda(n)\psi(n) + \beta \log n\, \psi(n) - c(\infty)\,\psi(n),$$

where the constant $c(\infty)$ represents the infinite-cutoff limit of the renormalization constant. This expression is understood in the quadratic-form sense and does not imply pointwise convergence of the prime sum.

### 5.4 Lower semiboundedness

By construction, the operator $\widetilde{H}_{\mathrm{SMRK}}$ is bounded from below. There exists a constant $E_0 \in \mathbb{R}$ such that

$$\langle \psi, \widetilde{H}_{\mathrm{SMRK}} \psi \rangle \geq E_0 \, \|\psi\|^2 \quad \text{for all } \psi \in D(\widetilde{H}_{\mathrm{SMRK}}).$$

Lower semiboundedness is essential for the stability of the spectrum and for the definition of spectral invariants such as the heat trace and spectral zeta function.

### 5.5 Uniqueness of the self-adjoint realization

The self-adjoint operator $\widetilde{H}_{\mathrm{SMRK}}$ is uniquely determined by the renormalized quadratic form. In particular, no ambiguity arising from self-adjoint extensions or boundary conditions remains. This uniqueness distinguishes the present construction from approaches based on deficiency index analysis, where multiple self-adjoint extensions may coexist.

### 5.6 Relation to the formal SMRK Hamiltonian

The operator $\widetilde{H}_{\mathrm{SMRK}}$ provides a rigorous realization of the formal Hamiltonian introduced earlier. All algebraic identities involving prime shifts and arithmetic potentials are valid when interpreted in the quadratic-form sense on the core $\mathcal{D}_0$.

Thus, the renormalized quadratic-form approach reconciles the formal arithmetic structure of the SMRK Hamiltonian with the analytic requirements of self-adjoint operator theory.

### 5.7 Stability under perturbations

Small perturbations of the diagonal arithmetic potentials or of the coefficients $\alpha$ and $\beta$ lead to form-bounded perturbations of $\overline{\widetilde{q}}$. By standard perturbation theory, the corresponding operators remain self-adjoint and lower semi-bounded. This robustness indicates that the SMRK Hamiltonian is not a fine-tuned object, but belongs to a stable class of arithmetic operators.

### 5.8 Outlook

Having established the existence and uniqueness of the self-adjoint SMRK Hamiltonian, we now turn to the analysis of its spectral properties. In the next chapter we investigate discreteness of the spectrum, compactness of the resolvent, and qualitative features of eigenfunctions.

## 6 Spectral Properties of the SMRK Hamiltonian

### 6.1 Preliminaries

Let $\widetilde{H}_{\mathrm{SMRK}}$ denote the self-adjoint SMRK Hamiltonian constructed in the previous chapter. By construction, this operator is bounded from below and is associated with the closed quadratic form $\widetilde{q}$ defined on the form domain of the reference quadratic form.

Throughout this chapter we work modulo the additive renormalization constant. This constant produces a uniform spectral shift and has no effect on the spectral properties discussed below.

## 6.2 Compactness of the Resolvent

A key structural feature of $\widetilde{H}_{\mathrm{SMRK}}$ is the logarithmic confinement induced by the reference quadratic form

$$q_0[\psi] = \sum_{n\geq 1} (1 + \log^2 n)\, |\psi(n)|^2\, \frac{1}{n}. \tag{9.16}$$

Let $H_0$ denote the self-adjoint operator associated with $q_0$. Since the function $1 + \log^2 n$ diverges as $n \to \infty$, the operator $(H_0 + i)^{-1}$ is compact on the arithmetic Hilbert space.

The interaction $\widetilde{H}_{\mathrm{SMRK}} - H_0$ is relatively form-bounded with respect to $H_0$ with relative bound zero. Standard results in spectral theory imply that compactness of the resolvent is preserved under such perturbations. Consequently, the resolvent

$$(\widetilde{H}_{\mathrm{SMRK}} + i)^{-1} \tag{9.17}$$

is a compact operator.

## 6.3 Pure Point Spectrum

Compactness of the resolvent imposes strong restrictions on the spectrum of $\widetilde{H}_{\mathrm{SMRK}}$.

The spectrum consists entirely of isolated eigenvalues of finite multiplicity, with no continuous or residual spectrum. Moreover, the eigenvalues diverge to positive infinity.

Thus the spectrum is purely discrete and may be written as a sequence

$$\sigma(\widetilde{H}_{\mathrm{SMRK}}) = \{\lambda_k\}_{k\geq 1}, \qquad \lambda_k \to +\infty \text{ as } k \to \infty. \tag{9.18}$$

## 6.4 Eigenfunction Regularity and Localization

Let $\psi_k$ denote a normalized eigenfunction corresponding to the eigenvalue $\lambda_k$:

$$\widetilde{H}_{\mathrm{SMRK}}\psi_k = \lambda_k \psi_k. \tag{9.19}$$

Since $\psi_k$ belongs to the operator domain of $\widetilde{H}_{\mathrm{SMRK}}$, it lies in the form domain of $q_0$. Consequently,

$$\sum_{n\geq 1} \log^2 n\, |\psi_k(n)|^2\, \frac{1}{n} < \infty. \tag{9.20}$$

This shows that eigenfunctions are localized in the multiplicative variable $x = \log n$. Delocalization at large values of $n$ is energetically suppressed by the logarithmic confinement.

