Representation Theory of Finite Groups and Applications

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Abstract

We develop the representation theory of finite groups, and do so through three main sections, standard results of group representations, character theory and finally induced representations. We give necessary definitions and theorems in the first section to reach Maschke's theorem and clarify the correspondence between representations and modules over group algebra. For the requirement of the uniqueness of Maschke's theorem, we introduce character theory and enhance the result on decomposition of representations. Moreover, we define induced representations to obtain a new representation that relates representations of subgroups with representations of the whole group and explain explicitly how induced representations connect with direct sums and tensor products. Finally, as an application of representation theory, we consider irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ for a prime power $q > 2$. We construct some irreducible representations from principal series representations and induced representations of subgroups of the general linear group.

Contents

List of Tables

List of Figures

1 Group representations

1.1 Basic notions

In what follows, we shall use group actions on sets and modules over a given ring. The reader is directed to J. J. Rotman (2010) for definitions of each of these terms. Respectively group actions are actions of a group on a set and modules are actions of a ring on an abelian group. We shall introduce a new type of action of groups on vector spaces which are called representations of groups. In particular, we focus on finite-dimensional vector spaces over $\mathbb{C}$ in this report.

Moreover, in the report we only consider representations of finite groups and thus we shall use G to represent a finite group unless otherwise stated. Particularly, in this section we introduce definitions and examples to construct foundations of the following sections.

Definition 1.1. *Let G be a finite group and let $\mathrm{GL}(V)$ denote the set of all invertible linear transformations from V to itself. A **representation** of G on V is a homomorphism $\sigma : G \to \mathrm{GL}(V)$ for some finite-dimensional vector space V. The vector space V is called the **representation space** of σ and its dimension is called the **degree** of σ.*

For convention, we usually only say that σ or V is a representation of G. If we only know the homomorphism σ precisely, we sometimes use V_σ to represent the representation space of σ. On the other hand, we may sometimes use σ_g to display $\sigma(g)$ for $g \in G$.

Example 1.1. Consider the group $G = \mathbb{Z}/n\mathbb{Z}$ for some positive integer n. Define the mapping $\varphi : G \to \mathrm{GL}(\mathbb{C}^2)$ by

$$\varphi(m) = \begin{pmatrix} \cos\left(\frac{2\pi m}{n}\right) & -\sin\left(\frac{2\pi m}{n}\right) \\ \sin\left(\frac{2\pi m}{n}\right) & \cos\left(\frac{2\pi m}{n}\right) \end{pmatrix}.$$

Then φ is a representation of G of degree 2 with representation space $\mathbb{C}^2$.

Definition 1.2. *Two representations $\sigma_1 : G \to \mathrm{GL}(V)$ and $\sigma_2 : G \to \mathrm{GL}(W)$ are said to be **equivalent** if there exists an isomorphism $T : V \to W$ such that $\sigma_1(g)T = T\sigma_2(g)$ for all $g \in G$. Namely, the diagram*

$$\begin{array}{ccc} V & \xrightarrow{\sigma_1(g)} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow{\sigma_2(g)} & W \end{array}$$

commutes. In this case, we write $\sigma_1 \cong \sigma_2$ or $V \cong W$.

Example 1.2. Let G be the group $\mathbb{Z}/n\mathbb{Z}$ for some positive integer n. Define the map $\psi : G \to \mathrm{GL}(\mathbb{C}^2)$ by

$$\psi(m) = \begin{pmatrix} e^{2\pi mi/n} & 0 \\ 0 & e^{-2\pi mi/n} \end{pmatrix}.$$

Clearly, ψ is a representation of G of degree 2 with representation space $\mathbb{C}^2$. Moreover, ψ is equivalent to the representation φ defined in Example 1.1 for the reason that $T^{-1}\varphi(m)T = \psi(m)$ where $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ and $m \in G$.

Normally we do not distinguish representations that are equivalent and we shall study properties of representations under this equivalence. Like subgroups, we can also define subrepresentations as follows.

Definition 1.3. *Let $\sigma : G \to \mathrm{GL}(V)$ be a representation of G. A subspace W of V is a **G-subspace** if for all $g \in G$ and $w \in W$, we have $\sigma_g(w) \in W$. We call W a **subrepresentation** of V.*

We denote the restriction of σ on W as σ_W. Then clearly σ_W is a representation of G. This is the reason why we call W a subrepresentation. In order to clarify relations between representations we introduce two operations of representations; namely, direct sum and tensor product which play an important role in representation theory.

Definition 1.4. *Suppose that $\sigma^{(1)} : G \to \mathrm{GL}(V_1)$ and $\sigma^{(2)} : G \to \mathrm{GL}(V_2)$ are representations of G. Then the **direct sum** of them $\sigma^{(1)} \oplus \sigma^{(2)} : G \to \mathrm{GL}(V_1 \oplus V_2)$, is given by*

$$\left(\sigma^{(1)} \oplus \sigma^{(2)}\right)_g ((v_1, v_2)) = \left(\sigma_g^{(1)}(v_1), (\sigma_g^{(2)}(v_2)\right), v_1 \in V_1 \text{ and } v_2 \in V_2.$$

Definition 1.5. *Suppose that $\sigma^{(1)} : G \to \mathrm{GL}(V_1)$ and $\sigma^{(2)} : G \to \mathrm{GL}(V_2)$ are representations of G. Then the **tensor product** of them $\sigma^{(1)} \otimes \sigma^{(2)} : G \to \mathrm{GL}(V_1 \otimes V_2)$, is given by*

$$\left(\sigma^{(1)} \otimes \sigma^{(2)}\right)_g (v_1 \otimes v_2) = \sigma_g^{(1)}(v_1) \otimes \sigma_g^{(2)}(v_2), v_1 \in V_1 \text{ and } v_2 \in V_2.$$

Recall that in number theory we have primes which describes how integers decompose. Similarly, one can define irreducible representations that almost play the same role in representation theory as primes in number theory.

Definition 1.6. *A non-zero representation $\sigma : G \to \mathrm{GL}(V)$ is said to be **irreducible** if the only G-subspaces of V are $\{0\}$ and V.*

Definition 1.7. *Let G be a group. A representation $\sigma : G \to \mathrm{GL}(V)$ is said to be **completely reducible** if $V \cong V_1 \oplus \cdots \oplus V_n$ for some $n \in \mathbb{N}$, where the V_i's are irreducible subrepresentations of V. We also write $\sigma \cong \sigma_1 \oplus \cdots \oplus \sigma_n$ if V_i is the representation space of σ_i for all $1 \leq i \leq n$.*

Notice that in Definition 1.7, V_i may be equivalent to V_j for some i, j. We therefore write V_i^n to represent n of V_i, thereby shortening our notation. Correspondingly, if we use σ_i to denote representations, then we shortly use $n\sigma_i$ to display n copies of σ_i.

Definition 1.8. *Let σ be a representation of G. If $\sigma \cong m_1\sigma_1 \oplus \cdots \oplus m_s\sigma_s$ for some distinct σ_i, i.e. $V_\sigma \cong V_{\sigma_1}^{m_1} \oplus \cdots \oplus V_{\sigma_s}^{m_s}$ with $V_{\sigma_i} \ncong V_{\sigma_j}$, for $i \neq j$, then m_i is called the **multiplicity** of σ_i in σ where $m_i \in \mathbb{N} \cup \{0\}$.*

Definition 1.9. *A non-zero representation σ of G is **decomposable** if $V_\sigma = V_1 \oplus V_2$ with V_1, V_2 non-zero G-subspaces. Otherwise, V is called **indecomposable**.*

To see the difference between indecomposable representations and irreducible representations, we consider the following example:

Example 1.3. Consider the additive group $G = (\mathbb{R}, +)$ and the $\mathbb{R}$-vector space $V = \mathbb{R}^2$. Define the map $\sigma : G \to \mathrm{GL}(V)$ by $\sigma_g((x, y)) = (x + gy, y)$ for $g, x, y \in \mathbb{R}$ where gy is the multiplication of real numbers. The reader may check that σ is a representation. Note that $\mathbb{R} \times \{0\}$ is the only G-subspace of V and $V \neq (\mathbb{R} \times \{0\})^2$; namely, σ is indecomposable. However, σ is not irreducible for the proper G-subspace $\mathbb{R} \times \{0\}$ of V.

So indecomposable representations are not the same as irreducible representations when the group G is infinite. Analogously, there are differences between completely reducible and decomposable. To keep the consistency of the report, we leave the definition of decomposable representations.

From the definition of irreducible representations (Definition 1.6), obviously we see that a vector space of dimension 1 has no proper subspace and therefore it cannot have non-trivial G-subspaces if it is a representation of G. We therefore have the result:

Proposition 1.1. *Any representation $\sigma : G \to \mathrm{GL}(V)$ of degree 1 is irreducible.*

Naturally, we want that reducibility of representations should be preserved under equivalence which is exactly the following result.

Proposition 1.2. *(B. Steinberg (2011), p.19) The property of a representation to be irreducible, completely reducible and decomposable is invariant under the equivalence of representations.*

Using the above proposition we can consider decomposition of representations under equivalence and avoid the case that two equivalent representations have different decomposition.

1.2 Maschke's theorem

This section is aimed at proving Maschke's theorem which states the connection between representations and irreducible representations. That is for every representation there is a decomposition of it into irreducible representations. To do this we state that every representation of a finite group is either irreducible or decomposable and then show it is indeed completely reducible. So Definition 1.7 gives a decomposition.

Theorem 1.1. *(B. Steinberg (2011), p.22) Let $\sigma : G \to \mathrm{GL}(V)$ be a non-zero representation of a finite group G. Then σ is either irreducible or decomposable.*

The proof of Theorem 1.1 involves unitary representations which are not intended to be covered in the report. The reader is direct to B. Steinberg (2011) for details.

Theorem 1.2 **(Maschke).** *Every representation of a finite group is completely reducible. Namely, every representation of a finite group can be decomposed as a direct sum of irreducible representations.*

Proof. Let $\sigma : G \to \mathrm{GL}(V)$ be a representation of a finite group G. We take induction on the degree of σ, i.e. $\dim(V) = n$. If $\dim(V) = 1$, then Proposition 1.1 implies that σ is irreducible. Assume that the statement is true for representations of dimension less than n. If V is irreducible then we are done. Otherwise, V is decomposable by Theorem 1.1, that is $V = V_1 \oplus V_2$ with $\dim(V_1), \dim(V_2) < n$. Then by inductive hypothesis, we obtain $V_1 = W_1 \oplus \cdots \oplus W_i, i \in \mathbb{N}$ and $V_2 = U_1 \oplus \cdots \oplus U_t, t \in \mathbb{N}$ for some irreducible representations W_j and $U_s, 0 \leq j \leq i, 0 \leq s \leq t$. Hence, we have the decomposition $V = V_1 \oplus V_2 = W_1 \oplus \cdots \oplus W_i \oplus U_1 \oplus \cdots \oplus U_t$ into irreducible representations. $\square$

1.3 Group algebras

To understand representations more precisely, we develop the concept of a group algebra $\mathbb{C}[G]$ and demonstrate a correspondence between representations and group algebras.

