Algebric Geometry

Def: Let X be a topological space. A presheaf $\mathcal{F}$ of abelian groups on X is functor $\mathcal{F}: \text{Open}^{op} \longrightarrow \text{Ab}$. i.e.

I) $\forall\, U \subseteq X$ open, $\mathcal{F}(U)$ is an abelian group

II) $\forall\, V \subseteq U$, $\rho_{UV}: \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

III) $\mathcal{F}(\emptyset) = 0$

IV) $\rho_{UU} = \text{id}$

V) $\forall\, W \subseteq V \subseteq U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Def: X topological space. A sheaf $\mathcal{F}$ on X is a presheaf s.t. the following sequence is exact

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_{\lambda \in I} \mathcal{F}(U_\lambda) \rightrightarrows \prod_{\lambda, \lambda' \in I} \mathcal{F}(U_\lambda \cap U_{\lambda'})$$

where $\{U_\lambda\}_{\lambda \in I}$ is an open covering of U and U is open in X.

Def: $\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$

Def: $\mathcal{F}, \mathcal{G}$ presheaves on X. A morphism $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ is a natural transformation of functors.

Proposition: Let $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on X. Then φ is an isomorphism iff the induced map $\varphi_x: \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism for any $x \in X$.

Prop (Sheafification): Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta: \mathcal{F} \longrightarrow \mathcal{F}^+$ with the property that for any sheaf $\mathcal{G}$ and any morphism $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ of presheaves, there is an unique morphism $\psi: \mathcal{F}^+ \longrightarrow \mathcal{G}$ s.t. $\varphi = \psi \circ \theta$. Moreover, $(\mathcal{F}^+, \theta)$ is unique up to isomorphism.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\varphi"] \arrow[dr, "\theta"'] & & \mathcal{G} \\ & \mathcal{F}^+ \arrow[ur, "\psi"'] & \end{tikzcd}$$

Def: A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ s.t. $\forall\, U \underset{\text{open}}{\subseteq} X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$.

Def: Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

a) $\operatorname{Ker}\varphi =$ the presheaf kernel of φ

b) $\operatorname{Im}\varphi =$ the sheafification of the presheaf image of φ

c) φ is injective if $\operatorname{Ker}\varphi = 0$

d) φ is surjective if $\operatorname{Im}\varphi = \mathcal{G}$.

e) A sequence $\cdots \to \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i+1} \to \cdots$ is exact if $\operatorname{Ker}\varphi_i = \operatorname{Im}\varphi_{i-1}$ for all i.

f) Let $\mathcal{F}'$ be a subsheaf of $\mathcal{F}$. The quotient sheaf $\mathcal{F}/\mathcal{F}'$ is the sheafification of the presheaf $U \longmapsto \mathcal{F}(U)/\mathcal{F}'(U)$.

Remark:
- $\mathcal{F}_x = \mathcal{F}^+_x$
- $\varphi: \mathcal{F} \to \mathcal{G}$ is injective iff $\varphi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every U open in X.
- $\varphi: \mathcal{F} \to \mathcal{G}$ is surjective iff $\varphi_x: \mathcal{F}_x \to \mathcal{G}_x$ is surjective for every $x \in X$.

Def: Let $f: X \to Y$ be continuous map of topological spaces.

a) direct image sheaf $f_*\mathcal{F}$: $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$

b) inverse image sheaf $f^{-1}\mathcal{G}$: the sheafification of the presheaf

$$U \longmapsto \varinjlim_{f(U) \subseteq V} \mathcal{G}(V)$$

Prop: $\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ i.e. f^{-1} is a left adjoint functor to f_*.

Definition: $Z \subseteq X$ closed subset, $i: Z \hookrightarrow X$ inclusion. $i^{-1}\mathcal{F}$ is called the restriction of $\mathcal{F}$ to Z, denoted by $\mathcal{F}|_Z$.

