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# Master's Project

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# Connective Algebraic K-theory of stable infinity categories

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**Abstract**

This report is aimed at presenting connective $K$-theory as a universal additive invariant. In the first part, we will also give an introduction to $\infty$-categories with listing results we need for proving our aim. The second part will be used to give a almost self-contained proof of our aim with assuming the knowledge of $\infty$-categories.

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# Introduction

In this note, we will introduce the algebraic $K$-theory functor. Classically, there are 4 basic constructions of this functors: plus constructions for rings, group completion construction for symmetric monoidal categories, Quillen's $Q$-construction for exact categories, and Waldhausen's $S$-construction for Waldhausen categories. They coincide for rings but can be useful in different settings. If people look at the definitions of exact categories and Waldhausen categories, then they can find they all were constructed to capture the characters of cofiber sequences, and define it gives split equivalence in $K_0$.

This idea is well-formulated in the modern setting of stable $\infty$-categories, which are pointed $\infty$-categories admitting finite limits, and its cofiber sequences and fiber sequences coincide. Hence, it is a very natural idea to study the algebraic $K$-theory functor over stable $\infty$-categories. It would not be the most general categories we can consider, but because of the stability, we can have a neat description of the functor. Blumberg-Gepner-Tabuada[1] proved that connective $K$-theory is a universal additive invariant with values in connective spectra. The main goal of this project is to prove this theorem, formulated in another form.

There is another perspective to look at this project. Infinity category is a thing that is generally used by algebraic topologists nowadays, so it is somehow fundamental, but there are rare courses that teach students how to use them. This report can be seen as a practice of $\infty$-categories. In the first part, we will cover an introduction to them with no proof given. The theorems will be used in the second part to approach the universality of algebraic $K$-theory functor. We try to give enough details so that we can say the proofs are in the $\infty$-categorical level rather than only in the 1-categorical level.

With accepting the fundamental theorems in $\infty$-categories, we also try to follow the philosophy that use $\infty$-categories model independently. Even though we give the definitions of $\infty$-categories using simplicial sets, it is explained in the end of the first section that we are considering infinity categories as categories enriched over An, which should be accepted as a packaged thing being the soul of the topological spaces. Hence, this report should be ideally readable for people who are familiar with 1-categories, algebraic topology, and some fundamental facts about simplicial sets. This note will basically follow Faibian's lecture note on Algebraic and Hermition $K$-theory [2], which is a lovely note with many interesting remarks.

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## Part I

# Preliminaries in infinity categories

In the first part, we shall introduce the world of infinity categories and also give most results we will use later with no proof given. One reason is that this is a project studying Algebraic $K$-theory. The other reason is that for mathematicians who are interested in Algebraic Topology, it is probably more important to see how infinity categories are used rather than the fundamental results in infinity categories. The results not referenced in this part can be found in Fabian's note.

## 1 Infinity categories, (co)limits and adjunctions

In this section, we will discuss $\infty$-categories and related concepts. To give precise definitions, we use the model of simplicial sets. However, one should always expect not to use simplicial sets in the actual use. The philosophy is that we will accept some theorems and always use $\infty$-category as a category enriched over 'spaces'.

**Definition 1.1.** *An* **$\infty$-category***, or* $(\infty, 1)$ *category* $C$ *is a simplicial set such that for all* $0 < i < n$ *and for all solid diagrams*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & C \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

*there exists a dashed arrow as in the diagram to make the diagram commute. Its objects are the* $0$*-simplices of* $C$*, and a morphism is a* $1$*-simplex in* $C$*. A simplicial set is a* **Kan complex** *if the above lifting also exists for* $i = 0, n$*.*

*Let* $C$*,* $D$ *be two infinity categories. A functor* $F : C \to D$ *is a map of simplicial set. The* **functor category** $\mathrm{Fun}(C, D)$ *is the simplicial set defined by*

$$\mathrm{Fun}(C, D)([n]) := \mathrm{Hom}_{\mathrm{sSet}}(C \times \Delta^n, D).$$

*Here, notice that we use a proposition that* $\mathrm{Fun}(C, D)$ *is an* $\infty$*-category if* $D$ *is an* $\infty$*-category. One defines a* **natural transformation** *between functors* $F, G : C \to D$ *is a* $1$*-simplex in* $\mathrm{Fun}(C, D)$*.*

Recall that there is a nerve construction $N : \mathrm{Cat} \to \mathrm{sSet}$, where the $n$-simplices of $N(C)$ consist of $n$ composable arrows in $C$. Moreover, it has the following proposition:

**Proposition 1.2.** *The nerve functor* $N : \mathrm{Cat} \to \mathrm{sSet}$ *is fully faithful with the essential image given by simplicial sets* $X$ *such that the dashed arrow in the definition above exists uniquely. Moreover, it is a right adjoint functor, whose left adjoint is the homotopy category functor* $\pi : \mathrm{sSet} \to \mathrm{Cat}$*.*

**Example 1.3.** *When $C$ is an $\infty$-category, $\pi(C)$ is a 1-category having $C_0$ as objects and homotopy classes of $C_1$ as morphisms.*

In this way, we can view any 1-category as an infinity category. We will drop the notion $N$, and always recognize a 1-category as a discrete $\infty$-category. Hence, in notations, $\Delta^n$ and $[n]$ are the same categories.

**Definition 1.4.** *Let $C$ be an $\infty$-category. For two objects $a, b$, we define their* **mapping space or hom space** *as the pullback of simplicial sets:*

$$\begin{tikzcd} \hom_C(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "(s,t)"] \\ \Delta^0 \arrow[r] & C \times C \end{tikzcd}$$

*, where $s, t$ are the source and target maps.*

**Definition 1.5.** *An $\infty$-category $C$ is an* **anima or $\infty$-groupoid***, if $\pi C$ is a groupoid in the 1-categorical sense. We also define a morphism in $C$ is an* **equivalence** *if it is an equivalence in $\pi C$.*

**Example 1.6.** *Recall for a 1-category $D$, we have a groupoid $\operatorname{core} D$ defined as the maximal sub-groupoid of $D$. Similarly, for a $\infty$-category $C$, we can define $\operatorname{core} C$ as the pullback of simplicial sets*

$$\begin{tikzcd} \operatorname{core} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "(s,t)"] \\ N(\operatorname{core} \pi C) \arrow[r] & N(\pi C) \end{tikzcd}$$

*,*

For a Kan complex, it is easy to check it is an $\infty$-groupid, but surprisingly, the converse direction is also true, and it leads to the first hard theorem in $\infty$-categories:

**Theorem 1.7.** *(Joyal) Every anima is a Kan complex*

Let us give some examples of infinity categories. The constructions of them needs a concept called coherent nerve, which we will not say here. It is more important for us that they exist.

**Example 1.8.** *We can define an $\infty$-category* An *consists of all animas.*

**Example 1.9.** *There is an $\infty$-category of infinity categories, which we denoted as $\mathrm{Cat}_\infty$, where we have*

$$\hom_{\mathrm{Cat}_\infty}(C, D) \simeq \operatorname{core} \mathrm{Fun}(C, D).$$

**Example 1.10.** *Let $R$ be a commutative ring. We can construct an $\infty$-category of chain complexes, which is denoted by $\mathcal{K}(R)$.*

In 1-categories, the main tools we use are colimits, limits and adjunctions. Here, we can also define them in infinity categories.

**Definition 1.11.** *Let $F : I \to C$ be a functor of $\infty$-categories. Then a* **cone over** *$F$ is a pair consisting of an object $y \in C$, together with a natural transformation $\eta : \mathrm{const}\, y \Rightarrow F$.*

*Given a cone $(y, \eta)$ over $F$ and $x \in C$, we have a canonical map*

$$\hom_C(x, y) \xrightarrow{\mathrm{const}} \hom_{\mathrm{Fun}(I,C)}(\mathrm{const}\, x, \mathrm{const}\, y) \xrightarrow{\eta_*} \hom_{\mathrm{Fun}(I,C)}(c_x, F)$$

*A cone $(y, \eta)$ over $F$ is called a* **limit cone** *if the map above is an equivalence of anima for all $x \in C$, and we write*

$$y := \lim_I F.$$

In a completely dual way, we have the definition of colimit:

**Definition 1.12.** *Let $F : I \to C$ be a functor of $\infty$-categories. Then a* **cone under** *$F$ is a pair consisting of an object $y \in C$, together with a natural transformation $\eta : F \Rightarrow \mathrm{const}\, y$.*

*Given a cone $(y, \eta)$ under $F$ and $x \in C$, we have a canonical map*

$$\hom_C(y, x) \xrightarrow{\mathrm{const}} \hom_{\mathrm{Fun}(I,C)}(\mathrm{const}\, y, \mathrm{const}\, x) \xrightarrow{\eta^*} \hom_{\mathrm{Fun}(I,C)}(F, \mathrm{const}\, x)$$

*A cone $(y, \eta)$ under $F$ is called a* **colimit cone** *if the map above is an equivalence of anima for all $x \in C$, and we write*

$$y := \operatorname*{colim}_I F.$$

Similarly as in 1-categories, we get all our beloved friends:

**Example 1.13.** *Let $I = \emptyset$. The colimit and limit gives initial object and terminal object respectively.*

**Example 1.14.** *Let $I$ be a discrete category. The colimit and limit gives coproducts and products respectively.*

**Example 1.15.** *Let $I$ be the diagram category $\bullet \leftarrow \bullet \rightarrow \bullet$. The colimit gives the pushout diagram.*

**Example 1.16.** *Let $I$ be the diagram category $\bullet \rightarrow \bullet \leftarrow \bullet$. The limit gives the pullback diagram.*

We can also define adjunctions of inifnity categories:

**Definition 1.17.** *Let $R : D \to C$ be a functor of $\infty$-categories:*

- *Let $y \in C$ and $x \in D$, and a morphism $\eta : y \to Rx$ in $C$, we say $\eta$ witnesses $x$ as a* **left adjoint object** *to $y$ under $R$ if the composite*

$$\hom_D(x, -) \xrightarrow{R} \hom_C(Rx, R-) \xrightarrow{\eta^*} \hom_C(y, R-)$$

  *is an equivalence.*

- *An* **adjunction** *between $R$ and a functor $L : C \to D$ is an equivalence:*

$$\hom_D(L-, -) \simeq \hom_C(-, R-)$$

  *as functors $C^{\mathrm{op}} \times C \to \mathrm{An}$.*

**Example 1.18.** *This example is a fundamental example of adjunctions for us. We have the inclusion* $\mathrm{Set} \to \mathrm{An}$ *admits a left adjoint, and the inclusion* $\mathrm{An} \to \mathrm{Cat}_\infty$ *admits both a left adjoint and a right adjoint. They can be shown in the following diagram:*

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "\text{incl.}"] & \mathrm{An} \arrow[l, bend right=60, "\pi_0"'] \arrow[r, "\text{incl.}"] & \mathrm{Cat}_\infty \arrow[l, bend right=60, "|-|"'] \arrow[l, bend left=60, "\mathrm{core}"]
\end{tikzcd}$$

*where $|-|$ is the left adjoint functor, and we define it as the* **geometric realization**.

Notice that because the form we define things are completely similar with 1-categories, so many properties about colimits, limits and adjunctions all hold for $\infty$-categories. In the following, we will list some fundamental results about $\infty$-categories. They are not hard to prove for 1-categories, but they are highly non-trivial for $\infty$-categories.

**Theorem 1.19.**
- *A functor $C \to D$ of $\infty$-categories is an equivalence (i.e. there exists $G : D \to C$ such that $GF \simeq \mathrm{id}_C$ and $FG \simeq id_D$ in functor categories) if and only if it is fully faithful (i.e. the induced map $\mathrm{map}_C(x, y) \to \mathrm{map}_D(F(x), F(y))$ is an equivalence) and essentially surjective (i.e. the induced functor $\pi_0 \operatorname{core} C \to \pi_0 \operatorname{core} D$ is essentially surjective).*
- *A natural transformation $\eta : F \Rightarrow G$ is a natural equivalence (i.e. an equivalence in functor categories) if and only if it induces equivalence in $D$ for all objects in $C$.*

**Theorem 1.20.** *(Yoneda's lemma) Let $C$ be an $\infty$-category. Given a functor $F : C \to \mathrm{An}$ and an object $x \in C$, the evaluation map*

$$\mathrm{ev}_{\mathrm{id}_x} : \mathrm{Nat}(\hom_C(x, -), F) \xrightarrow{\simeq} F(x)$$

*is an equivalence. Moreover, adjoining the functor $\hom_C : C^{\mathrm{op}} \times C \to \mathrm{An}$ gives a fully faithful functor*

$$Y^C : C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

**Definition 1.21.** *For an $\infty$-category $C$, we denote by $\mathcal{P}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ the infinity category of presheaves on $C$.*

Similarly as in 1-categories, $\mathcal{P}(C)$ also has the following universal property:

**Theorem 1.22.** *For every cocomplete $\infty$-category $D$ and every small $\infty$-category $C$, there is an equivalence*

$$\mathrm{Fun}^L(\mathcal{P}(C), D) \xrightarrow{\simeq} \mathrm{Fun}(C, D)$$

*where $\mathrm{Fun}^L(\mathcal{P}(C), D)$ is the full subcategory of $\mathrm{Fun}(C, D)$ spanned by left adjoint functors.*

The Yoneda embedding is also natural. This theorem has a short proof given by Maxime Ramzi [3]:

**Theorem 1.23.** *The Yoneda embedding can be lifted to a functor from small $\infty$-categories to large $\infty$-categories, which we call as the presheaf functor. There is a natural transformation from the inclusion to the presheaf functor, which is the Yoneda embedding pointwise.*

Before ending this section, let us comment on our statement in the start of the section. We use the model of simplicial sets to make things precisely defined, and people use them to prove the hard theorems we mentioned. But in the actual use, we should always think them as categories enriched over An. In most cases, knowing enough properties of anima will be enough in solving our problems, we will list the fundamental facts of anima:

**Theorem 1.24.** *The infinity category* An *has the following properties:*

- An *is complete and cocomplete*
- An *is freely generated under colimits by the point. More precisely, this means for any cocomplete $\infty$-category $C$, the evluation at the point induces an equivalence*

  $$\mathrm{Fun}^L(\mathrm{An}, C) \xrightarrow{\simeq} C$$

  *where* $\mathrm{Fun}^L(\mathrm{An}, C)$ *is the fully subcategory of* $\mathrm{Fun}(\mathrm{An}, C)$ *spanned by the colimit preserving functors.*
- *Given a morphism* $X \to Y \in \mathrm{An}$*, the pullback functor of over categories* $\mathrm{An}_{/Y} \to \mathrm{An}_{/X}$ *defined by* $Z \mapsto X \times_Y Z$ *preserves colimits and has a right adjoint.*
- *There is an adjunction*

  $$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "\text{inclu.}"] \arrow[r, phantom, "\perp"] & \mathrm{An} \arrow[l, shift left, "\pi_0"] \end{tikzcd}.$$

  *where* $\pi_0$ *also preserves filtered colimits.*
- *In the category* An*, filtered colimits commutes with finite limits.*
- *For any anima $X$, there is an straightening equivalence*

  $$\mathrm{An}_{/X} \xrightarrow{\simeq} \mathrm{Fun}(X, \mathrm{An})$$

  *We will explain this equivalence in the next section.*

Some properties in the above theorem can be encoded in the fact that An is an $\infty$-topos, which is a concept introduced in the Chapter 6 of Lurie's higher topos theory [4].

# 2 Straightening-Unstraightening

In this section, we will explain the Straightening-Unstraightening equivalence, which is one of Lurie's main results and also one of the central results in $\infty$-categories. One problem with $\infty$-categories is that functors are hard to define because they encode more coherence than functors of 1-categories, and Straightening-Unstraightening can turn studying functors into studying fibrations. This philosophy is one of the central techniques in proving the additivity of $K$-theory in the second part.

