Mathematics Meets Medicine: An Optimal Alignment

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Motivation

Image Registration

Given a reference image $\mathcal{R}$ and a template image $\mathcal{T}$,

find a reasonable transformation y, such that

the transformed image $\mathcal{T}[y]$ is similar to $\mathcal{R}$

reference $\mathcal{R}$

template $\mathcal{T}$

transformed template $\mathcal{T}[y]$

Motivation

Image Registration

Given a reference image $\mathcal{R}$ and a template image $\mathcal{T}$,

find a reasonable transformation y, such that

the transformed image $\mathcal{T}[y]$ is similar to $\mathcal{R}$

Questions:

- What is a transformed image $\mathcal{T}[y]$? $\rightsquigarrow$ image model $\mathcal{T}[y]$
- What is similarity of $\mathcal{T}[y]$ and $\mathcal{R}$? $\rightsquigarrow$ $\mathcal{D}[\mathcal{T}[y],\mathcal{R}]$
- What is reasonability of y? $\rightsquigarrow$ $\mathcal{S}[y]$

Image Registration: Variational Problem

$$\mathcal{D}[\mathcal{T}[y],\mathcal{R}] + \mathcal{S}[y - y_{\text{reg}}] \xrightarrow{y} \min, \qquad y_{\text{reg}}(x) = x$$

Outline

- Applications
- Variational formulation $\mathcal{D}[\mathcal{T}[y],\mathcal{R}]+\mathcal{S}[y]\xrightarrow{y}\min$
  - image models $T[y]$
  - distance measures $\mathcal{D}[T[y],R]$
  - regularizer $\mathcal{S}[y]$
- Numerical methods
- Constrained image registration
- Conclusions

People

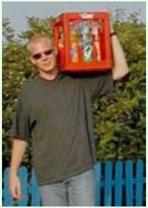
Bernd Fischer

Eldad Haber

Oliver Schmitt

Stefan Heldmann

Hanno Schumacher

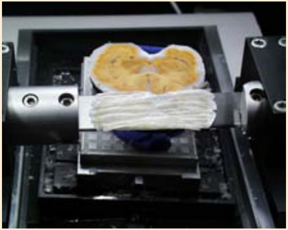


Nils Papenberg

Applications

HNSP: Sectioning

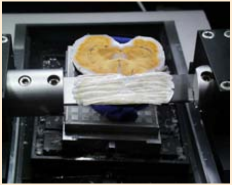
with Oliver Schmitt,
Institute of Anatomy, University Rostock, Germany

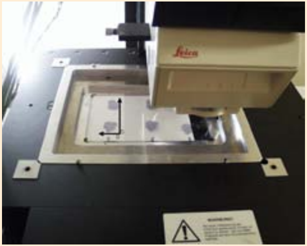


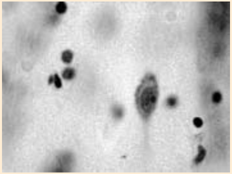
- sliced
- flattened
- stained
- mounted
- ...
- digitized

large scale digital images, up to 10.000×20.000 pixel

HNSP: Microscopy







HNSP: Deformed Images

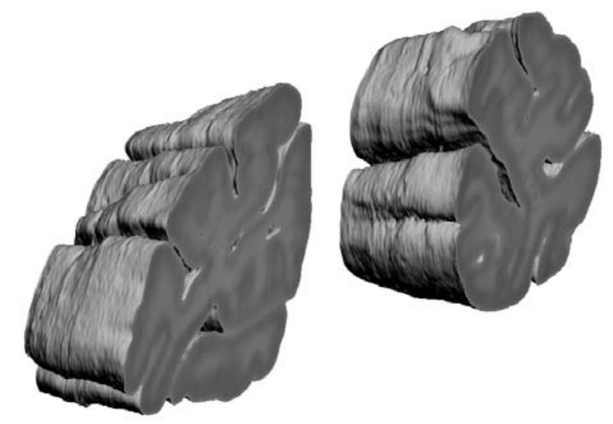
sections 3.799 and 3.800 out of about 5.000

human | affine linear | elastic

$|T_{\text{orig}} - R| = 100\%$ $\quad |T_{\text{linear}} - R| = 72\%$ $\quad |T_{\text{elastic}} - R| = 50\%$

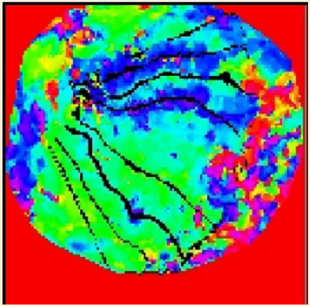
HNSP: Results

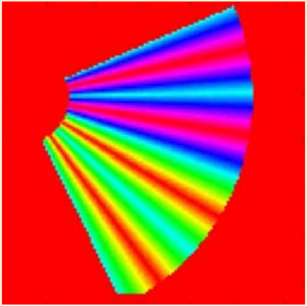
3D elastic registration of a part of the visual cortex
(two hemispheres; 100 sections á 512×512 pixel)



Neuroimaging (fMRI)

with Brian A. Wandell, Department of Psychology,
Stanford Vision Science and Neuroimaging Group

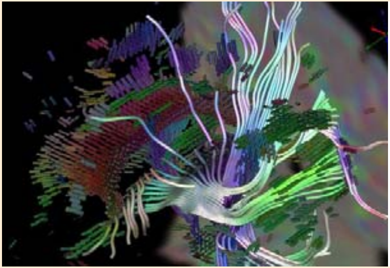


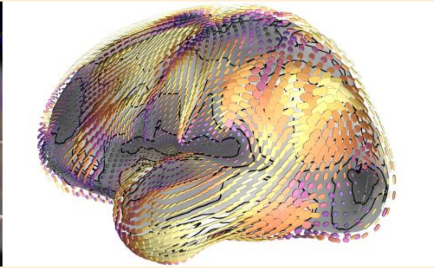


“flattened visual cortex”

DTI: Diffusion Tensor Imaging

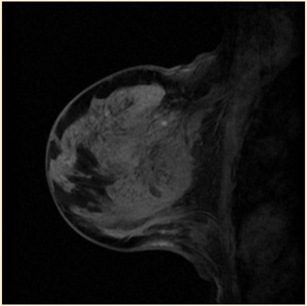
with Brian A. Wandell, Department of Psychology,
Stanford Vision Science and Neuroimaging Group



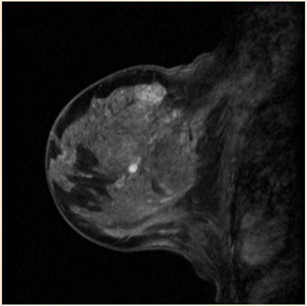


MR-mammography, biopsy (open MR)