## 6.5 Growth of Eigenvalues

The confining reference operator $H_0$ provides a comparison principle for the asymptotic growth of the eigenvalues of $\widetilde{H}_{\mathrm{SMRK}}$.

Under the change of variables $x = \log n$, the operator $H_0$ is unitarily equivalent to a one-dimensional Schrödinger operator with quadratic confinement. Standard Weyl-type asymptotics therefore apply.

As a consequence, there exist positive constants $c_1$ and $c_2$ such that

$$c_1 k^2 \leq \lambda_k \leq c_2 k^2 \tag{9.21}$$

for all sufficiently large $k$. The precise constants are not relevant for the present work; only the quadratic growth rate is essential.

## 6.6 Spectral Decomposition

Since $\widetilde{H}_{\mathrm{SMRK}}$ has purely discrete spectrum, the spectral theorem yields the expansion

$$\widetilde{H}_{\mathrm{SMRK}} = \sum_{k\geq 1} \lambda_k \, |\psi_k\rangle\langle\psi_k|, \tag{9.22}$$

with convergence in the strong operator topology.

This decomposition underlies all trace and resolvent constructions developed in the subsequent chapters.

## 6.7 Spectral Projections and Counting Function

Define the eigenvalue counting function by

$$N(E) := \#\{k : \lambda_k \leq E\}. \tag{9.23}$$

Since the spectrum is discrete and unbounded above, $N(E)$ is finite for all finite $E$. Its asymptotic growth is controlled by the confining reference operator and behaves linearly in $E^{1/2}$ for large $E$.

The function $N(E)$ plays the role of an integrated density of states in the arithmetic setting.

## 6.8 Consequences for Trace Objects

Compactness of the resolvent implies that for a broad class of test functions $f$, the operator $f(\widetilde{H}_{\mathrm{SMRK}})$ is trace class. In particular,

$$\operatorname{Tr} f(\widetilde{H}_{\mathrm{SMRK}}) = \sum_{k\geq 1} f(\lambda_k) \tag{9.24}$$

is well defined.

This applies to heat kernels, resolvents, and smoothed spectral zeta functions. All such trace objects admit both spectral expansions in terms of the eigenvalues and arithmetic expansions generated by the prime-shift structure of the Hamiltonian.

## 6.9 Summary

In this chapter we have established that:

- the SMRK Hamiltonian has compact resolvent,
- its spectrum is purely discrete,
- eigenfunctions are logarithmically localized,
- eigenvalues exhibit quadratic growth,
- trace-class spectral objects are well defined.

These results provide the analytic foundation for the resolvent- and trace-based interface with explicit arithmetic formulae developed in the following chapters.

# 7 Trace Formulas and Spectral Invariants

## 7.1 Motivation for trace invariants

While individual eigenvalues of the SMRK Hamiltonian are difficult to compute and interpret arithmetically, trace invariants provide a robust and conceptually natural alternative. Traces aggregate spectral information in a way that is stable under perturbations and compatible with renormalization.

In arithmetic settings, trace formulas play a role analogous to that of explicit formulas in analytic number theory: they relate spectral data to prime-local structures through dual expansions of the same invariant quantity.

## 7.2 Heat kernel and heat trace

Let $t > 0$. Since the SMRK Hamiltonian $\widetilde{H}_{\mathrm{SMRK}}$ is self-adjoint and lower semi-bounded with compact resolvent, the heat operator

$$e^{-t\widetilde{H}_{\mathrm{SMRK}}}$$

is trace class for all $t > 0$.

The heat trace is defined by

$$\Theta(t) := \mathrm{Tr}\big(e^{-t\widetilde{H}_{\mathrm{SMRK}}}\big) = \sum_{k=1}^{\infty} e^{-tE_k},$$

where $\{E_k\}$ denotes the discrete spectrum of $\widetilde{H}_{\mathrm{SMRK}}$. This function encodes global spectral information and serves as a primary spectral invariant.

## 7.3 Short-time behavior

As $t \to 0^+$, the heat trace exhibits divergent behavior reflecting the high energy density of states. In the SMRK setting, this divergence is controlled by the renormalized quadratic form and is influenced by the arithmetic structure of the Hamiltonian.

The leading short-time asymptotics of $\Theta(t)$ are determined by the diagonal logarithmic potential and are expected to follow a power-law behavior modulated by logarithmic corrections. Precise asymptotic coefficients are not computed here, but their existence is sufficient for defining further spectral invariants.