Prop: Let A be a ring and $(\mathrm{Spec}A, \mathcal{O})$ its spectrum.

a) $\mathcal{O}_\mathfrak{p} \cong A_\mathfrak{p} \quad \forall \mathfrak{p} \in \mathrm{Spec}A$

b) $\mathcal{O}(D(f)) \cong A_f \quad \forall f \in A$

c) $\Gamma(\mathrm{Spec}A, \mathcal{O}) \cong A$
$= \mathcal{O}(\mathrm{Spec}(A))$

$$\mathcal{O}(U) = \{ f: U \longrightarrow \coprod_{\mathfrak{p}\in U} A_\mathfrak{p} \text{ s.t. } f(\mathfrak{p}) \in A_\mathfrak{p} \text{ and } f \text{ is locally a fraction} \}$$

Def: • A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of topological space X and a presheaf of rings $\mathcal{O}_X$ on X.

• A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f: X \longrightarrow Y$ and a map $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ of sheave of rings on Y.

• The ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

• A morphism of locally ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism $(f, f^\#)$ of ringed spaces s.t. $\forall x \in X$ the induced map $f^\#_x: \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is a local homomorphism of local rings

local homomorphism: $f: A \longrightarrow B$ morphism of local rings s.t. $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$

$f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \Rightarrow f^\#(V): \mathcal{O}_Y(V) \longrightarrow (f_*\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V))$

$$\begin{array}{ccc}
\varinjlim_{f(x)\in V} \mathcal{O}_Y(V) & \longrightarrow & \varinjlim_{f(x)\in V} \mathcal{O}_X(f^{-1}(V)) \\
\| & & \| \\
\mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x}
\end{array}$$

Prop: a) A a ring, $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space.

b) If $\varphi: A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces

$$(f, f^\#): (\operatorname{Spec} B, \mathcal{O}_B) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_A)$$

c) If A and B are rings, then any morphism of locally ringed space from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ is induced by a homomorphism $\varphi: A \to B$.

Def: • An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring.

• A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighbourhood U s.t. $(U, \mathcal{O}_X|_U)$ is an affine scheme.

e.g. R: discrete valuation ring, K: fractional field of R.

Then $\operatorname{Spec} R = \{0, m\}$, $\operatorname{Spec} K = \{0\}$.

The morphism $\operatorname{Spec} K \xrightarrow{f} \operatorname{Spec} R$, $0 \mapsto 0$ is induced from $R \to K$, which is a morphism of locally ringed spaces.

$\operatorname{Spec} K \xrightarrow{\varphi} \operatorname{Spec} R$, $0 \mapsto m$; $f^\#: \mathcal{O}_R \to \mathcal{O}_K$ by $R \to K$.

φ is not a morphism of locally ringed spaces.

$\operatorname{Spec} K \longrightarrow \operatorname{Spec} R$

$0 \longmapsto 0$

$(f_* \mathcal{O}_{\operatorname{Spec} K})(V) = \mathcal{O}_{\operatorname{Spec} K}(f^{-1}(V))$

$V = 0$, $(f_* \mathcal{O}_{\operatorname{Spec} K})(V) = K$ $\qquad \mathcal{O}_{R,0} = K$

$V = \{0, m\}$, $(f_* \mathcal{O}_{\operatorname{Spec} K})(V) = K$ $\qquad \mathcal{O}_{K,0} = K$

$f^\#$: $K \longrightarrow K$, $V = 0$

$R \longrightarrow K$, $V = \{0, m\}$

$\operatorname{Spec} K \longrightarrow \operatorname{Spec} R$

$0 \longmapsto m$

$V = 0$, $(f_* \mathcal{O}_{\operatorname{Spec} K})(V) = 0$

$V = \{0, m\}$, $(f_* \mathcal{O}_{\operatorname{Spec} K})(V) = K$ $\qquad \mathcal{O}_{R,m} = R_m \cong R$

$f^\#$: $K \longrightarrow 0$

$R \longrightarrow K$ $\qquad \mathcal{O}_{K,0} = K$

Definition: $x \in X$ is a generic point if $\overline{\{x\}} = X$.

Proj functor.

S: graded ring

S_+: the ideal $\bigoplus_{d>0} S_d$

We define

$\operatorname{Proj} S := \{$homogeneous prime ideals $\mathfrak{p}$ which do not contain all $S_+\}$,

and define $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subseteq \mathfrak{p}\}$.