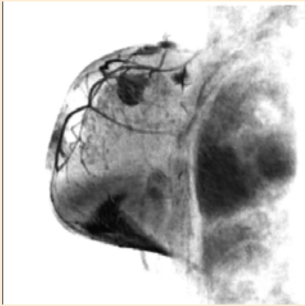
with Bruce L. Daniel,
Department of Radiology, Stanford University



pre contrast



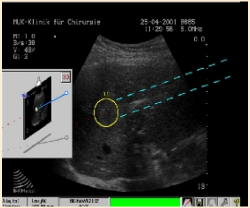
post contrast



3D

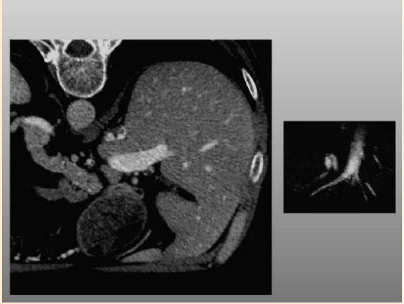
Virtual Surgery Planning

S. Bommersheim & N. Papenberg, SAFIR, BMBF/FUSION
Future Environment for Gentle Liver Surgery Using Image-Guided Planning and Intra-Operative Navigation



Results for 3D US/CT

with Oliver Mahnke, SAFIR, University of Lübeck
& MiE GmbH, Seth, Germany

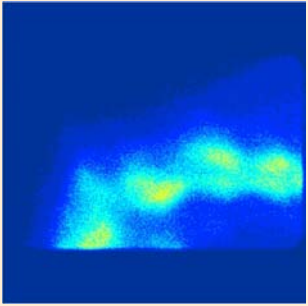


Motion Correction

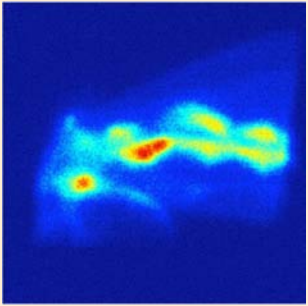
from Thomas Netsch,
Philips Research, Hamburg, Germany

SPECT: Single Photon Emissions CT

with Oliver Mahnke, SAFIR, University of Lübeck
& MiE GmbH, Seth, Germany



image



registered

Lester

Registration in Medical Imaging

- Comparing/merging/integrating images from different
  - times, e.g., pre-/post surgery
  - devices, e.g., CT-images/MRI
  - perspectives, e.g., panorama imaging
  - objects, e.g., atlas/patient mapping
- Template matching, e.g., catheter in blood vessel
- Atlas mapping, e.g., find 2D view in 3D data
- Serial sectioning, e.g., HNSP
- ...

Registration is not restricted to medical applications

Classification of Registration Techniques

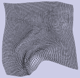
- feature space
- search space
- search strategy
- distance measure

- dimensionality of images $(d = 2, 3, 4, \ldots)$
- modality (binary, gray, color, . . .)
- mono-/multimodal images
- acquisition (photography, FBS, CT, MRI, . . .)
- inter/intra patient

Image Registration

Transforming Images

$$\mathcal{D}[\,\mathcal{T}[y],\mathcal{R}\,] + \alpha\mathcal{S}[\,y - y_{\mathrm{reg}}\,] \xrightarrow{y} \min$$



Variational Approach for Image Registration

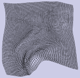
$$\mathcal{D}[\mathcal{T}[y],\mathcal{R}] + \alpha\mathcal{S}[y - y_{\mathrm{reg}}] \xrightarrow{y} \min$$

- Continuous models $\mathcal{R},\mathcal{T}$ for reference and template:

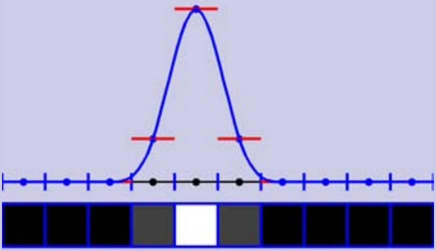
  discrete data $\mathtt{X},\mathtt{T} \quad \rightsquigarrow \quad \mathcal{T}(x) = \mathrm{interpolation}(\mathtt{X},\mathtt{T},x)$

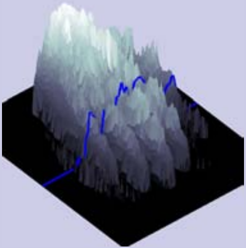
- Transformation $y:\mathbb{R}^d \longrightarrow \mathbb{R}^d$

  $\mathcal{T}[y](x) = \mathcal{T}(y(x)) = \mathrm{interpolation}(\mathtt{X},\mathtt{T},y(x))$

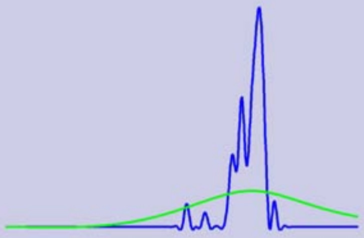


Interpolation

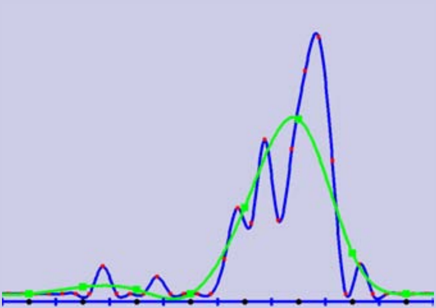




Multi-Scale



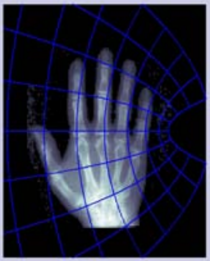
Multilevel

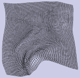


Transforming Images

$$\mathcal{T}[y](x) = \mathcal{T}(y(x)) = \text{interpolation}(\mathtt{X}, \mathtt{T}, y(x))$$

non-linear





Distance Measures

$$\mathcal{D}[\,\mathcal{T}[y],\mathcal{R}\,] + \alpha\mathcal{S}[\,y - y_{\mathrm{reg}}\,] \xrightarrow{y} \min$$

Distance Measures

- Feature Based
  (Markers / Landmarks / Moments / Localizer)

- L_2-norm, *Sum of Squared Differences (SSD)*

$$\mathcal{D}^{\mathrm{SSD}}[\,\mathcal{T}[y],\mathcal{R}\,] = \frac{1}{2}\int_{\Omega} [\,\mathcal{T}(y(x)) - \mathcal{R}(x)\,]^2\, dx,$$