Definition 1.10. *Let G be a group. The set of all formal sums $\sum\limits_{g\in G} a_g g$ forms a $\mathbb{C}$-algebra with scalar multiplication $\lambda \sum\limits_{g\in G} a_g g = \sum\limits_{g\in G} (\lambda a_g) g$ and the operations*

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g} (a_g + b_g) g \text{ and } \left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g,h\in G} a_g b_h gh,$$

*where $a_g, b_g \in \mathbb{C}$ for all $g \in G$, which is called the **group algebra** of G over $\mathbb{C}$. We denote the group algebra of G over $\mathbb{C}$ by $\mathbb{C}[G]$.*

Equivalently, for every formal sum $\sum\limits_{g\in G} a_g g$ we can correspondingly define a function $f : G \to \mathbb{C}$ that maps g to its coefficient, for every $g \in G$. Moreover, if G is a finite group we can define an inner product on the space. Thus, for finite group G we have the equivalently definition.

Definition 1.11. *Let G be a group and define $\mathbb{C}[G] = \{f | f : G \to \mathbb{C}\}$. Then $\mathbb{C}[G]$ is an inner product space with addition and scalar multiplication given by*

$$(f_1 + f_2)(g) = f_1(g) + f_2(g), \quad (\lambda f)(g) = \lambda \cdot f(g)$$

and with the inner product defined by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

*$\mathbb{C}[G]$ is called the **group algebra** of G.*

Theorem 1.3. *Let G be a finite group. Then there is a one-to-one correspondence between representations of G and left $\mathbb{C}[G]$-modules.*

Proof. Assume that $\sigma : G \to \mathrm{GL}(V)$ is a representation of G. We define the scalar multiplication on V by

$$\left(\sum_{g\in G} a_g g\right) v = \sum_{g\in G} a_g \sigma_g(v),$$

for every formal sum $\sum\limits_{g\in G} a_g g \in \mathbb{C}[G]$ and $v \in V$. Then one can check that the scalar multiplication satisfies the definition of modules. Therefore, V is a $\mathbb{C}[G]$-module with the scalar multiplication.

Conversely, let M be a $\mathbb{C}[G]$ module. We may regard every complex number $z \in \mathbb{C}$ as $z \cdot 1_G \in \mathbb{C}[G]$, where 1_G is the identity in G. Then it follows that M is a vector space over $\mathbb{C}$. Define the map $\sigma : G \to \mathrm{GL}(M)$ by $\sigma_g(v) = g \cdot v$, for every $g \in G$ and $v \in M$. Then σ is naturally a homomorphism by the definition of modules. $\square$

For a given module $\mathbb{C}[G]$-module M corresponding to a representation σ, there are submodules of M. By the correspondence in Theorem 1.3, we see that each submodule corresponds to a subrepresentation of σ. Therefore, it is possible to find irreducible representations of σ by looking at submodules. Combining this with Maschke's theorem leads to a construction of decomposition into irreducible representations.

1.4 Schur's lemma

For any algebraic structure, there are homomorphisms between them. Similarly, this section is going to build morphisms between representations. To apply them, we classify all irreducible representations of an abelian group by using Schur's lemma which is a foundation of representation theory.

Definition 1.12. *Let $\sigma^{(1)} : G \to \mathrm{GL}(V)$ and $\sigma^{(2)} : G \to \mathrm{GL}(W)$ be representations of G. A **morphism** from $\sigma^{(1)}$ to $\sigma^{(2)}$ is a linear transformation $T : V \to W$ such that $T\sigma^{(1)}_g = \sigma^{(2)}_g T$, for all $g \in G$. Namely, the diagram*

$$\begin{tikzcd} V \arrow[r, "\sigma_g^{(1)}"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\sigma_g^{(2)}"] & W \end{tikzcd}$$

commutes for all $g \in G$. The set of all morphisms from $\sigma^{(1)}$ to $\sigma^{(2)}$ is denote $\mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right)$.

Note that when T is an invertible linear transformation $\sigma^{(1)}$ and $\sigma^{(2)}$ are equivalent. So the equivalence works similarly as "isomorphism". Besides, $\mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right)$ is a subset of $\mathrm{Hom}(V, W)$. The following shows that it not only a subset but also a subspace.

Proposition 1.3. *Let $\sigma^{(1)} : G \to \mathrm{GL}(V)$ and $\sigma^{(2)} : G \to \mathrm{GL}(W)$ be representations of G. Then $\mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right)$ is a subspace of $\mathrm{Hom}(V, W)$.*

Proof. Let $T_1, T_2 \in \mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right)$ and $a \in \mathbb{C}$. Then

$$(aT_1)\sigma^{(1)}_g = a(T_1\sigma^{(1)}_g) = a(\sigma^{(2)}_g T_1) = \sigma^{(2)}_g (aT_1),$$

and

$$(T_1 + T_2)\sigma^{(1)}_g = T_1\sigma^{(1)}_g + T_2\sigma^{(1)}_g = \sigma^{(2)}_g T_1 + \sigma^{(2)}_g T_2 = \sigma^{(2)}_g (T_1 + T_2),$$

for all $g \in G$. $\square$

Lemma 1.1 (**Schur's lemma**). *(B. Steinberg (2011), p.29) Let $\sigma^{(1)}, \sigma^{(2)}$ be irreducible representations of G, and $T \in \mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right)$. Then either T is invertible or $T = 0$. More precisely:*

(i). If $\sigma^{(1)} \ncong \sigma^{(2)}$, then $\mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right) = 0$;

(ii). If $\sigma^{(1)} \cong \sigma^{(2)}$, then $T = \lambda I_n$ with $\lambda \in \mathbb{C}$, where $n = \deg(\sigma^{(1)})$ and I_n is the identity matrix of n dimension.

Note that Schur's lemma directly implies that the $\mathbb{C}$-vector space $\mathrm{Hom}_G\left(\sigma^{(1)}, \sigma^{(2)}\right)$ has dimension 1 if $\sigma^{(1)}$ and $\sigma^{(2)}$ are equivalent irreducible representations, and has dimension 0 otherwise. So Schur's lemma gives a necessary condition to see whether two irreducible representations are equivalent. On the other hand, as an application of Schur's lemma the following result holds. It gives an explicit description for irreducible representations of abelian groups.

Theorem 1.4. *Let G be an abelian group. Then any irreducible representation of G has degree 1.*

Proof. Suppose that σ is a representation of G with degree n. For a fixed element $h \in G$, let $T = \sigma_h$. Then for all $g \in G$,

$$T\sigma_g = \sigma_h\sigma_g = \sigma_{hg} = \sigma_{gh} = \sigma_g\sigma_h = \sigma_g T,$$

since G is abelian. It follows that $\sigma_h \in \mathrm{Hom}_G(\sigma, \sigma)$. Then by Schur's lemma(Lemma 1.1), $\sigma_h = \lambda_h I_n$ for some $\lambda_h \in \mathbb{C}$ depending on h. Let $v \in V_\sigma$ be a non-zero vector, then for any complex number $c \in \mathbb{C}$, $\sigma_h(cv) = \lambda_h cv \in \mathbb{C}v$. Therefore, $\mathbb{C}v$ is a G-subspace of V_σ as h is arbitrary. Since σ is irreducible, then it follows that $V_\sigma = \mathbb{C}v$ and hence, σ has degree 1. $\square$

2 Character theory and orthogonality relations

In this section we shall develop character theory and introduce orthogonality relations. By using these we shall show uniqueness of Maschke's theorem to see closely how representations decompose into irreducible representations.

2.1 Characters

Recall that every invertible linear transformation corresponds to an invertible matrix and vice visa. It follows that the group $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}_n(\mathbb{C})$ the set of invertible matrices over $\mathbb{C}$ of $n \times n$. We therefore, can regard $\mathrm{GL}(V)$ as $\mathrm{GL}_n(\mathbb{C})$, that is for a representation $\sigma : G \to \mathrm{GL}(V)$ the value σ_g of σ at $g \in G$ is determined by a matrix in the general linear group of V.

Suppose now that V is an n-dimensional complex vector space. Since $\mathrm{GL}(V)$ and $\mathrm{GL}_n(\mathbb{C})$ are isomorphic we shall not distinguish them and therefore, for each σ_g we can study its trace as follows.

Definition 2.1. *Let $\sigma : G \to \mathrm{GL}(V)$ be a representation. The **character** of σ is a mapping $\chi_\sigma : G \to \mathbb{C}$ defined by $\chi_\sigma(g) = \mathrm{Tr}(\sigma_g)$.*

In general we need to indicate the chosen basis of vector spaces. However, the trace of a matrix is the sum of all eigenvalues and eigenvalues are independent of the choice of basis. It follows that characters are independent of the choice of basis since we do not distinguish $\mathrm{GL}(V)$ and $\mathrm{GL}_n(\mathbb{C})$ for an n-dimensional vector space V.

Proposition 2.1. *Let $\sigma : G \to \mathrm{GL}(V)$ be a representation. Then $\chi_\sigma(1) = \deg(\sigma)$.*

Proof. Suppose that $\sigma : G \to \mathrm{GL}(V)$ is a representation. Let $n = \deg(\sigma)$. Then $\chi_\sigma(1) = \mathrm{Tr}(I_n) = n$, where I_n is the identity matrix of dimension n. $\square$

Example 2.1. Let φ and ψ be representations defined in Example 1.1 and Example 1.2 respectively. By the definition of characters, we see that $\chi_\varphi(m) = \chi_\psi(m) = 1$ for each $m \in \mathbb{Z}/n\mathbb{Z}$.

