

VIP Refresher: Probabilities and Statistics

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Introduction to Probability and Combinatorics

□ **Sample space** – The set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S .

□ **Event** – Any subset E of the sample space is known as an event. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E , then we say that E has occurred.

□ **Axioms of probability** – For each event E , we denote $P(E)$ as the probability of event E occurring. By noting E_1, \dots, E_n mutually exclusive events, we have the 3 following axioms:

$$(1) \quad 0 \leq P(E) \leq 1 \quad (2) \quad P(S) = 1 \quad (3) \quad P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$$

□ **Permutation** – A permutation is an arrangement of r objects from a pool of n objects, in a given order. The number of such arrangements is given by $P(n, r)$, defined as:

$$P(n, r) = \frac{n!}{(n-r)!}$$

□ **Combination** – A combination is an arrangement of r objects from a pool of n objects, where the order does not matter. The number of such arrangements is given by $C(n, r)$, defined as:

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Remark: we note that for $0 \leq r \leq n$, we have $P(n, r) \geq C(n, r)$.

Conditional Probability

□ **Bayes' rule** – For events A and B such that $P(B) > 0$, we have:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Remark: we have $P(A \cap B) = P(A)P(B|A) = P(A|B)P(B)$.

□ **Partition** – Let $\{A_i, i \in [1, n]\}$ be such that for all i , $A_i \neq \emptyset$. We say that $\{A_i\}$ is a partition if we have:

$$\forall i \neq j, A_i \cap A_j = \emptyset \quad \text{and} \quad \bigcup_{i=1}^n A_i = S$$

Remark: for any event B in the sample space, we have $P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$.

□ **Extended form of Bayes' rule** – Let $\{A_i, i \in [1, n]\}$ be a partition of the sample space. We have:

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

□ **Independence** – Two events A and B are independent if and only if we have:

$$P(A \cap B) = P(A)P(B)$$

Random Variables

□ **Random variable** – A random variable, often noted X , is a function that maps every element in a sample space to a real line.

□ **Cumulative distribution function (CDF)** – The cumulative distribution function F , which is monotonically non-decreasing and is such that $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow +\infty} F(x) = 1$, is defined as:

$$F(x) = P(X \leq x)$$

Remark: we have $P(a < X \leq b) = F(b) - F(a)$.

□ **Probability density function (PDF)** – The probability density function f is the probability that X takes on values between two adjacent realizations of the random variable.

□ **Relationships involving the PDF and CDF** – Here are the important properties to know in the discrete (D) and the continuous (C) cases.

| Case | CDF F | PDF f | Properties of PDF |
|------|---------------------------------------|------------------------|--|
| (D) | $F(x) = \sum_{x_i \leq x} P(X = x_i)$ | $f(x_j) = P(X = x_j)$ | $0 \leq f(x_j) \leq 1$ and $\sum_j f(x_j) = 1$ |
| (C) | $F(x) = \int_{-\infty}^x f(y) dy$ | $f(x) = \frac{dF}{dx}$ | $f(x) \geq 0$ and $\int_{-\infty}^{+\infty} f(x) dx = 1$ |

□ **Variance** – The variance of a random variable, often noted $\text{Var}(X)$ or σ^2 , is a measure of the spread of its distribution function. It is determined as follows:

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

□ **Standard deviation** – The standard deviation of a random variable, often noted σ , is a measure of the spread of its distribution function which is compatible with the units of the actual random variable. It is determined as follows:

$$\sigma = \sqrt{\text{Var}(X)}$$

Expectation and Moments of the Distribution – Here are the expressions of the expected value $E[X]$, generalized expected value $E[g(X)]$, k^{th} moment $E[X^k]$ and characteristic function $\psi(\omega)$ for the discrete and continuous cases:

| Case | $E[X]$ | $E[g(X)]$ | $E[X^k]$ | $\psi(\omega)$ |
|------|--------------------------------------|---|--|--|
| (D) | $\sum_{i=1}^n x_i f(x_i)$ | $\sum_{i=1}^n g(x_i) f(x_i)$ | $\sum_{i=1}^n x_i^k f(x_i)$ | $\sum_{i=1}^n f(x_i) e^{i\omega x_i}$ |
| (C) | $\int_{-\infty}^{+\infty} x f(x) dx$ | $\int_{-\infty}^{+\infty} g(x) f(x) dx$ | $\int_{-\infty}^{+\infty} x^k f(x) dx$ | $\int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx$ |

Remark: we have $e^{i\omega x} = \cos(\omega x) + i \sin(\omega x)$.

