# A criterion for transitivity of area preserving partially hyperbolic endomorphism on torus

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#### Abstract

We propose a criterion, referred to as *order-n transversality*, for transitivity of area preserving partially hyperbolic endomorphisms. Besides, we also give a further answer to the Gan's problem, as proposed in the work of Baolin He [6]

**Keywords:** Partially hyperbolic endomorphism, transitivity, stable ergodicity, stable ergodicity within skew product.

### 1 Introduction

In the study of volume-preserving partially-hyperbolic dynamical systems, stable ergodicity is an important property for conservative dynamics. We say that  $f \in \operatorname{Diff}^r(M)$  is  $C^r$ -stable ergodic if there is a  $C^r$ -neighborhood  $\mathcal U$  such that every volume-preserving diffeomorphism in  $\mathcal U$  is also ergodic. In this direction, there is a famous conjecture proposed by Pugh-Shub [9]:

Conjecture 1.1. (Pugh-Shub) Stable ergodicity is a  $C^r$ -dense property among  $C^r$  volume preserving partially hyperbolic diffeomorphisms on any compact connected manifold for all  $r \geq 2$ .

In their article, the authors proposed a geometrical condition, termed *accessibility*, as a possible tool for studying this conjecture. They suggested to divide the problem into the following two conjectures:

Conjecture 1.2. A  $C^2$  volume preserving partially hyperbolic differenomorphism which is essential accessible is ergodic.

**Conjecture 1.3.** Accessibility holds for an open and dense subset for  $C^r$  volume preserving (or not) partially hyperbolic difference or phisms, where  $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ .

Since then, there have been considerable efforts to develop this theory, see e.g. [4], [1], [3], [5], [10]. When the required regularity condition is restricted to  $C^1$ , there is an affrmative answer to the conjecture, see [1], expressed by the following theorem:

**Theorem 1.1.** Stable ergodicity is  $C^1$ -dense in the space of  $C^r$  partially-hyperbolic volume-preserving diffeomorphisms on a compact connected manifold, for any  $r \ge 1$ .

However, there exists no satisfying answer when the regularity is lifted up to  $C^r$  where  $r \geq 2$ . The authors of [10] have proved the following theorem, which confirms Pugh-Shub's conjecture for partially-hyperbolic diffeomorphisms with 1D centers:

**Theorem 1.2.** Stable ergodicity is  $C^r$  open and dense among volume preserving partially-hyperbolic diffeomorphisms with one dimensional center bundle, for all  $r \geq 2$ .

There is some progess in the more general cases. For conjecture 1.2, Keith Burns and Amie Wilkinson [4] proved it under a mild center bunching assumption, which is the best result so far. In comparison, the study of conjecture 1.3 is less comperhensive, one can see some of recent results in the article of [8] and [7].

These articles have verified the importance of accessibility in studying the conjecture. However, in the situation of partially-hyperbolic endomorphisms, there is no natural definition for the accessibility. Due to this, in this article we try to consider the problem from a different perspective, by considering a sufficient condition for transitivity. For volume-preserving partially-hyperbolic diffeomorphisms, accessibility implies transitivity [2], and we shall prove that order-n transversality also implies transitivity for endomorphisms. Notice that a quantitative version of this condition here was firstly introduced by Tsujii in the article [11].

In the second part of this article, we are going to address Gan's problem, i.e.

**Problem 1.1.** [6] Let  $f: \mathbb{T}^2 \to \mathbb{T}^2$  be the map as the following:

$$f(x,y) = (2x, y + \lambda \sin(2\pi x)), \quad \lambda \neq 0.$$

Is f is stably ergodic?

Baolin He has modified the definition of accessibility for partially hyperbolic endomorphisms and proved the accessibility of this map, but this does not imply stable ergodicity since the relation between ergodicity and accessibility has not been fully clarified. In this article, we are going to provide a direct answer to Gan's problem. Our result is based on using the theorem in the article of Zhang [13].

