Formal Groups and Lubin-Tate Theory

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Introduction

The main object of this note is the formal group. Roughly speaking, it is an abelian group object in the category of formal schemes. This definition shows that the note will be intertwined with algebraic geometry. This story goes back to Quillen, who observed that given an even periodic ring spectrum A, the formal scheme $\mathrm{Spf}(A^0(\mathbb{C}P^\infty))$ can be seen as a one-dimensional formal group over the ring $\pi_0(A)$. Furthermore, when A is Landweber exact, this formal group is a complete invariant up to homotopy equivalence. This shows that formal groups can potentially be exciting to study, and it turns out that we have an abundant theory about them.

The purpose of this note is to introduce the construction of the Lubin-Tate spectrum, which is also called the Morava E-theory. We will introduce two constructions. In the first construction, by studying the quasi-coherent sheaf over the moduli stack of formal groups, the Lubin-Tate theory can be constructed as a homotopy commutative diagram, and Goerss-Hopkins-Miller proved that it must have a unique E_∞-structure. The other construction is due to Lurie, who defines formal groups directly in the derived case and produces the Lubin-Tate spectrum with characterising universal properties.

In this note, we will assume the reader has a basic understanding of spectra and higher algebra and is familiar with the functor point viewpoint of schemes and quasi-coherent sheaf. For a reader unfamiliar with the latter part, there is a quick introduction in lecture 2 of Piotr Pstragowski's note and section 1.1 of Paul Goerss's note.

This note's content combines Lurie's paper Elliptic curve 2, Piotr's and Maxime's notes on chromatic homotopy theory. For people who are only interested in classical construction, they can read sections $2, 3, 4$, Theorem 5.2, and section 7. For people who are only interested in Lurie's construction, they can read sections $1, 2, 4, 5$ and 6.

Contents

1 Formal groups

In this section, we shall give an introduction to formal groups over E_∞-rings. Before giving the actual definitions, we will introduce smooth coalgebras over R. Intuitively, a smooth coalgebra over R is a coalgebra whose dual is locally isomorphic to a formal power series ring in several variables over R.

Let R be a commutative ring and M be a flat R-module. For each integer $n \geq 0$, the symmetric group Σ_n acts on the n-fold tensor product $M^{\otimes n}$. We define $\Gamma^n(M)$ to be the submodule of $M^{\otimes n}$ consisting of invariants for the Σ_n-action. This exhibits $\Gamma^*(M)$ as a graded module over R. When M admits a direct sum decomposition, $\Gamma^*(M)$ has a coalgebra structure over R, detail can be found in construction 1.1.11 of Lurie's paper.

A smooth coalgebra over a discrete ring R is a flat commutative coalgebra C over R such that C is isomorphic to $\Gamma^*(M)$ for some finite rank projective R-module M. Furthermore, if M has rank r, then we say C is of dimension r.

Definition 1.1. *Let R be an E_∞-ring. The ∞-**category of commutative coalgebras** over R is given by*

$$\mathrm{cCAlg}_R := \mathrm{CAlg}(\mathrm{Mod}_R^{op})^{op}.$$

*A coalgebra C over R is **flat** if it is flat as an R-module, which means*

- *$\pi_0(C)$ is a flat module over $\pi_0(R)$,*
- *for all integer n, the canonical map $\pi_n R \otimes_{\pi_0(R)} \pi_0(C) \to \pi_n(C)$ is an isomorphism.*

*The full subcategory of flat coalgebras over R is denoted as cCAlg_R^b. A coalgebra C over R is **smooth** if it is flat over R and $\pi_0(C)$ is smooth over $\pi_0(R)$. The full subcategory spanned by smooth coalgebras over R is denoted by cCAlg_R^{sm}. We say that C is of dimension r if $\pi_0(C)$ is of dimension r.*

Now we can define hyperplanes. This definition mimics the idea of transforming a ring R as a scheme $\mathrm{Spec}(R)$.

Definition 1.2. *Let R be a connective E_∞-ring and $C \in \mathrm{cCAlg}_R^b$. Then we can define*

$$\begin{aligned} \mathrm{cSpec}(C) : \mathrm{CAlg}_R^{cn} &\longrightarrow \mathrm{An} \\ A &\longmapsto \mathrm{Map}_{\mathrm{cCAlg}_A}(A, A \otimes C) \end{aligned}$$

The above defines a functor

$$\begin{aligned} \mathrm{cSpec} : \mathrm{cCAlg}_R^b &\longrightarrow \mathrm{Fun}(\mathrm{CAlg}_R^{cn}, \mathrm{An}) \\ C &\longmapsto \mathrm{cSpec}(C) \end{aligned}$$

Proposition 1.3. *The functor above restricts to a fully faithful embedding mapping from cCAlg_R^{sm} to $\mathrm{Fun}(\mathrm{CAlg}_R^{cn}, \mathrm{An})$.*

Definition 1.4. *Let R be a connective E_∞-ring. A **formal hyperplane** over R is a functor $X : \mathrm{CAlg}_R^{cn} \to \mathrm{An}$ such that it belongs to the essential image of the fully faithful embedding $\mathrm{cSpec} : \mathrm{cCAlg}_R^{sm} \to \mathrm{Fun}(\mathrm{CAlg}_R^{cn}, \mathrm{An})$ above. A **formal group** over R is a functor $G : \mathrm{CAlg}_R^{cn} \to \mathrm{Mod}_{\mathbb{Z}}^{cn}$ such that*

$$\mathrm{CAlg}_R^{cn} \xrightarrow{G} \mathrm{Mod}_{\mathbb{Z}}^{cn} \xrightarrow{\Omega^\infty} \mathrm{An}$$

is a formal hyperplane over R. We use $\mathrm{Hyp}(R)$ and $\mathrm{FGroup}(R)$ to denote the infinity categories of formal hyperplanes and formal groups over R respectively. Notice that from the definition, we have $\mathrm{Hyp}(R)$ is equivalent to cCAlg_R^{sm}.

Definition 1.5. *For a general E_∞-ring R, we define $\mathrm{Hyp}(R)$ as $\mathrm{Hyp}(\tau_{\geq 0}R)$ and $\mathrm{FGroup}(R)$ as $\mathrm{FGroup}(\tau_{\geq 0}R)$.*

Remark 1.6. *If we denote Lat as the category of free abelian group of finite rank. Then a formal group over R is equivalent to a finite product preserving functor mapping from Lat^{op} to $\mathrm{Hyp}(R)$. In short, we have $\mathrm{FGroup}(R) \simeq \mathrm{Fun}^{\Sigma}(\mathrm{Latt}^{op}, \mathrm{Hyp}(R))$.*

Using this remark, we can define Quillen formal groups, which is the fundamental example relating homotopy theory and formal groups. Let R be an E_∞ ring, and regard the ∞-category Mod_R as equipped with a symmetric monoidal structure given by smash product relative to R (see Definition 4.4.2.3 in HA), then there is an essentially unique symmetric monoidal functor $\mathrm{An} \to \mathrm{Mod}_R$ which preserves small colimit. We will denote this functor by $X \mapsto C_*(X; R)$; Here we have the property

$$\pi_*(C_*(X; R)) \simeq R_*(X).$$

Lurie proved the following theorem:

Theorem 1.7. *If $R \in \mathrm{CAlg}$ is* **complex periodic**, *which means R satisfies the following two conditions:*

1. **weakly 2-periodic:** *for all integer n, the canonical map $\pi_2(R) \otimes_{\pi_0(R)} \pi_n(R) \to \pi_{n+2}(R)$ is an isomorphism.*
2. **complex orientable:** *The unit map $e : S \to R$ factors as a composition:*

$$S \simeq \Sigma^{\infty-2}\mathbb{C}P^1 \to \Sigma^{\infty-2}\mathbb{C}P^\infty \to R$$

Then for every free abelian group M of finite rank $0 < r < \infty$, we have $C_*(K(M, 2); R)$ is a smooth coalgebra over R, where $K(M, 2)$ is the Eilenberg-MacLane space.

