《微分几何入门与广义相对论》(上册)
习题解答

第 1 章“拓扑空间简介”习题

˜1. 试证 $A-B=A\cap(X-B)$, $\forall\,A,B\subset X$.

证 只须证明等式两边互为包含.

(A) 设 $x\in A-B$, 则 $x\in A$, $x\notin B$. 前者 $x\in A$ 与 $A\subset X$ 结合得 $x\in X$. 再与后者 $x\notin B$ 结合得 $x\in X-B$. 结合前者 $x\in A$ 知 $x\in A\cap(X-B)$. 于是属于 $A-B$ 的元素都属于 $A\cap(X-B)$, 因而 $A-B\subset A\cap(X-B)$.

(B) 设 $x\in A\cap(X-B)$, 则 $x\in A$, $x\in X-B$. 后者导致 $x\in X$, $x\notin B$. $x\notin B$ 与前者 $x\in A$ 结合得 $x\in A-B$. 于是属于 $A\cap(X-B)$ 的元素必定属于 $A-B$, 因而 $A\cap(X-B)\subset A-B$.

˜2. 试证 $X-(B-A)=(X-B)\cup A$, $\forall\,A,B\subset X$.

证 只须证明等式两边互为包含.

(A) 设 $x\in X-(B-A)$, 则 $x\in X$, $x\notin B-A$. 后者导致 $x\notin B$ 或 $x\in A$. $x\notin B$ 与前者 $x\in X$ 结合得 $x\in X-B$. 现在是 $x\in A$ 或 $x\in X-B$, 即 $x\in(X-B)\cup A$. 于是属于 $X-(B-A)$ 的元素都属于 $(X-B)\cup A$, 因而 $X-(B-A)\subset(X-B)\cup A$.

(B) 设 $x\in(X-B)\cup A$, 则 $x\in X-B$ 或 $x\in A$. 前者导致 $x\in X$, $x\notin B$. $x\notin B$ 与后者 $x\in A$ 或 的结合, 即 $x\in A$ 或 $x\notin B$ 给出 $x\notin B-A$. 因此 $x\in X-(B-A)$. 于是属于 $(X-B)\cup A$ 的元素都属于 $X-(B-A)$, 因而 $(X-B)\cup A\subset X-(B-A)$.

˜3. 用“对”或“错”在下表中填空:

$f:RR\to RR$	是一一的	是到上的
$f(x)=x^3$	(对)	(对)
$f(x)=x^2$	(错, 正的 $f(x)$ 有两个逆像)	(错, 负的 $f(x)$ 没有逆像)
$f(x)=e^x$	(对)	(错, 0 和负的 $f(x)$ 没有逆像)
$f(x)=\cos x$	(错, $\lvert f(x)\rvert\in[0,1]$ 有无数个逆像)	(错, $\lvert f(x)\rvert\in(1,\infty)$ 没有逆像)
$f(x)=5,\ \forall x\in RR$	(错, 有无数个逆像)	(错, 除了 $f(x)=5$ 没有逆像)

˜4. 判断下列说法的是非并简述理由:

(a) 正切函数是 RR 到 RR 的映射;

答 不对. 因为 $x=n\pi+\frac{\pi}{2}$ (n 为整数) 没有像 $\tan x$.

(b) 对数函数是 RR 到 RR 的映射;

答 不对. 因为 $x\in(-\infty,0)$ 没有像 $\log x$.

(c) $(a,b] \subset RR$ 用 $\mathcal{T}_u$ 衡量是开集;

答 不对. 因为 $\mathcal{T}_u$ 的元素为 RR 的开区间或开区间之并, 故 $(a,b] \notin \mathcal{T}_u$. 只有 $\mathcal{T}_u$ 的元素才是开集, 所以 $(a,b]$ 用 $\mathcal{T}_u$ 衡量不是开集.

(d) $[a,b] \subset RR$ 用 $\mathcal{T}_u$ 衡量是闭集;

答 对. 因为 $[a,b] \subset RR$, $-[a,b] = RR - [a,b] = (-\infty, a) \cup (b, \infty) \in \mathcal{T}_u$, 所以根据定义 6 知道用 $\mathcal{T}_u$ 衡量 $[a,b]$ 是闭集.

[由 (c) 的结果和 (d) 的推理过程可以知道用通常拓扑 $\mathcal{T}_u$ 衡量, $(a,b]$ 既不是开集 也不是 闭集!]

~5. 举一反例证明命题" $(RR, \mathcal{T}_u)$ 的无限个开子集之交为开"不真.

解 设开区间族 $O_n = (0, 1/n) \in \mathcal{T}_u$, $(n = 1, 2, \cdots)$. 可知 $\cap_{n=1}^{\infty} O_n = \{0\}$. 而单点集 (区间) 不是开区间, 即 $\{0\} \notin \mathcal{T}_u$, 故"无限个开子集之交为开"不真. 事实上 $\{0\}$ 为闭集, 因为 $-\{0\} = RR - \{0\} = (-\infty, 0) \cup (0, \infty) \overset{定义\ 1(c)}{\in} \mathcal{T}_u$, 故根据 §1.2 定义 6 单点集 $\{0\}$ 是闭集.

~6. 试证 §1.2 例 5 中定义的诱导拓扑满足定义 1 的 3 个条件.

证 诱导拓扑的定义是 (1-2-2) 式: $\mathcal{S} := \{V \subset A | \ \exists O \in \mathcal{T}$ 使 $V = A \cap O\}$.

条件 1: 必须 存在 $O \supset A$, 这时 $V = A \cap O = A$, 即 A 是 $\mathcal{S}$ 的元素. 另外如果取 $O = \emptyset$, 则 $V = A \cap \emptyset = \emptyset$ 也是 $\mathcal{S}$ 的元素.

条件 2: 如果 $V_i = A \cap O_i \in \mathcal{S}$, 因为 $\cap_i V_i = \cap_i (A \cap O_i) \overset{结合律}{=} A \cap (\cap_i O_i)$, 显然有 $\cap_i V_i \subset A$. 另外由 §1.2 定义 1(b), $\cap_i O_i \in \mathcal{T}$, 故由诱导拓扑的定义知 $\cap_i V_i$ 也是 $\mathcal{S}$ 的元素.

条件 3: 如果 $V_\alpha = A \cap O_\alpha \in \mathcal{S} \ \forall \alpha$, 首先我们根据分配律证明 $\cup_\alpha V_\alpha = \cup_\alpha (A \cap O_\alpha) = A \cap (\cup_\alpha O_\alpha)$. 注意到 $(A \cap O_1) \cup (A \cap O_2) = A \cap (O_1 \cup O_2)$, 所以有 $(A \cap O_1) \cup (A \cap O_2) \cup (A \cap O_3) = [A \cap (O_1 \cup O_2)] \cup (A \cap O_3) = A \cap [(O_1 \cup O_2) \cup O_3] \overset{结合律}{=} A \cap [O_1 \cup O_2 \cup O_3]$. 推广得 $\cup_\alpha (A \cap O_\alpha) = A \cap (\cup_\alpha O_\alpha)$. 于是 $\cup_\alpha V_\alpha = A \cap (\cup_\alpha O_\alpha)$, 有 $\cup_\alpha V_\alpha \subset A$. 另外由 §1.2 定义 1(c), $\cup_\alpha O_\alpha \in \mathcal{T}$, 故由诱导拓扑的定义知 $\cup_\alpha V_\alpha$ 也是 $\mathcal{S}$ 的元素.

7. 举例说明 $(RR^3, \mathcal{T}_u)$ 中存在不开不闭的子集.

答 定义空间的某块内部的所有点以及部分表面点属于该子集 A, 那么 A 既不开也不闭. 不开是因为它不属于 $\mathcal{T}_u$, 即它不能通过有限的开球之交和无限的开球之并得到. 不闭是因为 $-A = RR^3 - A$ 也不属于 $\mathcal{T}_u$, 只有当 $-A \in \mathcal{T}_u$ 为开时, A 才是闭的.

~8. 常值映射 $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ 是否连续? 为什么?

答 连续. 所谓常值映射 (§1.1 定义 7) 是指 $\forall x \in X$, $x \mapsto f(x) = y_0 \in Y$, 只是 Y 的一个元素 (一点). 对 $\forall O \in \mathcal{S}$, 根据 §1.1 注 5(2) ②, 要么 $f^{-1}[O] = \emptyset$ (当 $y_0 \notin O \in \mathcal{S}$), 要么 $f^{-1}[O] = X$ (当 $y_0 \in O$). 而 $\emptyset$ 和 X 都是 $\mathcal{T}$ 的元素, 所以 $f^{-1}[O] \in \mathcal{T}$. 根据 §1.2 定义 3a, 这一映射连续.

~9. 设 $\mathcal{T}$ 为集 X 上的离散拓扑, $\mathcal{S}$ 为集 Y 上的凝聚拓扑,

(a) 找出从 $(X, \mathcal{T})$ 到 $(Y, \mathcal{S})$ 的全部连续映射;

(b) 找出从 $(Y, \mathcal{S})$ 到 $(X, \mathcal{T})$ 的全部连续映射.

解 $(X, \mathcal{T})$ 为离散拓扑空间 (§1.2 例 1), $\mathcal{T}$ 的元素为 X 的所有子集; $(Y, \mathcal{S})$ 为凝聚拓扑空间 (§1.2 例 2), $\mathcal{S}$ 的元素只有 2 个 — $\emptyset$ 和 Y. (如果 $X = Y = RR$, 则 $\mathcal{T}$ 的元素为 RR 的所有区间的集合, 无论开闭; 而 $\mathcal{S}$ 的元素也只有 2 个 — $\emptyset$ 和 RR.)

(a) 从 $(X, \mathcal{T})$ 到 $(Y, \mathcal{S})$ 的任何映射都是连续的. 因为只要是映射, 就有逆像, 而逆像的集合一定是 X 的子集, 又 X 的子集一定是离散拓扑 $\mathcal{T}$ 的元素. 因此根据 §1.2 定义 3a, 这一映射是连续的. 注意因为 $\mathcal{S}$ 为凝聚拓扑, 定义 3a 中现在的 O 只可能有两个: 一是 $O = \emptyset \in \mathcal{S}$, 则 $f^{-1}[\emptyset] = \emptyset \in \mathcal{T}$; 二是 $O = Y \in \mathcal{S}$, 则 $f^{-1}[Y] = X \in \mathcal{T}$. 由此推理过程可以看出, 只要 $(X, \mathcal{T})$ 是离散拓扑空间, $(Y, \mathcal{S})$ 可以是任何拓扑空间, 这一结论仍然成立. (如果 $X = Y = RR$, 就是说逆像的点的集合构成的区间无论开闭都是 RR 的子集, 也就是 $\mathcal{T}$ 的元素.)

(b) 从 $(Y, \mathcal{S})$ 到 $(X, \mathcal{T})$ 只有常值映射是连续的. 常值映射的连续性从习题 8 的结论可以直接看出, 下面要证明其他任何映射都不是连续的. 如果不是常值映射, 则至少有两个像 $x_1, x_2 \in X$, $x_1 \neq x_2$. 现在考虑 X 的单元素子集 $O_1 = \{x_1\}$ 和 $O_2 = \{x_2\}$. 因为 $\mathcal{T}$ 是离散拓扑, X 的所有子集都是它的元素, 所以 $O_1, O_2 \in \mathcal{T}$. 下面看它们的逆像 $f^{-1}[O_1] = f^{-1}[\{x_1\}]$ 和 $f^{-1}[O_2] = f^{-1}[\{x_2\}]$, 当然它们是 Y 的 非空 子集 (不然的话不会有像 x_1 和 x_2). 首先注意到 $f^{-1}[O_1] \cap f^{-1}[O_2] = \emptyset$, 因为否则的话必存在原像点 $y \in f^{-1}[O_1] \cap f^{-1}[O_2]$, 它有两个不同的像 x_1 和 x_2, 这与 §1.1 映射的定义 5 不符. 如果这一映射是连续的, 根据定义 3a, 要求 $f^{-1}[O_1], f^{-1}[O_2] \in \mathcal{S}$. 而 $\mathcal{S}$ 是凝聚拓扑, 非空元素只有 Y, 连续映射要求 $f^{-1}[O_1] = f^{-1}[O_2] = Y$, 这显然与它们的不相交性矛盾. 因此, 任何多于一个像的映射在此情形下都是不连续的. 与前面一样, 只要 $(Y, \mathcal{S})$ 是凝聚拓扑空间, $(X, \mathcal{T})$ 可以是任何拓扑空间, 这一结论仍然成立.

~10. 试证 §1.2 定义 3a 与 3b 的等价性.

证 (A) 从 3a 到 3b, 即要证明如果用 3a 定义的映射连续, 则用 3b 定义的任意点都连续; (B) 从 3b 到 3a, 即要证明如果用 3b 定义的任意点都连续, 则用 3a 定义的映射连续.

(A) 考虑任意一点的映射 $x \mapsto f(x)$. 在拓扑 $\mathcal{S}$ 中任意取两个元素 G' 和 G'', 使满足 $f(x) \in G'' \subset G'$. 因为映射是连续的，根据定义 3a 有 $G \equiv f^{-1}[G''] \in \mathcal{T}$, 当然 $x \in G$. 所以现在 $\exists G \in \mathcal{T}$ 使 $x \in G$ 且 $f[G] = G'' \subset G'$. 根据定义 3b, 映射在点 x 处连续. $x \in X$ 是任意的.

(B) 考虑任意的一个开集 $O \in \mathcal{S}$, 设它的元素为 y_α, 即 $\forall\, \alpha$, $y_\alpha \in O$. 如果 y_α 有逆像 $x_\alpha = f^{-1}(y_\alpha)$ 且映射是连续的, 则根据定义 3b, 一定存在开集 $G_\alpha \in \mathcal{T}$ 使 $x_\alpha \in G_\alpha$ 且 $f[G_\alpha] \subset O$. 可以证明 $\cup_\alpha G_\alpha = f^{-1}[O]$, 由定义 1(c) 知 $f^{-1}[O] \in \mathcal{T}$. 于是根据定义 3a, 映射连续. 最后我们证明 $\cup_\alpha G_\alpha = f^{-1}[O]$, 分两步骤: (i) 所有属于 $\cup_\alpha G_\alpha$ 的元素必属于 $f^{-1}[O]$; (ii) 所有不属于 $\cup_\alpha G_\alpha$ 的元素必不属于 $f^{-1}[O]$. (i) 设 $x \in \cup_\alpha G_\alpha$, 而 $f[\cup_\alpha G_\alpha] = \cup_\alpha f[G_\alpha] \overset{f[G_\alpha]\subset O}{\subset} O$, 有 $f(x) \in f[\cup_\alpha G_\alpha] \subset O$, 即 $f(x) \in O$. 根据注 5 ②的定义: $f^{-1}[O] = \{x \in X|\, f(x) \in O\}$, 所以这时 $x \in f^{-1}[O]$. (ii) 设 $x \notin \cup_\alpha G_\alpha$, 则 $x \in \cap_\alpha(X - G_\alpha) = \cap_\alpha(-G_\alpha)$, 有 $f(x) \in f[\cap_\alpha(-G_\alpha)] = \cap_\alpha f[-G_\alpha]$. 这时 x 必不属于 $f^{-1}[O]$, 因为否则的话有 $f(x) \in O$, 这会与 $f(x) \in \cap_\alpha f[-G_\alpha]$ 矛盾. 最后只要证明如下命题: $\forall\, \alpha, f[G_\alpha] \subset O$, 则 $O \cap (\cap_\alpha f[-G_\alpha]) = \emptyset$. 注意到 $f[-G_\alpha] = f[X - G_\alpha] = Y - f[G_\alpha]$, 有 $\cap_\alpha f[-G_\alpha] = \cap_\alpha(Y - f[G_\alpha]) \overset{\text{DM律}}{=} Y - \cup_\alpha f[G_\alpha] = -\cup_\alpha f[G_\alpha]$. 令 $A_\alpha \equiv f[G_\alpha]$, 即要证明 $\forall\, \alpha, A_\alpha \subset O$, 有 $O \cap (-\cup_\alpha A_\alpha) = \emptyset$, 但是显然这一结论不成立！

根据定义 3b 似乎可以得到如下定理：如果在所有点 $x \in X$ 上连续，则对 $\forall$ 满足 $f(x) \in G'$ 的 $G' \in \mathcal{S}$, $\exists G \in \mathcal{T}$ 使 $x \in G$ 且 $f[G] = G'$. 与定义的区别在于 $f[G] = G'$ 而不是 $f[G] \subset G'$. 现在考虑任意的一个开集 $O \in \mathcal{S}$, 设它的元素为 y_α, 即 $\forall\, \alpha$, $y_\alpha \in O$. 如果 y_α 有逆像 $x_\alpha = f^{-1}(y_\alpha)$ 且映射是连续的，则根据由定义 3b 推出的此定理，一定存在开集 $G_\alpha \in \mathcal{T}$ 使 $x_\alpha \in G_\alpha$ 且 $f[G_\alpha] = O$. 于是 $f^{-1}[O] = G_\alpha \in \mathcal{T}$, 于是根据定义 3a, 映射连续.

