Chapter 2
Parametrized Differential Equations

2.1 Parametrized Variational Problems

Let us first introduce a (suitably regular) physical domain $\Omega \in \mathbb{R}^d$ with boundary $\partial\Omega$, where $d = 1, 2$, or 3 is the spatial dimension. We shall consider only real-valued field variables. However, both scalar-valued (e.g., temperature in a Poisson conduction problems) and vector-valued (e.g., displacement in a linear elasticity problem) field variables $w : \Omega \to \mathbb{R}^{d_v}$ may be considered: here d_v denotes the dimension of the field variable; for scalar-valued fields, $d_v = 1$, while for vector-valued fields, $d_v = d$. We also introduce (boundary measurable) segments of $\partial\Omega$, Γ_i^D, $1 \le i \le d_v$, over which we will impose Dirichlet boundary conditions on the components of the field variable.

Let us also introduce the scalar spaces $\mathbb{V}_i$, $1 \le i \le d_v$,

$$\mathbb{V}_i = \mathbb{V}_i(\Omega) = \{v \in H^1(\Omega) \mid v|_{\Gamma_i^D} = 0\}, \qquad 1 \le i \le d_v.$$

In general $H_0^1(\Omega) \subset \mathbb{V}_i \subset H^1(\Omega)$, and for $\Gamma_i^D = \partial\Omega$, $\mathbb{V}_i = H_0^1(\Omega)$. We construct the space in which our vector-valued field variable shall reside as the Cartesian product $\mathbb{V} = \mathbb{V}_1 \times \cdots \mathbb{V}_{d_v}$; a typical element of $\mathbb{V}$ is denoted $w = (w_1, \ldots, w_{d_v})$. We equip $\mathbb{V}$ with an inner product $(w, v)_{\mathbb{V}}$, $\forall\, w, v, \in \mathbb{V}$, and the induced norm $\|w\|_{\mathbb{V}} = \sqrt{(w, w)_{\mathbb{V}}}$, $\forall\, w \in \mathbb{V}$: any inner product which induces a norm equivalent to the $(H^1(\Omega))^{d_v}$ norm is admissible. Therefore, $\mathbb{V}$ is a Hilbert space.

We finally introduce a (suitably regular) closed parameter domain $\mathbb{P} \in \mathbb{R}^P$, a typical parameter (or input) point, or vector, or P-tuple, denoted as $\mu = (\mu_{[1]}, \mu_{[2]}, \ldots, \mu_{[P]})$. We may thus define our parametric field variable as $u \equiv (u_1, \ldots, u_{d_v})\colon \mathbb{P} \to \mathbb{V}$; here, $u(\mu)$ denotes the field for parameter value $\mu \in \mathbb{P}$.

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2.1.1 Parametric Weak Formulation

Let us briefly introduce the general stationary problem in an abstract form. All of the working examples in this text can be cast in this framework. We are given parametrized linear forms $f : \mathbb{V} \times \mathbb{P} \to \mathbb{R}$ and $\ell : \mathbb{V} \times \mathbb{P} \to \mathbb{R}$ where the linearity is with respect to the first variable, and a parametrized bilinear form $a : \mathbb{V} \times \mathbb{V} \times \mathbb{P} \to \mathbb{R}$ where the bilinearity is with respect to the first two variables. Examples of such parametrized linear forms are given in the two examples of Sect. 2.3. The abstract formulation reads: given $\mu \in \mathbb{P}$, we seek $u(\mu) \in \mathbb{V}$ such that

$$a(u(\mu), v; \mu) = f(v; \mu), \qquad \forall\, v \in \mathbb{V}, \tag{2.1}$$

and evaluate

$$s(\mu) = \ell(u(\mu); \mu). \tag{2.2}$$

Here s is an output of interest, $s\colon \mathbb{P} \to \mathbb{R}$ is the input (parameter)-output relationship, and ℓ takes the role of a linear "output" functional which links the input to the output through the field variable $u(\mu)$.