Example 1.8. *Let R be a complex periodic E_∞-ring, the theorem above implies we have a functor $\mathrm{Lat}^{op} \to \mathrm{Hyp}(R)$ mapping M to $\mathrm{cSpec}(C_*(K(M^\vee, 2); R))$, and this functor commutes with the finite product. Hence, by the remark above, this defines a formal group over R. We denote this formal group as $G_R^{\mathcal{Q}}$ and refer to it as the* **Quillen formal group** *over R.*

Remark 1.9. *When R is a discrete commutative ring, our definition degenerates to the classical definition of formal groups, which means, we don't need to define over all E_∞ commutative algebra over R; It is enough to define over the ordinary commutative algebras over R.*

2 1-dimensional formal groups and formal group laws over discrete commutative rings

As suggested by the remark at the end of the last section, we can discuss 1-dimensional formal groups over discrete commutative rings in more detail. In the next two sections, everything is in a discrete setting. To start, let's define formal group laws, which are almost all the formal groups we will encounter.

Definition 2.1. *Let R be a commutative ring. A* **formal group law** *over R is a formal power series $f \in R[[x,y]]$ s.t.*

1. $f(x,y) = f(y,x)$.
2. $f(t,0) = t = f(0,t)$.
3. $f(f(x,y),z) = f(x,f(y,z))$.

We use $\mathrm{FGL}(R)$ *to denote the set of all formal group laws over R. Notice* $\mathrm{FGL} : \mathrm{CRing} \to An$ *is a scheme with respect to etale topology.*

Remark 2.2. *We can define the* **Lazard ring** $L = \mathbb{Z}[a_{ij} | i,j \in \mathbb{N}]/\sim$*, where the equivalence relations are given by the three conditions above. Then a formal group law over R is the same as a ring map $L \to R$.*

By the first two relations in the definition, it is easy to see that a formal group law looks like the following formal power series:

$$f(x,y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j.$$

It links to formal groups in the following way:

Definition 2.3. *Let f be a formal group law over R. Notice that* $\mathrm{CAlg}_R^{\heartsuit}$ *denotes the discrete commutative ring over R. Then the corresponding formal group G_F is the abelian group object* $G_f : \mathrm{CAlg}_R^{\heartsuit} \to \mathrm{Ab}$ *in the category of etale sheaves over* $\mathrm{Spec}(R)$ *defined by*

$$G_f(S) = \mathrm{Nil}(S)$$

the set of nilpotent elements in S with group structure given by $x +_f y = f(x,y)$.

Example 2.4. *Over R, we have a formal group law $f(x,y) = x + y$. We denote the corresponding formal group as the G_a.*

Example 2.5. *Over R, we always have a formal group law $f(x,y) = x + y + xy$. We denote the corresponding formal group as the G_m.*

Example 2.6. *Let E be a complex orientable multiplicative cohomology theory, one can prove*

$$E^*(\mathbb{C}P^\infty) \simeq E^*[[t]]$$

The multiplicative structure of $\mathbb{C}P^\infty$ will induce a formal group law over E^*. The associated formal group is also called the **Classical Quillen formal groups** *of E, denoted by $G_E^{\mathcal{Q}_0}$. If E is complex periodic E_∞-ring, this is the same formal group with the scalar extension $G_E^{\mathbb{Q}}$ in Example 1.8 along $E \to \pi_0(E)$.*

Notice that in the above example, we don't need to E to be E_∞ but only homotopy commutative. For example, when A is Morava K-theory spectrum at the prime 2, we cannot define a Quillen formal group, but we can still define a classical one, when $E = H\mathbb{Z}$ or KU, the above example will give additive formal group and multiplicative formal group respectively.

Using formal group laws, we can describe a 1-dimensional formal group over a discrete ring R as the following:

Proposition 2.7. *Let R be a commutative ring. The category of 1-dimensional formal group G over R is a full subcategory of abelian group objects of the category of etale sheaves over* $\mathrm{Spec}(R)$ *spanned by object which is locally in the Zariski topology on R of the form G_F, where F a formal group law.*

Let us explain the geometry picture of formal group law and formal group. Consider the ring $R[[x]]$ with ideal $I = (x)$, then $R[[x]]$ has an I-adic topology. We define the formal affine line (fileted colimit)

$$\widehat{A}^1_R := \varinjlim \mathrm{Spec}(R[[x]]/I^n)$$

From the functor viewpoint, the S-point of this formal scheme is

$$\widehat{A}^1_R(S) = \varinjlim \mathrm{Hom}_{\mathrm{CRing}}(R[[x]]/I^n, S) \simeq \mathrm{Hom}^{\mathrm{cont.}}_{\mathrm{CRing}}(R[[x]], S)$$

By a simple observation that

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spec}(S), \widehat{A}^1_R) \simeq \mathrm{Nil}(S)$$

We see the underlying etale sheaf of any formal group law is always a formal affine line, and as a geometry object, a formal group is pasted from formal affine lines.

Definition 2.8. *Let F, G be two formal group laws over R, a* **morphism of formal group law** *$\phi : F \to G$ is a power series in $R[[x]]]$ with no constant term such that*

$$\phi(F(x, y)) = G(\phi(x), \phi(y))$$

An **isomorphism** *is a morphism with the first coefficient being a unit in R, and it is* **strict** *if the first coefficient is 1.*

Example 2.9. *Over a $\mathbb{Q}$-algebra, all formal group laws are isomorphic to the additive one.*

Example 2.10. *Over any field of positive characteristic, G_a and G_m are not isomorphic. This will be explained in section 4.*

3 Moduli Stack of Formal Groups

Very importantly, we should study formal groups along with their morphisms, rather than recognizing them as discrete sets. Thus, they can be assembled into what algebraic geometers call the moduli stack of formal groups. The slogan here is that the geometry of this moduli stack controls the behaviour of stable homotopy theory.

Definition 3.1. *The* **moduli stack of formal groups** $\mathcal{M}_{fg} : \mathrm{CRing} \to \mathrm{An}$ *is the sheaf of spaces which associates to any ring R the groupoid of 1-dimensional formal groups over* $\mathrm{Spec}(R)$ *and their isomorphisms.*

Remark 3.2. *The moduli stack of formal groups can be defined over all $E_\infty-$rings as a functor* $\mathrm{CAlg} \to \mathrm{An}$ *mapping R to* $\mathrm{FGroup}(R)^{\simeq}$*. For the purpose of this note, we do not need this generality. Because in the derived case, it's more interesting to directly study the moduli stack of p-divisible groups, which is beyond the scope of this note.*

Let us firstly state Quillen and Lazard's theorem.

Theorem 3.3. *The Lazard ring L is isomorphic to MU_*, where MU is the complex cobordism spectra. MU_* is a polynomial ring with infinitely many variables.*

In this section, we want to make the slogan at the start a bit more precise by constructing the following functor:

Proposition 3.4. *There is a functor*

$$\mathcal{F} : \mathrm{Sp} \to \mathrm{Qcoh}(\mathcal{M}_{fg})$$

sending X to $\mathcal{F}_X$.

To approach our goal, we need to understand quasi-coherent sheaves over $\mathcal{M}_{fg}$. A common way to approach this is by providing a decent diagram of the sheaf so that its quasi-coherent sheaf will be a limit diagram of quasi-coherent sheaves over the sheaves in the diagram. From Quillen's theorem, one can guess that MU will be an approachable candidate. Indeed, we have the following proposition:

Proposition 3.5. *We define the* **moduli stack of formal group laws with strict isomorphisms** *as a functor as following*

$$\begin{aligned} \mathcal{M}^s_{fgl} : \mathrm{CRing} &\longrightarrow \mathrm{An} \\ R &\longmapsto \{\textit{Groupoid of formal group laws over } R \textit{ with strict isomorphisms}\} \end{aligned}$$

Then we have a canonical equivalence

$$\mathcal{M}^s_{fgl} \simeq \operatorname*{colim}_{\Delta^{op}} \mathrm{Spec}(\pi_*(MU^{\otimes n+1}))$$

We need to study the relation between $\mathcal{M}^s_{fgl}$ and $\mathcal{M}_{fg}$. They are connected by a group scheme action.