11. 试证任一开区间 $(a, b) \subset \mathbb{R}$ 与 $\mathbb{R}$ 同胚.

证 注意到映射 $f : x \mapsto \tan x$ 是 $(-\frac{\pi}{2}, \frac{\pi}{2}) \to (-\infty, \infty) = \mathbb{R}$ 的一一到上映射. 令 $y = \alpha x + \beta$, 我们解 $\begin{cases} -\frac{\pi}{2} = \alpha a + \beta \\ \frac{\pi}{2} = \alpha b + \beta \end{cases}$ 得 $\begin{cases} \alpha = \frac{\pi}{b-a} \\ \beta = -\frac{\pi}{2}\frac{a+b}{b-a} \end{cases}$, 于是 $y = \frac{\pi}{b-a}x - \frac{\pi}{2}\frac{a+b}{b-a} = \frac{\pi}{b-a}(x - \frac{a+b}{2})$.

构造映射 $f : x \mapsto \tan\,[\frac{\pi}{b-a}(x - \frac{a+b}{2})]$, 它是 $(a, b) \to \mathbb{R}$ 的一一到上映射. 下面证明 f 对 $((a,b), \mathcal{T}_u') \to (\mathbb{R}, \mathcal{T}_u)$ 连续以及 f^{-1} 对 $(\mathbb{R}, \mathcal{T}_u) \to ((a,b), \mathcal{T}_u')$ 连续，其中 $\mathcal{T}_u'$ 为 $\mathcal{T}_u$ 的诱导拓扑. 根据 §1.2 定义 3a, 从图像上看这一结论是不证自明的. 因此根据 §1.2 定义 4, 这是同胚映射，拓扑子空间 $((a,b), \mathcal{T}_u')$ 与拓扑空间 $(\mathbb{R}, \mathcal{T}_u)$ 同胚.

12. 设 X_1 和 X_2 是 $\mathbb{R}$ 的子集，$X_1 \equiv (1,2) \cup (2,3)$, $X_2 \equiv (1,2) \cup [2,3)$. 以 $\mathcal{T}_1$ 和 $\mathcal{T}_2$ 分别代表由 $\mathbb{R}$ 的通常拓扑在 X_1 和 X_2 上的诱导拓扑. 拓扑空间 $(X_1, \mathcal{T}_1)$

和 $(X_2, \mathcal{T}_2)$ 是否连通？

解 从直观图像我们可以知道 $(X_1, \mathcal{T}_1)$ 不连通而 $(X_2, \mathcal{T}_2)$ 连通，下面我们证明这一结论．首先根据诱导拓扑的定义 [§1.1 例 5 式 (1-2-2)] 知道 $\mathcal{T}_1$ 和 $\mathcal{T}_2$ 分别是包含于 X_1 和 X_2 的所有开区间以及开区间之并的集合．

(A) $(X_1, \mathcal{T}_1)$ 的不连通性．首先 $(1,2)$ 和 $(2,3)$ 都是开区间，所以它们都是 $\mathcal{T}_1$ 的元素，所以是开的．其次，因 $-(1,2) = X_1 - (1,2) = (2,3) \in \mathcal{T}_1$ 是开的，所以根据 §1.2 定义 6, $(1,2)$ 是闭的．同样可以说明 $(2,3)$ 也是闭的．因此拓扑空间 $(X_1, \mathcal{T}_1)$ (至少) 有 4 个既开又闭的子集 $\emptyset$ 、 X_1 、 $(1,2)$ 和 $(2,3)$, 所以根据 §1.2 定义 7, 它是不连通的．当然，这正是 §1.2 例 9 的一个特例．下面说明除了这 4 个，没有其他的既开又闭的子空间．属于 X_1 的子空间 (子区间、子集) 只有 4 种类型，如 (a) $(1.6, 1.7)$ 、 (b) $[1.6, 1.7]$ 、 (c) $(1.6, 1.7]$ 和 (d) $[1.6, 1.7)$. 很容易看出，其中 (a) 是开而不闭的 (本身属于 $\mathcal{T}_1$ 而补集不属于 $\mathcal{T}_1$), (b) 是闭而不开的 (补集属于 $\mathcal{T}_1$ 而本身不属于 $\mathcal{T}_1$), (c) 和 (d) 是既不开也不闭的 (本身不属于 $\mathcal{T}_1$ 而补集也不属于 $\mathcal{T}_1$). 因此没有其他的既开又闭的子集 (本身和补集都属于 $\mathcal{T}_1$).

(B) $(X_2, \mathcal{T}_2)$ 的连通性．首先注意到 $(1,2)$ 是开的而 $[2,3)$ 不是开的 (其实是闭的，见下). $[2,3)$ 的不开性是因为：根据诱导拓扑的定义，现在找不到 $O \in \mathcal{T}_u$, 能使 $[2,3) = X_2 \cap O = [(1,2) \cup [2,3)] \cap O$, 所以 $[2,3) \notin \mathcal{T}_2$. 其次，因 $-(1,2) = X_2 - (1,2) = [2,3) \notin \mathcal{T}_2$, 不是开的，所以根据 §1.2 定义 6, $(1,2)$ 不是闭的．而 $-[2,3) = X_2 - [2,3) = (1,2) \in \mathcal{T}_2$, 所以根据定义 6, $[2,3)$ 是闭的．也就是说 $(1,2)$ 是开而不闭的，而 $[2,3)$ 是闭而不开的．再根据 (A) 中最后的讨论，可知除了 $\emptyset$ 和 X_2, 没有其他的既开又闭的子集 (子区间), 因此根据 §1.2 定义 7, 它是连通的．

13. 任意集合 X 配以离散拓扑 $\mathcal{T}$ 所得的拓扑空间是否连通？

解 不连通，证明如下：所谓离散拓扑是指 X 的所有子集都是 $\mathcal{T}$ 的元素 (§1.2 例 1). 此时对任意的 $A \subset X$, 有 $-A = X - A \in \mathcal{T}$. 根据 §1.2 定义 6, A 是闭的．也就是说 X 的任何子集都是既开又闭的．所以根据 §1.2 定义 7, 离散拓扑空间不连通，这也正是“离散”的由来．

由此也可以看出凝聚拓扑 $\mathcal{T}$ 一定是连通的，因为它只有两个元素 $\emptyset$ 和 X, 其后果是对任何其他的 $A \subset X$, 有 $A \notin \mathcal{T}$ 及 $-A = X - A \notin \mathcal{T}$. 根据定义 6, A 既不是开的也不是闭的．只有 $\emptyset$ 和 X 是既开又闭的，根据定义 7, 故而连通．也是“凝聚”的由来．

~14. 设 $A \subset B$, 试证 (a) $\bar{A} \subset \bar{B}$; 提示： $A \subset B$ 表明 $\bar{B}$ 是含 A 的闭集． (b) $i(A) \subset i(B)$.

证 这两个结论从图像上来看是显然的．

(a) 由 §1.2 定理 1-2-3(a) ②, $B \subset \bar{B}$, 故 $A \subset B \subset \bar{B}$. 根据 §1.2 定义 8, 闭包 $\bar{A}$ 是 包含 A 的 最小 闭集，也就是包含 A 的 所有 闭集的交集．既然现在 $A \subset \bar{B}$, 而根据 §1.2 定理 1-2-3(a) ①, $\bar{B}$ 是闭集，所以 $\bar{B}$ 也是定义 8 中的 C_α 之一．又 $\bar{A} \subset C_\alpha\, \forall \alpha$, 故有 $\bar{A} \subset \bar{B}$. 还可以利用习题 15 的结果证明: $\forall x \in \bar{A}$, 取 x 的邻域 N, 根据 $\Rightarrow$, 知 $N \cap A$ 非空．而 $A \subset B$, 如果 $N \cap A$ 非空，必有 $N \cap B$ 非空．然后利用 $\Leftarrow$, 有 $x \in \bar{B}$. 于是属于 $\bar{A}$ 的元素必属于 $\bar{B}$, 根据 §1.1 的定义 1, $\bar{A} \subset \bar{B}$.

(b) 由 §1.2 定理 1-2-3(b) ②, $i(A) \subset A$, 故 $i(A) \subset A \subset B$. 根据 §1.2 定义 9, 内部 $i(B)$ 是 包含于 B 的 最大 开集，也就是包含于 B 的 所有 开集的并集．既然现在 $i(A) \subset B$, 而根据 §1.2 定理 1-2-3(b) ①, $i(A)$ 是开集，所以 $i(A)$ 也是定义 9 中的 O_α 之一．又 $O_\alpha \subset i(B)\, \forall \alpha$, 故有 $i(A) \subset i(B)$.

~15. 试证 $x \in \bar{A} \Leftrightarrow x$ 的 任一 邻域与 A 之交集非空. 对 $\Rightarrow$ 证明的提示: 设 $O \in \mathcal{T}$ 且 $O \cap A = \emptyset$, 先证 $A \subset X - O$, 再证 (利用闭包定义)$\bar{A} \subset X - O$.

证 这两个方向我们都通过等价的逆否命题来证明．$\Rightarrow$ 的逆否命题表述为: 如果 存在 x 的邻域，它与 A 的交集为空，则 $x \notin \bar{A}$; $\Leftarrow$ 的逆否命题表述为: 如果 $x \notin \bar{A}$, 则一定 存在 x 的邻域，它与 A 的交集为空.

$\Rightarrow$: 首先, 如果 $O \cap A = \emptyset$, 那么当 $x \in A$ 时, 必有 $x \notin O$. 当然 $x \in X$, 根据差集的定义 (§1.1 定义 2) 知这时必有 $x \in X - O$. 因此若 $O \cap A = \emptyset$, 则 $A \subset X - O$. 现在设 $O \in \mathcal{T}$ 为任一开集, 则 $X - (X - O) \overset{\text{习题 } 2}{=} (X - X) \cup O = \emptyset \cup O = O \in \mathcal{T}$. 根据 §1.2 定义 6 知 $X - O$ 是闭集．既然 $A \subset X - O$, 而 $X - O$ 是闭集，根据 §1.2 闭包的定义 8: $\bar{A}$ 是所有包含 A 的闭集的交集，自然 $X - O$ 是定义中的 C_α 之一，所以 $\bar{A} \subset X - O$. 至此我们证明了：对任一开集 $O \in \mathcal{T}$, 如果 $O \cap A = \emptyset$, 则有 $\bar{A} \subset X - O$. 下面我们令 O 是 x 的邻域，即 $x \in O$. 与 $\bar{A} \subset X - O$ 结合立即知道 $x \notin \bar{A}$. 因为如果 $x \in \bar{A}$ 的话，由 $\bar{A} \subset X - O$ 有 $x \in X - O$, 即有 $x \notin O$, 这与 $x \in O$ 矛盾．于是我们证明了：如果 x 存在某个开邻域 O, 它与 A 的交集为空，则 $x \notin \bar{A}$. 这就是 $\Rightarrow$ 的逆否命题．当然如果 O 非开的话，结论依然成立，因为非开的比开的要 “大”，如果非开的与 A 的交集为空，则开的肯定与 A 的交集为空．换句话说非开的至少带有部分边界，它如果与 A 不相交，那么它比开的要距离 A 更远． (但是证明过程中为何要用到 O 为开？)

$\Leftarrow$: 如果 $x \notin \bar{A}$, 根据 §1.2 定理 1-2-3(a) ② $A \subset \bar{A}$ 有 $x \notin A$. 这时必定存在 x 的某个邻域 N, $x \in N$, 使得 $N \cap A = \emptyset$. 证明如下：因为 $x \notin A$, 所以有 $x \in X - A$. 现在在 $X - A$ 内取 x 的某个邻域，即 $x \in N \subset X - A$. 这样得到的 N 必满足 $N \cap A = \emptyset$, 因为 $N \subset X - A$ 表明 N 的元素必不属于 A [属于 N 的元素必属于 $X - A$ (§1.1 定义 1), 属于 $X - A$ 的元素必不属于 A (§1.1 定义 2).], 因此 N 与 A 不相交．命题得证.

16. 试证 RR 不是紧致的.

证 以 NN 代表自然数集，则 $\{(-n,n)|n\in NN\}$ 是 RR 的开覆盖，它没有有限子覆盖.

附. 设 C 是拓扑空间 $(X,\mathcal{T})$ 的紧致子集， $A\subset C$ 且 A 是 $(X,\mathcal{T})$ 的闭子集，则 A 必紧致.

证 因 $C\subset X$, 如果 $C=X$, 则 X 为紧致，回到了定理 1-3-3 的情形. 结论成立. 下面设 C 是 X 的真子集，于是总可以找到 (?) 开子集 B 使满足 $C\subset B\subset X$. 首先证明 $B-A$ 为开集：考虑 $X-(B-A)\overset{\text{习题 2}}{=}(X-B)\cup A$. 因 B 为开，故 $X-B$ 为闭. 而 A 为闭，故由 §1.2 定理 1-2-2(b) 知 $(X-B)\cup A$ 为闭. 从而 $X-(B-A)$ 为闭，$B-A$ 为开. 然后设 $\{O_\alpha\}$ 为 A 的<u>任一</u>开覆盖，则用 $\{O_\alpha,B-A\}$ 可以开覆盖 $A\cup(B-A)=B$, 即用 $\{O_\alpha\}$ 覆盖 A, 用 $B-A$ 覆盖 $B-A$. 又注意到 $C\subset B$, 故而 $\{O_\alpha,B-A\}$ 也是 C 的一个开覆盖. 因 C 是紧致的，它的任何开覆盖都存在有限子覆盖，设为 $\{O_{\alpha_1},\cdots,O_{\alpha_n};B-A\}$. 因 $A\cap(B-A)=\emptyset$, 所以 $B-A$ 覆盖不到 A, 它只能覆盖 $C-A$. 也就是说 $\{O_{\alpha_1},\cdots,O_{\alpha_n}\}$ 是 A 的开覆盖. 因 $\{O_\alpha\}$ 为 A 的任一开覆盖，都有有限子覆盖 $\{O_{\alpha_1},\cdots,O_{\alpha_n}\}$, 故根据 §1.3 定义 1 知 A 是紧致的.

