SMOOTH REPRESENTATIONS OF TOTALLY
DISCONNECTED LOCALLY COMPACT GROUPS

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Term Paper

Field of Study: *01.04.01 «Mathematics»*

Master's Programme «Mathematics»

Supervisor
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Moscow 2022

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Abstract

In this report we develop representation theory of totally disconnected locally compact groups, and do so through two main sections, standard results of smooth representations and smooth representations of general linear group over a non-archimedean local field. Necessary definitions and results are given in the first section, including the notion of smooth representations and Hecke algebra, compact representations and so on. In the second section we only consider the general linear group over a non-archimedean local field. In order to describe its irreducible smooth representations decompositions and Jacquet functors are introduced. Finally, we reach to the result that irreducible smooth representations of such general linear group are obtained from induction of some cuspidal representation of a Levi subgroup.

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1 Introduction to Representation Theory

This section focus on basic notions in representation theory including l-groups, smooth representation, Hecke algebra etc.

1.1 Smooth Representations

In order to formulate representations of l-groups we define smooth representations of l-groups and study related properties which are the foundation of representations of l-groups.

Definition 1.1. An **l-space** is a Hausdorff topological space which is totally disconnected locally compact. Moreover, an **l-group** is a Hausdorff topological group which is totally disconnected locally compact.

Remark 1.1. If H is a closed subgroup of an l-group, then H is an l-group as well.

Example 1.1. Let F be a local field. F^+ is naturally a topological group with respect to addition. Moreover, a basis of compact open subsets is given by $\{\mathfrak{m}^i | i \in \mathbb{Z}\}$ where $\mathfrak{m}$ is the maximal ideal of the discrete valuation ring of F. It is also Hausdorff since it is a metric space. Therefore, F^+ is an l-group.

Example 1.2. Let n be a positive integer and F a local field. In what follows we are interested in the group $\mathrm{GL}_n(F)$ which is an l-group.

Consider the determinant map $\det : \mathrm{M}_n(F) \to F$ which is continuous as taking determinant is a polynomial of entries. Then $\mathrm{GL}_n(F)$ is open in $\mathrm{M}_n(F)$ since it is the preimage of $F^\times$. Then we can consider subspace topology of $\mathrm{M}_n(F) \cong F^{2n}$ which is clearly Hausdorff.

Now construct subsets K_i for $i \in \mathbb{N} \cup \{0\}$ by

$$K_i = \left\{ M \in \mathrm{GL}_n(F) | M \in I_n + \mathrm{M}_n(\mathfrak{m}^i) \right\}.$$

One can show that the subsets K_i's are compact open in $\mathrm{GL}_n(F)$ and form a basis at the identity. Hence, the group $\mathrm{GL}_n(F)$ is indeed an l-group. We shall focus on representations of the group.

Definition 1.2. Let (ρ, V) be a representation of a group G. The representation (ρ, V) is said to be **finitely generated** if there exists a finite collection $\{v_1, v_2, \dots, v_n\}$ of elements in V such that

$$V = \mathrm{Span}\left\{ g v_i | g \in G, 1 \leq i \leq n, n \in \mathbb{N} \right\}.$$

Definition 1.3. Let (ρ, V) be a representation of a group G. The representation (ρ, V) is said to be **semisimple** if V is a direct sum of irreducible representations.

Definition 1.4. Let (ρ, V) be a representation of a topological group G. A vector $v \in V$ is **smooth** if its stabilizer in G is open. We denote the set of all smooth vectors in V by V^∞. Moreover, if $V^\infty = V$ we say the representation is smooth.

Proposition 1.1. *Let (ρ, V) be a representation of an l-group G. Then the V^∞ is a smooth subrepresentation of V.*

Proposition 1.2. *Let G be an l-group. Assume that G is compact. Then every smooth representation of G is semisimple.*

Lemma 1.1. *If (ρ, V) is an irreducible smooth representation of an l-group G. Assume that G/K' is countable for some compact open subgroup K' of G, then $\dim_{\mathbb{C}} V$ is countable.*

Lemma 1.2 (**Schur's Lemma**)**.** *Let (ρ, V) be an irreducible smooth representation of an l-group G. Then $\mathrm{End}_G(V) \cong \mathbb{C}$ i.e., for any $f \in \mathrm{End}_G(V)$ we have $f = c \cdot \mathrm{id}_V$ for some constant $c \in \mathbb{C}$.*

Proof. Fix a nonzero element $v \in V$ and consider the linear transformation $\Phi : \mathrm{End}_G(V) \to V$ given by $\Phi(\varphi) = \varphi(v)$ for $\varphi \in \mathrm{End}_G(V)$. We have seen that V is generated by the sets $\{\rho(g)v | g \in G\}$. It follows that φ is uniquely determined by its value $\varphi(v)$. Therefore, Φ is injective. By Lemma 1.1 we have that $\dim_{\mathbb{C}} \mathrm{End}_G(V)$ is countable since $\dim_{\mathbb{C}} V$ is countable.