Definition 3.6. *We define the multiplicative group scheme by* $\mathbb{G}_m : \mathrm{CRing} \to \mathrm{Group}$ *by mapping* R *to its unit* $R^\times$*. The underlying scheme is the affine scheme* $\mathrm{Spec}(\mathbb{Z}[b^\pm])$*.*

The action of this group scheme is related to even graded rings.

Proposition 3.7. *For a ring* R*, the following piece of data are equivalent:*

(1) A $\mathbb{G}_m$*-action on the affine scheme* $\mathrm{Spec}(R)$*.*

(2) An even graded on R*; that is a choice of abelian groups* $R_{2n} \subset R$ *which forms a graded ring such that* $R \simeq \oplus_{n\in\mathbb{Z}} R_{2n}$*.*

Sketch of proof. An action $\mathbb{G}_m \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is equivalent to a coaction $\Delta : R \to R \otimes \mathbb{Z}[b^\pm] \simeq R[b^\pm]$. Then, we can set

$$R_{2n} = \{r \in R | \Delta(r) = rb^n\}.$$

It is not hard to check this gives an even grading of R.

Conversely, given an even grading, we define the coaction by $\Delta(r) = rb^n$ for $r \in R_{2n}$ and extend this to whole R by direct sum relation $R \simeq \oplus R_{2n}$. $\square$

Notice the fact that MU_* is an even graded ring and $(MU_*MU)^{\otimes n} \simeq \pi_*(MU^{\otimes(n+1)})$, we can rewrite the relation in Proposition 3.5 as

$$\mathcal{M}^s_{fgl} \simeq \operatorname*{colim}_{\Delta^{op}} \left(\mathrm{Spec}(MU_*) \underset{\leftarrow}{\leftarrow} \mathrm{Spec}(MU_*MU) \underset{\underset{\leftarrow}{\leftarrow}}{\leftarrow} \mathrm{Spec}(MU_*MU \otimes_{MU_*} MU_*MU) \cdots \right).$$

Since all the above rings are evenly graded, this passes to a $\mathbb{G}_m$-action on the simplicial diagram, so to the quotient $\mathcal{M}^s_{fgl}$.

Notice that the $\mathbb{G}_m$-action on $\mathrm{Spec}(MU_*)$ is quite visible. By Quillen's theorem, we know $\mathrm{Hom}(MU_*, R) \simeq \mathrm{FGL}(R)$. So for any commutative ring R, $\mathbb{G}_m$ gives an action $R^\times \times \mathrm{FGL}(R) \to \mathrm{FGL}(R)$ by sending $f(x, y)$ to $\lambda^{-1} f(\lambda x, \lambda y)$. From this, it tells us taking the stacky $\mathbb{G}_m$-quotient should give you the classification of formal group laws and general isomorphisms.

By the general theory of quotient stack, $\mathcal{M}^s_{fgl}/\mathbb{G}_m$ sends R to the groupoid of principal $\mathbb{G}_m$-bundles over $\mathrm{Spec}(R)$ together with a $\mathbb{G}_m$-equivariant map to $\mathcal{M}^s_{fgl}$. A principal $\mathbb{G}_m$-bundle is locally a map $\mathrm{Spec}(R_i) \times \mathbb{G}_m \to \mathrm{Spec}(R)$, so a $\mathbb{G}_m$-equivariant map to $\mathcal{M}^s_{fgl}$ is just a map $\mathrm{Spec}(R_i) \to \mathcal{M}^s_{fgl}$ over $\mathrm{Spec}(R)$, which gives a formal group law locally. Reviewing back to Proposition 2.7, we observe that the following proposition should be true:

Proposition 3.8. *There is a canonical equivalence*

$$\mathcal{M}^s_{fgl}/\mathbb{G}_m \simeq \mathcal{M}_{fg}.$$

Remark 3.9. *The above analysis is not a real proof, one should check Proposition 12.8 in Piotr Pstragowski's note.*

To achieve our final goal, we need to use the following lemma, which is not hard to prove by writing out the definitions and noticing the cosimplical ring is determined in degrees 0 and 1:

Lemma 3.10. *If R is a homotopy commutative ring spectrum such that R_*R is flat over R_*, then we have*

$$\mathrm{Comod}_{R_*R} \simeq \varprojlim \mathrm{Mod}_{\pi_*(R^{\otimes n})} . \tag{1}$$

Combining all the above analysis, we deduce the description we desired:

Theorem 3.11. *We have the canonical equivalence:*

$$\mathrm{Qcoh}(\mathcal{M}_{fg}) \simeq \mathrm{Comod}^{ev}_{MU_*MU}$$

Now, Proposition 3.4 is very easy to prove by observing for any spectrum X, $MU_{\mathrm{even}}(X)$ is a natural MU_*MU-comodule.

Remark 3.12. *Notice that there is a better way to tell this story. We can define graded formal groups from the very beginning, and everything can be discussed in the setting of Dirac Geometry. In this way, we do not need to discuss the $\mathbb{G}_m$-action. This will appear in Lars Hesselholt and Piotr Pstragowski's paper Dirac Geometry 2.*

4 Height Filtration

From Example 2.9 and more algebraic geometry, we can deduce the structure of $\mathcal{M}_{fg}$ over $\mathbb{Q}$-algebras:

Proposition 4.1. *There is an equivalence*

$$\mathrm{Spec}(\mathbb{Q}) \times \mathcal{M}_{fg} \simeq \mathrm{Spec}(\mathbb{Q}) \times B\mathbb{G}_m.$$

between the rational moduli stack of formal groups and the rational classifying stack of the multiplicative group.

The above proposition suggests we should now focus our attention with

$$\mathrm{Spec}(\mathbb{Z}_{(p)}) \times \mathcal{M}_{fg}$$

the moduli of formal group over $\mathbb{F}_p$-algebras. For that, we will introduce the concept of height, which gives the height filtration of $\mathcal{M}_{fg}$.

Suppose R is a commutative ring of characteristic p. Then we have the Frobenius homomorphism $\varphi_R : R \to R$ given by $\varphi_R(x) = x^p$. For an R-algebra A with structure map $f : R \to A$ and any integer $n \geq 0$, there is an R-algebra $A^{\frac{1}{p^n}}$ defined by being the same ring as A and having the structure map $A \xrightarrow{\varphi_R^n} A \xrightarrow{f} R$. Let X be a functor $\mathrm{CAlg}_R^{\heartsuit} \to \mathrm{Ab}$, where $\mathrm{CAlg}_R^{\heartsuit}$ denotes the discrete commutative ring over R. For each $n \geq 0$, X induces a functor $X^{(p^n)}: \mathrm{CAlg}_R^{\heartsuit} \to \mathrm{Ab}$ by defining $X^{(p^n)}(A) = X(A^{\frac{1}{p^n}})$. Moreover, the canonical map $A \to A^{\frac{1}{p^n}}$ induces a natural transformation

$$\varphi^n_{X/R} : X \to X^{(p^n)},$$

which we denote as the **relative Frobenius map**.

Definition 4.2. *Let R be a commutative ring, and let G be a 1-dimensional formal group over R. We have a map $[p] : G \to G$ induced by multiplication by p. For $n \geq 1$, we say G has* **height $\geq$n** *if $p = 0$ in R and $[p]$ factors through $\varphi^n_{G/R}$, as in the diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi^n_{G/R}"] \arrow[dr] & G^{(p^n)} \arrow[d, dashed] \\ & G. \end{tikzcd}$$

We also define all formal groups over R has height ≥ 0. Moreover, if G is of height at least n for all $n \geq 0$, we say G has height ∞.