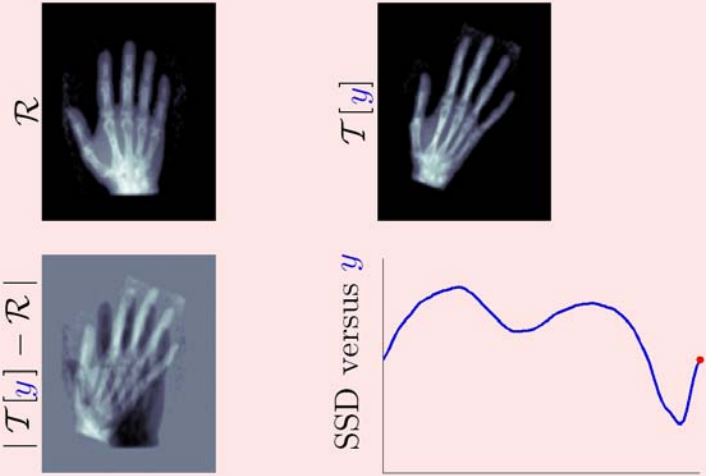
- correlation

- Mutual Information (multi-modal images)

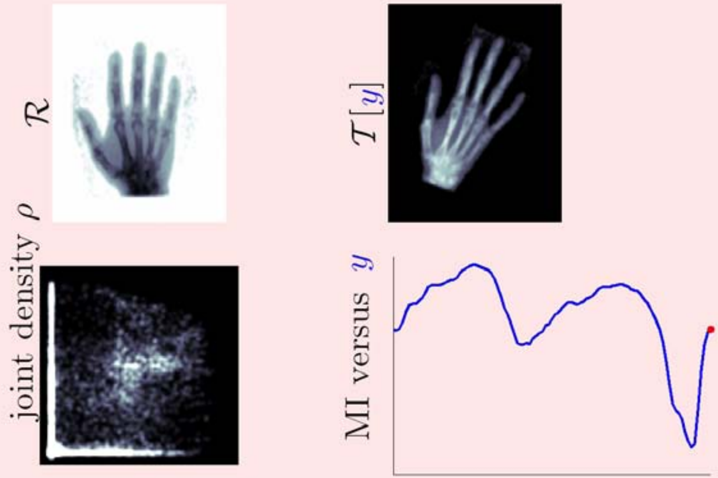
- Normalized Gradient Fields

- …

Sum of Squared Differences



Mutual Information



Regularization

$$\mathcal{D}[\,\mathcal{T}[y],\mathcal{R}\,] + \alpha\mathcal{S}[\,y - y_{\text{reg}}\,] \xrightarrow{y} \min$$

Transformation y

1	2	3
4	5	6
7	8	

1	2	3
4	5	8
7	6	

- Registration is severely ill-posed
- Restrictions onto the transformation y needed
- Goal: implicit physical restrictions

Implicit versus Explicit Regularization . . .

Registration is ill-posed $\rightsquigarrow$ requires regularization

- Parametric Registration
  - restriction to (low-dimensional) space (rigid, affine linear, spline,. . .)
  - regularized by properties of the space (implicit)
  - not physical or model based
- Non-parametric Registration
  - regularization by adding penalty or likelihood (explicit)
  - allows for a physical model
  - $\rightsquigarrow$ y is no longer parameterizable

… implicit versus explicit regularization

registration is ill-posed ⇝ requires regularization

- parametric registration

parametric registration

$\mathcal{D}[R,T;y] \stackrel{y}{=} \min$ s.t. $y \in \mathcal{Q} = \{x + \sum w_j q_j,\ w \in \mathbb{R}^m\}$

- non-parametric registration

non-parametric registration

$\mathcal{D}[R,T;y] + \alpha \mathcal{S}[y - y_{\text{reg}}] \stackrel{y}{=} \min$

References for Well-Posedness

- M. Droske and M. Rumpf.
  A variational approach to non-rigid morphological registration.
  SIAM Appl. Math., 64(2):668–687, 2004.
- B. Fischer and J. Modersitzki.
  A unified approach to fast image registration and a new curvature based registration technique.
  Linear Algebra and its Applications, 380:107–124, 2004.
- J. Weickert and C. Schnörr.
  A theoretical framework for convex regularizers in PDE-based computation of image motion.
  Int. J. Computer Vision, 45(3):245–264, 2001.
- ...

Regularizer $\mathcal{S}$

$y(x) = x + u(x)$, displacement $u : \mathbb{R}^d \to \mathbb{R}^d$

- "elastic registration" $\mathcal{S}^{\mathrm{elas}}[u] =$ elastic potential of u
- "fluid registration" $\mathcal{S}^{\mathrm{fluid}}[u] =$ elastic potential of $\partial_t u$
- "diffusion registration" $\mathcal{S}^{\mathrm{diff}}[u] = \frac{1}{2}\sum_{\ell=1}^{d} \int_\Omega \|\nabla u_\ell\|_{\mathbb{R}^2}^2 \, dx$
- "curvature registration" $\mathcal{S}^{\mathrm{curv}}[u] = \frac{1}{2}\sum_{\ell=1}^{d} \int_\Omega (\Delta u_\ell)^2 \, dx$
- ...

Elastic Registration

Transformation/displacement, $y(x) = x + u(x)$

$$\begin{aligned} \mathcal{S}^{\mathrm{elas}}[u] &= \text{elastic potential of } u \\ &= \int_\Omega \frac{\lambda+\mu}{2} \| \nabla \cdot u \|^2 + \frac{\mu}{2} \sum_{i=1}^{d} \| \nabla u_i \|^2 \, dx \end{aligned}$$

image painted on a rubber sheet

C. Broit.
Optimal Registration of Deformed Images.
PhD thesis, University of Pensylvania, 1981.

Bajcsy & Kovačič 1986, Christensen 1994, Bro-Nielsen 1996, Gee et al. 1997, Fischer & M. 1999, Rumpf et al. 2002, ...

Fluid Registration

Transformation/displacement, $y(x,t) = x + u(x,t)$

$$\mathcal{S}^{\text{fluid}}[u] = \text{elastic potential of } \partial_t u$$

image painted on honey

GE. Christensen.
Deformable Shape Models for Anatomy.
PhD thesis, Sever Institute of Technology, Washington University, 1994.

Bro-Nielsen 1996, Henn & Witsch 2002, …

Diffusion Registration

Transformation/displacement, $y(x) = x + u(x)$

$$\mathcal{S}^{\mathrm{diff}}[u] = \text{oszillations of } u = \frac{1}{2}\sum_{\ell=1}^{d} \int_\Omega \|\nabla u_\ell\|_{\mathbb{R}^2}^2 \, dx$$

heat equation

B. Fischer and J. Modersitzki.
Fast diffusion registration.
AMS Contemporary Mathematics, Inverse Problems, Image Analysis, and Medical Imaging, 313:117–129, 2002.

Horn & Schunck 1981, Thirion 1996, Droske, Rumpf & Schaller 2003, ...