第 2 章 “流形和张量场” 习题

~1. 试证 §2.1 例 2 定义的拓扑同胚映射 $\psi_i^\pm$ 在 $O_i^\pm$ 的所有交叠区上满足相容性条件，从而证实 S^1 确是 1 维流形.

证 设 (x,y) 是 RR^2 的自然坐标, 定义开半圆周 (不包含两端点) 如下：$O_i^+:=\{(x,y)\in S^1|\ x^i>0\}$, $O_i^-:=\{(x,y)\in S^1|\ x^i<0\}$, $i=x,y$, 分别对应左右和上下开半圆. 定义 $O_i^\pm$ 到 RR 的单位开区间 $V=(-1,1)$ 的同胚映射 $\psi_i^\pm$ 为如下的投影映射： $\psi_x^\pm(x,y)=y$, $\psi_y^\pm(x,y)=x$. 下面证明开半圆交叠区满足相容性条件. 我们就以第一象限的交叠区为例，它是 O_x^+ 和 O_y^+ 的交叠，即 $O_x^+\cap O_y^+\neq\emptyset$. 相应的映射为：

$O_x^+\to V_x^+=\psi_x^+[O_x^+]$:

$$\psi_x^+:(x,y)\mapsto x^1=y\,,$$

即为

$$x^1=\psi_x^+(x,y)=\psi_x^+\left(\sqrt{1-y^2},y\right)=y\,,$$

其逆映射为 $V_x^+\to O_x^+=(\psi_x^+)^{-1}[V_x^+]$:

$$(\psi_x^+)^{-1}:x^1\mapsto(x,y)=\left(\sqrt{1-(x^1)^2},x^1\right)\,,$$

即为

$$(x,y)=(\psi_x^+)^{-1}(x^1)=\left(\sqrt{1-(x^1)^2},x^1\right);$$

$O_y^+\to V_y^+=\psi_y^+[O_y^+]$:

$$\psi_y^+:(x,y)\mapsto x'^1=x\,,$$

即为

$$x'^1=\psi_y^+(x,y)=\psi_y^+(x,\sqrt{1-x^2})=x\,.$$

于是复合映射 $\psi_y^+\circ(\psi_x^+)^{-1}$ 为 $V_x^+\to\psi_x^+\cap\psi_y^+\to V_y^+$. 根据 §1.1 定义 8 知 $\psi_y^+\circ(\psi_x^+)^{-1}:x^1\mapsto x'^1$ 给出复合函数:

$$\begin{aligned}x'^1 &= (\psi_y^+\circ(\psi_x^+)^{-1})(x^1)=\psi_y^+\left((\psi_x^+)^{-1}(x^1)\right)\\ &= \psi_y^+\left(\sqrt{1-(x^1)^2},x^1\right)=\sqrt{1-(x^1)^2}\,.\end{aligned}$$

于是我们知道复合映射 $\psi_y^+\circ(\psi_x^+)^{-1}$ 的 1 个 1 元函数为:

$$x'^1=\phi^1(x^1)=\sqrt{1-(x^1)^2}\,,$$

显然在单位线段内 ($|x^1|<1$) 无限可微并连续，即是 C^∞ 的，因此光滑. 同样可以证明其他的复合映射 $\psi_\beta\circ\psi_\alpha^{-1}$ 也都是光滑的，于是图 (O_1^-,ψ_1^-) 与图 (O_2^+,ψ_2^+) 相交相容 (第二象限), 图 (O_1^-,ψ_1^-) 与图 (O_2^-,ψ_2^-) 相交相容 (第三象限), 图 (O_1^+,ψ_1^+) 与图 (O_2^-,ψ_2^-) 相交相容 (第四象限).

用同样方法可以证明例 3 的 S^2 是 2 维流形. 设 (x,y,z) 是 RR^3 的自然坐标，定义开半球面 (不包含圆周边界) 如下: $O_i^+:=\{(x,y,z)\in S^2|\ x^i>0\}$, $O_i^-:=\{(x,y,z)\in S^2|\ x^i<0\}$, $i=x,y,z$, 分别对应左右、前后、上下开半球面. 定义 $O_i^\pm$ 到 RR^2 的单位开圆盘 $D=\{(x,y)\in RR^2|\ \sqrt{x^2+y^2}<1\}$ 的同胚映射 $\psi_i^\pm$ 为如下的投影映射: $\psi_x^\pm(x,y,z)=(y,z)$, $\psi_y^\pm(x,y,z)=(x,z)$, $\psi_z^\pm(x,y,z)=(x,y)$. 下面证明开半球面交叠区满足相容性条件. 我们就以第一卦限的交叠区为例，它是 O_x^+ 、O_y^+ 和 O_z^+ 的交叠. 注意现在有 3 块坐标域交叠，要分别证明它们两两相容. 下面以 $O_x^+\cap O_y^+$ 为例. 相应的映射为:

$O_x^+\to D_x^+=\psi_x^+[O_x^+]$:

$$\psi_x^+:(x,y,z)\mapsto(x^1,x^2)=(y,z)\,,$$

即为

$$(x^1,x^2)=\psi_x^+(x,y,z)=\psi_x^+\left(\sqrt{1-y^2-z^2},y,z\right)=(y,z)\,,$$

其逆映射为 $D_x^+\to O_x^+=(\psi_x^+)^{-1}[D_x^+]$:

$$(\psi_x^+)^{-1}:(x^1,x^2)\mapsto(x,y,z)=\left(\sqrt{1-(x^1)^2-(x^2)^2},x^1,x^2\right),$$

即为

$$(x,y,z)=(\psi_x^+)^{-1}(x^1,x^2)=\left(\sqrt{1-(x^1)^2-(x^2)^2},x^1,x^2\right);$$

$O_y^+ \to D_y^+ = \psi_y^+[O_y^+]$:

$$\psi_y^+ : (x, y, z) \mapsto (x'^1, x'^2) = (x, z)\,,$$

即为

$$(x'^1, x'^2) = \psi_y^+(x, y, z) = \psi_y^+\Big(x, \sqrt{1 - x^2 - z^2}, z\Big) = (x, z)\,.$$

于是复合映射 $\psi_y^+ \circ (\psi_x^+)^{-1}$ 为 $D_x^+ \to \psi_x^+ \cap \psi_y^+ \to D_y^+$. 根据 §1.1 定义 8 知 $\psi_y^+ \circ (\psi_x^+)^{-1} : (x^1, x^2) \mapsto (x'^1, x'^2)$ 给出复合函数：

$$\begin{aligned}(x'^1, x'^2) &= (\psi_y^+ \circ (\psi_x^+)^{-1})(x^1, x^2) = \psi_y^+\Big((\psi_x^+)^{-1}(x^1, x^2)\Big) \\ &= \psi_y^+\Big(\sqrt{1 - (x^1)^2 - (x^2)^2}, x^1, x^2\Big) = \Big(\sqrt{1 - (x^1)^2 - (x^2)^2}, x^2\Big)\,.\end{aligned}$$

于是我们知道复合映射 $\psi_y^+ \circ (\psi_x^+)^{-1}$ 的 2 个 2 元函数为：

$$\begin{cases} x'^1 = \phi^1(x^1, x^2) = \sqrt{1 - (x^1)^2 - (x^2)^2}\,, \\ x'^2 = \phi^2(x^1, x^2) = x^2\,, \end{cases}$$

显然在单位圆盘内无限可微并连续，因此光滑.

2. 说明 n 维矢量空间可看作 n 维平庸流形.

答 因为 n 维矢量空间是能用一个坐标域覆盖的流形，所以是平庸流形.

3. 设 X 和 Y 是拓扑空间， $f : X \to Y$ 是同胚. 若 X 还是个流形，试给 Y 定义一个微分结构使 $f : X \to Y$ 升格为微分同胚.

答 根据 §1.2 定义 4, 如果 f 是拓扑空间之间的同胚映射，那么它是一一到上的，存在逆映射 $f^{-1} : Y \to X$, 且 f 和 f^{-1} 都连续. 现在 X 是个流形，那么 Y 可以通过 X 获得微分结构：对 X 用映射 ψ 获取坐标 $\psi : X \to RR^n$, 则对 Y 可通过映射 $\psi \circ f^{-1} : Y \to RR^n$ 获取坐标.

~4. 设 $\{x, y\}$ 为 RR^2 的自然坐标，$C(t)$ 是曲线，参数表达式为 $x = \cos t$, $y = \sin t$, $t \in (0, \pi)$. 若 $p = C(\pi/3)$, 写出曲线在 p 的切矢在自然坐标基的分量，并画图表出该曲线及该切矢.

解 在自然坐标下， $C(t) = (x(t), y(t)) = (\cos t, \sin t)$, 有 $\frac{d}{dt}C(t) = (\sin t, -\cos t)$. 在 p 点的切矢为 $v(p) = \frac{d}{dt}C(t)|_p = \frac{d}{dt}C(t)|_{t=\pi/3} = (\sin\frac{\pi}{3}, -\cos\frac{\pi}{3}) = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$, 即 p 点的切矢在自然坐标基的分量为 $v_x = \frac{\sqrt{3}}{2}$, $v_y = -\frac{\sqrt{1}}{2}$.

5. 设曲线 $C(t)$ 和 $C'(t) \equiv C(2t_0 - t)$ 在 $C(t_0) = C'(t_0)$ 点的切矢分别为 v 和 v', 试证 $v + v' = 0$.

证 因 $C'(t') = C(2t_0 - t')$, 根据定义，曲线 $C'(t')$ 的切矢为 $v'(t') = \frac{d}{dt'}C'(t') = -C^{(1)}(2t_0 - t')$, 这里 $C^{(1)}(x)$ 代表 $C(x)$ 的一阶导数. 于是在 $t' = t_0$ 点， $v'(t_0) = -C^{(1)}(t_0)$. 另一方面，曲线 $C(t)$ 的切矢为 $v(t) = \frac{d}{dt}C(t) = C^{(1)}(t)$. 于是在 $t = t_0$ 点， $v(t_0) = C^{(1)}(t_0)$. 故在这一点上有 $C(t_0) = C'(t_0)$, $v(t_0) + v'(t_0) = 0$.

~6. 设 O 为坐标系 $\{x^\mu\}$ 的坐标域, $p \in O$, $v \in V_p$, v^μ 是 v 的坐标分量, 把坐标 x^μ 看作 O 上的 C^∞ 函数, 试证 $v^\mu = v(x^\mu)$. 提示: 用 $v = v^\nu X_\nu$ 两边作用于函数 x^μ.

证 把坐标 x^μ 看作 O 上的 C^∞ 函数, 即 (2-2-1') 式中的 f, 以矢量式 $v = v^\nu X_\nu$ 作用上去后得到实数 $(V_p \to RR)$: $v(x^\mu) = v^\nu X_\nu(x^\mu)$. 这里 $v \in V_p$ 是 p 点的矢量, $v(x^\mu) \in RR$ 是个实数; p 点的坐标基矢 $X_\nu \in V_p$ 也是矢量, 而 $X_\nu(x^\mu) \in RR$ 也是个实数; $v^\nu \in RR$ 是个实数代表 v 的坐标分量. 然后再利用定义式 (2-2-1'), $X_\nu(x^\mu) = \frac{\partial}{\partial x^\nu} x^\mu = \delta_\nu{}^\mu = \delta^\mu{}_\nu = \delta^\mu_\nu$, 即有 $v(x^\mu) = v^\nu \delta^\mu_\nu = v^\mu$.

7. 设 M 是 2 维流形, (O, ψ) 和 (O', ψ') 是 M 上的两个坐标系, 坐标分别为 $\{x, y\}$ 和 $\{x', y'\}$, 在 $O \cap O'$ 上的坐标变换为 $x' = x$, $y' = y - \Omega x$ ($\Omega =$ 常数), 试分别写出坐标基矢 $\partial/\partial x$, $\partial/\partial y$ 用坐标基矢 $\partial/\partial x'$, $\partial/\partial y'$ 的展开式.

解 坐标基矢 $X_\mu = \frac{\partial}{\partial x^\mu}$ 的变换关系为 (2-2-5) 式: $X_\mu = \frac{\partial x'^\nu}{\partial x^\mu}|_p X'_\nu$. 所以现在

$$\begin{aligned} X_x &= \frac{\partial x'}{\partial x} X'_x + \frac{\partial y'}{\partial x} X'_y = X'_x - \Omega X'_y \,, \\ X_y &= \frac{\partial x'}{\partial y} X'_x + \frac{\partial y'}{\partial y} X'_y = X'_y \,, \end{aligned}$$

即

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x'} - \Omega \frac{\partial}{\partial y'} \,, \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \,. \end{aligned}$$

也可以这样得到: 因 $f(x, y) = f'(x', y')$, 故有

$$\begin{aligned} \frac{\partial}{\partial x}(f) &= \frac{\partial f}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial f'}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial f'}{\partial y'} = \frac{\partial f'}{\partial x'} - \Omega \frac{\partial f'}{\partial y'} \\ &= \left(\frac{\partial}{\partial x'} - \Omega \frac{\partial}{\partial y'}\right)(f) \,, \end{aligned}$$

此即上面的第一个关系.

~8. (a) 试证式 (2-2-9) 的 $[u, v]$ 在每点满足矢量定义 (§2.2 定义 2) 的两条件, 从而的确是矢量场. (b) 设 u, v, w 为流形 M 上的光滑矢量场, 试证

$$[[u, v], w] + [[w, u], v] + [[v, w], u] = 0 \quad (\text{此式称为 } \textbf{雅可比恒等式}) \,.$$

证 (a) $[u, v]$ 满足 §2.2 定义 2(a):

$$\begin{aligned} [u, v](\alpha f + \beta g) &\overset{\text{(2-2-9)}}{=} u(v(\alpha f + \beta g)) - v(u(\alpha f + \beta g)) \\ &\overset{\text{定义2}(a)}{=} u(\alpha v(f) + \beta v(g)) - v(\alpha u(f) + \beta u(g)) \\ &\overset{\text{定义2}(a)}{=} \alpha u(v(f)) + \beta u(v(g)) - \alpha v(u(f)) - \beta v(u(g)) \\ &= \alpha[u(v(f)) - v(u(f))] + \beta[u(v(g)) - v(u(g))] \\ &\overset{\text{(2-2-9)}}{=} \alpha[u, v](f) + \beta[u, v](g) \,. \end{aligned}$$

$[u,v]$ 满足 §2.2 定义 2(b):

$$
\begin{aligned}
[u,v](fg) &\overset{(2\text{-}2\text{-}9)}{=} u(v(fg))-v(u(fg)) \\
&\overset{定义2(b)}{=} u(f|_p v(g)+g|_p v(f))-v(f|_p u(g)+g|_p u(f)) \\
&\overset{定义2(a)}{=} f|_p u(v(g))+g|_p u(v(f))-f|_p v(u(g))-g|_p v(u(f)) \\
&= f|_p[u(v(g))-v(u(g))]+g|_p[u(v(f))-v(u(f))] \\
&\overset{(2\text{-}2\text{-}9)}{=} f|_p[u,v](g)+g|_p[u,v](f)\ .
\end{aligned}
$$

(b) 雅可比恒等式:

$$
\begin{aligned}
& [[u,v],w]+[[w,u],v]+[[v,w],u] \\
= & [uv-vu,w]+[wu-uw,v]+[vw-wv,u] \\
= & (uv-vu)w-w(uv-vu)+(wu-uw)v-v(wu-uw) \\
& +(vw-wv)u-u(vw-wv) \\
= & uvw-vuw-wuv+wvu+wuv-uwv-vwu+vuw \\
& +vwu-wvu-uvw+uwv \\
= & 0\ .
\end{aligned}
$$

~9. 设 $\{r,\varphi\}$ 为 RR^2 中某开集 (坐标域) 上的极坐标, $\{x,y\}$ 为自然坐标,

(a) 写出极坐标系的坐标基矢 $\partial/\partial r$ 和 $\partial/\partial\varphi$ (作为坐标域上的矢量场) 用 $\partial/\partial x$, $\partial/\partial y$ 展开的表达式.