Remark 4.3. *Notice that height is a concept depending on the prime we choose. But this confusion is very slight. Because if a formal group G is of height ≥ 1 for a fixed prime, then for any other prime, G cannot have positive height. There is essentially one meaningful height for a formal group.*

The definition of height can be very explicit for formal group laws. If $G = G_f$ is a formal group over R with formal group law f. Then $[p] : G \to G$ is equivalent a morphism of f, which gives a power series:

$$[p]_f(x) = f(x, f(x, f(x, \cdots))) = px + \text{higher terms},$$

where x appears p times. This means we add x p times using the formal group law f. Then $[p]$ factors through $\varphi^n_{G/R}$ is equivalent to $[p]_f(x)$ can be written as a formal power series in x^{p^n}, i.e.

$$[p]_f(x) = h(x^{p^n})$$

for some power series h. If we study the universal formal group law $F \in L[[x, y]]$, this can also produce a p-series:

$$[p]_F(x) = v_0 x + \cdots + v_1 x^p + \cdots + v_n x^{p^n} + \cdots$$

where $v_i \in L$ and $v_0 = p$. Then v_i can be understood as a function $\mathrm{FGL}(R) \to R$ by mapping f to the coefficient of x^{p^i} in the power series $[p]_f$. Moreover, we have the following non-trivial proposition:

Proposition 4.4. *Let G be a 1-dimensional formal group over R such that $G \simeq G_f$ for some formal group law $f \in \mathrm{FGL}(R)$. We define the n-the Landweber ideal of G as*

$$\mathcal{J}^G_n := (v_0(f), v_1(f), \cdots, v_{n-1}(f)) \subset R.$$

This is independent of the choice of the formal group law. We have G is of height $\geq n$ iff $\mathcal{J}^G_n = 0$ in R.

Definition 4.5. *Let G be a 1-dimensional formal group over R, we define the* **n-th Landweber ideal** $\mathcal{J}_n^G$ *of G as the unique ideal such that over any affine open* $\mathrm{Spec}(S)$ *such that* $G|_{\mathrm{Spec}(S)} \simeq G_f$ *for some formal group law* $f \in \mathrm{FGL}(S)$*, we have* $S \otimes \mathcal{J}_n^G = \mathcal{J}_n^{G|_{\mathrm{Spec}(S)}}$ *as ideals of S.*

The above construction also defines a quasi-coherent ideal sheaf $\mathcal{J}_n$ of $\mathcal{M}_{fg}$. Because for each R-point $\mathrm{Spec}(R) \to \mathcal{M}_{fg}$, which is the same as a formal group over R, the above construction defines a finitely generated ideal $\mathcal{J}_n^G$ in R. From general algebraic geometry, we know this gives a closed substack. Moreover, we have the following proposition:

Proposition 4.6. *Fix a prime p. Let $\mathcal{M}_{fg}^{\geq n}$ be the substack of $\mathcal{M}_{fg}$ mapping R to the groupoid of formal groups over R of height $\geq n$. Then the inclusion $\mathcal{M}_{fg}^{\geq n} \to \mathcal{M}_{fg}$ is the closed immersion defined by the quasi-coherent ideal sheaf $\mathcal{J}_n$.*

It follows that we have a filtration of $\mathcal{M}_{fg} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$, the moduli stack of formal groups over $\mathbb{Z}_{(p)}$ algebras, by closed substacks:

$$\mathcal{M}_{fg} \times \mathbb{Z}_{(p)} \hookleftarrow \mathcal{M}_{fg}^{\geq 1} \hookleftarrow \mathcal{M}_{fg}^{\geq 2} \hookleftarrow \cdots \hookleftarrow \mathcal{M}_{fg}^{\infty}$$

This is called the **height filtration**, and also called chromatic filtration. Using this, we can deduce a lot of interesting properties of spectra. One can check Maxime's note for an overview of this theory.

Using Landweber exact ideal, we can define formal groups of exact height:

Definition 4.7. *Let R be an E_∞-ring and let G be a 1-dimensional formal group over R. Then this induces a 1-dimensional formal group $\tilde{G}$ over $\pi_0(R)$. We define the* **n-th Landweber ideal** $\mathcal{J}_n^G$ *as* $\mathcal{J}_n^{\tilde{G}}$ *being an ideal of $\pi_0(R)$. Then G is said to have* **height $<$ n** *if we have $\mathcal{J}_n^G = \pi_0(R)$.*

Let R be a commutative ring and let G be a 1-dimensional formal group over R. Then G is said to have **height exact n** *if it has height $\geq n$ and has height $< n+1$. This is equivalent to $\mathcal{J}_{n+1}^G = \pi_0(R)$ and $\mathcal{J}_n^G = (0)$.*

Remark 4.8. *One might think if G has height $\geq n$, and does not have height $\geq n+1$, then G is height exact n. But this is not true in general. In fact, when $G = G_f$, the above condition implies*

$$[p]_f(x) = \lambda x^{p^n} + \textit{higher terms}, \quad \textit{where } \lambda \neq 0 \in R.$$

G_f is height exact n only if λ is a unit. This also implies that over a field, any formal group has height n for some integer $0 \leq n \leq \infty$.

At the end of this section, let us give some examples.

Example 4.9. *Consider the formal additive group G_a with $f(x,y) = x+y$. Then obviously, we have $[p]_f(x) = px$. Over any $\mathbb{F}_p$-algebra, G_a has height ∞.*

Example 4.10. *Consider the formal multiplicative group G_m with $f(x,y) = x + y + xy$. Then Obviously, we have $[p]_f(x) = px + x^p$. Over any $\mathbb{F}_p$-algebra, G_a has height* 1.

Combining the two examples, we can see that G_a and G_m are not isomorphic over any $\mathbb{F}_p$-algebra.

Example 4.11. *From Proposition 4.4, we can see that a formal group law has height ≥ 1 iff $v_0 = p = 0$. Hence, over any $\mathbb{F}_p$-algebra, any formal group is at least height 1, which means $\mathcal{M}_{fg}^{\geq 1} \simeq \mathcal{M}_{fg} \times \mathrm{Spec}(\mathbb{F}_p)$.*

5 p-divisble groups and Deformation theory

In the next two sections, we will formulate Lurie's construction of Lubin-Tate spectra. This is a theory I'm unfamiliar with, but hopefully, it can at least provide a rough idea of what the construction looks like.

Let us state the classical result of deformation theory, which will also be used in section 7.

Definition 5.1. *Let G_0 be a formal group defined over a field κ, and let A be a complete Noetherian ring A and a ring homomorphism $\rho_A : A \to k$ which induces an isomorphism $A/m_A \simeq \kappa$. A* **deformation of G_0 along** *ρ_A is a pair (G, α), where G is a formal group over A, and α is an isomorphism between G_0 and G_κ, where G_κ is the formal group obtained from G by extension of scalars along ρ_A. The collection of all such deformations can be organized into a category $\mathrm{Def}_{G_0}(A, \rho_A)$, which depends functorially on A.*

Lubin and Tate showed that in some situations, we have a universal deformation to describe all the possible deformations out of a formal group G_0:

Theorem 5.2. *Let κ be a perfect field of characteristic $p > 0$ and let G_0 be a 1-dimensional formal group of height $0 < n < \infty$. Then there exists a complete local Noetherian ring R_{LT}, a ring homomorphism $\rho_{LT} : R_{LT} \to \kappa$, which induces an isomorphism $R_{LT}/m_{LT} \simeq \kappa$, and a deformation (G, α) of G_0 along rho_{LT} satisfying the following property:*

For any complete local ring A with a map $\rho_A : A \to \kappa$ inducing isomorphism $A/m_A \simeq \kappa$, then extension of scalars induces an equivalence

$$\mathrm{Hom}_{/\kappa}(R_{LT}, A) \simeq \mathrm{Def}_{G_0}(A, \rho_A).$$

In particular, it tells us that the category $\mathrm{Def}_{G_0}(A, \rho_A)$ is always discrete.