In the above example φ and ψ are equivalent and we obtained that their characters are equal as well. In general, we have the following result holds:

Proposition 2.2. *If σ_1 and σ_2 are equivalent representations of a finite group G, then $\chi_{\sigma_1} = \chi_{\sigma_2}$.*

Proof. Suppose that σ_1 and σ_2 are representations from G to $\mathrm{GL}_n(\mathbb{C})$. Since they are equivalent, then there is an invertible matrix T in $\mathrm{GL}_n(\mathbb{C})$ such that $\sigma_1(g) = T\sigma_2(g)T^{-1}$ for all $g \in G$. Then

$$\chi_{\sigma_1}(g) = \mathrm{Tr}(\sigma_1(g)) = \mathrm{Tr}(T\sigma_2T^{-1}) = \mathrm{Tr}(TT^{-1}\sigma_2) = \mathrm{Tr}(\sigma_2(g)) = \chi_{\sigma_2}(g).$$

$\square$

One important observation is that characters are invariant on conjugacy classes which reduces a lot of computation. On the other hand, for two representations we defined their direct sum and tensor product. So in the following we describe how characters work on conjugacy classes and what happens to characters of tensor products and direct sums.

Definition 2.2. *A function $f : G \to \mathbb{C}$ is called a* ***class function*** *if $f(g) = f(hfh^{-1})$ for all $g, h \in G$.*

Proposition 2.3. *The character of a representation is a class function.*

Proof. Let σ be a representation and χ_σ its character. For $g, h \in G$, we have

$$\chi_\sigma(hgh^{-1}) = \mathrm{Tr}(\sigma_{hgh^{-1}}) = \mathrm{Tr}(\sigma_h\sigma_g\sigma_h^{-1}) = \mathrm{Tr}(\sigma_h\sigma_h^{-1}\sigma_g) = \mathrm{Tr}(\sigma_g) = \chi_\sigma(g).$$

$\square$

Proposition 2.4. *(W. Fulton, J. Harris (2013), p.13) Let σ_1 and σ_2 be representations of G. Then we have $\chi_{\sigma_1\oplus\sigma_2} = \chi_{\sigma_1} + \chi_{\sigma_2}$, and $\chi_{\sigma_1\otimes\sigma_2} = \chi_{\sigma_1}\chi_{\sigma_2}$.*

Since we are interested in the decomposition of representations into a direct sum of irreducible representations, Proposition 2.4 gives a convenient way to calculate characters of the decomposition. Conversely, comparing characters of a representations with sums of characters of its irreducible components, we may see how the representation decomposes.

2.2 Orthogonality relations and uniqueness of Maschke's theorem

Recall that in Definition 1.11 we made group algebra an inner product space. By looking at orthogonal relations, we can prove several nice results. One such result is the uniqueness of Maschke's theorem. Therefore, in this section our main goals are to introduce orthogonal relations and offer a proof of the uniqueness of Maschke's theorem based on these relations.

Definition 2.3. *Let G be a finite group and $\sigma : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G. The* ***coordinate functions*** *$\sigma_{ij} : G \to \mathbb{C}$ are defined by $\sigma_{ij}(g) = (\sigma_g)_{ij}$, for $1 \leq i, j \leq n$. Namely, $\sigma_{ij}(g)$ is the entry in i-th row and j-th column of the matrix σ_g.*

Theorem 2.1 (**Schur orthogonality relations**). *(B. Steinberg (2011), p.31) Suppose that $\sigma^{(1)} : G \to \mathrm{GL}_n(\mathbb{C})$ and $\sigma^{(2)} : G \to \mathrm{GL}_n(\mathbb{C})$ are irreducible representations of G that are not equivalent. Then for $1 \leq i, j, k, l \leq n$, we have*

$$\text{(i). } \langle \sigma^{(1)}_{ij}, \sigma^{(2)}_{kl} \rangle = 0; \qquad \text{(ii). } \langle \sigma^{(1)}_{ij}, \sigma^{(1)}_{kl} \rangle = \begin{cases} \dfrac{1}{n} & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5. *Let $\sigma^{(1)}, \sigma^{(2)}$ be irreducible representations of G. Then*

$$\langle \chi_{\sigma^{(1)}}, \chi_{\sigma^{(2)}} \rangle = \begin{cases} 1 & \sigma^{(1)} \cong \sigma^{(2)}, \\ 0 & \sigma^{(1)} \ncong \sigma^{(2)}. \end{cases}$$

Proof. Suppose that $\sigma^{(1)}$ and $\sigma^{(2)}$ have degree n, m respectively. By the definition of the inner product on group algebra(Definition 1.11), we calculate directly. Then

$$\begin{aligned} \langle \chi_{\sigma^{(1)}}, \chi_{\sigma^{(2)}} \rangle =& \frac{1}{|G|} \sum_{g \in G} \chi_{\sigma^{(1)}}(g) \overline{\chi_{\sigma^{(2)}}(g)} = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=1}^{n} \sigma^{(1)}_{ii}(g) \sum_{j=1}^{m} \overline{\sigma^{(2)}_{jj}(g)} \right) \\ =& \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{1}{|G|} \sum_{g \in G} \sigma^{(1)}_{ii}(g) \overline{\sigma^{(2)}_{jj}(g)} \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle \sigma^{(1)}_{ii}, \sigma^{(2)}_{jj} \right\rangle. \end{aligned}$$

By Theorem 2.1, it follows that $\langle \chi_{\sigma^{(1)}}, \chi_{\sigma^{(2)}} \rangle = 0$ if $\sigma^{(1)} \ncong \sigma^{(2)}$. Otherwise, by Proposition 2.2 we can assume that $\sigma^{(1)} = \sigma^{(2)}$. Then Theorem 2.1 implies that $\langle \chi_{\sigma^{(1)}}, \chi_{\sigma^{(2)}} \rangle = \sum_{i=1}^{n} \left\langle \sigma^{(1)}_{ii}, \sigma^{(1)}_{ii} \right\rangle = 1$. ☐

Lemma 2.1. *(B. Steinberg (2011), p.38) Let G be a finite group. Then the number of irreducible representations of G is at most equal to the number of conjugacy classes of G.*

To prove Lemma 2.1, we need techniques from unitary representations. In the report we are not interested in unitary representations, readers who are interested in this may check B. Steinberg (2011).

By the above lemma, we see that for a finite group it has finitely many irreducible representations and therefore we can choose a set of representatives under the equivalence of representations. Then we may assume that different representations can be decomposed into the same irreducible representations but with different multiplicities. Hereby, one is able to show the uniqueness of the Maschke's theorem.

Theorem 2.2 (Uniqueness of Maschke's theorem). *Suppose that σ is a representation of G. Let $\sigma_1, \cdots, \sigma_s$ be a complete set of representatives of irreducible representations of G. If $\sigma \cong m_1\sigma_1 \oplus \cdots \oplus m_s\sigma_s$, then $m_i = \langle \sigma, \sigma_i \rangle$ and therefore, the decomposition is unique up to equivalence and is determined by its character.*

Proof. By Proposition 2.4, we see that $\chi_\sigma = \sum\limits_{i=1}^{s} m_i \chi_{\sigma_i}$. Since the inner product is linear, then Proposition 2.5 implies that $\langle \sigma, \sigma_i \rangle = \sum\limits_{i=1}^{s} m_i \langle \chi_\sigma, \chi_{\sigma_i} \rangle = m_i$. □

Corollary 2.1. *Let σ be a representation of a finite group G, then σ is irreducible if and only if $\langle \chi_\sigma, \chi_\sigma \rangle = 1$.*

Proof. By the uniqueness of Maschke's theorem (Theorem 2.2), we can assume that σ has the decomposition $\sigma \cong m_1\sigma_1 \oplus \cdots \oplus m_s\sigma_s$ with σ_i distinct irreducible representations. It follows that $\langle \chi_\sigma, \chi_\sigma \rangle = \sum\limits_{i=1}^{s} m_i^2$. Since m_i's are non-negative integers, therefore $\langle \chi_\sigma, \chi_\sigma \rangle = 1$ if and only if σ has only one component and with multiplicity 1. □

Combining Proposition 2.1 with the following results, we can see that the order of a group has some connections with dimensions of all irreducible representations, which is also an application of the uniqueness of Maschke's theorem.

Definition 2.4. *Let G be a finite group. The* ***regular representation*** *of G is the homomorphism $L : G \to \mathrm{GL}(\mathbb{C}[G])$ defined by*

$$L_g\left(\sum_{g' \in G} c_{g'} g'\right) = \sum_{g' \in G} c_{g'} g g' = \sum_{x \in G} c_{g^{-1}x} x, \text{ for } g \in G.$$

Clearly, the group algebra $\mathbb{C}[G]$ as a vector space has dimension $|G|$ if G is a finite group as all elements of G form a basis of the group algebra.

Theorem 2.3. *(B. Steinberg (2011), p.43) Let L be a regular representation of a finite group G and $\{\sigma_1, \cdots, \sigma_s\}$ be a complete set of representatives of irreducible representations. Then L has the decomposition $L \cong d_1\sigma_1 \oplus \cdots \oplus d_s\sigma_s$, for irreducible representations σ_i with degree d_i, $1 \le i \le s$.*

Corollary 2.2. *Let $\{\sigma_1, \cdots, \sigma_s\}$ be a complete set of representatives of irreducible representations with dimension d_i respectively. Then the formula $|G| = \sum\limits_{i=1}^{s} d_i^2$ holds.*

Proof. By Proposition 2.4 and Theorem 2.3, we have $\chi_L = \sum\limits_{i=1}^{s} d_i \chi_{\sigma_i}$. Therefore, Proposition 2.1 shows that $|G| = \chi_L(1) = \sum\limits_{i=1}^{s} d_i \chi_{\sigma_i}(1) = \sum\limits_{i=1}^{s} d_i^2$. □

To see the importance of Corollary 2.2, let σ be a representation of G. Suppose that we found several different irreducible components of σ, say $\sigma_1, \ldots, \sigma_n$. If we can show that the sum of square of dimensions of $\sigma_1, \ldots, \sigma_n$ equals to the order of G, then it follows that $\sigma_1, \ldots, \sigma_n$ are all irreducible representations of G. We shall use this technique very often.

3 Induced representations

So far we have discussed representations of a dingle group. In this section, we want to discuss an approach of constructing a representation from a representation of subgroups. This kind of representations is called an induced representations. We shall give two explanation of induced representations which are equivalent so that the reader can have a clear horizon on what induced representations are.

3.1 Direct sum and induced representation

Definition 3.1. *Given a representation $\sigma : G \to \mathrm{GL}(V)$ and a subgroup H of G, the* ***restriction*** *of σ is the map $\sigma|_H$ which is a representation of H. We usually denote by $\mathrm{Res}_H^G\sigma$ or $\mathrm{Res}_H^G V$.*

The reader may check that the restriction is indeed a representation of H. As a representation the restriction has a character which is normally denoted by $\mathrm{Res}_H^G\chi_\sigma$ where χ_σ is the character of σ. To see the restriction more clearly, consider the following example:

Example 3.1. Let $G = \mathbb{Z}/2n\mathbb{Z}$ for some positive integer n. Define the one-dimensional representation $\sigma : G \to \mathbb{C}\backslash\{0\}$ by $m \mapsto e^{2\pi i m/2n}$ for every $m \in G$; namely, σ maps m to the m-th power of a $2n$-th primitive root of unity. Suppose that $H = \langle 2\rangle$ is a subgroup of G. Then the restriction of σ on H is given by $x \mapsto e^{2\pi i x/2n}$ i.e. x is sent to the x-th power of a n-primitive root of unity.

