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Master's Project

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An Introduction to Topological K-theory towards Hopf Invariant One Problem

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Abstract

This report is aimed at presenting Adams-Atiyah-Hopkin's solution to Hopf invariant one problem. We firstly give a gentle introduction to vector bundles and topological K-theory. Then, we calculate the cohomology of $U(n)$ and $BU(n)$ by spectral sequence. These results are used in defining Chern classes and Chern characters. Using them, we introduce how to transfer the original problem into a question in K-theory. Then, we answer the problem by studying Ext functors in the category of groups with Adams operations.

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Introduction

Let $f : S^{2n-1} \to S^n$ be a map for $n \geq 2$. A homotopy invariant called the Hopf Invariant can be defined with cohomology theory. There was a once famous question relating to this invariant that asked for which n such an invariant can be one.

Some examples are quickly recognized by people. Dimensions $n = 2, 4, 8$ have examples of Hopf invariant one, and the spaces are induced from the division algebra structure of $\mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively. For all other situations, Adams [1] answered in 1960 that there is no other n such that the invariant can be one. In his paper, he used Steenrod algebra and Adams spectral sequence to translate this problem as a question to determine when some differentials in the spectral sequence are non-zero.

Adams and Atiyah [2] gave another proof using the topological K-theory. An advantage of using K-groups is that the cohomology operation on them is over $\mathbb{Z}$, but Steenord operations can only be defined over $\mathbb{Z}/2\mathbb{Z}$. Hence, using K-groups would naturally give us more information, so it makes the solution simpler.

The way they presented their answer was a nice trick by observing the Adams operations. Hopkins [3] retold their method by describing the actual algebraic structure involved, which will also be our way to tell the story. Although this conceptual way means we make things a bit more complex, it can also produce the e-invariant and tell us what happens behind tricks.

We assumed readers are familiar with the material contained in Hatcher's Algebraic Topology [4]. Some homological algebra about Ext functors will be also used in section 10. During the paper, we will explain the theory of vector bundles and the topological K-theory involved, but the very rare proof will be given because they do not contribute to understanding the problem. A reader can find detailed proof in the reference we list.

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1 The Theory of Vector Bundles

We start from review some definitions and fundamental results about vector bundles.

Definition 1.1. *A vector bundle over $k(k = \mathbb{R}$ or $\mathbb{C})$ with a base space B and a total space E is a map $p : E \to B$ satisfying the following properties:*

1. *For every $b \in B$, the fiber $p^{-1}(b)$ is a vector space over k.*
2. *For every $b \in B$, there is a neighbourhood U of b and a non-negative integer n with a homeomorphism $\phi : U \times k^n \to p^{-1}(U)$ such that $p \circ \phi$ is the projection from $U \times k^n$ to U. Moreover, for any $x \in U$, the map ϕ restricting to a linear isomorphism $\{x\} \times k^n \to p^{-1}(x)$.*

A section of p is defined as a map $s : B \to E$ such that $p \circ s = id_B$.

From the definition, it's easy to observe that when B is connected, the dimension of fiber is fixed. In this case, we call a vector bundle has dimension (or rank) n. Fixing the base space B, all vector bundles over B can be made into a category Vect(B) with maps being maps of total spaces over B.

Example 1.2. *For any space B, the vector bundle $B \times k^n \to B$ is called the trivial bundle, we usually use ϵ^n to denote it when we do not need to clarify the base space.*

Example 1.3. *Let $B = \mathbb{C}P^n$. There is a tautological line bundle over B with total space*

$$E = \{(l, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | v \in l\},$$

where the projection map sends (l, v) to l.

We will describe some basic operations on vector bundles without giving full details. In the following, $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ will be two vector bundles.

Direct sum: The direct sum (also called the Whitney sum) is defined with the total space being

$$E = \{(x_1, x_2) \in E_1 \times E_2 | p_1(x_1) = p_2(x_2)\}$$

The projection to B is just sending (x_1, x_2) to $p_1(x_1) = p_2(x_2)$. One can easily see the fiber of a point $x \in B$ is a direct sum of $p_1^{-1}(x) \oplus p_2^{-1}(x)$.

Tensor product: As a set, the total space of a tensor product is defined as the disjoint union of all vector spaces $p_1(x) \otimes p_2(x)$ for all $x \in B$. It is given a topology by making the structure maps $\phi_1 \otimes \phi_2$ become homeomorphisms, and it can be checked this topology is independent of how we choose the structure maps.

Pullbacks: Let $p : E \to B$ be a vector bundle and $f : A \to B$ be a map. Then, we have a pullback diagram

$$\begin{tikzcd} p^*E \arrow[r] \arrow[d, "p'"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

It can be checked that p' is also a vector bundle. For any $a \in A$, the fiber $p'^{-1}(a)$ is isomorphic to the vector space $p^{-1}(f(a))$.

External Tensor Product: For two vector bundles $E_1 \to B_1$ and $E_2 \to B_2$, we can define an external tensor product over $B_1 \times B_2$. Noticing we have projections $p_i : B_1 \times B_2 \to B_i$ for $i = 1, 2$. The external product is defined as:

$$E_1 \otimes_{B_1 \times B_2} E_2 := p_1^* E_1 \otimes p_2^* E_2.$$

We use external products when base spaces are different, so when there is no misleading, $E_1 \otimes E_2$ can also mean the external product.

When the base space is compact and Hausdorff, every bundle has a complement in the following sense:

Proposition 1.4. *[5] Let B be a compact Hausdorff space and E be a vector bundle over B. Then there exists a vector bundle E' over B such that $E \oplus E'$ is a trivial bundle over B.*

The rest of this section will be used to introduce the classification of vector bundles. From now on, we will only consider complex vector bundles, which means vector bundles over $\mathbb{C}$, but the story we describe in the next few sections has a complete analogue for real vector bundles.

Definition 1.5. *Let G be a topological group. Suppose EG is any weakly contractible space with a continuous free G-action, then BG is defined as the quotient of EG by this action. This is called the classifying space of G. This definition is well-defined up to equivalence.*

Example 1.6. *From the definition, we can easily observe there is always a fibration sequence:*

$$G \to EG \to BG.$$

When $G = S^1$, this fiber sequence along with the long exact sequence of homotopy groups and $\mathbb{C}P^\infty$ is $K(G, 2)$ implies that $BG = \mathbb{C}P^\infty$.

Proposition 1.7. *[6] When $G = U(n)$, which is the group of all unitary $n \times n$-matrices, the classifying space $BU(n)$ is Gr_n, and $Gr_n := Gr_n(\mathbb{C}^\infty)$ is defined as the colimit of a sequence of Grassmannian manifolds:*

$$Gr_n(\mathbb{C}^{n+1}) \subset Gr_n(\mathbb{C}^{n+2}) \subset \cdots \subset Gr_n(\mathbb{C}^{n+k}) \subset \cdots$$

The classification of vector bundles is the following theorem:

Theorem 1.8. *[6] Let B be a paracompact space and Vect$^n(B)$ be the set of isomorphism classes of n-dimensional vector bundles. Then there is a bijection between Vect$^n(B)$ and the set of homotopy class of maps $[B, BU(n)]$.*

Proof. We will not give the whole proof, but let us give the maps between them. Camparing to the tautological line bundle, we can define a universal bundle γ_n over $BU(n) = Gr_n$ as:

$$\gamma_n = \{(\omega, v) \in Gr_n \times \mathbb{C}^\infty | v \in \omega\}.$$

Then this universal bundle induces a map:

$$\begin{aligned} \Phi : [B, BU(n)] &\to \text{Vect}^n(B) \\ f &\mapsto f^*\gamma_n \end{aligned}$$

By checking this map being both injective and surjective, we can finish the proof. □

2 Topological K-Theory

In this section, we shall give a short introduction to topological K-theory. We begin by introducing the definition of K-groups. From this section, we assume all the spaces are CW-complexes since our main concern is homotopy theory. Notice that CW-complexes are always Hausdorff.

Definition 2.1. *Let X be a compact CW-complex. We can consider the set of formal differences of vector bundles over X:*

$$L = \{E_1 - E_2 | E_1, E_2 \in \text{Vect}(X)\}$$

In this set, we define

$$(E_1 - E_2) + (E_1' - E_2') = E_1 \oplus E_1' - E_2 \oplus E_2',$$

and we define an equivalence relation on L by

$$E_1 - E_2 \sim E_1' - E_2' \qquad \textit{iff} \qquad \exists n \geq 0, s.t. E_1 \oplus E_2' \oplus \epsilon^n \cong E_1' \oplus E_2 \oplus \epsilon^n.$$

Then $K(X)$ is an abelian group $L/\sim$ with the addition above.