On the other hand, V is irreducible. φ is an isomorphism as $\mathrm{Ker}\varphi$ and $\mathrm{Im}\varphi$ is a subrepresentation of V. Suppose that $\varphi \in \mathrm{End}_G(V)$ satisfies $\varphi \notin \mathbb{C}$. Since $\mathbb{C}$ is algebraic closed, the field $\mathbb{C}(\varphi)$ is transcendental over $\mathbb{C}$ and so the elements $\frac{1}{\varphi - c}$ for all $c \in \mathbb{C}$ are linearly independent over $\mathbb{C}$. It shows that $\mathbb{C}(\varphi)$ is uncountable. However, we have $\mathbb{C}(\varphi) \subseteq \mathrm{End}_G(V)$ since any element in $\mathrm{End}_G(V)$ is invertible. Thus, $\dim_{\mathbb{C}} \mathrm{End}_G(V)$ is uncountable which yields a contradiction. Hence, $\mathrm{End}_G(V) \cong \mathbb{C}$. $\square$

Corollary 1.1. *Let G be an l-group and Z its center. If (ρ, V) is an irreducible smooth representation of G, then there is a smooth 1-dimensional representation χ_ρ of Z such that $\rho(z) = \chi_\rho(z)\mathrm{id}_V$ for all $z \in Z$.*

Corollary 1.2. *If G is an abelian l-group, then every irreducible smooth representation of G is 1-dimensional.*

Let (ρ, V) be a smooth representation of and l-group G. Consider $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ i.e., the dual vector space of V. We define an action

$$\rho^* : G \times V^* \to V^*$$

by $\rho^*(g, \lambda)(v) = \lambda(\rho(g)^{-1}v)$ for any $\lambda \in V^*, g \in G$ and $v \in V$. Clearly, ρ^* gives a representation, denote by (ρ^*, V^*) with

$$\rho^* : G \to \mathrm{GL}(V^*)$$

given by $\rho^*(g)(\lambda)(v) = \lambda(\rho(g)^{-1}v)$ for any $\lambda \in V^*, g \in G$ and $v \in V$.

Warning: The representation V^* of G may not be smooth in general.

In this report the most important is admissible representation. We want to see when a smooth representation is an admissible representation and properties it has.

Definition 1.5. Let $(\tilde{\rho}, \tilde{V})$ be the subrepresentation with representation space $\tilde{V} = (V^*)^\infty$ of V^*. The representation $\tilde{V}$ is called the **smooth dual** (or **contragredient**) of (ρ, V).

Example 1.3. Let F be a local field. Consider a smooth 1-dimensional representation $\chi : F^\times \to \mathbb{C}^\times$. Define a map

$$\rho : \mathrm{GL}_n(F) \to \mathbb{C}^\times$$

by $\rho(M) = \chi(\det(M))$ for any $M \in \mathrm{GL}(F)$. One can show that $(\rho, \mathbb{C})$ is a smooth representation. Then we have

$$\rho^* : \mathrm{GL}_n(F) \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^*$$

is given by $\rho^*(M) = \chi(\det(M)^{-1})$ for all $M \in \mathrm{GL}_n(F)$, which is smooth clearly. It follows that $(\rho^*, \mathbb{C}^*) \cong (\tilde{\rho}, \tilde{\mathbb{C}})$.

Definition 1.6. Let G be an l-group. A smooth representation (ρ, V) of G is **admissible** if for every compact open subgroup K of G, the space V^K is finite dimensional.

For a smooth representation (ρ, V) of G, we consider the canonical map $\Phi : V \to \tilde{\tilde{V}}$ given by

$$\langle \varphi(v), \tilde{v} \rangle = \langle \tilde{v}, v \rangle$$

for $v \in V, \tilde{v} \in \tilde{V}$. Clearly, Φ is an element in $\mathrm{Hom}_G(\rho, \tilde{\tilde{\rho}})$.

Theorem 1.1. *Let (ρ, V) be a smooth representation of an l-group G. (ρ, V) is admissible if and only if the canonical map Φ is an isomorphism.*

The key of the proof of Theorem 1.1 is to show $\tilde{\tilde{V}}^K \cong \left(V^K\right)^{**}$ for any compact open subgroup K of G. We omit details and the reader is referred to [**BH06**, p. 24]. On the other hand, Theorem 1.1 gives a criterion of admissible representations. However, finding double dual of a smooth representation is not easy as well.