Remark 5.3. *Intuitively, the above theorem is trying to describe the following phenomenon: For a formal group law of height n, we know its p-series has coefficient $v_0, v_1, \cdots, v_{n-1}$ being 0 and v_n being an invertible element of κ by the definition of height. Notice that there is a non-canonical equivalence $A \simeq W(\kappa)[[u_1, \cdots, u_{n-1}]]$, where $W(\kappa)$ is the ring of Witt vectors of κ. The theorem implies that the universal deformation ring gives the exact freedom for $v_1, \cdots, v_{n-1}$ to be non-zero. In a way, a deformation of a formal group is really a deformation of its p-series.*

Jacob Lurie extends this theory in the following three aspects:

(1) He replaces formal groups with p-divisible groups, a more general concept. As we will see, they contain information about informal groups. Moreover, p-divisible groups allow some finite dimension phenomenon exists, but formal groups have to have underlying infinite-dimensional polynomial algebra.

(2) He relaxes the assumption κ being a perfect field. In the theorem, we will start from a perfect $\mathbb{F}_p$ algebra, which means the Frobenius homomorphism is an isomorphism.

(3) The deformation range will also be much larger. In this case, he allows deforming over connective E_∞ rings rather than just discrete rings.

To begin with, let us define p-divisible groups:

Definition 5.4. *Let R be a connective E_∞ ring. A* **p-divisible group over** *R is a functor $\boldsymbol{G} : \mathrm{CAlg}_R^{cn} \to \mathrm{Mod}_{\mathbb{Z}}^{cn}$ with the following properties:*

1. *For every object $A \in \mathrm{CAlg}_R^{cn}$, we always have $\boldsymbol{G}(A)[1/p] \simeq 0$.*
2. *For every finite abelian p-group M, the functor $\mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}}}(M, \boldsymbol{G}(-)) : \mathrm{CAlg}_R^{cn} \to \mathrm{An}$ is corepresentable by a finite flat R-algebra.*
3. *The map $p : \boldsymbol{G} \to \boldsymbol{G}$ is locally surjective with respect to the finite flat topology. In other words, for every object $A \in \mathrm{CAlg}_R^{cn}$ and every element $x \in \pi_0(\boldsymbol{G}(A))$, there exists a finite flat map $A \to B$ for which $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is subjective, and the image of x in $\pi_0(G(B))$ is divisible by p.*

We let $BT^p(R)$ to denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R^{cn}, \mathrm{Mod}_{\mathbb{Z}}^{cn})$ spanned by p-divisible groups. For non-connective E_∞ ring R, we define $BT^p(R) \simeq BT^p(\tau_{\geq 0}R)$.

The connection between p-divisible groups and formal groups are reflected in the following theorems:

Theorem 5.5. *Let R be a p-complete E_∞ ring and let $\boldsymbol{G}$ be a p-divisible group over R. Then there exists an essentially unique formal group $\boldsymbol{G}^\circ \in \mathrm{FGroup}(R)$ with the following property:*

Let $\mathcal{E} \subset \mathrm{CAlg}_{\tau_{\geq 0}R}^{cn}$ be the full subcategory spanned by those algebras which are truncated and p-nilpotent. Then the functor $\boldsymbol{G}^\circ|_{\mathcal{E}}$ is given by the construction $A \mapsto \mathrm{fib}(\boldsymbol{G}(A) \to \boldsymbol{G}(A^{red}))$, where A^{red} is the derived reduced ring of A.

Definition 5.6. *Let R be a p-complete E_∞ ring and let $\boldsymbol{G}$ be a p-divisible group over R. Then the formal group $\boldsymbol{G}^\circ$ appearing in the theorem above is called the* **identity component** *of $\boldsymbol{G}$.*

In the case when p-divisible groups are connected, there is a stronger relationship with formal groups.

Definition 5.7. *Let R be a connective E_∞-ring and let $\boldsymbol{G}$ be a p-divisible group over R. We denote $\boldsymbol{G}[p]$ as the functor given by $A \mapsto \underline{\mathrm{Map}}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \boldsymbol{G}(A))$. By definition, we know the* *functor* $(\Omega^\infty \circ \boldsymbol{G}[p]) : \mathrm{CAlg}_R^{cn} \to \mathrm{An}$ *is corepresentable by a finite flat* R*-algebra* A*. If the underlying map of topological spaces* $|\operatorname{Spec}(A)| \to |\operatorname{Spec}(R)|$ *is bijective, then we say* $\boldsymbol{G}$ *is a* **connected p-divisible group***.*

More generally, if $\boldsymbol{G}$ *is a p-divisible group over an arbitrary* E_∞*-ring* R*, then we say it is connected if it is connected regarded as a p-divisible group over* $\tau_{\geq 0}R$*.*

Definition 5.8. *An adic* E_∞*-ring is a pair* (R, τ)*, where* R *is an* E_∞*-ring and* τ *is an adic topology on* $\pi_0(R)$*. Notice that the topology on* $\pi_0(R)$ *can be given by a finitely generated ideal* I*. An adic* E_∞*-ring* R *is called complete if for every* $x \in I$*, the ring* $\pi_0(R)$ *is* (x)*-complete.*

Theorem 5.9. *Let* R *be a complete adic* E_∞*-ring and suppose that p is topologically nilpotent in* $\pi_0(R)$*. Let* $BT^p(R)^{cd}$ *denote the full subcategory of* $BT^p(R)$ *spanned by the formally connected p-divisible groups over* R*. Then the construction* $\boldsymbol{G} \to \boldsymbol{G}^\circ$ *induces a fully faithful functor* $BT^p(R)^{cd} \to \mathrm{FGroup}(R)$*.*

We also have the following theorem to describe a part of the essential image of the functor above, which will be used in the next section:

Theorem 5.10. *Let* R *be a commutative* $\mathbb{F}_p$*-algebra and let* G *be a* 1*-dimensional formal group over* R *having exact height* n*. Then there exists a connected p-divisible group* $\boldsymbol{G}$ *over* R *of height* n *and an isomorphism* $G \simeq \boldsymbol{G}^\circ$*.*

Let us state Lurie's meaning of deformations. In the original definitions, we need to give a map from the field to the extended ring at first. Lurie generalized this by putting the map as a part of the data of deformation:

Definition 5.11. *Let* R_0 *be a commutative ring, and* A *be an adic* E_∞*-ring. We consider a p-divisible group* $\boldsymbol{G}_0$ *over* R_0 *and a p-divisible group* $\boldsymbol{G}$ *over* A*. A* **$\boldsymbol{G}_0$-tagging** *of* $\boldsymbol{G}$ *is a triple* (I, μ, α)*, where* $I \subset \pi_0(A)$ *is a finitely generated ideal,* $\mu : R_0 \to \pi_0(A)/I$ *is a ring homomorphism, and* $\alpha : (\boldsymbol{G}_0)_{\pi_0(A)/I} \simeq \boldsymbol{G}_{\pi_0(A)/I}$ *is an isomorphism of p-divisible groups over the commutative ring* $\pi_0(A)/I$*.*

We say that a pair of $\boldsymbol{G}_0$*-taggings* (I, μ, α) *and* (I', μ', α') *are equivalent if there exists a finitely generated ideal* $J \subset \pi_0(A)$ *containing both* I *and* I' *for which the diagram of ring homomorphisms*

$$\begin{tikzcd} R_0 \arrow[r, "\mu"] \arrow[d, "\mu'"'] & \pi_0(A)/I \arrow[d] \\ \pi_0(A)/I' \arrow[r] & \pi_0(A)/J \end{tikzcd}$$

commutes, and the isomorphisms α *and* α' *agree when restricted to* $\pi_0(A)/J$*.*

Example 5.12. *Let* R_0 *be a commutative ring and let* $\boldsymbol{G}_0$ *be a p-divisble group over* R_0*. Then* $\boldsymbol{G}_0$ *is equipped with the tautological* $\boldsymbol{G}_0$ *tagging* $(0, id_{R_0}, id_{\boldsymbol{G}_0})$*.*

Definition 5.13. *Let* $\boldsymbol{G}_0$ *be a p-divisible group defined over a commutative ring* R_0 *and let* A *be an adic* E_∞*-ring. A deformation of* $\boldsymbol{G}_0$ *over* A *consists of a p-divisble group* $\boldsymbol{G}$ *over* A *together with an equivalence class of* $\boldsymbol{G}_0$*-taggings of* $\boldsymbol{G}$*. The collection of deformations of*

$\boldsymbol{G}_0$ *over* A *can be organized into a* ∞*-category* $\mathrm{Def}_{G_0}(A)$. *More precisely, we let* $\mathrm{Def}_{\boldsymbol{G}_0}(A)$ *denote the filtered colimit*

$$\operatorname*{colim}_{I} BT^p(A) \times_{BT^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I).$$

where I *ranges over all finite generated ideals* $I \subset \pi_0(A)$. *Here* $\mathrm{Hom}(R_0, \pi_0(A)/I)$ *denote the set of ring homomorphisms from* R_0 *to* $\pi_0(A)/I$.

