An Introduction to Homology Theory Towards Spectral Sequence

Pengkun Huang
26823773

Department of Mathematics and Statistics
University of Reading

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Abstract

In this report, we shall give an introduction to homology theory, with a focus on the computation of homology groups. In the first section, we construct two essential types of homology groups: simplicial homology groups and singular homology groups, and their properties will be given. We then indicate that homotopic equivalent spaces have isomorphic singular homology groups. This intriguing property shows the potential of homology groups to connect algebra with topology. For readers who are not familiar with homotopy theory, we give the basic definitions and properties in appendix. In the third section, we develope two useful long exact sequences of singular homology groups, and as an application, homotopy groups of the sphere and the wedge sum are calculated. After we develop the essential theories, the homology groups of surfaces are considered in the fourth section, and the result shows that homology groups can be recognized as a character of surfaces. Following the idea of finding tools for computing homology groups, we describe one kind of spectral sequence for fibre bundles in the fifth section, and use this to calculate the homology groups of complex projective spaces.

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1 Simplicial Homology and Singular Homology

In this section, we shall introduce the concept of homology groups and their properties. Simplicial homology groups will be firstly defined, since it has the advantage of computability from their definitions. Then, we will construct the singular homology groups, which are the central objects of this report. In this report, a group will always mean an abelian group.

1.1 Simplicial Complexes and Simplicial Homology

Definition 1.1. Suppose there is a collection $C = \{C_n, \partial_n\}_{n\geq 0}$. For each $n \geq 0$, C_n is an abelian group and ∂_n is a group homomorphism, which is called the boundary map. They form the following sequence:

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

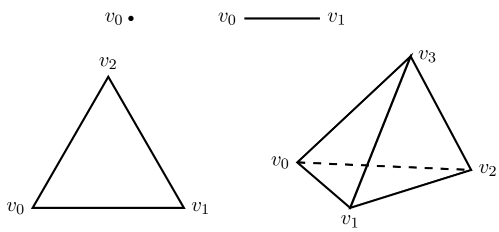
If $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$, we call C a **chain complex of abelian groups**. Since the equation implies that $\mathrm{Im}(\partial_{n+1})$ is a subgroup of $\mathrm{Ker}(\partial_n)$, the **n-th order homology group** can be defined by:

$$H_n(C) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})},$$

where the whole collection is written as $H(C)$. An element z in $\mathrm{Ker}(\partial_n) \subset C_n$ is called an **n-cycle** with its equivalence class written $\{z\}$, and an element in $\mathrm{Im}(\partial_{n+1}) \subset C_n$ is an **n-boundary**. Two cycles are said to be **homologous** if their difference is in $\mathrm{Im}(\partial_{n+1})$, written $z \sim z'$.

As we can see from definition, the main task to define a homology group is to find a chain complex. Here, we will introduce a special geometrical construction, simplicial complex, which can naturally associate to a chain complex. To properly state the definition of a simplicial complex, we firstly introduce some notations.

Let n be a non-negative integer and $\{v_0, v_1, ..., v_n\}$ be $n+1$ vertices in $\mathbb{R}^n$ such that the set $\{v_i - v_0\}_{i\geq 1}$ is linearly independent. Then the set of their convex combinations $\{\sum_i t_i v_i | \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$ is called an **n-simplex**, denoted by $[v_0, ..., v_n]$, and its dimension is defined to be n. Examples of dimension not more than 3 can be visualized as the figure in the right. A **(proper) face** of an n-simplex is a simplex generated by some non-empty (proper) subset of $\{v_0, v_1, .., v_n\}$. A **standard n-simplex** Δ^n is the set $\{(t_0, .., t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$. Among which, the union of the proper faces of Δ^n is called the **boundary** of Δ^n, written $\partial\Delta_n$, and the interior of Δ^n, which is $\Delta^n - \partial\Delta^n$, is denoted by $\mathring{\Delta}^n$. Notice that an n-simplex can be embedded into any Euclidean space in dimension larger than n. For example,Δ^2 is a 2-simplex in $\mathbb{R}^3$.



Definition 1.2. A **simplicial complex** K in $\mathbb{R}^n$ is a finite collection of simplices satisfying the following conditions:

1. Every face of a simplex in K is also in K.

2. For any two simplices in K, their intersection is either empty or a face of both the simplices.

A p-skeleton of a simplicial complex K is the union of all simplices in K of dimension not more than p, written K^p.

The following two definitions build a connection between simplicial complexes and topological spaces. This will be the foundation to give a relationship between simplicial homology groups and singular homology groups defined in the next section.

Definition 1.3. Given a simplicial complex K in $\mathbb{R}^n$, its **geometrical realization** $|K|$ is a topological space with a topology constructed in the following way: For any subset A of K, we define it to be closed if, for any simplex $\sigma \in K$, its intersection with A is closed in σ, where σ is recognized as a subspace of R^n.

Since K is a subset of $\mathbb{R}^n$, it can also be realized as a subspace of $\mathbb{R}^n$. In general, the subspace topology is not equivalent to the topology defined as above (Munkres 1984, Chapter 1), and the topology of $|K|$ is usually finer than the subspace topology.

Definition 1.4. A **simplicial complex structure** of a Hausdorff topological space X is a simplicial complex K, with a homeomorphism mapping from $|K|$ to X. A topological space which has a simplicial complex structure is said to be triangulable.

A simplicial complex structure is sometimes referred to as a **triangulation** of a topological space. Notice that there are plenty of topological spaces which are triangulable. In fact, it can be proven that every compact surface admits a simplicial complex structure. (Munkres 2000, Chapter 12).

Suppose K is a simplicial complex. We shall define a chain complex for K. Firstly, notice that for an integer $n \geq 1$, an n-simplex can have different representations corresponding to different orderings of its vertices. To make them unified, we define two orderings are equivalent if one ordering can be transformed into the other by an even permutation. Under this equivalence relation, an **oriented n-simplex** is defined as an equivalence class of orderings of an n-simplex and written as $[v_0, ..., v_n]$. Notice that an n-simplex corresponds exactly to two oriented n-simplices. We then define $C(K) = \{C_n(K), \partial_n\}_{n\geq 0}$ as follows:

1. $n = 0$: $C_0(K)$ is defined as a free abelian group generated by all 0-simplices. ∂_0 is the trivial homomorphism from $C_0(K)$ to 0.

2. $n \geq 1$: $C_n(K)$ is a free abelian group generated by all oriented n-simplices with a relation defined as follows: if σ_1 and σ_2 are two different equivalence classes of an n-simplex, then we ask $\sigma_1+\sigma_2 = 0$. A boundary map $\partial_n : C_n(K) \to C_{n-1}(K)$ is a homomorphism sending a generator $[v_0, ..., v_n] \in C_n(K)$ to $\sum_{k=0}^{n}(-1)^k[v_0, \cdots, \hat{v}_k, \cdots, v_n] \in C_{n-1}(K)$, where $[v_0, \cdots, \hat{v}_k, \cdots, v_n]$ means an oriented $n-1$ simplex generated by vertices $\{v_0, ..., v_{k-1}, v_{k+1}, ..., v_n\}$.

From the definition, $C_n(K)$ of any dimension is a free abelian group with its rank being the number of n-simplices in K. To check it is indeed a chain complex, we firstly notice that

$\partial_0 \circ \partial_1 = 0$ since ∂_0 is trivial. Then, for any $n \geq 2$ and any generator $[v_0, ..., v_n] \in C_n(K)$, it follows that:

$$\begin{aligned}\partial_{n-1} \circ \partial_n [v_0, ..., v_n] = &\sum_{0 \leq i < j \leq n} (-1)^i (-1)^j [v_0, \cdots, \hat{v}_i, \cdots \hat{v}_j, \cdots, v_n] \\ &+ \sum_{0 \leq i < j \leq n} (-1)^{i-1} (-1)^j [v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n].\end{aligned}$$

The two sums cancel since $[v_0, \cdots, \hat{v}_i, \cdots \hat{v}_j, \cdots, v_n]$ and $[v_0, \cdots, \hat{v}_j, \cdots \hat{v}_i, \cdots, v_n]$ represents the same oriented simplex, which means $\partial_{n-1} \circ \partial_n = 0$. Hence, we have obtained our desired chain complex $C(K)$, which is called an **oriented chain complex** of K. It can be represented by the following sequence:

$$\cdots \longrightarrow C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

As in Definition 1.1, this chain complex induces a collection of homology groups. Here we call them **simplicial homology groups** of K, and denote the collection by $H(K)$. To end this section, we give two examples indicating how to calculate simplicial homology groups, and their results will also be used in the later content.

Example 1.1. Suppose $K = \{v_0\}$ is a single 0-simplex. It follows that $C_0(K) = \mathbb{Z}$ and $C_n(K) = 0$ for $n \geq 1$, so we have that $H_0(K) = \mathbb{Z}$ and $H_n(K) = 0$ for $n \geq 1$.

Example 1.2. A torus T has a simplicial complex structure as follows:

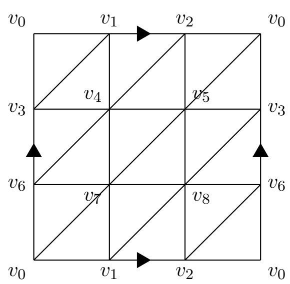


Figure 1: Triangulation of a torus.