(b) 求矢量场 $[\partial/\partial r,\partial/\partial x]$ 用 $\partial/\partial x$, $\partial/\partial y$ 展开的表达式.

(c) 令 $\hat{e}_r\equiv\partial/\partial r$, $\hat{e}_\varphi\equiv r^{-1}\partial/\partial\varphi$, 求 $[\hat{e}_r,\hat{e}_\varphi]$ 用 $\partial/\partial x$, $\partial/\partial y$ 展开的表达式.

解 (a) 因 $x=r\cos\varphi$, $y=r\sin\varphi$, 有 $r=\sqrt{x^2+y^2}$, $\tan\varphi=\frac{y}{x}$ 和 $\cos\varphi=\frac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi=\frac{y}{\sqrt{x^2+y^2}}$. 利用基矢的变换关系 (2-2-5) $e_\mu=\frac{\partial x'^\nu}{\partial x^\mu}e'_\nu$, 有

$$
\begin{aligned}
e_r &= \frac{\partial x}{\partial r}e_x+\frac{\partial y}{\partial r}e_y=\cos\varphi\, e_x+\sin\varphi\, e_y\ , \\
e_\varphi &= \frac{\partial x}{\partial \varphi}e_x+\frac{\partial y}{\partial \varphi}e_y=-r\sin\varphi\, e_x+r\cos\varphi\, e_y\ ,
\end{aligned}
$$

即为

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \cos\varphi\frac{\partial}{\partial x}+\sin\varphi\frac{\partial}{\partial y}=\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}\ , \\
\frac{\partial}{\partial \varphi} &= =-r\sin\varphi\frac{\partial}{\partial x}+r\cos\varphi\frac{\partial}{\partial y}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\ .
\end{aligned}
$$

也可如下推出：因 $f_{r\varphi}(r,\varphi)=f_{xy}(x,y)$, 故有

$$
\begin{aligned}
\frac{\partial}{\partial r}(f) &= \frac{\partial f_{r\varphi}(r,\varphi)}{\partial r}=\frac{\partial f_{xy}(x,y)}{\partial r}=\frac{\partial x}{\partial r}\frac{\partial f_{xy}(x,y)}{\partial x}+\frac{\partial y}{\partial r}\frac{\partial f_{xy}(x,y)}{\partial y}\\
&= \Big(\cos\varphi\,\frac{\partial}{\partial x}+\sin\varphi\,\frac{\partial}{\partial y}\Big)(f)\ ,\\
\frac{\partial}{\partial \varphi}(f) &= \frac{\partial f_{r\varphi}(r,\varphi)}{\partial \varphi}=\frac{\partial f_{xy}(x,y)}{\partial \varphi}=\frac{\partial x}{\partial \varphi}\frac{\partial f_{xy}(x,y)}{\partial x}+\frac{\partial y}{\partial \varphi}\frac{\partial f_{xy}(x,y)}{\partial y}\\
&= \Big(-r\sin\varphi\,\frac{\partial}{\partial x}+r\cos\varphi\,\frac{\partial}{\partial y}\Big)(f)\ ,
\end{aligned}
$$

于是有前面同样的结果.

(b) 利用 (a) 的结果，有

$$
\begin{aligned}
&\Big[\frac{\partial}{\partial r},\,\frac{\partial}{\partial x}\Big]=\Big[\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y},\,\frac{\partial}{\partial x}\Big]\\
=&\ \Big(\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big)\frac{\partial}{\partial x}-\frac{\partial}{\partial x}\Big(\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big)\\
=&\ -\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial x}\,\frac{\partial}{\partial x}-\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial x}\,\frac{\partial}{\partial y}\\
=&\ -\frac{y^2}{(x^2+y^2)^{3/2}}\,\frac{\partial}{\partial x}+\frac{xy}{(x^2+y^2)^{3/2}}\,\frac{\partial}{\partial y}\ .
\end{aligned}
$$

(c) 利用 (a) 的结果，有

$$
\begin{aligned}
\hat{e}_r &\equiv e_r=\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\ ,\\
\hat{e}_\varphi &\equiv \frac{1}{r}e_\varphi=-\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\ .
\end{aligned}
$$

于是

$$
\begin{aligned}
[\hat{e}_r,\,\hat{e}_\varphi] &= \Big[\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y},\,-\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big]\\
&= \Big(\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big)\Big(-\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big)\\
&\quad -\Big(-\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big)\Big(\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial}{\partial y}\Big)\\
&= \Bigg[-\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial x}-\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial y}\Bigg]\Big(\frac{\partial}{\partial x}\Big)\\
&\quad +\Bigg[\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial y}\Bigg]\Big(\frac{\partial}{\partial y}\Big)\\
&\quad -\Bigg[-\frac{y}{\sqrt{x^2+y^2}}\,\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial x}+\frac{x}{\sqrt{x^2+y^2}}\,\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial y}\Bigg]\Big(\frac{\partial}{\partial x}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\left[-\frac{y}{\sqrt{x^2+y^2}}\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial x}+\frac{x}{\sqrt{x^2+y^2}}\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial y}\right]\left(\frac{\partial}{\partial y}\right)\\
=\ &\left[\frac{x^2y}{(x^2+y^2)^2}-\frac{x^2y}{(x^2+y^2)^2}\right]\left(\frac{\partial}{\partial x}\right)\\
&+\left[\frac{xy^2}{(x^2+y^2)^2}-\frac{xy^2}{(x^2+y^2)^2}\right]\left(\frac{\partial}{\partial y}\right)\\
&-\left[-\frac{y^3}{(x^2+y^2)^2}-\frac{x^2y}{(x^2+y^2)^2}\right]\left(\frac{\partial}{\partial x}\right)\\
&-\left[\frac{xy^2}{(x^2+y^2)^2}+\frac{x^3}{(x^2+y^2)^2}\right]\left(\frac{\partial}{\partial y}\right)\\
=\ &\frac{y}{x^2+y^2}\frac{\partial}{\partial x}-\frac{x}{x^2+y^2}\frac{\partial}{\partial y}\\
=\ &\frac{y}{x^2+y^2}\,\hat{e}_x-\frac{x}{x^2+y^2}\,\hat{e}_y\ .
\end{aligned}
$$

此等式也可利用下题的结果获得.

~10. 设 u,v 为 M 上的矢量场，试证 $[u,v]$ 在任何坐标基底的分量满足

$[u,v]^\mu=u^\nu\partial v^\mu/\partial x^\nu-v^\nu\partial u^\mu/\partial x^\nu\ .$ 提示：用式 (2-2-3') 和 (2-2-3).

证 由 (2-2-3'), 矢量 $[u,v]$ 的第 μ 分量 $[u,v]^\mu$ 为矢量 $[u,v]$ 作用到函数 x^μ 上的值，即

$$
\begin{aligned}
[u,v]^\mu\ &=\ [u,v](x^\mu)\overset{\text{(2-2-9')}}{=}u(v(x^\mu))-v(u(x^\mu))\\
&\overset{\text{(2-2-3')}}{=}\ u(v^\mu)-v(u^\mu)\overset{\text{(2-2-3)}}{=}u^\nu X_\nu(v^\mu)-v^\nu X_\nu(u^\mu)\\
&\overset{\text{(2-2-1')}}{=}\ u^\nu\frac{\partial v^\mu}{\partial x^\nu}-v^\nu\frac{\partial u^\mu}{\partial x^\nu}\ .
\end{aligned}
$$

[根据此式可得上题 (b) 和 (c) 的结果. (b) 因 $\frac{\partial}{\partial r}=\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$, $\frac{\partial}{\partial x}=\frac{\partial}{\partial x}$, 于是

$$
\begin{aligned}
\left[\frac{\partial}{\partial r},\frac{\partial}{\partial x}\right]^x\ &=\ \left(\frac{\partial}{\partial r}\right)^x\frac{\partial(\frac{\partial}{\partial x})^x}{\partial x}+\left(\frac{\partial}{\partial r}\right)^y\frac{\partial(\frac{\partial}{\partial x})^x}{\partial y}-\left(\frac{\partial}{\partial x}\right)^x\frac{\partial(\frac{\partial}{\partial r})^x}{\partial x}-\left(\frac{\partial}{\partial x}\right)^y\frac{\partial(\frac{\partial}{\partial r})^x}{\partial y}\\
&=\ \frac{x}{\sqrt{x^2+y^2}}\frac{\partial 1}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial 1}{\partial y}-1\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial x}-0\frac{\partial\frac{x}{\sqrt{x^2+y^2}}}{\partial y}\\
&=\ -\frac{y^2}{(x^2+y^2)^{3/2}}\ ,\\
\left[\frac{\partial}{\partial r},\frac{\partial}{\partial x}\right]^y\ &=\ \left(\frac{\partial}{\partial r}\right)^x\frac{\partial(\frac{\partial}{\partial x})^y}{\partial x}+\left(\frac{\partial}{\partial r}\right)^y\frac{\partial(\frac{\partial}{\partial x})^y}{\partial y}-\left(\frac{\partial}{\partial x}\right)^x\frac{\partial(\frac{\partial}{\partial r})^y}{\partial x}-\left(\frac{\partial}{\partial x}\right)^y\frac{\partial(\frac{\partial}{\partial r})^y}{\partial y}\\
&=\ \frac{x}{\sqrt{x^2+y^2}}\frac{\partial 0}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial 0}{\partial y}-1\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial x}-0\frac{\partial\frac{y}{\sqrt{x^2+y^2}}}{\partial y}\\
&=\ \frac{xy}{(x^2+y^2)^{3/2}}\ ,
\end{aligned}
$$

即为 (b) 的结果

$$\left[\frac{\partial}{\partial r},\frac{\partial}{\partial x}\right]=-\frac{y^2}{(x^2+y^2)^{3/2}}\frac{\partial}{\partial x}+\frac{xy}{(x^2+y^2)^{3/2}}\frac{\partial}{\partial y}\,.$$

(c) 因 $\hat{e}_r=\frac{x}{\sqrt{x^2+y^2}}\,e_x+\frac{y}{\sqrt{x^2+y^2}}\,e_y$, $\hat{e}_\varphi=-\frac{y}{\sqrt{x^2+y^2}}\,e_x+\frac{x}{\sqrt{x^2+y^2}}\,e_y$, 于是

$$\begin{aligned}
[\hat{e}_r,\hat{e}_\varphi]^x &= (\hat{e}_r)^x\frac{\partial(\hat{e}_\varphi)^x}{\partial x}+(\hat{e}_r)^y\frac{\partial(\hat{e}_\varphi)^x}{\partial y}-(\hat{e}_\varphi)^x\frac{\partial(\hat{e}_r)^x}{\partial x}-(\hat{e}_\varphi)^y\frac{\partial(\hat{e}_r)^x}{\partial y}\\
&= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial\left(-\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial\left(-\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial y}\\
&\quad+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial x}-\frac{x}{\sqrt{x^2+y^2}}\frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial y}\\
&= \frac{x^2y}{(x^2+y^2)^2}-\frac{x^2y}{(x^2+y^2)^2}+\frac{y^3}{(x^2+y^2)^2}+\frac{x^2y}{(x^2+y^2)^2}\\
&= \frac{y}{x^2+y^2}\,,\\
[\hat{e}_r,\hat{e}_\varphi]^y &= (\hat{e}_r)^x\frac{\partial(\hat{e}_\varphi)^y}{\partial x}+(\hat{e}_r)^y\frac{\partial(\hat{e}_\varphi)^y}{\partial y}-(\hat{e}_\varphi)^x\frac{\partial(\hat{e}_r)^y}{\partial x}-(\hat{e}_\varphi)^y\frac{\partial(\hat{e}_r)^y}{\partial y}\\
&= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial y}\\
&\quad+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial x}-\frac{x}{\sqrt{x^2+y^2}}\frac{\partial\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial y}\\
&= \frac{xy^2}{(x^2+y^2)^2}-\frac{xy^2}{(x^2+y^2)^2}-\frac{xy^2}{(x^2+y^2)^2}-\frac{x^3}{(x^2+y^2)^2}\\
&= -\frac{x}{x^2+y^2}\,,
\end{aligned}$$

即为 (c) 的结果

$$[\hat{e}_r,\hat{e}_\varphi]=\frac{y}{x^2+y^2}\,\hat{e}_x-\frac{x}{x^2+y^2}\,\hat{e}_y\,.$$

另外注意对 $e_r\equiv\frac{\partial}{\partial r}=\frac{x}{\sqrt{x^2+y^2}}\,e_x+\frac{y}{\sqrt{x^2+y^2}}\,e_y$, $e_\varphi\equiv\frac{\partial}{\partial\varphi}=-y\,e_x+x\,e_y$, 其结果为

$$\begin{aligned}
[e_r,e_\varphi]^x &= (e_r)^x\frac{\partial(e_\varphi)^x}{\partial x}+(e_r)^y\frac{\partial(e_\varphi)^x}{\partial y}-(e_\varphi)^x\frac{\partial(e_r)^x}{\partial x}-(e_\varphi)^y\frac{\partial(e_r)^x}{\partial y}\\
&= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial(-y)}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial(-y)}{\partial y}+y\frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial x}-x\frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial y}\\
&= 0-\frac{y}{(x^2+y^2)^{1/2}}+\frac{y^3}{(x^2+y^2)^{3/2}}+\frac{x^2y}{(x^2+y^2)^{3/2}}\\
&= 0\,,\\
[e_r,e_\varphi]^y &= (e_r)^x\frac{\partial(e_\varphi)^y}{\partial x}+(e_r)^y\frac{\partial(e_\varphi)^y}{\partial y}-(e_\varphi)^x\frac{\partial(e_r)^y}{\partial x}-(e_\varphi)^y\frac{\partial(e_r)^y}{\partial y}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial(x)}{\partial x}+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial(x)}{\partial y}+y\frac{\partial\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial x}-x\frac{\partial\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial y}\\
&= \frac{x}{(x^2+y^2)^{1/2}}+0-\frac{xy^2}{(x^2+y^2)^{3/2}}-\frac{x^3}{(x^2+y^2)^{3/2}}\\
&= 0\ ,
\end{aligned}
$$

这正是定理 2-2-7 所保证的.]