**Definition 2.1.** *Let $p : C \to D$ be a functor of $\infty$-categories.*

- *A morphism $f : x \to y$ in $C$ is called $p$-***cocartesian** *if for all $z \in C$ the diagram*

$$\begin{tikzcd} \hom_C(y,z) \arrow[r, "f^*"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hom_C(x,z) \arrow[d, "p"] \\ \hom_D(p(y),p(z)) \arrow[r, "p(f)^*"'] & \hom_D(p(x),p(z)) \end{tikzcd}$$

  *is a pullback diagram in* An*.*

- *A morphism $f : x \to y$ in $C$ is called $p$-***cartesian** *if for all $z \in C$ the diagram*

$$\begin{tikzcd} \hom_C(z,x) \arrow[r, "f_*"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \hom_C(z,y) \arrow[d, "p"] \\ \hom_D(p(z),p(x)) \arrow[r, "p(f)^*"'] & \hom_D(p(z),p(y)) \end{tikzcd}$$

  *is a pullback diagram in* An*.*

Unwinding the definition out, one could understand $p$-cocartesian in the following way: Let $f : x \to y$ be an edge in $C$ with the following diagram

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"'] & z & & p(x) \arrow[r, "p(g)"] \arrow[d, "p(f)"'] & p(z) \\ y \arrow[ur, dashed] & & \overset{p}{\longmapsto} & p(y) \arrow[ur] & \end{tikzcd}$$

Then, $f$ being a $p$-cocartesian edge if and only if for any this kind of diagram, there is a unique lift as the dashed arrow fulfilling the diagram.

**Definition 2.2.** *Let $p : C \to D$ be a functor of $\infty$-categories.*

- *We define* p-Cocart *as the full subcategory of* $\mathrm{Fun}([1], C)$ *spanned by all $p$-cocartesian edges. Then $p$ is said to be a $p$-cocartesian fibration if the following diagram is a* *pullback diagram in* $\mathrm{Cat}_\infty$ :

$$\begin{tikzcd} \text{p-Cocart} \arrow[r, "p"] \arrow[d, "\text{inclu.}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], D) \arrow[d, "s"] \\ \mathrm{Fun}([1], C) \arrow[r, "p"'] & D \end{tikzcd}$$

*where* $s$ *is the source morphism mapping every* 1*-morphism to its source.*

- *We define* p-Cart *as the full subcategory of* $\mathrm{Fun}([1], C)$ *spanned by all p-cartesian edges. Then* $p$ *is said to be a p-cartesian fibration if the following diagram is a pullback diagram in* $\mathrm{Cat}_\infty$ :

$$\begin{tikzcd} \text{p-Cart} \arrow[r, "p"] \arrow[d, "\text{inclu.}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], D) \arrow[d, "t"] \\ \mathrm{Fun}([1], C) \arrow[r, "p"'] & D \end{tikzcd}$$

*where* $t$ *is the target morphism mapping every* 1*-morphism to its target.*

**Remark 2.3.** *In Fabian's note, he proved that* p-Cocart $\to \mathrm{Fun}([1], D) \times_D \mathrm{Fun}([1], C)$ *is always fully faithful. Hence, a functor* $p : C \to D$ *is p-cocartesian if and only if for any* 1*-morphism* $p(x) \to y$ *in* $D$ *with* $x \in C$ *and* $y \in D$*, there is a* 1*-morphism* $x \to y'$ *in* $C$ *such that it maps to* $p(x) \to y$*, and there is a similar statement for p-cartesian fibration.*

**Example 2.4.** *The target morphism* $t : \mathrm{Fun}([1], C) \to C$ *is a cocartesian fibration. Notice that an edge in* $\mathrm{Fun}([1], C)$ *is a square* $\sigma : f \to g$*:*

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\sigma"] & y \arrow[d, "g"] \\ x' \arrow[r] & y' \end{tikzcd}$$

*It can be shown that* $\sigma$ *is p-cocartesian if and only if the square above is a pullback. Using this, it is by definition of pullback that* $t$ *is cocartesian fibration.*

Let us state Lurie's Straightening-Unstraightening.

**Theorem 2.5.** *(Lurie) For any* $\infty$*-category* $C$*, we denote* $\mathrm{Cocart}(C)$ *by the infinity category of cocartesian fibrations over* $C$*. There is a diagram*

$$\mathrm{St}^{\mathrm{cocart}} : \mathrm{Cocart}(C) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{Fun}(C, \mathrm{Cat}_\infty) : \mathrm{Un}^{\mathrm{cocart}}$$

*where both functors* $\mathrm{St}^{\mathrm{cocart}}$ *and* $\mathrm{Un}^{\mathrm{cocart}}$ *are equivalences.*

It is hard to even define the two functors properly. Let us describe what they do informally. Let $p : \mathcal{E} \to C$ be a $p$-cocartesian fibration. Then the functor

$$\mathrm{St}^{\mathrm{cocart}}(p) : C \to \mathrm{Cat}_\infty$$

sends an object $c \in C$ to the preimage category $p^{-1}(c)$ of $p$. Given a morphism $c \to d$, $\mathrm{St}^{\mathrm{cocart}}(p)$ should give a functor $p^{-1}(c) \to p^{-1}(d)$ mapping in the level of objects as follows: For any object $c \in p^{-1}(c)$, $p$ being cocartesian fibration gives a unqiue lift in $\mathcal{E}$, the lifted object is the image of $c$.

The construction of $\mathrm{Un}^{\mathrm{cocart}}$ is by using universal cocartesian fibration. One can construct a universal cocartesian fibration over $\mathrm{Cat}_\infty$, which we denote by $\mathrm{Cocart} \to \mathrm{Cat}_\infty$. Then for any functor $F : C \to \mathrm{Cat}_\infty$, its unstraightening is defined as a pullback as follows:

$$\begin{tikzcd} \mathrm{Cocart}(C) \arrow[r] \arrow[d, "\mathrm{Un}^{\mathrm{cocart}}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cocart} \arrow[d] \\ C \arrow[r, "F"] & \mathrm{Cat}_\infty \end{tikzcd}$$

Similarly, one also has Straightening-Unstraightening for cartesian fibrations:

**Theorem 2.6.** *For any $\infty$-category $C$. We denote $\mathrm{Cart}(C)$ by the infinity category of cartesian fibrations over $C$. There is a diagram*

$$\mathrm{St}^{\mathrm{cart}} : \mathrm{Cart}(C) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty) : \mathrm{Un}^{\mathrm{cart}}$$

*where both functors $\mathrm{St}^{\mathrm{cart}}$ and $\mathrm{Un}^{\mathrm{cart}}$ are equivalences.*

This kind of Grothendick construction can be restricted to An. We need to introduce the following definitions:

**Definition 2.7.** *Let $p : C \to D$ be a cocartesian fibration. Then it is called a* **left fibration** *if one of the following equivalent conditions holds:*

- *$p$ is conservative, which means an edge $f$ in $C$ is an equivalence iff $p(f)$ is an equivalence in $D$.*
- *All morphisms in $C$ are $p$-cocartesian.*
- *$\mathrm{St}^{\mathrm{cocart}}(p) : C \to \mathrm{Cat}_\infty$ factors through $\mathrm{An}$.*

Dually, we have

**Definition 2.8.** *Let $p : C \to D$ be a cartesian fibration. Then it is called a* **right fibration** *if one of the following equivalent conditions holds:*

- *$p$ is conservative.*
- *All morphisms in $C$ are $p$-cartesian.*
- *$\mathrm{St}^{\mathrm{cart}}(p) : C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through $\mathrm{An}$.*

Then, we have the other version of Straightening-Unstraightening:

**Theorem 2.9.** *(Lurie) For any $\infty$-category $C$, we denote* $\mathrm{Left}(C)$ *by the infinity category of left fibrations over $C$. There is a diagram*

$$\mathrm{St}^{\mathrm{left}} : \mathrm{Left}(C) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{Fun}(C, \mathrm{An}) : \mathrm{Un}^{\mathrm{left}}$$

*where both functors* $\mathrm{St}^{\mathrm{left}}$ *and* $\mathrm{Un}^{\mathrm{left}}$ *are equivalences.*

Notice that by the definition of left fibrations, we don't need to specify the straightening functor. Here, the unstraightening functor has a nicer description. In the above, the universal cocartesian fibration is quite hard to describe, but here, let $\mathrm{An}_{*/}$ be the slice category of An under a point, then we have a pullback diagram

$$\begin{tikzcd}
\mathrm{Left}(C) \arrow[r] \arrow[d, "\mathrm{Un}^{\mathrm{left}}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{An}_{*/} \arrow[d] \\
C \arrow[r, "F"] & \mathrm{An}
\end{tikzcd}$$

For the sake of completeness, let us also formulate the Straightening-Unstraightening for right fibrations:

**Theorem 2.10.** *(Lurie) For any $\infty$-category $C$. We denote* $\mathrm{Right}(C)$ *by the infinity category of right fibrations over $C$. There is a diagram*

$$\mathrm{St}^{\mathrm{right}} : \mathrm{Right}(C) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \mathrm{Fun}(C, \mathrm{An}) : \mathrm{Un}^{\mathrm{right}}$$

*where both functors* $\mathrm{St}^{\mathrm{right}}$ *and* $\mathrm{Un}^{\mathrm{right}}$ *are equivalences.*

**Example 2.11.** *An anima $X$ is an $\infty$-groupid. It can be shown that* $\mathrm{Left}(X) \simeq \mathrm{An}_{/X}$*. Applying to the theorem above, we have the equivalence*

$$\mathrm{An}_{/X} \simeq \mathrm{Fun}(X, \mathrm{An}).$$

# 3 Kan Extensions

In 1-categories, we have seen Kan extension is a very useful tool. Hence naturally, one would expect we also have Kan extensions in $\infty$-categories. Indeed, in this section, we will formulate the related theorems about it.

**Theorem 3.1.** *If $C$ is a small $\infty$-category and $D$ cocomplete, then for any $f : C \to \mathcal{E}$ the functor*

$$f^* : \mathrm{Fun}(\mathcal{E}, D) \to \mathrm{Fun}(C, D)$$

*has a left adjoint* $\mathrm{Lan}_f$*, which has the pointwise formula*

$$\mathrm{Lan}_f F(e) \simeq \operatorname*{colim}_{c, f(c) \to e \in C_{/e}} F(c)$$

*where $C_{/e}$ is the pullback $C \times_D D_{/e}$. Similarly, when $D$ is complete, $f^*$ has a right adjoint with formula*

$$\operatorname{Ran}_f F(e) \simeq \lim_{c, e \to f(c) \in C_{e/}} F(c)$$

*In fact, even if $D$ is not complete or cocomplete, as long as the colimit formula or limit formula exists for all $e \in \mathcal{E}$, then they can be assemble into a functor $\operatorname{Lan}_f F : \mathcal{E} \to D$ or $\operatorname{Ran}_f F : \mathcal{E} \to D$ as a left or right adjont object of $F$, i.e. with properties*

$$\hom_{\operatorname{Fun}(\mathcal{E},D)}(\operatorname{Lan}_f F, -) \simeq \hom_{\operatorname{Fun}(C,D)}(F, f^*-)$$

*or*

$$\hom_{\operatorname{Fun}(C,D)}(f^*-, F) \simeq \hom_{\operatorname{Fun}(\mathcal{E},D)}(-, \operatorname{Ran}_f F).$$

*In this case, we call $\operatorname{Lan}_f F$ as the* **left or right Kan extension** *of $F$ along $f$.*

One particular case of Kan extensions we will use is to extend along fully faithful functors:

**Proposition 3.2.** *If $f : C \to \mathcal{E}$ is fully faithful, and let $F \in \operatorname{Fun}(C, D)$, then we have*

$$F \simeq \operatorname{Lan}_f F \circ f \quad \& \quad \operatorname{Ran}_f F \circ f \simeq F.$$

*When $D$ is complete or cocomplete, the above implies $\operatorname{Lan}_f$ and $\operatorname{Ran}_f$ are fully faithful functors.*

Let us discuss two examples.

**Example 3.3.** *If $D$ is a cocomplete $\infty$-category and $F \in \operatorname{Fun}(\Delta, D)$ is a cosimplicial object in $D$, then we can consider the following diagram*

$$\begin{tikzcd} \Delta \arrow[r, "F"] \arrow[d, "f"'] & D \\ \mathrm{sAn} \arrow[ur, dashed, "\operatorname{Lan}_f F := |-|_F"'] & \end{tikzcd}$$

*where $f : \Delta \to \mathrm{sAn}$ sends $[n]$ to $\Delta^n$. Moreover, by Theorem 1.22 applying to $\Delta$, we see $|-|_F$ is a left adjoint functor. We denote its right adjoint by $\operatorname{Sing}_F$, then there is an adjunction diagram*

$$|-|_F : \mathrm{sAn} \rightleftarrows D : \operatorname{Sing}_F$$

*In particular, when we take $D = \mathrm{An}$ and $F = \operatorname{colim} *$, the functor $|-|_F$ is $\operatorname{colim}_{\Delta^{op}}$. In this case, we write*

$$|-| : \mathrm{sAn} \to \mathrm{An}\,.$$

*Unfortunately, we use the same symbol for the geometric realization $|-| : \mathrm{Cat}_\infty \to \mathrm{An}$ and we will also call it geometric realization. This is because other people do this and we just follow them. But it turns out they are not far depending on your perspective. In fact, we can see a relation in the following example.*

Let us record a Proposition that we will use later:

**Proposition 3.4.** *The geometric realization*

$$|-|: \mathrm{sAn} \to \mathrm{An}$$

*preserves finite products.*

**Example 3.5.** *Similar as above, we consider $D = \mathrm{Cat}_\infty$ and $F : \Delta \to \mathrm{Cat}_\infty$ sending $[n]$ to itself. Then this induces an adjunction pair*

$$\mathrm{asscat} : \mathrm{sAn} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Cat}_\infty : N^r$$

*The right adjoint is called the* **Rezk Nerve** *and the left adjoint is called the* **associated category**. *When $X$ is a simplicial anima satisfying Segel condition (See Definition 5.1), we actually have*

$$|\,\mathrm{asscat}\, X| \simeq |X|.$$

About Rezk Nerve, we will need two properties for the second part. Let us formulate them here.

**Proposition 3.6.** *If $C$ is an $\infty$-category. Then we have an explicit description of Rezk Nerve, i.e.*

$$N_n^r(C) \simeq \hom_{\mathrm{Cat}_\infty}([n], C) \simeq \mathrm{core}\,\mathrm{Fun}([n], C).$$

*In particular, if $C$ is a 1-category, the above formular just gives us the ordinary Nerve functor.*

**Proposition 3.7.** *Let $C$ be an $\infty$-category. We can define its twisted arrow category as the pullback diagram*

$$\begin{tikzcd} \mathrm{TwAr}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{An}_{*/} \arrow[d] \\ C^{op} \times C \arrow[r, "\hom_C"] & \mathrm{An} \end{tikzcd}$$

*Then, we have*

$$N_n^r(\mathrm{TwAr}(C)) \simeq \hom_{\mathrm{Cat}_\infty}([n]^{\mathrm{op}} \star [n], C) \simeq \hom_{\mathrm{sAn}}((\Delta^n)^{\mathrm{op}} \star \Delta^n, N^r(C)),$$

*where $\star$ denotes the join of categories and the join of simplicial sets respectively.*

# 4 Localisation and Bousfield localisation

In this section, we will discuss localisation and the Bousfield localisation. Using them, we can see how to get derived category in the $\infty$-categorical sense.

**Definition 4.1.** *Let $C$ be an $\infty$-category and $S \in \pi_0\,\mathrm{core}\,\mathrm{Fun}([1], C)$ be a collection of 1-morphisms in $C$. For any $\infty$-category $D$, we define*

$$\mathrm{Fun}^W(C, D) \subset \mathrm{Fun}(C, D)$$

*to be the full subcategory spanned by functors sends morphisms in $W$ to* $\operatorname{core} D$*. Then we define a* **localisation** *of $C$ along $W$ is a functor $p : C \to C[W^{-1}]$ such that for any $\infty$-category $D$, the restriction functor $p^*$ defines an equivalence*

$$p^* : \operatorname{Fun}(C[W^{-1}], D) \xrightarrow{\simeq} \operatorname{Fun}^W(C, D).$$

Related to localisations, we give two theorems.