Remark 2.2. *For any element $E - E' \in K(X)$. Since X is compact Hausdorff, we know by Proposition 1.4 there exists $F' \in$ Vect (X) such that $E' \oplus F' \cong \epsilon^n$ for some n, so we have $E - E' = E \oplus F' - \epsilon^n$. Hence, every element of $K(X)$ can be written as the form $E - \epsilon^n$.*

Besides defining K-group on the set of formal differences, we can also define a reduced K-group on the actual set of vector bundles modulo some equivalence:

Definition 2.3. *Let X be a compact CW-complex. For any two vector bundles E, E' over X, we define $E \sim E'$ if there exists two trivial bundles ϵ^n and ϵ^m s.t. $E \oplus \epsilon^m \cong E' \oplus \epsilon^n$. The group $\tilde{K}(X)$ is defined to be the set of vector bundles over X modulo the equivalence relation with addition defined to be the direct sum of vector bundles.*

Remark 2.4. *Notice any two trivial bundles will be equivalent in $\tilde{K}(X)$, and it is the identity element of the group. The inverse elements exist by Proposition 1.4.*

Proposition 2.5. *Both K and $\tilde{K}$ are contravariant functors from the category of compact CW-complexes to the category of abelian groups. Moreover, with choosing a basepoint of X, we have*

$$\tilde{K}(X) \cong \textit{Ker}\,(K(X) \to K(*) \cong \mathbb{Z}),$$

which means $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$.

Proof. The fact that they are natural with a map between spaces will be clear when we describe K-groups from KU-spectrums. Let $f : X \to Y$ be a map between spaces, then the map from $K(Y) \to K(X)$ and $\tilde{K}(Y) \to \tilde{K}(X)$ are also defined in the level of vector bundles, which is simply by taking pullbacks.

The inclusion $* \to X$ then induces the following map:

$$\begin{aligned} K(X) \to \quad & K(*) \cong \mathbb{Z} \\ E - \epsilon^n \mapsto \quad & \text{dim}(E) - n \end{aligned}$$

Hence, the kernel of this map consists of elements $\{E - \epsilon^{\text{dim}(E)} \in K(X)\}$. If we omit the part of trivial bundles, then the equivalence relation of $K(X)$ becomes the equivalence relation in $\tilde{K}(X)$. Hence, the kernel is just $\tilde{K}(X)$ since they are the same set with the same operations. $\square$

The tensor product of vector bundles induces a natural ring structure on $K(X)$ defined as the following:

$$(E - E')(F - F') = E \otimes F + E' \otimes F' - E \otimes F' - E' \otimes F.$$

Using basic properties about tensor products and direct sums, it's not hard to check that $K(X)$ is a commutative ring with identity being the trivial line bundle ϵ^1. Then, by the above proposition, $\tilde{K}(X)$ is also a commutative ring, although identity may not exist in this situation. Notice that K and $\tilde{K}$ are still functors with the ring structures.

Besides the inner product, there is also an external product of K-groups:

Definition 2.6. *Let X, Y be two compact CW-complexes. We then can define the external product as follows:*

$$\begin{aligned} \mu : K(X) \otimes K(Y) \quad & \to K(X \times Y) \\ a \otimes b \quad & \mapsto p_X^*(a) p_Y^*(b) \end{aligned}$$

where p_X and p_Y are projections from $X \times Y$ to X and Y.

Let $X = S^2 = \mathbb{C}P^1$. We know there is a tautological line bundle H over X defined by

$$H = \{(v, x) \in \mathbb{C}P^1 \times \mathbb{C}^2 | x \in v\}.$$

Using clutching functions, which are explained in Hatcher [5], one can show that

$$H \otimes H \oplus \epsilon^1 = H \oplus H.$$

Write this relation in $K(S^2)$, we then have

$$(H-1)^2 = 0.$$

Using this line bundle and external product, we have the fundamental product theorem:

Theorem 2.7. *[5] The external product*

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$$

is an isomorphism for any compact CW-complexes X.

In particular, when X is a point, we have the K-group of S^2. This can be seen as the main technical step of Bott periodicity, which states that $\tilde{K}(X) \cong \tilde{K}(\Sigma^2 X)$. In the next section, we shall describe it from the perspective of the Ω-spectrum, and this theorem will be used to produce the isomorphism.

3 Bott Periodicity and Spectrum of K-theory

Recall that an Ω-spectrum is referred to as a sequence of CW-complexes indexing over $\mathbb{Z}$ such that for each $i \in \mathbb{Z}$, there is an equivalence $X_i \to \Omega X_{i+1}$. We have Brown's representability theorem describing the equivalence between Ω-spectrum and reduced cohomology theory. By reduced cohomology theory, we mean a collection of functors $h^i : CW_*^{op} \to$ AbGp with natural suspension isomorphisms $\sigma_i : h^{i+1}(\Sigma X) \to h^i(X)$ satisfying axioms of exactness, additivity and homotopy invariance. We do not give an explicit description, but they are easy to generalize from the singular cohomology. For more details, one can look up in section 4.3 of Hatcher's Algebraic Topology.

Theorem 3.1. *[4] (Brown's representability theorem) There is a one-to-one correspondence between Ω-spectrum and reduced cohomology theory. If $\{X_i\}_{i\in\mathbb{Z}}$ is an Ω-spectrum, then the corresponding ith cohomology maps a pointed CW-complex X to the pointed homotopy classes $\langle X, X_i \rangle$.*

We now move forward to Bott periodicity, which relies on essentially the following theorem:

Theorem 3.2. *[6] There is an equivalence*

$$\Phi : BU \times \mathbb{Z} \to \Omega U$$

where U is the group of unitary matrixes.

And we also need a simple lemma:

Lemma 3.3. *For any topological group G, we then have $\pi_{i+1}(G) \cong \pi_i(\Omega G)$ and $\pi_{i+1}(BG) \cong \pi_i(G)$ for all $i \geq 0$. In particular, we know ΩBG is equivalent to G.*

Proof. The relations of homotopy groups are just from long exact sequences of the following two fiber sequences:

$$G \to EG \to BG;$$
$$\Omega G \to PG \to G.$$

To prove the equivalence, we use the second fiber sequence on BG, and it gives

$$\Omega(BG) \to P(BG) \to BG.$$

Since EG is contractible, there is a homotopy $H : EG \times I \to EG$ with $H_0 = *$ and $H_1 = id$. This gives rise to a map $\Phi : EG \to P(BG)$ by $\Phi(x)(t) = [H(x,t)] \in BG$ for any $x \in EG$ and $t \in I$, and it induces a map of homotopy groups between the two fiber sequences. By comparing the long exact sequences, it's easy to see Φ is indeed an equivalence. $\square$

Using the above lemma and theorem, we can get Bott Periodicity:

Theorem 3.4. *For any $i \geq 0$, $\pi_i(U) \cong \pi_{i+2}(U)$*

Proof. By the above theorem and lemma, we simply have

$$\pi_i(U) \cong \pi_i(\Omega(BU)) = \pi_i(\Omega(BU \times \mathbb{Z})) \cong \pi_i(\Omega^2 U) \cong \pi_{i+2}(U).$$

$\square$

Moreover, the theorem and the lemma also indicate that we can define the following spectrum:

Definition 3.5. *We define an Ω-spectrum KU by*

$$KU_i = \begin{cases} BU \times \mathbb{Z} & i \text{ even} \\ U & i \text{ odd} \end{cases}$$

This is called the complex K-theory spectrum. By Theorem 3.1, we denote the associated reduced cohomology theory as $\tilde{K}^i(X)$ for any point space X, and it also induces an unreduced cohomology theory by defining $K^i(X) = \tilde{K}^i(X_+)$.

Using Theorem 3.1, this spectrum is associated with a reduced cohomology theory, which we denote as $\tilde{K}^i$. The following proposition says that it matches with the reduced K-group we defined in section 2:

Proposition 3.6. *[6] For any compact pointed CW-complex X, we have*

$$\tilde{K}(X) \cong \tilde{K}^0(X) = \langle X, BU \times \mathbb{Z} \rangle,$$

$$K(X) \cong \tilde{K}^0(X_+) = \langle X_+, BU \times \mathbb{Z} \rangle,$$

where X_+ means the disjoint union of X and a single point, and $< \cdot, \cdot >$ means the group of pointed homotopy class.

Notice that if we can prove the first one, the second one can be easily deduced because of Proposition 2.5 and the cofiber sequence $X \to X_+ \to *$. The first equality can be proved essentially by the classification of vector bundles since X being compact means that the image of any map in the right side lands in some $BU(n)$ for each component. One only needs to carefully check the correspondence in the classification theorem is compatible with the definition of reduced K-groups.