~11. 设 $\{e_\mu\}$ 为 V 的基底, $\{e^{\mu *}\}$ 为其对偶基底, $v\in V, \omega\in V^*$, 试证

$$\omega=\omega(e_\mu)e^{\mu *}\ ,\quad v=e^{\mu *}(v)e_\mu\ .$$

证 将 $\omega=\omega(e_\mu)e^{\mu *}$ 作用于 V 的任一基矢 e_ν, 注意这里的 $\omega, e^{\mu *}\in V^*$ 是对偶矢量和对偶矢量的基矢，它们都作用在 矢量 上而得到实数！ $\omega(e_\mu)\in RR$ 是实数. 有

$$
\begin{aligned}
\text{左边} &= \omega(e_\nu)\overset{(2\text{-}3\text{-}3)}{=}\omega_\nu\ ,\\
\text{右边} &= \omega(e_\mu)e^{\mu *}(e_\nu)\overset{(2\text{-}3\text{-}2)}{=}\omega(e_\mu)\delta^\mu{}_\nu=\omega(e_\nu)\ ,
\end{aligned}
$$

即 ω 和 $\omega(e_\mu)e^{\mu *}$ 作用到矢量 (任一基矢 e_ν) 都得到实数 $\omega_\nu\equiv\omega(e_\nu)$.

将对偶矢量 (任一对偶矢量的基矢)$e^{\nu *}$ 作用到矢量 $v=e^{\mu *}(v)e_\mu$, 这里 $e^{\mu *}(v)\in RR$ 是实数, $v, e_\mu\in V$ 是矢量和矢量的基矢，有

$$
\begin{aligned}
\text{左边} &= e^{\nu *}(v)\ ,\\
\text{右边} &= e^{\mu *}(v)e^{\nu *}(e_\mu)\overset{(2\text{-}3\text{-}2)}{=}e^{\mu *}(v)\delta^\nu{}_\mu=e^{\nu *}(v)\ ,
\end{aligned}
$$

即对偶矢量的任一基矢作用到矢量 v 和 $e^{\mu *}(v)e_\mu$ 得到同一个实数 $e^{\nu *}(v)$.

~12. 试证 $\omega'_\nu=\frac{\partial x^\mu}{\partial x'^\nu}\omega_\mu$ (定理 2-3-4).

证 根据矢量基矢的变换关系 (2-2-5) 式有 $e_\mu=\frac{\partial x'^\nu}{\partial x^\mu}e'_\nu$ 和 $e'_\mu=\frac{\partial x^\nu}{\partial x'^\mu}e_\nu$. 以对偶矢量 ω 作用到这两个矢量式，得到 $\omega(e_\mu)=\frac{\partial x'^\nu}{\partial x^\mu}\omega(e'_\nu)$ 和 $\omega(e'_\mu)=\frac{\partial x^\nu}{\partial x'^\mu}\omega(e_\nu)$, 利用定义 (2-3-3), 即为 $\omega_\mu=\frac{\partial x'^\nu}{\partial x^\mu}\omega'_\nu$ 和 $\omega'_\mu=\frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu$.

~13. 试证由式 (2-3-5) 定义的映射 $v\mapsto v^{**}$ 是同构映射. 提示：可利用线性代数的结论，即同维矢量空间之间的一一线性映射必到上.

证 如果 (2-3-5) 式 $v^{**}(\omega)=\omega(v)$ 成立，设 $v=v^\mu e_\mu$, 则有

$$\omega(v)=\omega(v^\mu e_\mu)=v^\mu\omega(e_\mu)\overset{(2\text{-}3\text{-}3)}{=}v^\mu\omega_\mu\ ,$$

其中 $v, e_\mu\in V$, $\omega\in V^*$, $v^\mu, \omega_u\in RR$. 另一方面，因 $\omega\overset{(2\text{-}3\text{-}4)}{=}\omega_\mu e^{\mu *}$, 有

$$v^{**}(\omega)=v^{**}(\omega_\mu e^{\mu *})=\omega_\mu v^{**}(e^{\mu *})=\omega_\mu v^{**\mu}\ ,$$

其中令 $v^{**}(e^{\mu *}) \equiv v^{**\mu} \in RR$, $\omega, e^{\mu *} \in V^*$, $v^{**} \in V^{**}$. 因为 ω 是任意的，即 n 个实数 ω_μ 是任意的，欲使该等式 $\omega_\mu v^{**\mu} = v^{**\mu}\omega_\mu = v^\mu \omega_\mu$ 成立，必有 $v^{**\mu} = v^\mu$, 这时这两个自然同构的矢量空间重合 $V^{**} = V$.

其实两个同维矢量空间之间的线性映射如果是一一的, 那么必定是到上的. 存在一一到上的线性映射，就保证了它们之间同构.

~14. 设 $\mathrm{C}_1^1 T$ 和 $(\mathrm{C}_1^1 T)'$ 分别是 $(2,1)$ 型张量 T 借两个基底 $\{e_\mu\}$ 和 $\{e'_\mu\}$ 定义的缩并，试证 $(\mathrm{C}_1^1 T)' = \mathrm{C}_1^1 T$.

证

$$
\begin{aligned}
(\mathrm{C}_1^1 T)' &\overset{\text{定义}}{=} T(e'^{\mu *}, \bullet; e'_\mu) \overset{\text{定理 2-3-2}}{=} T((A^{-1})^\mu{}_\rho e^{\rho *}, \bullet; A^\nu{}_\mu e_\nu) \\
&\overset{\text{线性性}}{=} A^\nu{}_\mu (A^{-1})^\mu{}_\rho T(e^{\rho *}, \bullet; e_\nu) \overset{\text{矩阵相乘}}{=} (AA^{-1})^\nu{}_\rho T(e^{\rho *}, \bullet; e_\nu) \\
&= \delta^\nu{}_\rho T(e^{\rho *}, \bullet; e_\nu) = T(e^{\nu *}, \bullet; e_\nu) \overset{\text{定义}}{=} \mathrm{C}_1^1 T \,.
\end{aligned}
$$

注意：考虑到 $\mathrm{C}_1^1 T$ 是个 $(1,0)$ 型张量，即是矢量 $\mathrm{C}_1^1 T \in V$, 故它按矢量方式变换，而不是不变的. 换句话说，如果上式中 $\bullet$ 填入了相应的量，等式并不成立！

*~15. 设 g 为 V 的度规，试证 $g: V \to V^*$ 是同构映射 (可参见第 13 题的提示).

证 度规 g 为 $(0,2)$ 型张量 $g(;\bullet,\bullet)$, 对矢量 $v \in V$ 的作用给出 $g(;v,\bullet)$ 或 $g(;\bullet,v)$, 都是对偶矢量, 因为它们再作用于矢量 $u \in V$ 后给出实数 $g(;v,u)$ 或 $g(;u,v)$, 于是 $g(;v,\bullet)$ 和 $g(;\bullet,v)$ 都属于 V^*. 因此可以将 g 看成是把一个矢量变成一个对偶矢量的映射 $g: V \to V^*$, 而且是线性映射. 另一方面，由于 g 是非退化的，这样的映射必定是一一的. 也就是说，对于任一像点 $\omega \in V^*$, 只有唯一的原像点 v, 满足 $g(;v,\bullet) = \omega$ (或 $g(;\bullet,v) = \omega$). 否则的话会与 g 的非退化性矛盾：如果有 $g(;v,\bullet) = \omega$ 和 $g(;v',\bullet) = \omega$, 且 $v \neq v'$, 根据 g 的线性性，两式相减有 $g(;v-v',\bullet) = 0$, g 退化. 最后根据线性代数，两个同维矢量空间的一一映射必定到上，而一一到上的线性映射保证这是同构映射.

~16. 试证线长与曲线的参数化无关.

证 设曲线 $C(t)$ 的重参数化曲线为 $C'(t')$, 即 $C(t) = C'(t')$, 而 $t' = \alpha(t)$ (见 §2.2.1 注 4). 考虑在 C 上 t_1 到 t_2 段的线长： $l = \int_{t_1}^{t_2} \frac{dC(\tau)}{d\tau} d\tau = C(t_2) - C(t_1)$, 那么在 C' 上的相应长度为 $l' = C'(t'_2) - C'(t'_1) = C(t_2) - C(t_1) = l$.

17. 设 $\{x, y\}$ 是 2 维欧氏空间的笛卡尔坐标系，试证由式 (2-5-14) 定义的 $\{x', y'\}$ 也是笛卡尔系.

证 由于 (其实就是张量变换关系定理 2-4-2)

$$
\delta\Big(\frac{\partial}{\partial x'^\alpha}, \frac{\partial}{\partial x'^\beta}\Big) = \delta\Big(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial}{\partial x^\mu}, \frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial}{\partial x^\nu}\Big) = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\delta\Big(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\Big) \,,
$$

如果 $\{x^\mu\}$ 是笛卡尔系，则 $\delta(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}) \overset{(2\text{-}5\text{-}12)}{=} \delta_{\mu\nu}$, 有 $\delta(\frac{\partial}{\partial x'^\alpha},\frac{\partial}{\partial x'^\beta}) = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}$. (注意笛卡尔系没有对求和 (缩并) 有上下标的要求 !) 现在，(2-5-14) 式的反变换为： $x = x'\cos\alpha - y'\sin\alpha$, $y = x'\sin\alpha + y'\cos\alpha$, 有 $\frac{\partial x}{\partial x'} = \cos\alpha$, $\frac{\partial x}{\partial y'} = -\sin\alpha$, $\frac{\partial y}{\partial x'} = \sin\alpha$, $\frac{\partial y}{\partial y'} = \cos\alpha$. 因此

$$\begin{aligned}
\delta\Big(\frac{\partial}{\partial x'},\frac{\partial}{\partial x'}\Big) &= \frac{\partial x}{\partial x'}\frac{\partial x}{\partial x'} + \frac{\partial y}{\partial x'}\frac{\partial y}{\partial x'} = (\cos\alpha)(\cos\alpha) + (\sin\alpha)(\sin\alpha) = 1\,,\\
\delta\Big(\frac{\partial}{\partial x'},\frac{\partial}{\partial y'}\Big) &= \frac{\partial x}{\partial x'}\frac{\partial x}{\partial y'} + \frac{\partial y}{\partial x'}\frac{\partial y}{\partial y'} = (\cos\alpha)(-\sin\alpha) + (\sin\alpha)(\cos\alpha) = 0\,,\\
\delta\Big(\frac{\partial}{\partial y'},\frac{\partial}{\partial x'}\Big) &= \frac{\partial x}{\partial y'}\frac{\partial x}{\partial x'} + \frac{\partial y}{\partial y'}\frac{\partial y}{\partial x'} = (-\sin\alpha)(\cos\alpha) + (\cos\alpha)(\sin\alpha) = 0\,,\\
\delta\Big(\frac{\partial}{\partial y'},\frac{\partial}{\partial y'}\Big) &= \frac{\partial x}{\partial y'}\frac{\partial x}{\partial y'} + \frac{\partial y}{\partial y'}\frac{\partial y}{\partial y'} = (-\sin\alpha)(-\sin\alpha) + (\cos\alpha)(\cos\alpha) = 1\,,
\end{aligned}$$

即 $\delta(\frac{\partial}{\partial x'^\alpha},\frac{\partial}{\partial x'^\beta}) = \delta_{\alpha\beta}$. 于是 $\{x', y'\}$ 也是笛卡尔系.

其实这一结论可以推广到任意 n 维笛卡尔系：设 $\{x^\mu\}$ 是笛卡尔系，通过正交变换与此系相联系的另一坐标系 $x'^\alpha = A^\alpha{}_\mu x^\mu$ 也必为笛卡尔系. 这里 A 是 n 维正交矩阵，具有性质 $A^{-1} = \tilde{A}$, 即 $(A^{-1})^\mu{}_\alpha = \tilde{A}^\mu{}_\alpha = A^\alpha{}_\mu$. 因逆变换为 $x^\mu = (A^{-1})^\mu{}_\alpha x'^\alpha = A^\alpha{}_\mu x'^\alpha$, 有 $\frac{\partial x^\mu}{\partial x'^\alpha} = (A^{-1})^\mu{}_\alpha = A^\alpha{}_\mu$, 于是由上面的结果

$$\delta\Big(\frac{\partial}{\partial x'^\alpha},\frac{\partial}{\partial x'^\beta}\Big) = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^\beta} = A^\alpha{}_\mu A^\beta{}_\mu = A^\alpha{}_\mu (A^{-1})^\mu{}_\beta = \delta^\alpha_\beta = \delta_{\alpha\beta}\,.$$

可见度规张量 δ 对 $\{x'^\mu\}$ 满足 (2-5-12) 式，故它也是笛卡尔系.

18. 设 $\{t, x\}$ 是 2 维闵氏空间的洛伦兹坐标系，试证由式 (2-5-20) 定义的 $\{t', x'\}$ 也是洛伦兹系.

证 由于 (其实就是张量变换关系定理 2-4-2)

$$\eta\Big(\frac{\partial}{\partial x'^\alpha},\frac{\partial}{\partial x'^\beta}\Big) = \eta\Big(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial}{\partial x^\mu},\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial}{\partial x^\nu}\Big) = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\eta\Big(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\Big)\,,$$

如果 $\{x^\mu\}$ 是洛伦兹系，则 $\eta(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}) = \eta_{\mu\nu}$, 有 $\eta(\frac{\partial}{\partial x'^\alpha},\frac{\partial}{\partial x'^\beta}) = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}$. 对于 2 维闵氏空间，有 $\eta_{xx} = -\eta_{tt} = +1$, $\eta_{tx} = \eta_{xt} = 0$. 现在， (2-5-20) 式的反变换为： $t = t'\cosh\lambda - x'\sinh\lambda$, $x = -t'\sinh\lambda + x'\cosh\lambda$, 有 $\frac{\partial t}{\partial t'} = \cosh\lambda$, $\frac{\partial t}{\partial x'} = -\sinh\lambda$, $\frac{\partial x}{\partial t'} = -\sinh\lambda$, $\frac{\partial x}{\partial x'} = \cosh\lambda$. 因此

$$\begin{aligned}
\eta\Big(\frac{\partial}{\partial t'},\frac{\partial}{\partial t'}\Big) &= \eta_{tt}\frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'} + \eta_{xx}\frac{\partial x}{\partial t'}\frac{\partial x}{\partial t'}\\
&= (-1)(\cosh\lambda)(\cosh\lambda) + (+1)(-\sinh\lambda)(-\sinh\lambda) = -1\,,\\
\eta\Big(\frac{\partial}{\partial t'},\frac{\partial}{\partial x'}\Big) &= \eta_{tt}\frac{\partial t}{\partial t'}\frac{\partial t}{\partial x'} + \eta_{xx}\frac{\partial x}{\partial t'}\frac{\partial x}{\partial x'}\\
&= (-1)(\cosh\lambda)(-\sinh\lambda) + (+1)(-\sinh\lambda)(\cosh\lambda) = 0\,,\\
\eta\Big(\frac{\partial}{\partial x'},\frac{\partial}{\partial t'}\Big) &= \eta_{tt}\frac{\partial t}{\partial x'}\frac{\partial t}{\partial t'} + \eta_{xx}\frac{\partial x}{\partial x'}\frac{\partial x}{\partial t'}\\
&= (-1)(-\sinh\lambda)(\cosh\lambda) + (+1)(\cosh\lambda)(-\sinh\lambda) = 0\,,
\end{aligned}$$

$$\begin{aligned}\eta\Big(\frac{\partial}{\partial x'},\frac{\partial}{\partial x'}\Big) &= \eta_{tt}\frac{\partial t}{\partial x'}\frac{\partial t}{\partial x'}+\eta_{xx}\frac{\partial x}{\partial x'}\frac{\partial x}{\partial x'}\\ &= (-1)(-\sinh\lambda)(-\sinh\lambda)+(+1)(\cosh\lambda)(\cosh\lambda)=+1\,,\end{aligned}$$

即 $\eta_{x'x'}=-\eta_{t't'}=+1$, $\eta_{t'x'}=\eta_{x't'}=0$. 于是 $\{t',x'\}$ 也是 2 维洛伦兹系.