Define a presheaf $\mathcal{O}$ on $\operatorname{Proj} S$ by

$$\mathcal{O}(U) := \{ f: U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \text{ s.t. } f \text{ is locally a fraction}\}.$$

Prop: S: graded ring

a) $\forall \mathfrak{p} \in \operatorname{Proj} S$, $\mathcal{O}_{\mathfrak{p}} \cong S_{(\mathfrak{p})}$

b) $\forall$ homogeneous $f \in S_+$, $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}$. Then $D_+(f)$ is open in $\operatorname{Proj} S$ and $\operatorname{Proj} S$ is covered by $\{D_+(f)\}_{f \in S_+}$ and $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$.

c) $\operatorname{Proj} S$ is a scheme.

Prop: $k = \bar{k}$. There is a natural fully faithful functor

$$t: \mathrm{Var}(k) \longrightarrow \mathrm{Sch}(k).$$

$$V \longmapsto \text{closed points of } t(V)$$

$t(V) := \{$irreducible closed subsets$\}$ with closed subsets given by

$t(Y)$ where $Y \subseteq V$ closed subset.

$\quad \|$

$\{$irr. closed subsets $\subseteq Y\}$

2. Properties of Schemes

Def: • A scheme is connected if its underlying topology is connected.
• ...... irreducible ...... irreducible

Def: • A scheme X is reduced if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotents ($\Leftrightarrow \mathcal{O}_{X,x}$ $\forall x \in X$ has no nilpotents)

• A scheme X is integral if $\forall \underset{\text{open}}{U \subseteq X}$, $\mathcal{O}_X(U)$ is an integral domain

Proposition: A scheme is integral iff it is both reduced and irreducible.

proof: $\Rightarrow$: integral $\Rightarrow$ reduced. Assume that X is not irreducible, then

$$\exists U_1, U_2 \text{ disjoint open subsets} \Rightarrow \mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$

$$\mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \text{ is not integral} \Rightarrow \rightarrow\leftarrow$$

$\Leftarrow$: Suppose that X is reduced and irr. Let $U \subseteq X$ be an open subset. Assume that $\exists f, g \in \mathcal{O}_X(U)$ s.t. $fg = 0$. Let $Y = \{x \in U \mid f_x \in \mathfrak{m}_x\}$ and let $Z = \{x \in U \mid g_x \in \mathfrak{m}_x\} \Rightarrow Y, Z$ are closed and $Y \cup Z = U$.

X irreducible $\Rightarrow U$ is irreducible $\Rightarrow$ one of Y, Z is U, say $Y = U$.

$\Rightarrow$ the restriction of f to any open affine subset of U will be nilpotent

$\Rightarrow f|_{\text{open affine}} \in \underset{\substack{\parallel \\ V(f)}}{I(\text{open affine})} \Rightarrow f|_{\dots} \in \sqrt{I} \Rightarrow (f|)^n \in I \Rightarrow f|_{\dots}$ nilpotent

$\Rightarrow f|_{\dots} = 0$. By conditions of sheaf $\Rightarrow f = 0 \Rightarrow X$ is integral.

Def: • A scheme X is locally noetherian if it can be covered by open affine subsets $\operatorname{Spec} A_i$, where each A_i is a noetherian ring.

• X is noetherian if it is locally noetherian and quasi-compact.
($\Leftrightarrow X$ can be covered by a finite number of $\operatorname{Spec} A_i$ where A_i is noetherian)

Remark: Noetherian scheme $\Rightarrow$ Noetherian topological space

$\not\Leftarrow$

Prop: A scheme X is locally noetherian iff $\forall$ open affine subset $U = \operatorname{Spec} A$, A is a noetherian ring. In particular, an affine scheme $X = \operatorname{Spec} A$ is a noetherian scheme iff A is a noetherian ring.

proof: $\Leftarrow$ ✓

$\Rightarrow$: if A is a noetherian ring, so is any localization A_f $\forall f \in A$

The open subsets $D(f) \cong \operatorname{Spec} A_f$ form a base for the topology.

$\Rightarrow \{D(f)\}$ form a base of X.