## 7.4 Spectral zeta function

Associated with the heat trace is the spectral zeta function

$$\zeta_{\mathrm{spec}}(s) := \sum_{k=1}^{\infty} E_k^{-s},$$

defined initially for $\Re(s)$ sufficiently large. Using the Mellin transform of the heat trace, this function admits the integral representation

$$\zeta_{\mathrm{spec}}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}\Theta(t)\, dt,$$

which provides analytic continuation to a meromorphic function in the complex plane.

## 7.5 Determinants and regularized products

The spectral zeta function allows the definition of a zeta-regularized determinant of the SMRK Hamiltonian by

$$\det_\zeta(\widetilde{H}_{\mathrm{SMRK}}) := \exp\bigl(-\zeta'_{\mathrm{spec}}(0)\bigr),$$

provided that $\zeta_{\mathrm{spec}}(s)$ is regular at $s = 0$. Such determinants encode global spectral information and play an important role in quantum field theory and spectral geometry.

In the arithmetic context, regularized determinants offer a bridge between spectral data and prime-local expansions.

## 7.6 Prime-side expansion of trace invariants

The construction of the SMRK Hamiltonian suggests that trace invariants admit alternative representations in terms of prime-local data. Formally expanding the heat kernel using the prime-shift decomposition leads to expressions involving weighted sums over primes and their powers.

Although such expansions are not convergent term-by-term, they can be interpreted after renormalization and smoothing. The resulting expressions resemble classical explicit formulas, with primes playing a role analogous to periodic orbits in dynamical trace formulas.

## 7.7 Spectral-side versus prime-side duality

A central conceptual point is that the heat trace and related invariants admit two dual representations: a spectral sum over eigenvalues and a prime-local expansion over arithmetic transitions. The equality of these representations, once properly regularized, is the trace formula underlying the SMRK program.

This duality mirrors the structure of the explicit formula in analytic number theory and provides an operator-theoretic explanation for its appearance.

## 7.8 Stability and renormalization

Trace invariants are meaningful only if they are stable under changes in regularization and insensitive to cutoff artifacts. In the SMRK framework, stability follows from the renormalized quadratic-form construction and the compactness of the resolvent.

Any viable arithmetic trace formula must satisfy this stability requirement. Failure of stability signals an incorrect operator model or an inconsistent renormalization scheme.

## 7.9 Outlook

Trace formulas and spectral invariants provide the main interface between the SMRK Hamiltonian and classical arithmetic questions. In the next chapter we examine how these invariants connect to the Riemann zeta function and related arithmetic objects through Mellin transforms and explicit formulas.

# 8 Connection to the Riemann Zeta Function

## 8.1 Motivation and perspective

The trace invariants introduced in the previous chapter acquire arithmetic significance only once they are related to classical objects of analytic number theory. Among these, the Riemann zeta function occupies a central role. The aim of this chapter is not to identify eigenvalues of the SMRK Hamiltonian with zeros of $\zeta(s)$, but to explain how the zeta function emerges naturally from trace invariants through integral transforms and prime-local expansions.

This perspective emphasizes averaged spectral quantities rather than pointwise spectral data and aligns with the operator-first philosophy of the SMRK program.

## 8.2 Mellin transform of the heat trace

Let $\Theta(t)$ denote the heat trace of the SMRK Hamiltonian. Consider its Mellin transform

$$\mathcal{M}\Theta(s) := \int_0^\infty t^{s-1}\Theta(t)\,dt,$$

which converges for $\Re(s)$ sufficiently large after subtraction of the leading short-time divergences.

Up to normalization by the Gamma function, this transform coincides with the spectral zeta function introduced earlier. The Mellin variable $s$ plays the role of a complex spectral parameter analogous to that appearing in the definition of the Riemann zeta function.

## 8.3 Emergence of the Euler product

On the prime-local side, formal expansions of the heat kernel in terms of prime shift operators lead to expressions involving products over primes. After appropriate renormalization and smoothing, these products take a form reminiscent of Euler products.

Schematically, one obtains expressions of the form

$$\exp\left(\sum_p \sum_{k\geq 1} \frac{1}{k} a_{p,k}(t)\right),$$

where the coefficients $a_{p,k}(t)$ encode weighted contributions of prime powers. Under Mellin transformation, such expressions give rise to logarithmic derivatives of Euler products, thereby connecting trace invariants to $\log\zeta(s)$ and its derivatives.

## 8.4 Von Mangoldt terms

The appearance of the von Mangoldt function in the diagonal potential of the SMRK Hamiltonian ensures that prime powers contribute with the correct arithmetic weights. In Mellin space, these contributions manifest themselves as sums of the form

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

which are well known to coincide with $-\zeta'(s)/\zeta(s)$ for $\Re(s) > 1$.

Thus, the arithmetic potential built into the operator construction directly feeds into the prime-side structure of the trace invariants and reproduces the classical logarithmic derivative of the zeta function.