Definition 3.2. *Let G be a group and H be a subgroup of G. Suppose that $\sigma : H \to \mathrm{GL}(V)$ is a representation of H. Then the* ***induced representation*** *$\mathrm{Ind}_H^G\sigma$, is the representation of G with representation space $\mathrm{Ind}_H^G V$ given by*

$$\mathrm{Ind}_H^G V = \{f : G \to V | f(hg) = \sigma(h)f(g), \textit{ for any } g \in G, h \in H\}$$

and the action $g'(f)(g) = f(gg')$, for all $g' \in G$.

As the restriction of a representation, we use $\mathrm{Ind}_H^G\chi_\sigma$ to denote the character of the induced representation from σ which is called the induced character.

By Theorem 1.3, we see that the vector space V can be regarded as a $\mathbb{C}[H]$-module. Since we defined an action on $\mathrm{Ind}_H^G V$, then by the definition of modules the reader may check that $\mathrm{Ind}_H^G V$ is indeed a $\mathbb{C}[G]$-module. Therefore, it is a $\mathbb{C}[H]$-module. To see how V can be embedded in $\mathrm{Ind}_H^G V$, we define, for each $v \in V$, a function $f_v : G \to V$ given by

$$f_v(g) = \begin{cases} \sigma(g)(v) & \text{if } g \in H; \\ 0 & \text{if } g \in G - H. \end{cases}$$

Clearly, f_v is an element in $\mathrm{Ind}_H^G V$. Now define a mapping $\varphi : V \to \mathrm{Ind}_H^G V$ given by $v \mapsto f_v$. We shall show that φ is actually an embedding of V into $\mathrm{Ind}_H^G V$.

Proposition 3.1. *The map $\varphi : V \to \mathrm{Ind}_H^G V$ given by $v \mapsto f_v$ is a $\mathbb{C}[H]$-embedding; namely, an injective homomorphism of $\mathbb{C}[H]$-modules.*

Proof. Let $v_1, v_2 \in V$ and $g \in G$. If $g \in H$, then

$$\varphi(v_1 + v_2)(g) = f_{v_1+v_2}(g) = \sigma(g)(v_1 + v_2) = \sigma(g)(v_1) + \sigma(g)(v_2) = \varphi(v_1) + \varphi(v_2).$$

If $g \notin H$, then $\varphi(v_1 + v_2)(g) = \varphi(v_1)(g) + \varphi(v_2)(g) = 0$. Suppose that $h \in H$ is arbitrary. Then it follows that if $g \in H$, then

$$\varphi(hv)(g) = \varphi(\sigma_h(v))(g) = f_{\sigma_h(v)}(g) = \sigma_g(\sigma_h(v)) = \sigma_{gh}(v).$$

On the other hand, we also have $h\varphi(v)(g) = h(f_v)(g) = f_v(gh) = \sigma_{gh}(v)$ by the definition of the action on the space $\mathrm{Ind}_H^G V$ (Definition 3.2). Therefore, we obtain that $\varphi(hv) = h\varphi(v)$ since everything is zero if $g \notin H$. It follows that φ is a homomorphism of $\mathbb{C}[H]$-modules.

Now consider $\mathrm{Ker}(\varphi)$. Suppose that $f_v \in \mathrm{Ker}(\varphi)$, then for every $g \in H$, $\sigma(g)(v) = 0$. In particular, we take $g = 1$ which forces $v = 0$. Hence, φ is injective. □

By the above result, we see that the image of V consists of all functions vanishing on $G - H$. Without loss of generality, we may assume that V is a subspace of $\mathrm{Ind}_H^G V$ which means we have the same action on V as the induced representation.

Theorem 3.1. *Let G be a group and H be a subgroup of G. Suppose that $\sigma : H \to \mathrm{GL}(V)$ is a representation of H. Then $\mathrm{Ind}_H^G V \cong \bigoplus_{r \in R} rV$, where R is a complete set of representatives of left cosets of H in G.*

Proof. For every $f \in \mathrm{Ind}_H^G V$ and every $r \in R$ we define a new function $f_r : G \to \mathbb{C}$ given by

$$f_r(g) = \begin{cases} f(g) & g \in Hr^{-1}; \\ 0 & \text{otherwise.} \end{cases}$$

For $h \in H, g \in G$, we have $f_r(hg) = \sigma(h)f_r(g)$ as $hg \in Hr^{-1}$ if and only if $g \in Hr^{-1}$. So f_r is an element of $\mathrm{Ind}_H^G V$. Now consider the functions $r^{-1}f_r$ for all $r \in R$. Suppose that $g \in G$ is an arbitrary element. Then we obtain that $r^{-1}f_r(g) = f_r(gr^{-1})$. Since $gr^{-1} \in Hr^{-1}$ if and only if $g \in H$, it follows that $r^{-1}f_r$ vanishes on $G - H$ and therefore, $r^{-1}f_r \in V$. Since R is a complete set of representatives of left cosets, then we can write $G = \bigcup_{r \in R} rH$. It follows that $f = \sum_{r \in R} r(r^{-1}f_r)$ and so $\mathrm{Ind}_H^G V \subseteq \bigoplus_{r \in R} rV$. Besides, the inverse inclusion is clearly true, then we get what we wished. □

Theorem 3.1 gives an explicit description of what an induced representation is. We shall see later that it is naturally a tensor product. Meanwhile, the isomorphic relation in Theorem 3.1 also gives a formula of the dimension for the induced representation.

Corollary 3.1. *Let G be a group and H be a subgroup of G. Suppose that $\sigma : H \to \mathrm{GL}(V)$ is a representation of H. Then $\dim\left(\mathrm{Ind}_H^G V\right) = [G : H]\dim(V)$.*

3.2 Tensor product and induced representation

In this part we describe induced representations from another perspective, that is tensor product. The reader who is interested in this may check page 73 of C. W. Curtis & I. Reiner (1966). Here we use the language of modules. If the reader are not familiar with these, the appendix on modules and tensor product is recommended.

Definition 3.3. *Let H be a subgroup of G, and let $\sigma : H \to \mathrm{GL}(V)$ be a representation of H. Then the $\mathbb{C}[G]$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is called the* ***induced representation*** *from σ and the action is given by*

$$g(\sum_i x_i \otimes v_i) = \sum_i gx_i \otimes v_i, \quad g \in G, x_i \in \mathbb{C}[G] \text{ and } v_i \in V,$$

where gx_i is the action when we regard $\mathbb{C}[G]$ as a left $\mathbb{C}[G]$-module.

For convenience we use V^G to denote the tensor product $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. In order to calculate V^G. We write $G = \bigcup\limits_{i=1}^{s} r_i H$ where $R = \{r_1 = 1, r_2, \cdots, r_s\}$ is a complete set of representatives of left cosets of H. So $s = [G : H]$ and every element of G is uniquely determined by H and R. Therefore, every element of $\mathbb{C}[G]$ can be uniquely expressed as $\sum\limits_{i=1}^{s} r_i h_i$, where $h_i \in \mathbb{C}[H]$, which implies that $\mathbb{C}[G] = \bigoplus\limits_{i=1}^{s} r_i \mathbb{C}[H]$. Hence, we have decomposition $V^G = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \bigoplus\limits_{i=1}^{s} r_i \mathbb{C}[H] \otimes_{\mathbb{C}[H]} V$, which can be written as

$$V^G = \bigoplus_{i=1}^{s} r_i \otimes V. \tag{$\sharp$}$$

Now compare the decomposition in Theorem 3.1 and the equation ($\sharp$). The action of r_i on V coincides with the tensor product $r_i \otimes V$, therefore, these two decomposition are indeed isomorphic as $\mathbb{C}[G]$-modules, which again clarifies the importance of Theorem 1.3.

We had two kinds representations i.e. restrictions and the induced representation. One natural question is that how these representations work together. Thanks to Frobenius, we have the following inner product relation and notice that the inner products are defined on different group algebras.

Theorem 3.2 (Frobenius reciprocity). *(B. Steinberg (2011), p.99) Suppose that H is a subgroup of G and let σ be a representation of H and ρ a representation of G, then $\langle \mathrm{Ind}_H^G \chi_\sigma, \chi_\rho \rangle = \langle \chi_\sigma, \mathrm{Res}_H^G \chi_\rho \rangle$.*

We have seen that induced representations are tensor products which are normally difficult to calculate. So we may not have a very explicit description of induced representations. Theoretically, our main interests are the reducibility and decomposition of induced representations. Frobenius reciprocity tells that an inner product on the group algebra $\mathbb{C}[G]$ can be reduced to an inner product on the group algebra $\mathbb{C}[H]$. Therefore, by Corollary 2.1 we can check the reducibility of induced representations if $\mathbb{C}[H]$ has nice properties. Besides, the uniqueness of Maschke's theorem tells that multiplicities of an irreducible component is determined by inner product. Together with Frobenius reciprocity, we can also reduce multiplicities of irreducible representations of G to multiplicities of these of H.

4 Irreducible representations of $\mathcal{U}, \mathcal{P}$ and $\mathcal{B}$

So far we have introduced basic notions and results in representation theory of finite groups. In what follows, we shall apply them to representations of $\mathrm{GL}_2(\mathbb{F}_q)$ and its subgroups for a prime power $q > 2$. Thus, we use G to denote the group $\mathrm{GL}_2(\mathbb{F}_q)$ for convenience. Readers who may interest in this are direct to I. Piatetski-Shapiro (1929).

In this section we construct irreducible representations of $\mathcal{U}, \mathcal{P}, \mathcal{B}$ which are subgroups of G. Many of the properties used are discussed in more detail in Appendix A. Thus in this section we use these properties directly without proof. The construction of irreducible representations of these three subgroups partially gives a foundation of discussions about irreducible representations of G. To begin our discussions, we define the following subgroups:

$$\mathcal{U} = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \middle| \beta \in \mathbb{F}_q \right\}, \quad \mathcal{P} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q^\times, \beta \in \mathbb{F}_q \right\},$$

$$\mathcal{Z} = \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \middle| \delta \in \mathbb{F}_q^\times \right\}, \quad \mathcal{A} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q^\times \right\}.$$

4.1 Representations of $\mathcal{U}$, $\mathcal{P}$

From Appendix A, we see that $\mathcal{U}$ is abelian and therefore, all irreducible representations of $\mathcal{U}$ have dimension one (Theorem 1.4). In Definition A.1, we defined that a character of a group is an one-dimensional representation of the group. We shall determine all characters of $\mathcal{U}$ by construction from a given non-unit character of $\mathcal{U}$ which is a character not equal to the identity map.