Curvature Registration

Transformation/displacement, $y(x) = x + u(x)$

$$\begin{aligned} \mathcal{S}^{\text{curv}}[\,u\,] &= \text{oscillations of } u \\ &= \tfrac{1}{2}\sum_{\ell=1}^{d} \int_\Omega \|\Delta u_\ell\|^2_{\mathbb{R}^2}\, dx \end{aligned}$$

bi-harmonic operator

B. Fischer and J. Modersitzki.
Curvature based image registration.
J. of Mathematical Imaging and Vision, 18(1):81–85, 2003.

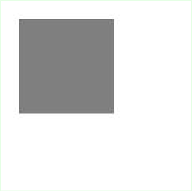
Stefan Henn.
A multigrid method for a fourth-order diffusion equation with application to image processing.
SIAM J. Sci. Comput., 2005.

Registration of a ■

Curvature Registration

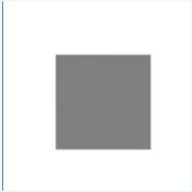
- **Goal:** do not penalize affine linear transformations
  $\mathcal{S}[Cx+b] \stackrel{!}{=} 0$ for all $C \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$
- **But:** $\mathcal{S}^{\text{diff,elas,fluid,}\dots}[Cx+b] \neq 0$!
- **Idea:** $\mathcal{S}^{\text{curv}}[y] = \sum_\ell \int_\Omega (\Delta y_\ell)^2 \, dx \;\Rightarrow\; \mathcal{S}^{\text{curv}}[Cx+b] = 0$

reference



“fluid”

“curvature”



Summary Regularization

- Registration is ill-posed $\rightsquigarrow$ requires regularization
- Regularizer controls reasonability of transformation
- Application conform regularization
- Enabling physical models
  (linear elasticity, fluid flow, . . .)
- $\rightsquigarrow$ high dimensional optimization problems

Numerical Methods for Image Registration

Optimize $\leftrightarrow$ Discretize

Image Registration

$$\mathcal{D}[\mathcal{T}[y],\mathcal{R}] + \alpha\mathcal{S}[y - y_{\mathrm{reg}}] \xrightarrow{y} \min, \qquad y_{\mathrm{reg}}(x) = x$$

Numerical Approaches:

- Optimize $\longrightarrow$ Discretize
- Discretize $\longrightarrow$ Optimize
- relatively large problems: 2.000.000 – 500.000.000 unknowns

Optimize $\longrightarrow$ Discretize: ELE

Image Registration

$$\mathcal{J}[y] = \mathcal{D}[\mathcal{T}[y], \mathcal{R}] + \alpha \mathcal{S}[y - y_{\mathrm{reg}}] \xrightarrow{y} \min, \qquad y_{\mathrm{reg}}(x) = x$$

- Euler-Lagrange eqs. (ELE) give necessary condition:
  $\mathcal{D}_y + \alpha \mathcal{S}_y = 0 \iff f[y] + \alpha \mathcal{A} y = 0$
  system of non-linear partial differential eqs. (PDE)
- outer forces f, drive registration
- inner forces $\mathcal{A}y$, tissue properties
- ELE $\rightsquigarrow$ PDE: balance of forces

Optimize $\rightarrow$ Discretize: Summary

Continuous Euler-Lagrange equations

$$f[y] + \alpha \mathcal{A}\, y = 0, \quad f[y^k] + \alpha \mathcal{A}\, y^{k+1} = 0, \quad f[y] + \alpha \mathcal{A}\, y = y_t$$

- all difficulties dumped into right hand side f
- spatial discretization straightforward
- efficient solvers for linear systems
- small controllable steps ($\rightsquigarrow$ movies)
- moderate assumptions on f and $\mathcal{A}$ (smoothness)

- no optimization problem behind
- non-linearity only via f
- small steps

- software: `http://www.math.uni-luebeck.de/SAFIR`

Discretize → Optimize: Summary

Discretization $\rightsquigarrow$ finite dimensional problem: $y^h \approx y(x^h)$

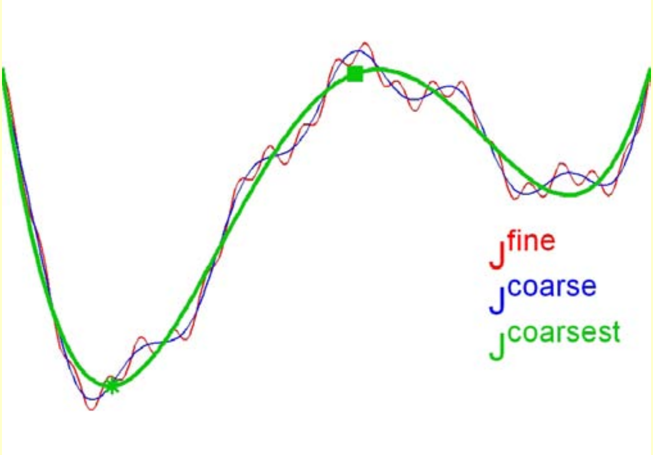
$$D(y^h) + \alpha S(y^h) \xrightarrow{y^h} \min, \quad y^h \in \mathbb{R}^n, \qquad h \longrightarrow 0$$

- efficient optimization schemes (Newton-type)
- linear systems of type $H\,\delta_y = -\mathrm{rhs}$,

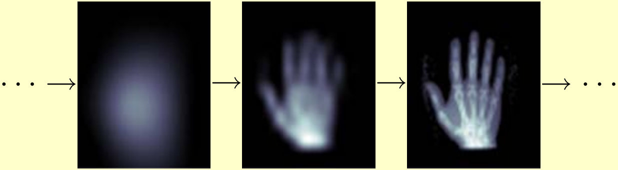
$$H = M + \alpha B^\top B, \quad M \approx D_{yy}, \quad \mathrm{rhs} = D_y + \alpha(B^\top B)y^h$$

- efficient multigrid solver for linear systems
- large steps
- discretization not straightforward (multigrid)
- all parts have to be differentiable (data model)

