1. K_0, K_1 & K_2

1. K_0 of Rings.

<u>Def</u>. M comm. monoid. $(\overset{M}{\mathbb{N}} \longrightarrow \overset{M^+}{\mathbb{Z}})$

M^+ is a group:

M^+ is the quotient of free ab. group generated by $[m]$, $m \in M$.

modulo relation $[m]+[n]-[m+n]$.

(UP): $\forall$ homo. $M \longrightarrow A$ $A \in \mathrm{Group}$. $\exists!$ group homo. $M^+ \to A$ s.t.
$$\begin{tikzcd} M \arrow[r] \arrow[dr] & A \\ & M^+ \arrow[u] \end{tikzcd}$$

<u>Def</u>. K_0 of ring. R ring with unit.

$P(R)$ = abelian monoid of f.g. proj module over R (with $\oplus$).

$K_0(R) = P(R)^+$

<u>Remark</u>. K_0 is a functor $R \to S \Rightarrow K_0(R) \to K_0(S)$.
$$P \longrightarrow S \otimes_R P$$

<u>Examples</u>. ① R = field, $K_0(R) = \mathbb{Z}$

② R = local ring ↗

③ O_K, K number field. (Milnor)
$$K_0(O_K) = \mathbb{Z} \oplus Cl(K)$$

Application (Wall finiteness obstruction)

X path-connection, CW-complex. & $\pi_1(X) := \pi$

Suppose X is a retract of finite CW complex.

Then $\exists\ x \in K_0(\mathbb{Z}[\pi])$ s.t. $x=0$ iff $X \simeq$ finite CW complex.

1.2 Topological K_0

<u>Def</u>. Vector bundle X top.

$KU^0_{top}(X) = \mathrm{Vect}_{\mathbb{C}}(X)^+$, $KO^0_{top}(X) = \mathrm{Vect}_{\mathbb{R}}(X)^+$

$\ \ \ \ ||$

$K^0(X)$ $\quad U = \operatorname{colim}_n U_n$ $\quad O = \operatorname{colim}_n O_n$

Classification of Vector bundles

$\mathbb{C}$: $\quad [X, BU_n] \longleftrightarrow \mathrm{Vect}_n(X)$, with $BU_n \simeq \mathrm{Gr}_n(\mathbb{C}^\infty)$

$\mathbb{R}$: $\quad [X, BO_n] \longleftrightarrow \mathrm{Vect}_n(X)$, with $BO_n \simeq \mathrm{Gr}_n(\mathbb{R}^\infty)$

<u>Theorem (Swan)</u> Let $\mathbb{F} = \mathbb{R}$ (or $\mathbb{C}$). Let X compact, Hausdorff.
Defien $C(X, \mathbb{F}) = \{\text{ring of conti. function}\}$
$\forall E \in \mathrm{Vect}_{\mathbb{F}}(X)$. $\Gamma(X, E) = \{s : X \to E \text{ conti. } p \circ s = id_X\}$. $C(X, \mathbb{R})$-module.

$$KO^0_{top}(X) \to K_0(C(X, \mathbb{R})) \qquad KU^0_{top}(X) \to K_0(C(X, \mathbb{C}))$$
$$E \longmapsto \Gamma(X, E) \qquad E \longmapsto \Gamma(X, E)$$

are iso.

<u>1.3 K_0 in Alg. Geo</u> $\quad X$ quasi-proj. variety.

① A quasi-coherent sheaf $\mathcal{F}$ on X is sheaf of $\mathcal{O}_X$-modules. s.t. $\forall U \subseteq X$ open. $\exists \{U_i \subseteq X \mid i \in I\}$ by affine open cover s.t. $\mathcal{F}|_{U_i}$ is a sheaf associated to an $\mathcal{O}_X(U_i)$-module M_i for each i.

② $\forall x \in X$, $\exists U$ open st $(\mathcal{O}_X|_U)^I \to (\mathcal{O}_X|_U)^J \to \mathcal{F}|_U \to 0$

Def. A coherent sheaf means in above it's locally a f.g. $\mathcal{O}_X(U)$-module.
An alg. vector bundle $\mathcal{E}$ is a locally free coherent sheaf. i.e. $\exists$ an open covering $\{U_i \mid i \in I\}$ of X s.t. $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{X|U_i}^{e_i}$ for each i.

Construction. M free A-module of rank r. then we can define

$$\mathrm{Sym}^{\cdot}_A(M) = \text{Symmetric poly. algebra.}$$

$$\pi : \mathrm{Spec}\, \mathrm{Sym}^{\cdot}_A(M) \to \mathrm{Spec} A$$
$$\|$$
$$\pi : \mathbb{A}^r \times \mathrm{Spec} A \longrightarrow \mathrm{Spec} A \quad \text{proj.}$$

$\mathcal{E}$ is alg. vector bundle, then

$$\pi_{\mathcal{E}} : \mathbb{V}_{\mathcal{E}} \equiv \mathrm{Spec}\, \mathrm{Sym}^{\cdot}_{\mathcal{O}_X}(\mathcal{E})^* \longrightarrow X.$$

Alt. def: A v.b. of rank r over an alg. variety X is an alg. variety F equipped with a morphism $\pi: F \to X$ s.t. $X = \bigcup_{i \in I} U_i$ open cover s.t.

(1) $\forall i \in I$ $\exists$ an iso. $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r$ satisfying

$$\pi \circ \psi_i^{-1} : U_i \times \mathbb{A}^r \to U_i \quad \text{proj.}$$

(2) $\forall i, j \in I$, $\exists$ an $(r \times r)$-matrix A_{ij} whose entries are regular functions in $U_i \cap U_j$ s.t.

$$\underline{\varphi_{ij} = \psi_j \circ \psi_i^{-1}|_{U_i \cap U_j}} : (U_i \cap U_j) \times \mathbb{A}^r \to \pi^{-1}(U_i \cap U_j) \to (U_i \cap U_j) \times \mathbb{A}^r$$
$$(x, v) \longmapsto (x, A_{ij}(x)\, v)$$

Example. Over $\mathbb{P}^n$. Consider F homo. poly. of degree d $\qquad \mathcal{O}_{\mathbb{P}^n}(d)$

transition $\frac{x_i^d}{x_j^d}$.

$\forall U_i$ with $x_i \neq 0$, consider f_i function on affine on U_i.

$$f_i \to f_j \quad \frac{f_i = \frac{F}{x_i^d}}{\text{multiply by } \frac{x_i^d}{x_j^d}}.$$

Prop. Each vector bundle on $\mathbb{P}^1$ has a unique decomp. as a finite direct sum of $\mathcal{O}_{\mathbb{P}^1}(k)$.

Serre's conjecture: Every alg. vector bundle on $\mathbb{A}^N$ is trivial.

$\Updownarrow$

every f.g. proj. module over $k[x_1, \cdots, x_n]$ is free

(proved by Quillen and Suslin).

Picard Group. Def. $\mathrm{Pic}(X)$ abelian group whose element are iso. classes of rank 1 alg. vector bundle on X. (by tensor product).

Fact.: $\mathcal{F}$ line bundle $\mathcal{F} \otimes \mathcal{F}^\vee = id$. $\quad \mathcal{F}^\vee = \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X^*)$

Def. X quasi-proj. variety $\qquad K_0(X)$ = quotient of free abelian group generated by $[\mathcal{E}]$

where $\mathcal{E}$ are alg. vector bundle over X. We modulo the relation $([\mathcal{E}], [\mathcal{E}_1]+[\mathcal{E}_2])$ for each exact seq. $0 \to [\mathcal{E}_1] \to [\mathcal{E}] \to [\mathcal{E}_2] \to 0$.

Prop. $K_0(\mathbb{P}^N)$ is a free abelian group of rank $N+1$. And it's generator is
$\forall k \in \mathbb{Z}$ $\mathcal{O}_{\mathbb{P}^N}(k), \ldots, \mathcal{O}_{\mathbb{P}^N}(k+N)$.

1.4 K_1 of rings. Let R be a ring.

$$K_1(R) \equiv GL(R) / [GL(R), GL(R)].$$

where. $GL(R) = \bigcup_n GL_n(R)$ – n-dim'l invertible matrix over R.

$$K_1(R) = H_1(GL(R), \mathbb{Z}).$$

Fact: ① $[GL(R), GL(R)] = \langle \text{elementary matrices} \rangle = E(R)$.
② It's perfect $E(R) = [E(R), E(R)]$.

Remark. ① R commutative: $\det : K_1(R) \longrightarrow R^\times$

Then we have a natural spliting $R^\times = GL(1,R) \to GL(R) \to K_1(R)$

$$K_1(R) = R^\times \times SK_1(R)$$

where $SK_1(R) = \ker\{\det\}$.

② If R is a field or local ring, then $SK_1(R) = 0$.

