



Numerical analysis/Mathematical problems in mechanics

# An improvement on geometrical parameterizations by transfinite maps



*Une amélioration pour les paramétrages géométriques par les transformations transfinies*

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## ARTICLE INFO

### Article history:

Received 31 July 2013

Accepted after revision 12 December 2013

Available online 10 January 2014

Presented by Philippe G. Ciarlet

## ABSTRACT

We present a method to generate a non-affine transfinite map from a given reference domain to a family of deformed domains. The map is a generalization of the Gordon–Hall transfinite interpolation approach. It is defined globally over the reference domain. Once we have computed some functions over the reference domain, the map can be generated by knowing the parametric expressions of the boundaries of the deformed domain. Being able to define a suitable map from a reference domain to a desired deformation is useful for the management of parameterized geometries.

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## R É S U M É

Nous présentons une méthode pour générer une transformation paramétrisée d'une géométrie de référence vers une famille de géométries déformées. La transformation est une généralisation de l'approche d'interpolation transfinie de Gordon–Hall et est définie globalement sur le domaine de référence. Une fois qu'on a calculé certaines fonctions sur le domaine de référence, la transformation peut être générée à partir des paramétrisations des bords du domaine déformé. Il est utile pour le maniement des géométries déformées d'être capable de définir une transformation appropriée d'un domaine de référence vers une déformation souhaitée.

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## Version française abrégée

De nombreux phénomènes physiques peuvent être décrit par des équations aux dérivées partielles (EDP). Les EDP paramétrisées sont des EDP dans lesquelles les propriétés physiques, les caractéristiques du matériaux, les propriétés géométriques (comme les déformations de domaine), ou encore les conditions au bord (comme les charges) dépendent d'un paramètre  $\mu$ . En faisant varier le paramètre  $\mu$ , les EDP paramétrisées permettent de modéliser une famille de systèmes physiques issus d'une même équation. Dans cet article, nous présentons une transformation permettant de manipuler un paramètre  $\mu$  qui rend compte de la déformation du domaine. Pour de nombreuses techniques numériques permettant de résoudre des EDP paramétrisées, il est utile de fixer un domaine de référence puis de définir une transformation entre ce domaine et le domaine déformé. En particulier, la méthode des bases réduites (MBR) est une technique efficace pour approximer les solutions d'EDP paramétrisées pour différentes valeurs de  $\mu$  de manière rapide et fiable. Dans cette méthode, les transformations

transfinies jouent un rôle important [3,7]. Dans cet article, nous proposons une nouvelle extension de la transformation transfinie (TT) présentée dans [5]. Cette extension peut être vue comme une généralisation de l'approche d'interpolation transfinie de Gordon–Hall pour les quadrilatères [2]. La transformation transfinie induit une paramétrisation géométrique non affine telle que la méthode d'interpolation empirique [1] est nécessaire pour retrouver l'affinité des formes linéaires et bilinéaires des problèmes considérés. La transformation transfinie généralisée proposée dans [5,6] et utilisée récemment dans [4] présente quelques problèmes critiques pour certaines configurations particulières du domaine, par exemple lorsque le domaine n'est pas centré à l'origine ou lorsqu'une arête du domaine est paramétrisée en plusieurs parties. C'est pour cette raison que nous en proposons une nouvelle version. Dans cet article, nous commençons par introduire les ingrédients en commun des deux transformations. Dans la section 2.2, nous rappelons les transformations transfinies généralisées, puis, dans la section 3, nous introduisons la transformation transfinie qui dépend du déplacement du bord (TT DDB). Pour terminer, nous présentons quelques résultats numériques pour comparer les deux transformations.