Remark 3.7. *By the above proposition and the adjunction $\Sigma \dashv \Omega$, the group $\tilde{K}^i(X)$ can also be written as*

$$\tilde{K}^i(X) = \begin{cases} \tilde{K}(X) & i \text{ even} \\ \tilde{K}(\Sigma X) & i \text{ odd} \end{cases}$$

From the perspective of Theorem 3.1, we really use this section to say that the reduced K-group is a reduced cohomology theory. Using this, one can actually show the external product on K-groups deduces an external product on reduced K-groups:

Theorem 3.8. *There is a commutative diagram as follows:*

$$\begin{tikzcd}
K(X) \otimes K(Y) \arrow[r, "\cong"] \arrow[d, "\mu"] & (\tilde{K}(X) \otimes \tilde{K}(Y)) \arrow[d, "\tilde{\mu}"] & \oplus \tilde{K}(X) \arrow[d, equal] & \oplus \tilde{K}(Y) \arrow[d, equal] & \oplus \mathbb{Z} \arrow[d, equal] \\
K(X \times Y) \arrow[r, "\cong"] & \tilde{K}(X \wedge Y) & \oplus \tilde{K}(X) & \oplus \tilde{K}(Y) & \oplus \mathbb{Z}
\end{tikzcd}$$

for any two compact CW-complexes X and Y. The map $\tilde{\mu}$ is just the external product of reduced K-groups, which is essentially the restriction of μ.

Proof. The first row is true because we have

$$K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}.$$

By writing the multiplication out, we can see the isomorphism. For the second row, we have a cofiber sequence

$$X \vee Y \to X \times Y \to X \wedge Y$$

by definition. This gives a long exact sequence as follows:

$$\tilde{K}(\Sigma(X \times Y)) \to \tilde{K}(\Sigma(X \vee Y)) \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)$$

Since $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$. By additivity, the sequence becomes

$$\tilde{K}(\Sigma(X \times Y)) \to \tilde{K}(\Sigma X) \oplus \tilde{K}(\Sigma Y) \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$$

We have in space level $X \to X \times Y \to X$ is id_x, where the first map is inclusion and the second map is projection. This and the similar map for Y induce a right inverse of the first map and the last map in the sequence above, which means they are surjective. Hence, we have the last three terms is a split short exact sequence, i.e.

$$\tilde{K}(X \times Y) \cong \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y)$$

Let $a \in \tilde{K}(X) = \text{Ker}(K(X) \to K(x_0))$ and $b \in \tilde{K}(Y) = \text{Ker}(K(Y) \to K(y_0))$. We want to show $\mu(a \otimes b) \in \tilde{K}(X \wedge Y)$. To do this, we denote i_X, i_Y as inclusions from X and Y to $X \vee Y \subset X \times Y$, and p_X, p_Y as projections from $X \times Y$ to X and Y. Then, we know

$$i_Y^* \circ p_X^*(a) = i_X^* \circ p_Y^*(b) = 0$$

Notice they are also ring isomorphisms, so we have

$$i_Y^*(p_X^*(a)p_Y^*(b)) = i_X^*(p_X^*(a)p_Y^*(b)) = 0$$

Hence, $\mu(a \otimes b) = p_X^*(a)p_Y^*(b)$ is 0 when projecting down to $\tilde{K}(X) \oplus \tilde{K}(Y)$, and similarly to the summand $\mathbb{Z} = \tilde{K}((x_0, y_0))$. Hence, by the direct sum relation, we finish the proof. □

Using the external product, we can give another version of Bott periodicity:

Theorem 3.9. *For any compact CW-complex X, we have an isomorphism:*

$$\begin{aligned} \beta : \tilde{K}(X) &\to \tilde{K}(S^2 \wedge X) \cong \tilde{K}^2(X) \\ a &\mapsto (H-1) * a \end{aligned}$$

Proof. Notice β is the composition:

$$\tilde{K}(X) \to \tilde{K}(S^2) \otimes \tilde{K}(X) \to \tilde{K}(S^2 \wedge X),$$

where the first map is just tensoring with $(H-1)$, and the second map is the external product. The first map is an isomorphism because $\tilde{K}(S^2) \cong \mathbb{Z}$, and the second map is an isomorphism because of Theorem 2.7. □

4 Adams Operations

In this section, we will construct Adams operations on K-groups. This is the key ingredient to attacking the Hopf invariant one problem. Using these operations, K-groups become sharper invariants. To construct them, we will firstly list some basic properties about exterior powers of vector bundles:

Proposition 4.1. *[5] For a vector bundle $p : E \to X$, we can construct its k-th exterior power $\lambda^k(E)$, whose fiber is the k-th exterior power of vector space $\lambda^k(p^{-1}(x))$ for every $x \in X$, and it satisfies the following basic properties:*

(1) $\lambda^k(E_1 \oplus E_2) = \oplus_i(\lambda^i(E_1) \otimes \lambda^{k-i}(E_2))$.

(2) $\lambda^0(E) = \epsilon^1$, which is the trivial line bundle.

(3) $\lambda^1(E) = E$.

(4) $\lambda^k(E) = 0$ for k greater than the maximal dimension of fibers of E.

We now list the splitting principle, which will be used to construct Adams operations. Later, we will also use it in the discussion of Chern classes and Chern characters.

Theorem 4.2. *[5] (Splitting Principle) Let E be a vector bundle over X with X being compact Hausdorff. Then there exists a compact Hausdorff space $F(E)$ with map $p : F(E) \to X$ such that $p^* : K(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

Let us discuss how the operation should be defined. Since $K(X)$ is a commutative ring, we would naturally want ψ^k to be a ring homomorphism. Furthermore, these operations should generalize the exterior power, which means the following property:

If L is a line bundle, then $\psi^k(L) = L^k$.

With these two desired properties, for $E = L_1 \oplus L_2 \oplus \cdots \oplus L_k$, we want

$$\psi^k(E) = L_1^k + \cdots + L_n^k \tag{1}$$

To inspire the definition, we will express the above equation in a polynomial of E. Then, in the spirit of the splitting principle, we know it would be the right definition.

We define $\lambda_t(E) = \sum_i \lambda^i(E)t^i \in K(X)[t]$. This is always a finite sum by Proposition 4.1. Moreover, the proposition says $\lambda_t(E_1 \oplus E_2) = \lambda_t(E_1)\lambda_t(E_2)$ Hence, in the situation $E = L_1 + \cdots + L_n$, we have

$$\lambda_t(E) = \lambda_t(L_1) \cdots \lambda_t(L_n) = \prod_{i=1}^{n}(1 + L_i t) = 1 + \sigma_1 t + \cdots + \sigma_n t^n,$$

where σ_i is the i-th elementary symmetric polynomial in $L_1, \cdots L_n$. so, we have

$$\lambda^k(E) = \sigma_k(L_1, \cdots, L_n).$$

From the theory of elementary symmetric polynomials, we know there is a collection of Newton polynomials s_k in k variables, characterized by the following property: for any $n \geq 1$,

$$s_k(\sigma_1(t_1, \cdots, t_n), \cdots, \sigma_k(t_1, \cdots, t_n)) = t_1^k + \cdots t_n^k.$$

Hence, we can rewrite $\psi^k(E)$ in the following equation:

$$\psi^k(E) = s_k(\lambda^1(E), \cdots, \lambda^k(E)).$$

This is exactly what we want to define:

Definition 4.3. *Let X be a compact CW-complex. we define for $k \geq 0$ a collection of maps $\psi^k : K(X) \to K(X)$ by*

(1) $\psi^0 = 0$

(2) $\psi^k(E - F) = s_k(\lambda^1(E), \cdots, \lambda^k(E)) - s_k(\lambda^1(F), \cdots, \lambda^k(F))$, for any $E, F \in K(X)$.

Notice that this definition is well-defined only if after checking its additivity, which is included in the following proposition:

Proposition 4.4. *For X being compact CW-complex, the operations ψ^k are natural ring homomorphisms satisfying the following properties:*

(1) $\psi^k(L) = L^k$ for any line bundles L;

(2) $\psi^k \psi^l = \psi^{kl}$;

(3) $\psi^p(\alpha) \equiv \alpha^p$ mod $pK(X)$ for any $\alpha \in K(X)$ and any prime p.

Proof. The naturality follows from the naturality of exterior towers. i.e. for any map $f : X \to Y$, we have $f^*\lambda^k(E) = \lambda^k(f^*(E))$.

When E_1 and E_2 can split as sums of line bundles, the additivity is clear since in this case, the Adams operation can be written as equation (1), and they clearly satisfy the additivity. For general cases, we use the splitting principle firstly on E_1, with map $p_1 : F(E_1) \to X$ making p^*E_1 splits. Then apply the principle to $p_1^*E_2$ and get a map $p_2 : F(E_2) \to F(E_1)$. Composing the two maps and denoting $p = p_1 \circ p_2$, we then know p^*E_1 and p^*E_2 both splits as line bundles. By naturality, we have

$$p^*(\psi^k(E_1 \oplus E_2)) = \psi^k(p^*E_1 \oplus p^*E_2) = \psi^k(p^*E_1) \oplus \psi^k(p^*E_2) = p^*(\psi^k(E_1) \oplus \psi^k(E_2))$$

Since p^* is injective, we then deduce the additivity. By the same procedure, we can observe that it's enough to show the rest properties for E splits as a sum of line bundles. And they are direct computations by expressing $\psi^k(E)$ in equation (1) $\square$

Remark 4.5. *Adams operations can also be defined on reduced K-groups by using naturality and Proposition 2.5. Moreover, Adams operations commute with the external product also by naturality.*

At the end of this section, we calculate Adams operations on even-dimensional spheres.