Theorem $\mathcal{O}_K$ ring of integers, then $SK_1(\mathcal{O}_K) = 0$ (Bass – Milnor – Serre)
K number field

Congruence Subgroup: Is every subgroup $H \subseteq SL(\mathcal{O}_K)$ of finite index a congruent subgroup?

(it has the form $\ker(SL(\mathcal{O}_K) \to SL(\mathcal{O}_K/P^n))$)

The answer is yes if the number field F admits a real embedding, and no otherwise.

Theorem (Dirichlet's Theorem) $\mathcal{O}_K^* = \mu(K) \oplus \mathbb{Z}^{r_1+r_2-1}$

1.5 K_2 of rings

$i, j \in \mathbb{N}$.

Def. Let $St(R)$ be the Steinberg group of R. which is generated by $X_{ij}(r)$ $i \neq j$, $r \in R$. subjects to following relation:

$$[X_{ij}(r), X_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k,\ i \neq l \\ X_{il}(rs) & \text{if } j = k,\ i \neq l. \\ X_{kj}(-rs) & \text{if } j \neq k,\ i = l. \end{cases}$$

$[a, b] = aba^{-1}b^{-1}$

Then we define

$$K_2(R) = \ker(St(R) \longrightarrow E(R))$$

$$X_{ij}(r) \longmapsto E_{ij}(r)$$

Remark. $K_2(R)$ has another presentation. That presentation inspires Milnor defines Milnor-K theory.

2. Classifying Space and higher K-theory

Def. Let (X,x) be a pointed space. for $n \geq 0$.

$$\pi_n(X,x) = [(S^n,\infty),(X,x)]$$

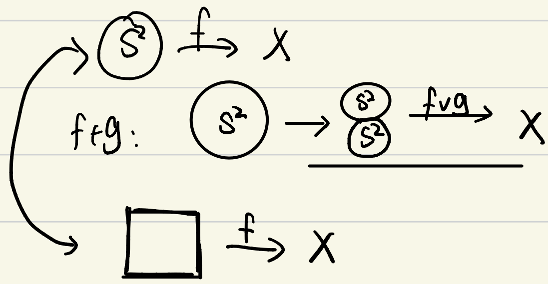
$$f:(S^n,\infty)\to(X,x)$$

$$f\simeq g \qquad H_t:(S^n,\infty)\to(X,x)$$

$$H: S^n\times I\to X \quad \text{s.t. } H_0=f,\ H_1=g$$

$\pi_0(X,x)$ is set of path-component.

$n\geq 1$, $\pi_n(X,x)$ is group / $n\geq 2$, $\pi_n(X,x)$ is abelian group.



$$f+g: \quad S^2 \to S^2\vee S^2 \xrightarrow{f\vee g} X$$

$$\boxed{f} + \boxed{g} = \boxed{f\,|\,g}$$

$$\boxed{f\,|\,g} \simeq \boxed{f\,|\,g} \simeq \boxed{g\,|\,f} \simeq \boxed{g\,|\,f}$$

X H-space $\Rightarrow \pi_0(X)$ monoid $\xrightarrow[\text{Completion}]{\text{Group}} \pi_0(X)$ group.

Thm. (Whitehead) If $f:X\to Y$ is a map of connected CW-complex s.t. $f_*:\pi_n(X,x)\to\pi_n(Y,f(x))$ is an isomorphism for $n\geq 1$, then f is a homotopy equivalence.

Weak equivalence $\Leftrightarrow$ homotopy equivalence

$\exists g$ s.t. $fg\simeq gf\simeq id$.

Def. (fibration) A map $f:X\to Y$ is called a fibration if it has (HLP):

$\forall A\in \text{Space}$

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ A\times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Thm. If $f:X\to Y$ is a fibration, Y connected. $\forall y\in Y$, we have an LES

$$\cdots\to\pi_n(f^{-1}(y),x_0)\to\pi_n(X,x_0)\to\pi_n(Y,y)\to\pi_{n-1}(f^{-1}(y),x_0)\to\cdots$$

Remark. Given a fibration, if Y connected, $\forall y_0,y_1$ $f^{-1}(y_0)\simeq f^{-1}(y_1)$, so in homotopy theory we often a fibration as $\underset{\text{fiber}}{F}\to X\to Y$

Thm. For every map $f: X \to Y$ we can always factor it as

$$X \underset{\text{w.e.}}{\xrightarrow{\simeq}} P(f) \underset{\text{fibration.}}{\longrightarrow} Y$$

We define $\mathrm{hofib}(X \to Y) := \mathrm{fib}(P(f) \to Y)$

example. $F \to X \to Y$ fiber bundle, Y CW-complex $\Rightarrow$ $X \to Y$ is a fibration.

$\downarrow$

$\forall y \in Y$, $\exists U$ open $y \in U$ st. $U \times F \cong f^{-1}(U)$.

2. BG

Def. Let G be a topological group and X topological space. then a principal G-bundle over X is a conti. map $p: E \to X$ toghether with a conti. action of G on E <u>over X</u> s.t. $\exists$ an open covering $\{U_i\}$ of X homeomorphism $G \times U_i \to p^{-1}(U_i) \subseteq E$ respecting G-actions. $(G \to E \to X)$

$$f: G \times U_i \to p^{-1}(U_i) \subseteq E$$
$$(g, x) \in G \times U_i$$

$\forall x \in E,\ g \in G$

$p(g x) = p(x)$

$e \in G \qquad e(g,x) := (eg, x)$

$$f(e(g,x)) = e f(g,x).$$

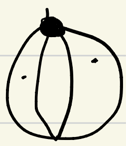
$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl] \\ & X & \end{tikzcd} \quad \forall g \in G.$$

centralizer of a point.

example. ① $U(n) \circlearrowright S^{2n-1}$ $\quad U(n-1) \subseteq U(n)$.

$U(n) = \{ M \text{ matrix of } \mathbb{C}^n \mid \text{st. } M \cdot M^* = I_n \}$

$$U(n)/U(n-1) = S^{2n-1}$$



$$\Downarrow$$

principal fiber bundle: $U(n-1) \to U(n) \to S^{2n-1}$

Thm. (Classfication of principal fiber bundle) (Milnor).

Let G be a topological group which is a CW-complex. Then there exists a connected C.W. complex BG and a principal fiber bundle $\pi: EG \to BG$ st. we have a bijection

contractible space

$$[X, BG] \xrightarrow{\simeq} \{\text{iso. class of principal } G\text{-bundle over } X\}$$

$$f: X \to BG \qquad \begin{tikzcd} f^*BG \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

Def. Let G be a topological group. EG is any weakly contractible space with continuous free G-action, then $BG = EG/G$.

if $g \neq h$, then $g(x) \neq h(x)$

example. $G = \mathbb{Z}$ $\quad \mathbb{Z} \curvearrowright \mathbb{R}$ $\quad E\mathbb{Z} = \mathbb{R}$ $\quad \mathbb{R}/\mathbb{Z} = B\mathbb{Z} = S^1$.

$G = S^1$. $\Rightarrow BS^1 = \mathbb{CP}^\infty$

$G = U(n)$ $\Rightarrow BU(n) = Gr_n(\mathbb{C}^\infty)$

(Plus construction)

<u>Theorem.</u> Let X be a connected CW-complex s.t. $P \subseteq \pi_1(X)$ is a normal perfect subgroup, then there exists X^+ with a map $X \to X^+$ s.t. for $f_*: \pi_1(X) \to \pi_1(X^+)$, $\ker(f_*) = P$ & $f_*: H_*(X) \to H_*(X^+)$ is an isomorphism.

<u>proof.</u> Firstly case when $H_1(X) = 0$ ($\Leftrightarrow \pi_1(X)$ is perfect). $[\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X)]$

$\pi_1(X) = [S^1, X] \Rightarrow$ Choose $\varphi_\alpha: S^1 \to X^{(1)}$ generating $\pi_1(X)$.

Then via φ_α, we attach e^2_α to form a simply-connected CW-complex X'.

$$X'/X = \vee S^2_\alpha$$

$$0 \to H_2(X) \to H_2(X') \to \underset{\substack{\| \\ \text{free abelian group.}}}{\overset{\substack{H_2(X'/X) \\ \|}}{H_2(X', X)}} \to 0^{= H_1(X)}$$

$$H_2(X') \cong H_2(X) \oplus H_2(X', X)$$

Hurewicz theorem $\Rightarrow H_2(X') \cong \pi_2(X') = [S^2, X']$

so we can represent a basis of $H_2(X', X)$ by <u>$\psi_\alpha: S^2 \to X'$</u>

then we can attach 3-cell e^α_3 along ψ_α, and get a space X^+.

We have $X \hookrightarrow X^+$ s.t. $\pi_1(X^+) = 0$. Moreover.