## 1. Introduction

Many physical phenomena can be described by partial differential equations (PDEs). Parameterized PDEs are a special case of PDEs, in which physical properties, material properties, geometrical properties (like domain deformations), or boundary conditions (like loads), are addressed to a parameter  $\mu$ . By varying the values of  $\mu$ , the parameterized PDEs are able to model the physical system for several settings of the same equation. In this work, we introduce a map that permits to deal with a parameter  $\mu$  addressing the deformation of the domain. For many numerical techniques used to solve the parameterized PDEs, it is crucial to define a suitable parametric map between the reference domain and the deformed one. In particular, for example, the reduced basis method (RBM) is an efficient technique for approximating the solutions of parameterized PDEs for many instances of  $\mu$  in a rapid and reliable way in which the parametric map plays an important role [3,7]. In this paper, we propose a new extension of the parameterized transfinite map (TM) proposed in [5]. This extension can be seen as a generalization of the Gordon–Hall transfinite interpolation approach for quadrilaterals [2]. The transfinite map induces a non-affine geometrical parameterization so that the empirical interpolation [1] method is necessary when we need to recover the affinity of the linear and bilinear forms of the considered problems. The generalized transfinite map proposed in [5,6] and used recently in [4] presents some critical issues when we deal with particular configurations of the domain, e.g. when the domain representation is not centered around the axis origin or when we want to consider an edge of the domain parameterized by sub parts. For that reason, we propose some improvements to overcome these problems. In this work, we start by introducing the ingredients that are common to both the generalized transfinite map and our proposed extension. In Section 2.2 we recall the generalized TM, then we introduce in Section 3 the new proposed extensions called boundary displacement-dependent transfinite map (BDD TM). Finally some numerical results are shown to compare the two maps.

## 2. Generalized transfinite maps: state of the art

The idea behind the transfinite maps (TMs) is to deform the interior points of the physical domain through a linear combinations of the deformed points belonging to the boundaries, that are easily parameterized through one-dimensional functions [2].

We assume a general two-dimensional domain  $\Omega$  and a reference domain  $\tilde{\Omega}$ , we suppose that both are polygons with the same number  $n$  of curved edges. Let  $\Gamma_i$  denote a generic edge in  $\Omega$ , while  $\tilde{\Gamma}_i$  denotes the corresponding edge in  $\tilde{\Omega}$ ; the edges are numbered clockwise.

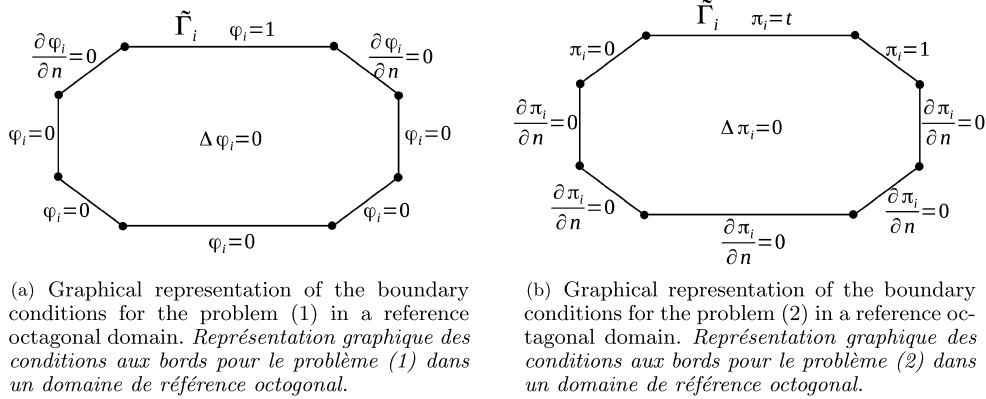
The common ingredients of the TMs are two functions that have to be found in correspondence of each edge of the domain  $\Omega$ ; the weight function  $\varphi_i$  and the projection function  $\pi_i$ . The computation of these functions is quite expensive but are independent of the parameters and can be obtained as solutions of proper Laplace problems on the reference domain  $\tilde{\Omega}$ . Due to this advantageous computational features, an offline/online computational decoupling procedure can be applied for the evaluation of the maps.