**Theorem 4.2.** *(Proposition 2.4.8 in [5]) For every $S \in \pi_0 \operatorname{core} \operatorname{Fun}([1], C)$, there exists a unique localisation of $C$ along $S$ up to equivalence.*

For a fibration $p : E \to S$, we denote by $\Gamma(p)$ the infinity category of sections given by the pullback

$$\begin{tikzcd} \Gamma(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Fun}(S, E) \arrow[d, "p_*"] \\ \{\mathrm{id}_S\} \arrow[r] & \operatorname{Fun}(S, S) \end{tikzcd}$$

**Theorem 4.3.** *(Lurie) Given a diagram $F : I \to \mathrm{Cat}_\infty$, we have*

$$\operatorname*{colim}_I F \simeq \mathrm{Un}^{\mathrm{cocart}(F)}[\{\textit{cocartesian edges}\}^{-1}]$$

$$\lim_I F \simeq \Gamma(\mathrm{Un}^{\mathrm{cocart}}(F))$$

*In particular, if $F$ is valued in* An*, then we have*

$$\operatorname*{colim}_I F \simeq |\operatorname{Un}^{\mathrm{left}}(F)|$$

*and*

$$\lim_I F \simeq \Gamma(\mathrm{Un}^{\mathrm{left}}(F))$$

Localisation has a very wild side, and we usually have very little control on the size of localisations. Localisations of locally small $\infty$-categories may be not locally small anymore. In fact, we have this surprising result:

**Theorem 4.4.** *(Theorem 6.3.7.1 in [6]) For an $\infty$-category $C$, there exists a relative poset $(P, W)$ and a functor $P \to C$ such that $C \simeq P[W^{-1}]$.*

For this reason, we usually tend to study another kind of localisation, which is the Bousfield localisation.

**Proposition 4.5.** *Let $F : C \rightleftarrows D : R$ be a pair of adjunction of $\infty$-categories. We define a morphism $f : x \to y$ in $C$ as a* **left $R$-local equivalence** *if the induced map*

$$f^* : \hom_C(y, Rd) \to \hom_C(x, Rd)$$

*is an equivalence for all $d \in D$. We have*

- *$R$ is fully faithful iff the counit $c : LR \Rightarrow \mathrm{id}_D$ is an eqivalence.*
- *For a morphism $f : x \to y$ in $C$, we have $L(f)$ is an equivalence iff $f$ is an $R$-local equivalence.*

*If $R$ is fully faithful in the adjunction, then:*

- *The unit $x \to RLx$ is an $R$-local equivalence for any $x \in C$.*
- *The functor $L : C \to D$ witness $D$ as the localisation of $C$ along left $R$-local equivalence.*

With this Proposition, we can define Bousfield localisation as follows:

**Definition 4.6.** *Given an adjunction pair $L \dashv R$ of infinity categories, if $R$ is fully faithful, then we call $L$ as a* **left Bousfield localisation**.

Similarly, one can give a dual Proposition to the proposition above. With that, it makes sense to define right Bousfield localisation as a right adjoint functor with a fully faithful left adjoint. A natural question after we define the Bousfield localisation is that when localisation is a Bousfield localisation and it can be answered in the following proposition:

**Proposition 4.7.** *Let $W$ be a collection of morphisms of $C$ with associated localisation*

$$p : C \to C[W^{-1}]$$

*If $p$ admits a right adjoint $R$, then the right adjoint is automatically fully faithful. Hence, in this situation, $p$ is a Bousfield localization.*

Now, let us give a characterisation of Bousfield localisation. We will use this several times in the second part. We should celebrate this result because it provides us with how to construct Bousfield localisations.

**Proposition 4.8.** *Let $L : C \to C$ be a functor toghether with a natural transformation $\eta : \mathrm{id} \Rightarrow L$ such that both maps*

$$\eta_{Lx} : Lx \xrightarrow{\simeq} LLx \qquad L\eta_x : Lx \xrightarrow{\simeq} LLx$$

*are equivalences for all $x \in C$. Then $L : C \to \mathrm{im}(L)$ is left adjoint to $\mathrm{im}(L) \subset C$ with unit $\eta$. In particular, this tells us that $L$ is a Bousfield localisation.*

**Example 4.9.** *Recall we have an adjunction diagram*

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "\text{incl.}"] & \mathrm{An} \arrow[l, bend right, "\pi_0"'] \arrow[r, "\text{incl.}"] & \mathrm{Cat}_\infty \arrow[l, bend right, "|-|"'] \arrow[l, bend left, "\text{core}"]
\end{tikzcd}$$

*Because inclusion is fully faithful. This tells $\pi_0$ and $|-|$ are left Bousfield localisations, and* core *is a right Bousfield localisation.*

In the end of this section, let us discuss the derived category of a ring $R$ in the $\infty$-categorical setting. Let $R$ be a ring. Recall that in Example 1.10, we have said there is an infinity category of chain complexes over $R$. We define the derived $\infty$-category $\mathcal{D}(R)$ as

$$\mathcal{D}(R) = \mathcal{K}(R)[\{\text{quasi-iso.}\}^{-1}]$$

as the localisation of $\mathcal{K}(R)$ along quasi-isomorphisms.

**Proposition 4.10.** *The infinity category $\mathcal{D}(R)$ has the following properties:*

- *The projection $\mathcal{K}(R) \to \mathcal{D}(R)$ admits both left and right adjoints, so it is a Bousfield localisation.*
- *$\mathcal{D}(R)$ is complete and cocomplete.*
- *Its homotopy category recover the usual derived 1-category $D(R)$.*

Given a ring homomorphism $\psi : R \to S$, we can look at the functor

$$S \otimes_R - : \mathrm{Ch}(R) \to \mathrm{Ch}(S).$$

This can be lift in the infinity category level:

$$S \otimes_R - : \mathcal{K}(R) \to \mathcal{K}(S).$$

Using this, one can make sense of derived tensor product by considering the following composition:

$$S \otimes_R - : \mathcal{D}(R) \to \mathcal{K}(R) \xrightarrow{S\otimes} \mathcal{K}(S) \to \mathcal{D}(S)$$

where the first arrow is the left adjoint of the projection $\mathcal{K}(R) \to \mathcal{D}(R)$. Similarly, one can make sense of $\hom_R(-, S) : \mathcal{D}(R) \to \mathcal{D}(S)$ and the the pullback functor $\psi^* : \mathcal{D}(S) \to \mathcal{D}(R)$. Putting the three functors together, we produce a diagram of adjunctions:

$$\begin{tikzcd} \mathcal{D}(R) \arrow[r, shift left, "S\otimes_R -"] \arrow[r, phantom, "\perp"] & \mathcal{D}(S) \arrow[l, shift left, "\psi^*"] \arrow[r, shift left, "\psi^*"] \arrow[r, phantom, "\perp"] & \mathcal{D}(R) \arrow[l, shift left, "\hom_R(S,-)"] \end{tikzcd}$$

This adjunction diagram tells us that we can define a derived localisation of rings.

**Definition 4.11.** *A ring map $\psi : R \to S$ is called a derived localisation if $S \otimes_R - : \mathcal{D}(R) \to \mathcal{D}(S)$ is a left Bousfield localisation. This is equivalent to the pullback functor $\psi^* : \mathcal{D}(S) \to \mathcal{D}(R)$ is fully faithful.*

# 5 Brave new categories

The k-theory functor that we will define takes a stable infinity category to a grouplike space. Up to now, we haven't defined the domain and the codomain. These will be the objects in this section. The reason we call this section as brave new categories is that they are the fundamental concepts towards higher algebra, which is also called brave new algebra. Although we won't reach that far, we still need the spirit of courage to carry on this journey.

We will start by defining Segal spaces. It is a bit random, but it is a definition that we want to use later

**Definition 5.1.** *A Segal space is a simplicial anima $X : \Delta^{\mathrm{op}} \to \mathrm{An}$ such that*

$$X_n \simeq \hom_{\mathrm{sAn}}(\Delta^n, X) \to \hom_{\mathrm{sAn}}(I_n, X) \simeq X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

*where $I_n \subset \Delta^n$ is the union of 1-simplices $0 \to 1, 1 \to 2, \cdots, n-1 \to n$.*

**Remark 5.2.** *From the definition of Segal space, we can inductively prove that the condition of Segal space is equivalent to for each $0 \leq i \leq n$, we have an equivalence:*

$$X_n \simeq X_i \times_{X_0} X_{n-i}.$$

**Definition 5.3.** *Let $\mathrm{Fin}_*$ be the category of finite pointed sets and basepoint preserving maps. The objects of $\mathrm{Fin}_*$ are sets $\langle n \rangle = \{*, 1, \cdots, n\}$. For $1 \leq i \leq n$, we denote $\rho^i : \langle n \rangle \to \langle 1 \rangle$ by the map sending $i$ to 1 and all other to 0*

*Let $C$ be an $\infty$-category with finite products. Then we define a* **commutative monoid** *$X$ in $C$ is a functor $\mathrm{Fin}_* \to C$ such that the morphism*

$$X(\langle n \rangle) \to \prod_{i=1}^{n} (\langle 1 \rangle)$$

*induced by $\rho^i$ is an equivalence. The condition above is called Segal condition. One can see they are defined in a similar philosophy as in the definition of Segal spaces. The commutative monoids form an $\infty$-category $\mathrm{CMon}(C)$ as the full subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, C)$*

Let us briefly discuss how this matches our intuition of commutative monoids. We should think $X(\langle 1 \rangle)$ as the underlying object of $X$. The higher structures are here to give the multiplication on $X(\langle 1 \rangle)$. For example, the multiplication map is given by

$$m : \prod_{i=1}^{n} (\langle 1 \rangle) \xleftarrow{\simeq} X(\langle n \rangle) \to X(\langle 1 \rangle).$$

Because here we need to choose an inverse, this multiplication is unique up to a contractible space of choices. Hence, in writing commutative monoids, we often only write the underlying object $M \in C$, and think it has a multiplication structure.

**Remark 5.4.** *Let $F : C \to D$ be a functor of $\infty$-categories. If $F$ preserves finite products and $D$ admits all finite products. Then one can see that $F$ factors through $\mathrm{CMon}(D)$ because any object in $D$ can be given a commutative monoidal structure by freely taking finite products to itself.*

**Definition 5.5.** *Let $X$ be a commutative monoid in $C$. It is called a* **commutative group** *in $C$ if the sheaf map*

$$(\mathrm{pr}_1, m) : X_1 \times X_1 \to X_1 \times X_1$$

*given by $(a, b) \mapsto (a, a * b)$ is an equivalence. Commutative groups in $C$ form an infinity category denoted by $\mathrm{CGrp}(C)$ as the full subcategory of $\mathrm{CMon}(C)$.*

The main example we want to consider is when $C = \mathrm{An}$. In this case, we have the following Proposition:

**Proposition 5.6.** *A commutative monoid $M$ in* An *is a commutative group if and only if $\pi_0(M)$ is a group.*

**Definition 5.7.** *Let $C$ be an $\infty$-category. We say $x \in C$ is a zero object if $X$ is both an initial and a terminal object.*

For an $\infty$-category $C$ with a zero object, we always have the loopspace functor defined by the pullback

$$\begin{aligned} \Omega : C &\to C \\ X &\mapsto 0 \times_X 0 \end{aligned}$$

and the suspension functor defined by the pushout

$$\begin{aligned} \Sigma : C &\to C \\ X &\mapsto 0 \cup_X 0 \end{aligned}$$

**Proposition 5.8.** *For any $\infty$-category $C$ admitting finite products, the $\infty$-category* $\mathrm{CMon}(C)$ *has a zero object and direct sums. In the case $C = \mathrm{An}$, the loopspace functor $\Omega$ defines a functor*

$$\mathrm{CMon}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{An})$$

Now, we introduce the stable $\infty$-categories.

**Proposition 5.9.** *Suppose $C$ is an $\infty$-category with a zero object. Then the following conditions are equivalent:*

- *$C$ has finite limits, and $\Omega : C \to C$ is an equivalence.*
- *$C$ has finite colimits, and $\Sigma : C \to C$ is an equivalence.*
- *$C$ has all finite limits and finite colimits, and a commutative square in $C$ is a pushout square iff it is a pullback square.*

**Definition 5.10.**

- *An $\infty$-category $C$ with a zero object satisfying one of the above equivalent conditions is called a* **stable $\infty$-category**.
- *A functor $F : C \to D$ between $\infty$-categories is called* **left exact** *if it preserves finite limits, and it is called* **right exact** *if it preserves finite colimits.*
- *We denote by $\mathrm{Cat}_\infty^{\mathrm{lex}} \subset \mathrm{Cat}_\infty$ the (non-full) sub-$\infty$-category spanned by $\infty$-categories with finite colimits and left exact functors between them. Similarly, we can also define $\mathrm{Cat}_\infty^{\mathrm{rex}} \subset \mathrm{Cat}_\infty$ as the sub-$\infty$-category spanned by the right exact functors.*

**Proposition 5.11.** *A functor $F : C \to D$ between stable $\infty$-categories preserves finite limits iff $F$ preserves finite colimits.*

Because of this Proposition, we can define the $\infty$-category of stable $\infty$-categories $\mathrm{Cat}_\infty^{\mathrm{st}}$ as either the full subcategory of $\mathrm{Cat}_\infty^{lex}$ or $\mathrm{Cat}_\infty^{rex}$ spanned by all stable $\infty$-categories. For an $\infty$-category, there is a procedure called stabilization, which will produce a stable $\infty$-category.

**Definition 5.12.** *let $C$ be an $\infty$-category with all finite limits. In particular, it has an terminal object $*$. One considers the category $C_{*/}$, then we can define a suspension functor $C_{*/} \to C_{*/}$ can be defined and we put*

$$\mathrm{Sp}(C) := \lim_{\mathbb{N}^{\mathrm{op}}}(\cdots \xrightarrow{\Omega} C_{*/} \xrightarrow{\Omega} C_{*/})$$

*This is called the $\infty$-category of* **spectrum objects** *in $C$. In $C = \mathrm{An}$, we define $\mathrm{Sp} := \mathrm{Sp}(\mathrm{An})$, and call it the $\infty$-category of* **spectra**.

A spectrum $X \in \mathrm{Sp}$ should be thought as a sequence of animas $X = (X_0, X_1, X_1, \cdots)$ with equivalences $X_i \simeq \Omega X_{i+1}$. Given a spectrum, we can define its homotopy groups as follows: For $i \leq 0$, we define $\pi_i(X) := \pi_0(X_{-i})$, and for $i \geq 0$, we define $\pi_i(X) := \pi_i(X_0)$. The homotopy groups are enough to detect equivalences of spectra.

**Proposition 5.13.** *Let $f : X \to Y$ be a map in $\mathrm{Sp}$. If it induces equivalence on all homotopy groups, then $f$ is an equivalence in $\mathrm{Sp}$.*

**Theorem 5.14.** *$\mathrm{Sp}$ is a symmetrical monoidal $\infty$-category, which means that $\mathrm{Sp} \in \mathrm{CMon}(\mathrm{Cat}_\infty)$. It has a symmetrical monoidal functor ($\mathrm{An}$ has is symmetrical monoidal $\infty$-category because it has finite products):*

$$\Sigma_+^\infty : \mathrm{An} \to \mathrm{Sp}$$

*Moreover, it has a lax symmetric monoidal right adjoint called the infinity loopspace functor:*

$$\Omega^\infty : \mathrm{Sp} \to \mathrm{An}\,.$$

*On the object level, it sends a spectrum $X = (X_0, X_1, X_2, \cdots)$ to a space $X_0$.*

Now, we can state the recognition principle:

**Theorem 5.15.** *A spectrum $X$ is called connective if $\pi_i(X) = 0$ for $i < 0$. All connective spectra forms a full sub-$\infty$-category of $\mathrm{Sp}$, denoted by $\mathrm{Sp}_{\geq 0}$. Then, the infinity loop space restricting to connective spectra induces an equivalence:*

$$\mathrm{Sp}_{\geq 0} \xrightarrow{\simeq} \mathrm{CGrp}(\mathrm{An}).$$

*and we denote its inverse as $B^\infty$.*

Before ending this section, we also mention that $\mathcal{D}(R)$ is a stable $\infty$-category. This in fact can be think as the universal property of $\mathcal{D}(R)$.

**Proposition 5.16.** *(Lecture 2 in [7]) Let $R$ be a ring. The $\infty$-category $\mathcal{D}(R)$ is a cocomplete stable $\infty$-category generated by a distinguished object $1 \in \mathcal{D}(R)$ by colimit with a distinguished identification $\pi_0 \hom(1,1) = R^{op}$ as ring, and $\pi_0 \hom(\Sigma^d 1, 1) \simeq 0$ for $d \in \mathbb{Z} \setminus \{0\}$.*

# 6 Extra Degeneracy Trick

This will be a short section where we want to introduce the extra degeneracy trick. It will be used in the next part. Moreover, it can be pretty helpful sometimes, so we record it here.