#### 2 Definitions and Notations

**Definition 2.1.** An endomorphism  $f: M \to M$  on a compact Riemannian manifold is a *partially-hyperbolic endomorphism* if there is a continuous cone

field  $C_u = \{C_u(x) \subset T_x M\}_{x \in M}$  and constants  $0 < \chi_c < \chi_u$  such that for all sufficiently large integer n > 0,

$$Df(\bar{C}_u) \subset C_u.$$

$$||Df^n(v)|| \ge e^{n\chi_u}||v||, \qquad \forall v \in C_u.$$

$$||Df^n(v)|| \le e^{n\chi_c}||v||, \qquad \forall v \in Df^{-n}(C_u^c).$$

In the following, we shall use M to indicate the torus.

**Definition 2.2.** For an area preserving partially-hyperbolic endomorphism, we say that it is  $C^r$ -stably ergodic if for any  $C^r$  area preserving  $g: M \to M$  that is sufficiently close to f in  $C^r$  topology, it is also ergodic with respect to the area form.

For a function  $\tau \in C^r(M) (r \geq 2)$ , we define  $f_\tau : M \to M$ :

$$f_{\tau}(x,y) = (2x, y + \tau(x)).$$

We say that  $f_{\tau}$  is  $C^r$ -stably ergodic within skew-products if for all  $\hat{\tau}$  sufficiently close to  $\tau$  in  $C^r$  topology,  $f_{\hat{\tau}}$  is also ergodic with respect to the area form

**Definition 2.3.** Given L > 1, Let  $\Gamma_L$  denote the collection of  $C^2$  curves  $\gamma : [0,1] \to M$  such that

$$\|\gamma\|_{C^2} < L.$$

$$\|\gamma'(t)\| > \frac{1}{L} \quad \forall t \in [0, 1].$$

$$\gamma'(t) \in C_u(\gamma(t)), \quad \forall t \in [0, 1].$$

Let  $\varrho_L$  be the set of  $C^1$  functions  $\rho:[0,1]\to\mathbb{R}_+$  such that

$$\|\rho\|_{C^1} < L.$$

$$\int \rho dm_{Leb} = 1,$$

where  $m_{Leb}$  denotes Lebesgue measure. We denote by  $\mathcal{M}_L$  the minimal closed convex set of probability measures on M containing  $m_{\gamma,\rho}$  for all  $\gamma \in \Gamma_L$  and  $\rho \in \varrho_L$  where

$$m_{\gamma,\rho} = \gamma_*(\rho dm_{Leb}).$$

**Definition 2.4.** Given an integer n > 0, we say that f satisfies order-n transversality condition if for any  $x \in M$ , there exist  $y, z \in f^{-n}(x)$  such that

$$D_z f^n(C_u(z)) \cap D_y f^n(C_u(y)) = \{0\}.$$

Given a continuous cone of fields, we denote by  $\mathcal{T}_n$  the set of all partially hyperbolic endomorphisms on M that satisfy order-n transversality condition.

## 3 Main Theorem

**Theorem 3.1.** For any n > 0, any area preserving  $f \in \mathcal{T}_n$  is transitive.

*Proof.* If f is not transitive, then there exist  $U, V \subset M$  such that  $f^n(V) \cap U = \emptyset$  for all positive integers n. We recall that the backward orbit of x is given by

$$G(x) = \{ y \in M | \exists n \ge 0, f^n(y) = x \}.$$

In order to analyze the cone fields on a tours, we first introduce a f-invariant set on the basis of

$$K_0 = \{ x \in M | G(x) \cap V = \emptyset \}.$$

Note that f is a submersion, which means that it is an open mapping, and  $K_0$  is a closed set. Given  $y \in f^{-1}(x)$  where  $x \in K_0$  and an arbitrary element z in G(y), there exists a positive integer  $n_0$  such that  $f^{n_0}(z) = y$ , so z is also an element of G(x) since  $f^{n_0+1}(z) = f(y) = x$ . We deduce that  $y \in K_0$ , and  $K_0$  is f-backward invariant. From the hypothesis that f is an area preserving map, it also follows that  $m(f^{-1}(K_0)) = m(K_0)$ .