In total, there are 9 0-simplexes, 27 1-simplexes, and 18 2-simplexes. We denote the simplicial complex described in the figure by $K(T)$. The structure induces the following chain complex:

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}^{18} \xrightarrow{\partial_2} \mathbb{Z}^{27} \xrightarrow{\partial_1} \mathbb{Z}^9 \xrightarrow{\partial_0} 0.$$

Undoubtedly, the computation is too complex by computing manually. By inputting the matrix of ∂_k into Excel and transforming them into a matrix in Matlab, we could get the rank by computer computation. Then, from the relation of the rank of a matrix with its kernel and image, we have

$$\mathrm{Ker}(\partial_2) = \mathbb{Z}, \quad \mathrm{Im}(\partial_2) = \mathbb{Z}^{17}, \quad \mathrm{Ker}(\partial_1) = \mathbb{Z}^{19} \quad \mathrm{Im}(\partial_1) = \mathbb{Z}^8, \quad \mathrm{Ker}(\partial_0) = \mathbb{Z}^9.$$

Hence, it indicates that $H_n(K(T)) = \mathbb{Z}$ for $n = 0, 2$, $H_1(K(T)) = \mathbb{Z}^2$ and $H_n(K(T)) = 0$ for $n > 2$. In the next section, we will define another kind of homology groups for a torus, and we will show that they are equivalent with the simplicial homology groups calculated as above.

1.2 Singular Homology

Now, we will see how to construct homology groups for topological spaces. To define them, we will generalize the definition of n-simplices to singular n-simplices, and use them as ingredients to construct a chain complex as in the last section. However, the generalization also makes the computation of homology groups much harder. We will see one computational method by indicating the relationship with simplicial homology groups, and the problem of computation will be the major motivation to most theories in this report.

Definition 1.5. Suppose X is a topological space. A **singular n-simplex** is a continuous map $\sigma : \Delta^n \to X$. By taking the set of all singular n-simplices as a basis, the free abelian group generated by the basis is denoted by $C_n(X)$. If we write Δ^n as $[e_0, e_1, ..., e_n]$, where e_i is the ith standard basis element in $\mathbb{R}^{n+1}$, then an **ith face** of a singular n-simplex is a singular $n-1$ simplex defined as follows:

$$\sigma^{(i)} : \Delta^{n-1} \xrightarrow{f} [e_0, ..., \hat{e}_i, ..., e_n] \xrightarrow{i} \Delta^n \xrightarrow{\sigma} X,$$

where f is the canonical linear homeomorphism and i is the inclusion mapping. By letting ∂_0 be a zero map and ∂_n for $n \geq 1$ be $\partial_n(\sigma) = \sum_{k=0}^{n}(-1)^k\sigma^{(k)}$, we obtain a chain complex of a topological space $C(X) = \{C_n(X), \partial_n\}_{n\geq 0}$. ∂_n is indeed a boundary map since it has the same form with the boundary maps of a simplicial complex. The induced homology groups are called **singular homology groups**.

From the above definition, we can see a direct computation of homology groups would be hard since a space can contain infinitely many n-simplices. However, we can still make some useful observations as follows:

Proposition 1.1. *Let X be a topological spaces with its path components denoted by $\{X_\alpha\}_{\alpha\in J}$. For any $n \geq 0$, $H_n(X)$ is isomorphic to $\oplus_{\alpha\in J}H_n(X_\alpha)$.*

Proof. Given a singular n-simplex σ, it maps Δ^n into one of the path components of X since σ is continuous. Hence, an element in $C_n(X)$ can be uniquely decomposed into the sum of elements in $C_n(X_\alpha)$, which means $C_n(X) = \oplus_{\alpha\in J}C_n(X_\alpha)$. Notice ∂_n preserves this decomposition, so the quotient $\text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$ splits into a direct sum of quotients defined from the chain complexes of its path components, and the result follows. □

Proposition 1.2. *(Spanier 1996, p. 175) Let X be a nonempty topological space. Then, $H_0(X)$ is a free abelian group with rank equating to the number of path components of X. In particular, we have that $H_0(X) = \mathbb{Z}$ when X is path-connected.*

Sketch of proof. Proposition 1.1 implies that it suffices to prove the situation when X is path-connected. We define a mapping $\epsilon : C_0(X) \to \mathbb{Z}$ by sending $\sum n_i\sigma_i$ to $\sum n_i$, where ϵ is surjective since X is nonempty. We claim that $\text{Im}(\partial_1) = \text{Ker}(\epsilon)$, and the proof is finished by noticing that $C_0(X)/\text{Ker}(\epsilon) \cong \mathbb{Z}$ and $H_0(X) = C_0(X)/\text{Im}(\partial_1)$. □

In the following, we shall introduce two extended constructions of singular homology groups. The first one can be seen as a generalization of the singular homology group, and the second one is an useful technique to simplify the computation.

Relative homology groups: Let A be a subspace of X. It follows that $C_n(A) \subset C_n(X)$ and ∂_n takes $C_n(A)$ to $C_{n-1}(A)$ since it is defined by restriction. Using $C_n(X, A)$ to denote $C_n(X)/C_n(A)$, there is a boundary map $\overline{\partial}_n : C_n(X, A) \to C_{n-1}(X, A)$ given by $\overline{\partial}_n(\{x\}) = \{\partial_n(x)\}$ for any $\{x\} \in C_n(X, A)$. They form a well-defined chain complex $C(X, A) := \{C_n(X, A), \overline{\partial}_n\}$, and the corresponding homology groups are called the **relative homology groups**, written $H(X, A)$. When $A = \emptyset$, we use $H(X, \emptyset)$ to denote $H(X)$. Notice that if we have a simplicial subcomplex K' of K, which means K' is a simplicial complex as a subset of K, then we could follow the exact same procedure to define relative simplicial homology groups $H_n(K, K')$.

Reduced homology group: Since $H_0(X) = \mathbb{Z}$ holds for any nonempty path-connected space, it will be convenient to omit this part when comparing multiple spaces. Reviewing the proof of Proposition 1.2, it hints us to define an **augmented chain complex** for a chain complex $C(X)$ in the following form:

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where ϵ is the map we defined in the proof of Proposition 1.2. One can check that $\epsilon \circ \partial_1 = 0$, so the above sequence is a chain complex, and the group $\mathbb{Z}$ is said to be a group of dimension -1. The homology groups induced from this chain complex in dimension not less than 0 are called **reduced homology groups**. Notice that ϵ is subjective, so it induces a map from $H_0(X)$ to $\mathbb{Z}$ having the kernel isomorphic to $\tilde{H}_0(X)$, which gives the following proposition:

Proposition 1.3. *For a nonempty topological space X, we have $\tilde{H}_n(X) \cong H_n(X)$ for any $n \geq 1$ and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.*

The construction of an augmented relative chain complex is different from above. It is defined by taking the quotient in each dimension of two reduced chain complexes. Hence, the group of dimension -1 is 0, which means $\tilde{H}_n(X, A) \cong H_n(X, A)$ holds for every $n \geq 0$. The reason for this definition will be clear when we discuss the reduced long exact sequence.

To end this section, we shall give two important theorems for singular homology. The first indicates a method for computing singular homology of a triangulable space X, and the second shows a connection of the relative homology group and the singular homology group of a quotient space.

Theorem 1.1. *(Spanier 1996, p. 191) Let X be a topological space with a simplicial complex structure K, and let A be a subspace of X such that it has a simplicial complex structure K' and K' is a subcomplex of K. It follows that*

$$H_n(X, A) \cong H_n(K, K')$$

for all $n \geq 0$. In particular, when A is taken to be an empty set, we have $H_n(X) \cong H_n(K)$.

Theorem 1.2. *(Hatcher 2002, p. 124) Suppose X is a topological space. Let A be a nonempty closed subspace such that there is an open subspace $U \subset X$ and A is a deformation retract of U (see Appendix B). Then for any $n \geq 0$, we have*

$$H_n(X, A) \cong \tilde{H}_n(X/A).$$

2 Homotopy Invariance

The aim of this section is to explain why the singular homology group is important. We will show that the groups are invariant under homotopic equivalence, so they can be used as indicators to classify topological spaces. We will use the basic definitions of homotopy theory. Readers who are not familiar with them can find an introduction in Appendix B. The definition of a chain map is firstly introduced, and this is the key to connect a continuous map between spaces with a homomorphism between groups.

Definition 2.1. Let $C = \{C_n, \partial_n\}$ and $D = \{D_n, \partial'_n\}$ be two chain complexes. A chain map $f_\# : C \to D$ is a collection of homomorphisms $\{f_n : C_n \to D_n\}_{n\geq 0}$ such that $f_{n-1} \circ \partial_n = \partial'_n \circ f_n$ for any $n \geq 1$, written $\partial' \circ f_\# = f_\# \circ \partial$. The commutative diagram can be illustrated as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, "\partial'_{n-1}"] & D_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose $f_\# : C \to D$ is a chain map. The commutativity of $f_\#$ indicates $f_n(\mathrm{Ker}(\partial_n)) \subset \mathrm{Ker}(\partial'_n)$ and $f_n(\mathrm{Im}(\partial_n)) \subset \mathrm{Im}(\partial'_n)$ for all $n \geq 0$, so $f_\#$ induces a group homomorphism $f_* : H_n(C) \to H_n(D)$ in each dimension, and we denote the whole collection by $f_* : H(C) \to H(D)$. Explicitly, let $\{\sigma\}$ be a homology class in $H(C)$, then we have $f_*\{\sigma\} = \{f \circ \sigma\}$.

On the other hand, a continuous map $f : X \to Y$ can give a chain map between $C(X)$ to $C(Y)$. Let σ be a generator of $C_n(X)$ for some $n \geq 0$, then $f_n(\sigma)$ is defined to be the composition $f \circ \sigma : \Delta^n \to Y \in C_n(Y)$. This implies the following calculation:

$$f_{n-1} \circ \partial_n(\sigma) = \sum_{k=0}^{n} (-1)^k f \circ \sigma^{(k)} = \sum_{k=0}^{n} (-1)^k (f \circ \sigma)^{(k)} = \partial_n \circ f_n(\sigma).$$

Hence, $f_\# = \{f_n\}$ is indeed a chain map, and we obtain the following theorem:

Theorem 2.1. *Let $f : X \to Y$ be a continuous map between spaces. Then it induces a collection of group homomorphism $f_* = \{f_n\}_{n\geq 0}$, where f_n maps from $H_n(X)$ to $H_n(Y)$.*

We can immediately deduce three basic properties about f_* from the construction above:

1. An identity map between spaces induces an identity map between homology groups.

2. We have $f_* \circ g_* = (f \circ g)_*$ for any two continuous maps $g : X \to Y$ and $f : Y \to Z$.

3. Let f be a homeomorphism from X to Y. Then its induced homomorphism $f_* : H(X) \to H(Y)$ is a group isomorphism in each dimension.