Multilevel



Multilevel



$\cdots \quad h^{\ell-1} \quad h^{\ell} \quad h^{\ell+1} \quad \cdots$

for $\ell = 1 : \ell_{\max}$ **do**

transfer images to level ℓ

approximately solve problem for y

prolongating y to finer level $\rightsquigarrow$ perfect starting point

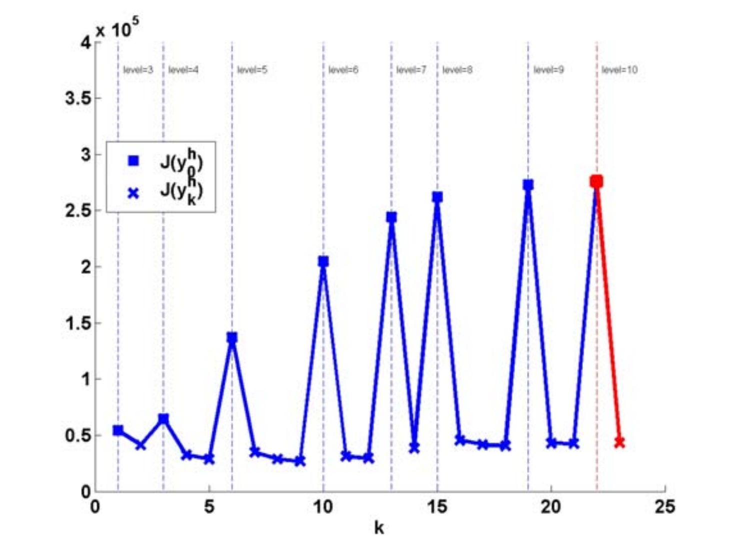
end for

Advantages of Multilevel Strategy

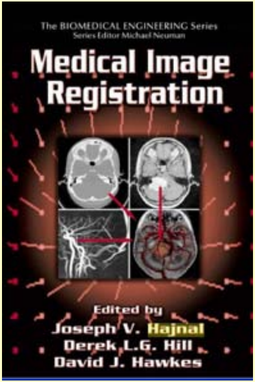
- Regularization
- Focus on essential minima
- Creates extraordinary starting value
- Reduces computation time

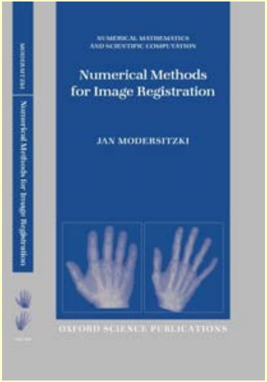
- ?

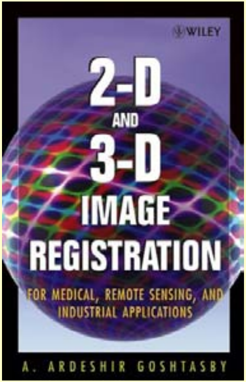
Example: Multilevel Iteration History



Literature







- Hajnal JV, Hill DLG, Hawkes DJ: Medical Image Registration, CRC 2001.
- Modersitzki J: Numerical Methods for Image Registration, OUP 2004.
- Goshtasby AA: 2-D and 3-D Image Registration, Wiley 2005.

Constrained Image Registration

Example: COLD

Combining Landmarks and Distance Measures

$\mathcal{R}$ $\quad$ $\mathcal{T}$ $\quad$ $\mathcal{T}[y^{\mathrm{LM}}]$ $\quad$ $\mathcal{T}[y^{\mathrm{COLD}}]$

Patent AZ 10253 784.4; Fischer & M., 2003

Adding Constraints

Constrained Image Registration

$$\mathcal{D}[\mathcal{T}[y],\mathcal{R}] + \alpha\mathcal{S}[y-y_{\mathrm{reg}}] + \beta\int_\Omega \psi\left(\mathcal{C}^{\mathrm{soft}}[y]\right)\,dx \xrightarrow{y} \min$$

subject to $\mathcal{C}^{\mathrm{hard}}[y](x) = 0$ for all $x \in \Omega_{\mathcal{C}}$

Example: landmarks/volume preservation

$$\begin{aligned} \mathcal{C}_i^{\mathrm{LM}}[y] &= \|y(r_i) - t_i\|, & \psi(\mathcal{C}) &= 0.5\|\mathcal{C}\|^2 \\ \mathcal{C}^{\mathrm{VP}}[y](x) &= \det(\nabla y(x)), & \psi(\mathcal{C}) &= (\log \mathcal{C})^2 \end{aligned}$$

- soft constraints (penalty)
- hard constraints
- both constraints

Rigidity Constraints

Soft Rigidity Constraints

FAIR with Soft Rigidity

$$\mathcal{D}[\mathcal{T}[y],\mathcal{R}] + \alpha\mathcal{S}[y - y_{\mathrm{reg}}] + \beta\mathcal{C}[y] \xrightarrow{y} \min$$

$\mathcal{C}$ soft constraints / penalty:

$$\mathcal{C}[y] = \tfrac{1}{2}\| \underbrace{r^{\mathrm{linear}}(y)}_{\text{linear}} \|_{\mathcal{Q}}^2 + \tfrac{1}{2}\| \underbrace{r^{\mathrm{orth}}(y)}_{\text{orthogonal}} \|_{\mathcal{Q}}^2 + \tfrac{1}{2}\| \underbrace{r^{\mathrm{det}}(y)}_{\text{orientation}} \|_{\mathcal{Q}}^2$$

$$r^{\mathrm{linear}}(y) = [\partial_{1,1}y_1, \ldots, \partial_{d,d}y_1,\ \partial_{1,1}y_2, \quad \ldots]$$

$$r^{\mathrm{orth}}(y) = \nabla y^\top \nabla y - I_d$$

$$r^{\mathrm{det}}(y) = \det(\nabla y) - 1$$

y rigid $\iff$ $[\ r^{\mathrm{linear}} = 0\ \wedge\ r^{\mathrm{orth}} = 0\ \wedge\ r^{\mathrm{det}} = 0\ ]$

The Weight $\mathcal{Q}$

- only locally rigid
- use weight function $\mathcal{Q}$
- regions to be kept rigid move with y

$$\| f \|_{\mathcal{Q}}^2 = \int_\Omega f(x)\ \mathcal{Q}(y(x))^2\, dx$$

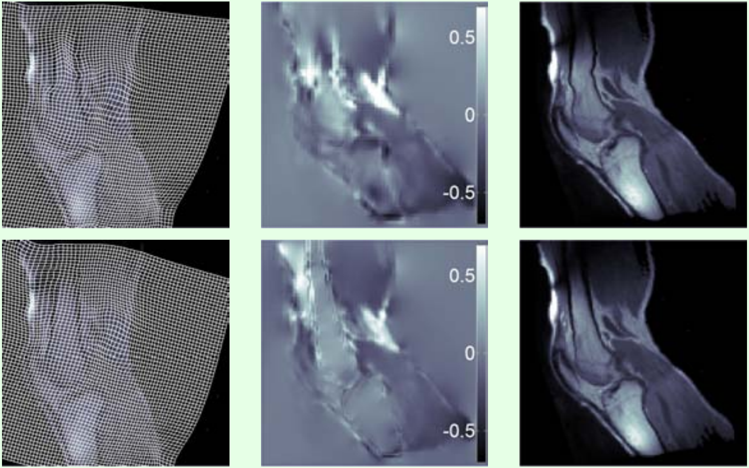
Numerical Scheme

- $Q(y^h) \approx \mathcal{Q}(y(x^h))$
- $r(y^h) = [\mathrm{diag}(Q(y^h))\; r_1(y^h), \ldots, \mathrm{diag}(Q(y^h))\; r_{\mathrm{end}}(y^h)]$
- $C(y^h) = \frac{1}{2} r(y^h)^\top r(y^h)$
- $C_y(y^h) =$ lengthy formula
- $D(y^h) + \alpha S(y^h) + \beta C(y^h) \xrightarrow{y^h} \min$
- Optimizer: Gauß-Newton type approach,

$$H \approx \text{“}\nabla^2 \mathcal{D}\text{”} + \alpha B^\top B + \beta {r_y}^\top r_y$$

Example: Knee

T & grid | $\det(\nabla y) - 1$ | $T(y)$



not penalized

penalized

Summary of Soft Rigidity Constraints

- Results are OK
- Implementation is straightforward
- Constraints are not fulfilled
- How to pick penalty (β, ψ)?