1.2 Hecke Algebra

In this subsection we introduce the Hecke algebra of an l-group. Via the Hecke algebra we study smooth representations and irreducible representations, namely the correspondence between smooth representations and non-degenerate module over the Hecke algebra. Finally, we introduce and prove the separation property. Unless otherwise stated, we assume that G is unimodular. Moreover, we shall omit some calculations and details. The reader is directed to [**BH06**] for more details.

We fix a Haar measure μ on an l-group G. For $f_1, f_2 \in \mathcal{C}_c^\infty(G)$, we define the **convolution** of f_1 and f_2 by

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

The reader may check that the convolution $*$ is indeed a binary operation on $\mathcal{C}_c^\infty(G)$ and it is associative. Therefore, the convolution makes $\mathcal{C}_c^\infty(G)$ an algebra over $\mathbb{C}$.

Definition 1.7. Let G be an l-group. The set $\mathcal{C}_c^\infty(G)$ is an algebra over $\mathbb{C}$ under convolution. We denote the algebra by $\mathcal{H}(G)$ and call it the **Hecke algebra** of G.

Remark 1.2. (1). In general, the Hecke algebra $\mathcal{H}(G)$ has no unity. But when G is compact, the Hecke algebra has unity.
(2). Clearly, the Hecke algebra depends on the choice of the Haar measure. One can obtain an isomorphism between Hecke algebras with respect to different Haar measures since the difference of two Haar measure is a multiple of a non-zero constant.

Even though the Hecke algebra $\mathcal{H}(G)$ has no unity in general one can study its idempotent elements. For any K compact open subgroup of G, we define a function $e_K \in \mathcal{H}(G)$ by

$$e_K(g) = \begin{cases} \dfrac{1}{\mu(K)} & \text{if } g \in K \\ 0 & \text{otherwise} \end{cases}, \quad \text{for any } g \in G.$$

Then we have the following properties for the function e_K.

Proposition 1.3. *Let G be an l-group and K a compact open subgroup of G. Then*

(1). The function e_K satisfies $e_K * e_K = e_K$.

(2). For any $f \in \mathcal{H}(G)$, f satisfies $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $k \in K, g \in G$.

(3). For any $f \in \mathcal{H}(G)$, f satisfies $f * e_K = f$ if and only if $f(gk) = f(g)$ for all $k \in K, g \in G$.

(4). The subset $e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$ with unity e_K.

Proof. (1)-(3) follow from direct calculation. (4) is a general result in ring theory. □

By the above proposition (Proposition 1.3) we note that $e_K * \mathcal{H}(G) * e_K$ is the subset consisting of $f \in \mathcal{H}(G)$ such that $f(k_1 g k_2) = f(g)$ for $g \in G, k_1, k_2 \in K$. We usually write $e_K * \mathcal{H}(G) * e_K$ as $\mathcal{H}(G, K)$ or simply $\mathcal{H}_K$.

Definition 1.8. A module M over $\mathcal{H}(G)$ is called **non-degenerate** if $\mathcal{H}(G)M = M$.

Since the Hecke algebra $\mathcal{H}(G)$ is the union of subalgebras $\mathcal{H}(G, K)$ for all compact open subgroups K of G, the module M is non-degenerate if and only if there exists a compact open subgroup K such that $e_K m = m$ for all $m \in M$.

Let $\mathrm{Rep}(G)$ denote the category of smooth representations and $\mathrm{Mod}(G)$ denote the category of non-degenerate modules over the Hecke algebra $\mathcal{H}(G)$. Then we are ready to characterize the relation between these two categories.

Theorem 1.2. *Let G be an l-group. Then there is an equivalence of categories between* $\mathrm{Rep}(G)$ *and* $\mathrm{Mod}(G)$.

Proof. (sketch) Suppose that (ρ, V) is a smooth representation of G. For $f \in \mathcal{H}(G)$, we define

$$f \cdot v = \int_G f(g)\rho(g) v d\mu(g). \qquad (\star)$$

The reader may check that $(\star)$ gives an action of $\mathcal{H}(G)$ on V. Explicitly, we can choose K a compact open subgroup of G and get

$$f \cdot v = \mu(K) \sum_{g \in G/K} f(g)\rho(g)v. \tag{$\star\star$}$$

It follows that $e_K \cdot v = v$ and so V is a non-degenerate $\mathcal{H}(G)$-module.

Conversely, let M be a non-degenerate $\mathcal{H}(G)$-module. We define a G-action on M by

$$\rho(g)m = \frac{1}{\mu(K)} \mathbb{1}_{gK} m,$$

where $\mathbb{1}_{gK}(x) = \begin{cases} 1 & \text{if } x \in gK \\ 0 & \text{otherwise} \end{cases}$. The action gives a representation (ρ, M) of G. Since M is non-degenerate, there is a compact open subgroup K of G such that $e_K m = m$ for all $m \in M$. Therefore, for $k \in K$ one can deduce

$$\rho(k)m = \frac{1}{\mu(K)} \mathbb{1}_K m = e_K m = m.$$

Namely, if $g \in \mathrm{Stab}_G(m)$ then gK is a compact open neighborhood of g in the stabilizer $\mathrm{Stab}_G(m)$. Thus, the stabilizer is open and the representation (ρ, M) is smooth. $\square$

Remark 1.3. The equation $(\star\star)$ in the proof of Theorem 1.2 is independent of the choice of the compact open subgroup K.