For the set to be f-forward invariant, let us consider the intersection of all f-backward image of  $K_0$ , and denote it by  $K_1$ :

$$K_1 = \bigcap_{m \ge 0} f^{-m}(K_0).$$

Let us prove that  $K_1$  is a f-invariant set. On the one hand, for any  $y = f(x) \in f(K_1)$ , the element  $f^n(y) = f^{n+1}(x)$  must be in  $K_0$  for all nonnegative integers n, so  $y \in K_1$ . On the other hand, for any  $y \in K_1$  and any point  $x \in f^{-1}(y)$ , it can be seen that  $x \in K_0$  as  $K_0$  is f-backward invariant and  $x \in f^{-m}(K_0)$  as  $f^m(x) = f^{m-1}(y) \in K_0$  for all  $m \ge 1$ . Therefore, x is an element of  $K_1$ , and  $y \in f(K_1)$ .

Notice also that  $K_1$  is closed, since it is built from a closed set  $K_0$ . Upon denoting by K the interior of  $K_1$ , we have that K is also a f-invariant set since f is an open mapping.

Let us now remind that U is an open subset of  $K_0$ . Since the measures of  $K_1$  and  $K_0$  are equal, U must also be a subset of  $K_1$ , which shows that K is non-empty. Then, a measure  $m_K$  can be constructed as the normalized Lebesgue measure on K. For any measurable set  $A \subset K$ ,

$$f^*m_K(A) = m_K(f^{-1}(A)) = \frac{m_{Leb}(f^{-1}(A) \cap K)}{m_{Leb}(K)} = \frac{m_{Leb}(f^{-1}(A))}{m_{Leb}(K)} = m_K(A),$$

where the last step is due to the area preserving property of f.

The above argument implies that  $m_K$  is a f-invariant measure. We now give a lemma that will be used later:

**Lemma 3.1.**  $m_K \in \mathcal{M}_L$  for some L > 0

*Proof.* The proof for this lemma follows from Corollary 3.8 in Tsujii [12] by noticing that class of admissible measure in [12] is contained in  $\mathcal{M}_L$  for some L > 0.

From now on, we fix the number L satisfying  $m_K \in \mathcal{M}_L$ 

**Lemma 3.2.** For any  $x \in \overline{K}$ , there is a curve  $\gamma : [0,1] \to M$  satisfying the following conditions:

- $\bullet \ \gamma(\frac{1}{2}) = x.$
- $1/L < ||\gamma'(t)|| < L$ .
- $\forall t \in [0,1], \ \gamma(t) \in \overline{K}$ .
- $\forall t \in [0,1], \ \gamma'(t) \in C_u(\gamma(t)).$

*Proof.* From Lemma 3.1 and the definition of  $\mathcal{M}_L$ ,  $m_K$  can be represented as

$$\int_{\Gamma_L \times \rho_L} m_{\gamma,\rho} d\mu,$$

where  $\mu$  is a measure on the space  $\Gamma_L \times \varrho_L$ . Then, for each integer n, we obtain a sequence of measures as

$$m_n = \int_{\Gamma_L \times \ \varrho_L} m_{\gamma,\rho}^n d\mu,$$

where  $m_{\gamma,\rho}^n$  is the measure  $m_{\gamma,\rho}$  restricted to the subset  $\gamma\left(\left[\frac{1}{n},1-\frac{1}{n}\right]\right)$ . Let  $\Omega_n = supp(m_n)$ , it follows that  $m_K(\Omega_n) \to m_K(\overline{K})$ . Moreover, by the above construction and Lusin theorem, for any  $y \in \Omega_n$ , there is a curve  $\gamma \in \Gamma_L$ , such that  $y \in \gamma(\left[\frac{1}{n},1-\frac{1}{n}\right])$ .

