Study Report on Exotic Sphere

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1 Introduction to Isoparametric Foliations

By making comparison with the disk bundle decomposition structure on unit sphere, we shall intevigate a similar description for closed simply-connected riemmanian manifold.

Definition 1.1. Given $(N^{n+1}, g, \mathcal{F} = \{L_p\})$, where L_p (denoted as a **leaf** through p) is complete submanifold and N is the union of all L_p, the triple will be denoted as an **isoparametric foliation** if the following conditions are satisfied:

(1) If there is a geodesic curve γ such that $\gamma \perp L_{p_0}$ for a point $p_0 \in \gamma$, then $\gamma \perp L_p$ for any $p \in \gamma$.

(2) For any tangent space T_pL_p, there is a family of local vector field $\{X_i\}$ s.t. $T_pL_p = \mathrm{Span}\{X_i|_p\}$.

(3) $\mathrm{Max}\{\dim(L_p)\} = n$ and the average curvature H on any leaf is a constant.

$(N^{n+1}, g, \mathcal{F} = \{L_p\})$ is a **transnormal system** if only (1) holds.
$(N^{n+1}, g, \mathcal{F} = \{L_p\})$ is a **singular foliation** if only (2) holds.
$(N^{n+1}, g, \mathcal{F} = \{L_p\})$ is a **singular Riemmanian foliation** if (1) and (2) hold.

Remark 1. In Definition 1.1, condition (1) means that the hypersurfaces are parallel to each other, and condition (3) refers to CMC condition defined on the situation of unit sphere.

When N is closed and simply connected with foliation of codimension 1, one can prove that N with an isoparametric foliation admits an isoparametric function, and vice versa. This fact shows that there are only two singular leaves $M_\pm$, such that they are focal manifolds and any regular leaf is a tube over them. In this situation, we denote the manifold has the following property:

Property 1.1. *The double disk bundle decomposition* **(DDBD)***:*

$$N = D\left(\nu(M_+)\right) \bigcup D\left(\nu(M_-)\right) \cong E_+ \cup_\varphi E_- := E_\varphi.$$

Conversely, the above property can also deduce an isoparametric foliation:

Theorem 1.1. *(Fundamental construction theorem) Given a DDBD E_φ with its canonical singular foliation $\mathcal{F}_\varphi$, there exists a metric g_φ on E_φ such that*

$$(E_\varphi, g_\varphi, \mathcal{F}_\varphi) \text{ is an isoparametric foliation.}$$

Hence, if we don't consider the metric, isoparametric foliation is equivalent to DDBD structure.

To classify manifold with foliation, we often use the following diffeomorphism to build up the equivalence classes:

Definition 1.2. $(N, \mathcal{F})$, $(N', \mathcal{F}')$ are said to be foliated diffeomorphic if there is a diffeomorphism

$$f : N \to N' \quad \text{s.t.} f(L) = L' \qquad \forall L \in \mathcal{F}(L' \in \mathcal{F}').$$

Here, we denote $(N, \mathcal{F}) \cong (N', \mathcal{F}')$.

2 Exotic Sphere

Exotic Sphere is firstly constructed by J. Milnor in 1956, who firstly discovered an exotic 7-sphere, which is a S^3-bundle over S^4. To begin with, we shall firstly review some related definitions:

Definition 2.1. A n-dimension smooth manifold Σ^n is called an **exotic sphere** if it is homeomorphic to S^n but not diffeomorphic to S^n.

A manifold having the same homotopy type as S^n is called a **homotopy n-sphere**

Before we study learning the exotic sphere, we shall firstly have a look of the generalized Poincaré conjure which stated as follows:

Conjecture 2.1. *A homotopy n-sphere is homeomorphic to S^n.*

When $n \geq 5$, Samle discovered that there is a special tool, h-cobordism theorem, and it deduced the authenticity of the conjecture. Later, Freedman proved the four dimensional situation in 1982. In his work, he gave a clear classification of the intersection form of 4-dimensional manifold. 3-dimensional case is the most complex one, and it was proved by Perelman in 2003, who invented a powerful tool, Ricci flow, as the main method in the proof.

Back to exotic sphere, we begin from a more detail review of the work of Milnor. His construction of the exotic sphere is based on the following one-to-one correspondence:

$$\begin{array}{ccc} S^3 \hookrightarrow \Sigma^7 & \Longleftrightarrow & m, n \in \pi_3 SO(4) \cong Z \oplus Z. \\ \quad\quad\;\downarrow & & \\ . \quad\quad S^4 & & \end{array}$$

Notice that $SO(4) \cong S^3 \times SO(3)$, the three order fundamental group can be viewed as mapping class of S^3 to $S^3 \times SO(3)$. Since the sphere bundle will induce a map $f \in \pi_3(S^3 \times SO(3))$, where f can be written as (f_1, f_2) mapping in its components. Then, (m, n) are assigned as the mapping degree of (f_1, f_2). Hence, we can write Σ^7 as $\Sigma^7_{m,n}$ to emphasize the correspondence.