Similar to the case of discrete rings that $\mathrm{Def}_{\kappa}(G)$ is discrete, we have the following proposition:

Proposition 5.14. *Let* R_0 *be a commutative ring and let* $\boldsymbol{G}_0$ *be a* p*-divisible group over* R_0. *For every complete adic* E_∞*-ring* A*, the* ∞*-category* $\mathrm{Def}_{\boldsymbol{G}_0}(A)$ *is an* ∞*-groupoid.*

Definition 5.15. *Let* R_0 *be a commutative ring and let* $\boldsymbol{G}_0$ *be a* p*-divisible group over* R_0. *Let* $\boldsymbol{G}$ *be a deformation of* R_0 *over a complete adic* E_∞*-ring* R. *We say* $\boldsymbol{G}$ *is a* **universal deformation** *if, for every complete adic* E_∞*-ring* A*, extension of scalars induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}^{ad}_{cpl}}(R, A) \to \mathrm{Def}_{\boldsymbol{G}_0}(A).$$

Notice that if $\boldsymbol{G}_0$ *admits a universal deformation, the pair of universal deformations is unique up to equivalence. We denote* R *as* $R^{un}_{\boldsymbol{G}_0}$ *and call it the* **spectral deformation ring** *of* $\boldsymbol{G}$.

The following is the theorem what we are going to use:

Theorem 5.16. *Let* R_0 *be a perfect* $\mathbb{F}_p$*-algebra and* $\boldsymbol{G}_0$ *be a* p*-divisible group over* R_0. *Then* $\boldsymbol{G}_0$ *admits a universal deformation such that the spectral deformation ring* $R^{un}_{\boldsymbol{G}_0}$ *is a connective* E_∞*-ring.*

Moreover, The canonical map $\rho : R^{un}_{\boldsymbol{G}_0} \to R_0$ *induces a surjective ring homomorphism* $\epsilon : \pi_0(R^{un}_{\boldsymbol{G}_0}) \to R_0$*, and* $R^{un}_{\boldsymbol{G}_0}$ *is complete with respect to* $ker(\epsilon)$.

6 Orientations and Lurie's construction of Lubin-Tate spectra

Lurie generalized orientations on formal groups. This is the major difference between his construction and classical construction. Here, he discovered that the functor of orientations is representable by some E_∞-ring. As we will see, this will turn out to be the Lubin-Tate spectra.

Definition 6.1. *Let* R *be an* E_∞*-ring. A* **pointed formal hyperplane over R** *is a functor* $X : \mathrm{CAlg}^{cn}_{\tau_{\geq 0}R} \to \mathrm{An}_*$ *with the property that the composition functor*

$$\mathrm{CAlg}^{cn}_{\tau_{\geq 0}R} \to \mathrm{An}_* \to \mathrm{An}$$

is a formal hyperplane over R.

Definition 6.2. *Let R be an E_∞-ring and let $X : \mathrm{CAlg}^{cn}_{\tau_{\geq 0}R} \to \mathrm{An}_*$ be pointed formal hyperplane over R. Then a* **preorientation** *of X is a map of pointed spaces*

$$e : S^2 \to X(\tau_{\geq 0}R).$$

We let $\mathrm{Pre}(X) = \Omega^2 X(\tau_{\geq 0}R)$ denote the ∞-groupoid of preorientations of X. A **preoriented formal hyperplane** *is a pair (X, e), where X is a pointed formal hyperplane over R and $e \in \mathrm{Pre}(X)$ is a preorientation.*

For a formal group G over R, a **preorientation** *of G is a preorientation of the underlying pointed formal hyperplane $X = \Omega^\infty G$ choosing the zero section as the base point. We let $\mathrm{Pre}(G) = \mathrm{Pre}(X)$ denote the ∞-groupoid of preorientations of G.*

Recall that in Example 1.8, we introduce the Quillen formal group for any complex periodic E_∞-ring R. A Quillen formal group $G_R^{\mathcal{Q}}$ has the underlying formal hyperplane $C_*(\mathbb{C}P^\infty, R)$. We have the following lemma:

Lemma 6.3. *Let R be a complex periodic E_∞-ring, and let $G_R^{\mathcal{Q}} \in \mathrm{FGroup}(R)$ denote the Quillen formal group and let G be any formal group over R. Then we have a canonical homotopy equivalence*

$$\mathrm{Pre}(G) \simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(G_R^{\mathcal{Q}}, G).$$

Proof. Let C denote the image of G under the equivalence

$$\mathrm{FGroup}(R) \simeq \mathrm{Ab}(\mathrm{Hyp}(R)) \simeq \mathrm{Ab}(\mathrm{cCAlg}_R^{sm}),$$

where the second functor is cSpec in Proposition 1.3. Then by Dold-Thom theorem and the induced-restriction adjunction between An_* and $\mathrm{Ab}(An)$, we have

$$\mathrm{Pre}(G) \simeq \mathrm{Map}_{\mathrm{An}_*}(S^2, \Omega^\infty G(\tau_{\geq 0}R)) \simeq \mathrm{Map}_{\mathrm{Ab}(S)}(\mathbb{C}P^\infty, \mathrm{Map}_{\mathrm{cCAlg}_R}(R, C)),$$

where the big Map means we are taking mapping spaces. Then Since the symmetrical monoidal structure on $\mathrm{Ab}(\mathrm{cCAlg}_R)$ is closed, we have

$$\mathrm{Map}_{\mathrm{Ab}(S)}(\mathbb{C}P^\infty, \mathrm{Map}_{\mathrm{cCAlg}_R}(R, C)) \simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{cCAlg}_R)}(C_*(\mathbb{C}P^\infty, R), C).$$

Using the fact that cSpec is an equivalence, we have

$$\mathrm{Map}_{\mathrm{Ab}(\mathrm{cCAlg}_R)}(C_*(\mathbb{C}P^\infty, R), C) \simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{FGroup}(R))}(\mathrm{cSpec}(C_*(\mathbb{C}P^\infty, R)), \mathrm{cSpec}(C)).$$

This implies

$$\mathrm{Pre}(G) \simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{FGroup}(R))}(G_R^{\mathcal{Q}}, G).$$

$\square$

Next, we will define the orientation of formal groups. We will give a different definition from Lurie to avoid talking about line bundles. Lurie defined orientations on the underlying pointed formal hyperplanes of formal groups. The two definitions are equivalent by Proposition 4.3.23 in Elliptic Cohomology 2.

Definition 6.4. *Let R be a complex periodic E_∞-ring and let G be a 1-dimensional formal group over R. A preorientation $e \in \mathrm{Pre}(G)$ is an* **orientation** *if the morphism $f : G_R^{\mathcal{Q}} \to G$ induced by e is an equivalence of formal groups. We denote $\mathrm{OrDat}(G)$ as the full subcategory of $\mathrm{Pre}(G)$ spanned by all isomorphisms of formal groups.*

Theorem 6.5. *Let R be a complex periodic E_∞-ring, and let G be a 1-dimensional formal group over R. Then there exists an E_∞-algebra $\mathcal{O}_G$ over R and an orientation $e \in \mathrm{OrDat}(G_{\mathcal{O}_G})$ which is universal in the following sense:*

For every object $R' \in \mathrm{CAlg}_R$, the evaluation on e induces a homotopy equivalence:

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathcal{O}_G, R') \to \mathrm{OrDat}(G_{R'})$$

Definition 6.6. *Let R be a complex periodic E_∞-ring, and let G be a 1-dimensional formal group over R. We define the E_∞-algebra $\mathcal{O}_G$ as the* **orientation classifier** *of G.*

There is one last thing we need to do before formulating Lurie's construction. We have to introduce Morava K-theory, which is another important spectrum in chromatic homotopy theory. Geometrically speaking, $K(n)$ represents the formal neighbourhood of a closed point in the moduli stack of formal groups.