Hard Rigidity Constraints

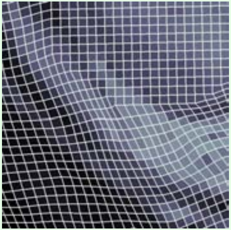
FAIR with Hard Rigidity

$\mathcal{D}[\mathcal{T}[y],\mathcal{R}]+\alpha\mathcal{S}[y-y_{\text{reg}}] \xrightarrow{y}$ min subject to y rigid on $\mathcal{Q}$

Eulerian $\rightarrow$ Lagrangian

computations of $\mathcal{D}$ and $\mathcal{S}$ involve $\det(\nabla y)$

rigidity in $\mathcal{T}$ domain $\rightsquigarrow$ “linear” constraints



$$y(x) = D_k x + t_k, \quad k = 1 : \#\text{segments}$$

Lagrangian Model of Rigidity (2D)

- rigid on segment i

$$y(x) = Q(x)w^i = \begin{pmatrix} \cos w_1^i & -\sin w_1^i \\ \sin w_1^i & \cos w_1^i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} w_2^i \\ w_3^i \end{pmatrix}$$

- $w = (w^1, \dots, w^m), \quad \mathcal{C} = (\mathcal{C}^1, \dots, \mathcal{C}^m), \quad m = \#\text{segments}$

$$\mathcal{C}^i[y, w] = y(x) - Q(x)w^i, \qquad i = 1, \dots, m$$

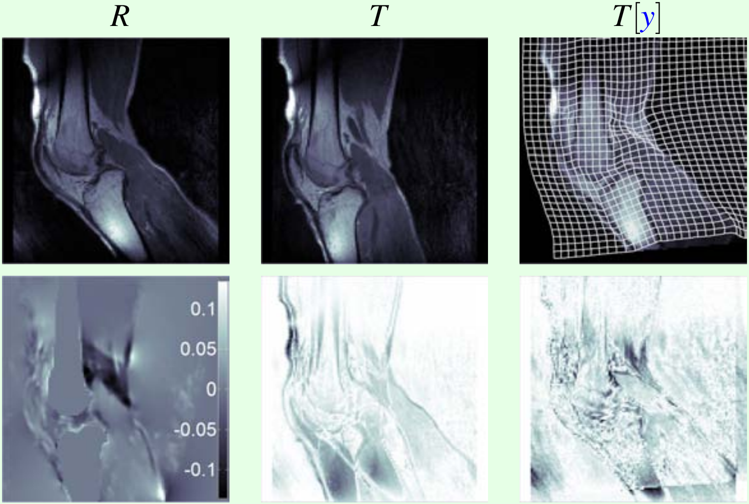
- Lagrangian:

$$L(y, w, p) = \mathcal{D}[y] + \alpha\mathcal{S}[y] + p^\top\mathcal{C}[y, w]$$

- Numerical Scheme:

Sequential Quadratic Programming

Rigidity as a Hard Constraint



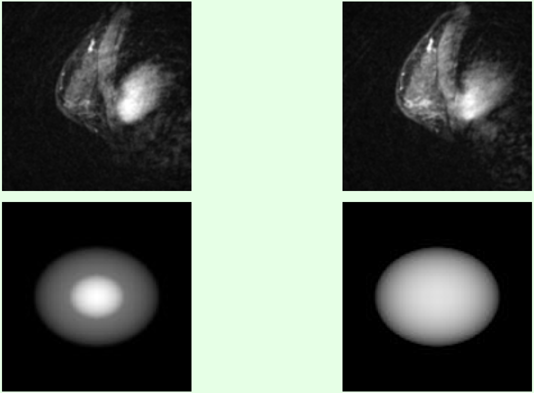
Summary of Hard Rigidity Constraints

- Results are OK
- Implementation is interesting
- Constraints are fulfilled
- No additional Parameters

Volume Preserving Image Registration

Example: Tumor Monitoring

MRI scans of a female breast, with Bruce L. Daniel
Department of Radiology, Stanford University



pre contrast post contrast
pre contrast post contrast

Volume Preserving Constraints

$$\int_{y(V)} dx = \int_V dx \quad \text{for all} \quad V \subset \Omega$$

assuming y to be sufficient smooth,

$$\det(\nabla y) = 1 \quad \text{for all} \quad x \in \Omega$$

Volume Preserving Constraints

$$\mathcal{C}[y](x) = \det(\nabla y(x)) - 1, \qquad x \in \Omega_{\mathcal{C}}$$

Approaches to VPIR

- Soft constraints (add a penalty)

$$\mathcal{D}[y] + \alpha\mathcal{S}[y] + \beta \int_\Omega \psi(\mathcal{C}[y])\, dx = \min$$

T. Rohlfing, CR. Maurer, DA. Bluemke, and MA. Jacobs.
Volume-preserving nonrigid registration of MR breast images using free-form deformation with an incompressibility constraint.
IEEE TMI, 22(6):730–741, 2003.

- Hard constraints

$$\mathcal{D}[y] + \alpha\mathcal{S}[y] = \min \quad \text{s.t.} \quad \mathcal{C}[y](x) = 0 \quad \text{for all } x \in \Omega_{\mathcal{C}}$$

E. Haber and J. Modersitzki.
Numerical methods for volume preserving image registration.
Inverse Problems, 20(5):1621–1638, 2004.