Revisiting the k^{th} moment – The k^{th} moment can also be computed with the characteristic function as follows:

$$E[X^k] = \frac{1}{i^k} \left[\frac{\partial^k \psi}{\partial \omega^k} \right]_{\omega=0}$$

Transformation of random variables – Let the variables X and Y be linked by some function. By noting f_X and f_Y the distribution function of X and Y respectively, we have:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Leibniz integral rule – Let g be a function of x and potentially c , and a, b boundaries that may depend on c . We have:

$$\frac{\partial}{\partial c} \left(\int_a^b g(x) dx \right) = \frac{\partial b}{\partial c} \cdot g(b) - \frac{\partial a}{\partial c} \cdot g(a) + \int_a^b \frac{\partial g}{\partial c}(x) dx$$

Chebyshev's inequality – Let X be a random variable with expected value μ and standard deviation σ . For $k, \sigma > 0$, we have the following inequality:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Jointly Distributed Random Variables

Conditional density – The conditional density of X with respect to Y , often noted $f_{X|Y}$, is defined as follows:

$$f_{X|Y}(x) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Independence – Two random variables X and Y are said to be independent if we have:

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Marginal density and cumulative distribution – From the joint density probability function f_{XY} , we have:

| Case | Marginal density | Cumulative function |
|------|---|---|
| (D) | $f_X(x_i) = \sum_j f_{XY}(x_i, y_j)$ | $F_{XY}(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} f_{XY}(x_i, y_j)$ |
| (C) | $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy$ | $F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(x', y') dx' dy'$ |

Distribution of a sum of independent random variables – Let $Y = X_1 + \dots + X_n$ with X_1, \dots, X_n independent. We have:

$$\psi_Y(\omega) = \prod_{k=1}^n \psi_{X_k}(\omega)$$

Covariance – We define the covariance of two random variables X and Y , that we note σ_{XY}^2 or more commonly $\text{Cov}(X, Y)$, as follows:

$$\text{Cov}(X, Y) \triangleq \sigma_{XY}^2 = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$$

Correlation – By noting σ_X, σ_Y the standard deviations of X and Y , we define the correlation between the random variables X and Y , noted ρ_{XY} , as follows:

$$\rho_{XY} = \frac{\sigma_{XY}^2}{\sigma_X \sigma_Y}$$

Remarks: For any X, Y , we have $\rho_{XY} \in [-1, 1]$. If X and Y are independent, then $\rho_{XY} = 0$.

Main distributions – Here are the main distributions to have in mind:

| Type | Distribution | PDF | $\psi(\omega)$ | $E[X]$ | $\text{Var}(X)$ |
|------|---|--|--|---------------------|-----------------------|
| (D) | $X \sim \mathcal{B}(n, p)$ Binomial | $P(X = x) = \binom{n}{x} p^x q^{n-x}$ $x \in \llbracket 0, n \rrbracket$ | $(pe^{i\omega} + q)^n$ | np | npq |
| | $X \sim \text{Po}(\mu)$ Poisson | $P(X = x) = \frac{\mu^x}{x!} e^{-\mu}$ $x \in \mathbb{N}$ | $e^{\mu(e^{i\omega} - 1)}$ | μ | μ |
| (C) | $X \sim \mathcal{U}(a, b)$ Uniform | $f(x) = \frac{1}{b-a}$ $x \in [a, b]$ | $\frac{e^{i\omega b} - e^{i\omega a}}{(b-a)i\omega}$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ |
| | $X \sim \mathcal{N}(\mu, \sigma)$ Gaussian | $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $x \in \mathbb{R}$ | $e^{i\omega\mu - \frac{1}{2}\omega^2\sigma^2}$ | μ | σ^2 |
| | $X \sim \text{Exp}(\lambda)$ Exponential | $f(x) = \lambda e^{-\lambda x}$ $x \in \mathbb{R}_+$ | $\frac{1}{1 - \frac{i\omega}{\lambda}}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ |

Parameter estimation

□ **Random sample** – A random sample is a collection of n random variables X_1, \dots, X_n that are independent and identically distributed with X .

□ **Estimator** – An estimator $\hat{\theta}$ is a function of the data that is used to infer the value of an unknown parameter θ in a statistical model.

□ **Bias** – The bias of an estimator $\hat{\theta}$ is defined as being the difference between the expected value of the distribution of $\hat{\theta}$ and the true value, i.e.:

$$\text{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta$$

Remark: an estimator is said to be unbiased when we have $E[\hat{\theta}] = \theta$.

□ **Sample mean and variance** – The sample mean and the sample variance of a random sample are used to estimate the true mean μ and the true variance σ^2 of a distribution, are noted \bar{X} and s^2 respectively, and are such that:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad s^2 = \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

□ **Central Limit Theorem** – Let us have a random sample X_1, \dots, X_n following a given distribution with mean μ and variance σ^2 , then we have:

$$\bar{X} \underset{n \rightarrow +\infty}{\sim} \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$