X^+/X = 3-cell attaching on 2-spheres

$$0 \to \underline{C_3} \overset{d_3}{\underset{\cong}{\to}} \underset{\substack{\| \\ \vee S^2.}}{\underline{C_2}} \to \underset{\substack{\| \\ 0.}}{\underline{C_1}} \to 0.$$

$$\Downarrow.$$

$$H_*(X^+/X) = 0$$

By LES, this it the construction we want.

Let $P \subseteq \pi_1(X)$ be the perfect normal subgroup.

we know there is

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d] \arrow[dl, "p"'] & \tilde{X}^+ \arrow[d] \\ X \simeq M_p \arrow[r] & X^+ \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$\nearrow$ s.t. $\pi_1(\tilde{X}) = P$.

$f: X \to Y$

$M_f = X \times I \cup Y / \{(x \times 1) \sim f(x)\}$.

$X \hookrightarrow M_f \simeq Y$

Since $X^+/M_p \cong \tilde{X}^+/\tilde{X}$ $\quad H_*(\tilde{X}^+, \tilde{X}) = 0 \Rightarrow H_*(X^+, M_p) = 0$.

& By Van-Kampen, we know $\pi_1(X^+) = \pi_1(X)/P$. $\square$

Def. For a ring R, let $\gamma: BGL(R) \to BGL(R)^+$ denote the Quillen plus construction. then we define

$$K_i(R) = \pi_i(BGL(R)^+) \quad \text{for } i > 0.$$

$$K_0(R) = \pi_0(BGL(R)^+) \times \mathbb{Z}.$$

Remark. when $n=1$, $K_1(R) = GL(R)/[GL(R), GL(R)]$

We know $\pi_1(BGL(R)) = GL(R)$ [When G is discrete. $\pi_1(BG) = G$].

$\pi_1(BGL(R)^+) = K_1(R)$ in the definition before.

Thm. Let $\mathbb{F}_q$ be a finite field. then for $i > 0$:

$$K_i(\mathbb{F}_q) = \mathbb{Z}/q^j - 1 \quad \text{if } i = 2j-1$$

$$K_i(\mathbb{F}_q) = 0 \quad \text{if } i = 2j.$$

$H^*(BG) = H^*(G)$ when G is discrete.

3. More def. of K-theory.

Recall. For $i \geq 1$, $K_i(R) = \pi_i(\underbrace{BGL(R)}^{+})$ $\quad B: \mathrm{Grp} \to \mathrm{Space}$.

$$\begin{cases} H_*(BGL(R)^+) = H_*(BGL(R)) \\ \pi_1(BGL(R)^+) = \pi_1(BGL(R)) / [\pi_1(BGL(R)), \pi_1(BGL(R))] \end{cases}$$

$$[GL(R), GL(R)] = \langle E(R) \rangle$$

Exact Category.

Def An Exact Cat. is a pair $(\mathcal{C}, \mathcal{E})$ where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a family of sequences in $\mathcal{C}$ of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \qquad (+)$$

satisfying the following conditions: there is a embedding of $\mathcal{C}$ as a full subcat. of an abelian category $\mathcal{A}$ s.t.

(1) $\mathcal{E}$ is the class of all sequences in $\mathcal{C}$ which are exact in $\mathcal{A}$.

(2) $\mathcal{C}$ is closed under extensions in $\mathcal{A}$ in the sense that if $(+)$ is an exact sequence in $\mathcal{A}$ and $B, D \in \mathcal{C}$, then C is isomorphic to an object in $\mathcal{C}$.

The seq. in $\mathcal{E}$ are called the short exact sequences of $\mathcal{C}$. We call a map in $\mathcal{C}$ an admissible monomorphism (admissible epimorphism) if it occurs as the mono. i (the epi. j) in a seq. in $\mathcal{E}$.

ex. ① $\mathcal{P}(R)$: all f.g. projective R-modules $\subseteq \mathrm{Mod}_R$.
② X scheme $\Rightarrow$ $VB(X)$: all algebraic vector bundle of X $\subseteq \mathcal{O}_X$-mod.

Def. If $\mathcal{C}$ is exact cat., then we define

$$K_0(\mathcal{C}) = \frac{\{\text{monoid of iso. classes of objects of } \mathcal{C} \text{ (w.r.t. } \oplus)\}}{[A_2] - [A_1] - [A_3] \text{ for every } (A_1 \to A_2 \to A_3) \text{ in } \mathcal{E}}$$

$\Rightarrow K_0(\mathcal{P}(R)) = K_0(R)$
$\quad K_0(VB(X)) = K_0(X)$.

Nerve and simplicial set

Def. Simplicial set $S_\cdot : \Delta^{op} \to \mathrm{Set}$.

ex. $\mathrm{Sing}_\cdot : \mathrm{Space} \to \mathrm{sSet}$

$X \longmapsto \mathrm{Sing}_\cdot(X)$.

Def. Geometric realization is the left adjoint of Sing:

$$\mathrm{sSet} \underset{\mathrm{Sing}_\cdot}{\overset{|\cdot|}{\rightleftarrows}} \mathrm{Space}$$

Def. ① Nerve. $N : \mathrm{Cat} \to \mathrm{sSet}$.

$$\mathcal{C} \longmapsto (N\mathcal{C}) : \Delta^{op} \to \mathrm{Set}$$

$$[n] \longmapsto \{C_0 \to \cdots \to C_n\}$$

composable chain

② B classifying space functor

$$B : \mathrm{Cat} \xrightarrow{N} \mathrm{sSet} \xrightarrow{|\cdot|} \mathrm{Space}.$$

example. when G is a discrete group, then we can define

$\mathcal{G}$: object: $\{*\}$

morphism: $\{$all element of $G\}$

then $B\mathcal{G} = BG$ ← defined before

Quillen Q-construction.

Def. Let P be an exact category and QP be the category obtained by applying Quillen Q-construction, then

$$K_i(P) = \pi_{i+1}(BQP) \quad \text{for } i \geq 0$$

Theorem. Let X be a scheme and $\mathrm{Vect}(P)$ be the exact cat. of algebraic vector bundles. Then $K_i(X) := K_i(\mathrm{Vect}(X)) = \pi_{i+1}(BQ\,\mathrm{Vect}(X))$

this agrees for $i=0$ with the group $K_0(X)$. For affine scheme $X = \mathrm{Spec} A$ then we know $K_i(X) = \pi_i(\underline{BGL(A)}^+) = K_i(A)$ for $i > 0$.

Construction: Let $\mathcal{P}$ be an exact category. We define the category $Q\mathcal{P}$ as follows. We set Obj $Q\mathcal{P}$ equal to Obj $\mathcal{P}$. For any $A, B \in \mathrm{Obj}\, Q\mathcal{P}$, we define

$$\mathrm{Hom}_{Q\mathcal{P}}(A,B) = \{ A \overset{p}{\twoheadleftarrow} X \overset{i}{\hookrightarrow} B;\ p \text{ (resp. } i) \text{ admissible epi (resp. mono.)} / \sim \}$$

where $(A \twoheadleftarrow X \hookrightarrow B) \sim (A \twoheadleftarrow X' \hookrightarrow B)$ if

$$\begin{tikzcd} A \arrow[d, equal] & X \arrow[l, two heads, "p"'] \arrow[r, hook, "i"] \arrow[d, "\simeq"] & B \arrow[d, equal] \\ A & X' \arrow[l, two heads, "p'"] \arrow[r, hook, "i'"] & B \end{tikzcd}$$

TOPOLOGICAL K-THEORY

X paracompact Hausdorff space.

$$K(X) = \{\text{Vector bundle on } X\} / E = E' + E'' \text{ if } E = E' \oplus E''$$
$$\shortparallel$$
$$K^0(X)$$

(Over $\mathbb{C}$)

Prop. ① Let X be a compact space. Then for $p: E \to X$ a vector bundle. There exists $N > 0$ s.t.

$$\begin{tikzcd} \mathbb{C}^{N+1} \times X \arrow[rr, "f"] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

surjective map of vector bundles.

② If $\begin{tikzcd} E \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$ is a surjective map of vector bundles. then $\exists F^{\perp} \to X$ vector bundle st. $E = F \oplus F^{\perp}$

③ (Classification of vector bundle) n-dimensional vector bundle over X.

$$[X, BU(n)] \xrightarrow{\simeq} \mathrm{Vect}_n(X)$$
$$f \longmapsto f^*\gamma_n$$

$$\begin{tikzcd} f^*\gamma_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_n = \{(x,y) \in Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty \mid y \in x\} \arrow[d, "\mathrm{proj}"] \\ X \arrow[r, "f"] & BU(n) \simeq Gr_n(\mathbb{C}^\infty) \end{tikzcd}$$

Prop. $BU = \varinjlim_n BU(n)$

$$K^0(X) = [X, BU \times \mathbb{Z}]$$

TOPOLOGY K-THEORY. (over $\mathbb{C}$)

X compact & Hausdorff

Def. $K(X) = \{E_1 - E_2 \mid E_1, E_2 \in \mathrm{Vect}(X)\}/\sim$ $\qquad \underset{\mathbb{Z}}{\overset{\{\varepsilon^n - \varepsilon^0\}}{\shortparallel}} \subseteq K(X)$

$\sim$: $E_1 - E_2 \sim E_1' - E_2'$ iff $E_1 \oplus E_2' \cong E_1' \oplus E_2$

$+$: direct sum.