The following theorem is the main result of this section, which is also an important theorem in algebraic topology. It shows that the homology group is indeed a homotopy invariance:

Theorem 2.2. *(Hatcher 2002, p. 111) Suppose $f, g : X \to Y$ are two homotopic continuous maps, then they induce the same homomorphisms of homology groups, i.e. $f_* = g_*$.*

As an application, we mention an interesting corollary that will be used later, which is a result of Example B.2, Example 1.1 and Theorem 1.1.

Corollary 2.1. *For any $m \geq 0$, $H_n(\mathbb{R}^m) = 0$ for any $n \geq 1$ and $H_0(\mathbb{R}^m) = \mathbb{Z}$.*

Before ending this section, we give a proposition indicating a relationship between homology and homotopy. The main reason to mention it here is because it will be a technique in the proof of the forth section. Notice the converse of this statement is not true in general.

Proposition 2.1. *Let $f, g : [0,1] \to X$ be two closed paths with $f(0) = f(1) = g(0) = g(1)$. If there exists a homotopy $F(t,s)$ such that $F(t,0) = f(t)$, $F(t,1) = g(t)$ and $F(0,s) = F(1,s) = f(0)$, then f and g are homologous 1-cycles, i.e. $\{f\} = \{g\} \in H_1(X)$.*

Sketch of proof. Since f and g are close paths, it is direct to verify that they are two 1-cycles in X with the same end points. Notice that F can be written as a sum of two singular 2-simplices by drawing a diagonal line in $[0,1] \times [0,1]$. Then we claim that $\partial(F) = f - g$, i.e. the 1-simplex $f - g$ is in the image of ∂_2, so f and g are homologous by definition. □

3 Exact Sequences in Homology Theory

Although Theorem 1.1 has given a method of calculating homology groups, the computation problem is far from solved since the method cannot work when a topological space is not triangulable, and the computation would become too complex when a topological space has a complex triangulation. Hence, in this section, we shall develop the theory of long exact sequence, which will provide an ingenious method for computation.

3.1 Exact Sequences

Below is a short introduction to the basic notations and properties of exact sequences, which are essential algebraic preliminaries for later content.

Definition 3.1. Suppose there is a sequence of abelian groups and homomorphism in the following form:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \cdots.$$

We say that the sequence is **exact** at C_n if $\mathrm{Ker}(f_n) = \mathrm{Im}(f_{n+1})$. If a sequence is exact at every C_n except for its end points (if they exist), then it is called an **exact sequence of abelian groups**. In particular, a **short exact sequence of abelian groups** is referred to as an exact sequence with the following form:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0. \tag{1}$$

Below, we introduce two basic properties and one important theorem for exact sequences.

1. Considering the sequence (1), exactness at A means f is injective, and exactness at C shows that g is surjective. Exactness at B is equivalent to that $B/A \cong C$.

2. Let $0 \to A \xrightarrow{f} B \to 0$ be an exact sequence. Then f is an isomorphism and $A \cong B$.

Theorem 3.1. *(Powell 2019, p. 11) Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be a short exact sequence with* C *a free abelian group. Then we have* $B \cong A \oplus C$.

Remark. A short exact sequence satisfying the condition $B \cong A \oplus C$ is said to be a **split** short exact sequence. In Powell (2019), there is a proof that a short exact sequence is split if and only if there is a group homomorphism $p : C \to B$ such that $g \circ p = Id_C$.

3.2 Long Exact Sequences of Singular Homology Groups

In the following, we aim to construct a long exact sequence involving the homology groups of a pair of spaces and their relative homology groups. Together with Theorem 1.2, the sequence will be used to calculate homology groups of spheres. And this computation provides a concise proof to the fact that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$, which would be extremely complex if one tries to prove the fact with general topology.

Let X be a topological space and A be a subspace of X. Then we have a short exact sequence $0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0$ for any $n \geq 0$, where i is the inclusion mapping and j is the canonical homomorphism. Since this holds for every $n \geq 0$, we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_{n-1}(A) \arrow[r] \arrow[d, "i"] & \cdots \\
... \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "j"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "j"] & C_{n-1}(X) \arrow[r] \arrow[d, "j"] & \cdots \\
... \arrow[r] & C_{n+1}(X, A) \arrow[r, "\partial"] \arrow[d] & C_n(X, A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X, A) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Notice that each square in the diagram is a commutative diagram, which can be checked easily by direct computation. More generally, for any three chain complexes, if we can connect them by a diagram as above and commutativity is satisfied in each square, then such a diagram is called a **short exact sequence of chain complexes.** We will show that it produces a long exact sequence as follows:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots .$$

The map i_* and j_* are straightforward to define from the chain maps on the diagram, but the definition of ∂_* requires some justification. Let $\{z\} \in H_n(X, A)$, where $z \in C_n(X, A)$ is an n-cycle. Since j is a surjective map, there exists $x \in C_n(X)$ such that $j(x) = z$. Since $j \circ \partial(x) = \partial \circ j(x) = \partial(z) = 0$, so $\partial(x) \in \mathrm{Ker}(j) = \mathrm{Im}(i)$, i.e. $\partial(x) = i(a)$ for some $a \in C_{n-1}(A)$. Notice that $i \circ \partial(a) = \partial \circ i(a) = \partial \circ \partial(x) = 0$ and i is injective, which means $a \in \mathrm{Ker}\partial$ and a is uniquely determined by x. We then define $\partial(\{z\}) = \{a\}$. To see the definition is well-defined, it requires reasoning in two steps:

1. If z is given, there may exist a different choice x' such that $j(x) = j(x') = z$, and x' gives an $a' \in C_{n-1}(A)$ such that $i(a') = \partial(x')$. The relation $j(x - x') = 0$ shows $i(k) = x - x'$ for some $k \in C_{n-1}(A)$. Commutativity implies that $i \circ \partial(k) = \partial \circ i(k) = \partial(x - x')$, so $\partial(k) = a - a'$ since i is injective, and this promises that z always induces the same element $\{a\} \in H_{n-1}(A)$.

2. Since $\{z\}$ also has different choices, we can take $z' \in C_n(X, A)$ such that $\partial(c) = z - z'$ for some $c \in C_{n+1}(X, A)$. The condition $c \in C_{n+1}(X, A)$ gives $j(k) = c$ for some $k \in C_{n+1}(X)$, then $j(\partial(k)) = z - z'$ and $i(0) = 0 = \partial \circ \partial(k)$, so we have $\partial_*\{z - z'\} = 0$. Suppose $j(b') = z'$ and $i(a') = \partial(b')$ for some $b' \in C_n(X)$ and $a' \in C_{n-1}(A)$, then it gives that $i(a - a') = \partial(b - b')$ and $j(b - b') = z - z'$. By step 1, $a - a'$ must be homologous to 0 since $j(\partial(k)) = j(b - b') = z - z'$. Hence, an equivalence class $\{z\}$ induces an unique equivalence class $\{a\}$, i.e., ∂_* is well-defined.

Notice that ∂_* can be easily checked as a group homomorphism since i, j and ∂ are all homomorphisms. In conclusion, we can properly state the following theorem:

Theorem 3.2. *(Spanier 1996, p. 182) Let X be a topological space and A be a subspace. Then we have the following long exact sequence of singular homology groups:*

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X, A) \longrightarrow 0.$$

In the proof given by Spanier, the main method is to check the exactness of every group by the properties of the short exact sequence of chain complexes, and it is essentially irrelevant to what the sequence is. Hence, for any short exact sequence of chain complexes, it can induce a long exact sequence by the same procedure. If we add $0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$ in the right of the above short exact sequence of chain complex, then its three rows are the augmented chain complexes, and we can obtain a long exact sequence of reduced singular homology groups as follows:

$$\cdots \longrightarrow \tilde{H}_1(X, A) \longrightarrow \tilde{H}_0(A) \longrightarrow \tilde{H}_0(X) \longrightarrow \tilde{H}_0(X, A) \longrightarrow 0.$$

Although we extend the chain complex to dimension -1, the homology groups of dimension -1 must be 0 since ϵ is surjective, so the long exact sequence ends at dimension 0. Recall that $\tilde{H}_n(X, A) \cong H_n(X, A)$ for all $n \geq 0$, Theorem 1.2 implies another sequence as follows:

Theorem 3.3. *If X is a space and A is a nonempty closed subsapce such that there is an open subspace $U \subset X$ and A is a deformation retract of U, then we have the following long exact sequence:*

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial_*} \tilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0. \quad (2)$$

As an application, we will compute the homology groups of spheres. The key to use the theorem is the following quotient structure of a sphere: A sphere S^n for any $n \geq 0$ can be written as a quotient space D^n/S^{n-1}, where D^n is a closed unit disk in $\mathbb{R}^n$.