Now we focus on irreducible representations and classify them via simple modules over $\mathcal{H}_K$. Recall that $\mathcal{H}_K = e_K * \mathcal{H}(G) * e_K$. By the proof of Theorem 1.2 we see that $V^K = e_K V$ for a smooth representation (ρ, V) of G.

Proposition 1.4. *(1). Let (ρ, V) be an irreducible representation of G. Then the space V^K is either 0 or a simple $\mathcal{H}_K$-module.*

(2). There is a bijection between equivalence classes of irreducible smooth representations (ρ, V) of G such that V^K is non-zero and isomorphism classes of simple $\mathcal{H}_K$-modules, which is given by the projection $V \to V^K$.

Proof. Assume that (ρ, V) is an irreducible smooth representation of G such that $V^K \neq 0$. Let W be a non-zero $\mathcal{H}_K$-submodule of V^K. Then $\mathcal{H}(G)W$ is a non-zero subspace of V. Since (ρ, V) is irreducible, we obtain $\mathcal{H}(G)W = V$. It follows that

$$V^K = e_K V = (e_K * \mathcal{H}(G))W = \mathcal{H}_K W \subseteq W.$$

Thus, we have $V^K = W$ and so V^K is simple $\mathcal{H}_K$-module, which proves (1).

Clearly, for each irreducible smooth representation of G such that $V^K \neq 0$, one has simple $\mathcal{H}_K$-module V^K. Conversely, let M be a simple $\mathcal{H}_K$-module. We construct an irreducible smooth representation (ρ, V) of G with $V^K = M$. Consider the representation (ρ, U) with $U = \mathcal{H}(G) \otimes_{\mathcal{H}_K} M$ where G acts by left translation. Then we have

$$U^K = e_K(\mathcal{H}(G) \otimes_{\mathcal{H}_K} M) = (e_K * \mathcal{H}(G) * e_K) \otimes_{\mathcal{H}_K} M \cong M.$$

By Zorn's Lemma, we can find a maximal G-subspace X of U such that $X \cap U^K = 0$ i.e. $X^K = 0$. Suppose that Y is another G-subspace satisfying $Y^K = 0$, then $(X + Y)^k = X^K + Y^K = 0$. The maximality of X implies that X is unique. Set $V = U/X$. If Z is a G-subspace of U containing X, it contains the simple $\mathcal{H}_K$-module U^K. Therefore, Z coincides with U and so V is irreducible. Moreover, the smoothness of V is obviously from the smoothness of U.

In order to check that the bijection is up to equivalence of representations and isomorphisms of modules, we can lift isomorphisms of modules to equivalence of representations via tensor product. Details are left to the reader. $\square$

Corollary 1.3. *Let (ρ, V) be a non-zero smooth representation of an l-group G. Then (ρ, V) is irreducible if and only if for any compact open subgroup K of G, V^K is either zero or a simple $\mathcal{H}_K$-module.*

As an application of Proposition 1.4, one can deduce the separation property:

Theorem 1.3 (**Separation Property**)**.** *Let G be an l-group and $f \in \mathcal{H}(G)$ such that $f \neq 0$. There is an irreducible smooth representation (ρ, V) of G satisfying $f \neq 0$(as a map $V \to V$).*

In order to prove the separation property we need a lemma about algebras of countable dimension over $\mathbb{C}$. But we are not going to give a proof. The reader interested in the proof is directed to [**BH06**] or [**Ber92**].

On the other hand, we can define the Hecke algebra from **distributions** which are linear functional on $\mathcal{C}_c^\infty(G)$. Let $\mathcal{D}(G)$ denote the space of distributions.

Definition 1.9. Let $\mathcal{E} \in \mathcal{D}(G)$ be a distribution. The **support** of $\mathcal{E}$ is the smallest closed subset Z such that $\mathcal{E}|_{G \setminus Z} = 0$.

Then we can define the Hecke algebra as the space of all locally constant, compactly supported distributions, denoted by $\mathcal{D}'(G)$. One can show that there is an isomorphism of algebras $\mathcal{H}(G) \to \mathcal{D}'(G)$ given by $f \mapsto F_f$, where F_f is the distribution

$$F_f(\varphi) = \int_G \varphi(g)f(g)d\mu(g), \quad \forall \varphi \in \mathcal{C}_c^\infty(G).$$

Therefore, we may and do consider the Hecke algebra as distributions sometimes.