This is a Abelian Group $\qquad$ id: $[E-E]$ $\quad$ purely formal

$\tilde{K}(X) = \{$All the vector bundles over $X\}/\sim$

$\sim$: $E_1 \sim E_2$ iff $\exists \varepsilon^n, \varepsilon^m$ s.t. $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$

$\varepsilon^n = X \times \mathbb{C}^n$

$+$: direct sum

This is Abelian Group. id: $[\varepsilon^n]$

inverse: exists because X is compact & Hausdorff

$\Downarrow$

$V \in \mathrm{Vect}(X)$ $\exists W$ s.t. $V \oplus W \cong \varepsilon^n$

Remark. $E_1 - E_2 \in K(X)$ $\qquad$ X c. H. $\Rightarrow$ $\exists W$ s.t. $E_2 \oplus W \cong \varepsilon^n$

$\Rightarrow$ $E_1 - E_2 = E_1 \oplus W - E_2 \oplus W = E_1 \oplus W - \varepsilon^n$.

$K(X) = \{E - \varepsilon^n\}$

Prop. $K, \tilde{K}$ are functors : $\mathrm{C.H.}^{op} \to \mathrm{AbGrps}$ (In fact, CRings)

$\tilde{K}(X) \cong \mathrm{Ker}(K(X) \to K(*))$ $\qquad * \hookrightarrow X$

Hence, $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$ $\qquad K(*) = \mathbb{Z}$.

proof. $f: X \to Y$ $\qquad f^*: K(Y) \to K(X)$ $\qquad E - F \to f^*E - f^*F$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$* \hookrightarrow X$: $\qquad K(X) \longrightarrow K(*) \cong \mathbb{Z}$

$E - \varepsilon^n \longmapsto \dim E - n$

$\mathrm{Ker} = \{E - \varepsilon^{\dim E}\}$ $\qquad E \in \mathrm{Vect}(X).$ $\qquad E - \varepsilon^{\dim E} \sim F - \varepsilon^{\dim F} \in \mathrm{Ker}$

$\mathrm{Ker} \cong \tilde{K}(X)$ $\qquad E - \varepsilon^{\dim E} \leftrightarrow E$ $\qquad E \oplus \varepsilon^{\dim F} \cong F \oplus \varepsilon^{\dim E}$

$\square$

$K(X)$ is a ring: $\quad (E-E')(F-F') = E\otimes F + E'\otimes F' - E'\otimes F - E\otimes F'$

$K(X)\in CRing$, Identity: ε^1

$\tilde{K}(X)$ is an ideal of $K(X)$ $\Rightarrow$ $\tilde{K}(X)\in CRing$

<u>Q:</u> $E, F\in \tilde{K}(X)$ $\quad E\times F \neq E\otimes F$

External Product of K-theory:

$$\mu.\ K(X)\otimes K(Y) \to K(X\times Y)$$
$$a\otimes b \longmapsto p_X^*(a)\, p_Y^*(b)$$

<u>example.</u> $X = \mathbb{CP}^1 = S^2$

We have tautological line bundle.

$$H = \{(U,x)\in \mathbb{CP}^1\times\mathbb{C}^2 \mid x\in U\}$$

$H \longrightarrow X$ is the bundle.

$$(H\otimes H)\oplus\varepsilon^1 = H\oplus H$$

Then in $K(X)$, $\quad H^2+1 = 2H \Rightarrow (H-1)^2 = 0.$

$$\mathbb{Z}[H]/(H-1)^2 \subseteq K(S^2).$$

<u>Thm.</u> (Fundamental Product Theorem)

$$\mu:\ K(X)\otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X\times S^2)$$

This is an isomorphism.

$X = * \Rightarrow K(S^2) = \mathbb{Z}[H]/(H-1)^2 = \mathbb{Z}\oplus\mathbb{Z}[H].$

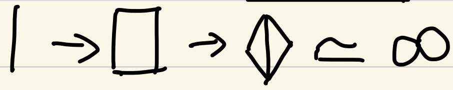
Ω-Spectrum: $\quad \{X_i\}_{i\in\mathbb{Z}}$ $\quad X_i\in CW_*$. equipped with h.e. (weak equivalence.) $X_i \xrightarrow{\simeq} \Omega X_{i+1}$

$$\Omega X = \{\gamma: I\to X \mid \gamma(0)=\gamma(1)=*\}$$

Infinite loop space. $\quad X$ s.t. $\exists Y_i$ $X = \Omega Y_1 = \Omega^2 Y_2 = \Omega^3 Y_3 = \cdots$

Reduced cohomology theory: $i\in\mathbb{Z}$ $\quad h^i: CW_*^{op}\longrightarrow AbGrp$, $\sigma^i: h^{i+1}(\Sigma X)\to h^i(X)$. n

$$\Sigma X = X\times I/\sim \quad X\times\{0\}\simeq *,\quad X\times\{1\}\simeq *.$$
$$x_0\times I \simeq *.$$



$\Sigma X \simeq SX = X\times I/\, X\times\{0\}\simeq *,\ X\times\{1\}\simeq *.$

$\Sigma \dashv \Omega$

$\langle \Sigma X, Y\rangle \cong \langle X, \Omega Y\rangle$

satisfy axioms: Additivity: $h^q(X \vee Y) \cong h^q(X) \oplus h^q(Y)$

h.e.: $f \simeq g \Rightarrow h^q(f) = h^q(g)$.

Exactness: $A \hookrightarrow X \longrightarrow X/A$

$$\cdots \to h^q(X/A) \to h^q(X) \to h^q(A) \to h^{q+1}(X/A) \to \cdots$$

dim. $h^q(*) = \begin{cases} G & q=0 \\ 0 & q \neq 0 \end{cases} \xrightarrow{\text{Thm.}} h^q(X) \cong H^q(X; G)$.

$\uparrow$ singular cohomology

<u>Thm.</u> (Brown). $\{\Omega\text{-Spectrum}\} \leftrightarrow \{\text{cohomology theory}\}$

$$E = (E_q)_{q \in \mathbb{Z}} \longmapsto E^q(X) = \langle X, E_q \rangle$$

$\uparrow$ point homotopy class

<u>Spectrum of K-theory.</u>

<u>Thm.</u> There is an equivalence

$$\Phi.\ BU \times \mathbb{Z} \to \Omega U$$

$$U = \operatorname{colim}_n U(n)$$

$\left(U \xrightarrow{\text{topolog.}} + \underset{\text{contr. + free } U\text{-action}}{\underset{\uparrow}{EU}} \Rightarrow EU/U = \overset{\text{classifying space.}}{\overset{\uparrow}{BU}} \right)$

$[X, BG] \longleftrightarrow \{\text{pri. } G\text{-bundle over } X\}$.

<u>Lemma.</u> G topological group $\pi_{q+1}(G) \cong \pi_q(\Omega G)$ for $q \geq 0$.

$\pi_{q+1}(BG) \cong \pi_q(G)$

In particular, $G \simeq \Omega BG$.

<u>proof.</u> $U \to EU \to BU \Rightarrow \cdots \to \pi_q(U) \to \underset{\underset{0}{\|}}{\pi_q(EU)} \to \underbrace{\pi_q(BU) \to \pi_{q-1}(U)}_{\overset{\uparrow}{\cong}} \to \cdots$

$\forall X \in Space$, $\Omega X \to PX \overset{\pi}{\to} X$

$PX = \{I \to X \mid I(0) = x_0\}$ PX contractible.