Proposition 4.6. *The operation $\psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ is multiplication by k^n.*

Proof. When $n = 0$, there are only trivial bundles, then it's obviously true by definitions. When $n = 1$, Theorem 3.9 shows $\tilde{K}(S^2) = \mathbb{Z}[H - 1]$ with H being the tautological line bundle. Hence, we have

$$\psi^k(H - 1) = H^k - 1 = (1 + H - 1)^k - 1 = 1 + k(H - 1) - 1 = k(H - 1)$$

since $(H - 1)^i = 0$ for $i \geq 2$. For higher dimensional cases, we know again by Theorem 3.9 that $\tilde{K}(S^{2n}) = \mathbb{Z}[(H - 1)^n]$ (Notice here the product is the external product). Using the fact that ψ^k commutes with the external product, we can easily prove the formula. $\square$

5 Cohomology of U(n)

In sections 5 and 6, we shall calculate the cohomology of $U(n)$ and $BU(n)$ by Serre spectral sequence, which is explained in [7]. These calculation results will be used to define characteristic classes later. People who are not familiar with spectral sequences can read the result of Theorem 6.2 and directly go to section 7.

Theorem 5.1.

$$H^*(U(n)) = \bigwedge\nolimits_{\mathbb{Z}}[x_1, x_3, \cdots, x_{2n-1}],$$

where the degree of x_i is i. To be more explicit, this means the cohomology ring $H^*(U(n))$ is an exterior algebra spanned by $x_1, \cdots, x_{2n-1}$.

Proof. We will prove this by induction on n. When $n = 1$, we know by definition $U(1) = S^1$, so it obviously satisfies the theorem. We then assume $H^*(U(n)) = \bigwedge_{\mathbb{Z}}[x_1, \cdots, x_{2n-1}]$, and prove the case for $n + 1$.

From the theory of Lie groups, we know $U(n + 1)$ acts transitively on S^{2n+1} with stabilizer $U(n)$, so we have $U(n + 1)/U(n) = S^{2n+1}$, so that we have a fibration

$$U(n) \to U(n + 1) \to S^{2n+1}.$$

Since S^{2n+1} is simply connected, the fiber sequence is associated with a Serre spectral sequence

$$E_2^{p,q} = H^p(S^{2n+1}; H^q(U(n))) \Rightarrow H^{p+q}(U(n + 1))$$

If we denote $H^{2n+1}(S^{2n+1}) = \mathbb{Z}[x]$, then the spectral sequence looks like the following:

$$\begin{array}{c|cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2n-1 & x_{2n-1} & 0 & 0 & \cdots & x_{2n-1}x & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
3 & x_3 & 0 & 0 & \cdots & x_3 x & 0 \\
2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & x_1 & 0 & 0 & \cdots & x_1 x & 0 \\
0 & 1 & 0 & 0 & \cdots & x & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
 & 0 & 1 & 2 & \cdots & 2n+1 & 2n+2
\end{array}$$

By degree reason, we know $d^k = 0$ for $k < 2n+1$. Similarly, we can also see in page $2n+1$,

$$d^{2n+1}(x_i) = 0, \qquad 1 \leq i \leq 2n-1.$$

For any element x in $E_2^{0,q}$ with $q \geq 2n$, it can be written as a sum of products in x_i's since $x \in \bigwedge_{\mathbb{Z}}[x_1, ..., x_{2n-1}]$. Then by the differential formula $d(rs) = (dr)s + (-1)^{|r|}r(ds)$, we deduce that $d^{2n+1} = 0$ from any group. This means $E^2_{p,q} = E^\infty_{p,q}$. Since there is no torsion in this spectral sequence, the cohomology group of $H^*(U(n+1))$ are just direct sums of the diagonals. Hence, we can still use $x_1, ..., x_{2n-1}$ to be the first $2n-1$ generators of $H^*(U(n+1))$. In the diagonal $p+q = 2n+1$, there is a inclusion map from $E_2^{2n+1,0}$ to $H^{2n+1}(U(n))$ induced by filtration, we take the image of x as x_{2n+1}, which is the last generator.

From this spectral sequence, we can easily see that $H^*(U(n+1))$ is a free $\mathbb{Z}$-module with the basis being finite products $x_{i_1} \cdots x_{i_k}$ with $i_1 < \cdots < i_k$. So to see they induce the exterior algebra, we only need to observe:

1. $x_i^2 = 0$ for $i = 1, \cdots, 2n+1$.
2. $x_i x_j = -x_j x_i$ for $i, j = 1, \cdots, 2n+1$.

Both are true because all the x_i's are in odd degrees. The commutativity of cup product implies the second equation. By further noticing there is no torsion in the cohomology group $H^*(BU(n))$, we know $2x_i^2 = 0$ implies $x_i^2 = 0$. □

6 Cohomology of BU(n)

Theorem 6.1. *(Thom-Gysin Sequence) Let $S^n \to E \to B$ be a Serre fibration over a simply connected CW-complex, then there exists $c \in H^{n+1}(E)$ s.t. it gives a long exact sequence of the form:*

$$\cdots \longrightarrow H^k(B) \xrightarrow{c\cup(-)} H^{1+k+n}(B) \xrightarrow{p^*} H^{1+k+n}(E) \longrightarrow H^{k+1}(B) \xrightarrow{c\cup(-)} \cdots$$

where $p : E \to B$ is the projection. Moreover, we know $c = d^{n+1}(x)$, where $d^{n+1} : E^{n+1}_{0,n} \to E^{n+1}_{n+1,0}$ is the differential map in the Serre spectral sequence, and x is the fundamental class of S^n.

Proof. Since B is simply connected, we can apply Serre cohomological spectral sequence to this fibration. which looks like the following graph:

$$
\begin{array}{c|ccccccc}
n+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n & x & xH^1 & xH^2 & \cdots & xH^{n+1} & xH^{n+2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 1 & H^1 & H^2 & \cdots & H^{n+1} & H^{n+2} & \cdots \\
\hline
 & 0 & 1 & 2 & \cdots & n+1 & n+2 & \cdots
\end{array}
$$

In the graph, H^i is the notation for $H^i(B)$. For each $k \in \mathbb{Z}$, we can get the following short exact sequences:

$$0 \longrightarrow \mathsf{Ker}(d_k^{n+1}) \longrightarrow H^k(B) \longrightarrow \mathsf{Im}(d_k^{n+1}) \longrightarrow 0;$$

$$0 \longrightarrow \mathsf{Im}(d_k^{n+1}) \longrightarrow H^{k+n+1}(B) \longrightarrow H^{k+n+1}(B)/\mathsf{Im}(d_k^{n+1}) \longrightarrow 0;$$

$$0 \longrightarrow H^{k+n+1}(B)/\mathsf{Im}(d_k^{n+1}) \longrightarrow H^{k+n+1}(E) \longrightarrow \mathsf{Ker}(d_{k+1}^{n+1}) \longrightarrow 0.$$

The first two sequences are direct, and the last one is by noticing d^{n+1} is the only possible non-zero differential in the spectral sequence, so the next page is the E_∞-page, and this short exact sequence is obtained by analyzing the construction of filtration. Since we have these three short exact sequences for all k, they all together induce the long exact sequence above. Now, we just need to see what are the maps.

$H^{k+n+1}(B) \to H^{k+n+1}(E)$: To see this is p^*, we use the following map of fibrations:

$$
\begin{tikzcd}
S^n \arrow[r] \arrow[d] & E \arrow[r, "p"] \arrow[d, "p"] & B \arrow[d, "id"] \\
* \arrow[r] & B \arrow[r, "id"] & B
\end{tikzcd}
$$

This induces a map of spectral sequences. On E^2-page, it is the identity for row 0. Passing to the infinity page, it gives the commutative square:

$$\begin{array}{ccc} H^{k+n+1}(B) & \xrightarrow{id.} & H^{k+n+1}(B) \\ \downarrow & & \downarrow p^* \\ H^{k+n+1}(B)/\mathrm{Im}(d_{n+1}^k) & \longrightarrow & H^{k+n+1}(E) \end{array}$$

Notice that the horizontal map means the map $E_\infty^{k+n+1,0} \hookrightarrow H^{k+n+1}$(total space). This proves the map in the long exact sequence is just p^*.