Let $\mathbb{F}_q^+$ be the additive group of $\mathbb{F}_q$ and $\mathbb{F}_q^\times$ be the multiplicative group of $\mathbb{F}_q$. Suppose that ψ is a non-unit character of $\mathbb{F}_q^+$. Since $\mathbb{F}_q^+ \cong \mathcal{U}$, we also consider ψ as s character of $\mathcal{U}$. For each $a \in \mathcal{A}$, we define a character ψ_a of $\mathcal{U}$ by $\psi_a(u) = \psi(aua^{-1})$, for any $u \in \mathcal{U}$.

Theorem 4.1. *All irreducible representations of $\mathcal{U}$ are given by ψ_a, for all element $a \in \mathcal{A}$.*

Proof. Let u_1, u_2 be elements of $\mathcal{U}$, then

$$\psi_a(u_1u_2) = \psi(au_1u_2a^{-1}) = \psi(au_1a^{-1}au_2a^{-1}) = \psi(au_1a^{-1})\psi(au_2a^{-1}) = \psi_a(u_1)\psi_a(u_2).$$

So ψ_a is a homomorphism from $\mathcal{U}$ to $\mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of $\mathbb{C}$. We then show that ψ_a and $\psi_{a'}$ are distinct if $a \neq a'$. Assume that $a = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, a' = \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, where $\alpha, \alpha' \in \mathbb{F}_q^\times, \beta \in \mathbb{F}_q$, then

$$\psi_a(u) = \psi(aua^{-1}) = \psi(\alpha\beta) \text{ and } \psi_{a'}(u) = \psi(aua^{-1}) = \psi(\alpha'\beta).$$

Suppose that $\psi_a = \psi_{a'}$, then $\psi(\alpha\beta) = \psi(\alpha'\beta)$ i.e. $\psi((\alpha - \alpha')\beta) = 0$ for all $\beta \in \mathbb{F}_q$. Hence, $\alpha = \alpha'$ which implies $a = a'$. It follows that all ψ_a's are distinct characters of $\mathcal{U}$. Since $|\mathcal{A}| = q - 1$, then Corollary 2.2 shows that these ψ_a's together with the unit character are all irreducible representations of $\mathcal{U}$. $\square$

Let μ be an arbitrary character of $\mathcal{A}$. By Proposition A.7, $\mathcal{P}$ is the semi-direct product of $\mathcal{U}$ by $\mathcal{A}$. Then we can define $\tilde{\mu}$ corresponding to μ as $\tilde{\mu}(ua) = \mu(a)$, for $u \in \mathcal{U}, p \in \mathcal{P}$.

Proposition 4.1. *The group $\mathcal{P}$ has $q-1$ different characters which are given by $\tilde{\mu}$ for all characters μ of $\mathcal{A}$.*

Proof. Clearly, $\mathcal{A}$ is abelian. Then by Proposition A.1 $\mathcal{A}$ has exactly $|\mathcal{A}| = q-1$ characters. For any $p_1 = u_1a_1, p_2 = u_2a_2$, we have $\tilde{\mu}(p_1p_2) = \tilde{\mu}(u_1a_1u_2a_2)$. By Proposition A.6, $\mathcal{U}$ is a normal subgroup of $\mathcal{P}$, then

$$\tilde{\mu}(p_1p_2) = \tilde{\mu}(u_1a_1u_2a_2) = \tilde{\mu}(u_1u'a_1a_2) = \mu(a_1a_2) = \mu(a_1)\mu(a_2) = \tilde{\mu}(p_1)\tilde{\mu}(p_2),$$

for some $u' \in \mathcal{U}$. Therefore, $\tilde{\mu}$ is a homomorphism. On the other hand, different characters of $\mathcal{A}$ have to induce distinct characters of $\mathcal{P}$ which completes the proof. □

In order to find irreducible representations of $\mathcal{P}$ having higher dimension, we need the following result and we still use the notations from above.

Lemma 4.1. *(I. Piatetski-Shapiro (1929), p.17) The identity* $\mathrm{Res}_{\mathcal{U}}^{\mathcal{P}}\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi \cong \bigoplus_{a\in\mathcal{A}} \psi_a$ *holds.*

Proof. For every $a \in \mathcal{A}$, we define a function $f_a \in \mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}V_\psi$ by

$$f_a(a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}, \text{ where } a' \in \mathcal{A}.$$

We claim that f_a is an eigenvector of $\mathcal{U}$ that belongs to the eigenvalue ψ_a with respect to $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$. In other words, we need to show

$$\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi(u)(f_a)(p) = \psi_a(u)(f_a)(p), \text{ for every } u \in \mathcal{U} \text{ and every } p \in \mathcal{P},$$

which is equivalent to

$$f_a(pu) = \psi_a(u)f_a(p), \text{ for every } u \in \mathcal{U} \text{ and every } p \in \mathcal{P}. \tag{$\star$}$$

Since $\mathcal{P}$ is the semi-direct product of $\mathcal{U}$ by $\mathcal{A}$, we write $p = u'a'$ with u' in $\mathcal{U}$ and a' in $\mathcal{A}$. By Definition 3.2, the identity $f_a(u'p') = \psi(u')f_a(p')$ holds for $p' \in \mathcal{P}$. Then

$$\begin{aligned} f_a(pu) &= f_a(u'a'u) = \psi(u')f_a(a'u) \\ \psi_a(u)f_a(p) &= \psi_a(u)f_a(u'a') = \psi_a(u)\psi(u')f_a(a'). \end{aligned}$$

So, in order to prove ($\star$) we only need to show

$$f_a(a'u) = \psi_a(u)f_a(a'). \tag{$\star\star$}$$

Direct calculation gives

$$\begin{aligned} f_a(a'u) =& f_a(a'ua'^{-1}a') = \psi(a'ua'^{-1})f_a(a') = \psi_{a'}(u)f_a(a') \\ =& \begin{cases} 0 & \text{if } a \neq a' \\ \psi_a(u)f_a(a') & \text{if } a = a' \end{cases}, \end{aligned}$$

which implies the identity ($\star\star$). Therefore, the equation ($\star$) is true.

Let V_ψ and V_{ψ_a} be the representation spaces of ψ and ψ_a respectively. We define the subspace V_a to be $\{\psi_a(u)f_a | u \in \mathcal{U}\}$. Clearly, V_a is isomorphic to $V_{\psi_a} = \{\psi_a(u) | u \in \mathcal{U}\}$.

Moreover, as ψ_a's are distinct and of dimension 1, then $\bigoplus_{a\in\mathcal{A}} V_{\psi_a} \cong \bigoplus_{a\in\mathcal{A}} V_a$ and $\bigoplus_{a\in\mathcal{A}} V_a$ has dimension $q-1$. Since $\mathrm{Res}_{\mathcal{U}}^{\mathcal{P}}\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}V_\psi$ is of dimension $q-1$, then $\mathrm{Res}_{\mathcal{U}}^{\mathcal{P}}\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}V_\psi \cong \bigoplus_{a\in\mathcal{A}} V_a$ and, therefore, $\mathrm{Res}_{\mathcal{U}}^{\mathcal{P}}\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}V_\psi \cong \bigoplus_{a\in\mathcal{A}} V_{\psi_a}$ which implies the lemma. □

Theorem 4.2. *The group $\mathcal{P}$ has q irreducible representations:*

(1) $q-1$ of them are given by $\tilde{\mu}$, where μ runs all characters of $\mathcal{A}$;

(2) one $(q-1)$-dimensional representations that is $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$.

Proof. By Proposition 4.1, (1) is clear. To prove (2), by Frobenius reciprocity (Theorem 3.2) and Lemma 4.1, we have

$$\begin{aligned}\langle \mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\chi_\psi, \mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\chi_\psi\rangle =& \langle \chi_\psi, \mathrm{Res}_{\mathcal{U}}^{\mathcal{P}}\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\chi_\psi\rangle \\ =& \langle \chi_\psi, \sum_{a\in\mathcal{A}}\chi_{\psi_a}\rangle = \langle \psi, \sum_{a\in\mathcal{A}}\psi_a\rangle = \langle\psi,\psi\rangle = 1,\end{aligned}$$

since ψ is an irreducible representation. It follows that $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$ is irreducible by Corollary 2.1. Moreover, by calculation we obtain

$$\sum_\mu (\dim(\tilde{\mu}))^2 + \left(\dim(\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi)\right)^2 = q-1+(q-1)^2 = |\mathcal{P}|.$$

So, all $\tilde{\mu}$'s together with $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$ are all irreducible representations of $\mathcal{P}$ by Corollary 2.2. □

Note that Lemma 4.1 implies that $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$ does not depend on ψ and so $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$ is uniquely determined. For convenience, we denote $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}}\psi$ as ϑ.

4.2 The Borel subgroup $\mathcal{B}$

In this section we use irreducible representations of $\mathcal{P}$ to construct irreducible representations of $\mathcal{B}$. Therefore, we shall use notations from the previous contents without explanation any more. Let $\mathcal{B}$ denote the Borel subgroup and $\mathcal{D}$ the quotient of $\mathcal{B}$ by $\mathcal{U}$ i.e.

$$\mathcal{B} = \left\{ \begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \middle| \alpha,\delta\in\mathbb{F}_q^\times; \beta\in\mathbb{F}_q \right\}, \quad \mathcal{D} = \left\{ \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix} \middle| \alpha,\delta\in\mathbb{F}_q^\times \right\}.$$

Theorem 4.3. *There is a one-to-one correspondence between all characters of $\mathcal{B}$ and pairs (μ_1,μ_2) of characters of $\mathbb{F}_q^\times$. More precisely, each character μ of $\mathcal{B}$ is given by*

$$\mu(b) = \mu_1(\alpha)\mu_2(\delta), \ \textit{for } b\in\mathcal{B},$$

where $b = \begin{pmatrix}\alpha & \cdot\\ 0 & \delta\end{pmatrix}$, $\alpha,\delta\in\mathbb{F}_q^\times$ and $\cdot$ represents an element in $\mathbb{F}_q$.