In this initial part of the text we assume that problems of interest are compliant. A compliant problem of the form (2.1)–(2.2) satisfies two conditions:

(i) $\ell(\cdot\,; \mu) = f(\cdot\,; \mu)$, $\forall\, \mu \in \mathbb{P}$—the output functional and load/source functional are identical.
(ii) The bilinear form $a(\cdot\,,\, \cdot\,; \mu)$ is symmetric for any parameter value $\mu \in \mathbb{P}$.

Together, these two assumptions greatly simplify the formulation, the a priori convergence theory for the output, and the a posteriori error estimation for the output. Though quite restrictive, there are many interesting problems fulfilling this requirement across mechanics and physics, e.g., material properties, geometrical parametrization, etc. However, we return to the more general non-compliant case in the final Chap. 6.

2.1.2 Inner Products, Norms and Well-Posedness of the Parametric Weak Formulation

The Hilbert space $\mathbb{V}$ is equipped with an intrinsic norm $\|\cdot\|_{\mathbb{V}}$. In many cases this norm coincides with, or is equivalent to, the norm induced by the bilinear form a for a fixed parameter $\bar{\mu} \in \mathbb{P}$:

$$\begin{aligned} (w, v)_{\mathbb{V}} &= a(w, v; \bar{\mu}), \qquad \forall\, w, v \in \mathbb{V}, \\ \|v\|_{\mathbb{V}} &= \sqrt{a(w, w; \bar{\mu})}, \quad \forall\, w \in \mathbb{V}. \end{aligned} \tag{2.3}$$

The well-posedness of the abstract problem formulation (2.1) can be established by the Lax-Milgram theorem [1], see also the Appendix. In order to state a well-posed problem for all parameter values $\mu \in \mathbb{P}$, we assume in addition to the bilinearity and the linearity of the parametrized forms $a(\cdot, \cdot; \mu)$ and $f(\cdot; \mu)$, that

(i) $a(\cdot, \cdot; \mu)$ is coercive and continuous for all $\mu \in \mathbb{P}$ with respect to the norm $\|\cdot\|_{\mathbb{V}}$, i.e., for every $\mu \in \mathbb{P}$, there exists a positive constant $\alpha(\mu) \geq \alpha > 0$ and a finite constant $\gamma(\mu) \leq \gamma < \infty$ such that

$$a(v, v; \mu) \geq \alpha(\mu) \, \|v\|_{\mathbb{V}}^2 \qquad \text{and} \qquad a(w, v; \mu) \leq \gamma(\mu) \, \|w\|_{\mathbb{V}} \, \|v\|_{\mathbb{V}}, \tag{2.4}$$

for all $w, v \in \mathbb{V}$.

(ii) $f(\cdot; \mu)$ is continuous for all $\mu \in \mathbb{P}$ with respect to the norm $\|\cdot\|_{\mathbb{V}}$, i.e., for every $\mu \in \mathbb{P}$, there exists a constant $\delta(\mu) \leq \delta < \infty$ such that

$$f(v; \mu) \leq \delta(\mu) \, \|v\|_{\mathbb{V}}, \qquad \forall \, v \in \mathbb{V}.$$

The coercivity and continuity constants of $a(\cdot, \cdot; \mu)$ over $\mathbb{V}$ are, respectively, defined as

$$\alpha(\mu) = \inf_{v \in \mathbb{V}} \frac{a(v, v; \mu)}{\|v\|_{\mathbb{V}}^2}, \qquad \text{and} \qquad \gamma(\mu) = \sup_{w \in \mathbb{V}} \sup_{v \in \mathbb{V}} \frac{a(w, v; \mu)}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}}, \tag{2.5}$$

for every $\mu \in \mathbb{P}$.

Finally, we also may introduce the usual energy inner product and the induced energy norm as

$$(w, v)_\mu = a(w, v; \mu), \qquad \forall \, w, v \in \mathbb{V}\,, \tag{2.6}$$

$$\|w\|_\mu = \sqrt{a(w, w; \mu)}, \quad \forall \, w \in \mathbb{V}\,, \tag{2.7}$$

respectively; note that these quantities are parameter-dependent. Thanks to the coercivity and continuity assumptions on a, it is clear that (2.6) constitutes a well-defined inner product and (2.7) an induced norm equivalent to the $\|\cdot\|_{\mathbb{V}}$-norm.