$H^k(B) \to H^{1+k+n}(B)$: From the first two short exact sequences, we can observe that this map is just d_{n+1}^k. Since its domain is $xH^k(B)$, we can write the generator as xe, where $e \in H^k(B)$. It follows that $d_{2n+1}(xe) = d(x)e + (-1)^n xd(e)$. Noticing $d(e) = 0$ by degree reason, we have proven the map is just the cup product.

□

Theorem 6.2.

$$H^*(BU(n)) = \mathbb{Z}[c_1, \cdots, c_n],$$

where the degree of c_i is $2i$.

Proof. Again, we will prove this by induction. When $n = 1$, we know $U(1) = S^1$, and it is well-known that $BU(1) = \mathbb{C}P^\infty$. So it satisfies the theorem. We assume $H^*(BU(n-1)) = \mathbb{Z}[c_1, \cdots, c_{n-1}]$ and prove the case for $BU(n)$. We need to prove the following fibration exists:

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

To see this, some information on how the $BU(n)$ is constructed is needed. By Proposition 1.7, we know $BU(n) \simeq Gr_n(\mathbb{C}^\infty)$, and it is associated with the universal bundle γ_n with fiber $\mathbb{C}^n$. From this, we can construct a sphere bundle

$$S(\gamma_n) = \{(\omega, v) \in \gamma_n | v \in \omega, ||v|| = 1\}$$

over Gr_n with fiber S^{2n-1}. This is a fiber bundle, so it is a fibration since Gr_n is paracompact. We have a natural projection $p : S(\gamma_n) \to Gr_{n-1}$ by defining $p(\omega, v) = \omega \cap v^\perp$. It defines a fiber bundle with fiber S^∞, because the preimage of a point consists of all unit vectors orthogonal to $\omega \cap v^\perp$. Since S^∞ is contractible, p induces isomorphisms on all homotopy groups, which means it is an equivalence. Substituting $S(\gamma_n)$ with Gr_{n-1} in the fiber bundle $S(\gamma_n) \to BU(n)$, we get the fibration we desired.

Now, we can apply the Gysin sequence to our fibration, and it induces a long exact sequence:

$$\cdots \longrightarrow H^k(BU(n)) \xrightarrow{c\cup(-)} H^{k+2n}(BU(n)) \xrightarrow{p^*} H^{k+2n}(BU(n-1))$$

$$\longrightarrow H^{k+1}(BU(n)) \xrightarrow{c\cup(-)} \cdots$$

When $k \leq -2$, the long exact sequence above implies p^* is an isomorphism. Since we assumed that $H^*(BU(n-1)) = \mathbb{Z}[c_1, ..., c_{n-1}]$, the generators $c_1, ..., c_{n-1}$ induce generators in $H^*(BU(n))$, and we will use the same name. For $k \geq -1$, since p^* is a ring homomorphism, and we know any element in $H^{k+2n}(BU(n-1))$ is a sum of products of c_i's, so p^* must be a surjective map, this implies the next map after p^* is the zero map and $c \cup (-)$ is an injective map. Hence, the long exact sequence is split into the following short exact sequences:

$$0 \longrightarrow H^k(BU(n)) \xrightarrow{c \cup (-)} H^{k+2n}(BU(n)) \xrightarrow{p^*} H^{k+2n}(BU(n-1)) \longrightarrow 0$$

If we denote c as $c_n \in H^{2n}(BU(n))$, it follows that $H^m(BU(n))$ is just the free module with the basis being all possible products $c_{i_1} \cdots c_{i_k}$, where $i_1 + \cdots + i_k = m$. Hence, we know $H^*(BU(n))$ is the polynomial algebra in $c_1, ..., c_n$. $\square$

7 Chern Classes and Chern Character

Let B be a paracompact space, which means the classification of vector bundles is true. We define Chern classes as follows:

Definition 7.1. *Let $E \to B$ be a complex n-dimensional vector bundle, with the classifying map $f : B \to BU(n)$. The* **i-th Chern class** *$c_i(E) \in H^{2i}(B; \mathbb{Z})$ is defined as*

(1) $c_0(E) = 1$;

(2) $c_i(E) = 0$ for $i > n$;

(3) $c_i(E) = f^*(c_i)$ for $1 \leq i \leq n$, where $c_1, \cdots, c_n$ are the generators of $H^*(BU(n))$.

The total Chern class $c(E)$ is defined as $c_0(E) + c_1(E) + c_2(E) + \cdots$.

This definition can be realized by a set of axioms, in which we state as the following theorem:

Theorem 7.2. *[5] There is a unique sequence of functions $c_0, c_1, c_2, \cdots$ assigning each complex vector bundle $E \to B$ to a class $c_i(E) \in H^{2i}(B; \mathbb{Z})$, depending only on the isomorphism class of E, such that*

(a) $c_0(E) = 1$ for any vector bundle E over B

(b)(Naturality) $c_i(f^*(E)) = f^*(c_i(E))$ for any map $f : B' \to B$.

(c)(Whitney sum formula)$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$ for $c = c_0 + c_1 + c_2 + \cdots \in H^*(B; \mathbb{Z})$.

(d) $c_i(E) = 0$ for $i > dimE$.

(e) For the universal line bundle $E \to \mathbb{C}P^\infty \simeq BU(1)$, $c_1(E)$ is a generator of $H^2(BU(1); \mathbb{Z})$.

By the above theorem, we can calculate some easy examples:

Example 7.3. *Let ϵ^k be a trivial vector bundle, we have $c(\epsilon^k) = 1$. This is because we always have the following pullback diagram:*

$$\begin{tikzcd}
\epsilon^k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}^k \arrow[d] \\
B \arrow[r] & \{*\}
\end{tikzcd}$$

By naturality and $H^i(*) = 0$ for $i \geq 1$, we get the result.

Example 7.4. *Combining the example above and the Whitney sum formula, we know for any vector bundle $E \to B$, $c_i(E) = c_i(E \oplus \epsilon^k)$.*

Example 7.5. *Let H be the tautological line bundle over $S^2 = \mathbb{C}P^1$. Then H is the pullback of the following diagram*

$$\begin{tikzcd}
H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_1 \arrow[d] \\
\mathbb{C}P^1 \arrow[r] & \mathbb{C}P^\infty
\end{tikzcd}$$

where the bottom map is inclusion. Then, using the definition, we know $c_1(E)$ is a generator of $H^2(S^2)$.

The total Chern class is a good invariant to dig useful information on vector bundles. However, one drawback is that it's not a ring homomorphism. That's why we want to define the Chern Character.

Definition 7.6. (Theorem) *[5] The Chern character is a natural ring homomorphism:*

$$ch : K^0(X) \to \prod_{i=0}^{\infty} H^{2i}(X; \mathbb{Q}) := H^{even}(X; \mathbb{Q}).$$

The ring $H^{even}(X; \mathbb{Q})$ should be thought of as a ring of formal power series, which means, an element in the ring is $x = a_0 + a_1 u + a_2 u^2 + \cdots$, where $a_i \in H^{2i}(X; \mathbb{Q})$ and u is a formal variable. This ring homomorphism is characterized by the following property:
For any line bundle L, we have

$$ch(L) = e^{uc_1} = 1 + c_1 u + \frac{c_1^2}{2!} u^2 + \cdots + \frac{c_1^n}{n!} u^n + \cdots$$

where c_1 is the first Chern class of L.

Proof. We will not give the whole proof, but it's not hard to derive the right definition of the Chern characters. When $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, we know

$$ch(E) = \sum_i e^{t_i u} = dim(E) + \left(\sum_i t_i\right) u + \cdots + \frac{\sum_i t_i^k}{k!} u^k + \cdots ,$$

where $t_i = c_1(L_i)$. By Whitney sum formula, we also know

$$\begin{aligned} c(E) &= (1+t_1)\cdots(1+t_n) \\ &= 1+\sigma_1+\sigma_2+\cdots\sigma_n \end{aligned}$$

,

where σ_i is the i-th elementary symmetric polynomial in the elements t_i's. The above equality shows that $\sigma_i = c_i(E)$. Now, if we use the Newton polynomial s_k, which is characterized by the equality $t_1^k + \cdots + t_n^k = s_k(\sigma_1, \cdots, \sigma_k)$, then we can rewrite $ch(E)$ as follows:

$$ch(E) = dim(E) + \sum_{k>0} \frac{s_k(c_1(E), \cdots, c_k(E))}{k!} u^k.$$

One can use this as the definition of the Chern character of vector bundles. In the sprit of splitting principle (Proposition 4.2), it's easy to check that this is exactly the definition we want. $\square$

Remark 7.7. *By naturality, the Chern character also induces a ring homomorphism between $\tilde{K}(X)$ and $\tilde{H}^{even}(X;\mathbb{Q})$.*

As an example, we calculate the Chern character of even-dimensional spheres:

Proposition 7.8. *For $n \geq 1$, the Chern character*

$$ch : \tilde{K}(S^{2n}) \to \tilde{H}^{even}(S^{2n};\mathbb{Q}) = \tilde{H}^{2n}(S^{2n};\mathbb{Q})$$

is an injective map, and its image is $\tilde{H}^{2n}(S^{2n};\mathbb{Z}) \subset \tilde{H}^{2n}(S^{2n};\mathbb{Q})$.