其实这一结论可以推广到任意 n 维笛卡尔系：设 $\{x^\mu\}$ 是洛伦兹系，通过洛伦兹变换与此系相联系的另一坐标系 $x'^\alpha=\Lambda^\alpha{}_\mu x^\mu$ 也必为洛伦兹系. 这里 Λ 是 n 维洛伦兹变换矩阵，具有性质 $\eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu=\eta_{\mu\nu}$, $\eta^{\mu\nu}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu=\eta^{\alpha\beta}$, 以及 $(\Lambda^{-1})^\mu{}_\alpha=\Lambda_\alpha{}^\mu=\eta_{\alpha\beta}\eta^{\mu\nu}\Lambda^\beta{}_\nu$. 因逆变换为 $x^\mu=(\Lambda^{-1})^\mu{}_\alpha x'^\alpha=\eta_{\alpha\beta}\eta^{\mu\nu}\Lambda^\beta{}_\nu x'^\alpha$, 有 $\frac{\partial x^\mu}{\partial x'^\alpha}=\eta_{\alpha\beta}\eta^{\mu\nu}\Lambda^\beta{}_\nu$, 于是由上面的结果

$$\begin{aligned}\eta\Big(\frac{\partial}{\partial x'^\alpha},\frac{\partial}{\partial x'^\beta}\Big) &= \eta_{\mu\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}=\eta_{\mu\nu}\,(\eta_{\alpha\gamma}\eta^{\mu\rho}\Lambda^\gamma{}_\rho)\,(\eta_{\beta\delta}\eta^{\nu\sigma}\Lambda^\delta{}_\sigma)\\ &= (\eta_{\mu\nu}\eta^{\mu\rho})\,\eta_{\alpha\gamma}\eta_{\beta\delta}\eta^{\nu\sigma}\Lambda^\gamma{}_\rho\Lambda^\delta{}_\sigma=\delta^\rho_\nu\,\eta_{\alpha\gamma}\eta_{\beta\delta}\eta^{\nu\sigma}\Lambda^\gamma{}_\rho\Lambda^\delta{}_\sigma\\ &= \eta_{\alpha\gamma}\eta_{\beta\delta}\,(\eta^{\nu\sigma}\Lambda^\gamma{}_\nu\Lambda^\delta{}_\sigma)=\eta_{\alpha\gamma}(\eta_{\beta\delta}\eta^{\gamma\delta})=\eta_{\alpha\gamma}\delta^\gamma_\beta\\ &= \eta_{\alpha\beta}\,.\end{aligned}$$

可见度规张量 η 对 $\{x'^\mu\}$ 满足 (2-5-18) 式，故它也是洛伦兹系.

~19. (a) 用张量变换律求出 3 维欧氏度规在球坐标系中的全部分量 $g'_{\mu\nu}$. (b) 已知 4 维闵氏度规 g 在洛伦兹系中的线元表达式为 $ds^2=-dt^2+dx^2+dy^2+dz^2$, 求 g 及其逆 g^{-1} 在新坐标系 $\{t',x',y',z'\}$ 的全部分量 $g'_{\mu\nu}$ 及 $g'^{\mu\nu}$, 该新坐标系定义如下：

$$t'=t\,,\quad z'=z\,,\quad x'=(x^2+y^2)^{1/2}\cos(\varphi-\omega t)\,,$$

$$y'=(x^2+y^2)^{1/2}\sin(\varphi-\omega t)\,,\quad \omega=\text{常数}\,,$$

其中 φ 满足 $\cos\varphi=y(x^2+y^2)^{-1/2}$, $\sin\varphi=x(x^2+y^2)^{-1/2}$. 提示：先求 $g'^{\mu\nu}$ 再求 $g'_{\mu\nu}$.

解 (a) 球坐标与直角坐标的关系 $x=r\cos\theta\cos\varphi$, $y=r\cos\theta\sin\varphi$, $z=r\sin\theta$. 由张量变换律，定理 2-4-2, $g'_{\mu\nu}=\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}=\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\alpha}{\partial x'^\nu}$, 于是有

$$\begin{aligned}g'_{rr} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r}+\frac{\partial y}{\partial r}\frac{\partial y}{\partial r}+\frac{\partial z}{\partial r}\frac{\partial z}{\partial r}\\ &= (\cos\theta\cos\varphi)^2+(\cos\theta\sin\varphi)^2+(\sin\theta)^2\\ &= 1\,,\\ g'_{\theta\theta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta}+\frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta}+\frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \theta}\\ &= (-r\sin\theta\cos\varphi)^2+(-r\sin\theta\sin\varphi)^2+(r\cos\theta)^2\\ &= r^2\,,\\ g'_{\varphi\varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \varphi}+\frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \varphi}+\frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \varphi}\end{aligned}$$

$$
\begin{aligned}
&= (-r\cos\theta\sin\varphi)^2 + (r\cos\theta\cos\varphi)^2 + (0)^2 \\
&= r^2\sin^2\theta\ , \\
g'_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\
&= (\cos\theta\cos\varphi)(-r\sin\theta\cos\varphi) + (\cos\theta\sin\varphi)(-r\sin\theta\sin\varphi) + (\sin\theta)(r\cos\theta) \\
&= 0\ , \\
g'_{r\varphi} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} \\
&= (\cos\theta\cos\varphi)(-r\cos\theta\sin\varphi) + (\cos\theta\sin\varphi)(r\cos\theta\cos\varphi) + (\sin\theta)(0) \\
&= 0\ , \\
g'_{\theta\varphi} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial \theta}\frac{\partial z}{\partial \varphi} \\
&= (-r\sin\theta\cos\varphi)(-r\cos\theta\sin\varphi) + (-r\sin\theta\sin\varphi)(r\cos\theta\cos\varphi) + (r\cos\theta)(0) \\
&= 0\ .
\end{aligned}
$$

因此求得 $g'_{rr}=1$, $g'_{\theta\theta}=r^2$, $g'_{\varphi\varphi}=r^2\sin^2\theta$, 非对角元都为零.

(b) 令 $r\equiv(x^2+y^2)^{1/2}$, $\Phi\equiv\varphi-\omega t$. 因 $\cos\varphi=\frac{y}{r}$, we have $-\sin\varphi\frac{\partial\varphi}{\partial x}=-\frac{xy}{r^3}\Rightarrow \frac{x}{r}\frac{\partial\varphi}{\partial x}=\frac{xy}{r^3}\Rightarrow\frac{\partial\varphi}{\partial x}=\frac{y}{r^2}$. 类似可得 $\frac{\partial\varphi}{\partial y}=-\frac{x}{r^2}$. 于是有

$$
\begin{aligned}
\frac{\partial\cos\Phi}{\partial x} &= -\sin\Phi\frac{\partial\varphi}{\partial x} = -\frac{y}{r^2}\sin\Phi\ , \\
\frac{\partial\cos\Phi}{\partial y} &= -\sin\Phi\frac{\partial\varphi}{\partial y} = \frac{x}{r^2}\sin\Phi\ , \\
\frac{\partial\sin\Phi}{\partial x} &= \cos\Phi\frac{\partial\varphi}{\partial x} = \frac{y}{r^2}\cos\Phi\ , \\
\frac{\partial\sin\Phi}{\partial y} &= \cos\Phi\frac{\partial\varphi}{\partial y} = -\frac{x}{r^2}\cos\Phi\ .
\end{aligned}
$$

下面需要用到

$$
\begin{aligned}
\frac{\partial t'}{\partial t} &= 1\ , \\
\frac{\partial t'}{\partial x} &= \frac{\partial t'}{\partial y} = \frac{\partial t'}{\partial z} = 0\ ; \\
\frac{\partial x'}{\partial t} &= (-r\sin\Phi)(-\omega) = r\omega\sin\Phi\ , \\
\frac{\partial x'}{\partial x} &= \frac{x}{r}\cos\Phi + r\frac{\partial\cos\Phi}{\partial x} = \frac{x}{r}\cos\Phi + r\Big(-\frac{y}{r^2}\sin\Phi\Big) \\
&= \sin\varphi\cos\Phi - \cos\varphi\sin\Phi = \sin(\varphi-\Phi) = \sin\omega t\ , \\
\frac{\partial x'}{\partial y} &= \frac{y}{r}\cos\Phi + r\frac{\partial\cos\Phi}{\partial y} = \frac{y}{r}\cos\Phi + r\Big(\frac{x}{r^2}\sin\Phi\Big) \\
&= \cos\varphi\cos\Phi + \sin\varphi\sin\Phi = \cos(\varphi-\Phi) = \cos\omega t\ , \\
\frac{\partial x'}{\partial z} &= 0\ ; \\
\frac{\partial y'}{\partial t} &= (r\cos\Phi)(-\omega) = -r\omega\cos\Phi\ ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial y'}{\partial x} &= \frac{x}{r}\sin\Phi + r\frac{\partial \sin\Phi}{\partial x} = \frac{x}{r}\sin\Phi + r\Big(\frac{y}{r^2}\cos\Phi\Big) \\
&= \sin\varphi\sin\Phi + \cos\varphi\cos\Phi = \cos(\varphi-\Phi) = \cos\omega t\ , \\
\frac{\partial y'}{\partial y} &= \frac{y}{r}\sin\Phi + r\frac{\partial \sin\Phi}{\partial y} = \frac{y}{r}\sin\Phi + r\Big(-\frac{x}{r^2}\cos\Phi\Big) \\
&= \cos\varphi\sin\Phi - \sin\varphi\cos\Phi = -\sin(\varphi-\Phi) = -\sin\omega t\ , \\
\frac{\partial y'}{\partial z} &= 0\ ; \\
\frac{\partial z'}{\partial t} &= \frac{\partial z'}{\partial x} = \frac{\partial z'}{\partial y} = 0\ , \\
\frac{\partial z'}{\partial z} &= 1\ .
\end{aligned}
$$

由张量变换律，定理 2-4-2, $g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}$ 得

$$
\begin{aligned}
g'^{tt} &= -\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial t} + \frac{\partial t'}{\partial x}\frac{\partial t'}{\partial x} + \frac{\partial t'}{\partial y}\frac{\partial t'}{\partial y} + \frac{\partial t'}{\partial z}\frac{\partial t'}{\partial z} \\
&= -(1)^2 + (0)^2 + (0)^2 + (0)^2 = -1\ , \\
g'^{xx} &= -\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial t} + \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial x'}{\partial y} + \frac{\partial x'}{\partial z}\frac{\partial x'}{\partial z} \\
&= -(r\omega\sin\Phi)^2 + (\sin\omega t)^2 + (\cos\omega t)^2 + (0)^2 \\
&= 1 - r^2\omega^2\sin^2\Phi = 1 - (x^2+y^2)\omega^2\sin^2(\varphi-\omega t)\ , \\
g'^{yy} &= -\frac{\partial y'}{\partial t}\frac{\partial y'}{\partial t} + \frac{\partial y'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial y'}{\partial y}\frac{\partial y'}{\partial y} + \frac{\partial y'}{\partial z}\frac{\partial y'}{\partial z} \\
&= -(-r\omega\cos\Phi)^2 + (\cos\omega t)^2 + (-\sin\omega t)^2 + (0)^2 \\
&= 1 - r^2\omega^2\cos^2\Phi = 1 - (x^2+y^2)\omega^2\cos^2(\varphi-\omega t)\ , \\
g'^{zz} &= -\frac{\partial z'}{\partial t}\frac{\partial z'}{\partial t} + \frac{\partial z'}{\partial x}\frac{\partial z'}{\partial x} + \frac{\partial z'}{\partial y}\frac{\partial z'}{\partial y} + \frac{\partial z'}{\partial z}\frac{\partial z'}{\partial z} \\
&= -(0)^2 + (0)^2 + (0)^2 + (1)^2 = 1\ ; \\
g'^{tx} &= -\frac{\partial t'}{\partial t}\frac{\partial x'}{\partial t} + \frac{\partial t'}{\partial x}\frac{\partial x'}{\partial x} + \frac{\partial t'}{\partial y}\frac{\partial x'}{\partial y} + \frac{\partial t'}{\partial z}\frac{\partial x'}{\partial z} \\
&= -(1)(r\omega\sin\Phi) + (0)(\sin\omega t) + (0)(\cos\omega t) + (0)(0) \\
&= -r\omega\sin\Phi = -(x^2+y^2)^{1/2}\omega\sin(\varphi-\omega t)\ , \\
g'^{ty} &= -\frac{\partial t'}{\partial t}\frac{\partial y'}{\partial t} + \frac{\partial t'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial t'}{\partial y}\frac{\partial y'}{\partial y} + \frac{\partial t'}{\partial z}\frac{\partial y'}{\partial z} \\
&= -(1)(-r\omega\cos\Phi) + (0)(\cos\omega t) + (0)(-\sin\omega t) + (0)(0) \\
&= r\omega\cos\Phi = (x^2+y^2)^{1/2}\omega\cos(\varphi-\omega t)\ , \\
g'^{tz} &= -\frac{\partial t'}{\partial t}\frac{\partial z'}{\partial t} + \frac{\partial t'}{\partial x}\frac{\partial z'}{\partial x} + \frac{\partial t'}{\partial y}\frac{\partial z'}{\partial y} + \frac{\partial t'}{\partial z}\frac{\partial z'}{\partial z} \\
&= -(1)(0) + (0)(0) + (0)(0) + (0)(1) \\
&= 0\ ; \\
g'^{xy} &= -\frac{\partial x'}{\partial t}\frac{\partial y'}{\partial t} + \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y} + \frac{\partial x'}{\partial z}\frac{\partial y'}{\partial z} \\
&= -(r\omega\sin\Phi)(-r\omega\cos\Phi) + (\sin\omega t)(\cos\omega t) + (\cos\omega t)(-\sin\omega t) + (0)(0)
\end{aligned}
$$

$$
\begin{aligned}
&= r^2\omega^2 \sin\Phi\cos\Phi = (x^2+y^2)\omega^2\sin(\varphi-\omega t)\cos(\varphi-\omega t)\ ,\\
g'^{xz} &= -\frac{\partial x'}{\partial t}\frac{\partial z'}{\partial t}+\frac{\partial x'}{\partial x}\frac{\partial z'}{\partial x}+\frac{\partial x'}{\partial y}\frac{\partial z'}{\partial y}+\frac{\partial x'}{\partial z}\frac{\partial z'}{\partial z}\\
&= -(r\omega\sin\Phi)(0)+(\sin\omega t)(0)+(\cos\omega t)(0)+(0)(1)\\
&= 0\ ;\\
g'^{yz} &= -\frac{\partial y'}{\partial t}\frac{\partial z'}{\partial t}+\frac{\partial y'}{\partial x}\frac{\partial z'}{\partial x}+\frac{\partial y'}{\partial y}\frac{\partial z'}{\partial y}+\frac{\partial y'}{\partial z}\frac{\partial z'}{\partial z}\\
&= -(-r\omega\cos\Phi)(0)+(\cos\omega t)(0)+(-\sin\omega t)(0)+(0)(1)\\
&= 0\ .
\end{aligned}
$$

因此我们求得分量矩阵

$$
g'^{\mu\nu} = \begin{pmatrix}
-1 & -r\omega\sin\Phi & r\omega\cos\Phi & 0\\
-r\omega\sin\Phi & 1-r^2\omega^2\sin^2\Phi & r^2\omega^2\sin\Phi\cos\Phi & 0\\
r\omega\cos\Phi & r^2\omega^2\sin\Phi\cos\Phi & 1-r^2\omega^2\cos^2\Phi & 0\\
0 & 0 & 0 & 1
\end{pmatrix}\ .
$$

因为 $g'_{\mu\nu}g'^{\nu\rho}=\delta^\rho_\mu$, 所以 $g'_{\mu\nu}$ 的分量矩阵是以上矩阵的逆矩阵:

$$
g'_{\mu\nu} = \begin{pmatrix}
-1+r^2\omega^2 & -r\omega\sin\Phi & r\omega\cos\Phi & 0\\
-r\omega\sin\Phi & 1 & 0 & 0\\
r\omega\cos\Phi & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{pmatrix}\ ,
$$

其中 $r=(x^2+y^2)^{1/2}$, $\Phi=\varphi-\omega t$.

