BA Jacobian矩阵推导

计算空间中第i个三维点在第j个相机投影的雅可比矩阵。为书写方便，以下省略掉空间三维点的下标i和相机的下标j。假设空间三维点P,坐标为$\boldsymbol{X}=[\mathrm{X},\mathrm{Y},\mathrm{Z}]^T$,在相机$C$中的投影点为$p$,坐标为 $\boldsymbol{x}=[u,v]^T$，其中与相机有关的参数是焦距f，径向畸变系数k_0,k_1，旋转矩阵(角轴法表示)$\boldsymbol{w}=[w_0,w_1,w_2]^T$，平移向量$\boldsymbol{t}=[t_0,t_1,t_2]^T$。$p$相对于相机参数以及三维点的雅可比矩阵可以表示为

$$\begin{bmatrix} \frac{\partial u(\boldsymbol{C},\boldsymbol{X})}{\partial \boldsymbol{C}} & \frac{\partial u(\boldsymbol{C},\boldsymbol{X})}{\partial \boldsymbol{X}} \\ \frac{\partial v(\boldsymbol{C},\boldsymbol{X})}{\partial \boldsymbol{C}} & \frac{\partial v(\boldsymbol{C},\boldsymbol{X})}{\partial \boldsymbol{X}} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial f},\frac{\partial u}{\partial k_0},\frac{\partial u}{\partial k_1},\frac{\partial u}{\partial w_0},\frac{\partial u}{\partial w_1},\frac{\partial u}{\partial w_2},\frac{\partial u}{\partial t_0},\frac{\partial u}{\partial t_1},\frac{\partial u}{\partial t_2},\frac{\partial u}{\partial \mathrm{X}},\frac{\partial u}{\partial \mathrm{Y}},\frac{\partial u}{\partial \mathrm{Z}} \\ \frac{\partial v}{\partial f},\frac{\partial v}{\partial k_0},\frac{\partial v}{\partial k_1},\frac{\partial v}{\partial w_0},\frac{\partial v}{\partial w_1},\frac{\partial v}{\partial w_2},\frac{\partial v}{\partial t_0},\frac{\partial v}{\partial t_1},\frac{\partial v}{\partial t_2},\frac{\partial v}{\partial \mathrm{X}},\frac{\partial v}{\partial \mathrm{Y}},\frac{\partial v}{\partial \mathrm{Z}} \end{bmatrix}$$

1. 相机投影过程

为了更为方便的求解每一项的偏导数，我们首先给出三维点到二维点的投影过程。

1.1 世界坐标系到相机坐标系的变换

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \boldsymbol{RX}+\boldsymbol{t} \ = \boldsymbol{R}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix}$$

1.2 归一化像平面

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x_c}{z_c} \\ \frac{y_c}{z_c} \end{bmatrix}$$

1.3 相机成像平面

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} fd(k_0,k_1,r^2)x \\ fd(k_0,k_1,r^2)y \end{bmatrix},$$

其中 $d(k_0,k_1,r^2)=1+(k_0+k_1r^2)r^2,$

$$r^2=x^2+y^2$$

2. 链式法则推到 Jacobian 矩阵

2.1 计算关于焦距的偏导数$\frac{\partial u}{\partial f},\frac{\partial v}{\partial f}$

$$\frac{\partial u}{\partial f}=d(k_0,k_1,r^2)x$$

$$\frac{\partial v}{\partial f}=d(k_0,k_1,r^2)y$$

2.2 计算关于径向畸变系数$\frac{\partial u}{\partial k_0},\frac{\partial u}{\partial k_1},\frac{\partial v}{\partial k_0},\frac{\partial v}{\partial k_1}$

引入中间变量

$$\frac{\partial u}{\partial d(k_0,k_1,r^2)}=fx$$

$$\frac{\partial v}{\partial d(k_0,k_1,r^2)}=fy$$

$$\frac{\partial d(k_0,k_1,r^2)}{\partial k_0}=r^2$$

$$\frac{\partial d(k_0,k_1,r^2)}{\partial k_1}=r^4$$

根据链式法则，我们可以得到

$$\frac{\partial u}{\partial k_0}=\frac{\partial u}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{\partial k_0}=fxr^2$$

$$\frac{\partial u}{\partial k_1}=\frac{\partial u}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{\partial k_1}=fxr^4$$

$$\frac{\partial v}{\partial k_0}=\frac{\partial v}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{\partial k_0}=fyr^2$$

$$\frac{\partial v}{\partial k_1}=\frac{\partial v}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{\partial k_1}=fyr^4$$