$\pi^{-1}(x_0) = \Omega X$

$\Rightarrow$ Similarly,

$X = BG$. $\Omega BG \to PBG \to BG$.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[r] \arrow[d, "f"] & BG \arrow[d, equal] \\ \Omega BG \arrow[r] & PBG \arrow[r] & BG \end{tikzcd} \Rightarrow f_*: \pi_q(G) \to \pi_q(\Omega BG) \text{ iso.}$$

□

Thm + Lemma

We can define: KU Ω-Spectrum

$$\{ \cdots \ U \quad \underset{(KU)_0}{\overset{}{BU\times\mathbb{Z}}} \quad \underset{(KU)_1}{U} \quad \underset{(KU)_2}{BU\times\mathbb{Z}} \ \cdots \}$$

$\Omega(KU_n) = (KU)_{n-1}$. $\leftarrow \left\{ \begin{array}{l} \Phi: BU\times\mathbb{Z} \to \Omega U \\ \Phi': U \xrightarrow{\simeq} \underline{\Omega BU = \Omega(BU\times\mathbb{Z})} \end{array} \right.$

Bott Periodicity (Version 1)

$$\pi_i U \cong \pi_{i+2} U$$

proof.

$$\pi_i U \cong \pi_i(\Omega BU) \cong \pi_i(\Omega(BU\times\mathbb{Z})) \cong \pi_i(\Omega(\Omega U)) \simeq \pi_i(\Omega^2 U) = \pi_{i+2} U$$

$\square$

With KU-Spectrum, we can define

$$K^i(X) = \langle X, KU_i \rangle = \begin{cases} \underline{\langle X, BU\times\mathbb{Z}\rangle} & i \text{ even} \\ \underline{\langle X, U\rangle} & i \text{ odd} \end{cases}$$

$\widetilde{K}(X)$ iso. class of vector bundles. $\updownarrow$ $\langle X, BU\times\mathbb{Z}\rangle$

$\uparrow$ This is a reduced cohomology theory.

Prop. $K^0(X) = \widetilde{K}(X)$ $\quad K^0(X_+) = K(X)$ $\quad X_+ = X \sqcup \{*\}$.

$$\langle X, \Omega Y\rangle = \langle \Sigma X, Y\rangle$$

Def. $\widetilde{K}^i(X) = \begin{cases} \widetilde{K}(X) & i \text{ even.} \\ \widetilde{K}(\Sigma X) & i \text{ odd.} \end{cases}$

$$\langle X, U\rangle = \overbrace{\langle X, \Omega(BU\times\mathbb{Z})\rangle}^{\text{lemma.}} = \langle \Sigma X, BU\times\mathbb{Z}\rangle = \widetilde{K}(\Sigma X).$$

$$X\wedge Y = X\times Y/(X\vee Y)$$

surjective, $\exists$ right inverse.

$$\underline{\widetilde{K}(\Sigma(X\times Y))} \to \widetilde{K}(\Sigma(X\vee Y)) \to \widetilde{K}(X\wedge Y) \to \widetilde{K}(X\times Y) \xrightarrow{i^*} \widetilde{K}(X\vee Y)$$

$\widetilde{K}(\Sigma(X\vee Y)) = \widetilde{K}(\Sigma X)\oplus\widetilde{K}(\Sigma Y)$

$\widetilde{K}(X\vee Y) = \widetilde{K}(X)\oplus\widetilde{K}(Y) \xrightarrow{i_X^*\oplus i_Y^*}$

$$P_X\circ i_X : \underbrace{X \hookrightarrow X\times Y \to X}_{\text{id.}}$$

$\Rightarrow \tilde{K}(X\times Y) = \tilde{K}(X\wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y)$

Prop.

$$\begin{array}{ccc} K(X)\otimes K(Y) = \tilde{K}(X)\otimes\tilde{K}(Y) & \oplus\ \tilde{K}(X)\otimes\tilde{K}(Y) \oplus \mathbb{Z} \\ \mu \downarrow \quad \Rightarrow \quad \mu' \downarrow & \quad \quad \quad \quad \| \\ K(X\times Y) = \tilde{K}(X\wedge Y) & \oplus\ \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \end{array}$$

a $\quad i_X^*$ $\quad i_Y^*$ $\quad i_{point}^*$

$\exists\ \tilde{K}(X)\otimes\tilde{K}(Y) \longrightarrow \tilde{K}(X\wedge Y)$ external product.

$a \in \tilde{K}(X) = \mathrm{Ker}(K(X) \to K(*))$

$b \in \tilde{K}(Y) = \mathrm{Ker}(K(Y) \to K(*))$

$p_X^*(a)\, p_Y^*(b) \in \tilde{K}(X\wedge Y)$

$i_X^*(p_X^*(a)\, p_Y^*(b)) = i_Y^*(p_X^*(a)\, p_Y^*(b)) = i_{point}^*(p_X^*(a)\, p_Y^*(b)) = 0$

$f(ab) = f(a)\cdot f(b)$ $\quad p_Y \circ i_X$ $\quad p_X \circ i_Y$ $\quad p_X \circ i_{point}$

$i_X^* p_Y^*(b) = 0$

$\|$

$(p_Y\circ i_X)^*(b) = 0$

$\square$

Bott Periodicity (Version 2).

$\Sigma^2 X = S^2 \wedge X$

$\beta:\ \tilde{K}(X) \longrightarrow \tilde{K}(\Sigma^2 X) = \tilde{K}(S^2\wedge X).$ $\qquad \tilde{K}^i(X) = \tilde{K}^{i+2}(X)$

$\alpha \longmapsto (H-1) * \alpha.$

this is an isomorphism.

$$\beta:\ \tilde{K}(X) \longrightarrow \underset{\substack{\| \\ \mathbb{Z}}}{\tilde{K}(S^2)} \otimes \tilde{K}(X) \xrightarrow{\ \mu'\ } \tilde{K}(S^2\wedge X) = \tilde{K}(\Sigma^2 X).$$

$\uparrow$ iso $\qquad \uparrow$ iso.

Adams Operation: $\forall k \in \mathbb{N},\quad \Psi^k:\ \tilde{K}(X) \to \tilde{K}(X)$

$\tilde{K}(X)$ is $\mathbb{Z}[\Psi^1, \Psi^2, \cdots]$-module.

Adams Operation + Hopf Invariant One.

X compact Hausdorff

$K(X) = \{E - E'\}$ $\quad\quad$ $\tilde{K}(X) = \{[E]\}$

$\tilde{K}(X) = [X, BU]$

$\tilde{K}$ is reduced cohomology theory:

$\tilde{K}$: CW-complex - AbGrp. $\quad$ δ: boundary functor

$\tilde{K}^{odd}(X) = \tilde{K}^0(X) = \tilde{K}(X)$

$\tilde{K}^{even}(X) = \tilde{K}^1(X) = \tilde{K}(\Sigma X) := [\Sigma X, BU] = [X, \Omega BU]$

Adams Operation: $\forall k > 0$: $\Psi^k : K(X) \to K(X)$ ring homo. $\quad$ X comp. Haus.

WANT TO: (1) Ψ^k is natural ring homomorphism

(2) $\Psi^k \Psi^l = \Psi^{kl}$ $\quad$ ($\leftrightarrow$ Ψ^p p prime)

(3) $\Psi^k(L) = L^k$ $\quad$ when L is a line bundle

(4) $\Psi^p(\alpha) = \alpha^p$ (mod p) $\quad$ $\forall \alpha \in K(X)$. $\quad$ p prime.

Ψ^k want to make exterior power be a ring homo.

$\underset{\sim}{V}$ $\quad$ $\Lambda^k V = V^{\otimes k}/\sim$ $\quad\quad$ $V = \operatorname{Span}\{a_1, \cdots, a_n\}$

$= \operatorname{Span}\{a_{i_1} \otimes \cdots \otimes a_{i_k}\}$ $\quad$ $i_1 < \cdots < i_k$

For vector bundle E we can also define $\Lambda^k E \in \operatorname{Vect}(B)$. called $\lambda^k(E)$

Power operation

(1) $\lambda^k(E_1 \oplus E_2) = \oplus_i (\lambda^i(E_1) \otimes \lambda^{k-i}(E_2))$

(2) $\lambda^0(E) = \varepsilon^1$ $\quad$ trivial line bundle.

(3) $\lambda^1(E) = E$

(4) $\lambda^i(E) = 0$ $\quad$ when $i > \dim E$

Thm (Splitting Principle) Let $E \in \operatorname{Vect}(X)$ with X cmHau. Then $\exists$ comHaus $F(E)$ with a map $p: F(E) \to X$ s.t. $p^*: K(X) \to K(F(E))$ is injective & p^*E splits as a sum of line bundle

$\boxed{E = L_1 \oplus \cdots \oplus L_n}$ $\quad\quad$ L line bundle $\Rightarrow$ $\Psi^k(L) = L^k$

$\Psi^k(E) = L_1^k + \cdots + L_n^k$

$\underline{\lambda_t(E) = \sum_i \lambda^i(E) t^i} \in K(X)[t]$

$\lambda_t(E_1 \oplus E_2) = \lambda_t(E_1)\lambda_t(E_2)$

$E = L_1 \oplus \cdots \oplus L_n$

$\lambda_t(E) = \lambda_t(L_1) \cdots \lambda_t(L_n)$

$= \prod_{i=1}^{n} (1 + L_i t)$

$\underline{= 1 + \sigma_1 t + \cdots + \sigma_n t^n}$

σ_i i-th elementary symm. poly. of L_i's.