2.2 Discretization Techniques

This section supplies an abstract framework of the discrete approximations of the parametric weak formulation (2.1) for conforming approximations, i.e., there is a discrete approximation space $\mathbb{V}_\delta$ in which the approximate solution is sought. This is a subset of $\mathbb{V}$, i.e., $\mathbb{V}_\delta \subset \mathbb{V}$. The conforming nature of the approximation space $\mathbb{V}_\delta$ is an essential assumption in the upcoming presentation of the method in Chap. 3, and for the error estimation discussed in Chap. 4.

As an example, the approximation space $\mathbb{V}_\delta$ can be constructed as a standard finite element method based on a triangulation and using piece-wise linear basis functions. Other examples include spectral methods or higher order finite elements, provided only that the formulation is based on a variational approach.

We denote the dimension of the discrete space $\mathbb{V}_\delta$ by $N_\delta = \dim(\mathbb{V}_\delta)$ and equip $\mathbb{V}_\delta$ with a basis $\{\varphi_i\}_{i=1}^{N_\delta}$. For each $\mu \in \mathbb{P}$, the discrete problem consists of finding $u_\delta(\mu) \in \mathbb{V}_\delta$ such that

$$a(u_\delta(\mu), v_\delta; \mu) = f(v_\delta; \mu), \quad \forall\, v_\delta \in \mathbb{V}_\delta, \tag{2.8}$$

and evaluate

$$s_\delta(\mu) = \ell(u_\delta(\mu); \mu).$$

This problem is denoted as the truth problem. It is a solver of choice in the case where the solution needs to be computed for one parameter value only and it is assumed that this solution, called the truth approximation, can be achieved with as high accuracy as desired.

The computation of the truth solution is, however, potentially very expensive since the space $\mathbb{V}_\delta$ may involved many degrees of freedom N_δ to achieve the desired accuracy level. On the other hand, it provides an accurate approximation $u_\delta(\mu)$ in the sense that the error $\|u(\mu) - u_\delta(\mu)\|_{\mathbb{V}}$ is acceptably small. This model is sometimes also referred to as high fidelity model.

We note that due to the coercivity and continuity of the bilinear form, and the conformity of the approximation space, we ensure the Galerkin orthogonality

$$a\big(u(\mu) - u_\delta(\mu), v_\delta; \mu\big) = 0, \quad \forall\, v_\delta \in \mathbb{V}_\delta,$$

to recover Cea's lemma. Indeed, let $v_\delta \in \mathbb{V}_\delta$ be arbitrary and observe that

$$\|u(\mu) - u_\delta(\mu)\|_{\mathbb{V}} \le \|u(\mu) - v_\delta\|_{\mathbb{V}} + \|v_\delta - u_\delta(\mu)\|_{\mathbb{V}},$$

by the triangle inequality. Furthermore, it holds that

$$\begin{aligned} \alpha(\mu)\|v_\delta - u_\delta(\mu)\|_{\mathbb{V}}^2 &\le a\big(v_\delta - u_\delta(\mu), v_\delta - u_\delta(\mu); \mu\big) = a\big(v_\delta - u(\mu), v_\delta - u_\delta(\mu); \mu\big) \\ &\le \gamma(\mu)\, \|v_\delta - u(\mu)\|_{\mathbb{V}}\, \|v_\delta - u_\delta(\mu)\|_{\mathbb{V}} \end{aligned}$$

by applying the coercivity assumption, the Galerkin orthogonality and the continuity assumption to obtain

$$\|u(\mu) - u_\delta(\mu)\|_{\mathbb{V}} \le \left(1 + \frac{\gamma(\mu)}{\alpha(\mu)}\right) \inf_{v_\delta \in \mathbb{V}_\delta} \|u(\mu) - v_\delta\|_{\mathbb{V}}.$$