Proof. Let μ_1,μ_2 be characters of $\mathbb{F}_q^\times$, then we define μ as $\mu(b)=\mu_1(\alpha)\mu_2(\delta)$, for $b\in\mathcal{B}$, where $b = \begin{pmatrix}\alpha & \cdot\\ 0 & \delta\end{pmatrix}$, $\alpha,\delta\in\mathbb{F}_q^\times$. It is not hard to show that μ is a homomorphism from $\mathcal{B}$ to $\mathbb{C}^\times$, hence, a character of $\mathcal{B}$.

Conversely, for every character μ of $\mathcal{B}$, we have

$$\mu\left(\begin{pmatrix}\alpha & \cdot \\ 0 & \delta\end{pmatrix}\right) = \mu\left(\begin{pmatrix}\alpha & \cdot \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \cdot \\ 0 & \delta\end{pmatrix}\right) = \mu\left(\begin{pmatrix}\alpha & \cdot \\ 0 & 1\end{pmatrix}\right)\mu\left(\begin{pmatrix}1 & \cdot \\ 0 & \delta\end{pmatrix}\right).$$

Define μ_1, μ_2 functions from $\mathbb{F}_q^\times$ to $\mathbb{C}^\times$ by

$$\mu_1(\alpha) = \mu\left(\begin{pmatrix}\alpha & \cdot \\ 0 & 1\end{pmatrix}\right), \mu_2(\delta) = \mu\left(\begin{pmatrix}1 & \cdot \\ 0 & \delta\end{pmatrix}\right).$$

For any $\alpha_1, \alpha_2 \in \mathbb{F}_q^\times$, the definition of μ_1 gives

$$\begin{aligned}\mu_1(\alpha_1\alpha_2) =& \mu\left(\begin{pmatrix}\alpha_1\alpha_2 & \cdot \\ 0 & 1\end{pmatrix}\right) = \mu\left(\begin{pmatrix}\alpha_1 & \cdot \\ 0 & 1\end{pmatrix}\right) = \mu\left(\begin{pmatrix}\alpha_2 & \cdot \\ 0 & 1\end{pmatrix}\right) \\ =& \mu_1(\alpha_1)\mu_1(\alpha_2).\end{aligned}$$

It follows that μ_1 is a character of $\mathbb{F}_q^\times$. Similarly, μ_2 is a character of $\mathbb{F}_q^\times$ as well. □

Since $\mathcal{Z}$ is abelian and $|\mathcal{Z}| = q-1$, then $\mathcal{Z}$ has $q-1$ characters by Proposition A.1. Suppose that υ is an arbitrary character of $\mathcal{Z}$. By Proposition A.5, $\mathcal{B}$ is the semi-direct product of $\mathcal{Z}$ by $\mathcal{P}$. Therefore, we can extend υ to a character $\tilde{\upsilon}$ of $\mathcal{B}$ by $\tilde{\upsilon}(zp) = \upsilon(z),$ for $z \in \mathcal{Z}, p \in \mathcal{P}$. The same techniques in the proof of Proposition 4.1 can be used to show that $\tilde{\upsilon}$ is a character of $\mathcal{B}$.

Let φ be the canonical homomorphism from $\mathcal{B}$ to $\mathcal{P}$ with kernel $\mathcal{Z}$, then the map $\tilde{\vartheta} := \vartheta \circ \varphi$ is an irreducible representation of $\mathcal{B}$ of dimension $q-1$. Because there is a proper subspace of V_ϑ if $\tilde{\vartheta}$ is not irreducible. We shall begin to construct irreducible representations of $\mathcal{B}$.

Lemma 4.2. *The tensor product $(\tilde{\upsilon}\otimes\tilde{\vartheta})(zp) = \upsilon(z)\vartheta(p),$ for $z \in \mathcal{Z}, p \in \mathcal{P}$, is an irreducible representation of $\mathcal{B}$ with dimension $q-1$.*

Proof. Obviously, $\dim(\tilde{\upsilon}\otimes\tilde{\vartheta}) = \dim(\tilde{\upsilon})\dim(\tilde{\vartheta}) = q-1$. Then we shall show that $\tilde{\upsilon}\otimes\tilde{\vartheta}$ is irreducible. Assume that $\tilde{\upsilon}\otimes\tilde{\vartheta}$ is not irreducible, then there exists a proper subspace W of $\mathrm{Ind}_{\mathcal{U}}^{\mathcal{P}} V_\psi$ such that $(\tilde{\upsilon}\otimes\tilde{\vartheta})(zp) = \upsilon(z)\vartheta(p),$ for $z \in \mathcal{Z}, p \in \mathcal{P}$, with $\tilde{\vartheta} \in \mathrm{Hom}_{\mathbb{C}}(W, W)$. Now, we take z to be 1. Then by the construction of $\tilde{\vartheta}$, it follows that W is a proper subrepresentation of ϑ which is contradictory with the irreducibility of ϑ. Hence, $\tilde{\upsilon}\otimes\tilde{\vartheta}$ is irreducible. □

Maintaining the same notations as Theorem 4.3 and Lemma 4.2 gives the result:

Theorem 4.4. *The group $\mathcal{B}$ has $(q-1)q$ irreducible representations:*

(1) $(q-1)^2$ of them are characters given by pairs (μ_1, μ_2) of characters of $\mathbb{F}_q^\times$;

(2) $q-1$ of them have dimension $q-1$ and they are given by $\tilde{\upsilon}\otimes\tilde{\vartheta}$, where $\tilde{\upsilon}$ runs all characters of $\mathcal{Z}$.

Proof. By the identity $(\tilde{\upsilon} \otimes \tilde{\vartheta})(zp) = \upsilon(z)\vartheta(p)$, then $\tilde{\upsilon} \otimes \tilde{\vartheta}$ are distinct when υ runs all characters of $\mathcal{Z}$. Moreover, by calculation the following equation holds:

$$\sum_{(\mu_1,\mu_2)} (\dim(\mu))^2 + \sum_{\upsilon} \left(\dim(\tilde{\upsilon} \otimes \tilde{\vartheta})\right)^2 = (q-1)^2 q = |\mathcal{B}|,$$

where μ is determined by the pair (μ_1, μ_2). It follows that μ's together with $\tilde{\upsilon} \otimes \tilde{\vartheta}$ are all irreducible representations of $\mathcal{B}$. □

5 Irreducible components of principal series representations

Let μ be a character of $\mathcal{B}$ defined in Section 4.2. So correspondingly we have a pair (μ_1, μ_2) of characters of $\mathbb{F}_q^\times$. We shall study the decomposition of a representation induced by μ which gives some information on irreducible representations of $G = \mathrm{GL}_2(\mathbb{F}_q)$. Precisely, we want to give a special representation and show that it can only decompose into two irreducible representations. Amazingly, the representation is irreducible if it does not contain an one-dimensional subrepresentation. So the left case is irreducible only. The reader may see I. Piatetski-Shapiro (1929) for a thorough explanation.

From Theorem 4.3, we know that $\mu\left(\begin{pmatrix} \alpha & \cdot \\ 0 & \delta \end{pmatrix}\right) = \mu_1(\alpha)\mu_2(\delta), \alpha, \delta \in \mathbb{F}_q^\times$. Define a new character μ_w of $\mathcal{B}$ by $\mu_w = \left(\begin{pmatrix} \alpha & \cdot \\ 0 & \delta \end{pmatrix}\right) = \mu_1(\delta)\mu_2(\alpha)$, where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The reason why we use w as the index is clear from the following result:

Proposition 5.1. *Let μ be a character of $\mathcal{B}$ related to the pair (μ_1, μ_2). Then $\mu_w = \mu(wdw)$ for every $d \in \mathcal{D}$ and $\mu_w = \mu$ if and only if $\mu_1 = \mu_2$.*

Proof. Suppose that $d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ is an element in $\mathcal{D}$. Then $wdw = \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}$, which implies that $\mu_w(d) = \mu_1(\delta)\mu_2(\alpha) = \mu(wdw)$.

Since characters of $\mathcal{B}$ are trivial on $\mathcal{U}$, then in order to show $\mu_w = \mu$, we only need to prove the equality on $\mathcal{D}$. Assume that $\mu_1 = \mu_2$, then $\mu_w(d) = \mu(wdw) = \mu_1(\alpha\delta) = \mu(d)$. Conversely, we have $\mu_w(d) = \mu_1(\delta)\mu_2(\alpha)$ and $\mu(d) = \mu_1(\alpha)\mu_2(\delta)$. Take δ to be 1, then $\mu_w = \mu$ implies that $\mu_1(\alpha) = \mu_2(\alpha)$ for every $\alpha \in \mathbb{F}_q^\times$. □

Definition 5.1. *Let μ be a character of $\mathcal{B}$. The induced representation $\mathrm{Ind}_{\mathcal{B}}^{G}\mu$ is called a* ***principal series representation*** *of G, denote by $\tilde{\mu}$.*

Definition 5.2. *Let σ be a representation of G. The* ***Jacquet Module*** *of the representation σ is defined as $J(V_\sigma) := \{v \in V_\sigma | \sigma(u)(v) = v, \text{ for every } u \in \mathcal{U}\}$.*

Proposition 5.2. *Let σ be a representation of G. If σ_1, σ_2 are two subrepresentations of σ, then $J(V_{\sigma_1} \oplus V_{\sigma_2}) = J(V_{\sigma_1}) \oplus J(V_{\sigma_2})$.*

Proof. Suppose that $v_1 \in V_{\sigma_1}, v_2 \in V_{\sigma_2}$. If $v_1 + v_2 \in J(V_{\sigma_1} \oplus V_{\sigma_2})$, then

$$\sigma(u)(v_1 + v_2) = \sigma(u)(v_1) + \sigma(u)(v_2) = v_1 + v_2,$$

for every $u \in \mathcal{U}$. Since σ_1, σ_2 are subrepresetnations of σ, then $\sigma(u)(v_1) \in V_{\sigma_1}$ and $\sigma(u)(v_2) \in V_{\sigma_2}$, which implies $J(V_{\sigma_1} \oplus V_{\sigma_2}) \subseteq J(V_{\sigma_1}) \oplus J(V_{\sigma_2})$. Conversely, it is clear that $J(V_{\sigma_1} \oplus V_{\sigma_2}) \supseteq J(V_{\sigma_1}) \oplus J(V_{\sigma_2})$. Hence, the equality holds. □

Since we pay much attention to the decomposition of representations into direct sums, Proposition 5.2 implies that we can focus on the Jacquet module of components. Furthermore, Jacquet module is not an arbitrary subset but indeed a representation.