Corollary 3.1. *For any $n \geq 0$,*

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \textit{otherwise.} \end{cases} \quad (3)$$

Proof. When $n = 0$, S^0 is just a space with two points, then Proposition 1.2 and 1.3 shows that $\tilde{H}_0(S^0) = \mathbb{Z}$, and its homology groups of other dimension are 0 since it is a 0-dimensional simplicial complex. When $n > 0$, we first notice that $\tilde{H}_k(D^n) = 0$ since D^n has its center as a deformation retract. Substituting this information into the sequence (2), the second fundamental property of exact sequence shows that $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$ for any $k \geq 1$. When $k > n$, we have $\tilde{H}_k(S^n) \cong \tilde{H}_{k-n}(S^0) = 0$ by iterating the above equality. When $n > k$, it follows that $\tilde{H}_k(S^n) \cong \tilde{H}_0(S^{n-k}) = 0$ since S^n is path-connected for any $n \geq 1$. Only the situation $k = n$ gives a nontrivial result $\tilde{H}_n(S^n) \cong \tilde{H}_0(S^0) = \mathbb{Z}$. The combination of the above arguments finishes our proof. □

As a result of this corollary, we have the following theorem:

Theorem 3.4. *Two Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

Proof. Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic, then their one-point compactification spaces S^n and S^m should also be homeomorphic (Munkres 2000, Chapter 3). Hence, their reduced homology groups should be the same in each dimension, which could only happen when $n = m$. □

Remark. The one-point compactification of $\mathbb{R}^n$ can be understood in the following way. For any sphere, the stereographic projection gives a homeomorphism between $S^n - \{(0, 0, 2)\}$ and $\mathbb{R}^n$. If we add ∞ to $\mathbb{R}^n$, then the projection can be naturally extended to S^n by mapping the north pole to ∞, and the one-point compactification of $\mathbb{R}^n$ is the union of $\mathbb{R}^n$ and $\{\infty\}$. The topology of its space is specially constructed such that the extended mapping will still be a homeomorphism.

3.3 The Mayer-Vietoris Sequences

In this section, we shall develop another useful long exact sequence for singular homology groups, and it can be especially useful when a topological space is known to be a union of subspaces. As we have argued in the last section, a short exact sequence of chain complexes gives a long exact sequence, so we need to first introduce such a sequence:

Theorem 3.5. *Let X be a topological space and A, B be two subspaces, then we have the following short exact sequence of chain complex:*

$$0 \longrightarrow C(A \cap B) \xrightarrow{\phi} C(A) \oplus C(B) \xrightarrow{\psi} C(A + B) \longrightarrow 0, \tag{4}$$

where $C(A+B)$ consists of the sum of singular n-simplices in A and B, $\phi(x) = (x, -x)$ for $x \in C_n(A \cap B)$ and $\psi(x, y) = x + y$ for $(x, y) \in C(A) \oplus C(B)$

Proof. We need to verify the exactness for an arbitrary $n \geq 0$. From the definition of ϕ, ψ and $C_n(A + B)$, it is obvious that ϕ is injective and ψ is surjective. $\mathrm{Im}(\phi) \subset \mathrm{Ker}(\psi)$ is a corollary that $\psi\phi(x) = x - x = 0$ for any $x \in C_n(A \cap B)$. Conversely, notice that if $(x, y) \in \mathrm{Ker}(\psi)$, then we have that $x = -y$, so $\phi(x) = (x, -x) = (x, y)$, which means $\mathrm{Ker}(\psi) \subset \mathrm{Im}(\phi)$. Each square in the large diagram is commutative since we can separate ϕ and ψ as mappings in $C(A)$ and $C(B)$, and the separated mappings are clearly commutative with ∂ becuase they are either Id or $-Id$. □

The above theorem induces a long exact sequence as follows:

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{\phi_*} H_n(A)\oplus H_n(B) \xrightarrow{\psi_*} H_n(A+B) \xrightarrow{\partial_*} H_{n-1}(A\cap B) \longrightarrow \cdots \longrightarrow 0.$$

In the sequence, $H_n(A+B)$ can be replaced by $H_n(X)$ if A and B satisfy an extra condition. To describe it, we shall state an important theorem in algebraic topology, which is usually referred to as the excision theorem:

Theorem 3.6. *(Spanier 1996, p. 178) Let X be a topological space and $\mathcal{U}$ be a collection of subspaces in X. $C_n(\mathcal{U})$ is defined to be a subgroup of $C_n(X)$ consisting of elements $\sum n_i\sigma_i$, where σ_i are singular n-simplices of subspaces in $\mathcal{U}$. If the union of the interiors of subspaces in $\mathcal{U}$ equals to X, the inclusion chain map $i : C(\mathcal{U}) \to C(X)$ induces an isomorphism on $H(\mathcal{U})$ and $H(X)$ in each dimension.*

If we take $\mathcal{U} = \{A, B\}$ such that $int(A)\cup int(B) = X$, then it gives the following long exact sequence, which is called the **Mayer-Vietoris sequence:**

$$\cdots \longrightarrow H_n(A\cap B) \xrightarrow{\phi_*} H_n(A)\oplus H_n(B) \xrightarrow{\psi_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A\cap B) \longrightarrow \cdots \longrightarrow 0.$$

Notice that we also have a similar reduced Mayer-Vietoris sequence for reduced singular homology groups by adding $0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$ in the right of the sequence (4). In the following, we will use the Mayer-Vietoris sequence to calculate the homology groups of wedge sum. From the definition below, we can see that the wedge sum is like a topology version of the direct sum in algebra, and it will be shown that they are indeed related.

Definition 3.2. A **wedge sum** $X \vee Y$ of two spaces X and Y with $x_0 \in X$ and $y_0 \in Y$ is a quotient space of the disjoint union of X and Y with x_0 and y_0 being defined as equivalent.

Corollary 3.2. *Let X and Y be two spaces and $X \vee Y$ be their wedge sum with an identification point α. If α has neighborhoods U, V in X, Y that have $\{\alpha\}$ as a deformation retract, then for any $n \geq 0$, we have*

$$\tilde{H}_n(X\vee Y) = \tilde{H}_n(X)\oplus\tilde{H}_n(Y).$$

Proof. Notice that the above condition means that $X \vee Y$ can be written as the union of interiors of $X \cup V$ and $Y \cup U$ such that they are homotopy equivalent to X and Y respectively. Since the intersection $V \cup U$ has $\{\alpha\}$ as a deformation retract and the reduced homology groups of $\{\alpha\}$ are 0 in each dimension, then we can get our result immediately by substituting this in the Mayer-Vietoris sequence. □

4 Homology Groups of Compact and Connected Surfaces

As we have said in the beginning of the second section, singular homology groups are important because they can be used for classification. In this section, we will explain this idea in detail by computing homology groups of surfaces. We will see that the first order homology group describes the number of holes of an orientable surface. Historically speaking, it is the urge to describe the number of holes of surfaces that motivates mathematicians to define the homology groups. Our analysis is based on the following classification theorem, where the operation connected sum (Definition A.9) is used:

Theorem 4.1. *(Munkres 2000, Chpater 12) Every compact and connected surface is homeomorphic to one of the following three situations:*

- *2-sphere S^2;*
- *Connected sum of n tori T^n;*
- *Connected sum of m projective planes P^m.*

Remark. The definition of a torus and a projective plane can be illustrated by the picture below. For example, to obtain a torus T, the two edges labeled with a can firstly be identified

Figure 2: Construction of a projective plane (left) and a torus (right).

as the marked direction, then the square becomes a tube. A torus is constructed by gluing the edges b with the direction shown in the figure. The construction of a projective plane P follows the same procedure, and reader can find a formal definition in Appendix A.

Although the theorem has classified the surfaces, we do not have an indicator to distinguish them, which means, if we are given an arbitrary surface, we do not know how to identify it with one of the three kinds. By finding that different kinds of surfaces have different homology groups, we will see that one answer to this problem is the homology groups.

Now, we shall investigate respectively the homology groups of the sphere, torus and projective plane, along with their connected sums. Since all the surfaces are 2-dimensional objects, their simplicial complex structures only contain simplexes of dimension no more than 2, i.e. homology groups of an arbitrary surface of dimension not less than 3 must be 0. Furthermore, since we have assumed surfaces to be connected, so their homology groups of order 0 must be $\mathbb{Z}$ by Proposition 1.2. In conclusion, the only homology groups we need to calculate are the groups of dimension 1 and 2.

Proposition 4.1. $H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \textit{otherwise}; \end{cases} \quad H_k(T) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ 0 & \textit{otherwise.} \end{cases}$

Proof. The singular homology groups of a sphere can be calculated by taking $n = 2$ in the Corollary 3.1, and the groups of a torus are a result of Example 1.2 and Theorem 1.1. □

Proposition 4.2.

$$H_k(P) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{5}$$

Proof. To use the Mayer-Vietoris sequence, we need another definition of the real projective plane. Considering the complex plane $\mathbb{C}$, P can be described as a quotient space of $D^2 = \{z \in \mathbb{C} | |z| \leq 1\}$ with an equivalence relation defined as $z \sim z^2$ for any $z \in S^1$. The equivalence classes again forms a circle, denoted by S.

Hence, P can be written as a union of the interiors of D^2 and a neighborhood M which has S as a deformation retract. Notice M can be seen as a mapping cylinder of the map $f(z) = z^2$ from S^1 to S^1, so its intersection with interiors of D^2 is homeomorphic to S^1. We apply the reduced Mayer-Vietoris sequence:

$$\cdots \xrightarrow{\partial_*} \tilde{H}_2(S^1) \xrightarrow{i_*} \tilde{H}_2(M) \oplus \tilde{H}_2(D^2) \xrightarrow{j_*} H_2(P) \xrightarrow{\partial_*} \tilde{H}_1(S^1)$$

$$\xrightarrow{i_*} \tilde{H}_1(M) \oplus \tilde{H}_1(D^2) \xrightarrow{j_*} \tilde{H}_1(P) \xrightarrow{\partial_*} \tilde{H}_0(S^1).$$

Then, by the knowledge of homology groups of D^2 and S^1, along with the fact that S is a deformation retract of M, the above sequence is deduced to:

$$0 \longrightarrow H_2(P) \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow H_1(P) \longrightarrow 0.$$

Since a circle S^1 turns two laps when it is embedded into M, it follows that $i_*(\alpha) = 2\beta$, where α is a generator of $H_1(S)$ and β is a generator of $H_1(M)$. Then, the exact sequence splits up into the following two exact sequences:

$$0 \longrightarrow H_2(P) \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H_1(P) \longrightarrow 0.$$

They complete our proof by exactness. $\square$

Proposition 4.3. *Let T^n be the connected sum of n tori for $n \geq 1$. We have*

$$H_k(T^n) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^{2n} & k = 1 \\ 0 & \text{others.} \end{cases} \tag{6}$$

