Appendix A
Irreducible Matrix and Dominant Eigenvalue

Definition: An $n \times n$ matrix $\mathbf{A}$ is irreducible if there is no permutation of coordinates such that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \tag{A.1}$$

where $\mathbf{P}$ is an $n \times n$ permutation matrix with each row and each column having exactly one element of 1 and all other elements of 0, $\mathbf{A}_{11}$ is $r \times r$, $\mathbf{A}_{22}$ is $(n-r) \times (n-r)$, and $\mathbf{A}_{12}$ is $n \times (n-r)$. That is, an irreducible matrix cannot be placed into block upper-triangular form by simultaneous row/column permutations.

Theorem: A nonnegative $n \times n$ matrix $\mathbf{A}$ is irreducible if and only if $(\mathbf{I}+\mathbf{A})^{n-1} \succ \mathbf{0}$, where $\mathbf{I}$ is an $n \times n$ identity matrix, and $\succ$ is element-wise larger than.

Definition: Let λ_i, $i = 1, 2, \ldots, n$, be the eigenvalues of an $n \times n$ matrix $\mathbf{A}$. Then the spectral radius of the matrix is defined as $\rho(\mathbf{A}) \stackrel{\text{def}}{=} \max_i(|\lambda_i|)$.

Perron-Frobenius theorem for irreducible matrices: If $\mathbf{A} = (a_{ij})$ is an $n \times n$ nonnegative and irreducible matrix, then

- one of its eigenvalues is positive and greater than or equal to (in absolute value) all other eigenvalues. Such an eigenvalue is called the "dominant eigenvalue" or Perron-Frobenius eigenvalue of the matrix;
- there is a positive eigenvector corresponding to that eigenvalue; and
- $\rho(\mathbf{A})$ is equal to the dominant eigenvalue of the matrix and satisfies

$$\min_i \sum_j a_{ij} \leq \rho(\mathbf{A}) \leq \max_i \sum_j a_{ij}.$$

References

1. Varga RS (1962) Matrix iterative analysis, Chapter 2, Prentice-Hall, Inc., Englewood Cliffs, N.J.

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Appendix B
Posynomial and Related Optimization Problems

Definition: A monomial is a function of the form

$$h(\mathbf{x}) = dx_1^{a^{(1)}} x_2^{a^{(2)}} \dots x_n^{a^{(n)}}, \tag{B.1}$$

where d is nonnegative, x_i's are positive real numbers, and $a^{(i)}$'s are real numbers. Monomials are closed under multiplication and division.

Definition: A posynomial is a sum of monomials and of the form

$$f(\mathbf{x}) = \sum_{k=1}^{K} d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \dots x_n^{a_k^{(n)}}, \tag{B.2}$$

where d_k's are nonnegative, x_i's are positive real numbers, and $a_k^{(i)}$'s are real numbers. Posynomials are closed under addition, multiplication, and nonnegative scaling.

Definition: A standard geometric programming (GP) problem is as follows

$$\min f_0(\mathbf{x}) \tag{B.3}$$

$$\text{s.t. } f_i(\mathbf{x}) \le 1, \ i = 1,2,\dots,m \tag{B.4}$$

$$h_l(\mathbf{x}) = 1, \ l = 1,2,\dots,n \tag{B.5}$$

where $f_i(\mathbf{x})$, $i = 0,1,\dots,m$, are posynomials,

$$f_i(\mathbf{x}) = d_{ik} x_1^{a_{ik}^{(1)}} x_2^{a_{ik}^{(2)}} \dots x_n^{a_{ik}^{(n)}}, \tag{B.6}$$

and $h_l(\mathbf{x})$, $l = 1,2,\dots,n$, are monomials.