$\Rightarrow U$ open subset can be covered by spectra of noetherian rings.

Want to show: X locally noetherian $\Rightarrow$ $U = \operatorname{Spec} A$, A is noetherian.

X locally noetherian $\Rightarrow X = \bigcup \operatorname{Spec} A_i$, A_i noetherian $\Rightarrow X = \bigcup(\bigcup \operatorname{Spec} A_{i,f_j})$

$\Rightarrow U$ can be covered by spectra of noetherian rings

$\Rightarrow$ We may assume $X = \operatorname{Spec} A$ is an affine scheme and X can be covered by spectra of noetherian rings.

$\forall x \in U, \exists \in$ ✓

$X = \bigcup U$, $U = \operatorname{Spec} B$, B noetherian.

$\{D(f)\}_{f \in A}$ form a base of $X \Rightarrow \exists f \in A$ s.t. $D(f) \subseteq \operatorname{Spec} B = U$

Let $\bar{f}$ be the image of f in B: $\begin{array}{ccc} D(f) & \hookrightarrow & U \\ \parallel & & \parallel \\ \operatorname{Spec} A_f & & \operatorname{Spec} B \end{array} \Rightarrow B \longrightarrow A_f$

$\Rightarrow A_f \cong B_{\bar{f}} \Rightarrow A_f$ noetherian

$\Rightarrow X = \bigcup U = \bigcup(\bigcup \operatorname{Spec} B_{\bar{f}}) = \bigcup \operatorname{Spec} A_f$

$X = \operatorname{Spec} A$ quasi-compact $\Rightarrow X = \bigcup_{\text{finite}} \operatorname{Spec} A_f$

Show: A is a ring. $f_1, \cdots, f_r$ are a finite number of element of A which generate A and each A_i is noetherian $\Rightarrow A$ noetherian.

$\mathfrak{a} \subseteq A$, $\varphi_i : A \longrightarrow A_{f_i}$. we have $\mathfrak{a} = \bigcap_i \varphi_i^{-1}(\varphi_i(\mathfrak{a}) A_{f_i})$

given a chain $\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$ in A gives a chain

$\varphi_i(\mathfrak{a}_0) A_{f_i} \subseteq \cdots$ in A_{f_i}

Def: A morphism $f: X \longrightarrow Y$ of schemes is locally of finite type if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ s.t. $\forall i$, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where A_{ij} is a finitely generated B_i-algebra. The morphism is of finite type if in addition each $f^{-1}(V_i)$ can be covered by a finite number of U_{ij}.

Def: A morphism $f: X \longrightarrow Y$ is a finite morphism if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ s.t. $\forall i$, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a f.g. B_i-module.

Def: • An open subscheme of a scheme X is a scheme U whose topological space is open in X and $\mathcal{O}_U \cong \mathcal{O}_X|_U$.

• An open immersion is a morphism $f: X \longrightarrow Y$ which induces an isomorphism of X with an open subscheme of Y.

Def: • A closed immersion is a morphism $f: Y \longrightarrow X$ of schemes s.t. topologically $f: Y \xrightarrow{\sim} f(Y)$ homeomorphism, $f(Y)$ closed subset of X. and furthermore the induced map $f^{\#}: \mathcal{O}_X \longrightarrow f_* \mathcal{O}_Y$ of sheaves on X is surjective.

• A closed subscheme of X is an equivalence class of closed immersions, where we say $f: Y \longrightarrow X$ and $f': Y' \longrightarrow X$ are equivalent if there is an isomorphism $i: Y' \longrightarrow Y$ s.t. $f' = f \circ i$.

$\mathcal{O}_Y \cong f^{-1}(\mathcal{O}_X / \mathcal{I})$ where $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$

$$\mathcal{O}_Y(U) = \left(\varinjlim_{f(U) \subseteq V} \mathcal{O}_X/\mathcal{I}(V) \right)$$

$$= \varinjlim_{f(U) \subseteq V} \left(\mathcal{O}_X(V) / \mathcal{I}(V) \right)$$

$\mathcal{I}$ ideal sheaf: $\mathcal{I}(U)$ ideal of $\mathcal{O}_X(U)$

$\mathcal{I}: U \longmapsto \mathcal{I}(U)$

3. Separatedness & Properness

Reduced induced closed subscheme structure: Let $X = \operatorname{Spec} A$ be an affine scheme and let Y be a closed subscheme. Set $\mathfrak{a} = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. Then we take the reduced induced structure on Y to be the one defined by $\mathfrak{a}$. $\operatorname{Spec} A/\mathfrak{a}$

In general, for an arbitrary scheme the reduced induced structure is obtained by gluing property.