Definition 6.7. *Fix a prime p and an integer $n > 0$. By Quillen and Lazard's theorem, we can assume*

$$\pi_* MU_{(p)} \simeq L_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, t_2, \cdots]$$

We can assume that $v_i = t^{p^i - 1}$ for each $i > 0$, and by convention we set $t_0 = p$. For each integer $k \geq 0$, we define

$$M(k) := \mathrm{fib}(\Sigma^{2k} MU_{(p)} \xrightarrow{v_i} MU_{(p)}).$$

Moreover, recall that

$$MU_{(p)}[v_n^{-1}] = \operatorname*{colim}_i (MU_{(p)} \xrightarrow{v_i} MU_{(p)} \xrightarrow{v_i} MU_{(p)} \cdots).$$

We define the **Morava K-theory** *$K(n)$ to be the smash product over $MU_{(p)}$ of $MU_{(p)}[v_n^{-1}]$ with $\bigotimes_{k \neq p^n - 1} M(k)$.*

Construction 6.8. *Let R_0 be a perfect $\mathbb{F}_p$-algebra and let G_0 be a 1-dimensional formal group of exact height n over R_0 (Definition 4.7).*

1. *By Theorem 5.10, there is a connected p-divisible group $\boldsymbol{G}_0$ over R_0 such that the identity component of $\boldsymbol{G}_0$ is G_0.*

2. *By Theorem 5.16, we have $R^{un}_{\boldsymbol{G}_0}$ and $\boldsymbol{G}$ being the spectral defomration ring and the universal deformation for $\boldsymbol{G}_0$.*

3. *By Theorem 6.5, there is an E_∞-algebra, denoted as $R^{or}_{G_0}$, which is the orientation classifier of the identity component $\boldsymbol{G}^\circ$ of $\boldsymbol{G}$ above.*

4. *We let $E(G_0)$ to denote the $K(n)$-localisation $L_{K(n)} R^{or}_{G_0}$, and refer to it as the* **Lubin-Tate spectrum of** G_0.

Remark 6.9. *In the case of perfect fields, we don't need to do the final step in the above construction because the orientation classifier would already be $K(n)$-local in step* 3.

Importantly, Lurie's construction also gives a universal property of the Lubin-Tate spectra. For that property, we need to fix some notations as follows:

Definition 6.10. *We define a* 1*-category $\mathcal{FG}$ as follows:*

- *The objects of $\mathcal{FG}$ are pairs (R, G), where R is a commutative ring and G is a* 1*-dimensional formal group over R.*
- *A morphism from (R, G) to (R', G') in $\mathcal{FG}$ is a pair (f, α), where $f : R \to R'$ is a ring homomorphism and $\alpha : G_{R'} \to G'$ is an isomorphism of formal groups over R'.*

Definition 6.11. *Let R be a complex periodic E_∞-ring. Recall from Example 1.8, we have the Quillen formal group $G_R^{\mathcal{Q}}$. This induces the classical Quillen formal group $G_R^{\mathcal{Q}_0}$ over $\pi_0(R)$. By Definition 4.7, we have the n-th Landweber exact ideal of $\mathcal{J}_n^{G_R^{\mathcal{Q}_0}}$, and we define*

$$\mathcal{J}_n^R := \mathcal{J}_n^{G_R^{\mathcal{Q}_0}}$$

Moreover, along the scalar extension $\pi_0(R) \to \pi_0(R)/\mathcal{J}_n^R$, we define

$$G_R^{\mathcal{Q}_n} := (G_R^{\mathcal{Q}_0})_{\pi_0(R)/\mathcal{J}_n^R}$$

Theorem 6.12. *Let R_0 be a perfect $\mathbb{F}_p$-algebra, and G_0 be a formal group of exact height n over R_0. Then we have the Lubin-Tate spectrum $E(G_0)$ satisfying the following universal property:*

For every complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring A, this is a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}}(E(G_0), A) \to \mathrm{Hom}_{\mathcal{FG}}((R_0, G_0), (\pi_0(A)/\mathcal{J}_n^A, G_A^{\mathcal{Q}_n}).$$

Definition 6.13. *Fix a prime p and a integer $n > 0$, the* **Morava E-theory** *$E(n)$ is defined as $E(G_0)$ for some* 1*-dimensional formal group of exact height n over $\mathbb{F}_p$.*

From the notation, one can see that $E(n)$ is not well defined since different formal groups will give different Lubin-Tate spectra. However, the Bousfield localization functor $L_{E(n)}$ is independent of the choice, which will be enough in many cases.

Before ending this section, let us comment on the relation between $E(n)$ and $K(n)$. From Remark 6.9, the definition of $E(n)$ is independent of $K(n)$, and using $E(n)$, one can have a simpler description of $K(n)$ as

$$K(n) \simeq E(n)/p \otimes E(n)/v_1 \otimes E(n)/v_2 \otimes \cdots \otimes E(n)/v_{n-1}.$$

7 Landweber Exactness and Classical construction of Lubin-Tate Spectra

Let M be a graded MU_*-module. Then we naturally have a functor

$$\begin{aligned} - \otimes M : \mathrm{Sp} &\to \mathrm{Mod}_{MU_*} \\ X &\mapsto MU_* X \otimes_{MU_*} M \end{aligned}$$

Towards our goal to build some spectra, a natural question we could ask is when this is an exact functor. One obvious answer is when R is a flat module. However, this would be too strong in our case since by Quillen's theorem, we know MU_* is a polynomial ring with infinitely many variables. For R to be flat implies that all the variables must act injectively on R. The Landweber exact functor theorem will give a better answer to this question.

Note that the association of a formal group law to a formal group induces a flat covering map:

$$p : \mathrm{Spec}(MU_*) \to \mathcal{M}_{fg}.$$

This induces an adjunction:

$$p^* : \mathrm{Qcoh}(\mathcal{M}_{fg}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Qcoh}(MU_*) : p_*$$

By Theorem 3.11 of section 3, this is the forgetful-cofree adjunction:

$$\mathrm{Comod}^{ev}_{MU_*MU} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Mod}^{ev}_{MU_*}$$

Recall that Proposition 3.4 tells us we have a functor $F : \mathrm{Sp} \to \mathrm{Qcoh}(\mathcal{M}_{fg})$ sending X to $\mathcal{F}_X$, where $\mathcal{F}_X$ corresponding to the even graded MU_*MU-comodule $MU_*^{ev}(X)$. Hence, we know $p^*\mathcal{F}_X \simeq MU_*^{ev}(X)$ as an MU_*-module. Suppose M is an even-graded MU_*MU-comdule, so we can assume $M \simeq p^*\mathcal{F}_M$ as MU_* modules, then, when $MU_*(X)$ is in even-degrees, we have

$$MU_*(X) \otimes_{MU_*} M \simeq p^*\mathcal{F}_X \otimes_{MU_*} p^*\mathcal{F}_M \simeq p^*(\mathcal{F}_X \otimes_{\mathcal{M}_{fg}} \mathcal{F}_M).$$

Here, we use that p^* is symmetric monoidal. In general, $MU_*(X)$ will determine two quasi-coherent sheaves of $\mathcal{M}_{fg}$ by twisting the odd part of its homology, and we can get similar formulas for each one separately. Also, when M is not even-graded, we can do a similar separating operation to treat the odd part and even part separately. The flatness of p implies p^* is an exact functor, and this gives us the following proposition:

Proposition 7.1. *Let M be an even graded MU_*MU-comdule such that the corresponding quasi-coherent sheaf $\mathcal{F}_M$ is flat over $\mathcal{M}_{fg}$. Then the functor $X \mapsto MU_*X \otimes_{MU_*} M$ determines a homology theory.*

