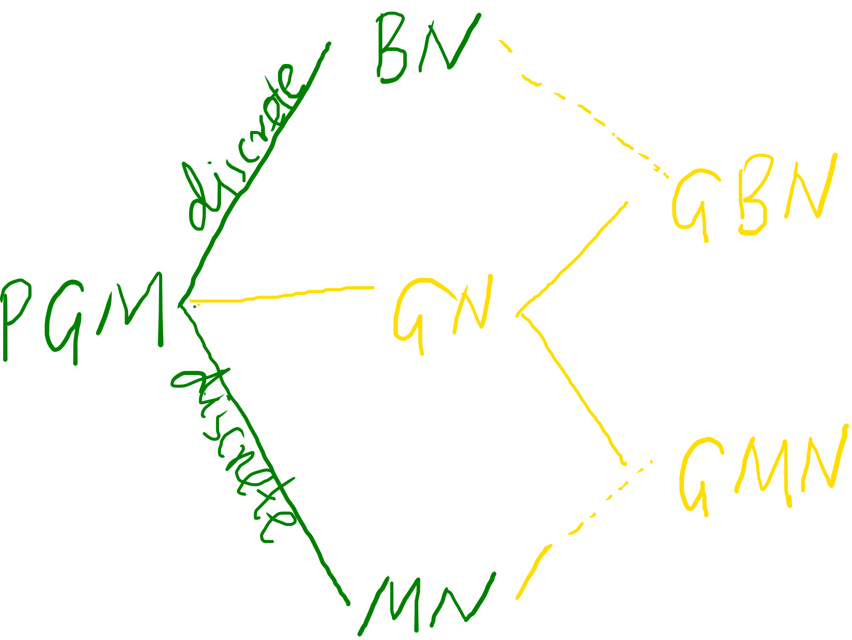
# Gaussian Network
# 高斯网络（高斯图模型）




PGM: probabilistic graphical model

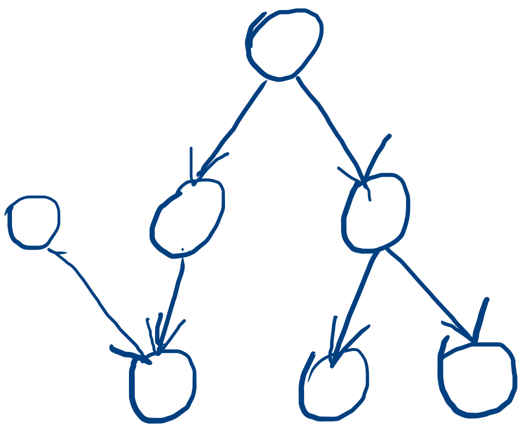
BN: bayesian network

MN: markov network

GN: Gaussian Network

# Gaussian Network

$x = (x_1, x_2, \cdots, x_p)^T$



$p(x_i) \sim N(\mu_i, \sigma_i^2)$

$p(x) \sim N(\mu, \Sigma)$

$$p(x) = \underbrace{\frac{1}{(2\pi)^{\frac{p}{2}} \cdot |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T \cdot \Sigma^{-1} (x-\mu)\right\}} \to \mu, \Sigma$$

$\Lambda = \Sigma^{-1} \to$ precision matrix

$\Lambda = (\lambda_{ij})$

条件独立性: $x_i \perp x_j \mid -\{i,j\}$.

$x_i \perp x_j \Leftrightarrow \sigma_{ij} = 0 \to$ 边缘独立性

$x_i \to$ 边缘概率

$$\Sigma = (\sigma_{ij}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & & & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{pmatrix}_{p \times p}$$