~20. 试证 3 维欧氏空间中球坐标基矢 $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\varphi$ 的长度依次为 1, r, $r\sin\theta$.

证 上题 (a) 中我们已经求得球坐标的 $g_{rr}=1$, $g_{\theta\theta}=r^2$, $g_{\varphi\varphi}=r^2\sin^2\theta$, 因此球坐标基矢的长度依次为 1, r, $r\sin\theta$.

~21. 用抽象指标记号证明 $T'^\mu{}_\nu=\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}T^\rho{}_\sigma$.

证 利用抽象指标记号, 张量式为 $T^a{}_b=T^\mu{}_\nu(e_\mu)^a(e^\nu)_b$ [见 (2-6-1) 式], 分量式为 $T^\mu{}_\nu=T^a{}_b(e^\mu)_a(e_\nu)^b$ [见 (2-6-2) 式], 于是

$$
\begin{aligned}
T'^\mu{}_\nu &= T^a{}_b(e'^\mu)_a(e'_\nu)^b = T^\rho{}_\sigma(e_\rho)^a(e^\sigma)_b(e'^\mu)_a(e'_\nu)^b\\
&= T^\rho{}_\sigma(e_\rho)^a(e'^\mu)_a(e^\sigma)_b(e'_\nu)^b\ ,
\end{aligned}
$$

其中

$$
\begin{aligned}
(e_\rho)^a(e'^\mu)_a &= \left(\frac{\partial}{\partial x^\rho}\right)^a(dx'^\mu)_a=\frac{\partial x'^\mu}{\partial x^\rho}\ ,\\
(e^\sigma)_b(e'_\nu)^b &= (dx^\sigma)_b\left(\frac{\partial}{\partial x'^\nu}\right)^b=\frac{\partial x^\sigma}{\partial x'^\nu}\ .
\end{aligned}
$$

因此得

$$
T'^\mu{}_\nu=\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\,T^\rho{}_\sigma\ .
$$

22. 以 g 和 g' 分别代表度规 g_{ab} 在坐标系 $\{x^\mu\}$ 和 $\{x'^\mu\}$ 的分量 $g_{\mu\nu}$ 和 $g'_{\mu\nu}$ 组成的两个 $n\times n$ 矩阵行列式，试证 $g'=|\partial x^\rho/\partial x'^\sigma|^2 g$, 其中 $|\partial x^\rho/\partial x'^\sigma|$ 是坐标变换 $\{x^\mu\}\mapsto\{x'^\mu\}$ 的雅可比行列式，即由 $\partial x^\rho/\partial x'^\sigma$ 组成的 $n\times n$ 行列式．注：本题表明度规的行列式在坐标变换下不是不变量．提示：取等式 $g'_{\rho\sigma}=(\partial x^\mu/\partial x'^\rho)(\partial x^\nu/\partial x'^\sigma)g_{\mu\nu}$ 的行列式.

证 将 $(0,2)$ 型张量的分量 $g_{\mu\nu}$ 和 $g'_{\mu\nu}$ 看成矩阵元，其中 μ 和 ν 分别是行和列指标．同样，将 $(1,1)$ 型张量的分量 $A_{\rho\sigma}\equiv\partial x^\rho/\partial x'^\sigma$ 也看成矩阵元，其中 ρ 和 σ 分别是行和列指标．于是，变换关系 $g'_{\rho\sigma}=(\partial x^\mu/\partial x'^\rho)(\partial x^\nu/\partial x'^\sigma)g_{\mu\nu}$ 可以写成 $g'_{\rho\sigma}=A_{\mu\rho}A_{\nu\sigma}g_{\mu\nu}=\tilde{A}_{\rho\mu}g_{\mu\nu}A_{\nu\sigma}=(\tilde{A}gA)_{\rho\sigma}$, 其中 $\tilde{A}$ 是 A 的转置矩阵，相应的矩阵等式为 $g'=\tilde{A}gA$. 两边取行列式则有 $\det g'=\det\tilde{A}\det g\det A=(\det A)^2\det g=|\partial x^\rho/\partial x'^\sigma|^2\det g$, 这正是要证的关系．以直角坐标到球坐标为例，在前两题中我们已经知道行列式 $|\partial x^\rho/\partial x'^\sigma|=r^2\sin\theta$, 故 $\det g_{\text{球坐标}}=r^4\sin^2\theta\det g_{\text{直角坐标}}=r^4\sin^2\theta$, 因为 $\det g_{\text{直角坐标}}=1$.

~23. 设 $\{x^\mu\}$ 是流形上的任一局域坐标系，试判断下列等式的是非：

(1) $(\partial/\partial x^\mu)^a(\partial/\partial x^\nu)_a=g_{\mu\nu}$, 其中 $(\partial/\partial x^\nu)_a\equiv g_{ab}(\partial/\partial x^\nu)^b$;

(2) $(dx^\mu)^a(dx^\nu)_a=g^{\mu\nu}$, 其中 $(dx^\mu)^a\equiv g^{ab}(dx^\mu)_b$;

(3) $(\partial/\partial x^\mu)_a=(dx^\mu)_a$;

(4) $(dx^\mu)^a=(\partial/\partial x^\mu)^a$;

(5) $v^\mu\omega_\mu=v_\mu\omega^\mu$;

(6) $g_{\mu\nu}T^{\nu\rho}S_\rho{}^\sigma=T_{\mu\rho}S^{\rho\sigma}$;

(7) $v^a u^b=v^b u^a$;

(8) $v^a u^b=u^b v^a$.

答 (1) 是； (2) 是； (3) 非； (4) 非； (5) 是； (6) 是； (7) 非； (8) 是.

如其中 (6) 式： $g_{\mu\nu}T^{\nu\rho}S_\rho{}^\sigma=T_\mu{}^\rho S_\rho{}^\sigma=T_{\mu\tau}g^{\rho\tau}S_\rho{}^\sigma=T_{\mu\tau}S^{\tau\sigma}=T_{\mu\rho}S^{\rho\sigma}$.

24. 设 T_{ab} 是矢量空间 V 上的 $(0,2)$ 型张量，试证 $T_{ab}v^av^b=0,\ \forall v^a\in V\ \Rightarrow T_{ab}=T_{[ab]}$. 提示：把 v^a 表为任意两个矢量 u^a 和 w^a 之和【有什么用？】.

证 我们证与其等价的分量式的命题：如果 $T_{\mu\nu}v^\mu v^\nu=0\ \ \forall v^\mu$, 则 $T_{\mu\nu}=T_{[\mu\nu]}$, 这里 $\mu,\nu=1,\cdots,n$. 首先，取 $v^\mu=(v,0,\cdots,0)$, 即 $v^1=v\in RR$, 其他 $\mu\neq 1$ 的分量都为零．等式变为 $T_{11}v^2=0\ \Rightarrow T_{11}=0$. 同样可以知道所有对角元素 $T_{\mu\mu}=0$. 下面取 $v^\mu=(v,v,0,\cdots,0)$, 即 $v^1=v^2=v$, 其他 $\mu\neq 1$ 和 2 的分量都为零．这时等式变为 $(T_{11}+T_{22}+T_{12}+T_{21})v^2=0$. 已知 $T_{11}=T_{22}=0$, 所以必有 $T_{12}+T_{21}=0$. 类似可证当 $\mu\neq\nu$ 时， $T_{\mu\nu}+T_{\nu\mu}=0$, 因此有 $T_{\mu\nu}=-T_{\nu\mu}=T_{[\mu\nu]}$, 命题得证.

25. 试证 $T_{abcd}=T_{a[bc]d}=T_{ab[cd]} \ \Rightarrow\ T_{abcd}=T_{a[bcd]}$.

注 (1) 推广至一般的结论是

$$T_{\cdots a\cdots b\cdots c\cdots}=T_{\cdots[a\cdots b]\cdots c\cdots}=T_{\cdots a\cdots[b\cdots c]\cdots} \quad\Rightarrow\quad T_{\cdots a\cdots b\cdots c\cdots}=T_{\cdots[a\cdots b\cdots c]\cdots}\ .$$

上式的前提中只有两个等号，关键是 $T_{\cdots[a\cdots b]\cdots c\cdots}$ 和 $T_{\cdots a\cdots[b\cdots c]\cdots}$ 中的指标 b 都在方括号内.

(2) 把前提和结论中的方括号改为圆括号，则推广前后的命题仍成立.

证 如果 $T_{abcd}=T_{a[bc]d}=\frac{1}{2}(T_{abcd}-T_{acbd})$ 和 $T_{abcd}=T_{ab[cd]}=\frac{1}{2}(T_{abcd}-T_{abdc})$, 则有 $T_{acbd}=-T_{abcd}$ 和 $T_{abdc}=-T_{abcd}$, 即交换中间两个指标和交换最后两个指标都会附加一负号. 于是

$$\begin{aligned} T_{a[bcd]} &= \frac{1}{6}\left(T_{abcd}-T_{abdc}+T_{acdb}-T_{acbd}+T_{adbc}-T_{adcb}\right)\\ &= \frac{1}{6}T_{abcd}\left[1-(-1)+(-1)^2-(-1)+(-1)^2-(-1)^3\right]\\ &= T_{abcd}\ . \end{aligned}$$

这一结论很容易推广，因为 $[a\cdots b]$ 和 $[b\cdots c]$ 内的反称化会导致 $[a\cdots b\cdots c]$ 内的反称化.

第 3 章 “黎曼 (内禀) 曲率张量” 习题

~1. 放弃 ∇_a 定义中的无挠性条件 (e),

(1) 试证存在张量 $T^c{}_{ab}$ (叫 **挠率张量**) 使

$$\nabla_a\nabla_b f-\nabla_b\nabla_a f=-T^c{}_{ab}\nabla_c f\ ,\qquad \forall f\in\mathscr{F}\ .$$

提示：令 $\tilde\nabla_a$ 为无挠算符，模仿定理 3-1-4 证明中的推导.

(2) 试证 $T^c{}_{ab}u^a v^b=u^a\nabla_a v^c-v^a\nabla_a u^c-[u,v]^c \quad \forall\, u^a,\, v^a\in\mathscr{F}(1,0)$.

证 (1) 因 (3-1-2), 可以令 $\omega_b=\nabla_b f=\tilde\nabla_b f$, 其中 $\tilde\nabla_b$ 为无挠导数算符. 根据定理 3-1-3 式 (3-1-6): $\nabla_a\omega_b=\tilde\nabla_a\omega_b-C^c{}_{ab}\omega_c$, 有 $\nabla_a\nabla_b f=\tilde\nabla_a\tilde\nabla_b f-C^c{}_{ab}\nabla_c f$. 交换指标得 $\nabla_b\nabla_a f=\tilde\nabla_b\tilde\nabla_a f-C^c{}_{ba}\nabla_c f$. 两式相减并利用 $\tilde\nabla_a$ 的无挠性，有 $\nabla_a\nabla_b f-\nabla_b\nabla_a f=-(C^c{}_{ab}-C^c{}_{ba})\nabla_c f=-T^c{}_{ab}\nabla_c f$, 其中已令 $C^c{}_{ab}-C^c{}_{ba}\equiv T^c{}_{ab}$ 为挠率张量.

(2) 将对易子作用于标量场 $f\in\mathscr{F}_M(0,0)$

$$[u,v](f)\ \overset{(2\text{-}2\text{-}9)}{=}\ u(v(f))-v(u(f))\ \overset{\text{定义 }1(\mathrm{d})}{=}\ u^a\nabla_a(v^b\nabla_b f)-v^a\nabla_a(u^b\nabla_b f)$$

$$
\begin{aligned}
&= u^a(\nabla_a v^b)(\nabla_b f) + u^a v^b(\nabla_a \nabla_b f) - v^a(\nabla_a u^b)(\nabla_b f) - v^a u^b(\nabla_a \nabla_b f) \\
&= [u^a(\nabla_a v^b) - v^a(\nabla_a u^b)](\nabla_b f) + u^a v^b[\nabla_a \nabla_b f - \nabla_b \nabla_a f] \\
&= [u^a \nabla_a v^c - v^a \nabla_a u^c](\nabla_c f) + u^a v^b[-T^c{}_{ab} \nabla_c f] \\
&= \Big(u^a \nabla_a v^c - v^a \nabla_a u^c - T^c{}_{ab} u^a v^b\Big)\nabla_c f\,.
\end{aligned}
$$

另一方面, $[u,v] \in \mathscr{F}_M(1,0)$ 本身是矢量场, 作用于 f 根据定义 1(d) 有 $[u,v](f) = [u,v]^c \nabla_c f$, 因此得

$$
u^a \nabla_a v^c - v^a \nabla_a u^c - T^c{}_{ab} u^a v^b = [u,v]^c
$$

~2. 设 v^a 为矢量场, v^ν 和 v'^ν 为 v^a 在坐标系 $\{x^\nu\}$ 和 $\{x'^\nu\}$ 的分量, $A^\nu{}_\mu \equiv \partial v^\nu/\partial x^\mu$, $A'^\nu{}_\mu \equiv \partial v'^\nu/\partial x'^\mu$, 试证 $A^\nu{}_\mu$ 和 $A'^\nu{}_\mu$ 的关系一般而言不满足张量分量变换律. 提示: 利用 v^ν 与 v'^ν 之间的变换规律.

证 矢量场 v^a 和 $(1,1)$ 型张量场 $T^a{}_b$ 在坐标系变换下满足的变换关系分别为:

$$
\begin{aligned}
v'^\mu &= v^a(e'^\mu)_a = v^\rho(e_\rho)^a(e'^\mu)_a = v^\rho(\partial/\partial x^\rho)^a(dx'^\mu)_a \\
&= v^\rho \frac{\partial x'^\mu}{\partial x^\rho}\,, \\
T'^\mu{}_\nu &= T^a{}_b(e'^\mu)_a(e'_\nu)^b = T^\rho{}_\sigma(e_\rho)^a(e^\sigma)_b(e'^\mu)_a(e'_\nu)^b \\
&= T^\rho{}_\sigma(\partial/\partial x^\rho)^a(dx'^\mu)_a(dx^\sigma)_b(\partial/\partial x'^\nu)^b \\
&= T^\rho{}_\sigma \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\,,
\end{aligned}
$$

即定理 2-4-2 的变换律. 现在根据定义

$$
\begin{aligned}
A'^\mu{}_\nu &= \frac{\partial}{\partial x'^\nu} v'^\mu = \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial}{\partial x^\sigma}\left(v^\rho \frac{\partial x'^\mu}{\partial x^\rho}\right) \\
&= \frac{\partial x^\sigma}{\partial x'^\nu}\left[\frac{\partial v^\rho}{\partial x^\sigma}\frac{\partial x'^\mu}{\partial x^\rho} + v^\rho\left(\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho}\right)\right] \\
&= A^\rho{}_\sigma \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho} + v^\rho \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho}\,,
\end{aligned}
$$

与张量的变换式比较, 显然右边第二项破坏了张量变换律. [但如果变换是线性的 (如洛伦兹变换), 那么第二项仍为零!]