Linear algebra box: The truth solver

We denote the stiffness matrix and the right hand side of the truth problem by $\mathbf{A}_\delta^\mu \in \mathbb{R}^{N_\delta \times N_\delta}$ and $\mathrm{f}_\delta^\mu \in \mathbb{R}^{N_\delta}$, respectively. Further, we denote by $\mathbf{M}_\delta \in \mathbb{R}^{N_\delta \times N_\delta}$ the matrix associated with the inner product $(\cdot,\cdot)_\mathbb{V}$ of $\mathbb{V}_\delta$, defined as

$$(\mathbf{M}_\delta)_{ij} = (\varphi_j, \varphi_i)_\mathbb{V}, \qquad (\mathbf{A}_\delta^\mu)_{ij} = a(\varphi_j, \varphi_i; \mu), \qquad \text{and} \qquad (\mathrm{f}_\delta^\mu)_i = f(\varphi_i; \mu),$$

for all $1 \leq i, j \leq N_\delta$. We recall that $\{\varphi_i\}_{i=1}^{N_\delta}$ is of a basis of $\mathbb{V}_\delta$. Then, the truth problem reads: for each $\mu \in \mathbb{P}$, find $\mathrm{u}_\delta^\mu \in \mathbb{R}^{N_\delta}$ s.t.

$$\mathbf{A}_\delta^\mu \, \mathrm{u}_\delta^\mu = \mathrm{f}_\delta^\mu.$$

Then, evaluate the output functional (in the compliant case)

$$s_\delta(\mu) = (\mathrm{u}_\delta^\mu)^T \, \mathrm{f}_\delta^\mu.$$

The field approximation $u_\delta(\mu)$ is obtained by $u_\delta(\mu) = \sum_{i=1}^{N_\delta} (\mathrm{u}_\delta^\mu)_i \, \varphi_i$ where $(\mathrm{u}_\delta^\mu)_i$ denotes the i-th coefficient of the vector u_δ^μ.

This implies that the approximation error $\|u(\mu) - u_\delta(\mu)\|_\mathbb{V}$ is closely related to the best approximation error of $u(\mu)$ in the approximation space $\mathbb{V}_\delta$ through the constants $\alpha(\mu)$, $\gamma(\mu)$. More details on the numerical analysis of unparametrized problems can be found in the Appendix.

The linear algebra box The truth solver illustrates the implementation of the truth solver on the level of linear algebra. The size of the unknown vector is N_δ and the size of the stiffness matrix is $N_\delta \times N_\delta$. Depending on the solver of choice to invert the linear system and the properties of the stiffness matrix, the operation count of the map $\mu \rightarrow s_\delta(\mu)$ is $\mathcal{O}({N_\delta}^\alpha)$, for $\alpha \geq 1$, but in any case dependent of N_δ.

2.3 Toy Problems

We want to consider simple parametrized examples, intended to be representative of larger classes of problems, to motivate the reader. We consider two model problems: a (steady) heat conduction problem with conductivity and heat flux as parameters; and a linear elasticity problem with load traction conditions as parameters.

We will present generalizations of these examples later in this text in Chaps. 5 and 6. We thus limit ourselves to the following problems only for the introduction of the topic and present some advanced examples later.

2.3.1 Illustrative Example 1: Heat Conduction Part 1

We consider a steady heat conduction problem (we refer the reader in need of more information about thermal problems to [2]) in a two-dimensional domain $\Omega = (-1, 1) \times (-1, 1)$ with outward pointing unit normal n on $\partial\Omega$. The boundary $\partial\Omega$ is split into three parts: the bottom $\Gamma_{\mathrm{base}} = (-1, 1) \times \{-1\}$, the top $\Gamma_{\mathrm{top}} = (-1, 1) \times \{1\}$ and the sides $\Gamma_{\mathrm{side}} = \{\pm 1\} \times (-1, 1)$. The normalized thermal conductivity is denoted by κ. Let Ω_0 be a disk centered at the origin of radius $r_0 = 0.5$ and define $\Omega_1 = \Omega \backslash \overline{\Omega_0}$. Consider the conductivity κ to be constant on Ω_0 and Ω_1, i.e.

$$\kappa|_{\Omega_0} = \kappa_0 \qquad \text{and} \qquad \kappa|_{\Omega_1} = 1.$$

The geometrical set-up is illustrated in Fig. 2.1.