Volume Preservation using Soft Constraints

$$\mathcal{D}[y] + \alpha\mathcal{S}[y] + \beta \int_\Omega \log^2\left(\mathcal{C}^{\text{soft}}[y] + 1\right) dx \xrightarrow{y} \min$$

Drawbacks of Soft Constraints

- constraints are generally not fulfilled
- small soft constraints might be large on small regions (tumor!)
- additional parameters
- bad numerics for $\beta \to \infty$

Continuous Framework, Hard Constraints

VPIR

$$\mathcal{D}[y] + \alpha\mathcal{S}[y] \xrightarrow{y} \min \quad \text{s.t.} \quad \mathcal{C}[y] = 0, \quad x \in \Omega_{\mathcal{C}}$$

- Distance measure $\mathcal{D}$ with Gâteaux derivative

$$d_{y,v}\mathcal{D}[y] = \int_\Omega \langle f(x, y(x)), v(x)\rangle_{\mathbb{R}^d} \; dx$$

- Regularizer $\mathcal{S}$ with Gâteaux derivative

$$d_{y,v}\mathcal{S}[y] = \int_\Omega \langle \mathcal{B}y(x), \mathcal{B}v(x)\rangle_{\mathbb{R}^d} \; dx$$

- Volume preserving constraints

$$\begin{aligned} \mathcal{C}[u] &= \det(\nabla y) - 1 \\ d_{y,v}\mathcal{C}[y] &= \det(\nabla y) \; \left\langle \nabla y^{-\top}, \nabla v \right\rangle_{\mathbb{R}^{d,d}} \end{aligned}$$

Example: Volume Preservation in 2D

$$\begin{aligned}
\mathcal{C}[\,x+u(x)\,] &= \det(I_2+\nabla u)-1\\
&= \partial_1 u_1+\partial_2 u_2+\partial_1 u_1\partial_2 u_2-\partial_2 u_1\partial_1 u_2\\
&= \nabla\cdot\; u+N[u]
\end{aligned}$$

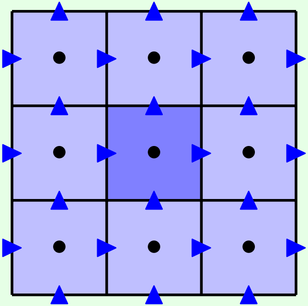
- N is nonlinear, $N[0]=0$
- linearization

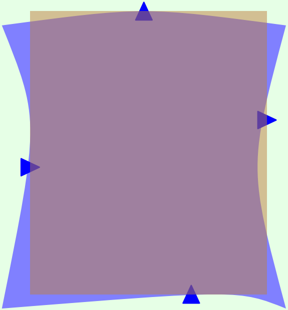
$$\mathcal{C}_y \approx \nabla\cdot \quad + [\xi(x)\cdot\partial_1 \quad \eta(x)\cdot\partial_2]$$

- $\rightsquigarrow$ Stokes problem, needs careful discretization to keep LBB conditions or h-ellipticity

Discretizing . . .

- $\mathcal{T}$ and $\mathcal{R}$ on cell center grid
- $y = [y_1, y_2]$ on staggered grids





- $\mathrm{vol}(V, y) = \int_{y(V)} dx \approx \mathrm{vol}(\mathrm{box}), \quad c_i = \mathrm{vol}(\mathrm{box}_i) - h^d$
- $C(y^h) = (c_i)_{i=1}^n, \quad C_y(y^h)$ straightforward but lengthy

. . . and Optimize

Discrete VPIR

Find y^h such that

$$D(y^h) + \alpha S(y^h) \xrightarrow{y^h} \min \quad \text{s.t.} \quad C_i(y^h) = 0, \quad i = 1 : \#\text{voxel}$$

- SQP: Sequential Quadratic Programming
- Lagrangian with multiplier p
$$L(y^h, p) = D(y^h) + \alpha S(y^h) + p^\top C(y^h)$$
- Necessary conditions for a minimizer: $\nabla L(y^h, p) = 0$
- Gauß-Newton type method, $H = \text{“}\nabla^2 D\text{”} + \alpha B^\top B$

$$\begin{pmatrix} H & C_y \\ C_y{}^\top & 0 \end{pmatrix} \begin{pmatrix} \delta y \\ \delta p \end{pmatrix} = - \begin{pmatrix} L_y \\ L_p \end{pmatrix}$$

Details . . .

- Solving the KKT system: MINRES

$$\begin{pmatrix} H & C_y \\ C_y^\top & 0 \end{pmatrix} \begin{pmatrix} \delta y \\ \delta p \end{pmatrix} = - \begin{pmatrix} L_y \\ L_p \end{pmatrix}$$

- with preconditioner

$$\begin{pmatrix} H & \\ & \hat{S} \end{pmatrix}, \qquad \begin{aligned} \hat{S} &\approx C_y\, H^{-1}\, C_y^\top \\ \hat{S}^{-1} &:= C_y^\dagger\, H\, (C_y^\dagger)^\top \\ C_y^\dagger &= (C_y\, C_y^\top)^{-1} C_y \end{aligned}$$

- Multigrid for H and $C_y\, C_y^\top$

. . . details

- Line search for $y^h \leftarrow y^h + \gamma \delta y^h$ based on merit function

$$\begin{aligned} \mathrm{merit}_{\mathrm{KKT}}(y^h) \quad &:= \quad D(y^h) + \alpha S(y^h) + \theta \|C(y^h)\|_1 \\ & \qquad \theta := \|p\|_\infty + \theta_{\min} \\ & \qquad p \quad \text{from} \quad \|D_y + \alpha S_y + {C_y}^\top p\| \xrightarrow{p} \min \end{aligned}$$

note, $$({C_y}\,{C_y}^\top)p = -C_y(D_y + \alpha S_y)$$

- For the projection step:

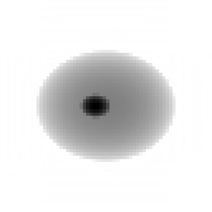
$$\mathrm{merit}_{\mathrm{C}}(y^h) := \|C(y^h)\|_2^2$$

- If $\mathrm{merit}_{\mathrm{C}}(y^h) > \mathrm{tol}$, solve for correction δy such that

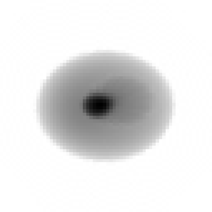
$$C(y^h + \delta y) \approx C(y^h) + C_y(y^h)\,\delta y = 0$$