1.3 Restriction and Induction

In representation theory of finite groups we have seen induced representations and Frobenius reciprocity. Similarly, for smooth representations of l-groups we have the same properties. Moreover, compact induction will be studied as well.

Let H be a closed subgroup of an l-group G. Then H is also and l-group. We may restrict G-modules to give H-modules. This implies a functor

$$\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$$

which is called **restriction**. As a functor, Res_H^G has a right adjoint Ind_H^G defined as follows:

Definition 1.10. Let (ρ, V) be a smooth representation of H, a closed subgroup of an l-group G. Define

$$W := \{f : G \to V | f(hg) = \rho(h)f(g), \forall h \in H, \forall g \in G\}$$

with the action $(gf)(x) = f(xg)$ for $x \in G$. Then the smooth part W^∞ is the **induced representation** of G obtained from (ρ, V), which is denoted by $\left(\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G V\right)$.

Therefore, we have a functor $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. It is called **induction** and is a right adjoint of Res_H^G. Moreover, by Definition 1.10 Ind_H^G is the space of functions $f : G \to V$ such that

(i) $f(hg) = \rho(h)f(g), \forall h \in H, \forall g \in G$,

(ii) f is right K-invariant for some compact open subgroup K of G.

Consider the map $\nu_\rho : \mathrm{Ind}_H^G V \to V$ given by $f \to f(1)$. Then the map ν_ρ is a H-morphism which give the Frobenius reciprocity:

Theorem 1.4 (**Frobenius Reciprocity**). *Let H be a closed subgroup of an l-group G. For a smooth representation (ρ, V) of H and a smooth representation (ρ', V') of G, the map*

$$\mathrm{Hom}_G(V', \mathrm{Ind}_H^G V) \to \mathrm{Hom}_H(\mathrm{Res}_H^G V', V)$$

given by $\varphi \mapsto \nu_\rho \circ \varphi$ is an isomorphism and is functorial.

Proof. For any $\psi \in \mathrm{Hom}_H(\mathrm{Res}_H^G V', V)$, we define $\tilde{\psi} : V \to V'$ by $\tilde{\psi}(g) = \psi(\rho'(g)v)$ which is a G-morphism. The map $\psi \mapsto \tilde{\psi}$ is the inverse of the map $\varphi \mapsto \nu_\rho \circ \varphi$. $\square$

Additionally, we also have compact induction:

Definition 1.11. Let (ρ, V) be a smooth representation of H, a closed subgroup of an l-group G. Define

$$\mathrm{ind}_H^G V := \left\{ f \in \mathrm{Ind}_H^G V | f \text{ has compact support modulo } H \right\}$$

which is called the **compact induction** of (ρ, V).

So we obtain another functor $\mathrm{ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. The relations between induction and compact induction are characterized by the following proposition:

Proposition 1.5. *Let H be a closed subgroup of an l-group G. Then we have the following properties.*

(i) Both Ind_H^G and ind_H^G are exact.

(ii) If $H \backslash G$ is compact, then $\mathrm{ind}_H^G = \mathrm{Ind}_H^G$.

(iii) If $H \backslash G$ is compact, then Ind_H^G maps admissible representations to admissible representations.

Proof. We refer (i) to [**BH06**, p. 18]. (ii) is clear from the definition of compact induction.

For (iii) let (ρ, V) be a smooth admissible representation of H and K a compact open subgroup of G. Then the double cosets HgK for all $g \in G$ form an open cover of $H \backslash G$. Since $H \backslash G$ is compact there is a finite subset $X \subseteq G$ such that $G = HXK$. Let $f \in \left(\mathrm{Ind}_H^G V\right)^K$ and $x \in X$. Since $H \cap xKx^{-1}$ is a compact open subgroup of H and (ρ, V) is admissible, we see that $\dim(V^{H \cap xKx^{-1}}) < \infty$. Define $V_0 := \bigoplus_{x \in X} V^{H \cap xKx^{-1}}$. Then V_0 is finite dimensional. Since $f \in \left(\mathrm{Ind}_H^G V\right)^K$ and $x \in G$, then we have

$$f(x) = f(xk) = f(xkx^{-1}x) = \rho(xkx^{-1})f(x), \forall k \in K \cap x^{-1}Hx.$$

It follows that $f(x) \in V^{H \cap xKx^{-1}}$ and so $f(X) \subseteq V_0$. Therefore, we have a map $\Phi : \left(\mathrm{Ind}_H^G V\right)^K \to \mathcal{C}(X, V_0)$ given by $f \mapsto f|_X$ where $\mathcal{C}(X, V_0)$ is the space of functions from X to V_0. By Definition 1.10, we see that Φ is one-to-one as f is determined by its values on X. Moreover, X is finite and $\dim(V_0) < \infty$. One can deduce $\dim\left(\mathcal{C}(X, V_0)\right) < \infty$. Thus, we have $\dim\left((\mathrm{Ind}_H^G V)^K\right) < \infty$. $\square$

1.4 Compact Representations

A special type of representations is compact representation. We will introduce the notion of compact representations and related properties. When we discuss cuspidal representations we shall use the notion and properties to deduce nice results.