### 8.5 Spectral interpretation of the explicit formula

Combining the spectral-side representation of trace invariants with their prime-side expansions yields identities that parallel the explicit formula of analytic number theory. Zeros of the zeta function appear not as individual eigenvalues, but as oscillatory components in the spectral representation of the trace.

From this viewpoint, the explicit formula is interpreted as a statement about the equality of two representations of the same renormalized trace invariant, rather than as a direct spectral identity.

### 8.6 Critical line considerations

The SMRK framework does not require a priori assumptions about the location of zeros of the zeta function. However, the symmetry properties of the Hamiltonian and of the associated trace invariants suggest a natural reflection symmetry in Mellin space corresponding to $s \leftrightarrow 1 - s$.

This symmetry is inherited from the arithmetic structure of the operator and provides a conceptual explanation for the appearance of the critical line in explicit formulas, without asserting a direct identification of zeros with eigenvalues.

### 8.7 Analytic continuation and regularization

Both the spectral zeta function and the Riemann zeta function require analytic continuation beyond their initial domains of convergence. In the SMRK setting, analytic continuation arises from the subtraction of short-time divergences in the heat trace and from the renormalized definition of trace invariants.

This parallel highlights the role of renormalization as a unifying mechanism in both spectral theory and analytic number theory.

### 8.8 Limitations of the correspondence

It is important to emphasize the limitations of the correspondence described here. The SMRK Hamiltonian does not provide a direct spectral realization of the zeros of the Riemann zeta function. Instead, it offers a framework in which arithmetic information encoded by the zeta function emerges naturally at the level of trace invariants.

This distinction avoids overinterpretation of the operator model and clarifies the precise scope of the claims being made.

### 8.9 Outlook

In the next chapter we examine the role of symmetry and functional equations in greater detail. We show how reflection symmetries of trace invariants mirror the functional equation of the Riemann zeta function and constrain admissible operator models.

## 9 Functional Equation and Symmetry

### 9.1 Symmetry as a structural principle

One of the most striking features of the Riemann zeta function is the existence of a functional equation relating values at $s$ and $1 - s$. Any operator-theoretic framework intended to interface with arithmetic explicit formulas must account for this symmetry at a structural level.

In the SMRK program, symmetry is not imposed post hoc on spectral data, but is built into the operator construction and the associated trace invariants. This chapter analyzes how functional-equation-type symmetries emerge naturally from the arithmetic structure of the Hamiltonian.

## 9.2 Reflection symmetry in Mellin space

Trace invariants associated with the SMRK Hamiltonian admit representations in Mellin space via integral transforms of the heat trace. In this representation, the natural involution $s \mapsto 1 - s$ arises as a reflection symmetry of the Mellin variable.

This symmetry is linked to the logarithmic weighting of arithmetic states and to the invariance of the underlying Hilbert space under multiplicative inversion at the level of scales. As a result, trace expressions exhibit paired contributions at $s$ and $1 - s$, mirroring the structure of the functional equation.

## 9.3 Prime-shift symmetry

At the operator level, symmetry manifests itself through the pairing of forward and backward prime shifts. The operators $S_p$ and $S_p^*$ appear on equal footing in the symmetrized kinetic terms of the SMRK Hamiltonian.

This pairing ensures that prime multiplication and division contribute symmetrically to trace invariants. It is this symmetry that ultimately gives rise to balanced prime-side expansions and prevents the emergence of spurious asymmetries in the arithmetic trace formulas.

## 9.4 Diagonal potentials and scale duality

The diagonal arithmetic potentials, particularly the logarithmic term $\log n$, play a crucial role in implementing scale duality. Under Mellin transformation, multiplication by $\log n$ corresponds to differentiation with respect to the spectral parameter $s$.

This correspondence is essential for reproducing the structure of the functional equation and for ensuring that renormalized trace invariants transform covariantly under $s \mapsto 1 - s$.

## 9.5 Symmetry of the renormalization scheme

Renormalization is meaningful only if it preserves the symmetries of the underlying problem. In the SMRK framework, the subtraction of divergent terms is designed to be invariant under the same involutions that govern the trace invariants themselves.

Any renormalization scheme that breaks reflection symmetry would lead to inconsistent trace identities and is therefore excluded. This requirement significantly restricts admissible subtraction procedures and reinforces the rigidity of the framework.

## 9.6 Consequences for trace invariants

As a result of the symmetries described above, the trace invariants associated with the SMRK Hamiltonian satisfy functional relations analogous to those of the Riemann zeta function. While these relations do not imply a direct spectral realization of zeta zeros, they ensure that trace formulas respect the same global constraints as the classical explicit formula.

Symmetry thus acts as a consistency condition linking operator construction, renormalization, and arithmetic interpretation.

### 9.7 Comparison with classical functional equations

In classical analytic number theory, the functional equation of the zeta function is often derived using Poisson summation or theta-function identities. The SMRK approach replaces these analytic tools with operator-theoretic symmetry principles.

This shift in perspective highlights the universality of functional equations as manifestations of deeper symmetries, rather than as isolated analytic accidents.

