Steenord Algebra

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1 Steenord Operations

This section is based on lectures 2 to 6 of Jacob Lurie's note on the Sullivan Conjecture. We will construct Steenord operations in the derived $\infty-$category $D(\mathbb{F}_2)$. Informally, this category contains objects as $\mathbb{F}_2$ chain complexes up to isomorphism, and the mapping space between two chain complexes A, B is $map(A, B) = RHom(A, B)$. Notice that we also have

$$D(\mathbb{F}_2) \cong \mathrm{Mod}_{\mathbb{F}_2}.$$

which means every chain complex can be viewed as a $\mathbb{F}_2$-module spectra. So we can also consider its homotopy groups. If V be a chain complex. then we know

$$\pi_n(V) = H_n(V).$$

Also, notice that one can always view a chain complex as a cochain complex by changing lower index to the minus of upper index, which means, V_n and V^n is the same thing. Since we're going to describe Steenord operations which are defined on co-chain complexes, the upper index will be more frequently used below.

For two cochain complexes V and W, we can consider their tensor product

$$(V \otimes W)^n = \bigoplus_{p+q=n} (V^p \otimes W^q),$$

and the differential will be the usual differential for tensor product of chain complexes. The symbol $V^{\otimes n}$ means V tensors with itself for n times.

The $\mathbb{F}_2$ module spectrum that we're interested are the cochain complexes of spaces, which we can denote as $C^*(X; \mathbb{F}_2)$. From classical algebraic topology, we know the cohomology has the cup product structure. Moreover, it comes from the composition on the cochain complexes:

$$C^*(X; \mathbb{F}_2) \otimes C^*(X; \mathbb{F}_2) \to C^*(X \times X; \mathbb{F}_2) \to C^*(X; \mathbb{F}_2).$$

The first map is the classical Alexander-Whitney morphism, and the second comes from the diagonal map. Notice that the first morphism is not compatible with Σ_2-action on both sides, which means the composite map

$$m : C^*(X; \mathbb{F}_2) \otimes C^*(X; \mathbb{F}_2) \to C^*(X; \mathbb{F}_2) \tag{1}$$

is not commutative until passing to cohomology. We will see that this non-commutativity produces the Steenord operations.

Definition 1.1. *Let V be a $\mathbb{F}_2$-module spectrum and $n \geq 0$ be a non-negative integer. We define the nth extended power of V*

$$D_n(V) = (V^{\otimes n})_{h\Sigma_n} = colim_{\Sigma_n} V^{\otimes n}.$$

Remark 1.2. *The homotopy orbit $D_n(V)$ has a general procedure to describe. Let M denote the vector space $\mathbb{F}_2$. Choose a projective resolution of M by free $\mathbb{F}_2[\Sigma_n]$-modules and denote it as $E\Sigma_n$. Then we have*

$$D_n(V) = (E\Sigma_n \otimes V^{\otimes n})/\Sigma_n.$$

On the complex $D_2(V)$, we can define

Definition 1.3. *A symmetric multiplication on V is a map*

$$D_2(V) \to V$$

for a complex V.

For any space X, since we know the map m defined in equation (1) is commutative up to homotopy, which means m factors through the homotopy orbit. We get our first example of symmetric multiplication. In the following, we will see this multiplication induces the Steenord operations.

Let n be an integer. We denote $\mathbb{F}_2[-n]$ as a chain complex whose module at cohomological degree n is $\mathbb{F}_2$, and its module at all other degrees are all 0. and we can denote its homology group as

$$H^n(\mathbb{F}_2[-n]) = \mathbb{F}_2 e_n$$

and all other cohomology groups are just 0. For this cochain complex, we have

$$\mathbb{F}_2[n]^{\otimes 2} = \mathbb{F}_2[-2n]$$

and Σ_2 acts trivially on this cochain complex. Consequently, we have

$$D_2(\mathbb{F}_2[-n]) = \mathbb{F}_2[-2n] \otimes (E\Sigma_2)_{\Sigma_2}.$$

Since one can identify chain complex with module spectrum, $(E\Sigma_2)_{\Sigma_2}$ really means the chain complex of $B\Sigma_2$, which is just $\mathbb{R}P^\infty$. Hence, we can express the cohomology groups of $D_2(\mathbb{F}_2[-n])$ as

$$H^k(D_2(\mathbb{F}_2[-n])) \cong H_{2n-k}(\mathbb{R}P^\infty; \mathbb{F}_2)e_{2n}$$

From Algebraic topology, we know $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[t]$. In homology groups, we use x_n to denote the generator in $H_n(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2$.