$\underline{\lambda^i(E) = \sigma_i(L_1, \cdots, L_n)}$ —

Poly theory: $\forall k \geq 0$, $\exists\, S_k$ in k-variables s.t. Newton Polynomial.

$S_k(\sigma_1(L_1, \cdots, L_n), \cdots, \sigma_k(L_1, \cdots, L_n)) = \underline{L_1^k + \cdots + L_n^k}$

Independent of n.

Def. $\quad \underline{\Psi^k(E) = S_k(\lambda^1(E), \cdots, \lambda^k(E))} \quad k \geq 1$.

$\Psi^0(E) = 1$.

Ψ^k natural. $\quad \widetilde{K}(X) = \mathrm{Ker}(K(X) \to K(*))$

$\Psi^k : \widetilde{K}(X) \to \widetilde{K}(X)$.

When $X = S^{2n}$. $\quad \Psi^k : \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$

$\widetilde{K}(S^{2n}) \cong \mathbb{Z}[(H-1)^n]$ As a group

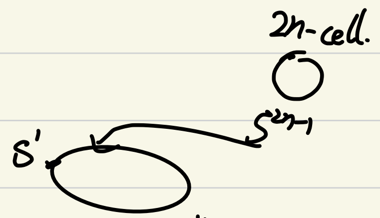
$\widetilde{K}(S^2) = \mathbb{Z}[H-1] \quad (H-1)^2 = 0$

$\Psi^k(H-1) = H^k - 1 = (H-1+1)^k - 1 \underset{(H-1)^2=0}{=} k(H-1)$

$n > 1$. $\quad \Psi^k((H-1)^n) = k^n (H-1)^n$.

Hopf Invariant One:

$\underline{n>1}$: Given $\underline{f : S^{2n-1} \to S^n} \rightsquigarrow C_f = e^{2n} \cup_f S^n$.



C_f has one $2n$-cell, one n-cell, one 0-cell.

cellular chain complex:

$$0 \to \underset{2n}{\mathbb{Z}} \to 0 \to \cdots \to \underset{n}{\mathbb{Z}} \to \cdots \to \underset{0}{\mathbb{Z}} \to 0$$

$$\Rightarrow H_*(C_f) = \begin{cases} \mathbb{Z} & \text{when } * = 0, n, 2n \\ 0 & \text{otherwise.} \end{cases}$$

when n odd,

cup product: $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$

$\beta^2 = 0$

$\beta^2 = -\beta^2 \Rightarrow 2\beta^2 = 0$
$\Rightarrow \beta^2 = 0$.

UCT $\Rightarrow H^*(C_f) = \begin{cases} \mathbb{Z} & \text{when } * = 0, n, 2n \\ 0 & \text{otherwise.} \end{cases}$

$H^n(C_f) = \mathbb{Z}[\alpha]$ $\quad H^{2n}(C_f) = \mathbb{Z}[\beta]$

we know there is cup product.

$\alpha^2 = k\beta$ for $k \in \mathbb{Z}$.

We call this k as the Hopf Invariant of f.

Question: For which n, there is f s.t. $H(f) = 1$?

when $n = 2, 4, 8$, there is

no for all other n.

$\mathbb{R}^n$ has structure of division algebra $\Rightarrow$ Hopf invariant one.

$\underline{\mathbb{C}, \mathbb{H}, \mathbb{O}}$

$k = \mathbb{C}$, $\quad \mathbb{C}^2$ 4-cell $\quad \rightsquigarrow \underline{\mathbb{CP}^2}$ 0-cell 2-cell 4-cell

$\underline{f: S^3 \to S^2}$ Hopf fibration.

$\mathbb{CP}^2$ manifold theory

$H^*(\mathbb{CP}^2) = \mathbb{Z}[\alpha]/\alpha^3 \quad |\alpha| = 2$.

$\underline{\text{Thm.}}$ When $n \neq 2, 4, 8$, $\quad H(f) \neq 1$.

From K-theory: $\underline{n = 2k}$ $\quad C_f/S^{2k} = S^{4k}$ $\quad \underline{n \text{ even}}$

$$\begin{array}{ccccccccc}
\tilde{K}^{-1}(S^{2k}) & \to & \underbrace{\tilde{K}^0(S^{4k})} & \to & \tilde{K}^0(C_f) & \to & \underbrace{\tilde{K}^0(S^{2k})} & \to & \tilde{K}^1(S^{4k}) \\
\| & & \| & & & & \| & & \| \\
0 & & \mathbb{Z} & & & & \mathbb{Z} & & 0
\end{array}$$

$\Downarrow$

$0 \to \mathbb{Z}[x] \longrightarrow \tilde{K}^0(C_f) \to \mathbb{Z}[y] \to 0 \,. \in \mathrm{Ext}^1_A(\tilde{K}(S^{4k}), \tilde{K}(S^{2k}))$

$x \longmapsto x'$ $\quad x' \longmapsto 0$

$y' \longmapsto y$

$(x')^2 = 0$ since $x^2 = 0$ & $(y')^2 = kx$

$k := \underline{h(f)}$

$\underline{\text{WTS}}$: $h(f)$ is independent of choice of y'. Suppose $y'-y''=mx$

$\hookrightarrow (y'')^2 = (y'+mx')^2$

$= \underline{h(f)}x + \underset{0}{\underset{\shortparallel}{\underline{2mx'y'}}} + \underset{0}{\underset{\shortparallel}{\underline{(x')^2}}}$

WTS: $x'y'=0$ $x'y' = k_1x' + k_2y'$.

$\tilde{K}_0(C(f))$ is torsion-free

$\begin{array}{ccc} x' & & y' \\ \downarrow & & \downarrow \\ 0 & & k_2y \end{array} \Rightarrow k_2=0$.

$y'x'y' = k_1x'y'$

$\Downarrow$

$k_1x'y' = x'(y')^2 = h(f)(x')^2 = 0$.

$\underline{\text{Prop}}$. $h(f) = H(f)$.

$\underline{\text{Thm}}$. When $n \neq 2,4,8$, $H(f) \neq 1$ $h(f) \neq 1$.

$\psi^m(y) = m^k y$ $\psi^m(x) = m^{2k}x$

$\Downarrow$ $\Downarrow$

$\underline{\psi^m(y') = m^k y' + \mu_m x'}$ $\underline{\psi^m(x') = m^{2k}x'}$

$\psi^2\psi^3(y') = \psi^3\psi^2(y')$

$\psi^2\psi^3(y') = \psi^2(\underline{3^k}y' + \mu_3 x')$

$= 2^k3^ky' + 3^k\mu_2x' + \mu_3\cdot 2^{2k}x'$

$\psi^3\psi^2(y') = \psi^3(2^ky' + \mu_2x')$

$= 2^k3^ky' + 2^k\mu_3x' + 3^{2k}\mu_2x'$

$\Rightarrow 2^k(1-2^k)\mu_3 = 3^k(1-3^k)\mu_2$

$\Rightarrow 2^k \mid \underline{3^k}(1-3^k)\underline{\mu_2}$.

$\Rightarrow 2^k \mid 1-3^k$

Suppose we have

$H(f)=1$

$(y')^2 = x'$

$\psi^2(y') \equiv (y')^2 \pmod 2$

$\mu_2x' \equiv x' \pmod 2$

$\Rightarrow \mu_2$ odd.

Lemma. If $2^k \mid 1-3^k$, then $k=0,1,2,4$

$\square$.