Proof. We take induction on the number n. When $n = 1$, the statement is true by Proposition 4.1. Suppose it is also true for the connected sum of $n-1$ tori, denoted as A, then the wedged sum $A \vee T$ is a quotient space $(A\#T)/S^1$, which is obtained by shrinking the circle generated from the construction of connected sum to a point. By Corollary 3.2,

$$H_k(A \vee T) = \begin{cases} \mathbb{Z}^2 & k = 0, 2 \\ \mathbb{Z}^{2n} & k = 1 \\ 0 & \text{others.} \end{cases}$$

Theorem 3.3 gives the reduced long exact sequence for the pair $(A\#T, S^1)$:

$$\cdots \xrightarrow{\partial_*} H_2(S^1) \xrightarrow{i_*} H_2(T^n) \xrightarrow{j_*} H_2(A \vee T) \xrightarrow{\partial_*} H_1(S^1)$$

$$\xrightarrow{i_*} H_1(T^n) \xrightarrow{j_*} H_1(A \vee T) \xrightarrow{\partial_*} \tilde{H}_0(S^1).$$

Substituting the known information reduces the sequence to:

$$0 \longrightarrow H_2(T^n) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \xrightarrow{i_*} H_1(T^n) \longrightarrow \mathbb{Z}^{2n} \longrightarrow 0.$$

Notice that the map i_* is a zero map. To see this, we can visualize the generating circle f as S^1 in the square below. It is then direct to see that f is homotopic to the path $a+b-a-b$. Proposition 2.1 then implies that $i_*\{f\} = \{a+b-a-b\} = 0$.

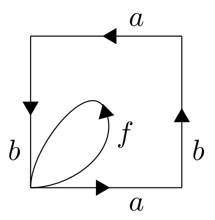


Figure 3: S^1 embedded in a torus

Hence, we can split the above sequence into two exact sequences:

$$0 \longrightarrow H_2(T^n) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

$$0 \longrightarrow H_1(T^n) \longrightarrow \mathbb{Z}^{2n} \longrightarrow 0.$$

For the second sequence, it proves that $H_1(T^n) \cong \mathbb{Z}^{2n}$ by exactness. For the first sequence, notice that $\mathbb{Z}$ is free, so Theorem 3.1 implies that $\mathbb{Z}^2 \cong H_2(T^n) \oplus \mathbb{Z}$, and this completes our proof by taking the quotient of the direct sum. ☐

Proposition 4.4. *Suppose P^m is a connected sum of m projective planes. It follows that*

$$H_k(P^m) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{m-1} & k = 1 \\ 0 & \textit{others.} \end{cases} \tag{7}$$

Proof. The idea of this proof is exactly the same with the last proposition, so we omit the first part. Following the procedure as before, we will eventually get two exact sequences:

$$0 \longrightarrow H_2(P^m) \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} H_1(P^m) \xrightarrow{j_*} \mathbb{Z}^{m-2} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Considering the simplicial complex structure of P^m, $H_1(P^m)$ must be a finitely generated abelian group, so we could suppose by the structure theorem of the finitely generated abelian group (Hungerford 1974, Chapter III) that $H_1(P^m) = \mathbb{Z}^k \oplus A$ where A is the torsion subgroup. Then, since i_* is injective, $\mathbb{Z}$ must be embedded into one $\mathbb{Z}$ of $\mathbb{Z}^k$. Furthermore, by using the same method in last proposition, we can see that i^* maps a generator α of S^1 to 2β, where β is a generator of the homology group $H_1(P^m)$. Hence, the kernel of j^* equals to the image of i_* as $2\mathbb{Z}$. By exactness, it follows that

$$\mathbb{Z}^{m-2} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \cong H_1(P^m)/\mathrm{Ker}(j_*) = \mathbb{Z}^{k-1} \oplus A \oplus \mathbb{Z}/2\mathbb{Z}.$$

The structure theorem then forces $k = m-1$ and $A = \mathbb{Z}/2\mathbb{Z}$. ☐

From the propositions of this section, we observe that $H_2(T^n) = H_2(S^2) = \mathbb{Z}$ and $H_2(P^m) = 0$, so the second order homology group distinguishes orientable and non-orientable surfaces. Furthermore, notice that a sphere can be recognized as a torus type surface without holes, and a connected sum of n tori has exactly n holes, so the first order homology group $\mathbb{Z}^{2n}$ indicates the number of holes by its rank. In conclusion, one can obtain the complete topology information of a surface by its algebra construction. It is the power of linking two subject fields that makes the study of homology theory thriving until these days.

5 Spectral Sequences

After finishing the fundamental theory of homology groups, we continue our exploration motivated by the desire to find more calculation methods for homology groups, and it leads to the goal of this section, spectral sequences. They were firstly introduced by Leray (1946), and had a reputation of being abstruse. However, they are powerful tools in homology theory, especially when dealing with fibre bundles. In this section, we will see how to build a spectral sequence from a fibre bundle, and use it to compute the homology groups of complex projective planes.

5.1 Homology with Coefficients, Fibre Bundles and Filtration

Before introducing spectral sequences, we shall introduce several concepts which will be necessary for understanding them. The operation tensor product will be used, and its definition can be found in Appendix C. We shall always use R as a principal ideal domain and G as an arbitrary R-module in this section. Notice this assumption could be loosened in some cases, but we drop the generality to simplify our discussion.

Definition 5.1. A **chain complex over** R is a chain complex where C_n of each dimension is an R-module and module homomorphisms ∂_n satisfies the condition $\partial_n \circ \partial_{n+1} = 0$. Let C be a chain complex over R, then there is an induced chain complex over R, denoted by $C \otimes G$, consisting of modules $C_n \otimes G$ and boundary maps $\partial_n \otimes 1$ for all $n \geq 0$. The corresponding homology module defined from this sequence is called a **homology module with coefficients** G, denoted by $H(C; G)$.

Notice that this generalization could be put right after the definition of homology groups, we wait until now only for the readability of the previous content. If we take $R = G = \mathbb{Z}$, then the homology module will be identical to the homology groups we previously defined. This is a corollary of Theorem C.1 in the appendix.

Definition 5.2. Given a quadruple (X, B, F, π), where X, B, F, π are respectively referred to as total space, base space, fibre, and bundle projection, we define the quadruple as a **fibre bundle** if the following conditions are fulfilled:

1. π is a surjective map;
2. For each $x \in B$, there is a neighborhood U of x and a homeomorphism ϕ mapping from $U \times F$ to $\pi^{-1}(U)$;

3. For each homeomorphism ϕ, the composite $U \times F \to \pi^{-1}(U) \to U$ is a projection from $U \times F$ to U, where the second arrow is the bundle projection π.

Notice that the map ϕ restricting to $\pi^{-1}(x) := F_x$ is a homeomorphism with F, and $\pi^{-1}(x)$ is called a fibre over x.

A vivid example of a fibre bundle would be a shoe brush. Its hairs can be seen as fibres, and the board would be a base space. By defining π sending all points in a fibre to the point on the board, we get a model of a fibre bundle.

Definition 5.3. A **filtration** of a topolgical space X is a sequence of subspace

$$\cdots \subset X_p \subset X_{p+1} \subset X_{p+2} \cdots$$

such that $\bigcup X_p = X$.

To proceed, we shall discuss the concept of a CW complex. Instead of outlining the formal definition (Hatcher 2002, p. 5), an intuitive description will be given. Roughly speaking, it is similar to a simplicial complex, but the ingredients are q-cells (which are n-dimensional disks) here. Starting from dimension 0, we define a set of points as X_0. Inductively, an n-skeleton X_n is obtained by attaching n-cells via a continuous map f from the boundary of S^{n-1} to X_{n-1}. A k-dimensional CW complex X is defined as a union of n-skeletons such that the maximal dimension of its cells is k. From this construction, we can obtain a filtration of X by letting X_p be the p-skeleton of X for $p \geq 0$ and 0 otherwise.

A CW complex can have different CW complex structures, and an example is S^n. On the one hand, it is a union of a 0-cell and one n-cell, where the attaching map identifies the whole boundary of D^n as the point. On the other hand, we can also say it contains two m-cells for each dimension $m \leq n$. (For example, S^1 consists of two semi-circles as two 1-cells, and their endpoints connect them as two 0-cells.)

5.2 The Serre Spectral Sequence

The purpose of this section is to discuss the concept of spectral sequences, and see how to build one from a topological space with a filtration. Recalling that the Mayer-Vietoris sequence is defined for $X = A \cup B$, a spectral sequence can be recognized as a generalization of this since it is defined for $X = \cup X_i$, where X_i are the elements in the filtration. Then we will show that a fibre bundle induces such a sequence, where the sequence in this situation is usually referred to as the Serre spectral sequence to acknowledge his contribution.

Definition 5.4. A **homological spectral sequence** is a collection $\{E^r, d^r\}_{r \geq 1}$ such that:

- For each $r \geq 1$, E^r is a collection of R-modules $\{E^r_{p,q}\}$ where $p, q \in \mathbb{Z}$, and d^r is a collection of homomorphisms $\{d^r_{p,q}\}$ mapping from $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$;
- For each $r \geq 1$ and $p, q \in \mathbb{Z}$, we have $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$, so we can define a homology module in each node (p, q) by $H_{p,q}(E^r) := \mathrm{Ker}(d^r_{p,q})/\mathrm{Im}(d^r_{p+r,q-r+1})$;
- For each $r \geq 1$ and $p, q \in \mathbb{Z}$, we have $E^{r+1}_{p,q} \cong H_{p,q}(E^r)$.

If we also have that $E^r_{p,q} \neq 0$ implies $p, q \geq 0$, then such a homological spectral sequence will be called a first quadrant homological spectral sequence. In the following text, a spectral sequence always means a first quadrant homological spectral sequence. The main advantage for this hypothesis is the following property: if we focus on one node (p, q), the maps entering and leaving (p, q) node will eventually become trivial with r increasing, which is because the non-zero modules are only in the first quadrant. If this trivialization happens at $r = r_0$, then $E^{r_0}_{p,q} \cong E^{r_0+1}_{p,q} \cong ...$ and they shall be denoted by $E^\infty_{p,q}$.