Definition 1.4. *Let V be a complex and $v \in H^n V$. Then v can be represented by a homotopy class of maps*

$$\eta : \mathbb{F}_2[-n] \to V.$$

For $i \leq n$, we let

$$\overline{Sq}^i(v) \in H^{n+i}(D_2(V))$$

be the image of

$$x_{n-i} \otimes e_{2n} \in H_{n-i}(\mathbb{R}P^\infty; \mathbb{F}_2)e_{2n} \cong H^{n+i}(D_2(\mathbb{F}_2[-n]))$$

under the induced map

$$D_2(\eta) : D_2(\mathbb{F}_2[-n]) \to D_2(V).$$

By convention (or $x_{n-i} = 0$), we define $\overline{Sq}^i(v) = 0$ for $i > n$.

When V is equipped with a symmetric multiplication, we define the Steenord operation Sq^i : $H^n(V) \to H^{n+i}(V)$ as the composition of $\overline{Sq}^i$ and the induced map of multiplication.

Example 1.5. *When $V = C^*(X; \mathbb{F}_2)$ for some space X, we recover the classical definition of Steenord operations $Sq^i : H^n(X; \mathbb{F}_2) \to H^{n+i}(X; \mathbb{F}_2)$ using the multiplication discussed in the start of the chapter.*

2 Properties of Steenord Operations

In this section, we shall try to prove all the important properties of Steenord Operations on spaces. They will be the actual important thing later. For start, we list all the properties:

1. $Sq^i \circ f^* = f^* \circ Sq^i$ for $f : X \to Y$.
2. $Sq^i(\alpha) = \alpha^2$ for $i = |\alpha|$.
3. $Sq^i(\alpha + \beta) = Sq^i(\alpha + \beta)$.
4. $Sq^i \circ \sigma = \sigma \circ Sq^i$, where $\sigma : H^n(X; \mathbb{F}_2) \to H^{n+1}(\Sigma X; \mathbb{F}_2)$ is the canonical suspension isomorphism. (Stability)
5. $Sq^0 = id$.
6. $Sq^i(\alpha) \neq 0$ only if $0 \leq i \leq |\alpha|$
7. $Sq^i(\alpha \cup \beta) = \Sigma_j Sq^j(\alpha) \cup Sq^{i-j}(\beta)$ (Cartan formula).
8. $Sq^a Sq^b(v) = \Sigma_k (2k - a, b - k - 1) Sq^{b+k} Sq^{a-k}(v)$, where for $i, j \in \mathbb{Z}$, we have the notation

   $$(i,j) = \begin{cases} \begin{pmatrix} i+j \\ i \end{pmatrix} = \begin{pmatrix} i+j \\ j \end{pmatrix} = \dfrac{(i+j)!}{i!j!} & i, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   (Adem relation)

One can see property 1 is an immediate result since our construction is functorial. The second property is also not hard. From the definition, when $i = n$, we can see that there is a commutative diagram:

$$\begin{tikzcd}
\mathbb{F}_2[-n] \arrow[r] \arrow[d, "\eta"] & \mathbb{F}_2[-n] \otimes \mathbb{F}_2[-n] \arrow[r] \arrow[d, "\eta\otimes\eta"] & D_2(\mathbb{F}_2[-n]) \arrow[d, "D_2(\eta)"] \\
V \arrow[r] & V \otimes V \arrow[r] & D_2(V)
\end{tikzcd} \tag{2}$$

which means $Sq.(v)$ is just the composition

$$V \otimes V \to D_2(V) \to V$$

In the case of topological spaces, this simply means the square cup product. We now turn to prove property 3. For that, we need the following lemma:

Lemma 2.1. *Let V be a complex in $D(\mathbb{F}_2)$, and that $\{v_i\}_{i\in I}$ is a basis of H_*V, where $v_i \in H^{n_i}V = H_{-n_i}V$. Then $H_*D_2(V)$ has a basis of the following collection*

$$\{v_iv_j\}_{i<j} \cup \{Sq^n v_i\}_{n\leq n_i, i\in I}$$

Proof. using the fact that D_2 commutes with filtered colimit, we reduce to the case of finite generators. Then, we can use induction. When H_*V contains a single basis vector, this is obvious since $V \simeq \mathbb{F}_2[-n]$ and we have concrete description for $D_2[\mathbb{F}_2[-n]]$. For the induction step, one only need to notice the following formula:

$$D_2(V \oplus W) \simeq (V \oplus W)^{\otimes 2}_{h\Sigma_2} \simeq V^{\otimes 2}_{h\Sigma_2} \oplus (V \otimes W) \oplus W^{\otimes 2}_{h\Sigma_2}.$$

$\square$

Proposition 2.2. *The Steenord operation preserves additions. Let V be a complex and $v, v' \in H^nV$. Then we have*

$$\overline{Sq}^i(v + v') = \overline{Sq}^i(v) + \overline{Sq}^i(v')$$

Hence, when V is equipped with a symmetric multiplication, the Steenord Operations commutes with additions.

Proof. When $i > n$, both sides are 0 by definition, so there is nothing to prove.

When $i = n$, we again use the commutative diagram (2) above. Notice that in the level of $V \otimes V$, we have

$$(v + v') \otimes (v + v') = v \otimes v + v' \otimes v' + v \otimes v' + v' \otimes v$$

When passing to $D_2(V)$, we know the map $V \otimes V \to D_2(V)$ is commutative up to homotopy by definition, so the last two terms become $2vv' = 0$. The first two terms are exact $\overline{Sq}^i(v) + \overline{Sq}^i(v')$.

When $i < n$, by naturality, it suffices to prove the case $V \simeq \mathbb{F}_2[-n] \oplus \mathbb{F}_2[-n]$. This induces canonical map $V \to \mathbb{F}_2[-n]$ for each factors, and it then induces a map

$$H^m D_2(V) \to H^m D_2(\mathbb{F}_2[-n]) \times H^m D_2(\mathbb{F}_2[-n])$$

This is an injective map by lemma 2.1. Notice in the definition of Steenord operations, we need to transfer v as maps η_v. In our situations, we have a commutative diagram:

$$\begin{tikzcd} & \mathbb{F}_2[-n] \arrow[dl, "\eta_{v+w}"'] \arrow[dr, "\eta_v"] & \\ V \arrow[rr, "p"] & & \mathbb{F}_2[-n] \end{tikzcd}$$

This implies $\overline{Sq}^i(v+v')$ maps to $\left(\overline{Sq}^i(v), \overline{Sq}^i(v')\right) \in H^m D_2(\mathbb{F}_2[-n]) \times H^m D_2(\mathbb{F}_2[-n])$, and so is $\overline{Sq}^i(v) + \overline{Sq}^i(v')$. Injectivity then finishes the proof. $\square$

Now, we turn our attention to prove stability of Steenord operations. Let V be a $\mathbb{F}_2$-module spectrum. Recall that we have the loop functor Ω, where ΩV is the pullback of the following dirgram:

$$\begin{tikzcd} \Omega V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & V \end{tikzcd}$$

Notice this is a pullback in $D(\mathbb{F}_2)$, so translate in the classical description, this means the homotopy pullback. Think V as chain complex, we then know in cohomological indexing,

$$(\Omega V)^n \simeq V^{n-1};$$

$$H^n \Omega V \cong H^{n-1} V.$$

Notice that the functor D_2 takes acyclic object to acyclic objects, so applying D_2 to the pullback diagram above, it induces a canonical morphism:

$$\phi : D_2(\Omega V) \to \Omega(D_2 V).$$

To show the stability, we need the following lemma, which basically states the loop functor can commute with $\overline{Sq}^i$. We will omit its proof, one can find a detailed one in lecture 3 of Lurie's note.