We consider $P = 2$ parameters and $\mathbb{P} = [\mu_{[1]}^{\min}, \mu_{[1]}^{\max}] \times [\mu_{[2]}^{\min}, \mu_{[2]}^{\max}]$ in this model problem. The first one is related to the conductivity in Ω_0, i.e. $\mu_{[1]} = \kappa_0$. We can write $\kappa_\mu = \mathbf{1}_{\Omega_1} + \mu_{[1]} \mathbf{1}_{\Omega_0}$, where $\mathbf{1}$ is the characteristic function of the corresponding set in the sub-script. The second parameter $\mu_{[2]}$ reflects the constant heat flux over Γ_{base}. Our parameter vector is thus given by $\mu = (\mu_{[1]}, \mu_{[2]})$.

The scalar field variable $u(\mu)$ is the temperature that satisfies Poisson's equation in Ω; homogeneous Neumann (zero flux, or insulated) conditions on the side boundaries Γ_{side}; homogeneous Dirichlet (temperature) conditions on the top boundary Γ_{top}; and parametrized Neumann conditions along the bottom boundary Γ_{base}.

The output of interest is the average temperature over the base made up by Γ_{base}. Note that we consider a non-dimensional formulation in which the number of physical parameters has been kept to a minimum.

The strong formulation of this parametrized problem is stated as: for some parameter value $\mu \in \mathbb{P}$, find $u(\mu)$ such that

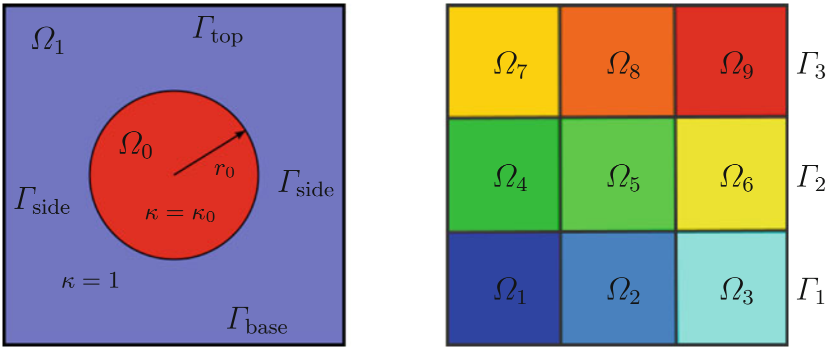

Fig. 2.1 Geometrical set-up (*left*) of the heat conductivity problem, illustrative Example 1, and (*right*) the elasticity problem, illustrative Example 2

$$\begin{cases} \nabla \cdot \kappa_\mu \nabla u(\mu) = 0 & \text{in } \Omega, \\ u(\mu) = 0 & \text{on } \Gamma_{\text{top}}, \\ \kappa_\mu \nabla u(\mu) \cdot n = 0 & \text{on } \Gamma_{\text{side}}, \\ \kappa_\mu \nabla u(\mu) \cdot n = \mu_{[2]} & \text{on } \Gamma_{\text{base}}. \end{cases}$$

The output of interest is given as

$$s(\mu) = \ell(u(\mu); \mu) = \mu_{[2]} \int_{\Gamma_{\text{base}}} u(\mu).$$

We recall that the function space associated with this set of boundary conditions is given by $\mathbb{V} = \{v \in H^1(\Omega) \mid v|_{\Gamma_{\text{top}}} = 0\}$: the Dirichlet boundary conditions are essential; the Neumann boundary conditions are natural.

The weak parametrized formulation then reads: for some parameter $\mu \in \mathbb{P}$, find $u(\mu) \in \mathbb{V}$ such that

$$a(u(\mu), v; \mu) = f(v; \mu) \qquad \forall\, v \in \mathbb{V},$$

with

$$a(w, v; \mu) = \int_\Omega \kappa_\mu \nabla w \cdot \nabla v \qquad \text{and} \qquad f(v; \mu) = \mu_{[2]} \int_{\Gamma_{\text{base}}} v,$$

for all $v, w \in \mathbb{V}$. We endow the space $\mathbb{V}$ with the scalar product

$$(v, w)_{\mathbb{V}} = a(v, w; \bar{\mu}) = \int_\Omega \nabla w \cdot \nabla v, \qquad \forall\, w, v \in \mathbb{V},$$