Proposition 5.3. *Let σ be a representation of G. Then $J(V_\sigma)$ is a representation of $\mathcal{B}$.*

Proof. Let $v \in J(V_\sigma), b \in \mathcal{B}$ and $u \in \mathcal{U}$. Since $\mathcal{U}$ is a normal subgroup of $\mathcal{B}$, then $b^{-1}ub = u'$, for some $u' \in \mathcal{U}$. Hence

$$\sigma(u)\left(\sigma(b)(v)\right) = \sigma(ub)(v) = \sigma(b)\left(\sigma(b^{-1}ub)(v)\right) = \sigma(b)\left(\sigma(u')(v)\right) = \sigma(b)(v).$$

□

From now on we can study principal series representations by using Jacquet module. So, firstly, we describe the dimension of Jacquet module and show this by finding a basis:

Lemma 5.1. *If μ is a character of $\mathcal{B}$, then the Jacquet module $J(V_{\tilde{\mu}})$ has dimension 2.*

Proof. By Definition 3.2 and Definition 5.2, $J(V_{\tilde{\mu}})$ consists of all functions $f : G \to \mathbb{C}$ such that $f(bg) = \mu(b)f(g)$ and $f(gu) = f(g)$, for all $b \in \mathcal{B}, g \in G$ and $u \in \mathcal{U}$. Particularly, we have $f(b) = \mu(b)f(a)$ and $f(bwu) = \mu(b)f(wu) = \mu(b)f(w)$, where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the Bruhat decomposition (Theorem A.1) implies that f is uniquely determined by its values at 1 and w.

Consider two functions f_1, f_2 satisfying $f_1(1) = 1, f_1(w) = 0$ and $f_2(1) = 0, f_2(w) = 1$. Assume that $a_1 f_1 + a_2 f_2 = 0$ for some complex number a_1 and a_2. Then $a_1 f_1(1) + a_2 f_2(1) = 0$ and $a_1 f_1(w) + a_2 f_2(w) = 0$, which implies that $a_1 = a_2 = 0$. In other words, f_1 and f_2 are linear independent.

Let f be an arbitrary function in $J(V_{\tilde{\mu}})$. By the above discussion, f is uniquely determined by $f(1)$ and $f(w)$, namely, $f(b) = \mu(b)f(1)$ and $f(bwu) = \mu(b)f(w)$. Therefore, we have $f = f(1)f_1 + f(w)f_2$ which means that f can be written as a linear combination of f_1 adn f_2. Hence, the dimension of $J(V_{\tilde{\mu}})$ is 2 as f_1 and f_2 are actually basis. □

The connection between Jacquet modules and principal series representations reflects on inner product and whether the Jacquet module is trivial.

Lemma 5.2. *Let σ be a representation of G. Then $J(V_\sigma) \neq 0$ if and only if there exist a character μ of $\mathcal{B}$ such that $\langle \chi_\sigma, \chi_{\tilde{\mu}} \rangle \neq 0$.*

Proof. Suppose that $J(V_\sigma) \neq 0$. Then $J(V_\sigma) \neq 0$ can be regarded as a representation of $\mathcal{B}/\mathcal{U}$ since it is a representation of $\mathcal{B}$ (Proposition 5.3). But $\mathcal{B}/\mathcal{U}$ is abelian. Then by Theorem 1.4, $J(V_\sigma)$ can be decomposed as a direct sum of 1-dimensional $\mathcal{B}/\mathcal{U}$-subspaces. Since every character of $\mathcal{B}$ acts trivially on $\mathcal{U}$, it follows that there exists a character μ of $\mathcal{B}$ such that

μ is a subrepresentation of $\mathrm{Res}_{\mathcal{B}}^{G}\sigma$. Therefore, we have $\langle \mu, \mathrm{Res}_{\mathcal{B}}^{G}\chi_\sigma \rangle \neq 0$. By the Frobenius reciprocity (Theorem 3.2), $\langle \chi_{\tilde{\mu}}, \chi_\sigma \rangle$ is non-zero.

Conversely, suppose that $\langle \chi_\sigma, \chi_{\tilde{\mu}} \rangle \neq 0$. Then again the Frobenius reciprocity (Theorem 3.2) gives $\langle \mathrm{Res}_{\mathcal{B}}^{G}\chi_\sigma, \mu \rangle \neq 0$. Thus, there exists a non-zero element $v \in V_\sigma$ such that $\sigma(v)(v) = \mu(b)v$, for $b \in \mathcal{B}$, which implies that $v \in J(V_\sigma)$. Hence, $J(V_\sigma) \neq 0$. □

Lemma 5.3. *(I. Piatetski-Shapiro (1929), p.25) Let μ and μ' be two distinct characters of $\mathcal{B}$. Then $\langle \chi_{\tilde{\mu}}, \chi_{\tilde{\mu'}} \rangle \neq 0$ if and only if $\mu' = \mu_w$.*

Lemma 5.4. *If μ is a character of $\mathcal{B}$, then $\tilde{\mu}$ has at most two irreducible components.*

Proof. Assume that $\tilde{\mu} = \sigma_1 \oplus \cdots \oplus \sigma_n$ is a decomposition of $\tilde{\mu}$ into irreducible representations for some positive integer n. Then Proposition 5.2 shows that $J(V_{\tilde{\mu}}) = J(V_{\sigma_1}) \oplus \cdots \oplus J(V_{\sigma_n})$. By Lemma 5.2, we obtain that $J(V_{\sigma_i}) \neq 0$, for $i = 1, 2, \cdots, n$. Then $\dim(J(V_{\sigma_1}) \oplus \cdots \oplus J(V_{\sigma_n})) \geq n$. However, Lemma 5.1 gives that $\dim(J(V_{\tilde{\mu}})) = 2$. Hence, $n \leq 2$. □

Lemma 5.5. *If μ is a character of $\mathcal{B}$ and $\mu = \mu_w$, then $\tilde{\mu}$ has a 1-dimensional component.*

Proof. By Proposition 5.1, we obtain $\mu_1 = \mu_2$ if μ corresponds to the pair (μ_1, μ_2). Thus if $b = \begin{pmatrix} \alpha & \cdot \\ 0 & \delta \end{pmatrix} \in \mathcal{B}$, then $\mu(b) = \mu_1(\alpha)\mu_2(\delta) = \mu_1(\det(b))$. Now we define a function $f : G \to \mathbb{C}$ given by $f(g) = \mu_1(\det(g))$. Then clearly, $f(bg) = \mu_1(\det(bg)) = \mu(b)f(g)$, which implies that $f \in V_{\tilde{\mu}}$. On the other hand, $(\tilde{\mu}(g)(f))(g) = f(gh) = \mu_1(\det(h))f(g)$, for every $h, g \in G$. It follows that the character $\mu_1 \circ \det$ of G is an eigenvalue of G with respect to $\tilde{\mu}$. Hence, $V_{\tilde{\mu}}$ has a 1-dimensional G-subspace. □

Lemma 5.6. *If μ is a character of $\mathcal{B}$, then $\tilde{\mu}$ has at most one 1-dimensional component.*

Proof. Assume that $\tilde{\mu}$ has two 1-dimensional components, say σ_1 and σ_2. By Lemma 5.4, we deduce that $\tilde{\mu} = \sigma_1 \oplus \sigma_2$. It follows that $\dim(\tilde{\mu}) = q + 1 = 2$, which is impossible. □

Lemma 5.7. *(I. Piatetski-Shapiro (1929), p.25) If μ is character of $\mathcal{B}$ and $\tilde{\mu}$ is completely reducible, then $\tilde{\mu}$ has a 1-dimensional component. Moreover, we have $\mu = \mu_w$.*

Remark that Lemma 5.7 also holds if we replace the condition "completely reducible" by "decomposable". In order to indicate that we are working on representation of finite groups, we use "completely reducible".

Lemma 5.8. *Let μ and μ' be two distinct characters of $\mathcal{B}$. Then $\tilde{\mu} = \tilde{\mu'}$ if and only if $\mu' = \mu_w$.*

Proof. Assume that $\mu' = \mu_w$. Since $\mu \neq \mu'$, then $\mu \neq \mu_w$. Note that $(\mu_w)_w = \mu$, then $\mu \neq \mu'$ also implies that $\mu' \neq \mu'_w$. By Lemma 5.7, $\tilde{\mu}$ and $\tilde{\mu'}$ are not completely reducible. Hence, $\tilde{\mu}$ and $\tilde{\mu'}$ are both irreducible. Then $\tilde{\mu} = \tilde{\mu'}$ as Lemma 5.3 says that $\langle \chi_{\tilde{\mu}}, \chi_{\tilde{\mu'}} \rangle \neq 0$. □

Finally, by the above lemmas we are able to classify all irreducible components of principal series representations.

Theorem 5.1. *The irreducible representations of G, that are components of the principal representation $\tilde{\mu} = \mathrm{Ind}_{\mathcal{B}}^{G}\mu$ where μ is a character of $\mathcal{B}$, can be divided into three classes:*

(i) $q-1$ of them are 1-dimensional; *(ii) $q-1$ of them are q-dimensional;*

(iii) $\frac{1}{2}(q-1)(q-2)$ of them ae $(q+1)$-dimensional.

Proof. Case 1: $\mu = \mu_w$. By Proposition 5.1, we have that $\mu_1 = \mu_2$. From Lemma 5.4, Lemma 5.6 and Lemma 5.7, we see that $\tilde{\mu}$ is a direct sum of a 1-dimensional representation and an irreducible q-dimensional representation. Respectively, we denote them as $\sigma'(\mu_1, \mu_1)$ and $\sigma(\mu_1, \mu_1)$. Since μ_1 is arbitrary and $\mathbb{F}_q^{\times}$ is abelian of order $q-1$, then there are $q-1$ $\sigma'(\mu_1, \mu_1)$'s and $q-1$ $\sigma(\mu_1, \mu_1)$'s.

Case 2: $\mu \neq \mu_w$. Again Proposition 5.1 gives that $\mu_1 \neq \mu_2$. By Lemma 5.5 and Lemma 5.7, $\tilde{\mu}$ is an irreducible representation of dimension $q+1$ and we denote it by $\sigma(\mu_1, \mu_2)$. Counting μ_1 and μ_2 implies that the number of $\sigma(\mu_1, \mu_2)$'s is $\frac{1}{2}(q-1)(q-2)$.

Hence, $\sigma'(\mu_1, \mu_1)$'s, $\sigma(\mu_1, \mu_1)$'s and $\sigma(\mu_1, \mu_2)$'s are these three type of irreducible representations. □

6 Conclusion

In this report we introduced fundamental definitions and results in representation theory of finite groups including Maschke's theorem and its uniqueness, Frobenius reciprocity, induced representations and so on. By these foundations we constructed the structure of group representations.