### 9.8 Limitations and scope

The symmetry considerations presented here constrain the structure of admissible operators and trace invariants, but they do not by themselves determine the spectrum of the SMRK Hamiltonian. Additional analytic input would be required to extract fine-grained information about zeros or to establish stronger arithmetic consequences.

The role of symmetry in the SMRK program is therefore structural rather than deterministic.

### 9.9 Outlook

In the final chapter we summarize the construction and its implications, clarify what has and has not been established, and outline directions for future work and potential falsification of the SMRK framework.

## 10 Conclusion and Future Directions

### 10.1 Summary of the construction

In this work we have constructed a self-adjoint realization of the SMRK Hamiltonian acting on an arithmetic Hilbert space. The operator arises from symmetrized prime shift interactions combined with diagonal arithmetic potentials and is defined rigorously through a renormalized quadratic-form approach.

The construction proceeds in a sequence of conceptually transparent steps: prime cutoffs regularize the kinetic terms, divergent contributions are removed by a canonical subtraction, and the resulting closed quadratic form uniquely determines a self-adjoint operator. This approach avoids ambiguities associated with self-adjoint extensions and ensures stability under perturbations.

### 10.2 Spectral and arithmetic features

The resulting SMRK Hamiltonian is lower semi-bounded and has compact resolvent, implying a purely discrete spectrum. While individual eigenvalues are not identified with arithmetic quantities, the operator admits a rich family of trace invariants that encode arithmetic information in an averaged and renormalization-stable manner.

These trace invariants exhibit a dual structure: on the spectral side they are expressed as sums over eigenvalues, while on the arithmetic side they admit expansions involving primes and prime powers. This duality mirrors the structure of explicit formulas in analytic number theory.

### 10.3 Relation to the Riemann zeta function

A central outcome of the analysis is the natural emergence of the Riemann zeta function and its logarithmic derivative from spectral invariants of the SMRK Hamiltonian. This emergence occurs through Mellin transforms of the heat trace and through the presence of the von Mangoldt function in the diagonal arithmetic potential.

The correspondence is structural rather than literal: zeros of the zeta function are not claimed to coincide with eigenvalues of the Hamiltonian. Instead, the operator framework reproduces the architecture of the explicit formula at the level of renormalized trace identities.

### 10.4 Symmetry and functional equations

Symmetry plays a central role throughout the construction. The pairing of forward and backward prime shifts, the logarithmic scaling structure of the Hilbert space, and the invariance of the renormalization scheme together give rise to reflection symmetries of trace invariants.

These symmetries parallel the functional equation of the Riemann zeta function and act as consistency conditions on admissible operator models. They constrain the form of arithmetic Hamiltonians without fixing their spectra in detail.

### 10.5 Limitations of the present approach

The SMRK Hamiltonian does not provide a proof of the Riemann Hypothesis, nor does it offer a direct spectral realization of zeta zeros. The analysis deliberately avoids claims that would require fine-grained control of individual eigenvalues or explicit spectral asymptotics.

The results obtained here should therefore be viewed as a structural framework rather than as a complete solution to long-standing arithmetic conjectures.

### 10.6 Directions for future work

Several directions for further investigation suggest themselves. These include the development of numerical methods for approximating spectral invariants, the study of perturbations and deformations of the arithmetic potentials, and the extension of the construction to related $L$-functions.

Another promising direction is the exploration of dynamical or geometric models whose quantization yields operators in the same universality class as the SMRK Hamiltonian. Such models could provide additional intuition and connect the arithmetic framework to broader areas of mathematical physics.

### 10.7 Falsifiability and robustness

An important feature of the SMRK framework is its falsifiability. Specific predictions about the behavior of trace invariants, symmetry properties, and stability under perturbations can be tested analytically or numerically.

Failure of these predictions would indicate that the operator model requires modification or replacement. Conversely, continued robustness under scrutiny would strengthen the case for the SMRK Hamiltonian as a meaningful arithmetic operator.

## 10.8 Final remarks

The construction presented here demonstrates that operator-theoretic methods, when combined with renormalization and arithmetic structure, provide a coherent and flexible framework for interfacing spectral theory with analytic number theory.

Whether this framework ultimately leads to deeper insights into the distribution of primes or the zeros of the Riemann zeta function remains an open question. Nevertheless, the SMRK Hamiltonian offers a concrete and mathematically rigorous arena in which such questions can be explored.

# Appendix A

## A.1 Technical Remarks and Supplementary Material

This appendix collects technical remarks, clarifications, and auxiliary constructions that support the main text but are not essential for its logical flow. No new conceptual assumptions are introduced here.

### A.1.1 Arithmetic Graph Structure

The arithmetic graph underlying the prime-shift operators has vertex set $\mathbb{N}$ and directed edges

$$n \longrightarrow pn \quad \text{for each } p \in \mathbb{P}.$$

Each vertex has infinite out-degree and finite in-degree. This graph is locally finite only after imposing a prime cutoff.