$\tilde{K}(X)$ adams opera. $A = \mathbb{Z}[\psi_1, \psi_2, \psi_3, \cdots, \psi_p, \cdots]$

$\tilde{K}(X) \in \mathrm{Mod}_A$. $\mathrm{Ext}_A(\tilde{K}(S^{4k}), \tilde{K}(S^{k})) \cong \mathbb{Z}/\gcd$

Chern class and $F\Psi^q$

Chern class

$$U(n) \rightsquigarrow BU(n) \simeq Gr_n(\mathbb{C}^\infty)$$

Thm. $H^*(BU_n) \cong \mathbb{Z}[c_1, \cdots, c_n]$ $\qquad Vect_n(B) \simeq [B, BU(n)]$

Def. (Chern class) Let $E \to B$ be a complex vector bundle. Then by classification of vector bundles, we know it associates to a map

n-dim'l

$$f: B \to BU(n) \Rightarrow f^*: H^*(BU(n)) \to H^*(B)$$

Hence, we can define a series of functions $c_0, c_1, \cdots, c_n, \cdots$ from iso. classes of vector bundles to $H^*(B)$ s.t.

$$c_0(E) = 1$$

$$c_i(E) = f^*(c_i) \qquad c_i \text{ means generator of } H^*(BU(n))$$

$$c_i(E) = 0 \qquad \text{for } i > \dim E.$$

we define total chern class as

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \cdots$$

Whitney sum formula: $E, E' \to B$

$$\Rightarrow c(E \oplus E') = c(E) \cup c(E')$$

$$\Downarrow$$

$$c_n(E \oplus E') = \sum_{i+j=n} c_i(E) \cup c_j(E').$$

$K(\mathbb{F}_q)$ $\qquad \mathbb{F}_p$ finite field. $q = p^n$

Plan:

$$K_i(\mathbb{F}_q) = \pi_i(BGL(\mathbb{F}_q)^+) \qquad \text{for } i \geq 1.$$

H-space

$\uparrow$ homological iso.

$$BGL(\mathbb{F}_q) \xrightarrow[\text{iso.}]{\text{homological}} F\Psi^q$$

H-space

Thm. A homological iso. between H-spaces is a homotopy equivalence.

$$K_i(\mathbb{F}_q) = \pi_i(F\Psi^q) \leftarrow \text{simple!}$$

reduced K-group $\uparrow$ iso. classes of vector bundles $/ [E]+[F] \cong [E \oplus F]$

$\tilde{K}(X) = [X, BU]$ for X compact. —(*)

$X = BU^n$ X non-compact

Prop. $[BU^n, BU] = \{$ Natural trans $(\tilde{K})^n \to \tilde{K} \}$

proof. If (*) true for X non-compact, Yoneda require X- both compact & non-compact.

$$Nat((\tilde{K})^n, \tilde{K}) = Hom([-, BU^n], [-, BU]) \cong [BU^n, BU]$$

BU^n is the limit of spaces X_m with only even-dim'l cells.

$BU^n = \varinjlim X_m$ X_m even CW-complex

$\Downarrow$ Milnor exact seq.

$$0 \to R^1 \lim K^{-1}(X_m) \to [BU^n, BU] \xrightarrow{\cong} \varprojlim [X_m, BU] \to 0 \qquad \tilde{K}(BU^n)$$

$K^{-1}(X_m) = 0 \quad \forall m \Rightarrow R^1 \lim K^{-1}(X_m) = 0$ $\qquad \varprojlim \tilde{K}(X_m) = \tilde{K}(BU^n)$

$\Downarrow$

$Hom(Hom(-, X), F) \cong F(X)$

$$\varprojlim [X_m, BU] = [\varinjlim X_m, BU] = [(BU)^n, BU] \qquad [X, (BU)^n]$$

$$\| $$

$$\varprojlim Nat([-, X_m], [-, BU])$$

$$Nat(\varinjlim [-, X_m], \tilde{K})$$

$Nat(\tilde{K}^n, \tilde{K})$

$= Nat([-, (BU)^n], \tilde{K})$

cat. of compact spaces.

$= Nat(\varinjlim [-, X_m], \tilde{K})$

$= \varprojlim Nat([-, X_m], \tilde{K})$

$= \varprojlim \tilde{K}(X_m)$

$q = p^n$

$\Psi^q : \tilde{K}(X) \to \tilde{K}(X) \Rightarrow \sigma : BU \to BU$

$$\begin{array}{ccccc} F\Psi^q & \xrightarrow{\theta} & BU^I & & \gamma \\ \downarrow \phi & & \downarrow \Delta & & \downarrow \\ BU & \xrightarrow{(id, \sigma)} & BU \times BU & & (\gamma(0), \gamma(1)) \end{array}$$

$Map(I, BU) = [BU^n, BU]$

$$F\Psi^q = \{ (x, \gamma) \in BU \times BU^I \mid \gamma(0) = x, \ \gamma(1) = \sigma(x) \}$$

$= \{ \gamma : I \to BU \mid \gamma(1) = b \}$

$$\begin{array}{ccccccc} F\Psi^q & \xrightarrow{\theta} & BU^I & \xrightarrow{m} & BU^I \times_{BU} \{b\} & & \gamma \\ \downarrow \phi & & \downarrow \Delta & & \downarrow n & & \downarrow \\ BU & \xrightarrow{(id, \sigma)} & BU \times BU & \xrightarrow{d} & BU & & \gamma(0) \end{array}$$

$d : BU \times BU \to BU$ represents difference on $\tilde{K}$.

$d(x, b) = x, \ d(x, x) = b, \ \sigma(b) = b$

$$pull(d, n) = \{ (x, y, \gamma) \in BU \times BU \times BU^I \times_{BU} \{b\} \mid \gamma(0) = \underset{x-y}{d(x,y)}, \ \gamma(1) = \underset{b}{b} \}$$

$\downarrow + y$ $\qquad \gamma : x \to y$

$$\mathrm{pull}(d,n) \xrightarrow{+y} BU^I \quad \text{h.e.}$$

$$(x,y,\gamma) \longmapsto \gamma_{+y}$$

$$\Psi^q$$

$$d(\mathrm{id},\sigma)(x) = d(x,\sigma(x)) \qquad d(\mathrm{id},\sigma) \overset{\text{represents}}{\rightsquigarrow} 1-\Psi^q$$

$BU^I \times_{BO} \{b\}$ contractible $\xrightarrow{\text{endpoint}}$ contractible.

$$\begin{array}{ccc} F\Psi^q & \longrightarrow & * \\ \downarrow & & \downarrow \\ BU & \xrightarrow{1-\Psi^q} & BU \end{array}$$

$$F\Psi^q = \mathrm{hofib}(BU \xrightarrow{1-\Psi^q} BU) \qquad \mathrm{fib}(X\to Y) \to X \xrightarrow{f} Y$$

$$\Downarrow \qquad\qquad \Downarrow \text{LES on } \pi_q.$$

$$\cdots \to \pi_k(BU) \xrightarrow{1-\Psi^q} \pi_k(BU) \xrightarrow{\partial} \pi_{k-1}(F\Psi^q) \to \cdots$$

Bott Periodicity: $\pi_k(BU) = \pi_{k+2}(BU)$

$$\underset{\mathbb{Z}}{\overset{\pi_0(BU)}{\|}} \quad \underset{0}{\overset{\pi_1(BU)}{\|}} \qquad \pi_0(BU) = \tilde{K}(S^0) = \mathbb{Z}.$$

$$\tilde{K}(S^0) = [S^0, BU]$$

$$\Rightarrow \pi_k(F\Psi^q) = \begin{cases} 0 & \text{when } k \text{ even} \\ \mathbb{Z}/(q^k-1) & \text{when } k \text{ odd.} \end{cases}$$

$$Sq^k : H^n(X;\mathbb{F}_2) \longrightarrow H^{n+k}(X;\mathbb{F}_2)$$

$Sq^1, Sq^2, \ldots,$

A coalgebra

$A \to A\otimes A$

Algebra: $A\otimes A \to A$

Recall $[BU^n, BU] \longleftrightarrow \{K^n \to K \text{ natural trans}\}$.

$-: K\times K \longrightarrow K \quad \Rightarrow d: BU^2 \to BU$

$\Psi^q: K \longrightarrow K \quad \Rightarrow \sigma: BU \to BU$

$$\begin{tikzcd} F\Psi^q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & BU^I \arrow[d, "\Delta"] & \gamma \arrow[d, mapsto] \\ BU \arrow[r, "(id,\sigma)"] & BU\times BU & (\gamma(0),\gamma(1)) \end{tikzcd}$$

$\Rightarrow$ $F\Psi^q$ is the homotopy fiber of $d(id,\sigma): BU \to BU$

$$\Rightarrow \pi_k(F\Psi^q) = \begin{cases} \mathbb{Z}/q^i-1 & k=2i-1 \\ 0 & k=\text{even} \end{cases}$$

Goal: E complex rep. $\Rightarrow BG \to F\Psi^q$

$G \longrightarrow GL(\mathbb{C}^n)$ $\quad$ G Compact Lie Group (only use finite group)

$R(G) \longrightarrow [BG, F\Psi^q]$

<u>Lemma</u>. Let X be a space with $[X, \Omega BU] = 0$. Then the pushforward of $\phi: F\Psi^q \longrightarrow BU$, $(x,\gamma) \longmapsto x$ gives a bijection

$$\phi_*: [X, F\Psi^q] \longrightarrow [X, BU]^{\Psi^q}$$
$$(f, F) \longmapsto f$$

where $[X, BU]^{\Psi^q}$ is the space of homotopy classes invariant under composition with map $\sigma: BU \to BU$ representing Ψ^q.

proof:

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & & \\ & F\Psi^q \arrow[r] \arrow[d, "\phi"] & BU^I \arrow[d, "\Delta"] & \gamma \arrow[d, mapsto] \\ & BU \arrow[r] & BU\times BU & (\gamma(0),\gamma(1)) \end{tikzcd}$$

$$\overset{Map(I,BU)}{f_2}: X \to BU^I \longleftrightarrow F_2: X\times I \to BU$$
$$(x \mapsto \gamma) \longleftrightarrow ((x,t) \mapsto \gamma(t))$$

$f \leftrightarrow (f_1, f_2)$ s.t. $(id, \sigma)\circ f_1 = \Delta\circ f_2$

$(F_2(x,0), F_2(x,1))$
$= \Delta\circ f_2$
$= (id, \sigma)\circ f_1$
$= (f_1(x), \sigma\circ f_1(x))$

$f: X \longrightarrow F\Psi^q \iff (f, F)$ $\quad f: X\to BU$ & $F: X\times I \to BU$ representing $f \simeq \sigma\circ f$

ϕ_* is well-defined ✓

ϕ_* is surjective ✓

ϕ_* is injective: $f, g: X \longrightarrow F\Psi^q$ so that $\phi\circ f \simeq \phi\circ g$.