VPIR example: Blobs

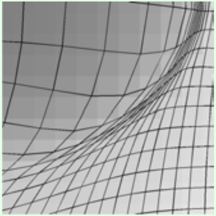
R: pre contrast



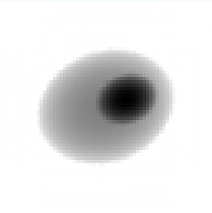
unconstrained



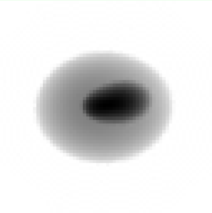
grid detail



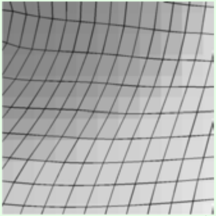
T: post contrast



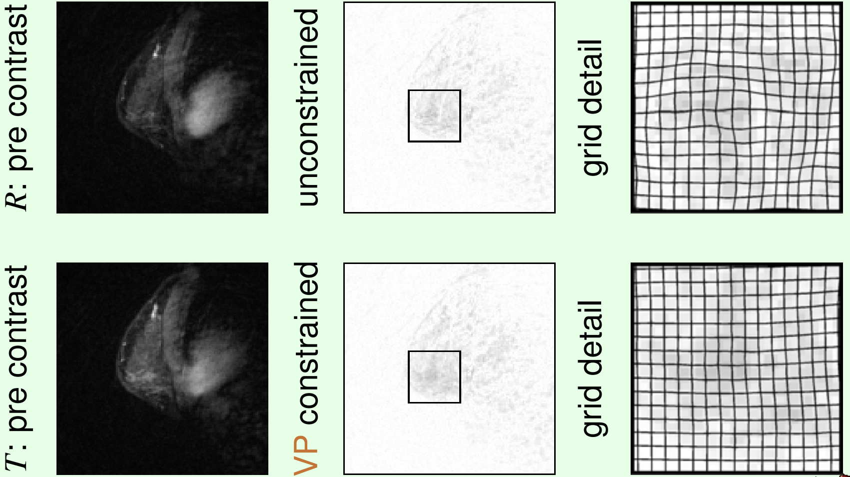
VP constrained



grid detail

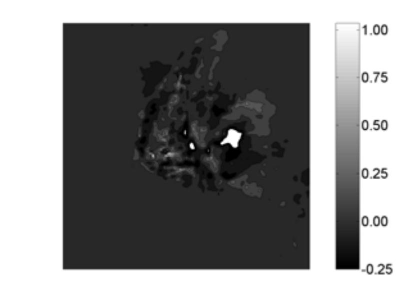


VPIR Example: Tumor Monitoring

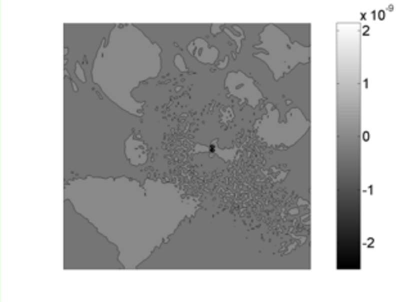


VPIR Example: Tumor Monitoring

unconstrained



VP constrained



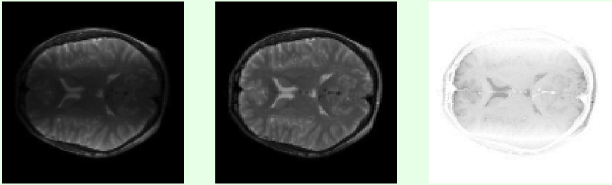
Registration and Intensity Correction

Distance Measures

- images features (moments, landmarks, markers, . . .)
- sum of squared differences (SSD)
- mutual information (MI)

Problem: sophisticated distance measures enable registration, but do not correct intensities

R $\qquad$ T $\qquad$ $|T - R|$



http://www.sci.utah.edu/stories/2002/sum_mri-epi.html

RIC: Registration and Intensity Correction

Registration and Intensity Correction

$$\mathcal{J}[y,s] = \mathcal{D}[\,\mathcal{T}[y], s\,\mathcal{R}\,] \;+\; \mathcal{S}[\,y - y_{\text{reg}}\,] + \text{Hom}(s) \xrightarrow{y,s} \min$$

$$\mathcal{D}[\,\mathcal{T}[y], s\,\mathcal{R}\,] \overset{\text{e.g.}}{=} \tfrac{1}{2}\|\mathcal{T}[y] - s\,\mathcal{R}\|_{L_2}^2 = \tfrac{1}{2}\int \Big(\mathcal{T}(y(x)) - s(x)\mathcal{R}(x)\Big)^2\,dx$$

intensity correction needs to be regularized
(excludes trivial solutions $s = \mathcal{T}/\mathcal{R}$, $s \equiv 1$)

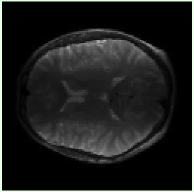
choices: $\text{Hom}(s) = \int |\nabla s|^p\,dx, \qquad |\nabla s| = \sqrt{(\partial_1 s)^2 + (\partial_2 s)^2}$

- diffusivity for $p = 2$
- total variation for $p = 1$
- Mumford-Shah penalty
- …

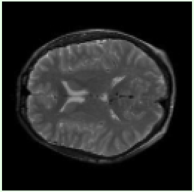
MRI inhomogeneities

Images from Samsonov, Whitaker, & Johnson, University of Utah

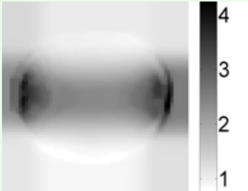
R



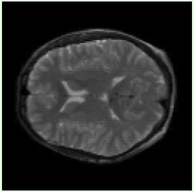
$T(y^{\mathrm{RIC}})$



TV correction s



$s\ R$



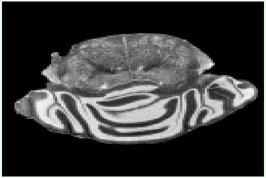
$|T(y^{\mathrm{RIC}}) - s\ R|$

$|T(y^{\mathrm{plain}}) - R|$

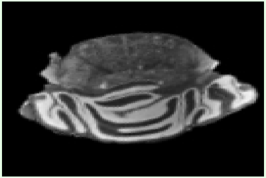
Staining Artifacts

Images from Oliver Schmitt, Anatomy, University Rostock

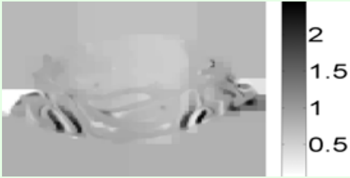
R



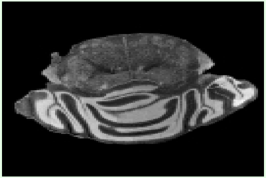
$T(y^{\mathrm{RIC}})$



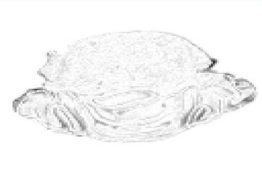
TV correction s



$s\ R$



$|T(y^{\mathrm{RIC}}) - s\ R|$



$|T(y^{\mathrm{plain}}) - R|$

Summary

Summary

- Introduction to image registration:
  important, challenging, interdisciplinary
- General framework based on a variational approach:
  $\mathcal{D}[\,\mathcal{T}[y],\mathcal{R}\,] + \alpha\mathcal{S}[\,y - y_{\mathrm{reg}}\,] \xrightarrow{y} \min$
- Discussion of various building blocks:
  - image model $\mathcal{T}[y]$
  - distance measures $\mathcal{D}$
  - regularizer $\mathcal{S}$
- Numerical methods:
  multilevel, optimize $\leftrightarrow$ discretize
- Constraints $\mathcal{C}$:
  landmarks, local rigidity, intensity correction, ...

Solutions and Algorithms For Image Registration