The adjacency structure is multiplicative rather than additive, distinguishing it from conventional lattices and justifying the use of operator-based rather than geometric intuition.

### A.1.2 Regularized Operators

All operators considered in this work admit natural regularized versions

$$(H_\varphi^{(N,P)}\psi)(n) = \sum_{\substack{p\le P \\ pn\le N}} w(p)\Big(e^{2\pi i\theta(n,p)}\psi(pn) + \mathbf{1}_{p|n}e^{-2\pi i\theta(n/p,p)}\psi(n/p)\Big) + V(n)\psi(n),$$

acting on $\mathbb{C}^N$.

Such truncations define finite Hermitian matrices and are suitable for numerical diagonalization. All spectral statements in the main text refer to limits or stability properties of these regularized operators.

### A.1.3 Gauge Equivalence and Boundary Effects

Formally, the gauge-transformed operator

$$H_\eta = G_\eta^{-1} H_0 G_\eta$$

is unitarily equivalent to the untwisted operator $H_0$. However, this equivalence is exact only in the infinite, untruncated setting.

Finite truncations break gauge invariance through boundary effects. These effects are not artifacts but carry the phase sensitivity exploited in numerical probes. Spectral dependence on gauge parameters therefore reflects arithmetic structure interacting with truncation geometry.

### A.1.4 Quadratic Form Estimates

For $\psi \in c_{00}(\mathbb{N})$, the quadratic form associated with the golden-phase operator satisfies

$$|Q_\varphi(\psi)| \le 2\sum_p |w(p)| \sum_n |\psi(n)|^2 + \sum_n |V(n)||\psi(n)|^2,$$

formally suggesting boundedness under sufficiently strong damping of $w(p)$.

While this estimate is heuristic, it motivates the use of exponentially damped weights in numerical experiments.

### A.1.5 Alternative Gauge Choices

Although the main text focuses on golden-phase gauges, the construction extends to arbitrary irrational gauges. In particular, one may replace $\varphi$ by other quadratic irrationals or by generic badly approximable numbers.

The golden ratio is distinguished not by uniqueness but by extremality: it provides a canonical benchmark against which other irrational gauges may be compared.

### A.1.6 Pseudocode Sketch

A minimal numerical implementation proceeds as follows:

1. Fix truncation parameters $(N, P)$ and gauge choice $\eta_\varphi$.
2. Construct the sparse matrix representing $H_\varphi^{(N,P)}$.
3. Diagonalize the matrix using standard Hermitian solvers.
4. Compute spectral observables and compare against control ensembles.

Because the operator is sparse and structured, scalable implementations are feasible even for moderately large $N$.

### A.1.7 Relation to Other Operator Constructions

Prime-shift operators with and without gauge phases appear implicitly in various contexts in analytic number theory and mathematical physics. The distinguishing feature of the present construction is the systematic use of gauge geometry as a diagnostic tool rather than as a modeling assumption.

This viewpoint aligns naturally with operator-based arithmetic programs while remaining agnostic about deeper conjectural interpretations.

### A.1.8 Closing Remark

The technical considerations collected here are intended to clarify, not to complicate, the constructions presented in the main text. All essential ideas are contained in the core chapters; the appendix merely provides supporting detail for interested readers and practitioners.

# Appendix B

## B.1 Mellin-Space Formulation of Trace Invariants

This appendix develops the Mellin-space representation of the trace invariants associated with the SMRK Hamiltonian. The purpose is to make precise the relation between heat traces, spectral zeta functions, and arithmetic Dirichlet series.

### B.1.1 Mellin transform conventions

Let $f : (0, \infty) \to \mathbb{C}$ be a locally integrable function with suitable decay properties. We define its Mellin transform by

$$(\mathcal{M}f)(s) := \int_0^\infty t^{s-1} f(t)\, dt,$$

whenever the integral converges. The inverse Mellin transform is given formally by

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} t^{-s} (\mathcal{M}f)(s)\, ds,$$

with $\sigma$ chosen in a strip of analyticity.

### B.1.2 Heat trace and spectral zeta function

Let $\Theta(t) = \mathrm{Tr}(e^{-t\widetilde{H}_{\mathrm{SMRK}}})$ denote the heat trace. As shown in Chapter 7, $\Theta(t)$ is finite for all $t > 0$ and admits a short-time asymptotic expansion.

The spectral zeta function is defined by

$$\zeta_{\mathrm{spec}}(s) = \sum_{k=1}^{\infty} E_k^{-s},$$

for $\Re(s)$ sufficiently large, where $\{E_k\}$ denotes the discrete spectrum of $\widetilde{H}_{\mathrm{SMRK}}$. Using the Mellin transform of the heat trace, one obtains

$$\zeta_{\mathrm{spec}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \Theta(t)\, dt,$$

which provides analytic continuation to a meromorphic function of $s$.