Definition 1.12. Let (ρ, V) be a smooth representation of an l-group G. (ρ, V) is called **compact** if for every $v \in V$ and every compact open subgroup K of G, the function $\mathcal{D}_{v,K} : G \to V$ given by

$$g \mapsto e_K \rho(g^{-1})v$$

has compact support.

Definition 1.13. Let (ρ, V) be a smooth representation of an l-group G and $(\tilde{\rho}, \tilde{V})$ the smooth dual. For $v \in V, \tilde{v} \in V$, the function $m_{\tilde{v},v} : G \to \mathcal{C}$ given by $m_{\tilde{v},v}(g) = \langle \tilde{v}, \rho(g^{-1}v) \rangle$ is called a **matrix coefficient** of (ρ, V).

With the definition of matrix coefficient(Definition 1.13), there is an equivalent definition of compact representations:

Theorem 1.5. *Let (ρ, V) be a smooth representation of an l-group G. Then (ρ, V) is a compact representation if and only if all matrix coefficients of (ρ, V) are compactly supported.*

Proof. Let K be a compact open subgroup of G such that $\tilde{v} \in \tilde{V}$. Then we can deduce that $\mathrm{Supp}(m_{\tilde{v},v}) \subseteq \mathrm{Supp}(\mathcal{D}_{v,K})$. Therefore, if (ρ, V) is compact, them all matrix coefficients have compact support.

For the converse, we give a hint: show that

$$\mathrm{Supp}(\mathcal{D}_{v,K}) \subseteq \mathrm{Supp}\left(\bigcup_{i=1}^{k} m_{\tilde{v},v}\right)$$

for some $\tilde{v_1}, \dots, \tilde{v_k} \in \tilde{V}^K$. $\square$

Proposition 1.6. *Let (ρ, V) be a smooth representation of an l-group G. If (ρ, V) is finitely generated and compact, then (ρ, V) is admissible.*

Proof. Suppose that V is generated by $v_1, \dots, v_t$. Namely,

$$V = \mathrm{Span}\left\{\rho(g)v_i | 1 \leq i \leq t\right\}.$$

For a compact open subgroup K of G, we have that

$$V^K = \mathrm{Span}\left\{e_K \rho(g)v_i | 1 \leq i \leq t\right\}.$$

Since (ρ, V) is a compact representation, there re only finitely many linear independent vectors $e_K\rho(g)v_i$. Therefore, V^K is finite dimensional i.e. is admissible. $\square$

Observe that irreducible smooth representations are generated by any non-zero vector. So immediately we have the following corollary:

Corollary 1.4. *Any irreducible smooth representation of an l-group is admissible.*

2 Representations of GL_n

In this section we pay attention to smooth representations of $\mathrm{GL}_n(F)$ where F is a non-archimedean local field with the local ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. We first introduce decomposition in $\mathrm{GL}_n(F)$, then move to Jacquet functors. Finally, we show the most important result on irreducible smooth representation. However, the following contents are correct for general case and proofs are similar as well. Therefore, without specific mention we assume that $G = \mathrm{GL}(F)$.

2.1 Decomposition of GL_n

To consider representations of general linear groups we need to know some decomposition of general linear groups. In this part, we basically demonstrates related results without proofs. For proofs and explanations, the reader is directed to [**Hum12**] or Chapter II of [**Ber92**].

Recall that in Example 1.1 the subgroups K_i for $i \in \mathbb{N} \cup \{0\}$ are compact open in G and form a basis at the identity.

The **Weyl chamber** Λ^+ is given by $\Lambda^+ := \{(r_1, \cdots, r_n) | r_1 \leq r_2 \leq \cdots \leq r_n\}$. Equivalently, Λ^+ can be characterized as

$$\Lambda^+ := \{\mathrm{diag}(\pi^{r_1}, \cdots, \pi^{r_n}) | r_1 \leq r_2 \leq \cdots \leq r_n\},$$

where π is an uniformizer of $\mathcal{O}$ i.e. a generator of the ideal $\mathfrak{m}$. Then the Cartan decomposition can be described as follows:

Cartan Decomposition: $G = K_0\Lambda^+K_0$.

Let P be a closed subgroup of G. We say that P is **parabolic** if G/P is projective as a variety. More generally, suppose that G is a reductive group.