We want to define a 1-category $\Delta_{-\infty}$ as follows:

- The objects are integers $n \leq -1$.
- The hom set of $\hom_{\Delta_\infty}([m],[n])$ is the set of nondecreasing maps $\alpha : [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ such that $\alpha(-\infty) = -\infty$.

In the category $\Delta_{-\infty}$, we can also define a subcategory $\Delta_+$, where the latter has the same objects with $\Delta_{-\infty}$, and a map $\alpha : [m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ belongs to $\delta_+$ if and only if $\alpha^{-1}(-\infty) = \{-\infty\}$. In other words, $\Delta_+$ is the category extends from $\Delta$ and we add one object $[-1]$, also set $\hom([-1],[n]) = *$. Hence, we have an inclusion of categories:

$$\Delta \subset \Delta_+ \subset \Delta_{-\infty}$$

**Definition 6.1.** *Let $C$ be an $\infty$-category. A simplicial object $X : \Delta^{\mathrm{op}} \to C$ is called an* **augmented simplicial object** *if it admits an extension to $\Delta_+^{\mathrm{op}}$. $X$ is called a* **split simplicial object** *if $X$ admits an extension to $\Delta_{-\infty}^{\mathrm{op}}$.*

**Proposition 6.2.** *Let $C$ be an $\infty$-category and $X : \Delta^{\mathrm{op}} \to C$ be a simplicial object of $C$. If $X$ is a spilt simplicial diagram which means that it extends to a diagram $X^s : \Delta_{-\infty}^{\mathrm{op}} \to C$, then $X^s|_{\Delta_+^{\mathrm{op}}}$ is a limit diagram in $C$, i.e.*

$$\lim_{\Delta^{\mathrm{op}}} X \simeq X^s([-1]).$$

*Proof.* [Lemma 6.1.3.16, HTT] $\square$

# Part II

# Algebraic K-theory

## 7 The Q-construction and The S-construction

In this section, we want to explain the two equivalent constructions of the algebraic $K$ theory functor. In the modern setting, people usually refer to the $S$-construction as the definition of $K$-theory because it can be defined over a general class of categories called Waldhuasen categories. For our purpose of studying and using higher algebra, we maintain our interest in stable infinity categories, which have already given abundant examples.

To motivate our definition, let us recall for a ring $R$, its $K_0$-group $K_0(R)$ is defined as the free abelian group on symbols $[X]$ where $X$ ranges over all isomorphism classes of finitely generated projective modules, modulo the relation $[X] + [Y] - [X \oplus Y]$.

We want to give a proposition to see how to generalize the classical definition to the infinite categories. Recall that $\mathcal{D}(R)$ is the derived $\infty$-category of $R$-modules, and we let $\mathcal{D}^{perf}(R)$ denote the full sub-$\infty$-subcategory of $\mathcal{D}(R)$ generated by all complexes of finitely generated projective modules, where $\mathcal{D}^{perf}(R)$ is full subcategories generated by compact objects of $\mathcal{D}(R)$.

**Proposition 7.1.** *We can define an equivalence relation on $\pi_0 core(\mathcal{D}^{perf}(R))$ generated by $Z \sim X \oplus Y$ iff $X \to Z \to Y$ is a fiber sequence in $\mathcal{D}^{perf}(R)$. Then there is an isomorphism*

$$\pi_0 \operatorname{core}(\mathcal{D}^{perf}(R)) \xrightarrow{\simeq} K_0(R),$$

*where we sends $[P_n \to P_{n-1} \to \cdots \to P_{-m}]$ to $\sum_{k=-m}^{n} (-1)^k [P_k]$.*

*Proof.* By writing out the definition, the left hand is just the homotopy equivalent class of finite chain complexes of finitely generated projective modules modulo the equivalence relation. The well-definedness of the map can be boiled down to show the two homotopy equivalent chain complexes maps to the same element in $K_0(R)$, which is a standard proof by splitting the long exact sequence into many short ones. Surjectivity is also very easy because any finitely generated projective module $P$ gives $P[0]$ and $P[1]$ in $\mathcal{D}^{perf}(R)$ that maps to $[P]$ and $-[P]$ respectively.

The injectivity is also not very hard. Because of the additivity of the map, it's enough to show that $K$ mapping to 0 implies $K \sim 0$. If we write $K \simeq P_n \to P_{n-1} \to \cdots \to P_m$, then the equivalence relation implies

$$K \simeq \bigoplus_{k=-m}^{n} P_k[k].$$

Notice for $-m \leq k \leq n$, we have fiber sequence

$$P_k[i-1] \to 0 \to P_k[i].$$

Adding them inductively, we get

$$P_k[even] \oplus P_k[1] \simeq P_k[even] \oplus (P_k[even-1] \oplus P_k[even-2]) \oplus \cdots \oplus P_k[1] \simeq 0$$

and

$$P_k[odd] \oplus P_k[0] \simeq P_k[odd] \oplus (P_k[odd-1] \oplus P_k[odd-2]) \oplus \cdots \oplus P_k[0] \simeq 0$$

Adding them with $K$, we see $K$ is equivalent to a complex concentrated on degree 0 and degree 1. Hence, we can assume

$$K \simeq P_1[1] \oplus P_0[0].$$

Now, $K$ mapping to 0 implies $[P_1] \simeq [P_0]$. By the Grothendieck group construction of $K_0(R)$, there is a finitely projective module $Q$ such that $P_1 \oplus Q \simeq P_0 \oplus Q$. Hence, we have

$$K \simeq (P_0[0] \oplus Q[0]) \oplus (P_1[1] \oplus Q[1]) \simeq 0.$$

□

From the proposition, it's easy to see $K_0$ of a stable infinity category should be

**Definition 7.2.** *Let $C$ be a (small) stable $\infty$-category, then we define*

$$K_0(C) := \pi_0 \operatorname{core}(C)/\simeq,$$

*where $\simeq$ is the equivalence relation generated by defining $Z \sim X \oplus Y$ iff $X \to Z \to Y$ is a fiber sequence in $C$.*

The definition of $K_0$ also gives us an intuition on how a higher $K$ theory should be defined. We want to construct a functor mapping from $\mathrm{Cat}_\infty^{st}$ to $\mathrm{An}_*$, where the homotopy groups give all the $K$-groups, and its fundamental group should be $K_0$-group in particular. In the following, we think spaces as Kan complexes. Firstly, it is reasonable to have one vertex as the 0-simplex to define canonical homotopy groups. Then, each finite projective complex should provide a 1-complex, and a fiber sequence $X \to Y \to Z$ should give a 2-simplex looking like the following:

$$\begin{tikzcd}
 & \bullet \arrow[dr, "X"] & \\
\bullet \arrow[ur, "Z"] \arrow[rr, "Y"'] & & \bullet
\end{tikzcd}$$

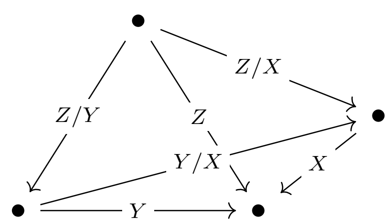
Moving to higher homotopies, we want to define some relations similar to the relation above. If we are given two homomorphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

It determines two diagrams like the following, in which each column and row is a cofiber sequence:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id"'] & Y \arrow[r] \arrow[d, "g"'] & Y/X \arrow[d] \\
X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z/X \arrow[d] \\
\bullet \arrow[r] & Z/Y \arrow[r, "id"] & Z/Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "f"'] & X \arrow[r] \arrow[d] & \bullet \arrow[d] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & Z/Y \arrow[d] \\
Y/X \arrow[r] & Z/Y \arrow[r] & Z/Y
\end{tikzcd}$$

They both determine the equation$[Z] = [X] + [Z/Y] + [Y/X]$ with two different homotopies. In the Kan complex we want to build, this corresponds to a boundary of $\Delta^3$ looks like the following:




Hence, to mimic how we define $K_0$-group, we should ask that there is a filling of the $\partial\Delta^3$ for any pair of this kind of diagram. In other words, the 3-simplices of the simplicial set we want to build can be visualized as follows:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
& Y/X \arrow[r] & Z/X \arrow[d] \\
& & Z/Y
\end{tikzcd}$$

This can be easily generalized to a composition of $n$ homomorphisms, and it motivates us to define the $S$-construction as the following:

**Definition 7.3.** *(Waldhuasen $S$-construction) Let $C$ be a stable infinity category and $[n] \in \Delta^{\mathrm{op}}$, and let $\mathrm{Ar}([n])$ be its arrow category whose*

- *objects are arrows of $[n]$, which we denote as $i \leq j$.*
- *morphism between $a_0 \leq a_1$ and $b_0 \leq b_1$ is a diagram as follows:*

$$\begin{tikzcd}
a_0 \arrow[r] \arrow[d] & b_0 \arrow[d] \\
a_1 \arrow[r] & b_1
\end{tikzcd}$$

*A square in* $\mathrm{Ar}([n])$ *is defined as a diagram looking like*

$$\begin{tikzcd}
i \leq j \arrow[r] \arrow[d] & i \leq l \arrow[d] \\
k \leq j \arrow[r] & k \leq l
\end{tikzcd}$$

*. Then, we define*

$$S_n(C) \subset \mathrm{Fun}(\mathrm{Ar}[n], C)$$

*to be the full* $\infty$*-subcategory containing functors* $F$ *satisfying the following two conditions:*

- *For all* $0 \leq i \leq n$*, we let* $F(i \leq i) = 0$*, where* $0$ *is the pointed object of* $C$*.*
- $F$ *sends all squares of* $\mathrm{Ar}([n])$ *to pullback squares in* $C$*.*

From our definition, it is easy to see that the objects of $S_n(C)$ is a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r, dotted, no head] \arrow[d] & \bullet \arrow[d] \\
& 0 \arrow[r] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r, dotted, no head] \arrow[d] & \bullet \arrow[d] \\
& & 0 \arrow[r] & \bullet \arrow[r] \arrow[d, dotted, no head] & \bullet \arrow[r, dotted, no head] \arrow[d, dotted, no head] & \bullet \arrow[d, dotted, no head] \\
& & & 0 \arrow[r] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d] \\
& & & & 0 \arrow[r] & \bullet \arrow[d] \\
& & & & & 0
\end{tikzcd}$$

where each square is a pullback diagram, and from the two conditions, it is easy to see the whole diagram is determined by the information of the first row. Hence, we have

$$S_n(C) \simeq \mathrm{Fun}([n-1], C).$$

This implies $S_n(C)$ is a stable $\infty$-category again. We get a functor

$$S : \mathrm{Cat}_\infty^{st} \to \mathrm{sCat}_\infty^{st}.$$

Combining all our discussions, we can finally define the algebraic $K$ theory

**Definition 7.4.** *For a stable* $\infty$*-category* $C$*, we define*

$$k(C) \simeq \Omega|\operatorname{core} S(C)|.$$

*Recall that* $|-|$ *denotes taking colimit over* $\Delta^{\mathrm{op}}$*.*

**Remark 7.5.** *The first thing we should observe is that $k$ actually has the image in* $\mathrm{CGrp}(\mathrm{An})$*. Because levelwise, $S_n$ is $\mathrm{Fun}([n-1], -)$, which obviously preserves products. By Proposition 3.4, we know $|-|$ preserves products. Because* core *is the right adjoint of* incl. $: \mathrm{An} \to \mathrm{Cat}_\infty$*,* core *also preserves products. Hence, by Remark 5.4, we see before taking $\Omega$, $k(C)$ lands in* $\mathrm{CMon}(\mathrm{An})$*. Then, applying $\Omega$ gives a commutative group.*

**Definition 7.6.** *As we have seen in Theorem 5.15. There is an equivalence functor:*

$$B^\infty : \mathrm{CGrp}(\mathrm{An}) \to \mathrm{Sp}_{\geq 0}.$$

*Combining with the Remark above, we can define the* **Algebraic K-theory Spectrum:**

$$K : \mathrm{Cat}_\infty^{\mathrm{st}} \xrightarrow{k} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{B^\infty} \mathrm{Sp}_{\geq 0}$$

*This gives a connective $K$-theory spectrum.*

This definition explains the title of our project. However, we will not use the spectrum viewpoint. Moreover, as the name suggested, one can imagine there is a non-connective $K$-theory, which recovers connective $K$-theory by taking connective covers. In a way, that should be the only right definition.

After giving this definition, the first thing to do should be checking its $\pi_0$-group indeed gives $K_0$ as before. One can find its proof in Lemma 4.7 of Fabian's note [2]:

**Proposition 7.7.** *There is a canonical equivalence*

$$K_0(C) \simeq \pi_0 k(C).$$

To prove the additivity, we will introduce Quillen's $Q$-construction. They're equivalent constructions, as we will see.

**Definition 7.8.** *(Quillen's Q-construction) Let $C$ be a stable $\infty$-category and $[n] \in \Delta^{\mathrm{op}}$, and let* $\mathrm{TwAr}([n])$ *be its twisted arrow category whose*

- *objects are arrows of $[n]$, which we denote as $i \leq j$.*
- *morphism between $a_0 \leq a_1$ and $b_0 \leq b_1$ is a diagram as follows:*

$$\begin{tikzcd} a_0 & b_0 \arrow[l] \arrow[d] \\ a_1 \arrow[u, leftarrow] \arrow[r] & b_1 \end{tikzcd}$$

*A square in* $\mathrm{TwAr}([n])$ *is defined as a diagram looking like*

$$\begin{tikzcd} & i \leq l \arrow[dl] \arrow[dr] & \\ i \leq k \arrow[dr] & & j \leq l \arrow[dl] \\ & j \leq k & \end{tikzcd}$$

*We define*

$$Q_n(C) \subset \mathrm{Fun}(\mathrm{TwAr}([n])^{\mathrm{op}}, C)$$

*to be the full $\infty$-subcategory spanned by functors $F$ sending squares to pullback squares in $C$.*

The objects of $Q_n(C)$ can be visualized as diagrams of the following form (n=2):

$$\begin{tikzcd}[column sep=small]
 & & F(0 \leq 2) \arrow[dl] \arrow[dr] & & \\
 & F(0 \leq 1) \arrow[dl] \arrow[dr] & & F(1 \leq 2) \arrow[dl] \arrow[dr] & \\
F(0 \leq 0) & & F(1 \leq 1) & & F(2 \leq 2)
\end{tikzcd}$$

Notice that all the $Q_n$'s also form a functor:

$$Q : \mathrm{Cat}_\infty^{st} \to \mathrm{sCat}_\infty^{st}$$

Even though $Q_n(C)$ is a big diagram, similarly as $S_n(C)$, it admits a smaller description by the following proposition:

**Proposition 7.9.** *Let $C$ be a stable $\infty$-category. Then the simplicial anima*

$$\mathrm{core}(Q(C)) : \Delta^{\mathrm{op}} \to An$$

*is a Segal space. In fact, we know for all $n$,*

$$Q_n(C) \simeq Q_1(C) \times_{Q_0(C)} \times \cdots \times_{Q_0(C)} Q_1(C)$$

*Proof.* We will show that $Q(C)$ satisfies Segal condition as a simplicial $\infty$-category. Then because core is a right adjoint, which means it preserves limits, the simplicial anima core $Q(C)$ will be a Segel anima automatically.

By definition of $\mathrm{TwAr}[n]$, its 0-simplices are given by posets of $[n]$. We define $J_n \subset \mathrm{TwAr}[n]$ as the full subcategory spanned by $i \leq i$ and $i \leq i+1$. In pictures, $J_n$ is the following diagram category:

$$\begin{tikzcd}[column sep=small]
 & \bullet \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\
0 & & 1 & & n-1 & & n
\end{tikzcd}$$

By right Kan extension along $J_n \subset \mathrm{TwAr}[n]$, we have a functor:

$$R : \mathrm{Fun}(J_n, C) \to \mathrm{Fun}(\mathrm{TwAr}[n], C)$$

Because $J_n \to \mathrm{TwAr}[n]$ is a fully faithful embedding, one can see $R$ is also fully faithful by Proposition 3.2, and we claim that its essential image is $Q_n(C)$. To see this, for $H \in \mathrm{Fun}(J_n, C)$, recall the pointwise formula for the right Kan extension is

$$RH(e) \simeq \lim_{(c, e \to H(c)) \in H_{e/}} H(c)$$

Notice that we can do the right Kan extension step by step. By this, we mean we firstly do the right Kan extension to the spot $0 \leq 2$, and then $0 \leq 3$, all to the right, and move to the upper line. By continuing this process, we can get $RH$. One can observe that in each step, the arrows in the domain of the limit all factor through two maps. For example, for $0 \leq 2$, the limit can be concentrated on the arrows $0 \leq 2 \to 0 \leq 1$ and $0 \leq 2 \to 1 \leq 2$, and these two maps coincide after composing to $1 \leq 1$. Hence, the pointwise formula reduces to take limit in the form $\bullet \leftarrow \bullet \to \bullet$, which is taking pullback. From the above analysis, one sees that $F$ in the image of the right Kan extension precisely means the squares in the diagram are pullback squares.