Given a topological space X and a filtration $\{X_p\}$, they decide a spectral sequence. The construction can be found in chapter 9 of the book of Spanier (1996), and the first section of the book written by Hatcher (2004). Note that the authors use different methods which induce two essentially equivalent spectral sequences.

Here, we shall briefly discuss Hatcher's idea. He used a more general definition of the spectral sequence. An exact couple is a diagram consists of modules and maps as shown in the graph, where we require it is exact at each node. If we let $d = j \circ k$, then $d \circ d = 0$ and a homology module E' can be defined as $\mathrm{Ker}(d)/\mathrm{Im}(d)$. Furthermore, if we let $A' = i(A)$, $i' = i|_{A'}$, $k'(e) = k(e)$ for some $e \in \mathrm{Ker}(d)$, and let $j'(i(a)) = j(a)$ for some $i(a) \in A'$. Then one can check that they are well-defined and form a new exact couple. In the new couple, we could again let $d' = j' \circ k'$, and the above process can be repeated infinitely many times. Denote the modules $E, E', ...$ as $E^1, E^2, ...$ and the differentials $d, d', ...$ as $d^1, d^2, ...,$ the collection $\{E^r, d^r\}$ is called a spectral sequence.

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Let X be a CW complex with X_p as its p-skeleton, then a filtration of X is $\{X_p\}$ and $X_p = 0$ when $p < 0$. We could have a large diagram as follows (Hatcher 2004):

$$\begin{tikzcd}
 & \arrow[d] & & \arrow[d] & & \arrow[d] & \\
\arrow[r] & H_{n+1}(X_p) \arrow[r] \arrow[d] & H_{n+1}(X_p, X_{p-1}) \arrow[r] & H_n(X_{p-1}) \arrow[r] \arrow[d] & H_n(X_{p-1}, X_{p-2}) \arrow[r] & H_{n-1}(X_{p-2}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H_{n+1}(X_{p+1}) \arrow[r] \arrow[d] & H_{n+1}(X_{p+1}, X_p) \arrow[r] & H_n(X_p) \arrow[r] \arrow[d] & H_n(X_p, X_{p-1}) \arrow[r] & H_{n-1}(X_{p-1}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H_{n+1}(X_{p+2}) \arrow[r] \arrow[d] & H_{n+1}(X_{p+2}, X_{p+1}) \arrow[r] & H_n(X_{p+1}) \arrow[r] \arrow[d] & H_n(X_{p+1}, X_p) \arrow[r] & H_{n-1}(X_p) \arrow[r] \arrow[d] & {} \\
 & {} & & {} & & {} &
\end{tikzcd}$$

Notice that the grey part is a long exact sequence for the pair (X_p, X_{p-1}). Here, by defining A as the collection $\{A^1_{p,q}\} = \{H_{p+q}(X_p)\}$, E as the collection $\{E^1_{p,q}\} = \{H_{p+q}(X_p, X_{p-1})\}$, we can got our initial exact couple with the maps induced from the long exact sequences. By carefully checking, we will eventually get the spectral sequence as in the Definition 5.4.

As we have promised in the beginning of this section, we now give the theorem to relate a fibre bundle and a spectral sequence:

Theorem 5.1. *(Hatcher 2004, p. 527) Suppose (X, B, F, π) is a fibre bundle and B is a simply conneced CW complex. Let B_k denote the k-skeletons of B, then the inverse images $\{\pi^{-1}(B_k)\}$ form a filtration of X. It gives a spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ satisfying the following properties:*

1. $E^2_{p,q} \cong H_p(B; H_q(F; G))$;

2. *For any integer $n \geq 0$, there is a filtration $0 \subset F_n^0 \subset F_n^1 \subset \cdots \subset F_n^n = H_n(X;G)$ of $H_n(X,G)$ such that $F_n^p/F_n^{p-1} \cong E_{p,n-p}^{\infty}$, where $E_{p,n-p}^{\infty}$ is the stable term of $E_{p,n-p}^r$. Furthermore, we have $F_n^p = Im(H_n(\pi^{-1}(B_p);G) \xrightarrow{i^*} H_n(X))$, where i^* is the induced map of the inclusion mapping $i : \pi^{-1}(B_p) \to \pi^{-1}(B) = X$.*

Remark. Notice that there is some difference with the original theorem stated in Hatcher's book. The first one is that we state the theorem for a fibre bundle instead of a fibration. There is no essential difference since the projection map π can be converted into a fibration using a pathspace construction, and it induces an isomorphism on the homology modules from the fibre bundle to the fibration. Hatcher gives a detailed explanation in his book. The second difference is that we require B is a CW complex and simply connected, and these conditions could be loosened as stated in the book. Here, we use the CW complex structure for giving a filtration of X.

5.3 An Example: Complex Projective Space

To explain how to use Theorem 5.1, we will calculate the singular homology groups of an n-dimensional complex projective space $\mathbb{C}P^n$ in this section. Notice that we only use homology modules with coefficient $\mathbb{Z}$ in the following, so our discussion is restricted to the homology groups. An **n-dimensional complex projective space** is a quotient space of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ with the equivalence relation defined as follows:

$$(z_0, \cdots, z_n) \sim (u_0, \cdots, u_n) \quad \text{if and only if} \quad \exists \lambda \in S^1, (u_0, \cdots, u_n) = \lambda(z_0, \cdots, z_n). \tag{8}$$

A CW complex structure of $\mathbb{C}P^n$ can be described as the union of $e^0 \cup e^2 \cup \cdots \cup e^{2n}$, where e^k is a k-dimensional disk. Hence, when $n = 1$, $\mathbb{C}P^1$ is the union $e^0 \cup e^2$, and notice that the only possible way to attach the e^2 cell is to identify its boundary with e^0, so $\mathbb{C}P^1$ is homeomorphic to S^2 since they have the same CW complex structure.

From our definition,we can get a fibre bundle $(S^{2n+1}, \mathbb{C}P^n, S^1, \pi)$, where π maps $(z_0, \cdots, z_n)$ to its equivalence class $[z_0, \cdots, z_n]$. To define the local homeomorphism, we can take an arbitrary open set V of $\mathbb{C}P^n$. Since it belongs to S^{2n+1}, there is an always-non-zero coordinate z_i for any point $(z_0, \cdots, z_n) \in p^{-1}(V)$. Hence, by defining the local map $h : p^{-1}(V) \to V \times S^1$ as $h(z_0, \cdots, z_n) = ([z_0, \cdots, z_n], z_i/|z_i|)$, one can check it fulfils the requirements in the definition. Schultz (2016) showed in his note that $\mathbb{C}P^n$ is simply connected. Theorem 5.1 then implies that there is a spectral sequence for the fibre bundle, and we have

$$E_{p,q}^2 = H_p(\mathbb{C}P^n; H_q(S^1)) = \begin{cases} H_p(\mathbb{C}P^n) & q = 0, 1 \\ 0 & q \geq 2. \end{cases} \tag{9}$$

Notice that $H_0(\mathbb{C}P^n) = \mathbb{Z}$ since $\mathbb{C}P^n$ is path-connected, so the E^2 page looks like:

2	0	0	0	0	0
1	$\mathbb{Z}$	$H_1(\mathbb{C}P^n)$	$H_2(\mathbb{C}P^n)$	$E_{3,1}^2$	$E_{4,1}^2$
0	$\mathbb{Z}$	$H_1(\mathbb{C}P^n)$	$H_2(\mathbb{C}P^n)$	$E_{3,0}^2$	$E_{4,0}^2$
	0	1	2	3	4

Observation 1: If it is known that $E^r_{p,q} = 0$ for some integers r, p and q, then $E^n_{p,q}$ is also 0 for any $n \geq r$. This is a result from the inductive construction of spectral sequence, where a group in the next page is the quotient of a subgroup of the group on the last page. Hence, for any page E^r with $r \geq 2$, the only possible non-zero group must lie in the row 0 and row 1 of the first quadrant.

Observation 2: When $n \geq 3$, since $d^n_{p,q}$ maps from $E^n_{p,q}$ to $E^n_{p-n,q-n+1}$, the difference of their row numbers will be at least 2. Observation 1 implies that $d^n_{p,q}$ must either start from 0 or end at 0, so the definition of spectral sequence gives $E^\infty_{p,q} = E^3_{p,q}$, for any $p, q \in \mathbb{Z}$.

Observation 3: As we have calculated, a homology group of $H_k(S^{2n+1})$ is non-trivial if and only if k is either 0 or $2n+1$. Hence, by Theorem 5.1, the only possible non-zero stable terms are $E^\infty_{0,0}$, $E^\infty_{2n+1,0}$, and $E^\infty_{2n,1}$.

Proposition 5.1. *The stable term of the spectral sequence is*

$$E^\infty_{p,q} = \begin{cases} \mathbb{Z}, & (p,q) = (0,0) \text{ or } (2n,1) \\ 0, & \text{others} \end{cases}$$

Proof. From Observation 3, we only need to calculate the groups $E^\infty_{0,0}$, $E^\infty_{2n+1,0}$, and $E^\infty_{2n,1}$, where $E^\infty_{0,0}$ is obviously 0 since the differential maps $d^2_{0,0}$ and $d^2_{2,-1}$ are both trivial maps.