For each set  $\Omega_n$ , we shall show that there is an integer  $k_n$  satisfying the following condition: for any  $x \in f^{k_n}(\Omega_n)$ , there is a curve  $\gamma_x \in \Gamma_L$  that passes through x at  $t = \frac{1}{2}$ . The idea is to restrict and reparametrize a curve passing through x. Noticing that the map f expands the tangent vectors exponentially, a large enough positive integer  $k_n$  may be found such that the tangent vectors of  $f^{k_n}(\gamma)$  are enlarged sufficiently for any curve  $\gamma$  in  $\Omega_n$ . Let us pick  $y \in \Omega_n$  where  $f^{k_n}(y)$  equals to x, a curve  $\gamma_y$  can be obtained from the argument of last paragraph, then the thesis follows easily by restricting and reparameterizing  $f^{k_n}(\gamma_y)$ .

Finally, since  $m_K$  is a f-invariant measure, it follows that  $m_K(\Omega_n) \leq m_K(f^{k_n}(\Omega_n))$ , and the union of all  $f^{k_n}(\Omega_n)$  is a full measure set in  $\overline{K}$ . Recalling the definition of  $m_K$ , it is obvious that  $\overline{K} = supp(m_K)$ . Hence, for any  $x \in \overline{K}$ , there is a sequence of points  $x_n \in f^{k_n}(\Omega_n)$  such that the sequence converges to x, and thus a sequence of curves  $\gamma_n \in \Gamma_L$  passing through  $x_n$  can be constructed as in the above paragraph, where the curves have the property that  $\gamma_n(\frac{1}{2}) \to x$ . Furthermore, the fact  $\|\gamma_n\|_{C^2} < L$  indicates that

there is a subsequence  $\gamma_{n_k}$  converging in  $C^1$  topology by using Arzelá–Ascoli theorem, and it is not hard to see that the limiting curve satisfies all the conditions as we want.

For any  $x' \in \overline{K}^c$ , there is a corresponding point  $x \in \overline{K}$  such that  $d(x, x') = d(x', \overline{K})$ . In the following proof, we fix a point  $x'_0 \in K^c$  where the correspondence  $x_0$  lies in one coordinate chart as  $x'_0$ . By Lemma 3.2, there is a curve  $\gamma$ , entirely in  $\overline{K}$ , that passes through  $x_0$ .

Since  $x_0$  and  $x_0'$  are contained in one coordinate chart, they can be viewed as points in the Euclidean space  $\mathbb{R}^2$ . Take a circle C with center  $x_0'$  and radius  $d(x_0', \overline{K})$ , then a curve  $\gamma$  obtained from Lemma 3.2 must be tangent to the image of circle under coordinate map. Otherwise, it passes through the inner part of the circle, and this produces a point on  $\gamma$  that is closer to  $x_0'$  than  $x_0$ , which is a contradiction. Furthermore, for any differentiable curve  $\tilde{\gamma}$  which is totally in  $\overline{K}$  with  $\tilde{\gamma}(\frac{1}{2}) = x$ , we must have  $\gamma'(\frac{1}{2}) = \tilde{\gamma}'(\frac{1}{2})$  by the same argument.

Recalling that  $f \in \mathcal{T}_n$ , there exist  $y, z \in f^{-n}(x)$  such that

$$D_z f^n(C_u(z)) \cap D_y f^n(C_u(y)) = \{0\}. \tag{1}$$

Given that  $\overline{K}$  is f-backward invariant, there are curves  $\gamma_y$  and  $\gamma_z$  that pass through y and z respectively by Lemma 3.2, where the tangent vectors  $T_y$  and  $T_z$  at y, z are in their own cone fields. Since  $f^n(\gamma_y)$  and  $f^n(\gamma_z)$  are curves that pass through x and they both lie in  $\overline{K}$ ,  $Df^n(T_y)$  and  $Df^n(T_z)$  must coincide and equal to  $\gamma'(\frac{1}{2})$ , which yields a contradiction with equation (1).