In the last paragraph, we have shown $R : \mathrm{Fun}(J_n, C) \to Q_n(C)$ is an equivalence. Using $J_n \to \mathrm{TwAr}[n]$ is fully faithful, it is also easy to see the inverse of $R$ is the restriction functor. With this equivalence, it will be easy to show that $Q(C)$ satisfies the Segal condition. Because we have

$$J_n \simeq J_1 \times_{J_0} J_1 \times_{J_0} \cdots \times_{J_0} J_1$$

Applying $\mathrm{Fun}(-, C)$ to the diagram above gives

$$Q_n(C) \simeq Q_1(C) \times_{Q_0(C)} \times_{Q_0(C)} \cdots \times_{Q_0(C)} Q_1(C).$$

This finishes our proof. □

**Remark 7.10.** *One can further show* $\mathrm{core}\, Q(C)$ *is actually a complete Segal anima. which means,*

$$\begin{tikzcd}
\mathrm{core}\, Q_0(C) \arrow[r, "\Delta"] \arrow[d] & \mathrm{core}\, Q_0(C) \times \mathrm{core}\, Q_0(C) \arrow[d, "{(s,s)}"] \\
\mathrm{core}\, Q_3(C) \arrow[r, "{(d_{0,2}, d_{1,3})}"'] & \mathrm{core}\, Q_1(C) \times \mathrm{core}\, Q_1(C)
\end{tikzcd}$$

*is a pullback square.*

In the rest of this section, we will discuss the equivalence between $Q$-construction and $S$-construction. Its proof involves the edgewise subdivision.

**Definition 7.11.** *Let us consider a functor* $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ *that mapps* $[n]$ *to* $[n] \star [n]^{\mathrm{op}}$*, where* $[n] \star [n]^{\mathrm{op}}$ *denote the join of two categories. It induces a functor:*

$$(-)^{edg} : \mathrm{sAn} \to \mathrm{sAn}\,.$$

*We call this functor an edgewise subdivision.*

**Proposition 7.12.** *For a simplicial anima* $X$*, we have*

$$|X| \simeq |X^{edg}|.$$

*Proof.* Notice that $|-|$ is colimit-preserving because it is defined as a colimit, the functor $(-)^{edg}$ is also colimit-preserving because the colimit of preshaves is calculated levelwise. Hence, the functor on both sides is colimit-preserving. Since An is generated by $\Delta^n$ by colimits, we only need to check the two functors induce equivalence on $\Delta^n$.

The left hand is clearly contractible. For the right hand, by Proposition 3.6 and Proposition 3.7 , we know

$$|(\Delta^n)^{edg}| \simeq |(N^r([n]))^{edg}| \simeq |N^r(TwAr([n]))|$$

Because $\mathrm{TwAr}([n])$ has an initial object, we know the right hand is also contractible. $\square$

Notice that $[n] \star [n]^{\mathrm{op}} \simeq [2n-1]$, and we can formulate this category as the following poset:

$$0_l \leq 1_l \cdots \leq n_l \leq n_r \leq n-1_r \leq \cdots \leq 0_r$$

where $l$ denotes left and $r$ denotes right.

**Theorem 7.13.** *Let $C$ be a stable $\infty$-category. The map $\mathrm{TwAr}([n])^{\mathrm{op}} \to \mathrm{Ar}([n] \star [n]^{\mathrm{op}})$ sending $i \leq j$ to $i_l \leq j_r$ induces an equivalence*

$$\tau : S(C)^{edg} \xrightarrow{\simeq} Q(C)$$

*In particular, this implies that*

$$k(C) \simeq \Omega|\operatorname{core} S(C)^{edg}| \simeq \Omega|\operatorname{core} Q(C)|$$

*Proof.* The map $\mathrm{TwAr}([n])^{\mathrm{op}} \to \mathrm{Ar}([n] \star [n]^{\mathrm{op}})$ described in the theorem induces a map between simplicial $\infty$-categories $\mathrm{TwAr}(-)^{\mathrm{op}} \to \mathrm{Ar}([-] \star [-]^{\mathrm{op}})$, applying $\mathrm{Fun}(-, C)$ and restricting to $S(C)^{edg}$ gives us the map $\tau$. We need to check that it is an equivalence for each $n$.

Let $I_n$ denote the image of $\mathrm{TwAr}([n])^{\mathrm{op}}$ in $\mathrm{Ar}([n] \star [n]^{\mathrm{op}})$. For $n = 1$, one can see that $\tau$ sends an $S$-construction diagram to the blue part below:

$$\begin{tikzcd}
0_l \leq 0_l \arrow[r] & 0_l \leq 1_l \arrow[r] \arrow[d] & 0_l \leq 1_r \arrow[r] \arrow[d] & 0_l \leq 0_r \arrow[d] \\
& 1_l \leq 1_l \arrow[r] & 1_l \leq 1_r \arrow[r] \arrow[d] & 1_l \leq 0_r \arrow[d] \\
& & 1_r \leq 1_r \arrow[r] & 1_r \leq 0_r \arrow[d] \\
& & & 0_r \leq 0_r
\end{tikzcd}$$

To show it is an equivalence, we want to see that the whole diagram is determined by the blue part. Now we let $\Delta^n_l$ denote the part of the diagram spanned by $\{i_l \leq i_l\}$, and let $\Delta^n_r$

denote the part of the diagram spanned by $\{i_r \leq i_r\}$, and let $J_n$ denote the part of diagram spanned by $\{i_l \leq j_l, i_r \leq j_r\}$. Then the inverse morphism is constructed as follows:

$$
\begin{tikzcd}
Fun(I_n, C) \arrow[rr, dashed, "\iota"] \arrow[d, "\mathrm{Lan}"] & & Fun(Ar([n] \star [n]^{\mathrm{op}}), C) \\
Fun(I_n \cup \Delta_n^l, C) \arrow[r, "\mathrm{Ran}"] & Fun(I_n \cup \Delta_n^l \cup \Delta_n^R, C) \arrow[r, "\mathrm{Ran}"] & Fun(J_n \cup I_n, C) \arrow[u, "\mathrm{Lan}"']
\end{tikzcd}
$$

, where the union above means we take the subposet of $\mathrm{Ar}([n] \star [n]^{\mathrm{op}})$ generated by the two subposets in the diagram. We use colors to indicate the part we extend in each step from a $Q$-construction diagram to a diagram in $\mathrm{Fun}([n] \star [n]^{\mathrm{op}}, C)$. Notice that the functors in all the steps are Kan extensions along fully faithful embeddings, so we have the Kan extensions in the diagrams are all fully faithful embeddings and $\tau \circ \iota \simeq id_{Fun(I_n, C)}$ by Proposition 3.2. Hence, it is enough to show the essential image of $\iota$ is $S_n(C)^{edg}$.

Recall the formulas of Kan extensions are the followings:

$$\mathrm{Lan}_f(e) \simeq \operatorname*{colim}_{(c, f(c) \to e) \in C_{/e}} F(c); \qquad \mathrm{Ran}_f(e) \simeq \lim_{(c, e \to f(c)) \in C_{e/}} F(c).$$

From them, we see in the first step that the red parts are all 0 because there is no map mapping to the spots, and we are taking a colimit over an empty diagram. Similarly, in the second step, we get all the yellow parts being 0. In the third step, the right Kan extension can be seen as taking right Kan extensions vertices by vertices, where the order is to extend from the right line to the left line, and in each line, we extend from the bottom to the top. For a start, at the spot $n - 1_l \leq n_r$, the right Kan extension formula reduces to take a limit over the three spots $n_l \leq n_l$, $n_l \leq n_r$ and $(n-1)_l \leq (n-1)_r$, and taking limit precisely tells us that it gives a pullback square. In the order we describe, the value of each point is extended such that the square we filled at that step is a pullback square so that the whole diagram we extended only consists of pullback squares. Similarly, in step 4, we see that the formula for left Kan extensions implies each square is a pushout square, which is also pullback because $C$ is stable. From the analysis, we see that

$$\iota(Q(C)) \simeq S(C)^{edg}$$

Hence, we can conclude $\iota$ is an equivalence and $\tau$ is an inverse which is also an equivalence. □

# 8 Exact sequences of stable infinity categories

In this section, we need to introduce the concept of Verdier sequences. We will discuss the fibers and cofibers of functors in $\mathrm{Cat}_\infty^{\mathrm{st}}$. Using them, we can give a characterisation of Verdier sequences. This characterisation gives an important class of Verdier sequences given by localisations.

**Definition 8.1.** *In the category* $\mathrm{Cat}_\infty^{\mathrm{st}}$*, we define the following:*

- *A sequence $A \to B \to C$ in $\mathrm{Cat}_\infty^{st}$ is called a* **Verdier sequence** *if it is both a fiber sequence and cofiber sequence. It is called* **left/right split** *if both functors admit left/right adjoint. It is called* **split** *if it is both left and right split.*

- *In a Verdier sequence $A \to B \to C$, we say $B \to C$ is a* **Verdier projection** *and $A \to B$ is a* **Verdier injection**. *If the sequence is left/right split, then we say its Verdier projection or injection is* **left/right split**.

- *A pullback square in $Cat_\infty^{st}$ is called a* **Verdier square** *if its vertical maps are Verdier projections. A Verdier square is* **left/right split** *if its Verdier projections are left/right split.*

- *Given a $\infty$-category $E$ that admits finite limits, we say a functor $F : \mathrm{Cat}_\infty^{\mathrm{st}} \to E$ is* **additive** *if it takes split Verdier squares to pullback square in $E$, an additive functor is called* **grouplike** *if it factors through $\mathrm{CGrp}(E)$. We define $\mathrm{Fun}^{add}(\mathrm{Cat}_\infty^{\mathrm{st}}, E)$ and $\mathrm{Fun}^{grp}(\mathrm{Cat}_\infty^{\mathrm{st}}, E)$ to be the full subcategory of $\mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{st}}, E)$ spanned by additive functors and grouplike functors respectively.*

For a functor between stable infinity categories, its fiber and cofiber can be constructed explicitly. Using them, we will give a very useful characterization of Verdier sequences.

**Proposition 8.2.** *Let $F : C \to D$ be an exact functor between stable infinity categories. Its fiber $\mathrm{fib}(F)$ can be described as the full subcategory*

$$\mathrm{fib}(F) \simeq \{c \in C | F(c) \simeq 0\}.$$

*which means it has the universal property that the inclusion $\mathrm{fib}(F) \to C$ is terminal among exact functors over $C$ that vanish after composing with $F$.*

*Proof.* Because $F$ is an exact functor, $\mathrm{fib}(F)$ is closed under finite direct sum, pullbacks and pushouts, so as a subcategory of $C$, it is also stable. The fact that it satisfies the universal property is to check we have equivalence for any stable infinity category $E$

$$\hom^{ex}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(E, \mathrm{fib}(F)) \simeq \hom^{ex}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(E, C) \times_{\hom^{ex}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(E,D)} 0$$

By definition, both sides are path components of $\hom_{\mathrm{Cat}_\infty^{\mathrm{st}}}(E, C)$ containing functors that vanish after composing with $D$, so they must coincide. ☐

**Proposition 8.3.** *Let $F : C \to D$ be an exact functor between stable infinity categories. We define*

*W:= {mod-C equivalences}:={ maps $\phi : x \to y$ in $D$ where their fiber/cofiber are retract of objects in the essential image of $F$.}*

*Then, we have*

$$\mathrm{cofib}(F) \simeq D[W^{-1}]$$

*and we denote $\mathrm{cofib}(F)$ by $D/C$.*

The proof that $D/C$ has the correct universal property is the same as the Proposition above. The main difficulty is to prove that it is a stable $\infty$-category. For this, we reference the following lemma:

**Lemma 8.4.** *(Theorem 1.3.3(ii) [8]) Let $C \subset D$ be a stable sub-$\infty$-category and let $D[W^{-1}]$ be the localisation of $D$ with respect to* $\operatorname{mod} -C$ *equivalences. Then, for all $x, y \in D$,*

$$\hom_{D[W^{-1}]}(x, y) \simeq \operatorname*{colim}_{c \in C_{/y}} \hom_D(x, \operatorname{cofib}(c \to y)).$$

Using the lemma, the proof of the Proposition is not complicated:

*Proof.* Since we can always start from the essential image of $C$ in $D$, it is enough to show the lemma when $C$ is a stable sub-$\infty$-category of $D$, and $F$ is the inclusion.

Let us first show $C_{/y}$ is filtered. This means that given any finite simplicial set $K$ with a map $K \to C_{/y}$, there is an extension to the join of a simplicial set $K \star \Delta^0 \to C$. The extension can be done by taking the colimit of $K \to C$, and it canonically maps to $y$ because $C \subset D$ preserves finite colimits.

Then, we want to show $p : C \to D/C$ preserves finite limits and finite colimits. Notice for this, we need to use the fact that finite limits commute with filtered colimits in An by Theorem 1.24. Let $c = \operatorname{colim}_{i \in I} F(i)$ in $D$. We want to show

$$p(c) \simeq \operatorname*{colim}_{i \in I} pF(i)$$

For any $x \in D/C$, we have

$$\begin{aligned}
\hom_{D/C}(p(c), x) &\simeq \operatorname*{colim}_{t \in C_{/x}} \hom_D(c, \operatorname{cofib}(t \to x)) \\
&\simeq \operatorname*{colim}_{t \in C_{/x}} \lim_{i \in I} \hom_D(F(i), \operatorname{cofib}(t \to x)) \\
&\simeq \lim_{i \in I} \hom_{D/C}(pF(i), x)
\end{aligned}$$

This shows $p(c)$ has the universal property of being a colimit of $pF$.

To show $C$ is stable, one needs to check $C$ has finite limits and finite colimits, and pullback squares coincide with pushout squares. To see it has finite limits, it is enough to see it has a terminal object and pullback squares. Its terminal object comes from $D$, and given a diagram $x \to y \leftarrow z$ in $D/C$, it can be represented by a diagram $x \to \operatorname{cofib}(c \to y) \leftarrow z$ in $D$ for some $c \in C_{/y}$ because the hom formula is over $C_{/y}$, which is a filtered category. By representing, we mean the diagram is $x \to \operatorname{cofib}(c \to y) \leftarrow z$ maps to $x \to y \leftarrow z$ in $D[W^{-1}]$ under $p$. Because $D$ admits finite limits and $p$ preserves finite limits, we see $D/C$ have pullbacks, and so are finite limits. Similarly, one can see that it admits finite colimits. Also, in our proof, we have seen that pullback squares and pushout squares in $D/C$ must coincide because they are lifted to pullback-pushout squares in $D$. $\square$

Using them, we can deduce the theorem we need:

**Theorem 8.5.** *Let $F : C \to D$ be an exact functor between stable infinity categories.*

*(a) $F$ is a Verdier projection iff it is a localisation.*

*(b) $F$ is a left/right split Verdier projection iff it admits a left/right fully faithful adjoint.*

*(c) $F$ is a Verdier inclusion iff it is fully faithful and its essential image is closed under retracts of $D$.*

*(d) $F$ is a left/right split Verdier inclusion iff it is fully faithful and admits a left/right adjoint.*

*Proof.* **(a):** For a Verdier sequence $B \to C \to D$, we see

$$D \simeq \mathrm{cofib}(B \to C) \simeq C[\ \{\text{mod-B equivalences}\}^{-1}]$$

Hence, a Verdier projection is always a localisation. On the other hand, let us assume $D \simeq C[W^{-1}]$, where $W$ is a collection of morphisms in $C$. Then, we want to show the fiber sequence

$$\mathrm{fib}(F) \to C \to D$$

is also a cofiber sequence. Notice that every morphism in W is mod(fib(F))-equivalence because they are sent to equivalence, which has fiber 0. Conversely, because $\mathrm{fib}(F)$ is closed under retract, mod(fib(F))-equivalences are precisely the morphisms in $C$ such that whose fiber are in fib(F), which means F sends the morphism to equivalence in $D$. Hence, by the description of cofiber, we conclude it is also a cofiber sequence.