$E^\infty_{2n+1,0}$: Recall the CW complex structure of $\mathbb{C}P^n$ is $e^0 \cup e^2 \cup \cdots \cup e^{2n}$, so the $2n$-skeleton of $\mathbb{C}P^n$ is equal to the $2n+1$-skeleton and to the base space itself. Hence, Theorem 5.1 implies that $F^{2n}_{2n+1} = F^{2n+1}_{2n+1} = H_{2n+1}(S^{2n+1}) = \mathbb{Z}$, which shows

$$E^\infty_{2n+1,0} = F^{2n+1}_{2n+1}/F^{2n}_{2n+1} = 0.$$

$E^\infty_{2n,1}$: Since there is only one 0-cell of $\mathbb{C}P^n$ and the fibre is S^1, we obtain the pre-image $\pi^{-1}(e^0)$ is a loop homeomorphic to S^1, so we have $H_{2n+1}(\pi^{-1}(e^0)) = 0$, and this implies that the map $i^* : H_{2n+1}(\pi^{-1}(e^0)) \to H_{2n+1}(S^{2n+1})$ is trivial, which means that $F^0_{2n+1} = 0$. From Observation 3 that $E^\infty_{0,2n+1} = ... = E^\infty_{2n-1,2} = 0$, the filtration subgroups $F^0_{2n+1} = ... = F^{2n-1}_{2n+1} = 0$. This proves that

$$E^\infty_{2n,1} = F^{2n}_{2n+1}/F^{2n-1}_{2n+1} = \mathbb{Z}.$$

□

Proposition 5.2. $H_1(\mathbb{C}P^n) = E^2_{1,0} = E^2_{1,1} = 0.$

Proof. By the definition of spectral sequence, we have

$$E^3_{1,0} = \frac{\mathrm{Ker}(d^2_{1,0} : E^2_{1,0} \to E^2_{-1,1})}{\mathrm{Im}(d^2_{3,-1} : E^2_{3,-1} \to E^2_{1,0})} = \frac{\mathrm{Ker}(d^2_{1,0} : E^2_{1,0} \to 0)}{\mathrm{Im}(d^2_{3,-1} : 0 \to E^2_{1,0})} = E^2_{1,0}.$$

From Observation 3 and 2, we know that $E^3_{1,0} = E^\infty_{1,0} = 0$, which completes the proof. □

Proposition 5.3. *For any $2 \leq k \leq 2n$ and $k > 2n+2$, the map $d^2_{k,0}$ is an isomorphism.*

Proof. On the one hand, from Observation 3 and 2, the group $E^3_{k,0} = E^\infty_{k,0}$ is trivial. which means that

$$0 = E^3_{k,0} = \frac{\mathrm{Ker}(d^2_{k,0} : E^2_{k,0} \to E^2_{k-2,1})}{\mathrm{Im}(d^2_{k+2,-1} : E^2_{k+2,-1} \to E^2_{k,0})} = \frac{\mathrm{Ker}(d^2_{k,0} : E^2_{k,0} \to E^2_{k-2,1})}{\mathrm{Im}(d^2_{k+2,-1} : 0 \to E^2_{k,0})} = \mathrm{Ker}(d^2_{k,0}).$$

Hence, $d^2_{k,0}$ must be an invevtive homomorphism. On the other hand, we also have the group $E^3_{k-2,1} = E^\infty_{k-2,1}$ is trivial, and that gives

$$0 = E^3_{k-2,1} = \frac{\mathrm{Ker}(d^2_{k-2,1} : E^2_{k-2,1} \to E^2_{k-4,2})}{\mathrm{Im}(d^2_{k,0} : E^2_{k,0} \to E^2_{k-2,1})} = \frac{\mathrm{Ker}(d^2_{k-2,1} : E^2_{k-2,1} \to 0)}{\mathrm{Im}(d^2_{k,0} : E^2_{k,0} \to E^2_{k-2,1})} = \frac{E^2_{k-2,1}}{\mathrm{Im}(d^2_{k,0})}.$$

It follows that $E^2_{k-2,1} = \mathrm{Im}(d^2_{k,0})$ and the map is surjective. □

Proposition 5.4. *For $0 \le k \le 2n$, the homology group of $\mathbb{C}P^n$ is*

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & k \quad \textit{even} \\ 0 & k \quad \textit{odd.} \end{cases} \tag{10}$$

Proof. Concluding the last two propositions, it gives the following diagram:

$$\begin{tikzcd}
1 & \mathbb{Z} & 0 & E^2_{2,1} & E^2_{3,1} & E^2_{4,1} \\
0 & \mathbb{Z} & 0 & E^2_{2,0} \arrow[ull] & E^2_{3,0} \arrow[ull] & E^2_{4,0} \arrow[ull] \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

Since the arrows are all isomorphisms, our conclusion can be easily deduced from an inductive argument. □

For readers who are familiar with cellular homology, it is obvious that $H_k(\mathbb{C}P^n) = 0$ for $k \ge 2n+1$ since its CW complex structure has dimension $2n$. Here, we give another proof for this fact by using the spectral sequence.

Proposition 5.5. *For $k \ge 2n+1$, the homology group of $\mathbb{C}P^n$ is*

$$H_k(\mathbb{C}P^n) = 0. \tag{11}$$

Proof. Notice that Proposition 5.3 is essentially deduced from the condition $E^\infty_{k,0} = E^\infty_{k-2,1} = 0$, so $d^2_{2n+1,0}$ is also an isomorphism since $E^3_{2n+1,0}$ is 0 by Proposition 5.1 and $E^3_{2n-1,1}$ is 0 by Observation 3. Hence, the inductive argument extends to $k = 2n+1$ and $H_{2n+1}(\mathbb{C}P^n) = 0$. We can also show that $E^2_{2n+2,0} = 0$ by the result $E^\infty_{2n,1} = \mathbb{Z}$ and the following computation:

$$\mathbb{Z} = E^3_{2n,1} = \frac{\mathrm{Ker}(d^2_{2n,1} : E^2_{2n,1} \to E^2_{2n-2,2})}{\mathrm{Im}(d^2_{2n+2,0} : E^2_{2n+2,0} \to E^2_{2n,1})} = \frac{\mathrm{Ker}(d^2_{2n,1} : \mathbb{Z} \to 0)}{\mathrm{Im}(d^2_{2n+2,0} : E^2_{2n+2,0} \to E^2_{2n,1})} = \frac{\mathbb{Z}}{E^2_{2n+2,0}},$$

where $d^2_{2n+2,0}$ is injective since $E^3_{2n+2,0} = 0$. So we have that $H_{2n+2}(\mathbb{C}P^n) = E^2_{2n+2,0} = 0$. For $k > 2n+2$, the homology groups are all 0 by using the inductive argument as in the Proposition 5.4. □

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A Topological Space

In this section we shall provide some definitions and constructions of topological spaces for readers who are not familiar with them.

Definition A.1. Let X be a set. We call a collection $\mathcal{U}$ of subsets of X a **topology** of X if

- $\emptyset$ and X are elements in $\mathcal{U}$;
- Let $\{X_\alpha\}_{\alpha\in J}$ be a subset of $\mathcal{U}$, then their union $\cup_{\alpha\in J}X_\alpha$ is also an element of $\mathcal{U}$;
- The intersection of finitely many elements of $\mathcal{U}$ is an element in $\mathcal{U}$.

If we are given a topology $\mathcal{U}$, a subset $U \subset X$ is called **open** if $U \in \mathcal{U}$, and a subset of X is **closed** if its complementary set in X is open. The couple $(X, \mathcal{U})$ is called a **topological space**.

Notice that we will omit $\mathcal{U}$ when referring to a topological space if this would not cause any confusion.

Definition A.2. Suppose X, Y are two topological spaces. A map $f : X \to Y$ is **continuous** if for any open subset $U \subset Y$, its preimage $f^{-1}(U)$ is also open. If f is a continuous bijective map, and its inverse mapping f^{-1} is also continuous, then we call f is a **homeomorphism**. X and Y are said to be **homeomorphic** if a homeomorphism exists.

If two spaces are homeomorphic, then we can identify them when we are only interested in topological properties, which are referred to as properties that will be invariant under homeomorphism, such as path-connectedness, compactness and being Hausdorff. Before giving definitions of these properties, we shall first define some standard constructions of topological spaces:

Subspace: Considering a topological space X and a subset A of X, we can define A as a topological space with the following topology: $U \subset A$ is said to be open in A if there exists an open set V in X such that $V \cap A = U$. One check directly that this definition indeed gives a topology of A. A subset with the above topology is called a **subspace** of X.

If A itself is open in X, $U \subset A$ is open in A if and only if U is open in X. However, this is not true in general. For example, $(0, 1) \times 0$ is an open interval in $\mathbb{R}^1$ as a subspace of $\mathbb{R}^2$, but $(0, 1) \times 0$ is not open in $\mathbb{R}^2$.

Product space: Let $X_1, ..., X_n$ be n topological spaces. We can define a topology for their Cartesian product $X_1\times\cdots\times X_n$: A set U is open in $\prod_{i=1}^n X_i$ if for any point $(x_1, \cdots, x_n) \in U$, there exists a set $U_1 \times \cdots \times U_n \subset U$ such that $x \in \prod_{i=1}^n U_i$ and U_i is open in X_i.

An Euclidean space $\mathbb{R}^n$ can either have a topology defined as a metric space or as a product space of n copies of $\mathbb{R}$, one can prove that they coincide, i.e. a subset in $\mathbb{R}^n$ is open in the metric space if and only if it is open in the product space. Notice that we only define product space when finitely many spaces are given, and this definition cannot be simply generalized for an arbitrary collection of topological spaces. We usually use the word product space of infinitely many topological spaces to denote another kind of construction, Munkres (2000) describes it in section 19.

Quotient space: Suppose X is a topological space and $\sim$ is an equivalence relation on X, then its equivalence classes form a partition of X, denoted as X^*. We can define a projective map $p : X \to X^*$ mapping $x \in X$ to its equivalence class $[x] \in X^*$, then p induces a topology on X^*: $U \subset X^*$ is defined to be open if its preimage $p^{-1}(U)$ is open in X. The topological space is usually denoted as $X/\sim$.

The definition of a projective plane (Figure 1) is an example of a quotient space. Considering a subspace $[0,1]\times[0,1]$ of $\mathbb{R}^2$, we define $(x,0) \sim (1-x,1)$, $(0,1-y) \sim (1,y)$ and $(x,y) \sim (x,y)$ for any $x, y \in [0,1]$, which gives an equivalence relation on X. The topological space generated by this relation is a projective plane.