### 2.1. Offline stage

For each edge  $\tilde{\Gamma}_i$ ,  $i = 1, \dots, n$ , of the reference domain  $\tilde{\Omega}$  (with  $n$ -sides), we define a weight function  $\varphi_i$  by solving the following Laplace problem:

$$\begin{cases} \Delta \varphi_i = 0 & \text{in } \tilde{\Omega}, \\ \varphi_i = 1 & \text{on } \tilde{\Gamma}_i, \\ \frac{\partial \varphi_i}{\partial n} = 0 & \text{on } \tilde{\Gamma}_j, \quad j = i-1, i+1, \\ \varphi_i = 0 & \text{on } \tilde{\Gamma}_j, \quad j \neq i-1, i, i+1. \end{cases} \quad (1)$$

We represent in Fig. 1(a) a scheme concerning the boundary conditions for the case of an octagonal domain. We use the notational convention that if  $i = 1$ ,  $\tilde{\Gamma}_{i-1} = \tilde{\Gamma}_n$ , and if  $i = n$ ,  $\tilde{\Gamma}_{i+1} = \tilde{\Gamma}_1$ . To define the generalized transfinite map, we also



**Fig. 1.** Boundary conditions for the two problems (1) and (2) in a reference octagonal domain.

**Fig. 1.** Conditions aux bords pour les deux problèmes (1) et (2) dans un domaine de référence octogonal.

need to define an operator that “projects” the internal part of the reference domain onto each side  $\tilde{\Gamma}_i$ . For that, we compute the projection function  $\pi_i$  associated with the side  $\tilde{\Gamma}_i$  by solving the Laplace problem:

$$\begin{cases} \Delta \pi_i = 0 & \text{in } \tilde{\Omega}, \\ \pi_i = t & \text{on } \tilde{\Gamma}_i, \\ \pi_i = 0 & \text{on } \tilde{\Gamma}_{i-1}, \\ \pi_i = 1 & \text{on } \tilde{\Gamma}_{i+1}, \\ \frac{\partial \pi_i}{\partial n} = 0 & \text{on } \tilde{\Gamma}_j, \quad j \neq i-1, i, i+1, \end{cases} \quad (2)$$

where the Dirichlet boundary condition along  $\tilde{\Gamma}_i$  corresponds to a linear function of the arc-length  $t$  ranging from 0 to 1. On the sides adjacent to  $\tilde{\Gamma}_i$  we set  $\pi_i$  equal to either 0 or 1, and on the remaining sides we impose homogeneous Neumann boundary conditions (see Fig. 1(b)).

Thus, for each side of the reference domain, we associate one weight function  $\varphi_i$  with one projection function  $\pi_i$  by solving the problems (1) and (2), respectively. For a domain with  $n$  sides, we have to solve  $2n$  elliptic problems; however, these computations are independent of the deformation (and so of the parameter  $\mu$ ) and they could be included in the offline stage (computed just once) to guarantee computational efficiency.

## 2.2. Generalized transfinite map – online stage

Let  $\Omega = \Omega(\mu)$  be a parameterized domain. We suppose that  $\Omega$  is a curved polygonal with the same number  $n$  of edges. Let each edge  $\Gamma_i$ ,  $i = 1, \dots, n$  be parameterized through the parameter  $\mu \in \mathcal{D}$  by a bijective map  $\psi_i : [0, 1] \times \mathcal{D} \rightarrow \Gamma_i$  such that  $\psi_i(1, \mu) = \mathbf{x}_i$ , where  $\mathbf{x}_i$  denotes the vertex shared by  $\Gamma_i$  and  $\Gamma_{i+1}$ , and  $\psi_i(0, \mu) = \mathbf{x}_{i-1}$ , where  $\mathbf{x}_{i-1}$  denotes the vertex shared by  $\Gamma_i$  and  $\Gamma_{i-1}$ . We denote by  $\tilde{\mathbf{x}}$  a generic point of the reference domain  $\tilde{\Omega}$  and by  $\mathbf{x}$  a generic point of the parameterized domain  $\Omega$ . The transfinite map proposed in [5] is defined as:

$$\mathbf{T}(\tilde{\mathbf{x}}, \mu) = \sum_{i=1}^N [\phi_i(\tilde{\mathbf{x}}) \psi_i(\pi_i(\tilde{\mathbf{x}}), \mu) - \phi_i(\tilde{\mathbf{x}}) \phi_{i+1}(\tilde{\mathbf{x}}) \mathbf{x}_i]. \quad (3)$$

The advantage of such a map is that for each parameterized domain  $\Omega(\mu)$ , we need only to compute the boundary expressions  $\psi_i(\pi_i(\tilde{\mathbf{x}}), \mu)$  and to perform the linear combination in (3).