$\Rightarrow F: X\times I \to BU$ witnesses $\underline{\phi\circ f} \simeq \underline{\phi\circ g}$

$$F\Psi^q \hookrightarrow BU \xrightarrow{d(id,\sigma)} BU$$

Recall: $F\Psi^q$ is the fiber of $\underline{BU} \xrightarrow{d(id,\sigma)} BU$ $\qquad F\Psi^q \hookrightarrow BU \xrightarrow{d(id,\sigma)} BU$

$\phi: F\Psi^q \to BU$ we can understand it as inclusion up to homotopy.

$\Rightarrow d(id,\sigma)\circ F: X\times I \to BU$

$d(id,\sigma)\circ F: X \to BU^I$

$\forall x\in X$, $\underline{d(id,\sigma)}\circ F(x)$ is a based loop in BU.

$F|_0 = f(x) \quad F|_1 = g(x)$

$d(id,\sigma) = id - \sigma \qquad\qquad f \simeq \sigma\circ f$

$d(id,\sigma)\, F(x,0) = f(x) - \sigma\circ f(x) =$ basepoint.

$d(id,\sigma)\, F(x,1) = g(x) - \sigma\circ g(x)$ //

$d(id,\sigma)\circ F: X \to \Omega BU$

$[X, \Omega BU] = 0 \quad \Rightarrow \phi_*$ is injective.

$F \hookrightarrow \underline{E \xrightarrow{d} B}$ fibration

$H: X\times I \to E$ homotopy

$dH: X\times I \to B \Rightarrow dH: X \to \Omega B$, $[X, \Omega B] = 0$

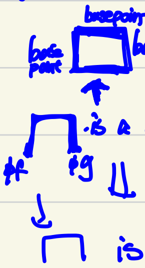
$$\begin{tikzcd} X\times I \arrow[r, "H"] \arrow[d] & E \arrow[d, "d"] \\ X\times I\times I \arrow[r, "\widetilde{dH}"'] \arrow[ur, dashed] & B \end{tikzcd}$$

$\Downarrow$

$dH: X\times I \to \Omega B$ constant space

$\Downarrow$

$\widetilde{dH}: X\times I\times I \to B$



basepoint

base point $\qquad$ basepoints $\to B$

lifting $\qquad$ is a homotopy in the basepoint

$f \qquad g \qquad \Downarrow$

is a homotopy in the fiber

$\square$

$\Psi^q \Rightarrow \sigma: BU \to BU$

G finite group $\qquad \rho: G \to \underline{GL(\mathbb{C}^n)}$

Given a rep. G on $\mathbb{C}^n \Rightarrow$ Vector bundle over BG s.t. an element h of $\pi_1(BG) = G$ acts on the fibers by $\rho(h)$.

$EG\times\mathbb{C}^n/\sim$ where $(x,v)\sim(g(x), \rho(g)^{-1}(v))$ $\forall x\in EG, v\in\mathbb{C}^n, g\in G$.

G topological group $\Rightarrow EG$ contractible space with free G-action

$Vect(BG) \leftrightarrow [BG, BU] \qquad\qquad \Rightarrow BG = EG/G$.

Classification of vectorbundles $\Rightarrow$ $\quad EG\times\mathbb{C}^n/\sim \quad \Rightarrow \rho_*: BG \to BU$

$\Rightarrow \qquad R(G) \to K(BG)$

$\rho \longmapsto \rho_*: BG \to BU$

Remark. If ρ is Ψ^q- invariant. $\Rightarrow$ $\rho_+ : [BG, BU]^{\Psi^q}$.

$\Downarrow$ lemma.

$\rho^{\#} : BG \to F\Psi^q$

Suppose Our representation is over finite field k with q elements. $\bar{k}$.

$\iota : \bar{k}^* \to \mathbb{C}^*$

For a rep. E over $\bar{k}$, we can consider Brauer character χ_E

$$\chi_E(g) = \sum \iota(\lambda_i)$$

$\{\lambda_i\}$ $\rho(g)$ eignvalues (with multiplicity) $E(g)$ auto. of $\bar{k}^n$.

Theorem 14. *Let $x \mapsto E(x)$ be a representation of a finite group by invertible $n \times n$ matrices $E(x)$ with coefficients in a finite field $\mathbb{F}$. Suppose that $\mathbb{F}$ contains the 'latent roots' (eigenvalues) $\lambda_1(x), \ldots, \lambda_n(x)$ of $E(x)$. Then if*

$$\iota : \mathbb{F}^* \to \mathbb{C}^*$$

is any homomorphic embedding, and S is any symmetric polynomial in n variables with integer coefficients, then the function

$$\chi_E(x) = S(\iota(\lambda_1(x)), \ldots, \iota(\lambda_n(x)))$$

is the character of a virtual complex representation of G.

Proof. [Gre55], Theorem 1. $\chi_E : G \to \mathbb{C}$ □

We consider $S = t_1 + \cdots + t_n$. Brauer lift

Hence Any rep. over $\bar{k}$ $\Rightarrow$ rep. over $\mathbb{C}$.

We consider E rep. over k.

$\Downarrow$

$\bar{E}$ rep over $\bar{k}$ $\bar{E} = E \otimes_k \bar{k}$.

$\rho(\Psi^q \chi)(g) = \chi(g^q)$ χ character of $\bar{E}$.

The set of eignvalues of automophism $E(g)$ of k^n is invariant under Frobenius iso. $x \longmapsto x^q$

$\Rightarrow$ Brauer lift of $\bar{E}$ & $\Psi^q E$ is the same.

E rep. over k $\Rightarrow$ rep over $\mathbb{C}$ is Ψ^q- invariant.

Lemma. G compact Lie Group $K_1(BG) = [BG, \Omega BU] = 0$

$\Rightarrow$ there is a well-defined between

$$R(G, \mathbb{F}_q) \longrightarrow [BG, F\Psi^q]$$

$$E \longmapsto E^{\#}$$

The Brauer lift of $k(\mu_\ell)$ $\quad \ell$ prime number $\quad k = \mathbb{F}_q$

C = multiplicative group of the finite field extension $k(\mu_\ell)$

$\Rightarrow \quad C \cong \mathbb{Z}/q^r - 1 \quad r$ is the minimal integer s.t. $\ell \mid q^r - 1$

C has natural rep. L over k by multiplication

$k(\mu_\ell)$ k-vector space of dim r

Brauer lifting of L: we have a vector space homo.

$$k(\mu_\ell) \otimes \bar{k} \to \bar{k}^r$$

given by $\quad z \otimes w \longmapsto (w, z^q w, z^{q^2} w, \cdots, z^{q^{r-1}} w)$

In $\bar{k}^r$, we see in the world of $\bar{k}^r$, we see the action of an element z is by multiplication $(1, z^q, \cdots, z^{q^{r-1}})$

$$\chi_L(z) = \sum_{k=0}^{r-1} l(z^{q^k})$$

$$l: C \to \mathbb{C}^* \quad l(1) = e^{2\pi i/(q^r-1)} := \zeta$$

$$W = \zeta \oplus \zeta^q \oplus \zeta^{q^2} \oplus \cdots \oplus \zeta^{q^{r-1}}$$

<u>Group.</u> $GL_n k$ it has a natural rep. over k by matrix multiplication

$$GL_n k \xrightarrow{id.} GL_n k.$$
$$\Downarrow$$
$$BGL_n k \longrightarrow F\Psi^q$$
$$\Downarrow n \to \infty$$
$$\underline{BGL(k)^+} \leftarrow BGL(k) \longrightarrow \underline{F\Psi^q}$$

$\uparrow$ this is a homology iso.

$$K_i(\mathbb{F}_q) = \pi_i(BGL(k)^+)$$

$$BGL(k)^+ \simeq F\Psi^q$$

$$H_*(BGL(k)^+) \cong H_*(BGL(k)) \cong H_*(F\Psi^q)$$