Proof. Let H denote the tautological line bundle over S^2. Then we know from K-theory that $\tilde{K}(S^{2n}) = \mathbb{Z}[H-1]$. By naturality of the Chern character, we get the following commutative diagram:

$$\begin{array}{ccc} \tilde{K}(X) & \xrightarrow{\cong} & \tilde{K}(S^2X) \\ \big\downarrow{\scriptstyle ch} & & \big\downarrow{\scriptstyle ch} \\ \tilde{H}^*(X;\mathbb{Q}) & \xrightarrow{\cong} & \tilde{H}^*(S^2X;\mathbb{Q}) \end{array}$$

The upper map is isomorphism because of the Bott periodicity, and we also know the map is just the external product with $H-1$. The lower map is an isomorphism even restricting to the cohomology group with coefficient $\mathbb{Z}$. Now, we can prove this proposition by induction. When $n=1$, we have $ch(H-1) = ch(H) - ch(1) = 1 + c_1(E) - 1 = c_1(E)$. By Example 7.5, this is a generator of $\tilde{H}^2(S^2;\mathbb{Z})$. Take X to be S^{2n} in the diagram above, then the induction step is also clear. $\square$

8 Hopf Invariant One Problem

In this section, we will introduce how the problem is defined and how is translated into a K- theory problem. Let

$$f : S^{2n-1} \to S^n$$

be a continuous map for $n > 1$. This map induces a cell complex $C_f = e_{2n} \cup_f S^n$. Then because of its cell structure, we know $H_*(C_f) = \mathbb{Z}$ for $* = 0, n, 2n$, and 0 for $* \neq 0, n, 2n$. By universal coefficient theorem, the cohomology group of C_f is

$$H^k(C_f) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}[\beta] & k = n, \\ \mathbb{Z}[\alpha] & k = 2n, \\ 0 & \text{others}, \end{cases}$$

where α and β are generators we fix. Then, we know $\beta \cup \beta = k\alpha$ for some $k \in \mathbb{Z}$. We denote k as $H(f)$, and it is called the Hopf invariant. This invariant depends on the choice of generators, but only up to a sign. The Hopf invariant one problem refers to the following question:

For which values of n, there exists f such that $H(f) = 1$?

The above construction can also be done in the context of K-theory. Firstly, notice that there is no problem to answer when n is odd. This is because in this situation $\alpha^2 = -\alpha^2 = 0$, so $H(f)$ is always 0 when n is odd, so we can suppose $n = 2k$ in the following text.

For the map f above, we can deduce a cofiber sequence

$$S^{2k} \to C_f \to C_f/S^{2k} = S^{4k}.$$

This induces a short exact sequence of reduced K groups by the spectrum KU:

$$\tilde{K}^{-1}(S^{2k}) \to \tilde{K}^0(S^{4k}) \to \tilde{K}^0(C_f) \to \tilde{K}^0(S^{2k}) \to \tilde{K}^1(S^{4k}).$$

Since we know $\tilde{K}^0(S^{even}) = \mathbb{Z}$ and $\tilde{K}^1(S^{even}) = 0$, this gives a short exact sequence

$$0 \to \mathbb{Z}[x] \to \tilde{K}^0(C_f) \to \mathbb{Z}[y] \to 0,$$

where x and y are generators of $\tilde{K}^0(S^{4k})$ and $\tilde{K}^0(S^{2k})$ separately. We denote the image of x as x' and choose a preimage of y as y'. In the sequence, we know $(y')^2$ maps to 0 since $y^2 = 0$ and the maps in the sequence are ring homomorphisms. By exactness, there exists $k \in \mathbb{Z}$ such that $(y')^2 = kx'$. We denote this k as $h(f)$. This $h(f)$ does not depend on the choice of preimage. To see this, let y'' be another preimage of y so that $y' - y'' = mx'$ for some $m \in \mathbb{Z}$, then we have

$$\begin{aligned}(y'')^2 &= (y' + mx')^2 \\ &= h(f)x' + 2mx'y' + (x')^2\end{aligned}$$

It's easy to observe $(x')^2 = 0$ since $x^2 = 0$ in $\tilde{K}^0(S^{2k})$. We assume $x'y' = k_1x' + k_2y'$. The image of this element in $\tilde{K}^0(S^{2k})$ is $0 = k_2y$, which means $k_1x' = x'y'$. By multiplying y' on both sides, we have

$$k_1x'y' = x'(y')^2 = h(f)(x')^2 = 0.$$

Because $\tilde{K}^0(C_f)$ is torsion-free, we can conclude $x'y' = 0$ and $h(f)$ is an invariant that does not depend on the choice of preimages. However, as before, it also relies on the choice of x and y, so $h(f)$ is well-defined up to a sign.

Proposition 8.1. *h(f) is the Hopf invariant defined above.*

Proof. Using Chern character, we can build the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
\tilde{K}^0(S^{4k}) \arrow[r] \arrow[d, "ch_{4k}"] & \tilde{K}^0(C_f) \arrow[r] \arrow[d, "ch_f"] & \tilde{K}^0(S^{2k}) \arrow[d, "ch_{2k}"] \\
\tilde{H}^{even}(S^{4k}; \mathbb{Q}) \arrow[r] & \tilde{H}^{even}(C_f; \mathbb{Q}) \arrow[r] & \tilde{H}^{even}(S^{2k}; \mathbb{Q})
\end{tikzcd}$$

We denote x and y are generators of $\tilde{K}^0(S^{4k})$ and $\tilde{K}^0(S^{2k})$ separately. As above, x' is the image of x in $\tilde{K}^0(C_f)$ and we choose y' as a preimage of y.

By Proposition 7.8, we can observe that $ch_{4k}(x)$ and $ch_{2k}(y)$ are the generators of $\tilde{H}^{even}(S^{4k}; \mathbb{Z})$ and $\tilde{H}^{even}(S^{4k}); \mathbb{Z})$. This implies the image of $ch_{4k}(x)$ in $\tilde{H}^{even}(C_f; \mathbb{Q})$ is a generator of $\tilde{H}^{4k}(C_f; \mathbb{Z}) \cong \mathbb{Z}$. By commutativity, we know $ch_f(x')$ is a generator of $\tilde{H}^{4k}(C_f; \mathbb{Z})$. For the other part of the diagram, the map $\tilde{H}^{even}(C_f; \mathbb{Q}) \to \tilde{H}^{even}(S^{2k}; \mathbb{Q})$ is induced by the inclusion $S^{2k} \to C_f$. Hence, if we write

$$\tilde{H}^{even}(C_f; \mathbb{Z}) = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]$$

where α is $ch_f(x')$, and β is a generator mapping to $ch_{2k}(y)$, then we have the following description:

$$\begin{aligned} \tilde{H}^{even}(C_f; \mathbb{Q}) &\to \tilde{H}^{even}(S^{2k}; \mathbb{Q}) \\ k_1\alpha + k_2\beta &\mapsto k_2 ch_{2k}(y) \end{aligned}$$

for any $k_1, k_2 \in \mathbb{Q}$. By commutativity, it implies that

$$ch_f(y') = k\alpha + \beta \quad \text{for some } k \in \mathbb{Q}.$$

Moreover, we have

$$(ch_f(y'))^2 = (k\alpha + \beta)^2 = k^2\alpha^2 + 2k\alpha\beta + \beta^2 = \beta^2.$$

The first two terms are zero simply for degree reasons. Since ch_f is a ring homomorphism, it gives the following computation:

$$\beta^2 = (ch_f(y'))^2 = ch_f((y')^2) = ch_f(h(f)x') = h(f)ch_f(x') = h(f)\alpha.$$

Looking back to the definition of Hopf invariant, we can conclude that $h(f) = H(f)$. $\square$

9 The Ext groups in the Category of Abelian Groups with Adams Operations

In the last section, we stated the problem and transferred it to a question in topological K-theory. Now, we shall see how K-theory helps to attack the question. The key point we want to use is that there are Adams operations for K-groups, which will provide more information about the K-groups compared with cohomology groups. Using some properties of Adams operations, the Hopf invariant one problem can be answered smartly. Here, we calculate the Ext groups to provide a more conceptual viewpoint to the problem.