计算关于平移向量和旋转向量的偏导数略微复杂，为了方便采用链式法则进行求导，引入一些中间量

$$\frac{\partial x}{\partial x_c}=\frac{1}{z_c},\frac{\partial x}{\partial y_c}=0,\frac{\partial x}{\partial z_c}=-\frac{x_c}{z_c^2}=-\frac{x}{z_c}$$

$$\frac{\partial y}{\partial x_c}=0,\frac{\partial y}{\partial y_c}=\frac{1}{z_c},\frac{\partial y}{\partial z_c}=-\frac{y_c}{z_c^2}=-\frac{y}{z_c}$$

$$\frac{\partial u}{\partial x}=fd(k_0,k_1,r^2)$$

$$\frac{\partial v}{\partial y}=fd(k_0,k_1,r^2)$$

$$\frac{\partial d(k_0,k_1,r^2)}{\partial r^2}=k_0+2k_1r^2$$

$$\frac{\partial r^2}{\partial x_c}=2x\frac{\partial x}{\partial x_c}=\frac{2x}{z_c}$$

$$\frac{\partial r^2}{\partial y_c}=2y\frac{\partial y}{\partial y_c}=\frac{2y}{z_c}$$

$$\frac{\partial r^2}{\partial z_c}=2x\frac{\partial x}{\partial z_c}+2y\frac{\partial y}{\partial z_c}=-2x\frac{x}{z_c}-2y*\frac{y}{z_c}$$

$$=-2\frac{x^2+y^2}{z_c}=-2\frac{r^2}{z_c}$$

$$\frac{\partial d(k_0,k_1,r^2)}{\partial x_c}=\frac{\partial d(k_0,k_1,r^2)}{\partial r^2}\frac{\partial r^2}{\partial x_c}=(k_0+2k_1r^2)\frac{2x}{z_c}$$

$$\frac{\partial d(k_0,k_1,r^2)}{\partial y_c}=\frac{\partial d(k_0,k_1,r^2)}{\partial r^2}\frac{\partial r^2}{\partial y_c}=(k_0+2k_1r^2)\frac{2y}{z_c}$$

$$\frac{\partial d(k_0,k_1,r^2)}{\partial z_c}=\frac{\partial d(k_0,k_1,r^2)}{\partial r^2}\frac{\partial r^2}{\partial z_c}=-(k_0+2k_1r^2)\frac{2r^2}{z_c}$$

$$\frac{\partial u}{\partial x_c}=\frac{\partial u}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{x_c}+\frac{\partial u}{\partial x}\frac{\partial x}{x_c}$$

$$\frac{\partial u}{\partial y_c}=\frac{\partial u}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{y_c}+\frac{\partial u}{\partial x}\frac{\partial x}{y_c}$$

$$\frac{\partial u}{\partial z_c}=\frac{\partial u}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{z_c}+\frac{\partial u}{\partial x}\frac{\partial x}{z_c}$$

$$\frac{\partial v}{\partial x_c}=\frac{\partial v}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{x_c}+\frac{\partial v}{\partial y}\frac{\partial y}{x_c}$$

$$\frac{\partial v}{\partial y_c}=\frac{\partial v}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{y_c}+\frac{\partial v}{\partial y}\frac{\partial y}{y_c}$$

$$\frac{\partial v}{\partial z_c}=\frac{\partial v}{\partial d(k_0,k_1,r^2)}\frac{\partial d(k_0,k_1,r^2)}{z_c}+\frac{\partial v}{\partial y}\frac{\partial y}{z_c}$$

2.3 计算关于平移向量的偏导数$\frac{\partial u}{\partial t_0},\frac{\partial u}{\partial t_0},\frac{\partial u}{\partial t_0},\frac{\partial u}{\partial t_0},\frac{\partial u}{\partial t_0},\frac{\partial u}{\partial t_0}$

引入中间变量

$$\frac{\partial x_c}{\partial t_0}=1,\frac{\partial x_c}{\partial t_1}=0,\frac{\partial x_c}{\partial t_2}=0$$

$$\frac{\partial y_c}{\partial t_0}=0,\frac{\partial y_c}{\partial t_1}=1,,\frac{\partial y_c}{\partial t_2}=0$$

$$\frac{\partial z_c}{\partial t_0}=0,\frac{\partial z_c}{\partial t_1}=0,\frac{\partial z_c}{\partial t_2}=1$$