Let X be a topological space and A be a subspace. The notation X/A means a quotient space of X, where we define all the elements in A are equivalent to one point.

Definition A.3. For a topological space X, a collection of open subsets $\mathcal{A}$ is said to be an open cover of X if the union of all the elements in $\mathcal{A}$ is X. A topological space is said to be **compact** if any open cover of X has finitely many elements such that their union is X.

There is an import result for compact spaces, which is called Tychonoff theorem (Munkres 2000, chpater 5). It states that a product space of an arbitrary collection of compact spaces is still a compact space. Although this is not hard to prove for finite dimension situation, it is quite remarkable to see that we have this result for infinitely many spaces since a character of an infinite dimensional normed space is that a closed unit ball is always not compact (Ciarlet 2013, Chapter2).

Definition A.4. Let X be a topological space, a path from x to y is a continuous mapping from $[0,1]$ to X such that $f(0) = x$ and $f(1) = y$. If for any $x, y \in X$, there is a path connecting them , then we call X is **path-connected**. For a topological space X, we define $x \sim y$ if there is path from x to y. One can check this is an equivalence relation on X, and the equivalence classes are called **path components** of X.

From the definition of path components, we have that a topological space is path-connected if and only if it has only one path component.

Definition A.5. Suppose X is a topological space. We call X a **connected** space if it cannot be written as a union of two disjoint open subsets of X.

Notice that a path-connected space is always a connected space, but the converse is not true in general. If we suppose that X is locally homeomorphic to $\mathbb{R}^n$ for some n, i.e., there is an open subset U for every $x \in X$ such that U is homeomorphic to $\mathbb{R}^n$, then X is path-connected if and only if it is a connected space (Munkres 2000, Chapter 3).

Definition A.6. A topological sapce X is **Hausdorff** if for any pair $x, y \in X$, there are two disjoint open subsets $U, V \subset X$ such that $x \in U$ and $y \in V$.

This condition is satisfied for all the spaces we studied, and it is generally asked to be true because it guarantees a critical property: If a sequence has a limit, it must be unique.

Definition A.7. A **basis** of a topological space X is a collection $\mathcal{B}$ of open subsets in X such that any open set in X can be written as a union of some elements in $\mathcal{B}$. If X has a countable basis, then we call X is **second countable**. A Hausdorff and second countable space is called a nn**-manifold** for some positive integer n if X is locally homeomorphic to $\mathbb{R}^n$. In particular, a **2-manifold** is said to be a **surface**.

From what we have discussed before, a manifold is connected if and only if it is path-connected. In the end of this section, we list two constructions of topological spaces that are mentioned in the report.

Definition A.8. Suppose $\{A_i\}_{i\in J}$ is an indexed family of sets, we say a disjoint union of $\{A_i\}_{i\in J}$ is

$$\bigsqcup_{i\in J} A_j = \{(a,k)|a \in A_k \text{ and } k \in J\}.$$

Definition A.9. The connected sum is an operation between two surfaces which can be roughly described as the following. Given two surfaces, we first delete two small disks from each of them, and then a new surface is obtained by homeomorphiclly identifying the circles from the deleting step. This operation can be defined for n surfaces using an inductive construction. One can find a rigorous definition in chapter 1 of the book of Massey (1991). The picture below gives an intuitive description of the connected sum.

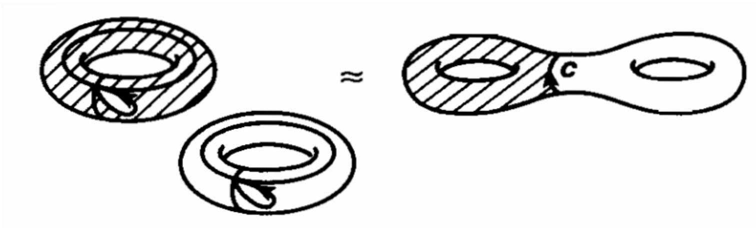

Figure 4: Connected sum of two tori (Munkres 2000)

B Homotopy and Deformation Retraction

Let $f, g : X \to Y$ be two continuous maps between topological spaces. A continuous map $F : X \times [0,1] \to Y$ is said to be a **homotopy** between f and g if $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for any $x \in X$. For maps f and g such that a homotopy exists, we say that they are **homotopic**, written as $f \sim g$.

For a continuous map $f : X \to Y$, it is a **homotopy equivalence** if there is another continuous map $g : Y \to X$ such that $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$. Here, g is called a **homotopy inverse** of f. Two topological spaces X and Y are called **homotopy equivalent** if there is a homotopy equivalence between them, and we write $X \simeq Y$.

Suppose A is a subspace of X, we call a continuous map $r : X \to A$ a **retraction** if $r(x) = x$ for any $x \in A$. The space A is said to be a **deformation retract** of X if the inclusion map $i : A \to X$ is a homotopy inverse of r. The homotopy between $i \circ r$ and Id_X is called a **deformation retraction**. Notice that in this case, A is homotopy equivalent to X.

Example B.1. Suppose $f : X \to Y$ is a continuous map. A **mapping cylinder** M_f is a quotient space of the disjoint union (see Definition A.8) of $X \times [0,1]$ and Y with an equivalence relation $(x,0) \sim y$ if $f(x) = y$. Then the subspace Y is always a deformation retract of M_f via the following deformation retraction:

$$F(s,t) = \begin{cases} (x, t \cdot t_0) & \text{for } (s,t) = (x,t_0) \in X \times [0,1] \\ y & \text{for } s = y \in Y. \end{cases}$$

Notice that this is a homotopy between $i \circ r$ and Id_{M_f}, where $r((x,t)) = f(x)$ for any $x \in X \times [0,1]$ and $r(y) = y$ for $y \in Y$.

Example B.2. Let $\{0\}$ be a single point subspace of $\mathbb{R}^n$ and $F(x,t) = tx$ for any $(x,t) \in \mathbb{R}^n \times [0,1]$, then F is a homotopy between $Id_{\mathbb{R}^n}$ and $i \circ r$, where $r(x) = 0$ for any $x \in \mathbb{R}^n$ as a retraction. Hence, $\mathbb{R}^n$ is homotopic equivalent to one point space $\{0\}$.

Notice that the relations defined by homotopy and homotopy equivalence are both equivalence relations. A homeomorphism between spaces is always a homotopy equivalence since it has an inverse mapping. Conversely, there are spaces which are homotopy equivalent but not homeomorphic. For example, as we have shown, a point is homotopy equivalent to a line, but they are not homeomorphic since there is no bijection between them.

C Module and Tensor Product

Since our report always requires commutativity for any algebraic structure we discussed, we also ask R to be a commutative ring. The definition of a module we give is usually called a bi-module, and a general definition can be found in the book of Hungerford (1974).

Definition C.1. Suppose R is a commutative ring. An abelian group A is called an **R-module** if there are two operations $\bullet : R \times A \to A$ and $\bullet : A \times R \to A$, called left (right) scalar product, such that $r \cdot a = a \cdot r$ for all $r \in R$ and $a \in A$, and the left scalar product satisfies the following conditions:

- $r \cdot (a + b) = r \cdot a + r \cdot b$,
- $(r + s) \cdot a = r \cdot a + s \cdot a$,
- $r \cdot (s \cdot a) = (rs) \cdot a$,

where $r, s \in R$, and $a, b \in A$. Notice that right scalar product also satisfies these conditions since it is defined to be identical with the left one. We usually omit the dot when it does not cause any confusion. If R has an identity, then A is defined as an **unitary R-module** if $1_R \cdot a = a$ for any $a \in A$. In particular, A is called a **vector space** if R is a field.

Notice that R itself can be recognized as an R-modulue, where the scalar product is defined to be the product of R.

Definition C.2. Considering A, B as two R-modules, an R-module homomorphism is a mapping $f : A \to B$ with the following properties:

- $f(a+b) = f(a) + f(b)$,
- $f(ra) = rf(a)$,

where $r \in R$ and $a, b \in A$.

Definition C.3. Suppose A and B are two modules are over a ring R. We use F to denote the free abelian group generated by the set $A \times B$, let K be a subgroup of F generated by the union of the following sets:

- $\{(a+a', b) - (a, b) - (a', b) | a, a' \in A \text{ and } b \in B\}$;
- $\{(a, b+b') - (a, b) - (a, b') | a \in A \text{ and } b, b' \in B\}$;
- $\{(a, rb) - (ra, b) | a \in A,\ b \in B \text{ and } r \in R\}$.

Then, the quotient group F/K is called the **tensor product** of A and B over R, denoted by $A \otimes_R B$, and we can also write $A \otimes B$ if it would not cause any confusion. An equivalence class of $(a, b) \in F$ is denoted as $a \otimes b$.

Since F is a free abelian group, a typical element in F should be $\sum_{i=1}^{n} n_i(a_i, b_i)$, where n_i are some integers. Hence, a general element in $A \otimes B$ should be written as $\sum_{i=1}^{n} n_i(a_i \otimes b_i)$. From the definition of tensor product, we have the following basic properties for all $a, a' \in A$, $b, b' \in B$ and $r \in R$:

$$(a + a') \otimes b = a \otimes b + a' \otimes b;$$
$$a \otimes (b + b') = a \otimes b + a \otimes b';$$
$$ar \otimes b = a \otimes rb.$$

By the last property, we can have a well-defined module structure on $A \otimes B$ by defining $r(\sum_{i=1}^{n} n_i(a_i \otimes b_i)) = \sum_{i=1}^{n} n_i((ra_i) \otimes b_i)$. The following theorem is a property of tensor product, which is used in the report:

Theorem C.1. *(Hungerford 1974, Chapter IV) If R is a commutative ring with identity and A, B are two unitary R-modules, then we have the following isomorphisms:*

$$A \otimes B \cong B \otimes A; \quad A \otimes R \cong A.$$