#### 4 Gan's Problem

We prove the following

**Theorem 4.1.** Let  $f: M \to M$  be the map as the following:

$$f(x,y) = (2x, y + \lambda \sin(2\pi x)), \quad \lambda \neq 0.$$

Then f is  $C^{20}$ -stably ergodic

*Proof.* At first, let us state the following lemma, which follows by putting together Proposition 2 and Theorem 1 of in [13].

**Lemma 4.1.** For the map  $f_{\tau} = (2x, y + \tau(x))$ , if  $\tau \in C^{20}(M)$  and  $f_{\tau} \in \mathcal{T}_n$  for some integer  $n \geq 1$ , then  $f_{\tau}$  is  $C^{20}$ -stably ergodic.

As shown in [6], f is an area preserving partially hyperbolic endomorphism. We choose a cone field as follows for all points  $x = (x_1, x_2) \in M$  such that it fulfills the requirements in Definition 2.1:

$$C_u(x) = \{(t, t\eta) | t \in \mathbb{R}, |\eta| < 4\pi |\lambda| \}.$$

Without loss of generality, we shall suppose  $\lambda > 0$ . Notice that for any point  $x = (x_1, x_2) \in M$ , the set of first coordinate of pre-image  $f^{-n}\{x\}$  is  $\{\frac{x_1+k}{2^n}|0 \le k \le 2^n-1\}$ . Furthermore, the following holds:

$$Df_{x}^{n}((t,t\eta) = \left(2^{n}t, t\eta + \sum_{k=0}^{n} 2^{k}t\pi\lambda\cos(2^{k}\pi x)\right),$$
$$Df_{x}^{n}(C_{u}(x)) = \left\{(t,t\eta)||\eta - h_{n}(x_{1})\pi\lambda| < \frac{1}{2^{n-2}}\pi\lambda\right\},$$

where  $h_n(x_1) = \sum_{k=1}^n \frac{\cos(2^k\pi y_1)}{2^{n-k}}$ . Hence, order-n transversality follows by checking that there exists a positive integer n, such that for any  $x \in [0,1)$ , we can find  $y, z \in \{\frac{k+x}{2^n} | 1 \le k \le 2^n - 1\}$  where  $|h_n(y) - h_n(z)| > \frac{1}{2^{n-3}}$  holds. This can be easily proved by letting n = 6 which  $\cos(\frac{\pi}{2^{n-3}}) > 0.9$ . Then, for an arbitrary element  $x \in [0,1)$ , we pick  $y = \frac{x}{2^n}$  and  $z = \frac{2^{n-3} + x}{2^n} = \frac{1}{2^3} + y$ ,

$$h_n(y) - h_n(z) \ge \frac{2\cos(\frac{\pi}{2^{n-3}})}{2^{n-3}} + \frac{\cos(\frac{\pi}{2^{n-2}})}{2^{n-2}} + \frac{\cos(\frac{\pi}{2^{n-1}})}{2^{n-1}} - \frac{\cos(\frac{\pi}{4})}{2^{n-1}}$$
$$\ge \frac{1.8}{2^{n-3}} + \frac{2.7 - \frac{\sqrt{2}}{2}}{2^{n-1}} > \frac{1}{2^{n-3}}.$$

i.e.  $f \in \mathcal{T}_6$  and the proof is concluded by using Lemma 4.1.

Remark 1. If  $\tau$  is not  $C^{20}$  smooth, but it satisfies the conditions for ordern transversality, we can still make some observation. On the one hand, by Theorem 3.1, it must be transitive. On the other hand, we have the following proposition, which is essentially proved by the Proposition 2 in [13]:

Proposition 4.1.  $f_{\tau} \in \mathcal{T}_n$  for some integer  $n \geq 1$  if and only if  $f_{\tau}$  is  $C^2$ -stably ergodic within skew products.

Hence, for skew product on the torus, order-n transversality can be recoginzed as an equivalent condition for  $C^2$ -stale ergodicity within skew products.

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