Fix $g \in G$ and consider subgroups:

$$P_g := \{x \in G | \{\text{Ad}(g^n), n \geq 0\} \text{ is relatively compact in } G\}$$
$$U_g := \{x \in G | \text{Ad}(g^n)(x) \to 1 \text{ as } n \to \infty\}$$

where Ad is the adjoint representation of G. The subgroups P_g are called to be **parabolic**.

Moreover, if $M = P_g \cap P_{g^{-1}}$, then we can deduce that $P_g = MU_g$ and $P_{g^{-1}} = MU_{g^{-1}}$. Therefore, for a fixed parabolic subgroup P, P can be written as $P = MU$. The subgroup M is called a **Levi subgroup** and we obtain a decomposition:

Levi Decomposition:Let P be a parabolic subgroup, then $P = MU$.

With the notion of parabolic subgroup, we also have the decomposition:

Iwasawa Decomposition: Let P be a parabolic subgroup and $K_0 = \text{GL}_n(\mathcal{O})$. Then we have $G = K_0P = PK_0$. In particular, G/P is compact.

2.2 Jacquet Functors

In order to study irreducible smooth representations, we introduce Jacquet module and two functors. By checking if the Jacquet module is zero we divide irreducible representations into two classes, namely, irreducible representations obtained from cuspidal representations and principal series representations.

Let (ρ, V) be a smooth representation of $G = \text{GL}_n(F)$. More generally, G could be a reductive group. But here we focus on the general linear group. Suppose that P is a parabolic subgroup of G, then we have $P = MU$ where M is a Levi subgroup and U is the unipotent radical. Set

$$V(U) := \text{Span}\{\rho(u)v - v | v \in V, u \in U\} \text{ and } V_U := V/V(U).$$

Obviously, $V(U)$ is a U-invariant space and so U acts trivially on V_U. We denote V_U by $J_U(V)$, called the **Jacquet module** of V. Equivalently, we can define $J_U(V)$ to be $\mathbb{C}_U \otimes_U V$ where $\mathbb{C}_U$ is the trivial representation of U. Since M normalize U, $J_U(V)$ is a smooth representation of M. Therefore, one can define a functor

$$r_M^G : \text{Rep}(G) \to \text{Rep}(M)$$

by $r_M^G(V) = J_U(V)$. Moreover, the functor r_M^G is not the only functor in which we are interested and its adjoint functor is also intriguing.

Theorem 2.1. *Let P be a parabolic subgroup of G with Levi decomposition $P = MU$. Suppose that (ρ, V) is a smooth representation of M, then the functor r_M^G has a right adjoint functor*

$$i_M^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$$

defined by $i_M^G = \mathrm{ind}_P^G V$ where V extends trivially on U.

Proof. Suppose that W, V are smooth representations of M and G respectively. By Frobenius Reciprocity(Theorem 1.4) we have

$$\mathrm{Hom}_G(V, i_M^G(W)) \cong \mathrm{Hom}_G(V, \mathrm{Ind}_P^G W) \cong \mathrm{Hom}_P(\mathrm{Res}_P^G V, W).$$

Since U acts trivially, we deduce

$$\begin{aligned} \mathrm{Hom}_G(V, i_M^G(W)) &\cong \mathrm{Hom}_P(V/V(U)), W) \\ &\cong \mathrm{Hom}_P(r_M^G(V), W) \\ &\cong \mathrm{Hom}_M(r_M^G(V), W) \end{aligned}$$

which completes the proof. □

Proposition 2.1. *Suppose that M is a Levi subgroup of G. Then we have the following properties.*

(i) i_M^G maps admissible representations to admissible representation.

(ii) The functors r_M^G, i_M^G are both exact.

(iii) r_M^G maps finitely generated representations to finitely generated representations.

Proof. (i) is direct from Proposition 1.5 since G/P is compact by Iwasawa decomposition. (ii) follows from exactness of ind_P^G and the characterization of the Jacquet module via tensor product.

For (iii) let (ρ, V) be finitely generated smooth representation of G. For any $v \in V$, we consider $Gv = \{gv | g \in G\}$. Since (ρ, V) is finitely generated, we may assume that

$$V = \mathrm{Span}\left\{ \bigcup_{1 \le i \le t} Gv_t \right\}$$

for some $v_1, \ldots, v_t$ in V. Fix a compact open subgroup K of G such that $v_i \in V^K$ for all $1 \le i \le t$. This can be done since t is a positive integer. It follows that, by the compactness of G/P,

$$P \backslash G / K = \bigcup_{1 \le i \le s} P g_i K$$

for some $g_1, \ldots, g_s$ in G. Therefore, we deduce that

$$V = \mathrm{Span}\left\{ \bigcup_{i,j} P\rho(g_i)v_j | 1 \leq i \leq s, 1 \leq j \leq t \right\}.$$

Hence, $V/V(U)$ is finitely generated as a representation of P and so finitely generated as a representation of M since U acts trivially. $\square$

The importance about reasons why we pay attention to the functor r_M^G is clear from the following theorem. Namely, irreducible representations of $\mathrm{GL}_n(F)$ can be divided into two groups: irreducible representations with $r_M^G \neq 0$ for some M and irreducible representations with $r_M^G = 0$ for all M.