for $\bar{\mu} = (\bar{\mu}_{[1]}, \bar{\mu}_{[2]})$ such that $\bar{\mu}_{[1]} = 1$. For the problem to be well-posed, we assume that $\mu_{[1]}^{\min} > 0$ so that $\kappa_\mu \geq \min(1, \mu_{[1]}^{\min}) > 0$ and coercivity of the bilinear form a follows. Further, continuity of the forms a and f can be easily obtained using the Cauchy-Schwarz inequality; and linearity and bilinearity can be easily verified as well. We can therefore apply the Lax-Milgram theorem to guarantee existence and uniqueness of the solution $u(\mu) \in \mathbb{V}$ for any parameter value $\mu \in \mathbb{P}$.

A conforming discretization introduces a finite-dimensional subspace $\mathbb{V}_\delta \subset \mathbb{V}$, for instance a standard finite element space. Following the Galerkin approach we obtain the following discrete problem: for some parameter $\mu \in \mathbb{P}$, find $u_\delta(\mu) \in \mathbb{V}_\delta$ such that

$$a(u_\delta(\mu), v_\delta; \mu) = f(v_\delta : \mu) \qquad \forall\, v_\delta \in \mathbb{V}_\delta.$$

In the following illustration, the finite element method, employing piece-wise linear elements, has been chosen as the truth model. The mesh is illustrated in Fig. 2.2 (left) featuring 812 elements. The chosen ranges for the parameters are

$$\mu = (\mu_{[1]}, \mu_{[2]}) \in \mathbb{P} = [0.1, 10] \times [-1, 1].$$

In Fig. 2.3, four representative solutions—snapshots—are depicted for different values of the parameters.

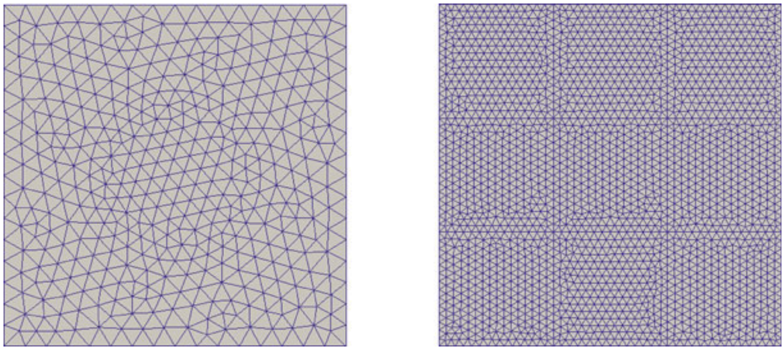

Fig. 2.2 Finite element mesh for Example 1 (*left*) and Example 2 (*right*)