Theorem: $X \longrightarrow S$, $Y \longrightarrow S$, $X \times_S Y$ exists. (unique up to isomorphisms)

Definition: Let $f: X \longrightarrow Y$ be a morphism of schemes and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y and let $\operatorname{Spec} k(y) \longrightarrow Y$ be natural morphism. Then we define the fiber of the morphism f over the point y to be the scheme $X_y := X \times_Y \operatorname{Spec} k(y)$. $\quad k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & \operatorname{Spec} k(y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Base extension: $X \longrightarrow S$, $S' \longrightarrow S$ schemes over S. $X \times_S S' \longrightarrow S'$ is a scheme over S'.

Remark: • A morphism $X \longrightarrow S$ being of finite type is stable under base extension.

• A scheme being integral is NOT stable under base extension.

Separatedness

Def: Let $f: X \longrightarrow Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $\Delta: X \longrightarrow X \times_Y X$ whose composit with both projection $p_1, p_2: X \times_Y X \longrightarrow X$ is the identity map of $X \longrightarrow X$. We say that the morphism is separated if Δ is a closed immersion. In this case we say X is separated over Y. X is separated if X is separated over $\operatorname{Spec} \mathbb{Z}$.

e.g.: k: a field

X: affine line with doubled origin.

$X \times_k X$ is the affine plane with doubled axes and four origins. The image of

Δ is the usual diagonal with two of those origins. This is not closed because all four origins are in closure of $\Delta(X)$.

Prop: If $f: X \to Y$ is any morphism of affine schemes, then f is separated.

proof: Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. $\Rightarrow A$ is a B-algebra and $X \times_Y X$ is affine, given by $\operatorname{Spec}(A \otimes_B A)$. The diagonal morphism Δ comes from the diagonal homomorphism $A \otimes_B A \to A$ defined by $a \otimes a' \mapsto aa'$. $A \otimes_B A \to A$ is surjective $\Rightarrow$ Δ is closed immersion.

Corollary: An arbitrary morphism $f: X \to Y$ is separated iff the image $\Delta(X)$ is closed in $X \times_Y X$.

proof: $\Rightarrow$ ✓

$\Leftarrow$: $\Delta(X)$ closed implies Δ closed immersion.

$p_1 \circ \Delta = id_X \Rightarrow \Delta$ is a homeomorphism.

$p_1 : X \times_Y X \to X$, $\Delta^\# : \mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective.

Want to show: $\forall x \in X$, let U be an open affine nbhd of x. U is small enough so that $f(U)$ is contained in an open affine subset V of Y.

$\Rightarrow \Delta(x) \in U \times_V U$ open affine

By Proposition, $\Delta : U \to U \times_V U$ is closed immersion. $\Rightarrow \Delta^\#$ is surjective.

Lemma 1: Let R be a valuation ring of a field K. Let $T = \operatorname{Spec} R$ and let $U = \operatorname{Spec} K$. To give a morphism of U to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subseteq K$. To give a morphism of T to X is equivalent to giving two points x_0, x_1 in X with [1] x_0 a specialization of x_1 and an inclusion of fields $k(x_1) \subseteq K$ such that [2] R dominates the local ring $\mathcal{O}$ of x_0 on the subscheme $Z = \overline{\{x_1\}}$ of X with its reduced structure.

1: $x_0 \in \overline{\{x_1\}}$

2: $R \subseteq \mathcal{O}$

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & X \\ \| & & \\ \{0, m\} & \longmapsto & 0 \mapsto x_1 \\ & & m \mapsto x_0 \end{array}$$

$R \subseteq \mathcal{O}_{X, x_0}$

$\overline{\{0\}} \ni m$ $\qquad$ $x_0 \in \overline{\{x_1\}}$

Lemma 2: Let $f: X \to Y$ be a quasi-compact morphism of schemes. Then the subset $f(X)$ of Y is closed iff it is stable under specialization.