In order to apply representation theory of finite groups, we move our attention to representations of subgroups of 2-dimensional general linear group over finite fields. Then by defining Jacquet module and applying results from the study of representations of subgroups, we classified all irreducible components of principal series representations of the general linear group.

However, the irreducible components we got are not all irreducible representations of the general linear group. The other irreducible representations are usually called complementary representations or cuspidal representations. To find all other irreducible representations, we need much more results of representation theory which are not covered in this report. The reader who is crazy about this is direct to I. Piatetski-Shapiro (1929).

7 Bibliography

B. Steinberg (2011), *Representation Theory of Finite Groups: An Introductory Approach*, Springer Science & Business Media

C. W. Curtis & I. Reiner (1966), *Representation Theory of Finite Groups and Associative Algebras*, American Mathematical Soc.

I. Piatetski-Shapiro (1929), *Complex Representations of GL(2,K) for Finite Fields K*, American Mathematical Soc.; v. 16

J. J. Rotman (2010), *Advanced Modern Algebra*, American Mathematical Soc.; v. 114

W. Fulton & J. Harris (2013), *Representation Theory: A First Course*, Springer Science & Business Media

A Subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$

In this section we introduce several subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$ and properties of them.

Definition A.1. *Let G be a group. A* ***character*** *of G is an one-dimensional representation of G.*

Note that we have characters of representations as well. However, in these two cases we call both characters. So the reader should notice whether the characters is with respect to a group or a representation. Moreover, one can give a precise result on the number of characters of a group which states as follows.

Proposition A.1. *(B. Steinberg (2011), p.77) Let G be a finite group. If G' is the commutator subgroup of G, then there is a bijection between characters of G and irreducible representations of the abelian group G/G'. Hence G has $|G/G'| = [G : G']$ characters.*

Let $\mathbb{F}_q^\times$ denote the multiplicative group of $\mathbb{F}_q$ and suppose that $q > 2$. The **Borel subgroup** $\mathcal{B}$ of $\mathrm{GL}_2(\mathbb{F}_q)$ consists of all upper triangular matrices , namely,

$$\mathcal{B} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \middle| \alpha, \delta \in \mathbb{F}_q^\times ; \beta \in \mathbb{F}_q \right\}.$$

Clearly, we have $|\mathcal{B}| = (q-1)^2 q$ by counting possible values of α, β and δ.

Proposition A.2. *The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, together with the matrices $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \gamma \in \mathbb{F}_q$ form a system of representatives of the left cosets of $\mathrm{GL}_2(\mathbb{F}_q)$ modulo $\mathcal{B}$.*

Proof. Let g be an element of $\mathrm{GL}_2(\mathbb{F}_q)$. If g has the form $g = \begin{pmatrix} 0 & \delta \\ \alpha & \beta \end{pmatrix}, \alpha, \delta \in \mathbb{F}_q^\times$ and $\beta \in \mathbb{F}_q$, then $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$. Otherwise, g has the form $\begin{pmatrix} \alpha & \beta \\ \theta & \delta \end{pmatrix}$, where $\alpha \in \mathbb{F}_q^\times$ and $\det(g) \neq 0$. Then $g = \begin{pmatrix} 1 & 0 \\ \theta\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta - \theta\alpha^{-1}\beta \end{pmatrix}$. Since $\det(g) = \alpha\delta - \beta\theta \neq 0$, then the determinant of the matrix $h := \begin{pmatrix} \alpha & \beta \\ 0 & \delta - \theta\alpha^{-1}\beta \end{pmatrix}$ is $\alpha\delta - \beta\theta \neq 0$. It follows that h is an element of $\mathcal{B}$.

Let $h_1 = \begin{pmatrix} 1 & 0 \\ \gamma_1 & 1 \end{pmatrix}, h_2 = \begin{pmatrix} 1 & 0 \\ \gamma_2 & 1 \end{pmatrix}, b_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \delta_1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \delta_2 \end{pmatrix}$. Assume that $h_1 b_1 = h_2 b_2$, then by comparing each entry we deduce that $h_1 = h_2$ which completes the proof. □

By the Proposition A.2, we see that $[\mathrm{GL}_2(\mathbb{F}_q) : \mathcal{B}] = q + 1$ and therefore we obtain the following result.

Corollary A.1. *The order of $\mathrm{GL}_2(\mathbb{F}_q)$ is $(q-1)^2 q(q+1)$.*

Let $\mathbb{F}_q^+$ be the additive group of $\mathbb{F}_q$. Consider the subgroup of all unipotent upper triangular matrices

$$\mathcal{U} = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \middle| \beta \in \mathbb{F}_q \right\},$$

which is a normal abelian subgroup of $\mathcal{B}$. Indeed, the subgroup $\mathcal{U}$ is isomorphic to $\mathbb{F}_q^+$ by mapping the matrix $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ to β. Therefore, we may sometimes do not distinguish $\mathcal{U}$ and $\mathbb{F}_q^+$.

As $\mathcal{U}$ is a normal subgroup of $\mathcal{B}$, then the quotient $\mathcal{B}/\mathcal{U}$ does have meaning and the quotient is isomorphic to the group

$$\mathcal{D} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| \alpha, \delta \in \mathbb{F}_q^\times \right\},$$

which is also called the **Cartan group**. It is not hard to see that the group $\mathcal{D}$ is isomorphic to $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ and hence $\mathcal{D}$ is abelian and $|\mathcal{D}| = (q-1)^2$. With the above notations we have the following results by calculation.

Proposition A.3. *The group $\mathcal{B}$ is the semi-direct product of $\mathcal{U}$ through $\mathcal{D}$.*

Proposition A.4. *$\mathcal{U}$ is the derived(commutator) subgroup of $\mathcal{B}$.*

Notice that we need the condition $q > 2$ to prove Proposition A.4. Otherwise, the derived subgroup of $\mathcal{B}$ would be trivial which is not what we wish.

We define the following groups

$$\mathcal{P} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q^\times, \beta \in \mathbb{F}_q \right\},$$

$$\mathcal{Z} = \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \middle| \delta \in \mathbb{F}_q^\times \right\},$$

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \middle| \alpha \in \mathbb{F}_q^\times \right\}.$$

We can check that $\mathcal{P}$, $\mathcal{Z}$ are normal subgroups of $\mathcal{B}$ and $\mathcal{P}$ has order $(q-1)q$ and thus has index $q-1$ in $\mathcal{B}$.

Via doing the same calculation as Proposition A.3 and Proposition A.4, we see the consequences as follows.

Proposition A.5. *The group $\mathcal{B}$ is the semi-product of $\mathcal{Z}$ and $\mathcal{P}$.*

Proposition A.6. *The group $\mathcal{U}$ is also the derived subgroup of $\mathcal{P}$.*

Proposition A.7. *The group $\mathcal{P}$ is the semi-direct product of $\mathcal{U}$ through $\mathcal{A}$.*

To see the relations between $\mathrm{GL}_2(\mathbb{F}_q)$ and subgroups $\mathcal{B}$, $\mathcal{U}$, we use the **Bruhat's Decomposition** which states as follows.

Theorem A.1. ***(Bruhant's Decompostion)*** *An element of* $\mathrm{GL}_2(\mathbb{F}_q)$ *is either in* $\mathcal{B}$ *or in* $\mathcal{B}w\mathcal{U}$*, where* $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*. Namely,* $\mathrm{GL}_2(\mathbb{F}_q) = \mathcal{B} \cup \mathcal{B}w\mathcal{U}$*.*

Proof. We firstly show that $\mathcal{B}$ and $\mathcal{B}w\mathcal{U}$ are disjoint. Let $b = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \alpha, \delta \in \mathbb{F}_q^\times, \beta \in \mathbb{F}_q$ be an element of $\mathcal{B}$ and $u = \begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix}, \beta' \in \mathbb{F}_q$ an element of $\mathcal{U}$. Then

$$bwu = \begin{pmatrix} \beta & \beta\beta' + \alpha \\ \delta & \delta\beta' \end{pmatrix}$$

is apparently not an element in $\mathcal{B}$ as $\delta \neq 0$ which implies that $\mathcal{B}$ and $\mathcal{B}w\mathcal{U}$ are disjoint.

Suppose that $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in $\mathrm{GL}_2(\mathbb{F}_q)$, namely, $\alpha\delta - \beta\gamma \neq 0$. If $\gamma = 0$, then obviously g lies in $\mathcal{B}$. Otherwise, when $\gamma \neq 0$, we have the decomposition

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \beta - \alpha\gamma^{-1}\delta & \alpha \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix} := bwu.$$

Since $\alpha\delta - \beta\gamma \neq 0$, then $\det(b) = \beta\gamma - \alpha\delta$. It follows that $\beta - \alpha\gamma^{-1}\delta$ is in $\mathbb{F}_q^\times$, i.e. $b \in \mathcal{B}$. Note that u is clearly in $\mathcal{U}$, then the theorem holds. □

B Modules and tensor product

Definition B.1. *Let* R *be a ring. An abelian group* M*, written additively, is called a* ***left*** R***-module*** *if, for each* $r \in R$ *and* $m \in M$*, a product* $rm \in M$ *is defined such that*

$$r(m_1 + m_2) = rm_1 + rm_2, \quad (r_1 + r_2)m = r_1 m + r_2 m,$$
$$(r_1 r_2)m = r_1(r_2 m), \quad 1m = m,$$

for all r *in* R*,* m *in* M*.*

Definition B.2. *Let* R *be a ring and let* M, M' *be* R*-modules. A mapping* $f : M \to M'$ *is a* R***-homomorphism*** *if it satisfies*

$$f(m_1 + m_2) = f(m_1) + f(m_2) \text{ and } f(rm) = rf(m),$$

for any $m, m_1, m_2 \in M$ *and* $r \in R$*. In particular, if* f *is a bijection then* f *is a* R***-isomorphism*** *and we denote* $M \cong M'$*.*

Definition B.3. *The* ***tensor product*** $V \otimes_K W$ *of vector spaces* V *and* W *over a field* K *is the quotient of the space* $V \times W$ *by the relations*

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$
$$av \otimes w = a(v \otimes w),$$
$$v \otimes aw = a(v \otimes w),$$

where $v, v_1, v_2 \in V$ *and* $w, w_1, w_2 \in W$*.*

The tensor product is indeed a vector space over the field K. Moreover, if both V and W are finite dimensional, we have an explicit formula to calculate the dimension of the tensor product.

Proposition B.1. *Let V, W be finite dimensional vector spaces over a field K, then* $\dim(V \otimes_K W) = \dim(V) \cdot \dim(W)$.