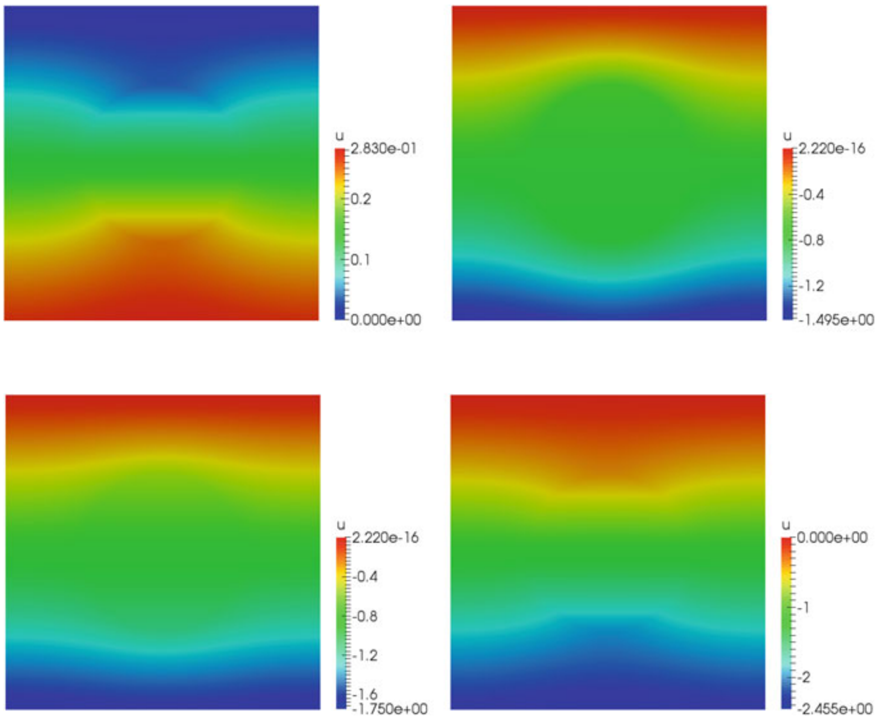

Fig. 2.3 Four different representative solutions for the parametrized conductivity problem (Example 1)

2.3.2 Illustrative Example 2: Linear Elasticity Part 1

We consider a linear elasticity example [3, 4] in the two-dimensional domain $\Omega = (0, 1) \times (0, 1)$, shown in Fig. 2.1 with 9 mini-blocks Ω_i, where the Young's modulus on each mini-block is denoted by E_i and the Poisson's ratio is set to $\nu = 0.30$. The outward pointing unit normal on $\partial\Omega$ is denoted by n.

We consider $P = 11$ parameters: the 8 Young's moduli with respect to the reference value $E = E_9 = 10$ set in Ω_9 and the 3 horizontal traction/compression load conditions at the right border of the elastic block. Our parameter vector is thus given by $\mu = (\mu_{[1]}, \ldots, \mu_{[P]})$ and we choose for our parameter domain $\mathbb{P} = [\mu_{[1]}^{\min}, \mu_{[1]}^{\max}] \times \cdots \times [\mu_{[P]}^{\min}, \mu_{[P]}^{\max}]$ where

$$[\mu_{[p]}^{\min}, \mu_{[p]}^{\max}] = [1, 100], \qquad p = 1, \ldots, 8,$$
$$[\mu_{[p]}^{\min}, \mu_{[p]}^{\max}] = [-1, 1], \qquad p = 9, \ldots, 11.$$

The local Young's moduli are given by $E_i = \mu_{[i]} E$.

Our vector field variable $u(\mu) = (u_1(\mu), u_2(\mu))$ is the displacement of the elastic block under the applied load: the displacement satisfies the plane-strain linear elasticity equations in Ω in combination with the following boundary conditions: homogeneous Neumann (load-free) conditions are imposed on the top and bottom boundaries Γ_{top} and Γ_{base} of the block; homogeneous Dirichlet (displacement) conditions on the left boundary Γ_{left} (the structure is clamped); and parametrized inhomogeneous Neumann conditions on the right boundary $\Gamma_{\text{right}} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with zero shear. The non-trivial (inhomogeneous) boundary conditions are summarized as follows

$$\begin{cases} n \cdot u = \mu_{[9]} & \text{on } \Gamma_1, \\ n \cdot u = \mu_{[10]} & \text{on } \Gamma_2, \\ n \cdot u = \mu_{[11]} & \text{on } \Gamma_3, \end{cases}$$

representing traction loads. The output of interest $s(\mu)$ is the integrated horizontal (traction/compression) displacement over the full loaded boundary Γ_{right}, given by

$$s(\mu) = \int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} u_1(\mu).$$

This corresponds to the compliant situation as we will see later.

The function space associated with this set of boundary conditions is given by

$$\mathbb{V} = \left\{ v \in (H^1(\Omega))^2 \;\middle|\; v|_{\Gamma_{\text{left}}} = 0 \right\}.$$

Hence, the Dirichlet interface and boundary conditions are essential and the Neumann interface and boundary conditions are natural. We then define the load (and also output) functional

$$f_i(v; \mu) = \int_{\Gamma_i} v_1, \qquad \forall\, v = (v_1, v_2) \in \mathbb{V} \quad \text{and} \quad i = 1, 2, 3,$$

such that

$$f(v; \mu) = \sum_{i=1}^{3} \mu_{[i+8]} f_i(v; \mu).$$

The bilinear form associated with the left-hand-side of the problem is given by:

$$a(w, v; \mu) = \sum_{p=1}^{8} \mu_{[p]} E \int_{\Omega_p} \frac{\partial v_i}{\partial x_j} C_{ijkl} \frac{\partial w_k}{\partial x_l} + E \int_{\Omega_9} \frac{\partial v_i}{\partial x_j} C_{ijkl} \frac{\partial w_k}{\partial x_l},$$

where $\mu_{[p]}$ is the ratio between the Young modulus in Ω_p and Ω_9 and Einstein's summation convention is used for the indices i, j, k and l.