**(b)** If we are given a left/right split Verdier projection, then $F$ has a left/right adjoint by definition, and we also know it is localisation. By Proposition 4.7, we see its left/right adjoint is automatically fully faithful. On the other hand, if $F$ admits a fully faithful left/right adjoint, it is a Bousfield localisation. By $(a)$, this automatically gives a Verdier sequence

$$\mathrm{fib}(F) \to C \to D.$$

We only need to check $\mathrm{fib}(F) \subset C$ has a left/right adjoint. We will do the left case. Let $H : D \to C$ be the left adjoint of $F$ with counit $c : HF \Rightarrow id_C$. Then, we claim that the

$$G \simeq \mathrm{cofib}(c : HF \Rightarrow id_C)$$

is the left adjoint of $\mathrm{fib}(F) \to C$. We want to show that $G$ is a left Bousfield localisation onto $\mathrm{fib}(F)$. Firstly, notice that $F(G(a)) \simeq \mathrm{cofib}(F(c) : FHF(a) \to F(a))$. Using triangular identity and unit is equivalence, we see $F(c)$ must be an equivalence so that $F(G(a)) \simeq 0$. Hence, we learn that $G$ maps into $\mathrm{fib}(F)$.

By Proposition 4.8, we need to check there is a natural transformation $\eta : id \Rightarrow G$ such that $\eta_{Gx}$ and $G(\eta_x)$ are equivalences for all $x$, and also check $\mathrm{Im}(G) = \mathrm{fib}((F)$. The natural transformation $\eta$ exists because $G(x)$ is defined via cofiber. Because of the cofiber sequence:

$$0 \simeq GF(G(x)) \xrightarrow{c_{g(x)}} G(x) \xrightarrow{\eta_{g(x)}} G(G(x))$$

we see $\eta_{g(x)}$ must be an equivalence. For $G(\eta_x)$, we can consider a morphism of cofiber sequences:

$$\begin{tikzcd} GF(x) \arrow[r, "c"] \arrow[d, "GF(\eta_x)"] & x \arrow[r, "\eta_x"] \arrow[d, "\eta_x"'] & G(x) \arrow[d, "g(\eta_x)"] \\ 0 \simeq GF(G(x)) \arrow[r, "c_{g(x)}"] & G(x) \arrow[r, "\eta_{g(x)}"] & G(G(x)) \end{tikzcd}$$

Notice the left square is a pushout, so it induces equivalence on fibres, so $G(\eta_x)$ must be an equivalence. We have already proven $G$ is a left Bousfield localisation onto its image, which consists of all $x$ such that $\eta_x$ is an equivalence. But by definition, all the fiber of $F$ satisfy this condition, so we finish the proof of (b).

**(c):** If $F$ is a Verdier inclusion, then by our description of fiber, the only if part is clear. Conversely, we will prove the following claim, which finishes the proof immediately.

**Claim:** Let $F : C \to D$ be a functor of stable infinity categories. Then $\mathrm{fib}(p : D \to \mathrm{cofib}(F))$ is the full subcategory of $D$ spanned by objects that are retracts of objects in $C$.

If $d$ is a retracts of an object of $C$, then the map $0 \to d$ is sent to equivalence by our description of cofiber, so the inclusion from right to left is clear. On the other hand, for $x \in \ker(p)$, let us consider the exact functor $\phi_x : D \to \mathrm{Sp}$ given by the formula

$$\phi_x(y) = \operatorname*{colim}_{\beta : z \to y \in C_{/y}} \hom_D(x, \mathrm{cofib}(\beta))$$

where $C_{/y} := C \times_D D_/y$ is the sliced category over $y \in D$. Then if $y \in C$, which means it is in the image of $C \to D$, we know $id_y : y \to y$ is the terminal object in $C_{/y}$. This implies $\phi_x(y) \simeq \hom_D(x, \mathrm{cofib}(id_y)) \simeq 0$. Proposition A.1.5 from [9] tells us that any exact functor $D \to \mathrm{Sp}$ vanishes on $C$ also vanishes on $\ker(p)$. Hence, the above discussion implies that $\phi_x$ vanishes on $\ker(p)$. In particular, we have $\phi_x(x) = 0$, i.e.

$$\operatorname*{colim}_{\beta : z \to x \in C_{/x}} \hom_D(x, \mathrm{cofib}(\beta) \simeq 0.$$

As we have seen in the proof of Proposition 8.4, $C_{/x}$ is a sliced category, and by Theorem 1.24, we know the functor $\pi_0$ commutes with filtered colimits, so that we can pass the colimit above to $\pi_0$, $id_x \in \hom(x, \mathrm{cofib}(0 \to x))$ must vanish in some stage because it is a colimit over filtered category. In other words, there is $\beta : z \to x$ for some $z \in C$ such that $id_x$ is in the kernel of the map

$$\pi_0 \hom_D(x, \mathrm{cofib}(0 \to x)) \to \pi_0 \hom_D(x, \mathrm{cofib}(\beta))$$

By the universal property of cofiber, we see that $id_x$ factors through $z$ and hence $x$ is a retract of z. This finishes the proof of the claim and also $(c)$.

**(d):** The only if part is trivial, so let us assume $F : C \to D$ is fully faithful and admits a left adjoint $H : D \to C$ with unit $\eta : id_D \Rightarrow FH$. Completely dual to $(b)$, one can show

$$G := \mathrm{fib}(\eta : id_D \Rightarrow FH)$$

is a right Bousfield localisation onto its image $\mathrm{im}(G)$. We only need to show that $\mathrm{im}(G) \simeq D/C$. By Proposition 4.5, we can see that $\mathrm{im}(G)$ is a localisation at morphisms $\phi : x \to y$ for which

$$\phi_* : \hom_D(G(d), x) \to \hom_D(G(d), y)$$

is an equivalence for all $d \in D$. This condition is equivalent to

$$* \simeq \hom_D(G(d), \mathrm{fib}(\phi)) \simeq \hom_{\mathrm{im}(G)}(G(d), G(\mathrm{fib}(\phi))).$$

By Yoneda lemma, this is equivalent to $G(\mathrm{fib}(\phi)) \simeq 0$. From the definition of $G$, we have $\mathrm{fib}(\phi)$ in the essential image of $C$, so $\phi$ is a mod $-C$ equivalence. On the other hand, one can see from the definition that $G$ inverts all mod $-C$ equivalences. This finishes our proof that $\mathrm{im}(G) \simeq D/C$. □

The following corollary will be useful in the next section. It also tells us that the split Verdier sequence is the right concept to consider.

**Corollary 8.6.** *A pullback of a split Verdier projection is still a split Verdier projection. In particular, to check a pullback square is a Verdier square, it is enough to check on its right leg.*

*Proof.* From the Proposition, we need to check that the property of having left/right fully faithful adjoint is preserved under pullback. Let

$$\begin{tikzcd} A \arrow[r, "L"] \arrow[d, "G'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, shift right, "G"'] \\ C \arrow[r, "T"] & D \arrow[u, shift right, "F"'] \arrow[u, phantom, "\vdash"] \end{tikzcd}$$

be a pullback square of stable $\infty$-categories with $F$ being fully faithful. Then this gives the square

$$\begin{tikzcd} C \arrow[r, "T"] \arrow[d, "\mathrm{id}"'] & D \arrow[r, "F"] & B \arrow[d, "G"] \\ C \arrow[rr, "L"] & & D \end{tikzcd}$$

which commutes because $GF \simeq \mathrm{id}_D$. The universal property of pullback defines the functor $F' : C \to A$ with natural equivalence $\eta : \mathrm{id}_C \Rightarrow G'F'$. Hence, we only need to show $F'$ is the left adjoint of $G'$. We want to construct:

$$c : F'G' \Rightarrow id_A.$$

By the universal property of pullback, we have an equivalence

$$\mathrm{Fun}(A, A) \simeq \mathrm{Fun}(A, C) \times_{\mathrm{Fun}(A,D)} \mathrm{Fun}(A, B).$$

Because $F'G'$ maps to $G'F'G' \times_{TG'F'G'} LF'G' \simeq G' \times_{GL} FGL$ and $\mathrm{id}_A$ maps to $G' \times_{GL} L$. To define $c$, it is equivalent to define a natural transformation

$$G' \times_{GL} FGL \Rightarrow G' \times_{GL} L,$$

and this comes from the counit of $F \dashv G$. To check triangular identity, we only need to factor $\eta$ out via the pullback diagram

$$\mathrm{Fun}(C, C) \simeq \mathrm{Fun}(C, A) \times_{\mathrm{Fun}(C,D)} \mathrm{Fun}(C, B)$$

and write out the functors, it will be easy to see that the triangular identities hold. $\square$

One important example of the split Verdier sequence is the derived localisation of rings. We will state this Proposition, and reference it back to Fabian's note since we will not use it later.

**Proposition 8.7.** *(Lemma 4.16(a) [2]) Let $f : R \to S$ be a derived localisation of rings, then there is a split Verdier sequence*

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "f^*"] & \mathcal{D}(R) \arrow[l, bend right=50, "S\otimes_R -"'] \arrow[l, bend left=50, "\hom_R(S,-)"] \arrow[r, "I\otimes_R -"] & \mathcal{D}(R)^{\text{S-tors}} \arrow[l, bend right=50, "\text{incl.}"'] \arrow[l, bend left=50, "\hom_R(I,-)"] \end{tikzcd}$$

*where $I$ is defined as the fiber of $f$, and $\mathcal{D}(R)^{\text{S-tors}}$ is the full stable sub-$\infty$-category spanned by $R$-modules $M$ such that $S \otimes_R M \simeq 0$.*

# 9 Algebraic K-theory is a universal additive invariant

With the discussion of all the former sections, we can finally discuss our main result, which is the following:

**Theorem 9.1.** *There is a left Bousfield localisation:*

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}_{\infty}, \mathrm{An}) \arrow[r, shift left, "(-)^{\mathrm{grp}}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}^{\mathrm{grp}}(\mathrm{Cat}^{\mathrm{st}}_{\infty}, An). \arrow[l, shift left, "\text{inclusion}"] \end{tikzcd}$$

*where $F^{\mathrm{grp}(-)} := \Omega|FQ(-)|$. In particular, we have*

$$k \simeq \mathrm{core}^{\mathrm{grp}}$$

*This means that $k$ is the initial grouplike functor under* $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}}_{\infty} \to \mathrm{An}$.

The proof of the theorem divides into two parts. In the first part, we want to show $k$ is indeed an additive functor. For this, we will prove a more general theorem that for any additive functor $F$, $|FQ(-)|$ is also an additive functor. In the second step, we will prove the construction indeed gives a left Bousfield localisation.

## 9.1 Waldhausen's additivity theorem

**Theorem 9.2.** *If $F : \mathrm{Cat}_\infty^{\mathrm{st}} \to$ An is an additive functor, then so is*

$$|FQ(-)| : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{An}\,.$$

As a corollary, it implies that

**Corollary 9.3.** *We have $k : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{An}$ is additive.*

*Proof.* To see this, we notice Proposition 7.13 gives that

$$k(C) \simeq \Omega|\operatorname{core}(Q(C))|$$

Hence, we only need to see $\Omega$ and core are additive, but core is right adjoint to the inclusion $\mathrm{An} \to \mathrm{Cat}_\infty$ so it preserves pullback, and $\Omega$ defined as a pullback also preserves pullback. Hence, they're obviously additive. $\square$

*Proof of Theorem 9.2.* We will state what we are going to prove in the lemmas below and see how it finishes the proof:

- In Lemma 9.4, we will prove the $FQ(C)$ is a Segal space.
- In Corollary 9.6, we will see that a split Verdier projection between stable $\infty$-categories is a bicartesian fibration.
- In Definition 9.7, we will define bicartesian fibration of Segal spaces and prove that $FQ(p)$ is bicartesian fibration in Lemma 9.10.
- In Lemma 9.11, we will give a reference to the fact that geometric realization preserves pullback squares of Segal spaces with one leg being bicartesian fibration.

The proof now becomes simple. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be a split Verdier square. Because we know $Q_n(-) \simeq \mathrm{Fun}(J_n, -)$ from the proof of Proposition 7.9, so $Q_n(-)$ preserves pullback squares. It also preserves split Verdier projection because it extends adjunctions and we use Proposition 8.5. Then using the assumption that $F$ is additive, we see we have a pullback square of Segal spaces:

$$\begin{tikzcd} FQ(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & FQ(B) \arrow[d] \\ FQ(C) \arrow[r] & FQ(D) \end{tikzcd}$$

The first three steps of our plan show that the legs of the square are bicartesian fibrations, and the final step shows that we get a pullback after applying $|-|$. $\square$

**Lemma 9.4.** *Let $C$ be a stable infinity category and $F : \mathrm{Cat}^{\mathrm{st}}_{\infty} \to An$ is an additive functor. Then we have $F(Q(C)) \in sAn$ is a Segal anima.*

*Proof.* By Remark 5.2, we need to prove that for each $0 \leq i \leq n$ with square

$$\begin{tikzcd}
{[0]} \arrow[r] \arrow[d] & {[n-i]} \arrow[d] \\
{[i]} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & {[n]}
\end{tikzcd}$$

Applying $F(Q)$ will give us a pullback square. Since we know $Q(C)$ satisfies the Segal condition by Proposition 7.9, applying $Q$ to the square above gives a pullback square. If we can show the square is a split Verdier square, then the proof will be finished by the additivity of $F$.

Hence, it is enough to show that given an inclusion $i : [k] \to [l]$, the map $i^* : Q_l(C) \to Q_k(C)$ is a split Verdier projection. As we have discussed in the proof of Proposition 7.9,

$$Q_n(C) \simeq Fun(J_n, C),$$

so $i^*$ is equivalent to the functor induced by the inclusion $J_k \to J_l$. Since $i$ is fully faithful, its left and right Kan extensions are fully faithful left and right adjoints of $i^*$ respectively. Then by Theorem 8.5, we know $i^*$ is a split Verdier projection. $\square$

**Lemma 9.5.** *Let $F\colon C \rightleftarrows D : G$ be an adjunction pair between infinity categories with counit $FG \Rightarrow id_C$. Then,*

1. *A morphism $\psi : x \to y$ in $D$ is $G$-cocartesian iff the square*

$$\begin{tikzcd}
FG(x) \arrow[r, "FG(\psi)"] \arrow[d, "c"'] & FG(y) \arrow[d, "c"] \\
x \arrow[r, "\psi"'] \arrow[ur, phantom, "\ulcorner", very near end] & y
\end{tikzcd}$$

   *is a pushout in $C$.*

2. *If $D$ admits pushout and $G$ preserves pushout, and $F$ is fully faithful, then $G$ is cocartesian fibration.*

*Proof.* For 1, notice that by definition, $\psi$ is $g$-cocartesian iff for every $z \in D$, the following square is a pullback

$$\begin{tikzcd}
\hom(y, z) \arrow[r, "\psi^*"'] \arrow[d, "G"] & \hom(x, z) \arrow[d, "G"'] \\
\hom(G(y), G(z)) \arrow[r, "G(\psi)^*"] & \hom(G(x), G(z))
\end{tikzcd}$$

But by adjunction, we have a natural diagram

$$\begin{tikzcd} & \hom(G(y), G(z)) \arrow[d] \\ \hom(y,z) \arrow[ur, "G"] \arrow[r, "c^*"] & \hom(FG(y), z) \end{tikzcd}$$

so the first square is a pullback iff the following square is a pullback

$$\begin{tikzcd} \hom(y,z) \arrow[r, "\psi^*"'] \arrow[d, "c^*"] & \hom(x,z) \arrow[d, "c^*"'] \\ \hom(FG(y), FG(z)) \arrow[r, "FG(\psi)^*"] & \hom(FG(x), FG(z)) \end{tikzcd}$$

By definition, this is equivalent to the condition in 1.