根据链式法则求取关于平移向量的偏导数

$$\frac{\partial u}{\partial t_0}=\frac{\partial u}{\partial x_c}\frac{\partial x_c}{\partial t_0}=\frac{\partial u}{\partial x_c}$$

$$\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial y_c}\frac{\partial y_c}{\partial t_1}=\frac{\partial u}{\partial y_c}$$

$$\frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial z_c}\frac{\partial z_c}{\partial t_2}=\frac{\partial u}{\partial z_c}$$

$$\frac{\partial v}{\partial t_0}=\frac{\partial v}{\partial x_c}\frac{\partial x_c}{\partial t_0}=\frac{\partial v}{\partial x_c}$$

$$\frac{\partial v}{\partial t_1}=\frac{\partial v}{\partial y_c}\frac{\partial y_c}{\partial t_1}=\frac{\partial v}{\partial y_c}$$

$$\frac{\partial v}{\partial t_2}=\frac{\partial v}{\partial z_c}\frac{\partial z_c}{\partial t_2}=\frac{\partial v}{\partial z_c}$$

2.4 计算关于旋转(角轴向量)的偏导数$\frac{\partial u}{\partial w_0},\frac{\partial u}{\partial w_1},\frac{\partial u}{\partial w_2},\frac{\partial v}{\partial w_0},\frac{\partial v}{\partial w_1},\frac{\partial v}{\partial w_2}$

根据 Rodrigues 公式

$$\boldsymbol{R}=\cos\|\boldsymbol{w}\|\boldsymbol{I}_{3\times 3}+\frac{[\boldsymbol{w}]_\times}{\|\boldsymbol{w}\|}sin(\|\boldsymbol{w}\|)+\frac{[\boldsymbol{w}]_\times^2}{\|\boldsymbol{w}\|^2}(1-\cos(\|\boldsymbol{w}\|))$$

对于微小变化，$\cos(\|\boldsymbol{w}\|)\approx 1,\sin(\|\boldsymbol{w}\|)\approx\|\boldsymbol{w}\|$

$$\delta\boldsymbol{R}\approx\boldsymbol{I}_{3\times 3}+[\boldsymbol{w}]_\times$$

其中

$$[\boldsymbol{w}]_\times=\begin{bmatrix}0,-w_2,w_1\\ w_2,0,-w_0\\ -w_1,w_0,0\end{bmatrix}$$

在世界坐标系向相机坐标系转换过程中添加左扰动

$$\begin{bmatrix}x_c\\ y_c\\ z_c\end{bmatrix}=\delta\boldsymbol{R}\,\boldsymbol{R}\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}+T=(\boldsymbol{I}_{3\times 3}+[\boldsymbol{w}]_\times)\boldsymbol{RX}+\boldsymbol{T}$$

$$=\begin{bmatrix}x_c\\ y_c\\ z_c\end{bmatrix}+\begin{bmatrix}0,-w_2,w_1\\ w_2,0,-w_0\\ -w_1,w_0,0\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_0\boldsymbol{X}\\ \boldsymbol{r}_1\boldsymbol{X}\\ \boldsymbol{r}_2\boldsymbol{X}\end{bmatrix}$$

$$= \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} + \begin{bmatrix} -w_2\boldsymbol{r}_1\boldsymbol{X} + w_1\boldsymbol{r}_2\boldsymbol{X} \\ w_2\boldsymbol{r}_0\boldsymbol{X} - w_0\boldsymbol{r}_2\boldsymbol{X} \\ -w_1\boldsymbol{r}_0\boldsymbol{X} + w_0\boldsymbol{r}_1\boldsymbol{X} \end{bmatrix}$$

其中$\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2$是旋转矩阵$\boldsymbol{R}$的第 1-3 行。

引入中间变量

$$\frac{\partial x_c}{\partial w_0} = 0, \frac{\partial x_c}{\partial w_1} = \boldsymbol{r}_2\boldsymbol{X}, \frac{\partial x_c}{\partial w_2} = -\boldsymbol{r}_1\boldsymbol{X}$$

$$\frac{\partial y_c}{\partial w_0} = -\boldsymbol{r}_2\boldsymbol{X}, \frac{\partial y_c}{\partial w_1} = 0, \frac{\partial y_c}{\partial w_2}\boldsymbol{r}_0\boldsymbol{X}$$

$$\frac{\partial z_c}{\partial w_0} = \boldsymbol{r}_1\boldsymbol{X}, \frac{\partial z_c}{\partial w_1} = -\boldsymbol{r}_0\boldsymbol{X}, \frac{\partial z_c}{\partial w_2} = 0$$

