Algebraic Cycles

Conventions: • $k=\bar{k}$

• scheme: algebraic scheme i.e. $X \to \operatorname{Spec}(k)$ is of finite type

• variety: reduced and irreducible scheme.

Definition: Let X be a scheme. A q-cycle on X is a finite formal sum

$$\sum_i n_i [V_i]$$

where each V_i is a subvariety of dimension q. We denote the set of all q-cycles on X by $Z_q(X)$, which is a free abelian group.

Proper push-forward: Let $f: X \to Y$ be a proper morphism and V a q-dimensional subvariety. We define

$$f_*[V] = \begin{cases} [k(V) : k(f(V))]\,[f(V)] & \text{if } \dim(f(V)) = q. \\ 0 & \text{if } \dim(f(V)) < q. \end{cases}$$

Then f_* extends to a homomorphism

$$f_* : Z_q(X) \to Z_q(Y)$$
$$\sum_i n_i [V_i] \mapsto \sum_i n_i f_*[V_i].$$

Flat pull-back: Let $f: X \to Y$ be a flat morphism. Then every fiber of f has dimension $n = \dim X - \dim Y$. Let $V \subseteq Y$ be a q-dimensional subvariety. We define $f^*[V] = [f^{-1}(V)]$. Then f^* extends to a homomorphism

$$f^* : Z_q(Y) \to Z_q(X)$$
$$\sum_i n_i [V_i] \mapsto \sum_i n_i [f^{-1}(V_i)]$$

Remark: The proper push-forward and flat pull-back are functorial i.e. $(f\circ g)_* = f_*\circ g_*$ and $(f\circ g)^* = f^*\circ g^*$.

Exterior product: Let $V\subseteq X$, $W\subseteq Y$ be subvarieties of dimension q, p. We define

$$\times : Z_q(X)\times Z_p(Y) \longrightarrow Z_{q+p}(X\times Y)$$

by $[V]\times[W] = [V\times W]$. Since $k=\bar{k}$, $V\times W$ is irreducible, so $\times$ is well-defined. Then $\times$ extends bilinearly to a homomorphism

$$\otimes : Z_q(X)\otimes Z_p(Y) \longrightarrow Z_{q+p}(X\times Y).$$

Rational equivalence: Let X be a variety, $W\subseteq X$ a subvariety of $\dim = q+1$, $r\in k(W)^\times$. We define a q-cycle $[\operatorname{div}(r)]$ on X by

$$[\operatorname{div}(f)] := \sum_{\operatorname{codim} V=1} \operatorname{ord}_V(r)[V],$$

where ord_V is the order function on $k(W)^\times$ defined by the local ring $\mathcal{O}_{V,W}$.

i.e. $\operatorname{ord}_V(f) = \begin{cases} \operatorname{length}_{\mathcal{O}_{V,W}}(\mathcal{O}_{V,W}/(f)) & \text{if } f\in\mathcal{O}_{V,W} \\ \operatorname{ord}_V(f_1)-\operatorname{ord}_V(f_2) & \text{if } f=\frac{f_1}{f_2} \text{ with } f_1,f_2\in\mathcal{O}_{V,W}. \end{cases}$

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~~there~~ Definition: (1) A q-cycle α is rationally equivalent to zero, written $\alpha\sim 0$ if there are finite number of $(q+1)$-dimensional subvariety W_i of X, and $r_i\in k(W_i)^\times$ such that $\alpha = \sum_i [\operatorname{div}(r_i)]$.

(2) "Chow groups": $A_q(X) := Z_q(X)/\sim$.

Notations: $Z_*(X) := \bigoplus_i Z_i(X)$, $A_*(X) := \bigoplus_i A_i(X)$.

Proposition 1: A cycle $\alpha \in Z_q(X)$ is rationally equivalent to zero iff there are $(q+1)$-dim subvarieties $V_1, \cdots, V_t$ of $X \times \mathbb{P}^1$ such that the projections from V_i to $\mathbb{P}^1$ are dominant, with $\alpha = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$ in $Z_q(X)$.

Proposition 2: Let $f: X \to Y$ be a proper, surjective morphism of varieties and let $r \in k(X)^\times$. Then

(a) $f_*[\mathrm{div}(r)] = 0$ if $\dim(Y) < \dim(X)$,

(b) $f_*[\mathrm{div}(r)] = [\mathrm{div}(N(r))]$ if $\dim(Y) = \dim(X)$

where $N(r)$ is the norm of r.

Remark: The proper push-forward and flat pull-back preserve rational equivalence.