For our isotropic material, the elasticity tensor is given by

$$C_{ijkl} = \lambda^1 \delta_{ij} \delta_{kl} + \lambda^2 \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right),$$

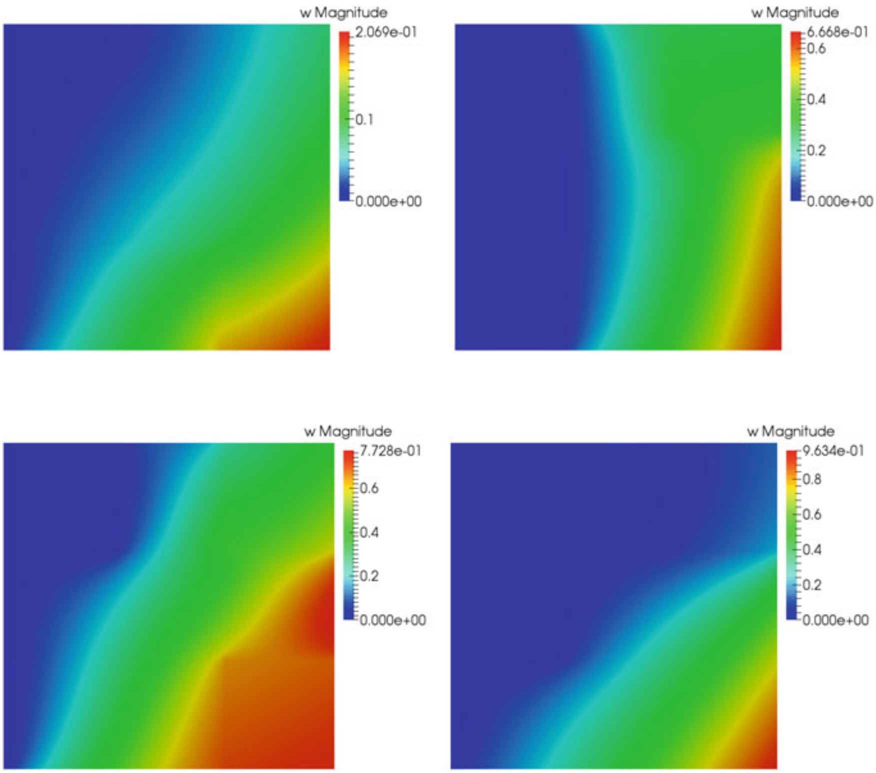

Fig. 2.4 Four representative solutions for the laterally loaded (traction/compression) elastic block (Example 2)

where

$$\lambda^1 = \frac{\nu}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \lambda^2 = \frac{1}{2(1+\nu)} \ ,$$

are the Lamé constants for plane strain. We recall that the Poisson's ratio is set to $\nu = 0.30$. The weak form is then given by (2.1)–(2.2). The inner product is specified by (2.3),

$$(v, w)_{\mathbb{V}} = a(v, w; \bar{\mu}) = E \int_{\Omega} \frac{\partial v_i}{\partial x_j} \, C_{ijkl} \, \frac{\partial w_k}{\partial x_l}, \qquad \forall \, w, v \in \mathbb{V},$$

for some $\bar{\mu} \in \mathbb{P}$ satisfying $\bar{\mu}_{[p]} = 1$, for all $1 \leq p \leq 8$.

We can now readily verify our hypotheses. First, it is standard to confirm that f is indeed bounded. Second, we readily confirm by inspection that a is symmetric, and we further verify by application of the Korn inequality [5] and the Cauchy-Schwarz inequality that a is coercive and continuous, respectively.

The finite element method, employing piece-wise elements, has been chosen as the truth model. In Fig. 2.2 (right) the mesh is represented, featuring 4,152 elements. In Fig. 2.4, four representative solutions are illustrated.

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