For 2, we want to see that given $\psi' : G(x) \to y'$ in $C$ where $x \in D$, there is a p-cocartesion edge $\psi : x \to y$ in $D$ such that $G(\psi) \simeq \psi'$. We consider the following pushout:

$$\begin{tikzcd} FG(x) \arrow[r, "c"] \arrow[d, "F(\psi')"'] & x \arrow[d] \\ F(y') \arrow[r] & y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We define the right vertical edge to be $\psi$. Applying $G$ gives us the following diagram

$$\begin{tikzcd} G(x) \arrow[r, "\cong"] \arrow[d, "\psi'"'] & GFG(x) \arrow[r, "G(c)"] \arrow[d, "GF(\psi')"] & G(x) \arrow[d, "G(\psi)"] \\ y' \arrow[r, "\cong"] & GF(y') \arrow[r] & G(y) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The two isomorphisms in the diagram are because $F$ is fully faithful so the unit is equivalence. Combining triangular identities, we learn that the big square is a pushout square with horizontal arrows being the identity. In other words, we have $G(\phi) \simeq \phi'$. Then, the first square and the first part of the lemma imply that $\psi$ is a $G$-cocartesion edge, which finishes the proof. □

**Corollary 9.6.** *If $p : C \to D$ is a split Verdier projection of stable infinity categories then $p$ is a bicartesian fibration.*

*Proof.* By Proposition 8.5, we see that $p$ has a fully faithful left adjoint. Because we are considering maps between stable infinity categories, the fact that $p$ preserves pullbacks implies that it preserves pushout. The above lemma shows that $p$ is cocartesian fibration. A completely dual version of the lemma above shows $p$ is also cartesian fibration. □

For the following, we define a version of (co)cartesian fibrations for Segal spaces.

**Definition 9.7.** *Let $p : X \to Y$ be a map of Segal spaces. We define* $\mathrm{map}(x, y) \in An$ *as*

$$\begin{tikzcd} \mathrm{map}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\ * \arrow[r, "{(x,y)}"] & X_0 \times X_0 \end{tikzcd}$$

*Then, we say a morphism $f : x \to y \in X_1$ is $p$-**cocartesian** if we have a pullback square for every $z \in X_0$:*

$$\begin{tikzcd} \mathrm{map}(y,z) \arrow[r, "f^*"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{map}(x,z) \arrow[d] \\ \mathrm{map}(p(y),p(z)) \arrow[r, "p(f)^*"] & \mathrm{map}(p(x),p(z)) \end{tikzcd}$$

*Moreover, $p : C \to D$ is called a* **cocartesian fibration** *if any edge of $D$ with a fixed starting point admits a cocartesian lift.*

**Remark 9.8.** *The definition here is slightly imprecise because we did not explain how to define the precomposition of Segal Anima. To define $f^*$, we consider the diagram*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X_0 \times X_0 \arrow[d, "{(f,id)}"'] & X_1 \arrow[l] \arrow[d, "h"'] \\ * \arrow[r] & X_0 \times X_0 & X_1 \arrow[l] \end{tikzcd}$$

*where $h$ is defined as*

$$X_1 \simeq * \times_{X_0} X_1 \xrightarrow{(f,id)} X_1 \times_{X_0} X_1 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1,$$

*where the equivalence is given by the Segal condition, and we pick a canonical inverse.*

Similarly, one can define cartesian fibration and bicartesian fibration for maps between Segal spaces.

**Lemma 9.9.** *Let $p : C \to D$ be a bicartesian fibration in $\mathrm{Cat}_\infty^{\mathrm{st}}$. Then a morphism $f : x \leftarrow y \to z$ in $\mathrm{core}Q(C)$ is $\mathrm{core}Q(p)$-cocartesian if $y \to x$ is $p$-cartesian and $y \to z$ is $p$-cocartesian.*

*Dually, a morphism in $\mathrm{core}Q(C)$ is $\mathrm{core}Q(p)$-cartesian if its left edge is $p$-cocartesian and its right edge is $p$-cartesian.*

*Proof.* Let us first make some reductions. The proof of $p$-cartesian is completely dual from the proof of $p$-cocartesian, so we will only do the first case. Moreover, any edge $x \leftarrow y \to z \in Q_1(C)$ can be decomposed as $x \leftarrow y = y$ composed with $y = y \to z$, so we only need to check the two situations separately. It will turn out that the proof of the second one is dual to the proof of the first one. Hence, we are left to prove the following claim:

**Claim: Let $x \leftarrow y = y \in Q_1(C)$ be a edge such that $x \leftarrow y$ is $p$-cartesian. Then $x \leftarrow y = y$ is $\operatorname{core} Q(p)$-cocartesian.**

To prove the claim, we want to see for any $z \in \operatorname{core} Q_0(C)$, we have

$$\begin{tikzcd}
\operatorname{map}(y,z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{map}(x,z) \arrow[d] \\
\operatorname{map}(p(y),p(z)) \arrow[r] & \operatorname{map}(p(x),p(z))
\end{tikzcd}$$

In the diagram, we have

$$\begin{tikzcd}
 & & w \arrow[dl, dashed] \arrow[dr, dashed, "(1)\simeq"] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & & & & & & p(w) \arrow[dl] \arrow[dr, "\simeq"] \arrow[dd, phantom, "\lrcorner"] & & \\
 & y \arrow[dl] \arrow[dr, "id_y"] & & w \arrow[dl, dashed] \arrow[dr, dashed] & & \longmapsto & & & p(y) \arrow[dl] \arrow[dr, "id_{p(y)}"] & & p(w) \arrow[dl] \arrow[dr] & \\
x & & y & & z & & & p(x) & & p(y) & & p(z)
\end{tikzcd}$$

Because the square in the first diagram must be a pullback, the assumption of $id_y$ implies the arrow (1) must be an equivalence. From the definition of mapping spaces, we have

$$\begin{tikzcd}
\hom_{\operatorname{core}(C)}(w,z) \times \hom_{\operatorname{core}(C)}(w,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{map}_{\operatorname{core} Q(C)}(y,z) \arrow[d, "\text{middle point}"] \\
* \arrow[r, "w"'] & C
\end{tikzcd}$$

Hence, we know the maps $\operatorname{map}(y,z) \to \operatorname{map}(x,z)$ and $\operatorname{map}(p(y),p(z)) \to \operatorname{map}(p(x),p(z))$ commute with the map picking the middle point. Hence, to see the square above is a pullback, it is enough to check over each fiber. Fix a $w$, the square we need to check becomes

$$\begin{tikzcd}
\hom_{\operatorname{core}(C)}(w,y) \times \hom_{\operatorname{core}(C)}(w,z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \hom_{\operatorname{core}(C)}(w,x) \times \hom_{\operatorname{core}(C)}(w,z) \arrow[d] \\
\hom_{\operatorname{core}(D)}(p(w),p(y)) \times \hom_{\operatorname{core}(D)}(p(w),p(z)) \arrow[r] & \hom_{\operatorname{core}(D)}(p(w),p(x)) \times \hom_{\operatorname{core}(C)}(p(w),p(z))
\end{tikzcd}$$

This can be factored as a square time the map $\hom(w,z) \to \hom(p(w),p(z))$, and we only need to check the square left over is a pullback, but this is our assumption that $w \to y$ is $p$-cartesian and core preserves cartesian edges. $\square$

**Lemma 9.10.** *If $F : \mathrm{Cat}_\infty^{\mathrm{st}} \to An$ is an additive functor and $p : C \to D$ is a split Verdier projection, then $FQ(p)$ is a bicartesian fibration.*

*Proof.* By Lemma 9.6, one sees $p$ is a bicartesian fibration. The main proof of this lemma is to prove that we have the following two split Verdier squares:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "d_1"] \arrow[d, "Q_1(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ Q_1(D) \arrow[r, "d_1"'] & D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \times_{Q_1(C)} Q_2(C) \arrow[r, "(id_{\mathcal{E}}, d_1)"] \arrow[d, "Q_2(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \times_C Q_1(C) \arrow[d] \\ Q_2(D) \arrow[r, "(d_2, d_1)"'] & Q_1(D) \times_D Q_1(D) \end{tikzcd}$$

where $\mathcal{E}$ is the full subcategory of $Q_1(C)$ spanned by all diagrams where the left edge is $p$-cartesian and the right edge is $p$-cocartesian. Let us first see how to use them to finish the proof. Because $F$ is additive, we see that it preserves the two pullback squares above. Moreover, $F$ also commutes with pullback squares in each spot of the second diagram because as we have seen in the proof of Lemma 9.4, the functor $Q_n(C) \to Q_m(C)$ is also split Verdier projection. Now, just following from the definition of $p$-cocartesian edge, one sees that the second square being pullback says that the image of $F(\mathcal{E}) \to F(Q_1(C))$ consists of $FQ(p)$-cocartesian edges, and the first square says any 1-morphism in $FQ_1(D)$ admits a $p$-cocartesian lift.

Let us show the two squares are pullbacks and one leg of each square is a split Verdier projection.

**The first diagram:** On the object level, whenever we have a 1-morphism $(p(x) \leftarrow y \to z) \in Q_1(D)$, because $p$ is bicartesian, we see that there is always a choice $(x \leftarrow y \to z) \in \mathcal{E}$ unique up to a contractible space . Hence, we only need to show, for $x_1 \leftarrow y_1 \to z_1, x_2 \leftarrow y_2 \to z_2 \in \mathcal{E}$, we have an equivalence

$$\mathrm{map}_{\mathcal{E}}(x_1 \leftarrow y_1 \to z_1, x_2 \leftarrow y_2 \to z_2) \simeq \tag{1}$$

$$\mathrm{map}_C(x_1, x_2) \times_{\mathrm{map}_D(p(x_1), p(x_2))} \mathrm{map}_{Q_1(D)}(p(x_1) \leftarrow p(y_1) \to p(z_1), p(x_2) \leftarrow p(y_2) \to p(z_2))$$

Notice the pushout $\Delta^1 \cup_{\Delta^0} \Delta^1 \simeq J_1$ where $\mathrm{Fun}(J_1, C) \simeq Q_1(C)$ gives us a pullback square

$$\begin{tikzcd} Q_1(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\Delta^1} \arrow[d] \\ C^{\Delta^1} \arrow[r] & C \end{tikzcd}$$

This implies that

$$\mathrm{map}_{Q_1(D)}(p(x_1) \leftarrow p(y_1) \to p(z_1), p(x_2) \leftarrow p(y_2) \to p(z_2)) \simeq$$

$$\mathrm{map}_{D^{\Delta^1}}(p(x_1) \leftarrow p(y_1), p(x_2) \leftarrow p(y_2)) \times_{\mathrm{map}_D(p(y_1), p(y_2))} \mathrm{map}_{D^{\Delta^1}}(p(y_1) \to p(z_1), p(y_2) \to p(z_2))$$

Using that $x_1 \leftarrow y_1, x_2 \leftarrow y_2$ are $p$-cartesian, we have an equivalence

$$\mathrm{map}_{C^{\Delta^1}}(x_1 \leftarrow y_1, x_2 \leftarrow y_2) \simeq \mathrm{map}_C(x_1, x_2) \times_{\mathrm{map}_D(p(x_1), p(x_2))} \mathrm{map}_{D^{\Delta^1}}(p(x_1) \leftarrow p(y_1), p(x_2) \leftarrow p(y_2))$$

Plug the above two equivalence into equation (1), one can see

$$\text{LHS of (1)} \simeq \mathrm{map}_{C^{\Delta^1}}(x_1 \leftarrow y_1, x_2 \leftarrow y_2) \times_{\mathrm{map}_D(p(y_1), p(y_2))} \mathrm{map}_{D^{\Delta^1}}(p(y_1) \to p(z_1), p(y_2) \to p(z_2))$$

Then, we plug $\times_{\mathrm{map}_C(y_1,y_2)} \mathrm{map}_C(y_1,y_2)$ into the equation and use $y_1 \to z_1, y_2 \to z_2$ are $p$-cocartesian to reduce to the equivalence

$$\text{LHS of (1)} \simeq \mathrm{map}_{C^{\Delta^1}}(x_1 \leftarrow y_1, x_2 \leftarrow y_2) \times_{\mathrm{map}_C(y_1,y_2)} \mathrm{map}_{C^{\Delta^1}}(y_1 \to z_1, y_2 \to z_2)$$

Now, it is easy to see that this will be equivalent to the left hand. The fact that it is a split Verdier square follows from the assumption that $p : C \to D$ is a split Verdier projection and Proposition 8.6.

**The second diagram:** Although the second diagram seems more complicated, the fact it is a pullback will be easier to see because $Q_2(C) \to Q_1(C) \times_C Q_1(C)$ is an equivalence by Segal condition. One only needs to show

$$\mathcal{E} \times_{Q_1(C)} Q_2(C) \to \mathcal{E} \times_C Q_1(C)$$

is an equivalence. We consider the following cube

$$\begin{tikzcd}
 & & \mathcal{E} \times_{Q_1(C)} Q_2(C) \arrow[dll] \arrow[d, dashed] \arrow[drr] & & \\
\mathcal{E} \arrow[d, "id"] \arrow[drrr] & & \mathcal{E} \times_C Q_1(C) \arrow[dll, dashed] \arrow[drr, dashed] & & Q_2(C) \arrow[dl] \arrow[d] \\
\mathcal{E} \arrow[drrr] & & & Q_1(C) \arrow[d] & Q_1(C) \arrow[dl] \\
 & & & C &
\end{tikzcd}$$

It is easy to see the top face, the bottom face and the right face are all pullbacks, so the left face is also a pullback, then $\mathcal{E} \to \mathcal{E}$ being equivalence implies the result. Now, we are left to show

$$\mathcal{E} \times_C Q_1(C) \to Q_1(D) \times_D Q_1(D)$$

is a split Verdier projection. For this, we draw the following cube:

$$\begin{tikzcd}
 & & \mathcal{E} \times_C Q_1(C) \arrow[dll] \arrow[d, dashed] \arrow[drr] & & \\
Q_1(C) \arrow[d] \arrow[drrr] & & Q_1(D) \times_D Q_1(D) \arrow[dll, dashed] \arrow[drr, dashed] & & \mathcal{E} \arrow[dl] \arrow[d] \\
Q_1(D) \arrow[drrr] & & & C \arrow[d] & Q_1(D) \arrow[dl] \\
 & & & D &
\end{tikzcd}$$

The top and bottom faces are pullbacks, and the right face is the first diagram, so it is also a pullback. Hence, we have $\mathcal{E} \times_C Q_1(C) \to Q_1(D) \times_D Q_1(D)$ is a pullback of $Q_1(C) \to Q_1(D)$. Because $Q_1(C)$ is equivalent to $\mathrm{Fun}(J_1, C)$, we see that the Bousfield localisation between $C$ and $D$ extends to a Bousfield localisation between $Q_1(C)$ and $Q_1(D)$. This shows $Q_1(p)$ is a split Verdier projection and we finish the proof. $\square$

**Lemma 9.11.** *The functor* $|-|$ : sAn $\to$ An *sends pullback squares with one leg being bicartesian fibration to pullback square in* An.

*Proof.* See Theorem 2.11 of [10]. $\square$

## 9.2 Proof of the main theorem

In this subsection, we want to finish the proof of Theorem 9.1. This establishes the main goal of this project report. As in the last subsection, we still need some techniques in $\infty$-categories. The main techniques we will use here are equifibred maps and Null construction.

**Definition 9.12.** *Let $I$ be any $\infty$-category. A map $\tau : C \to D$ in* $\mathrm{Fun}(I, \mathrm{An})$ *is called equifibred if for $i \to j$ in $I$, we have a pullback square*

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_j \arrow[d] \\ D_i \arrow[r] & D_j \end{tikzcd}$$

If you are given an equifibred map, then any two fibers of $C_i \to D_i$ are equivalent from the definition, then loosely speaking, we should see the fiber of $\mathrm{colim}\, C \to \mathrm{colim}\, D$ is also equivalent to the fiber above. It turns this property uniquely characterize $\mathrm{colim}\, C$ when $\tau : C \to D$ is an equifibred map.