Several applications of the generalized transfinite map have been performed with different geometries [4,5] and we have observed some limits in particular configurations [3]. In particular, the position of the reference domain  $\tilde{\Omega}$  in the plane  $\mathbb{R}^2$  affects the performances of the transfinite map (3) by stretching the triangles of the mesh and often by creating an overlapping of the mesh nodes. This effect can be minimized by placing the center of the geometry on the origin of the axis, but it does not disappear completely. In order to show this phenomenon, we define  $\tilde{\Omega}$  and  $\Omega$  being an octagon depending on four parameters  $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$  as shown in Fig. 2(a). We consider the geometry centered in the axis origin; moreover, we even choose  $\mu = \mu_{\text{ref}}$ , this setting should represent the best configuration that minimizes the undesired behavior of the TM. We expect that, in this case,  $\mathbf{T}(\cdot, \mu_{\text{ref}}) : \tilde{\Omega} \rightarrow \Omega_{\text{ref}}$  defines the identity map, but unfortunately is not the case. In order to observe it, we define a mesh into the reference domain  $\tilde{\Omega}$  and we map it by (3) onto  $\Omega$ . Fig. 2(b) shows the two meshes, the reference one and the deformed one. We note that the grid points are pushed away from the origin. This behavior is much more evident when the geometry is not centered in the origin of the coordinates and when we really deform the reference domain ( $\mu \neq \mu_{\text{ref}}$ ).

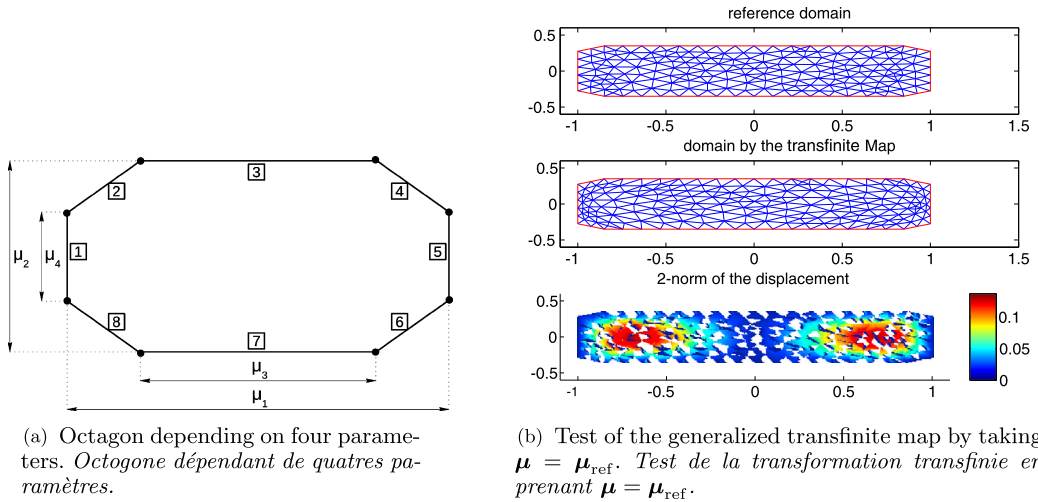


Fig. 2. (Color online.) First test for the generalized TM on an octagonal geometry.

Fig. 2. (En couleur sur Internet.) Premier test pour la transformation transfinie dans une géométrie octogonale.