For $\Sigma^7_{m,n}$, J.Milnor proved the following theorem:

Theorem 2.1.

$$m + n = 1 \Leftrightarrow \Sigma^7_{m,n} \xrightarrow{homeomo.} S^7$$
$$(m-n)^2 \neq 1 \ mod\ 7 \Rightarrow \Sigma^7_{m,n} \xrightarrow{not\ diffeo.} S^7.$$

The idea of the proof of the first equivalence is to construct a morse function from $\Sigma^7_{m,n} \to \mathbb{R}$ with 2 critical points, which guarantees the existence of homeomorphism.

For the second equivalence, the main ingredient is the Pontyajin number. The number is an invariant for diffeomorphic manifolds, and it must be integer for any smooth manifold. By taking the fiber cone, $\Sigma^7_{m,n}$ can be extended to a disk bundle N^8. If $\Sigma^7_{m,n}$ is diffeomorphic to S^7 via ϕ, then we can realize a smooth manifold $N^8 \cup_\phi D^8$. However, if $(m-n)^2$ is not congruent to 1 mod 7, the Pontyajin number cannot be a integer, which yields a contradiction.

If m, n satisfy the both conditions in the above theorem, then we can get an exotic sphere. For example, when $m = 2$, $n = -1$, $\Sigma^7_{2,-1}$ is an exotic sphere, and it also called as Gromoll-Meyer sphere. The name is due to the biquotient structure discovered by them on the exotic sphere, which gave the first example of metric structure with non-negative sectional curvature on exotic sphere.

To study further in exotic sphere, J.Milnor defines a natural group of spheres with the connected sum as the operation:

$$\Theta_n = \{n\text{-dimensional homotopy spheres}\}.$$

When $n = 7$, $\Theta_7 \cong \mathbb{Z}_{28}$. Notice that we distinguished the orientation in the above group. J.Milnor proved that there are 14 exotic 7-spheres, and 10 of which can be represented as S^3-bundle over S^4, they are denoted as Milnor spheres.

There is a special group of Θ_n which is denoted as bP_{n+1}. A sphere belongs to bP_{n+1} if and only if it is a boundary of a parallelizable manifold (A manifold is parallelizable if its tangent bundle is a trivial bundle). It has been proved that bP_{n+1} is 0 for n even and bP_{n+1} is finite cyclic for n odd. For instance, we have $bP_8 = \Theta_8$. Due to its nice property, the sphere can have a simpler structure, especially when concerning with problems related to curvature.

We mentioned that Samle used h-cobordism theorem to prove Poincaré conjure in dimension not less than 5. Another consequence of the theorem is that it also gives a construction of exotic spheres. Given two n-disks and a mapping ϕ identifying the boundaries of them, the quotient space derived from the identification is called as twist sphere. The work of Smale proved that all exotic spheres with dimension $n \geq 5$ are twist spheres. However, when $n = 4$, Cerf proved that any twist sphere is differeomorphic to the standard sphere S^4. Ge and Tang generalized the Cerf theorem by proving that twist disk bundle also cannot produce an exotic sphere when $n = 4$.

3 Isoparametric Foliations on Exotic Spheres

From the fundamental construction theorem in the first section, we know that the study of isoparametric foliation on exotic sphere is equivalent to the study of DDBD structure on exotic sphere. Following this idea, Ge generalized the work of Samle, and proved that an isoparametric foliation on exotics sphere can be realized as an isoparametric foliation on the standard sphere with suitable metric:

Theorem 3.1. *For each exotic n-sphere Σ^n $(n > 4)$, there exists the following one-to-one correspondence:*

$$\left\{[\Sigma_n, \widetilde{\mathcal{F}}]\right\} \Longleftrightarrow \{[S^n, \mathcal{F}]\},$$

*where $[,]$ denotes the equivalence class up to the foliated diffeomorphism defined in the first section. In the correspondence, $[\Sigma^n, \widetilde{\mathcal{F}}]$ has the same disk bundle as $[S^n, \mathcal{F}]$ but different identification mapping. **

The one-to-one correspondence has the naturality with foliated diffeomorphism:

$$\begin{array}{ccc}
\Sigma^n_\phi = (E_+ \bigcup_{d_\phi \circ \varphi} E_-, \mathcal{F}_{d_\phi\circ\varphi}) & \xleftarrow[1-1]{} & S^n = (E_+ \bigcup_{\varphi} E_-, \mathcal{F}_{\varphi}) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\Sigma^n_\phi = (E_+ \bigcup_{d_\phi \circ \psi} E_-, \mathcal{F}_{d_\phi\circ\psi}) & \xleftarrow[1-1]{} & S^n = (E_+ \bigcup_{\psi} E_-, \mathcal{F}_{\psi}).
\end{array}$$

Notice that the correspondence itself is also explained in the commutative diagram, with the mapping d defined in the article of Ge (2018). The above structure is also called a **DDBD-Twisted Sphere**. We have seen that when dim> 4, an exotic sphere are both twist sphere and DDBD-twisted sphere.