根据链式法则求取对于旋转向量的偏导数

$$\frac{\partial u}{\partial w_0} = \frac{\partial u}{\partial y_c}\frac{\partial y_c}{\partial w_0} + \frac{\partial u}{\partial z_c}\frac{\partial z_c}{\partial w_0}$$

$$\frac{\partial u}{\partial w_1} = \frac{\partial u}{\partial x_c}\frac{\partial x_c}{\partial w_1} + \frac{\partial u}{\partial z_c}\frac{\partial z_c}{\partial w_1}$$

$$\frac{\partial u}{\partial w_2} = \frac{\partial u}{\partial x_c}\frac{\partial x_c}{\partial w_2} + \frac{\partial u}{\partial y_c}\frac{\partial y_c}{\partial w_2}$$

$$\frac{\partial v}{\partial w_0} = \frac{\partial v}{\partial y_c}\frac{\partial y_c}{\partial w_0} + \frac{\partial v}{\partial z_c}\frac{\partial z_c}{\partial w_0}$$

$$\frac{\partial v}{\partial w_1} = \frac{\partial v}{\partial x_c}\frac{\partial x_c}{\partial w_1} + \frac{\partial v}{\partial z_c}\frac{\partial z_c}{\partial w_1}$$

$$\frac{\partial v}{\partial w_2} = \frac{\partial v}{\partial x_c}\frac{\partial x_c}{\partial w_2} + \frac{\partial v}{\partial y_c}\frac{\partial y_c}{\partial w_2}$$

2.5 求关于三维点的偏导数

将世界坐标系到相机坐标系的变换写成如下形式

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \boldsymbol{RX} + \boldsymbol{t} \ = \boldsymbol{R}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix}$$
$$= \begin{bmatrix} R_{00}, R_{01}, R_{02} \\ R_{10}, R_{11}, R_{12} \\ R_{20}, R_{21}, R_{22} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_0 \\ t_1 \\ t_2 \end{bmatrix}$$
$$= \begin{bmatrix} R_{00}X + R_{01}Y + R_{02}Z + t_0 \\ R_{10}X + R_{11}Y + R_{12}Z + t_1 \\ R_{20}X + R_{21}Y + R_{22}Z + t_2 \end{bmatrix}$$

我们可以得到

$$\frac{\partial x_c}{\partial X}=R_{00},\frac{\partial x_c}{\partial Y}=R_{01},\frac{\partial x_c}{\partial Z}R_{02}$$

$$\frac{\partial y_c}{\partial X}=R_{10},\frac{\partial y_c}{\partial Y}=R_{11},\frac{\partial y_c}{\partial Z}R_{12}$$

$$\frac{\partial z_c}{\partial X}=R_{20},\frac{\partial z_c}{\partial Y}=R_{21},\frac{\partial z_c}{\partial Z}R_{22}$$

根据链式法则，可以得到

$$\frac{\partial u}{\partial \mathrm{X}}=\frac{\partial u}{\partial x_c}\frac{\partial x_c}{\partial X}+\frac{\partial u}{\partial y_c}\frac{\partial y_c}{\partial X}+\frac{\partial u}{\partial z_c}\frac{\partial z_c}{\partial X}$$

$$\frac{\partial u}{\partial \mathrm{Y}}=\frac{\partial u}{\partial x_c}\frac{\partial x_c}{\partial Y}+\frac{\partial u}{\partial y_c}\frac{\partial y_c}{\partial Y}+\frac{\partial u}{\partial z_c}\frac{\partial z_c}{\partial Y}$$

$$\frac{\partial u}{\partial \mathrm{Z}}=\frac{\partial u}{\partial x_c}\frac{\partial x_c}{\partial Z}+\frac{\partial u}{\partial y_c}\frac{\partial y_c}{\partial Z}+\frac{\partial u}{\partial z_c}\frac{\partial z_c}{\partial Z}$$

$$\frac{\partial v}{\partial \mathrm{X}}=\frac{\partial v}{\partial x_c}\frac{\partial x_c}{\partial X}+\frac{\partial v}{\partial y_c}\frac{\partial y_c}{\partial X}+\frac{\partial v}{\partial z_c}\frac{\partial z_c}{\partial X}$$

$$\frac{\partial v}{\partial \mathrm{Y}}=\frac{\partial v}{\partial x_c}\frac{\partial x_c}{\partial Y}+\frac{\partial v}{\partial y_c}\frac{\partial y_c}{\partial Y}+\frac{\partial v}{\partial z_c}\frac{\partial z_c}{\partial Y}$$

$$\frac{\partial v}{\partial \mathrm{Z}}=\frac{\partial v}{\partial x_c}\frac{\partial x_c}{\partial Z}+\frac{\partial v}{\partial y_c}\frac{\partial y_c}{\partial Z}+\frac{\partial v}{\partial z_c}\frac{\partial z_c}{\partial Z}$$