Definition 9.1. *We define a polynomial ring*

$$\mathcal{A} = \mathbb{Z}[\psi_1, \psi_2, \psi_3, \psi_5, \cdots, \psi_p, \cdots]$$

where p ranges over all possible primes. These ψ_p's are representatives of Adams operations. For any compact complex X, the group $K(X)$ is naturally an $\mathcal{A}$-module. The category of $\mathcal{A}$-modules can be called the category of abelian groups with Adams operations.

Let S^{2k} be an even-dimensional sphere. We have calculated in Proposition 4.6 that

$$\begin{aligned} \psi^m : \tilde{K}(S^{2k}) \to \quad & \tilde{K}(S^{2k}) \\ x \mapsto \quad & m^k x. \end{aligned}$$

For simplicity, we use $\mathbb{Z}(k)$ to denote the $\mathcal{A}-$module $\tilde{K}(S^{2k})$. Then, the short exact sequence

$$0 \to \tilde{K}^0(S^{4k}) \to \tilde{K}^0(C_f) \to \tilde{K}^0(S^{2k}) \to 0$$

can be rewritten as

$$0 \to \mathbb{Z}(2k) \to E \to \mathbb{Z}(k) \to 0.$$

The sequence is an element of $\mathrm{Ext}^1(\mathbb{Z}(k), \mathbb{Z}(2k))$, where Ext^1 should be understood in the category of $\mathrm{Mod}_{\mathcal{A}}$. To solve the Hopf invariant one problem, it's natural to obtain a better understanding of this group.

To do this in a slightly more general situation, we will calculate the group $\mathrm{Ext}^1(\mathbb{Z}(k), \mathbb{Z}(k+n))$, where n is a positive integer. For start, we observe that there is a short exact sequence

$$0 \to \mathbb{Z}(k+n) \to \mathbb{Q}(k+n) \to \mathbb{Q}/\mathbb{Z}(k+n) \to 0.$$

It induces a long exact sequence in Ext groups. The part we care about is:

$$\begin{aligned} \cdots \to \mathrm{Hom}(\mathbb{Z}(k), \mathbb{Q}(k+n)) \to \mathrm{Hom}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n)) \to \mathrm{Ext}^1(\mathbb{Z}(k), \mathbb{Z}(k+n)) \\ \to \mathrm{Ext}^1(\mathbb{Z}(k), \mathbb{Q}(k+n)) \to \cdots \end{aligned} \tag{2}$$

Using this, we will see the following proposition hold:

Proposition 9.2. *The map Hom*$(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n)) \to$ *Ext*$^1(\mathbb{Z}(k), \mathbb{Z}(k+n))$ *is an isomorphism.*

Proof. To see Hom$(\mathbb{Z}(k), \mathbb{Z}(k+n)) = 0$, we assume $f(1) = a \in \mathbb{Q}$. Then, being compatible with Adams operations implies that

$$l^k a = f(l^k) = f(\psi^l(1)) = \psi^l(f(1)) = l^{n+k} a$$

This can happen in $\mathbb{Q}$ only if when $a = 0$. To see Ext$^1(\mathbb{Z}(k), \mathbb{Q}(k+n)) = 0$, we will firstly observe that

$$\text{Ext}^1_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}(k+n)) \cong \text{Ext}^1_{\mathcal{A}\otimes\mathbb{Q}}(\mathbb{Q}(k), \mathbb{Q}(k+n))$$

This is due to an adjunction as follows:

$$\begin{tikzcd} \text{Mod}_{\mathcal{A}} \arrow[r, bend left, "\otimes\mathbb{Q}"] \arrow[r, phantom, "\perp"] & \text{Mod}_{\mathcal{A}\otimes\mathbb{Q}} \arrow[l, bend left, "U"] \end{tikzcd}$$

If we pick a resolution cochain complex of $\mathbb{Z}(k)$ as $Q.(\mathbb{Z}(k))$, then the adjunction implies that

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(Q.(\mathbb{Z}(k)), \mathbb{Q}(k+n)) &\cong \text{Hom}_{\mathcal{A}\otimes\mathbb{Q}}(Q.(\mathbb{Z}(k)) \otimes \mathbb{Q}, \mathbb{Q}(k+n)) \\ &\cong \text{Hom}_{\mathcal{A}\otimes\mathbb{Q}}(Q.(\mathbb{Q}(k)), \mathbb{Q}(k+n)). \end{aligned}$$

Hence the Ext groups are isomorphic as the cohomology groups of two isomorphic chain complexes. The group Ext$^1_{\mathcal{A}\otimes\mathbb{Q}}(\mathbb{Q}(k), \mathbb{Q}(k+n))$ consists of short exact sequences of the following form:

$$0 \to \mathbb{Q}(k+n) \to F \to \mathbb{Q}(k) \to 0$$

We want to see the sequence splits in the category of Mod$_{\mathcal{A}\otimes\mathbb{Q}}$. Let us choose e_1, e_2 being two generators of $F \cong \mathbb{Q} \oplus \mathbb{Q}$ such that e_1 is the image of a basis element of $\mathbb{Q}(k+n)$, and e_2 projects to a generator of $\mathbb{Q}(k)$. Then

$$\psi^m = \begin{pmatrix} m^{n+k} & * \\ 0 & m^k \end{pmatrix}$$

This matrix is diagonalizable. Moreover, since all Adams operations are commutative, the knowledge from linear algebra implies we can choose a new basis for E such that all ψ^l's are diagonalized, which means the short sequence splits as $\mathcal{A} \otimes \mathbb{Q}$-modules. Hence, we have

$$\text{Ext}^1_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}(k+n)) \cong \text{Ext}^1_{\mathcal{A}\otimes\mathbb{Q}}(\mathbb{Q}(k), \mathbb{Q}(k+n)) = 0$$

Substituting the above equation and Hom$(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n)) = 0$ in the sequence (2) finishes the proof. □

By the above proposition, we can calculate the Ext group:

Proposition 9.3.

$$\textit{Ext}^1(\mathbb{Z}(k), \mathbb{Z}(k+n)) \cong \mathbb{Z}/\textit{g.c.d.}(p^k(p^n - 1))$$

where p ranges over all possible primes. For simplicity, we will write this great common divisor as Gcd in the following context.

Proof. By the above proposition, we only need to find all the possible group homomorphisms from $\mathbb{Z}(k)$ to $\mathbb{Q}/\mathbb{Z}(k+n)$. A group homomorphism of this form is determined by the value of 1. Since it is compatible with all Adams operations, this is equivalent to, for all primes p,

$$p^k x = p^{k+n} x \in \mathbb{Q}/\mathbb{Z}$$

which means

$$p^k(p^n - 1)x \in \mathbb{Z}$$

Hence, such an x must divide all possible $p^k(p^n - 1)$, so that it divides the number Gcd. On the other hand, any such number arises as a homomorphism. Now, it's easy to observe all such x in $\mathbb{Q}/\mathbb{Z}$ form a subgroup which is isomorphic to $\mathbb{Z}/Gcd$. $\square$

Now we will give a more explicit description of the isomorphism Hom$(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n)) \to$ Ext$^1(\mathbb{Z}(k), \mathbb{Z}(k+n))$ in Proposition 9.2. From homological algebra, it can be written out in the following way:

Let $f \in$ Hom$(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n))$, we can build the following diagram:

$$\begin{tikzcd}
\mathbb{Z}(k+n) \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}(k+n) \arrow[d] \\
E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Q}(k+n) \arrow[d] \\
\mathbb{Z}(k) \arrow[r, "f"] & \mathbb{Q}/\mathbb{Z}(k+n)
\end{tikzcd}$$

The lower square is obtained by taking pullback. Because $\mathbb{Q}(k+n) \to \mathbb{Q}/\mathbb{Z}(k+n)$ is a surjection, this square is also a pushout. Hence, the kernel and cokernel of the map $E \to \mathbb{Z}(k)$ are isomorphic to those of the map $\mathbb{Q}(k+n) \to \mathbb{Q}/\mathbb{Z}(k+n)$, so the left three maps indeed form a short exact sequence, and this is the image of f in Ext$^1(\mathbb{Z}(k), \mathbb{Z}(k+n))$. A natural question one would ask is that what its Adams operations are, and this is discussed in the following proposition.

Proposition 9.4. *The inverse map of*

$$\mathit{Hom}_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n)) \to \mathit{Ext}^1_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Z}(k+n))$$

can be described as follows: For an element in Ext$^1(\mathbb{Z}(k), \mathbb{Z}(k+n))$ *having the form*

$$0 \to \mathbb{Z}(k+n) \to E \to \mathbb{Z}(k) \to 0. \tag{3}$$

We denote α, β *as a basis of* $E \cong \mathbb{Z} \oplus \mathbb{Z}$, *where* α *is the image of the generator of* $\mathbb{Z}(k+n)$ *and* β *maps to the generator of* $\mathbb{Z}(k)$. *Then, for any prime* p, *we have*

$$\psi^p(\alpha) = p^{k+n}(\alpha);$$
$$\psi^p(\beta) = p^k\beta + a_p\alpha.$$

The inverse map maps a sequence (3) to the number $\frac{a_p}{p^{k+n} - p^k} \in \mathbb{Q}/\mathbb{Z}$, *which represents an element in Hom*$_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n))$.