### 3. Boundary displacement-dependent transfinite maps

Motivated by the results of the previous section, we introduce an extension of the generalized TMs, with the aim of keeping the suitable properties of the TMs, but solving their critical issues. More precisely, we define a map independent of the position of the geometry in the plane  $\mathbb{R}^2$ . The basic idea of the *Boundary Displacement-Dependent Transfinite Map* (BDD TM) is to keep into account the original positions of the points in the reference domain  $\tilde{\Omega}$  and to move them by weighting only the difference between the original boundaries and the deformed ones. The convenient online/offline computational decoupling can still be maintained. We define the weight functions  $\phi_i$  and the projection functions  $\pi_i$  as before and still each of the boundaries in the reference domain is parameterized by a function  $\tilde{\psi}_i : [0, 1] \rightarrow \tilde{\Gamma}_i$ . We introduce a further function, the *displacement function*. Let  $\mathcal{D}$  be the parameter domain, we define the displacement function  $\mathbf{d}_i : [0, 1] \times \mathcal{D} \rightarrow \tilde{\Gamma}_i$  as:

$$\mathbf{d}_i(t, \mu) = \tilde{\psi}_i(t, \mu) - \tilde{\psi}_i(t).$$

For each point on the boundary, this function gives us the relative displacement between the new and the old position of the boundaries. This new ingredient characterizes the new map. If  $\tilde{\mathbf{x}}$  is a generic point in the reference domain  $\tilde{\Omega}$ , the idea of the BDD TM is to displace it through the quantity  $\tilde{\mathbf{x}} + \sum_{i=1}^N \phi_i(\tilde{\mathbf{x}}) \mathbf{d}_i(\pi_i(\tilde{\mathbf{x}}), \mu)$ . As in the previous map, in every term we have to subtract a correction term, which is, in this case,  $\phi_i(\tilde{\mathbf{x}}) \phi_{i+1}(\tilde{\mathbf{x}}) \mathbf{d}_i(1, \mu)$ , such that the BDD Transfinite Map is defined as:

$$\mathbf{S}(\tilde{\mathbf{x}}, \mu) = \tilde{\mathbf{x}} + \sum_{i=1}^N [\phi_i(\tilde{\mathbf{x}}) \mathbf{d}_i(\pi_i(\tilde{\mathbf{x}}), \mu) - \phi_i(\tilde{\mathbf{x}}) \phi_{i+1}(\tilde{\mathbf{x}}) \mathbf{d}_i(1, \mu)], \quad (4)$$

where  $\phi_{N+1} = \phi_1$ . As we can see in formula (4), every point  $\tilde{\mathbf{x}} \in \tilde{\Omega}$  is displaced only if we want to deform the geometry, i.e. if there exists an  $i \in \{1, \dots, N\}$  such that  $\mathbf{d}_i(\pi_i(\tilde{\mathbf{x}}), \mu) \neq 0$ . So it is easy to see that if  $\mu = \mu_{\text{ref}}$ , then the BDD TM (4) just defines the identity map.

Before we compare the two transfinite maps, we introduce a geometry representing a stenosis. This geometry can be interpreted as a two-dimensional model of a pipe and is defined by eight parameters  $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8)$  (see Fig. 3). Through the parameters  $\mu_5$  and  $\mu_8$  we can either “inflate” or “compress” the pipe. The other six parameters determine the size of the pipe as well as the position and the length of the deformations.

In the following, we illustrate some deformations of the octagonal and the stenosis geometry, obtained with both the generalized TM and the BDD TM. Figs. 4(a) and (b) are plots of the two reference geometries containing a quasi-uniform mesh obtained with the BDD TM. Figs. 5(a) and (b) show some relatively small deformations of the geometries, while the geometries in Figs. 5(c) and (d) show big deformations of the domains. We can observe that in both cases the meshes deformed by the BDD TM are much more regular and there is no overlapping between the triangles of the mesh. Moreover, the mesh regularity is ensured by the positivity of the determinant of the mesh Jacobian matrix.

### 4. Conclusion

In this work, we have presented an extension of the transfinite maps (i.e. the Gordon–Hall and the generalized transfinite maps), which is able to improve and to solve their critical issues. With respect to the previous versions, the proposed

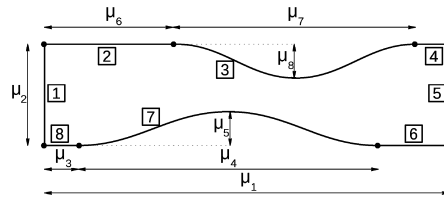


Fig. 3. The parameters defining the stenosis geometry.