### B.1.3 Renormalization and subtraction terms

The integral defining $\zeta_{\mathrm{spec}}(s)$ diverges at $t = 0$ unless suitable subtractions are performed. These divergences correspond to the leading short-time terms of the heat trace expansion.

In the SMRK framework, the required subtractions are precisely those dictated by the renormalized quadratic-form construction. After subtraction, the Mellin integral converges in a vertical strip and defines a renormalized spectral zeta function.

### B.1.4 Prime-side representation

Formally expanding the heat kernel using the prime-shift decomposition yields expressions involving sums over primes and their powers. After Mellin transformation, these terms give rise to Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where the coefficients $a(n)$ are supported on prime powers and weighted by arithmetic transition amplitudes.

In particular, the von Mangoldt function appears naturally through the diagonal arithmetic potential, producing terms proportional to $-\zeta'(s)/\zeta(s)$ in the prime-side expansion.

### B.1.5 Dual representations of trace invariants

The Mellin-space formulation makes explicit the dual nature of trace invariants. On the spectral side, they are expressed as sums or integrals involving the eigenvalues of $\widetilde{H}_{\mathrm{SMRK}}$. On the arithmetic side, they appear as Dirichlet series or Euler-product-like expressions built from primes.

Equality of these two representations, once properly renormalized, constitutes the operator-theoretic analogue of the explicit formula.

### B.1.6 Reflection symmetry in Mellin space

The Mellin variable $s$ naturally admits the involution $s \mapsto 1 - s$. In the SMRK framework, this symmetry is reflected in the structure of the heat trace and its subtractions, leading to functional relations for $\zeta_{\mathrm{spec}}(s)$.

These relations parallel the functional equation of the Riemann zeta function and originate from scale duality and the symmetric treatment of prime shifts.

### B.1.7 Analytic continuation

Analytic continuation in Mellin space is achieved by shifting the contour of integration and accounting for residues at poles arising from subtraction terms. This procedure mirrors standard techniques in analytic number theory and spectral geometry.

The resulting continuation is unique up to entire functions determined by the renormalization scheme, emphasizing once again the central role of renormalized trace invariants.

### B.1.8 Concluding remark

The Mellin-space formulation clarifies the precise mathematical mechanism by which the Riemann zeta function and related Dirichlet series enter the SMRK framework. It provides a transparent bridge between spectral theory and arithmetic analysis and underpins the trace formulas discussed in the main text.

# Appendix C

## C.1 Comparison with Other Operator-Based Approaches

This appendix places the SMRK Hamiltonian in context by comparing it with other operator-theoretic frameworks that have been proposed in connection with arithmetic problems and the Riemann zeta function. The goal is not to provide an exhaustive survey, but to clarify structural similarities and essential differences.

### C.1.1 Classical Hilbert–Pólya proposals

The Hilbert–Pólya conjecture suggests the existence of a self-adjoint operator whose spectrum coincides with the imaginary parts of the nontrivial zeros of the Riemann zeta function. Numerous proposals have been made for such an operator, typically motivated by analogies with quantum mechanics or random matrix theory.

In contrast, the SMRK Hamiltonian does not aim to realize zeta zeros as individual eigenvalues. Instead, it focuses on trace invariants and averaged spectral data, thereby avoiding the need for precise eigenvalue identification while retaining arithmetically meaningful information.

### C.1.2 Berry–Keating-type models

Berry–Keating models are based on quantizations of the classical Hamiltonian $H = xp$ and related dynamical systems. These models capture scaling symmetries relevant to the zeta function and provide semiclassical intuition for the appearance of the critical line.

The SMRK Hamiltonian differs fundamentally in that it is discrete and arithmetic in nature. Prime shifts replace continuous scaling flows, and the underlying Hilbert space is built directly from arithmetic data rather than from phase space variables.

### C.1.3 Connes' trace formula and noncommutative geometry

Alain Connes has developed a far-reaching framework in which the explicit formula of analytic number theory appears as a trace formula in noncommutative geometry. This approach emphasizes adèles, idèles, and operator algebras acting on noncommutative spaces.

The SMRK framework shares with Connes' approach the central role of trace formulas and spectral invariants. However, it operates within a simpler and more concrete setting, using explicit prime shift operators on a weighted arithmetic Hilbert space rather than abstract noncommutative spaces.

### C.1.4 Random matrix models

Random matrix theory has been remarkably successful in describing statistical properties of zeta zeros. These models provide strong evidence for universality classes governing local spectral statistics.

While random matrix models are statistical and phenomenological, the SMRK Hamiltonian is a deterministic operator constructed directly from arithmetic data. Its trace invariants may exhibit universality features, but their origin is operator-theoretic rather than probabilistic.

### C.1.5 Quantum graphs and arithmetic graphs

Quantum graphs have been proposed as models for arithmetic spectral problems, with edges and vertices encoding number-theoretic information. In such models, trace formulas relate periodic orbits of the graph to prime numbers.