Proof. We will calculate Adams operations of the exact sequence induced by $f \in \text{Hom}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n))$. Notice that f can be represented by a number $q \in \mathbb{Q}/\mathbb{Z}$, and this q is understood as a fixed representative in the following computation. By the definition of pullback, we know:

$$\begin{aligned} E &= \{(x,y) \in \mathbb{Q} \times \mathbb{Z} | y = qx \in \mathbb{Q}/\mathbb{Z}\} \\ &= \{(x,y) \in \mathbb{Q} \times \mathbb{Z} | y - qx \in \mathbb{Z}\} \\ &= \mathbb{Z}[(0,1)] \oplus \mathbb{Z}[(1,q)]. \end{aligned}$$

The set $\{(0,1),(1,q)\}$ is a basis because we have known for every $(x,y) \in E$, we have $(x,y) = (y-qx)(0,1)+x(1,q)$, and they're clearly linearly independent. Let us call $(0,1) = \alpha$ and $(1,q) = \beta$. Then, for the map $E \to \mathbb{Z}(k)$, it maps α to 0, and β to 1. The Adams operation on E can now be easily calculated: for any prime p,

$$\begin{aligned} &\psi^p(\alpha) = (0, \Psi^p(1)) = p^{k+n}(0,1) = p^{k+n}\alpha \\ &\psi^p(\beta) = (\Psi^p(1), \Psi^p(q)) = (p^k, qp^{k+n}) = p^k\beta + q(p^{k+n} - p^k)\alpha. \end{aligned}$$

This number of course depends on the choice of generators. However, since we have known the homomorphism we are describing is an isomorphism, we can conversely identify q from Adams operations of the short exact sequence. Hence, for any prime number p, q can always be written as

$$q = \frac{a_p}{p^{k+n} - p^k} \in \mathbb{Q}/\mathbb{Z},$$

where a_p is as defined in the statement of the theorem. □

This proposition implies us to define the following invariant:

Definition 9.5. *We define the e-invariant map:*

$$e : \textit{Ext}^1_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Z}(k+n)) \to \textit{Hom}_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(k+n))$$

as the inverse of the natural isomorphism that we described in Proposition 9.2. More concretely, an e-invariant for a short exact sequence:

$$0 \to \mathbb{Z}(n+k) \to E \to \mathbb{Z}(k) \to 0$$

is

$$e = \frac{a_p}{p^{k+n} - p^k} \in \mathbb{Q}/\mathbb{Z}$$

where $a_p \in \mathbb{Z}$ is the number such that $\psi^p(\beta) = p^k\beta + a_p\alpha$, as explained in Proposition 9.4.

10 Answering Hopf Invariant One Problem

In this section, we shall give a complete answer to the Hopf invariant one problem. Recall our map is

$$f : S^{4k-1} \to S^{2k}$$

for $k \geq 1$. There are positive answers for some k:

Theorem 10.1. *For $k = 1, 2, 4$, there exists f such that $H(f) = 1$.*

Proof. For $k = 1$, we use the complex projective plane $\mathbb{C}P^2$. Its CW complex structure gives an attaching map from its 4-cell to 2-skeleton, which is S^2. Hence, this map is just a map $f : S^3 \to S^2$, and by definition, we have $C_f = \mathbb{C}P^2$. Since we know $H^*(\mathbb{C}P^2) = \mathbb{Z}[x]/(x^3)$ with $|x| = 2$, the map f has Hopf invariant one. For $k = 2, 4$, we do the exact similar proof by using quaternionic projective space $\mathbb{H}P^2$ and octonionic projective plane $\mathbb{O}P^2$. One can also check [5] for a more detailed treatment discussing the relation between Hopf invariant and division algebra structure on $\mathbb{R}^n$. $\square$

The main theorem we are going to prove is the following one:

Theorem 10.2. *For $k \neq 1, 2, 4$, the Hopf invariant cannot be 1.*

Proof. Suppose such an f exists, we have the following short exact sequence:

$$0 \to \mathbb{Z}(4k) \to E \to \mathbb{Z}(2k) \to 0,$$

where $E = \tilde{K}(C_f)$. As we stated in Proposition 9.4, E is generated by α and β such that

$$\psi^2(\beta) = 2^k\beta + a_2\alpha.$$

On the other hand, the properties of Adams operations imply

$$\psi^2(\beta) \equiv \beta^2 = \alpha \qquad (\text{mod } 2)$$

This implies a_2 must be an odd number when $H(f) = 1$. We shall prove this cannot be true when $k \neq 1, 2, 4$. By the calculations in the last section, we get the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Z}(2k)) \arrow[r, "\cong"] \arrow[d, "U"] & \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(2k)) \arrow[d, "U"] \\
\mathrm{Ext}^1_{\mathbb{Z}[\psi^2]}(\mathbb{Z}(k), \mathbb{Z}(2k)) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Z}[\psi^2]}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(2k))
\end{tikzcd}$$

In the diagram, U means the forgetful functor from $\mathrm{Mod}_{\mathcal{A}}$ to $\mathrm{Mod}_{\mathbb{Z}[\psi^2]}$. The lower row means we only consider Adams operations of the form ψ^{2^m}. Notice that in the last section, we didn't use multiple Adams operations in the proofs, so for the category $\mathrm{Mod}_{\mathbb{Z}[\Psi_2]}$, we can get a similar description of the lower isomorphism in the graph as for the top one. We only need to replace Gcd with $2^k(2^k - 1)$.

By Proposition 9.3 and 9.4, the upper map in the diagram can be written as follows:

$$\begin{array}{rcl}
\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Z}(2k)) \to & \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(2k)) & \to \mathbb{Z}/Gcd \\
E \mapsto & a_p/(p^k(p^k-1)) & \mapsto a_p/(p^k(p^k-1)) \times Gcd
\end{array}$$

By multiplying with Gcd, the image of E is an integer modulo $Gcd = g.c.d.(p^k(p^k - 1))$. Similarly, we interpret the lower horizontal map in the diagram as the following map:

$$\begin{array}{rcl}
\mathrm{Ext}^1_{\mathbb{Z}[\psi^2]}(\mathbb{Z}(k), \mathbb{Z}(2k)) \to & \mathrm{Hom}_{\mathbb{Z}[\psi^2]}(\mathbb{Z}(k), \mathbb{Q}/\mathbb{Z}(2k)) & \to \mathbb{Z}/(2^k(2^k-1)) \\
E \mapsto & a_2/(2^k(2^k-1)) & \mapsto a_2
\end{array}$$

The right vertical map is recognized as:

$$U : \mathbb{Z}/Gcd \to \mathbb{Z}/(2^k(2^k-1))$$
$$1 \mapsto (2^k(2^k-1))/Gcd$$

Since a_2 is an odd number, the above computations imply that the image of U must contain an odd number. This requires the number $(2^k(2^k-1))/Gcd$ to be an odd one. We will prove it is not possible when $k \neq 1, 2, 4$. Since Gcd=g.c.d.$(p^k(p^k-1))$, for $p = 3$, we can write $3^k - 1 = 2^{v(k)} \times$odd by prime decomposition, then the power of 2 of Gcd is not greater than $v(k)$. To finish our proof, we only need to show for $k \neq 1, 2, 4, v(k) < k$, which is proven in the lemma below. □

Lemma 10.3. *Let k be a positive integer. Then we can write*

$$3^k - 1 = 2^{v(k)} \times \textit{odd}$$

For $k \neq 1, 2, 4$, we always have $v(k) < k$.

Proof. If k is an odd number, we have

$$3^k - 1 \equiv 2 \qquad \text{mod } 4$$

Hence, it implies $v(k) < 2$. Also, notice that $3^k - 1$ is an even number, so we have $v(k) = 1$. If k is an even number, we can write $k = 2^r j$, where j is an odd number. Then, we have

$$3^k - 1 = (3^j - 1)\prod_{l=0}^{r-1}(3^{2^l j} + 1)$$

As we have seen, the power of 2 of $3^j - 1$ is 1. Moreover, we know

$$3^j + 1 \equiv 0 \qquad \text{mod } 4$$
$$3^j + 1 \equiv 4 \qquad \text{mod } 8$$

They imply that the power of 2 of $3^j + 1$ is exactly 2. For all $l > 0$, we know

$$3^{2^l j} + 1 \equiv 2 \qquad \text{mod } 4$$

so the power of 2 of $3^{2^l j} + 1$ is 1. By adding them up, we then have

$$v(k) = r + 2.$$

Then it is to obvious that $r + 2 \geq 2^r j$ only if when $j = 1$ and $r = 1, 2, 4$. □

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