**Proposition 9.13.** *Suppose $\tau : C \to D$ is an equifibred map in* $\mathrm{Fun}(I, \mathrm{An})$*. Let $K \in$* An *with $\eta : B \Rightarrow constK$ together with a map $K \to$* $\mathrm{colim}\, D_i$ *such that we have a square for all $i \in I$*

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & K \arrow[d] \\ D_i \arrow[r] & \mathrm{colim}\, D \end{tikzcd}$$

*Then they are all pullback diagrams if and only if $\eta$ induces an equivalence* $\mathrm{colim}\, C \xrightarrow{\simeq} K$*.*

*Proof.* Using the straightening functor, we consider functors $F_i \simeq \mathrm{St}^{\mathrm{left}}(C_i \to D_i)$. The main difficulty of this proposition is to prove that they assemble into a functor

$$F\colon\ \mathrm{colim}\, D \to \mathrm{An}\,.$$

Notice that the natural transformation $\tau : C \to D$ induces a map

$$\tilde{\tau}\colon I \to \mathrm{Fun}([1], \mathrm{An}),$$

and it being equifibred is equivalent to it sends morphisms in $I$ to pullback squares in An. Let $t : \mathrm{Fun}([1], \mathrm{An}) \to \mathrm{An}$ be the target morphism. In Example 2.4, we see it is a cartesian fibration, and the $t$-cartesian edge in $\mathrm{Fun}([1],\mathrm{An})$ is precisely the pullback square. Then we denote

$$\mathrm{An}/(-) \simeq \mathrm{Un}^{\mathrm{cart}}(t) : \mathrm{An}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

The paragraph above tells us that $\tilde{\tau}$ induces a morphism

$$I \to I \times_{\mathrm{An}} \mathrm{Fun}([1],\mathrm{An}).$$

Moreover, because $\tilde{\tau}$ sends morphisms in $I$ to $t$-cartesian edges in $\mathrm{Fun}([1],\mathrm{An})$, the above map is a map of cartesian fibrations over $I$. By doing unstraightening, it gives a natural transformation:

$$(\mathrm{const}\, * \Rightarrow \mathrm{An}/D_{(-)}) : I^{\mathrm{op}} \to \mathrm{An}\,.$$

Notice that we use the symbol $\mathrm{An}/D_{(-)}$ because the unstraightening of $I \times_{\mathrm{An}} \mathrm{Fun}([1],\mathrm{An}) \to I$ sends $i$ to $\mathrm{An}/D_i$, and it sends $i \to j$ to the functor $- \times_{D_j} D_i : \mathrm{An}/D_j \to \mathrm{An}/D_i$. The natural transformation above induces a map $* \to \lim_{I^{\mathrm{op}}} \mathrm{An}/D_i$. Now, we use

$$\lim_{I^{\mathrm{op}}} \mathrm{An}/D_i \simeq \lim_{I^{\mathrm{op}}} \mathrm{Fun}(D_i,\mathrm{An}) \simeq \mathrm{Fun}(\operatorname*{colim}_I D_i, \mathrm{An}).$$

It says that we have obtained the $F$ we want from the beginning.

By definition, we have $\mathrm{Un}^{\mathrm{left}}(F) \to \mathrm{colim}\, D$ is a left fibration over the anima $\mathrm{colim}\, D$, so $\mathrm{Un}^{\mathrm{left}}(F)$ is also an anima. Because of the Bousfield localization inclu. $\dashv |-|$, we learn that $\mathrm{Un}^{\mathrm{left}}(F) \simeq |\mathrm{Un}^{\mathrm{left}}(F)|$. Then, Proposition 4.3 tells that

$$\begin{aligned}
\mathrm{Un}^{\mathrm{left}}(F) &\simeq \operatorname*{colim}_{\mathrm{colim}\, D} F \\
&\simeq \operatorname*{colim}_{i\in I} \operatorname*{colim}_{D_i} F_i \\
&\simeq \operatorname*{colim}_{i\in I} \mathrm{Un}^{\mathrm{left}}(F_i) \\
&\simeq \operatorname*{colim}_{i\in I} B_i
\end{aligned}$$

From our construction and functoriality of Straightening-Unstraightening, we see for each $i$, there is a diagram

$$\begin{tikzcd}
C_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Un}^{\mathrm{left}}(F) \arrow[d] & \\
D_i \arrow[r] & \mathrm{colim}\, D \arrow[r] & \mathrm{An}
\end{tikzcd}$$

Hence, when $K \simeq \mathrm{colim}\, B$, we naturally have the squares in the Proposition are all pullbacks. On the other hand, if $K$ makes all the squares become pullback, then we see

$$(D_i \to \mathrm{colim}\, D \xrightarrow{\mathrm{St}^{\mathrm{left}}(K \to \mathrm{colim}\, D)} \mathrm{An}) \simeq (D_i \xrightarrow{F_i} \mathrm{An}).$$

so we must have that $K \simeq \mathrm{Un}^{\mathrm{left}}(F) \simeq \mathrm{colim}\, B$. $\square$

Using the above Proposition, it will not be hard to see the following lemma:

**Lemma 9.14.** *Let $I$ be any $\infty$-category. If there is a pullback square in* $\mathrm{Fun}(I, \mathrm{An})$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\sigma"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\tau"] \\ B \arrow[r] & D \end{tikzcd}$$

*with the map $C \to D$ being equifibred, then the diagram*

$$\begin{tikzcd} \mathrm{colim}_{i\in I} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{colim}_{i\in I} C \arrow[d] \\ \mathrm{colim}_{i\in I} B \arrow[r] & \mathrm{colim}_{i\in I} D \end{tikzcd}$$

*is a pullback diagram in* An.

*Proof.* We firstly see $\sigma$ is also equifibred. For every $i \to j \in I$, we have the following diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & A_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{colim}\, C \arrow[d, <->] \\ B_i \arrow[r] & B_j \arrow[r] & D_j \arrow[r] & \mathrm{colim}\, D \end{tikzcd}$$

The second square is a pullback because $\tau$ is equifibred, and the third square is a pullback by the Proposition above. Then, the composition of the second and third squares is a pullback. For the same reason, we see the big outer square is also a pullback. Hence, by 2-out-of-3, we see the first square is a pullback square. This proves $\sigma$ is equifibred. Now, we can consider the following diagram:

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & \mathrm{colim}\, B \times_{\mathrm{colim}\, D} \mathrm{colim}\, C \arrow[r] \arrow[d] & \mathrm{colim}\, C \arrow[d] \\ B_j \arrow[r] & \mathrm{colim}\, B \arrow[r] & \mathrm{colim}\, D \end{tikzcd}$$

By the argument above, we see the second square and the big outer square are both pullback squares. Again, by 2-out-of-3, we see the first square is a pullback for all $i \in I$. Hence, by the Proposition above, we see that $\mathrm{colim}\, B \times_{\mathrm{colim}\, D} \mathrm{colim}\, C \simeq \mathrm{colim}\, A$, which finishes the proof. $\square$

There is one last thing we need to discuss before going to the proof. That is the Null-construction.

**Definition 9.15.** *Let $C$ be a stable $\infty$-category. Consider the functor $[0] \star - : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$. It induces a functor, which we call as* **décalage***:*

$$\text{déc} : \mathrm{sCat}_\infty^{st} \to \mathrm{sCat}_\infty^{st}.$$

*The inclusion* $[0] \subset [0] \star [n]$ *induces a natural transformation*

$$p : \text{déc} \Rightarrow \text{const}\, \text{ev}_0$$

*We define*

$$\text{Null}(C) := \text{fib}_0(p : \text{déc}\, Q(C) \to \text{const}\, C).$$

To be more explicit, one can unwind the definition of $\text{Null}(C)$ and see that $\text{Null}_n(C)$ is the full subcategory of $Q_{n+1}(C)$ spanned by diagrams of the following form:

$$\begin{tikzcd}
& & & & \circ \arrow[dl, dashed] \arrow[dr, dashed] & & & & \\
& & & \circ \arrow[dl, dashed] \arrow[dr, dashed] & & \circ \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
& & \circ \arrow[dl, dashed] \arrow[dr, dashed] & & \circ \arrow[dl, dashed] \arrow[dr, dashed] & & \circ \arrow[dl, dashed] \arrow[dr, dashed] & & \\
& \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\
0 & & \bullet & & \bullet & & \bullet & & \bullet
\end{tikzcd}$$

For $\text{Null}(C)$, we have the following lemma:

**Lemma 9.16.** *Let* $C$ *be a stable* $\infty$*-category, and* $F : \text{Cat}_\infty^{\text{st}} \to \text{An}$ *be an additive functor. Then one has a split Verdier sequence*

$$F(\text{Null}(C)) \to \text{déc}\, FQ(C) \to \text{const}\, F(C)$$

*and we have*

$$|F\, \text{Null}(C)| \simeq *.$$

*Proof.* To prove the first claim, it is enough to show that $p$ is a levelwise split Verdier projection, and one just needs to use the additivity of $F$ on the split exact sequence. Notice that $\text{déc}_n(C) \simeq Q_{n+1}(C) \simeq \text{Fun}(J_{n+1}, C)$, we see $J_0 \subset J_{n+1}$ induces left and right Kan extensions $\text{const}\, C \to \text{déc}\, Q(C)$, it is easy to check this defines $p$ as left and right Bousfield localisation (see a detailed discussion in the proof of Theorem 7.13). Then Proposition 8.5 shows that $p$ is a split Verdier projection.

For the second claim, we need to use the extra degeneracy trick as described in section 6. We want to show $F\, \text{Null}(C)$ extends to a diagram over $\Delta_{-\infty}$ with its value on $[-1]$ is 0. Because $F\, \text{Null}(C)$ is described as the fiber of $F\, \text{déc}(C) \to F\, \text{const}\, C$, it is enough to show $F\, \text{déc}(C)$ and $\text{const}\, C$ are split, and the extending maps are compatible. To see this, we consider the diagram below:

$$\begin{tikzcd}
\Delta^{\text{op}} \arrow[r] \arrow[d] & \Delta^{\text{op}} \arrow[r] & \text{Cat}_\infty^{\text{st}} \arrow[r, "F"] & \text{An} \\
\Delta_{-\infty} \arrow[ur, dashed] & & &
\end{tikzcd}$$

Hence, we only need to see the maps $[n] \mapsto [n+1]$ and $[n] \mapsto [0]$ can extend to $\Delta_{-\infty}$ and the extended maps are still compatible. Because they are 1-categories, one can explicitly write out the needed maps for them to be split, and there is an obvious choice for both maps. We have checked that $F\,\mathrm{Null}(C)$ is a split simplicial diagram, by Proposition 6.2, we conclude that

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} F\,\mathrm{Null}(C) \simeq *.$$

□

The key step towards our main theorem is the following lemma:

**Lemma 9.17.** *Let* $F : \mathrm{Cat}^{\mathrm{st}}_{\infty} \to \mathrm{An}$ *be grouplike and let* $C$ *be a stable* $\infty$*-category. Then there is a pullback of simplicial animas*

$$\begin{tikzcd}
\mathrm{const}\, F(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Null}(C)) \arrow[d] \\
\mathrm{const}\, * \arrow[r] & F(Q(C))
\end{tikzcd}$$

*Applying* $|-| : \mathrm{sAn} \to \mathrm{An}$ *to the diagram gives us the pullback square:*

$$\begin{tikzcd}
F(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
* \arrow[r] & |F(Q(C))|
\end{tikzcd}$$

*Proof.* The inclusion $[n] \subset [0] * [n]$ induces a natural transformation $\mathrm{déc} \Rightarrow \mathrm{id}$, which restricts to a natural transformation

$$d_0 : \mathrm{Null} \Rightarrow \mathrm{id}\,.$$

The fiber of this map is easy to describe. Level-wisely, the pre-image of 0 of the map $d_0$ are diagrams of the following form:

$$\begin{tikzcd}[column sep=small]
 & & & & x \arrow[dl, dashed] \arrow[dr, dashed] & & & & \\
 & & & x \arrow[dl, dashed] \arrow[dr, dashed] & & 0 \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
 & & x \arrow[dl, dashed] \arrow[dr, dashed] & & 0 \arrow[dl, dashed] \arrow[dr, dashed] & & 0 \arrow[dl, dashed] \arrow[dr, dashed] & & \\
 & x \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & \\
0 & & 0 & & 0 & & 0 & & 0
\end{tikzcd}$$

so, we can see $\mathrm{fib}_0(d_0) \simeq \mathrm{const}\, C$. In other words, the first diagram without $F$ is a pullback. Now, we only need to see that $d_0$ is a split Verdier projection. This follows the exact same proof of $p$ being a split Verdier projection, which we proved in the lemma above. Now, we are only left to show the right vertical arrow of the first square is equifibred. Indeed, if we have proven that, then combining Lemma 9.14 and Lemma 9.16, we can finish the proof.

We want to show for any $[m] \to [n] \in \Delta^{\mathrm{op}}$, there is a pullback square:

$$\begin{tikzcd} F(\mathrm{Null}_m(C)) \arrow[r] \arrow[d, "F(d_0)"] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathrm{Null}_n(C)) \arrow[d, "F(d_0)"] \\ F(Q_m(C)) \arrow[r] & F(Q_n(C)) \end{tikzcd}$$

Here, we need to use that $F$ is a grouplike functor. Because the spaces in the diagram above are all $E_\infty$-groups, any fiber of the vertical maps is either empty or is equivalent to the fiber of 0. As we have shown above, we have a split Verdier sequence

$$C \to \mathrm{Null}_m(C) \xrightarrow{d_0} Q_m(C).$$

This induces a fiber sequence in CGrp(An):

$$F(C) \to F(\mathrm{Null}_m(C)) \xrightarrow{d_0} F(Q_m(C)).$$

Either the fully faithful right adjoint or the left adjoint of $d_0 : \mathrm{Null}_m(C) \to Q_m(C)$ induces a split of $F(d_0)$. Hence, this implies $F(d_0)$ is subjective on $\pi_0$. so the fibers cannot be empty, and this tells that all the fiber of $F(d_0)$ is equivalent to the fiber at 0. To check the diagram is a pullback, it is enough to check the fibers of 0 of the two vertical maps are equivalent, so we have finished our proof. □

Let us recall the main theorem we want to prove:

**Theorem 9.1.** *There is a left Bousfield localisation:*

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}}, \mathrm{An}) \underset{\text{inclusion}}{\overset{(-)^{\mathrm{grp}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Fun}^{\mathrm{grp}}(\mathrm{Cat}_\infty^{\mathrm{st}}, An).$$

*where $F^{grp}(-) := \Omega|FQ(-)|$. In particular, we have*

$$k \simeq \mathrm{core}^{\mathrm{grp}}$$

*This means that $k$ is the initial grouplike functor under* $\mathrm{core} : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{An}$.

*Proof.* In the first half of this section, we have shown $|FQ(-)|$ is an additive functor. Because it preserves split Verdier squares, in particular, it preserves products. Hence, for any stable infinity category $C$, we know $|FQ(C)| \in \mathrm{CMon}(\mathrm{An})$. From Proposition 5.8, we know $\Omega$ is a functor $\mathrm{CMon}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{An})$. Hence, one sees that $F^{\mathrm{grp}} \simeq \Omega|F \circ Q|$ indeed is a grouplike functor. Let us denote:

$$L \simeq (-)^{\mathrm{grp}} : \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}}, \mathrm{An}) \to \mathrm{Fun}^{\mathrm{grp}}(\mathrm{Cat}_\infty^{\mathrm{st}}, An)$$

We want to show this is a left Bousfield localisation, which can be done using Proposition 4.8. Firstly, we need to construct a natural transformation $\eta : \mathrm{id} \Rightarrow L$. Notice that in the proof of Lemma 9.17, we don't need $F$ to be grouplike, so one can still obtain the second diagram, which provides the natural transformation:

$$\eta : \mathrm{id} \Rightarrow L$$

Because $LF$ is grouplike functor, we see $\eta_L F$ is an equivalence as we proved. On the other hand, $L(\eta_F)$ is obtained by applying $L$ to the diagram, and it gives

$$\begin{tikzcd} \Omega|FQ(C)| \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & |\Omega|FQ^2(C)|| \end{tikzcd}$$

This is a pullback diagram again because $\Omega|FQ(-)|$ is a grouplike functor. Hence, this proves that $F\eta$ is an equivalence. Because $L$ indeed maps into grouplike functors, and we have seen for any grouplike functor, $\eta$ is an equivalence. this gives that

$$\mathrm{im}(L) \simeq \mathrm{Fun}^{\mathrm{grp}}(\mathrm{Cat}^{\mathrm{st}}_{\infty}, \mathrm{An}).$$

This concludes our proof. $\square$

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