Proposition 3: Let Y be a closed subscheme of a scheme X and $U = X \setminus Y$. Let $i: Y \to X$, $j: U \to X$ be inclusions. Then the sequence

$$A_q(Y) \xrightarrow{i_*} A_q(X) \xrightarrow{j^*} A_q(U) \longrightarrow 0$$

is exact for all q.

proof: Note that any subvariety V of U extends to a subvariety $\overline{V}$ of X, then the sequence

$$Z_q(Y) \xrightarrow{i_*} Z_q(X) \xrightarrow{j^*} Z_q(U) \longrightarrow 0 \quad (*)$$

is exact. Let $\alpha \in Z_q(X)$ and $j^*\alpha \sim 0$, then

$$j^*\alpha = \sum_i [\mathrm{div}(r_i)]$$

for $r_i \in k(W_i)^\times$, W_i subvarieties of U. Since $k(W_i) = k(\overline{W_i})$, r_i corresponds to $\overline{r_i}$ in $k(\overline{W_i})^\times$. Then

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$$j^*(\alpha) - \sum_i [\mathrm{div}(\bar{r}_i)]) = 0$$

in $Z_q(U)$. Therefore, $\alpha - \sum_i [\mathrm{div}(r_i)] = i_*\beta$ for some $\beta \in Z_q(Y)$ since $(*)$ is exact.

Proposition 4: Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fiber square with g flat, f proper. Then g' is flat, f' is proper and for all $\alpha \in Z_*(X)$, $f'_* g'^* \alpha = g^* f_* \alpha$ in $Z_* Y'$.

Proposition 5: Let $p: E \to X$ be an affine bundle of rank n. Then the flat pull-back

$$p^*: A_q(X) \longrightarrow A_{q+n}(E)$$

is surjective for all q.

proof: choose a closed subscheme Y of X so that $U = X \setminus Y$ is affine open set over which E is trivial. There is a commutative diagram

$$\begin{array}{ccccccc} A_q(Y) & \longrightarrow & A_q(X) & \longrightarrow & A_q(U) & \longrightarrow & 0 \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ A_{q+n}(p^{-1}(Y)) & \longrightarrow & A_{q+n}(E) & \longrightarrow & A_{q+n}(p^{-1}(U)) & \longrightarrow & 0 \end{array}$$

with exact rows and vertical maps are flat pull-backs. By Five Lemma, it suffices to show that α and γ are surjective. By induction on $\dim(Y)$ it is enough to show the result for $X = U$ only. Therefore, we may assume $E = X \times \mathbb{A}^n$. Since the projection factors

$$X\times\mathbb{A}^n \longrightarrow X\times\mathbb{A}^{n-1}\longrightarrow X,$$

we can assume $n=1$. Namely, we ~~want~~ need to show

$$[V]\in p^*A_q(X),\ \forall\ V\subseteq E \text{ with } \dim(V)=q+1.$$

Proposition 4 guarantees that we can assume $X=\overline{p(V)}$ i.e. p maps V dominantly to X. Let $A=k[X]$, $K=k(X)$ and $\mathfrak{p}$ the prime ideal in $A[t]$ corresponding to V. If $\dim(X)=q$, then $p^*[X]=V$ and we are done. Otherwise, $\dim(X)=q+1$. Since $p: V\to X$ is dominant and $V\neq E$, the ideal $\mathfrak{p}K[t]$ is non-trivial. Let $r\in K[t]$ generate $\mathfrak{p}K[t]$, then

$$[V]-[\operatorname{div}(r)]=\sum_i n_i[V_i],\quad V_i\subseteq E \text{ with } \dim V_i=q+1.$$

So we have $[V]=[\operatorname{div}(r)]+\sum_i n_i\, p^*[W_i]$ where $W_i=p(V_i)$, i.e. $[V]\in p^*A_q(X)$.

Proposition 6: (a) If $\alpha\sim 0$ or $\beta\sim 0$, then $\alpha\times\beta\sim 0$.

(b) Let $f: X'\to X$, $g: Y'\to Y$ be morphisms, $f\times g$ the induced morphism from $X'\times Y'$ to $X\times Y$.

(i) If f, g are proper, so is $f\times g$ and

$$(f\times g)_*(\alpha\times\beta)=f_*\alpha\times g_*\beta.$$

(ii) If f, g are flat, so is $f\times g$ and

$$(f\times g)^*(\alpha\times\beta)=f^*\alpha\times g^*\beta.$$

Proof: We first prove (b). Factor $f\times g$ into $(f\times id_Y)\circ(id_X\times g)$ reduces to the easy case where f or g is identity. Since we have commutative diagram

$$\begin{tikzcd}
X'\times Y' \arrow[r] \arrow[d] & X'\times Y \arrow[r] \arrow[d] & X' \arrow[d] \\
X\times Y' \arrow[r] & X\times Y \arrow[r] & X
\end{tikzcd},$$

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then (b) follows from Proposition 4.

For (a), if $\alpha \sim 0$ we may assume $\beta = [W]$ for some subvariety $W \subseteq Y$. By (i) of (b) we may assume $W = Y$. ($f = \mathrm{id}$, $g = j : W \hookrightarrow Y$). In this case, we have $\alpha \times \beta = p^*(\alpha)$ where $p : X \times W \longrightarrow X$ is the projection. Then the statement holds since flat pull-pack preserves ~~ra~~ rational equivalence. Similarly, we can do the same trick for $\beta \sim 0$.

Remark: Proposition 6 gives the exterior products

$$\otimes : A_q(X) \otimes A_p(Y) \longrightarrow A_{q+p}(X \times Y)$$

satisfying formulas $(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$ and $(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta$.