Fig. 3. Les paramètres définissant la géométrie de la sténose.

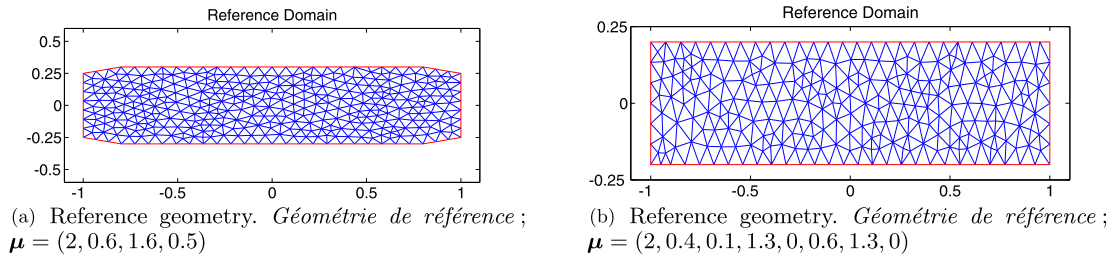


Fig. 4. (Color online.) Reference geometries.

Fig. 4. (En couleur sur Internet.) Géométries de référence.

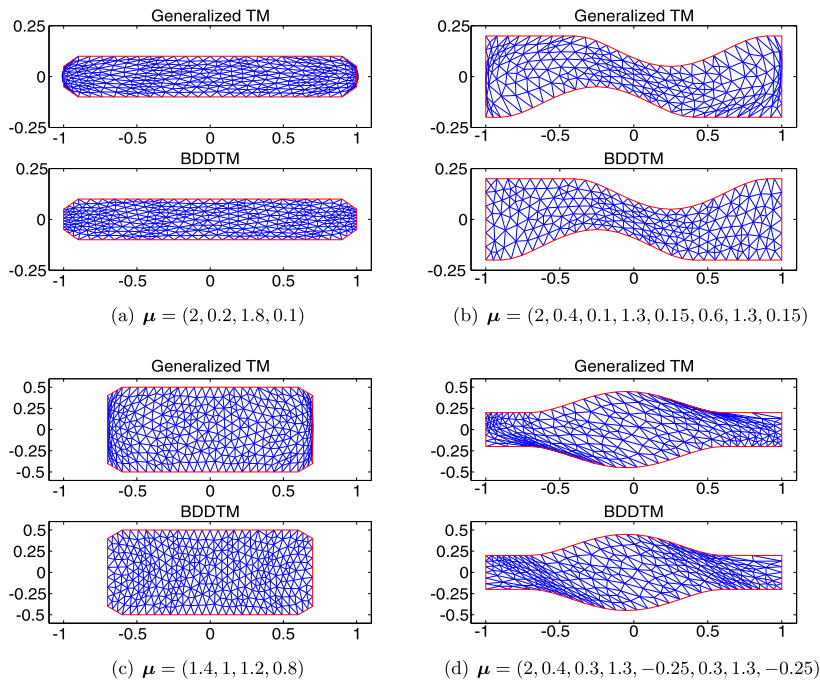


Fig. 5. (Color online.) Several deformations of the two reference geometries.

Fig. 5. (En couleur sur Internet.) Plusieurs déformations des deux géométries de références.

boundary displacement-dependent transfinite map (BDD TM) is able to perform suitable domain deformations for a larger set of configurations. In particular, with the BDD TM, the position of the domain in the Cartesian coordinate plane does not affect the effectiveness of the map. Moreover, the new proposed map allows us to deal with more complex parameterizations of the domain and consequently bigger deformations of the geometry without producing any overlapping phenomena between the triangles of a mesh defined in the domain. In order to compare the regularities of the maps, we have defined two geometries (a stenosis and an octagon) and we have shown the effectiveness of the BDD TM deformations with respect to that of the generalized TM.

In conclusion, the BDD TM represents an efficient technique to parameterize the computational domain involved in several numerical models. In particular, it has proved to be a robust map [3] in the reduced basis methods [7]. The BDD TM represents a powerful technique to recover an important range of deformations of a parameterized domain.

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