A Note on Density of Dust Sphere

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This simple article is aimed to calculate the density of a dust sphere and show some interesting features.

Imagine a dust sphere in space. It has a Schwarzschild since it's spherical symmetric,

$$\mathrm{d}s^2 = -(1 - \frac{2m(r)}{r})c^2\mathrm{d}t^2 + (1 - \frac{2m}{r})^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{1}$$

To get the density, we have to calculate the volume. Use a time orthonormal space and integrate over a shpere.

$$V(R, M) = \int \frac{1}{\sqrt{1 - \frac{2m(r,M,R)}{r}}} r^2 \sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \tag{2}$$

$$= 4\pi^2 \int \frac{1}{\sqrt{1 - \frac{2m(r)}{r}}} r^2\mathrm{d}r \tag{3}$$

where $m(r, M, R)$ is the mass distribution, M is the total mass of the sphere, R is the total radius (in unit of the coordinate of Shwarzschild metric) of the sphere.

Choose a specific mass uniform distribution,

$$m(r, M, R) = \frac{4/3\pi r^3}{4/3\pi R^3} = (\frac{r}{R})^3 M \tag{4}$$

Then the volume can be simplified,

$$V = 4\pi^2 \int \frac{1}{\sqrt{1 - 2M\frac{r^2}{R^3}}} r^2\mathrm{d}r \tag{5}$$

Finally, the density of the dust can be calculated using the defination$\rho = M/V$

It would help us to understand the behavior of this density if we plot the $\rho \sim R$ and $\rho \sim M$ figure.

Figure (1) shows the density of the dust with $R = 1$ while change total mass M.

Now it is clear that for a raduis 1 sphere, the density it increases at first and then reaches it's peak value 0.816 at point $M = 0.427$ if we add mass to it and keeps it's mass distribution not

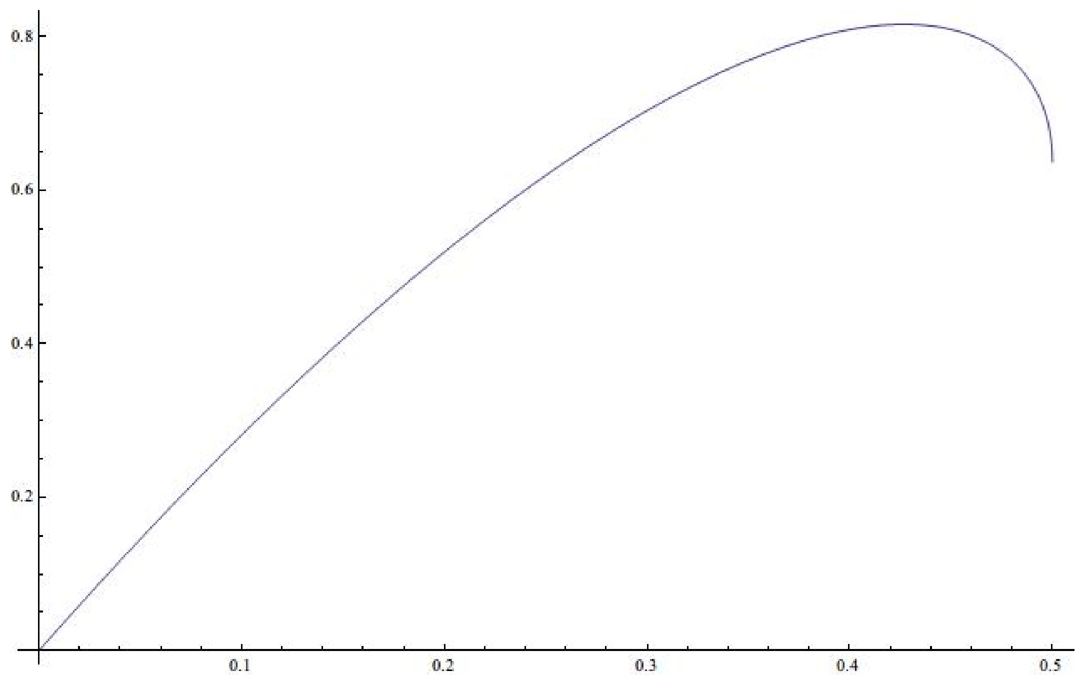

Figure 1: A dust sphere with radius 1 and varying mass.

changed. Then the density drop to $2/\pi$ when $M = 0.5$. It forms a black hole if we add more mass. It is really clear from this figure that the slope could be infinite at $M = 0.5$ and a calculation confirms this guess.

This is weird. If the density really goes to zero, where does the mass go? How does the volume change when the dust forms a black hole? From the expression of density, the volume becomes infinity. However, this is a result of gravity. Can we reform the expression so that the effect is on mass?

The explicit density is

$$\rho(M,R) = \frac{-2R\sqrt{M(-2MR^2+R^3)} + \sqrt{2}R^3 \arcsin(\sqrt{2\frac{M}{R}})}{8(\frac{M}{R})^{3/2}} \tag{6}$$

In equation (6), the mass should be gravitational mass because it changes the background metric through gravity. If the gravitational mass does change, then the gravitational mass should be zero too. This is not easy to model this gravity since we have to keep the expression of both mass, volume and density consistent. I have no idea on this now.

One more thing, the gravity theory should fit the observable universe.

Figure (2) is the change of density when the radius is changed for a $M = 1$ dust sphere.

The peak point of figure (2) is ($R = 2.062$, $\rho = 0.178$). At the critical point occurs at $R = 2$, where the density is $\rho = 2/\pi$. This is consistent with figure (1).