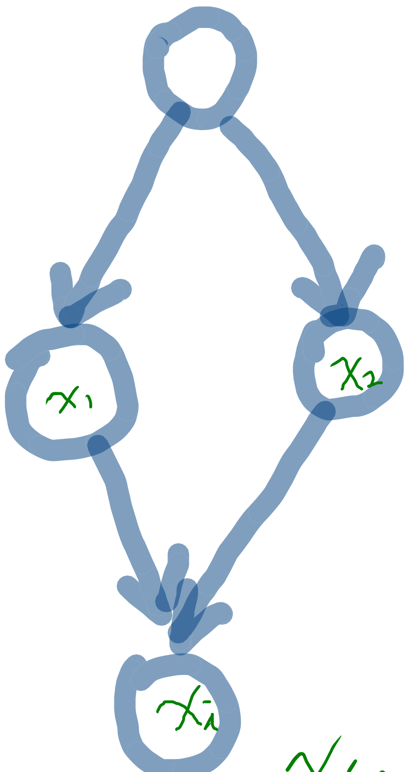
# Gaussian Network

连续型的 PGM

↳ 有向：GBN

$$P(x) = \prod_{i=1}^{p} P(x_i \mid x_{pa(i)}) \longrightarrow \text{BN的因子分解}$$

$x_{pa(i)}$ ↳ 一个集合（父亲）



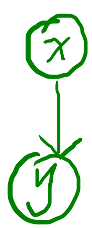
$x_{pa(i)} \rightarrow set = \{x_1, x_2\}$

GBN is based on Linear Gaussian Model

GBN ↓ global model

Linear Gaussian Model ↓ local model

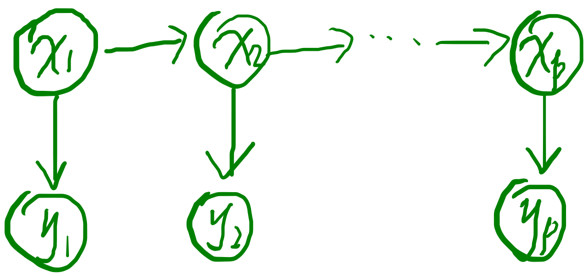
$$\begin{cases} p(x) = N(x \mid \mu_x, \Sigma_x) \\ p(y \mid x) = N(y \mid Ax + b, \Sigma_y) \end{cases}$$



## Kalman Filter (HMM)

$\lambda = (\pi, A, B)$

$P(x_t | x_{t-1})$ $\quad$ $P(y_t | x_t)$




$$\begin{cases} P(x_t | x_{t-1}) = N(x_t | A x_{t-1} + B, \; Q) \\ P(y_t | x_t) = N(y_t | C x_t + D, \; R) \end{cases}$$

$$\begin{cases} x_t = A x_{t-1} + B + \varepsilon, & \varepsilon \sim N(0, Q) \\ y_t = C x_t + D + \delta, & \delta \sim N(0, R) \end{cases}$$

GBN

# Gaussian Bayesian Network

$x = (x_1, x_2, \cdots, x_p)^T$

$x \sim N(\mu, \Sigma)$ $\quad P(x) = \prod_{i=1}^{p} P(x_i \mid x_{pa(i)})$ $\quad x_{pa(i)} = (x_1, x_2, \cdots, x_k)^T$

$P(x_i \mid x_{pa(i)}) = N(x_i \mid \mu_i + W_i^T x_{pa(i)}, \sigma_i^2)$

$x_i$ 是一维的

$x_i = \mu_i + \sum_{j \in x_{pa(i)}} W_{ij} \cdot (x_j - \mu_j) + \sigma_i \cdot \varepsilon_i$

$\varepsilon_i$ is r.v.

$\varepsilon_i \sim N(0,1)$

$\mu = (\mu_1, \mu_2, \cdots, \mu_p)^T$

$x_i - \mu_i = \sum_{j \in x_{pa(i)}} W_{ij} (x_j - \mu_j) + \sigma_i \varepsilon_i$

$W = [W_{ij}]$

$\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)^T$

$x - \mu = W \cdot (x - \mu) + S \cdot \varepsilon$

$S = \mathrm{diag}(\sigma_i)$

$(I - W) \cdot (x - \mu) = S \cdot \varepsilon$

$x - \mu = (I - W)^{-1} S \cdot \varepsilon$

$\Sigma = \mathrm{cov}(x) = \mathrm{cov}(x - \mu) = \mathrm{cov}((I - W)^{-1} S \cdot \varepsilon)$

$x = (x_1, x_2, \cdots, x_p)^T$

# Gaussian Bayesian Network

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$P(x) = \prod_{i=1}^{p} P(x_i \mid x_{pa(i)})$

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