Lemma 2.3. *Let V be a chain complex and i be an integer. There is an commutative diagram as follows:*

$$\begin{tikzcd} H^*(\Omega V) \arrow[rr, "\simeq"] \arrow[d, "\overline{Sq}^i"] & & H^{*-1}(V) \arrow[d, "\overline{Sq}^i"] \\ H^{*+i}(D_2(\Omega V)) \arrow[r, "\phi"] & H^{*+i}(\Omega(D_2(V))) \arrow[r, "\simeq"] & H^{*+i-1}(D_2(V)) \end{tikzcd}$$

If V is a complex with symmetric multiplication $m : D_2(V) \to V$, then ΩV also has a symmetric multiplication given by

$$\overline{m} : D_2(\Omega V) \xrightarrow{\phi} \Omega D_2(V) \xrightarrow{\Omega(m)} \Omega V$$

By this construction, we have the following diagram:

$$\begin{array}{ccccc} H^{*+1}(D_2(\Omega V)) & \xrightarrow{\phi} & H^{*+1}(D_2(V)) & \xrightarrow{\simeq} & H^*(D_2(V)) \\ \downarrow{\scriptstyle \overline{m}} & & & & \downarrow{\scriptstyle m} \\ H^{*+1}(\Omega V) & & \xrightarrow{\simeq} & & H^*(V) \end{array}$$

Combining this and the diagram in Lemma 2.3, we have the following corollary:

Proposition 2.4. *Let V be a complex equipped with a symmetric multiplication, then it induces a symmetric multiplication on ΩV. Moreover, the canonical isomorphism*

$$H^*(V) \simeq H^*(\Omega V)$$

commutes with Steenord operations.

In particular, for a space X, since $C^*(\Sigma X; \mathbb{F}_2) \simeq \Omega C^*(X; \mathbb{F}_2)$, we establish the stability result of Steenord operations.

Using stability, we can calculate the Steenord operation on spheres:

Example 2.5. *For $X = S^n$ and v being the generator in the top cohomology. We have*

$$Sq^k v = \begin{cases} v & k = 0 \\ 0 & \textit{otherwise} \end{cases}$$

This is true because we use the above Proposition to reduce to the case $n = 0$. When $n = 0$, Sq^0 is cup product as property 2 of Steenord operations.

Now, we can prove property 5 and 6 all together. Notice that for property 6, we do not need to prove $Sq^k(\alpha) = 0$ for $k > |\alpha|$ since it is defined as 0 in our construction.

Proposition 2.6. *Let X be a topological space and $v \in H^n(X; \mathbb{F}_2)$. We have $Sq^0(v) = 0$ and $Sq^k(v) = 0$ if $k < 0$.*

Proof. Recall that in Algebraic Topology, we proved that there is a cohomology class

$$\chi \in H^n(K(\mathbb{F}_2, n); \mathbb{F}_2)$$

such that the following map is an isomorphism for any space X:

$$\begin{aligned} \pi_0 Map(X, K(\mathbb{F}_2; n)) \to &\quad H^n(X; \mathbb{F}_2) \\ f \mapsto &\quad f^*(\chi) \end{aligned}$$

Using this, we can reduce to the case $X = K(\mathbb{F}_2, n)$ and $v = \chi$. This is because for any $v \in H^n(X;\mathbb{F}_2)$, there is a map $f : X \to K(\mathbb{F}_2, n)$ satisfying the following diagram:

$$\begin{tikzcd} H^n(K(\mathbb{F}_2,n);\mathbb{F}_2) \arrow[r, "Sq^k"] \arrow[d, "f^*"] & H^{n+k}(K(\mathbb{F}_2,n);\mathbb{F}_2) \arrow[d, "f^*"] \\ H^n(X;\mathbb{F}_2) \arrow[r, "Sq^k"] & H^{n+k}(X;\mathbb{F}_2) \end{tikzcd}$$

Let $v \in H^n(S^n;\mathbb{F}_2)$ be a cohomology class as in Example 2.5. Again, there is a map $f : S^n \to K(\mathbb{F}_2; n)$ making the diagram commutes. This map induces isomorphism

$$f^* : H^{n+k}(K(\mathbb{F}_2; n);\mathbb{F}_2) \to H^{n+k}(S^n;\mathbb{F}_2)$$

for $k \leq 0$ by Hurewicz theorem. (When k=0, both sides are $\mathbb{F}_2$ and f^* maps generator to generator by definition.) Hence, we finish the proof by referring Example 2.5. $\square$

Remark 2.7. *The above results are not true for a general chain complex V. This holds really because of the results from algebraic topology.*