Theorem 2.2. *Let (ρ, V) be a smooth irreducible representation of G. Suppose that there is a parabolic subgroup $P = MU$ such that $r_M^G(V) \neq 0$, then there is a representation W of M such that V is a subrepresentation of $\mathrm{Ind}_P^G W$.*

Proof. We refer the proof to [**RFB16**, p. 68]. $\square$

2.3 Cuspidal Representations

Under the Jacquet functors defined in Section 2.2, we can now describe cuspidal representations which play an important role in representations of $\mathrm{GL}_n(F)$. For some important results we shall not give proof and the reader may check [**Ber92**] for details.

Definition 2.1. A smooth representaion (ρ, V) of G is called **quasi-cuspidal** if for any standard Levi subgroup M except $M = G$ we have $r_M^G(V) = 0$.

Definition 2.2. A smooth representation (ρ, V) of G is called **cuspidal** if it is quasi-cuspidal and finitely generated.

Recall that we defined compact representations(Definition 1.12). Now we generalize the notion as follows.

Definition 2.3. Let (ρ, V) be a smooth representation of G and K a compact subgroup of G. (ρ, V) is called **compact modulo center** if for any $v \in V$, the function $\mathcal{D}_{v,K}$ has compact support modulo center.

Under the notion of representations compact modulo center, a characterization of quasi-cuspidal representation is given below:

Theorem 2.3. *Let (ρ, V) be a smooth representation of G. Then (ρ, V) is quasi-cuspidal if and only if it is compact modulo center.*

Consider the subgroup $G^\circ = \{g \in G | \det(g) \in \mathcal{O}^\times\}$. The subgroup G° is a dense open normal subgroup of G with $G/G^\circ \cong \mathbb{Z}$. On the other hand, G° contains all compact subgroups of G.

Remark 2.1. $Z(G)G^\circ$ is an open subgroup of finite index in G.

In order to study irreducible representations we need to pay attention to representations of G°. For the case of G° one has an important theorem:

Theorem 2.4 (**Harish-Chandra**). *Smooth representations of G° are quasi-cuspidal if and only if they are compact.*

Directly from Harish-Chandra's Theorem, we obtain the admissibility of some particular irreducible representations.

Corollary 2.1. *Any irreducible cuspidal smooth representation of G is admissible.*

Proof. Suppose that (ρ, V) is such a representation. Since $Z(G)G^\circ$ has finite index, $V|_{Z(G)G^\circ}$ is finitely generated. The irreducibility of V implies that $Z(G)$ acts as multiplication by scalars according to Schur's Lemma(Lemma 1.2). Therefore, V is finitely generated as a representation of G°. Then Harish-Chandra's Theorem(Theorem 2.4) shows that $V|_{G^\circ}$ is compact. By Proposition 1.6, $V|_{G^\circ}$ is admissible. Since G° contains all compact subgroups, then V itself is admissible. $\square$

Lemma 2.1. *Let (ρ', V') be an irreducible smooth representation of G. Then there is a parabolic subgroup $P = MU$ and an irreducible cuspidal representation (ρ, V) of M such that there is an embedding from V' into $i_M^G(V)$.*

Note that the above lemma is much stronger than Theorem 2.2. With Lemma 2.1 we want to show the admissibility of irreducible smooth representations.

Theorem 2.5. *Any irreducible smooth representation of G is admissible.*

Proof. With the notations in Lemma 2.1, V is irreducible and cuspidal. By Corollary 2.1, (ρ, V) is admissible. Since i_M^G maps admissible representations to admissible representations, $i_M^G(V)$ is admissible as well. Therefore,V' is admissible as W can be embedded into $i_M^G(V)$. $\square$

Apart from the admissibility of irreducible smooth representations, another useful result is the uniform admissibility theorem which states as follows:

Theorem 2.6. *Let K be a compact open subgroup of G. Then there is a constant c depending on both K and G such that $\dim(V^K) \leq c$ for any irreducible smooth representation V of G.*

We have showed that in Proposition 1.4 that V^K is either zero or an irreducible representation of $\mathcal{H}_K$ and there is an one-to-one correspondence between them. Therefore, the uniform admissibility theorem can be also described as:

all irreducible representations of $\mathcal{H}_K$ have dimension bounded by the constant c.

We remark that as mentioned in the beginning of the section the above results on cuspidal representations hold in general for reductive groups. However, in this report we only discussed the case for $\mathrm{GL}_n(F)$. For general case the reader is directed to [**Ber92**].

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