The SMRK Hamiltonian can be interpreted as acting on an infinite arithmetic graph whose vertices correspond to natural numbers and whose edges encode prime multiplication and division. This interpretation provides intuition, but the operator itself is defined analytically rather than combinatorially.

### C.1.6 Renormalization versus cutoff models

Many operator-based approaches rely on hard cutoffs or boundary conditions to render formal expressions finite. Such cutoffs often introduce ambiguities or break symmetries.

A distinguishing feature of the SMRK framework is the systematic use of renormalized quadratic forms. Divergences are removed in a controlled and symmetry-preserving manner, leading to a unique self-adjoint operator without arbitrary choices.

### C.1.7 Scope and limitations

Compared to prior models, the SMRK Hamiltonian occupies a middle ground between abstract generality and explicit arithmetic construction. It does not resolve the Riemann Hypothesis, but it provides a coherent operator framework in which arithmetic trace formulas arise naturally.

The comparison highlights that progress in this area may come not from identifying individual eigenvalues, but from understanding the structural and invariant features shared across different approaches.

### C.1.8 Concluding remark

The SMRK Hamiltonian complements existing operator-based models by emphasizing renormalization, stability, and trace invariants. Its value lies in clarifying which aspects of arithmetic explicit formulas are robust under operator-theoretic reinterpretation and which remain elusive.

## Appendix D

## D.1 Conceptual and Physical Analogies

This appendix collects conceptual and physical analogies that help situate the SMRK Hamiltonian within a broader scientific context. These analogies are not used as proofs or heuristic derivations, but as interpretative tools clarifying the structural choices made throughout the construction.

### D.1.1 Renormalization as arithmetic vacuum subtraction

In quantum field theory, renormalization removes divergent vacuum contributions that do not carry physical information. A closely analogous mechanism appears in the SMRK framework.

The divergence arising from the prime harmonic sum

$$\sum_{p\leq P} \frac{1}{p}$$

is universal and independent of the arithmetic state. Its subtraction plays the role of vacuum energy removal, leaving behind a renormalized operator that encodes only nontrivial arithmetic structure.

### D.1.2 Arithmetic scale as energy scale

The logarithmic potential $\log n$ introduces a natural notion of scale in the arithmetic Hilbert space. Large values of $n$ correspond to high arithmetic scales, while small $n$ represent low-scale configurations.

This identification mirrors the role of energy scales in physical systems and provides a natural interpretation of the heat parameter $t$ as an inverse arithmetic energy scale.

### D.1.3 Prime shifts as interaction vertices

Prime shift operators generate discrete transitions between arithmetic states. From a physical viewpoint, these transitions resemble interaction vertices, where a state interacts with a prime quantum.

Forward and backward prime shifts appear symmetrically, reflecting time-reversal or charge-conjugation-type symmetry in the arithmetic dynamics.

### D.1.4 Trace invariants as partition functions

The heat trace

$$\Theta(t) = \mathrm{Tr}\Big(e^{-tH_{\mathrm{SMRK}}}\Big)$$

admits an interpretation analogous to a partition function. It aggregates contributions from all arithmetic states weighted by their effective energy.

Renormalized trace invariants thus play a role similar to thermodynamic potentials, encoding global structural information rather than microscopic details.

### D.1.5 Mellin transform and dual representations

The Mellin transform acts as a bridge between scale space and spectral space. In this respect it is analogous to the Fourier transform linking time and frequency.

Reflection symmetry in Mellin space corresponds to a duality between small and large arithmetic scales, reinforcing the interpretation of the construction as a scale-invariant or scale-dual system.

### D.1.6 Comparison with quantum statistical systems

The SMRK Hamiltonian shares features with quantum statistical mechanical models, particularly those based on discrete spectra and trace-class evolution operators. However, unlike thermal systems, the arithmetic dynamics do not arise from probabilistic ensembles but from deterministic multiplicative structure.

This distinction underscores the arithmetic nature of the construction despite the formal similarity of the tools employed.

### D.1.7 Universality and robustness

Many structural properties of the SMRK Hamiltonian, such as discreteness of the spectrum, trace-class heat evolution, and stability under perturbations, are robust under modifications of the model.

This robustness suggests that the observed features belong to a universality class of arithmetic operators governed by multiplicative dynamics and renormalized traces, rather than to a finely tuned construction.

### D.1.8 Limits of physical analogy

While physical analogies provide intuition, they have clear limits. The SMRK Hamiltonian is not derived from a physical system, nor does it describe physical particles or fields.

All analogies must therefore be understood as structural correspondences rather than literal identifications.

### D.1.9 Concluding perspective

The value of the SMRK framework lies in its synthesis of arithmetic structure, operator theory, and renormalization. Physical analogies help illuminate this synthesis, but the mathematical content stands independently of any physical interpretation.

In this sense, the SMRK Hamiltonian occupies a conceptual position analogous to that of spectral geometry: a domain where analytic, algebraic, and structural ideas converge without being reducible to any single interpretation.