A set U is stable under specialization if $\forall\, y_1 \in U$, $y_0 \in \overline{\{y_1\}} \Rightarrow y_0 \in U$.

Theorem 1 (Valuative Criterion of Separatedness): Let $f: X \to Y$ be a morphism of schemes and assume that X is noetherian. Then f is separated iff the following condition holds. For any field K and for any valuation ring R with $\operatorname{Frac} R = K$. Let $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, $i: U \to T$ the morphism induced by the inclusion $R \subseteq K$. Given a morphism T to Y, and given a morphism of U to X which makes a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K = U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\operatorname{Spec} R = T \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

there is at most one morphism of T to X making the whole diagram commutative.

Assume that all schemes are noetherian in the following statements.

Corollary: Assume that all schemes are noetherian.

(a) Open and closed immersions are separated.

(b) A composition of two separated morphisms is separated.

(c) Separated morphisms are stable under base extension.

(d) If $X \to Y$ and $f': X' \to Y'$ are separated morphisms of schemes over S, then $f \times f': X \times_S X' \to Y \times_S Y'$ is also separated.

(e) If $f: X \to Y$ and $g: Y \to Z$ are morphisms and $g \circ f$ is separated then so is f.

(f) $f: X \to Y$ being separated is a local property.

$Y = \bigcup V_i$ $\quad$ $f^{-1}(V_i) \to V_i$ separated $\forall i$

proof of (c): $f: X \to S$ separated $\quad S' \to S$

Want to show: $f': X' = X \times_S S' \to S'$ separated

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X' \arrow[r, "p_1"] \arrow[d, "f'"] & X \arrow[d] \\
T \arrow[r] \arrow[ur, "\alpha"] \arrow[ur, "\beta"'] & S' \arrow[r] & S
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] \arrow[ur, "p_1 \circ \alpha"] \arrow[ur, "p_1 \circ \beta"'] & S
\end{tikzcd}$$

$$\xrightarrow[\text{of } X \to S]{\text{separatedness}} \quad p_1 \circ \alpha = p_1 \circ \beta$$

$$\begin{tikzcd}
T \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[rr, bend left, "p_1\circ\alpha = p_2\circ\beta"] \arrow[dr, bend right] & X' \arrow[r] \arrow[d, "f"] & X \arrow[d] \\
& S' \arrow[r] & S
\end{tikzcd}$$

$$\xrightarrow[\text{property}]{\text{universal}} \alpha = \beta$$

Def: A morphism $f: X \to Y$ is proper if it is separated, of finite type and universally closed.

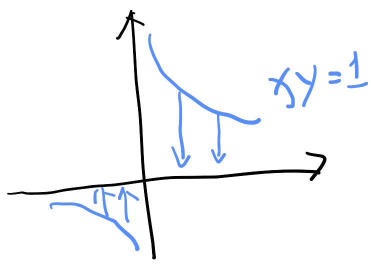
$f: X \to Y$ universally closed if $f': X \times_S S' \to S'$ is closed for any $S' \to S$.

e.g.: k: a field

X: affine line over k

X is separated and of finite type, but it is not proper over k.

$X \to \operatorname{Spec} k$, $\quad X \times_k X \xrightarrow{\Delta} X$ is a projection of affine plane to the affine line



$\Delta(\text{hyperbola}) = \mathbb{A} \setminus \{0\}$

Theorem 2 (Valuative Criterion for Properness): $X \xrightarrow{f} Y$ of finite type, X noetherian. . . . .

f is proper iff . . . .

. . . .

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

there exists a unique morphism $T \to X$ . . . . . .

Corollary: noetherian

(a) A closed immersion is proper

(b) composition is proper

(c) stable under base extension

(d) as (d) in Corollary of Theorem 1

(e) $f: X \to Y$, $g: Y \to Z$, $g \circ f$ is proper, g is separated

$\Rightarrow$ f is proper

(f) Properness is a local property