The full behaviour of the density is shown in figure (3).

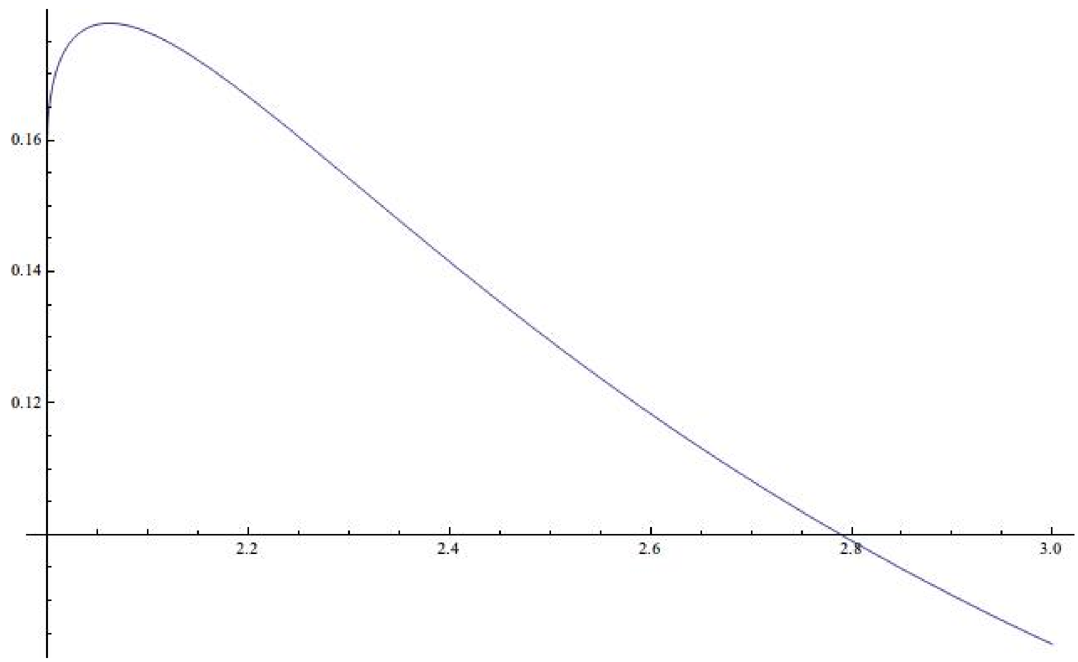

Figure 2: Dust with mass $M = 1$.

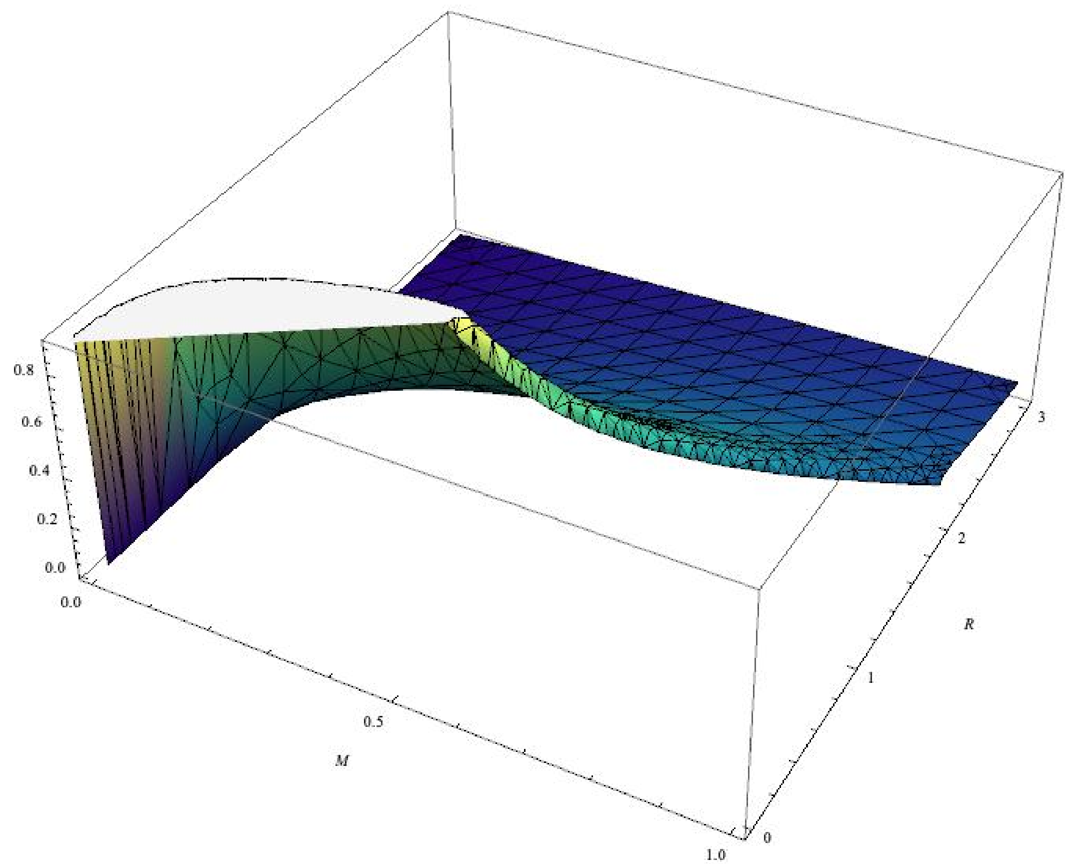

Figure 3: Full behaviour of the density.