When concerned with isoparametric foliations on Σ^n, there is a natural classification problem of them up to foliated diffeomorphism. By theorem 3.1, we see that it is sufficient to solve the

Classification problem of isoparametric foliations on S^n.

Since S^n now has a general metric rather than the standard sphere, this problem is still far from completed. However, there is a possible path towards the solution of the problem, which is described as follows (let $N = S^n$):

Step 1: Classify disk bundles $E_\pm$, s.t. $N = E_+ \bigcup_\varphi E_-$

Step 2: Classify isotopy classes of the identification mapping φ, i.e. compute the subset

$$G_N(E_\pm) := \{[\varphi] \in \pi_0(\mathrm{Diff}(\partial E_+ \to \partial E_-))|S^n \cong E_\varphi\}.$$

Step 3: Compute the action:

$$\begin{aligned} \beta: \quad \pi_o(\mathrm{Isom}_b(\partial E_\pm)) \times G_N(E_\pm) &\to G_N(E_\pm)) \\ ([f_\pm], \varphi) &\to [f_- \circ \varphi \circ f_+^{-1}] \end{aligned}$$

Notice that in step 2 and 3, $[,]$ refers to the isotopy class of the mapping. From the notation in step 2, we could see that there is a fact that any two mappings belongs to the same connected component are isotopic. When $E_\pm$ are disks, Smale proved that $\pi_0(\mathrm{Diff}^+(S^{n-1} \to S^{n-1})) \cong \Theta_n$, but this doesn't hold for disk bundles, that is why we need to compute the subgroup $G_N(E_\pm)$.

In step 3, the notation $\mathrm{Isom}_b(\partial E)$ refers to the restriction of bundle maps of E preserving Euclidean metric to ∂E. Ge proved the following theorem establishing the feasibility of step 3:

Theorem 3.2. *For φ_0,φ_1: $\partial E_+ \to \partial E_-$:*

$$(E_{\varphi_0}, \mathcal{F}_{\varphi_0}) \cong (E_{\varphi_1}, \mathcal{F}_{\varphi_1})$$
$$\Updownarrow$$
$$\exists f_\pm \in Isom_b(\partial E_\pm) s.t. [\varphi_1] = [f_- \circ \varphi_0 \circ f_+^{-1}].$$

Then, when fixes $E_\pm$, the number of equivalence classes of isoparametric foliations on N is $|G_N(E_\pm)/\beta|$.

The subset $G_N(E_\pm)$ is intriguing, since we do not know if there exists a diffeomorphism from E_{φ_0} to E_{φ_1} such that it maps the disk bundle to disk bundle. Be more specific, we can define the following set:

$$\Upsilon_\varphi := \left\{ [h_- \circ \varphi \circ h_+^{-1}] | [h_\pm] \in \pi_0(\mathrm{Diff}_{E_\pm}(\partial E_\pm)) \right\},$$

where the notation $\pi_0(\mathrm{Diff}_{E_\pm}(\partial E_\pm))$ refers to the diffeomorphism of ∂E that extendable to E. This is a subset of $G_N(E_\pm)$ and we want to study the difference between them. Due to the disk theorem of Palais, we have known that $\Upsilon_\varphi = G_N(E_\pm)$ if one of $E_\pm$ is D^n. Notice that there is a smaller subset of Υ_φ which could be mentioned sometimes during people's research:

$$\Gamma_\varphi = \{[\varphi'] | \varphi' \text{ pseduo-isotopic to } \varphi\} \subset \Upsilon_\varphi.$$

When $E_+ = E_- = D^n$, Hatcher proved that $\pi_0(O(n)) \cong \pi_0(\mathrm{Diff}_{D^n}(S^{n-1})) \cong \mathbb{Z}_2$ if $n \neq 5$. Based on this fact, Ge proved in 2016 that pseduo-isotopy is equivalent to isotopy on S^{n-1}. Furthermore, we have

$$G^+_{S^n}(D^n, D^n) = \Upsilon^+_\varphi = \Gamma_\varphi = [\varphi], \quad n \neq 5.$$

Hence,

$$|G_{S^n}(D^n, D^n)/\beta| = 1.$$

This means the following fact: when dim $\neq 5$, any two isoparametric foliations whose focal submanifolds are two points are equivalent on sphere. When dim$=$ 5, it is known that $\pi_0(\mathrm{Diif}(S^4)) \cong \pi_0(\mathrm{Diff}_{D^5}(S^4))$, but if it is isomorphic to $\pi_0(O(n))$ is still unknown.