The above proposition inspires us to study flat quasi-coherent sheaves over $\mathcal{M}_{fg}$. And for this, we have the Landweber exactness theorem:

Theorem 7.2. *Let M be an even graded MU_*MU-comdule. Then the corresponding sheaf $\mathcal{F}_M$ is flat over $\mathcal{M}_{fg}$ if and only if for each prime p, the sequence $(p, v_1, v_2, \cdots)$ is a regular sequence of M as a MU_*-module, which means that $v_i \in MU_*$ acts injectively on $M/(p, v_1, \cdots, v_{n-1})$.*

Notice that when M is only a MU_*-module, the conditions in the above theorem can also be satisfied. In this case, we can consider the cofree comodule $p_*M \simeq MU_*MU \otimes_{MU_*} M$, and one can prove that $(p, v_1, \cdots)$ is still a regular sequence. Hence, by the above two results, we have $X \mapsto MU_*MU \otimes_{MU_*} M \otimes_{MU_*} MU_*(X)$ determines a homology theory. Using the fact that MU_*MU is a faithfully flat right MU_*-module, we can have the following form of Landweber exact functor theorem:

Theorem 7.3. *Let M be a graded MU_*-module such that for each prime p, we have $(p, v_0, v_1, \cdots)$ is a regular sequence for M. Then we have the functor $X \mapsto MU_*X \otimes_{MU_*} M$ defines a homology theory.*

Using this, we can build homology theory out of formal groups. Let F be a formal group law over R, which is equivalent to giving a ring homomorphism $MU_* \to R$. Then Theorem 7.2 can be translated to the following: R as a MU_*-module satisfies the conditions of Theorem 7.2 if and only if the map $\mathrm{Spec}(R) \to \mathrm{Spec}(L) \to \mathcal{M}_{fg}$ is flat. Because being flat is a local condition, we can give the following proposition:

Proposition 7.4. *Let G be a 1-dimensional formal group over R so that it is classified by a map $\mathrm{Spec}(R) \to \mathcal{M}_{fg}$. The map $\mathrm{Spec}(R) \to \mathcal{M}_{fg}$ is flat is equivalent to the following condition: There is a covering of G by formal group laws such that each formal group law determines a module over MU_* satisfying the condition that $(p, v_1, v_2, \cdots)$ is a regular sequence for each prime. In this case, we call G a* **Landweber exact** *formal group.*

We want to see how this will give a homology theory. Let us focus on formal group laws. Note that to get a homology theory, it is not enough to provide a MU_*-module. Instead, we need to provide a graded MU_*-module over the graded ring MU_*. A formal group law does not come with a grading naturally. However, we can fix this issue by the following construction:

Construction 7.5. *Let R be a ring and $L \to R$ a formal group law, where $L \cong MU_*$ is the Lazard ring. The grading of L is as follows: Notice that L is generated by all the formal coefficients $a_{i,j}$. We put a grading on L by require $|a_{i,j}| = 2(i + j - 1)$. Quillen's theorem proves that this grading is compatible with the natural grading of MU_*. Viewing L as a graded ring, there exists a unique factorization*

$$L \to R[u^{\pm}] \to R,$$

where the second map sends u to 1, and the first map is a map of graded rings with R in degree 0 and $|u| = 2$. This determines a graded formal group law over $R[u^{\pm}]$.

Now, using the above construction and Proposition 7.3, we see that for any formal group law F over R such that G_F is Landweber exact, there is an even periodic homology theory given by

$$(E_F)_*(X) := R[u^{\pm}] \otimes_{MU_*} MU_*(X)$$

Let X be a spectrum, and then we have

$$(E_F)_0(X) \simeq R \otimes_{MU_*} MU_*^{ev}(X)$$

The fact that G_F is associated with F can be translated into the following diagram:

$$\begin{tikzcd} & \mathrm{Spec}(MU_*) \arrow[d, "p"] \\ \mathrm{Spec}(R) \arrow[ur, "q"] \arrow[r] & \mathcal{M}_{fg} \end{tikzcd}$$

In the diagram, q is the map classifying the formal group law F, and p is the covering map. Recall from the beginning of the section that we can write $MU_*^{ev}(X) \simeq p^*\mathcal{F}_X$. We then have

$$(E_G)_0(X) \simeq q^*p^*\mathcal{F}_X.$$

Notice that the composition $p \circ q$ is independent of the choice of F, but only depends on the isomorphism class of the formal groups. Hence, we can generalize our construction to formal groups:

Definition 7.6. *Let G be a Landweber exact formal group over R classified by the map $g : \mathrm{Spec}(R) \to \mathcal{M}_{fg}$. Then the corresponding* **Landweber exact homology theory** *is given by*

$$(E_G)_n X := g^*\mathcal{F}_{\Sigma^{-n}X}.$$

By Brown's representability theorem, the corresponding spectrum is called the **Landweber spectra** *of G, denoted as E_G.*

One can prove the spectra have the following property:

Proposition 7.7. *Let G be a Landweber exact formal group over R. Then E_G is a weakly 2-periodic (see Theorem 1.7) homotopy commutative spectra.*

The definition above defines a covariant functor $(R, G) \mapsto E_G$ mapping from the full subcategory of $\mathcal{FG}$ (see Definition 6.10) containing Landweber exact formal groups to the homotopy category of Spectra.

Remark 7.8. *Notice that our definition only defines a functor to homology theories. Because Brown's representability only said the homology theory could be lifted to a spectrum unique up to non-unique equivalence. In general, a map of homology theories does not lift to a unique map of spectra, even up to homotopy. The above Proposition says that in our case, the map of homology theories at least lifts to a unique map of spectra up to homotopy.*

The construction above seems promising for us to construct spectra in chromatic homotopy theory. But the conditions are somehow still restrictive. For example, since we know $v_0 = p$, the condition that p acts injectively has ruled out all the formal groups over rings of positive characteristics. This can be seen as a motivation to develop the deformation theory. Extending our ring a bit gives us more possibilities to find Landweber exact formal groups. In our case, we have the following proposition:

Proposition 7.9. *Let κ be a perfect field of characteristic $p > 0$ and let G_0 be a 1-dimensional formal group of height $n < \infty$. By Theorem 5.2, there is a universal deformation $G \to A$. This is a Landweber exact formal group.*

Finally, we can state the classical construction of the Lubin-Tate spectrum:

Construction 7.10. *Let R_0 be a perfect field of characteristic $p > 0$ and let G_0 be a 1-dimensional formal group of exact height $n < \infty$ over R_0 (Definition 4.7).*

1. *By Theorem 5.2, there is a universal deformation ring, denoted by $E_0(G_0)$, and a universal deformation $G \to \mathrm{Spec}(E_0(G_0))$.*
2. *By Proposition 7.9 and Proposition 7.7, the universal deformation induces a weakly even periodic Landweber exact spectrum, which we denote as $E(G_0)$. It is called the* **Lunbin-Tate spectrum** *of (G, κ).*

Up to this place, we have only constructed a homotopy commutative spectrum, which is different from Lurie's construction which produces a E_∞ algebra. It is proven by Goerss-Hopkins-Miller that $E(G_0)$ has a unique E_∞-algebra structure.

Theorem 7.11. *(Goerss-Hopkins-Miller) The Lubin-Tate spectrum $E(G_0)$ admits a unique E_∞-ring structure compatible with the ring structure on its homotopy groups, and it is functorial as an E_∞-ring spectrum in the choice of $G_0 \to \mathrm{Spec}(\kappa)$.*

One can find its proof in Piotr Pstragowskia and Paul VanKoughnett's paper *Abstract Goerss-Hopkins theory*. In their paper, they use synthetic spectra to transform the problem of admitting E_∞-structure into an obstruction theorem. Also, Maxime has a new proof of this theorem, which uses the separability of E_1-algebra and some cool techniques in higher algebra.

We have finished the goal of this note. After knowing what the main actors are in chromatic homotopy theory, a good step to go further is to read how they interact with finite spectra. For this, I really recommend Maxime's note. It shows how fun chromatic homotopy theory could be.

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