Consider the following optimization problem

$$\min f_0(\mathbf{x}) \tag{B.7}$$

$$\text{s.t. } f_i(\mathbf{x}) \leq 1,\ i = 1,2,\ldots,m \tag{B.8}$$

Case 1: When $f_i(\mathbf{x})$ for $i = 0,1,\ldots,m$ are all posynomials of the form (B.6), the problem is a standard GP problem, which in general is not convex, but can be transformed into a convex problem. With a change of variables: $y_i = \log x_i$ and $b_{ik} = \log d_{ik}$, (B.7) can be converted into convex form [2]:

$$\min \tilde{f}_0(\mathbf{y}) = \log\left(\sum_k e^{\mathbf{a}_{0k}^T \mathbf{y} + b_{0k}}\right) \tag{B.9}$$

$$\text{s.t. } \tilde{f}_i(\mathbf{y}) = \log\left(\sum_k e^{\mathbf{a}_{ik}^T \mathbf{y} + b_{ik}}\right) \leq 0,\ i = 1,2,\ldots,m, \tag{B.10}$$

where $\mathbf{a}_{ik} = (a_{ik}^{(1)}, a_{ik}^{(2)},, \ldots, a_{ik}^{(n)})^T$. Since the functions $\tilde{f}_i(\mathbf{x})$ are convex, this problem is a convex optimization problem, which can be solved globally and efficiently through the interior point primal dual method [2] with polynomial running time.

Case 2: When $f_0(\mathbf{x})$ is convex, and $f_i(\mathbf{x})$, $1 \leq i \leq m$, is in the format of a ratio of posynomials, i.e., $f_i(\mathbf{x}) = s(\mathbf{x})/g(\mathbf{x})$, the optimization problem is not convex and difficult to solve directly. A successive approximation method is designed in [1], where the basic idea is to solve such a problem by a series of approximations, each of which can be optimally solved in an easy way.

The problem can be turned into a geometrical programming (GP) problem by approximating the denominator of the ratio of posynomials, $g(\mathbf{x})$, with a monomial $\tilde{g}(\mathbf{x})$, but leaving the numerator $s(\mathbf{x})$ unchanged. It is proved in [1] that if $g(\mathbf{x}) = \sum_i u_i(\mathbf{x})$ is a posynomial, then

$$g(\mathbf{x}) \geq \tilde{g}(\mathbf{x}) = \prod_i \left[\frac{u_i(\mathbf{x})}{\alpha_i}\right]^{\alpha_i}. \tag{B.11}$$

If, in addition, $\alpha_i = \frac{u_i(\mathbf{x}_0)}{g(\mathbf{x}_0)}$, $\forall i$, for any fixed positive $\mathbf{x}_0$, then $\tilde{g}(\mathbf{x}_0) = g(\mathbf{x}_0)$, and $\tilde{g}(\mathbf{x}_0)$ is the best local monomial approximation to $g(\mathbf{x}_0)$ near $\mathbf{x}_0$ in the sense of the first order Taylor approximation. It is further proved in [1] that the approximation of a ratio of posynomials $f_i(\mathbf{x}) = s(\mathbf{x})/g(\mathbf{x})$ with $\tilde{f}_i(\mathbf{x}) = s(\mathbf{x})/\tilde{g}(\mathbf{x})$ satisfies the Karush-Kuhn-Tucker (KKT) conditions:

(1) $f_i(\mathbf{x}) \leq \tilde{f}_i(\mathbf{x})$ for all $\mathbf{x}$,
(2) $f_i(\mathbf{x}_0) = \tilde{f}_i(\mathbf{x}_0)$ where $\mathbf{x}_0$ is the optimal solution of the approximated problem in the previous iteration, and
(3) $\nabla f_i(\mathbf{x}_0) = \nabla \tilde{f}_i(\mathbf{x}_0)$.

With the above process, the denominator of $f_i(\mathbf{x})$ is approximated as a monomial, and $f_i(\mathbf{x})$ is then approximated as a posynomial. An iterative method as follows is then proposed in [1] to solve the original optimization problem:

Step 0: Choose an initial feasible point $\mathbf{x}^{(0)}$ and set $j = 1$.
Step 1: Approximate $g_i(x)$ with $\tilde{g}_i(\mathbf{x})$ around the previous point $\mathbf{x}^{(j-1)}$.
Step 2: Solve the approximated problem and obtain solution $\mathbf{x}^{(j)}$.
Step 3: Increase j by 1 and go back to Step 2 until the solution converges.

The convergency of this method is guaranteed by the KKT conditions in the approximation.

References

1. Chiang M (2005) Geometric programming for communication systems. Foundations and Trends in Communications and Information Theory 2(1-2): 1-154.
2. Boyd S, Vandenberghe L (2004) Convex Optimization. Cambridge University Press.