~3. 试证定理 3-1-7.

证 由定理 3-1-5 式 (3-1-7) 和定义 2: $\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c$, 而

$$
\begin{aligned}
v^\nu{}_{;\mu} &= \nabla_a v^b (e^\nu)_b (e_\mu)^a = (\partial_a v^b + \Gamma^b{}_{ac} v^c)(e^\nu)_b (e_\mu)^a \\
&= \partial_a v^b (e^\nu)_b (e_\mu)^a + \Gamma^b{}_{ac} v^c (e^\nu)_b (e_\mu)^a \\
&= \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu{}_{\mu\sigma} v^\sigma = v^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\sigma} v^\sigma\,.
\end{aligned}
$$

由定理 3-1-3 式 (3-1-6) 和定义 2: $\nabla_a\omega_b = \partial_a\omega_b - \Gamma^c{}_{ab}\omega_c$, 而

$$\begin{aligned}
\omega_{\nu;\mu} &= \nabla_a\omega_b(e_\nu)^b(e_\mu)^a = (\partial_a\omega_b - \Gamma^c{}_{ab}\omega_c)(e_\nu)^b(e_\mu)^a \\
&= \partial_a\omega_b(e_\nu)^b(e_\mu)^a - \Gamma^c{}_{ab}\omega_c(e_\nu)^b(e_\mu)^a \\
&= \frac{\partial\omega_\nu}{\partial x^\mu} - \Gamma^\sigma{}_{\mu\nu}\omega_\sigma = \omega_{\nu,\mu} - \Gamma^\sigma{}_{\mu\nu}\omega_\sigma \ .
\end{aligned}$$

即为定理 3-1-7 的 (3-1-11) 中的两式.

4. 用下式定义 $\Gamma^\sigma{}_{\mu\nu}$: $(\frac{\partial}{\partial x^\nu})^b\nabla_b(\frac{\partial}{\partial x^\mu})^a = \Gamma^\sigma{}_{\mu\nu}(\frac{\partial}{\partial x^\sigma})^a$, 试证

(a) $\Gamma^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\nu\mu}$ (提示：利用 ∇_a 的无挠性和坐标基矢间的对易性.);

(b) $v^\nu{}_{;\mu} = v^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\beta}v^\beta$ (注：这其实是克氏符的等价定义.).

证 为简洁起见，以下都用基矢符号表示坐标基底的基矢，即矢量的基矢 $(\frac{\partial}{\partial x^\mu})^a \equiv (e_\mu)^a$, 对偶矢量的基矢 $(dx^\mu)_a \equiv (e^\mu)_a$. 于是定义式为：

$$(e_\nu)^b\nabla_b(e_\mu)^a = \Gamma^\sigma{}_{\mu\nu}(e_\sigma)^a \ .$$

(a) 根据定理 3-1-9 式 (3-1-13), 对于无挠微分算符 ∇_a 成立

$$[u,v]^a = u^b\nabla_b v^a - v^b\nabla_b u^a \ .$$

而对于坐标基底的基矢 $\{(e_\mu)^a\}$, 它们互相对易，因此有

$$0 = [e_\mu, e_\nu]^a = (e_\mu)^b\nabla_b(e_\nu)^a - (e_\nu)^b\nabla_b(e_\mu)^a \ ,$$

即 $(e_\mu)^b\nabla_b(e_\nu)^a = (e_\nu)^b\nabla_b(e_\mu)^a$. 以对偶基矢 $(e^\rho)_a$ 作用定义式：

$$(e^\rho)_a(e_\nu)^b\nabla_b(e_\mu)^a = \Gamma^\sigma{}_{\mu\nu}(e^\rho)_a(e_\sigma)^a = \Gamma^\sigma{}_{\mu\nu}\delta^\rho{}_\sigma = \Gamma^\rho{}_{\mu\nu} \ .$$

利用对易关系，于是有

$$\Gamma^\rho{}_{\mu\nu} = (e^\rho)_a(e_\nu)^b\nabla_b(e_\mu)^a = (e^\rho)_a(e_\mu)^b\nabla_b(e_\nu)^a = \Gamma^\rho{}_{\nu\mu}$$

(b) 两边作用 $(e^\nu)_c$ 于定义式： $(e^\nu)_c(e_\nu)^b[\nabla_b(e_\mu)^a] = \delta^b{}_c[\nabla_b(e_\mu)^a] = \nabla_c(e_\mu)^a = (e^\nu)_c\Gamma^\sigma{}_{\mu\nu}(e_\sigma)^a$, 即为

$$\nabla_a(e_\mu)^b = \Gamma^\sigma{}_{\mu\nu}(e^\nu)_a(e_\sigma)^b$$

$$\begin{aligned}
v^\nu{}_{;\mu} &= (e_\mu)^a(e^\nu)_b\nabla_a v^b = (e_\mu)^a(e^\nu)_b\nabla_a[v^\rho(e_\rho)^b] \\
&= (e_\mu)^a(e^\nu)_b\Big\{(\nabla_a v^\rho)(e_\rho)^b + v^\rho[\nabla_a(e_\rho)^b]\Big\} \\
&\overset{(3\text{-}1\text{-}1)}{=} (e_\mu)^a(e^\nu)_b\Big\{(dv^\rho)_a(e_\rho)^b + v^\rho[\Gamma^\sigma{}_{\rho\lambda}(e^\lambda)_a(e_\sigma)^b]\Big\} \\
&= (e_\mu)^a(dv^\rho)_a(e^\nu)_b(e_\rho)^b + v^\rho\Gamma^\sigma{}_{\rho\lambda}(e_\mu)^a(e^\lambda)_a(e^\nu)_b(e_\sigma)^b \\
&= (e_\mu)^a(dv^\rho)_a\delta^\nu{}_\rho + v^\rho\Gamma^\sigma{}_{\rho\lambda}\delta^\lambda{}_\mu\delta^\nu{}_\sigma \\
&= (e_\mu)^a(dv^\nu)_a + v^\rho\Gamma^\nu{}_{\rho\mu}
\end{aligned}$$

其中 $(e_\mu)^a(dv^\nu)_a = dv^\nu(e_\mu) = dv^\nu(\frac{\partial}{\partial x^\mu}) = \frac{\partial v^\nu}{\partial x^\mu} = v^\nu{}_{,\mu}$. 再利用 (a) 的结果得

$$v^\nu{}_{;\mu} = v^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\rho} v^\rho \ .$$

~5. 判断是非：

(1) $\nabla_a (dx^\mu)_b = 0$；

(2) $v^\nu{}_{;\mu} = (\nabla_a v^b)(\partial/\partial x^\mu)^a (dx^\nu)_b$；

(3) $v^\nu{}_{,\mu} = (\partial_a v^b)(\partial/\partial x^\mu)^a (dx^\nu)_b$；

(4) $v^\nu{}_{;\mu} = (\partial/\partial x^\mu)^a \nabla_a v^\nu$；

(5) $v^\nu{}_{,\mu} = (\partial/\partial x^\mu)^a \nabla_a v^\nu$.

答 (1) 错．见上题 (a) 中的结果，对无挠导数算符有 $\nabla_a(dx^\mu)_b = \nabla_b(dx^\mu)_a$.

(2) 对．为定义式.

(3) 对．也为定义式.

(4) 错. 因为 $\nabla_a v^\nu = \nabla_a[v^b(dx^\nu)_b] = (\nabla_a v^b)(dx^\nu)_b + v^b[\nabla_a(dx^\nu)_b]$, 所以此式右边为 $(\partial/\partial x^\mu)^a(\nabla_a v^b)(dx^\nu)_b + (\partial/\partial x^\mu)^a v^b[\nabla_a(dx^\nu)_b] = v^\nu{}_{;\mu} + (\partial/\partial x^\mu)^a v^b[\nabla_a(dx^\nu)_b] \neq v^\nu{}_{;\mu}$. 另外从 (5) 的结果知右边其实是 $v^\nu{}_{,\mu}$, 它一般不等于 $v^\nu{}_{;\mu}$.

(5) 对．因为如果把分量 v^μ 看成标量函数，则由 (3-1-2) 式知 $\nabla_a v^\nu = \partial_a v^\nu = (dv^\nu)_a$. 于是 $(\partial/\partial x^\mu)^a \nabla_a v^\nu = (\partial/\partial x^\mu)^a \partial_a v^\nu = (\partial/\partial x^\mu)^a (dv^\nu)_a = (dv^\nu)(\partial/\partial x^\mu) \overset{(2\text{-}3\text{-}7)}{=} \partial v^\nu/\partial x^\mu = v^\nu{}_{,\mu}$. 也可以这样看：因 $\nabla_a v^\nu = \partial_a v^\nu = \partial_a[v^b(dx^\nu)_b] = (\partial_a v^b)(dx^\nu)_b + v^b[\partial_a(dx^\nu)_b] \overset{(3\text{-}1\text{-}10)}{=} (\partial_a v^b)(dx^\nu)_b$, 于是右边 $(\partial/\partial x^\mu)^a \nabla_a v^\nu = (\partial/\partial x^\mu)^a (\partial_a v^b)(dx^\nu)_b$, 根据定义它就是 $v^\nu{}_{,\mu}$ [见 (3)].

~6. 设 $C(t)$ 是 $\{x^\mu\}$ 的坐标域内的曲线， $x^\mu(t)$ 是 $C(t)$ 在该系的参数表达式，v^a 是 $C(t)$ 上的矢量场，令 $Dv^\mu/dt \equiv (dx^\mu)_a(\partial/\partial t)^b \nabla_b v^a$, 试证

$$Dv^\mu/dt \equiv dv^\mu/dt + \Gamma^\mu{}_{\nu\sigma} v^\sigma dx^\nu(t)/dt \ .$$

证 由定义

$$\begin{aligned}
\frac{Dv^\mu}{dt} &= (dx^\mu)_a \frac{Dv^a}{dt} \overset{(3\text{-}2\text{-}13)}{=} (dx^\mu)_a T^b \nabla_b v^a = (dx^\mu)_a \Big(\frac{\partial}{\partial t}\Big)^b \nabla_b v^a \\
&\overset{(3\text{-}1\text{-}7)}{=} (dx^\mu)_a \Big(\frac{\partial}{\partial t}\Big)^b (\partial_b v^a + \Gamma^a{}_{bc} v^c) \\
&\overset{(3\text{-}1\text{-}10)}{=} \Big(\frac{\partial}{\partial t}\Big)^b \Big(\partial_b[(dx^\mu)_a v^a] + (dx^\mu)_a \Gamma^a{}_{bc} v^c\Big) \\
&= \Big(\frac{\partial}{\partial t}\Big)^b \Big(\partial_b[v^\mu] + \Gamma^\mu{}_{b\sigma} v^\sigma\Big) \\
&= \Big(\frac{\partial}{\partial t}\Big)^\nu \Big(\partial_\nu v^\mu + \Gamma^\mu{}_{\nu\sigma} v^\sigma\Big) ,
\end{aligned}$$

其中 $(\frac{\partial}{\partial t})^\nu$ 为曲线的切矢 $(\frac{\partial}{\partial t})^b$ 的坐标分量 $(\frac{\partial}{\partial t})^\nu \overset{(2\text{-}2\text{-}7)}{=} \frac{dx^\nu(t)}{dt}$，或者 $(\frac{\partial}{\partial t})^\nu = (\frac{\partial}{\partial t})^a (dx^\nu)_a = dx^\nu(\frac{\partial}{\partial t}) \overset{(2\text{-}3\text{-}7)}{=} \frac{\partial x^\nu}{\partial t}$. 于是

$$
\begin{aligned}
\frac{Dv^\mu}{dt} &= \frac{dx^\nu(t)}{dt}\left(\frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\sigma} v^\sigma\right) \\
&= \frac{dv^\mu}{dt} + \Gamma^\mu{}_{\nu\sigma} v^\sigma \frac{dx^\nu(t)}{dt} \,.
\end{aligned}
$$

~7. 求出 3 维欧氏空间中球坐标系的全部非零 $\Gamma^\sigma{}_{\mu\nu}$.

证 根据 (3-2-10') 式 $\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$. 对于求坐标，只有 [见前一章习题 19(a)]

$$
g_{rr} = 1\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\varphi\varphi} = r^2 \sin^2\theta\,.
$$

所以有

$$
g^{rr} = 1\,, \quad g^{\theta\theta} = r^{-2}\,, \quad g^{\varphi\varphi} = r^{-2}\sin^{-2}\theta\,.
$$

求导后得

$$
\begin{aligned}
& g_{rr,r} = g_{rr,\theta} = g_{rr,\varphi} = 0\,; \\
& g_{\theta\theta,r} = 2r\,, \quad g_{\theta\theta,\theta} = g_{\theta\theta,\varphi} = 0\,; \\
& g_{\varphi\varphi,r} = 2r\sin^2\theta\,, \quad g_{\varphi\varphi,\theta} = r^2\sin 2\theta\,, \quad g_{\varphi\varphi,\varphi} = 0\,.
\end{aligned}
$$

代入公式得到

$$
\begin{aligned}
\Gamma^\sigma{}_{rr} &= \frac{1}{2} g^{\sigma\rho}(g_{\rho r,r} + g_{\rho r,r} - g_{rr,\rho}) = \frac{1}{2} g^{\sigma r}(g_{rr,r} + g_{rr,r} - g_{rr,r}) \\
&= 0\,, \\
\Gamma^\sigma{}_{r\theta} &= \frac{1}{2} g^{\sigma\rho}(g_{\rho r,\theta} + g_{\rho\theta,r} - g_{r\theta,\rho}) = \frac{1}{2} g^{\sigma\theta} g_{\theta\theta,r} = \frac{1}{2}(\delta^{\sigma\theta} r^{-2})(2r) \\
&= \delta^{\sigma\theta} r^{-1}\,, \\
\Gamma^\sigma{}_{r\varphi} &= \frac{1}{2} g^{\sigma\rho}(g_{\rho r,\varphi} + g_{\rho\varphi,r} - g_{r\varphi,\rho}) = \frac{1}{2} g^{\sigma\varphi} g_{\varphi\varphi,r} = \frac{1}{2}(\delta^{\sigma\varphi} r^{-2}\sin^{-2}\theta)(2r\sin^2\theta) \\
&= \delta^{\sigma\varphi} r^{-1}\,; \\
\Gamma^\sigma{}_{\theta\theta} &= \frac{1}{2} g^{\sigma\rho}(g_{\rho\theta,\theta} + g_{\rho\theta,\theta} - g_{\theta\theta,\rho}) = -\frac{1}{2} g^{\sigma r} g_{\theta\theta,r} = -\frac{1}{2}(\delta^{\sigma r})(2r) \\
&= -\delta^{\sigma r} r\,, \\
\Gamma^\sigma{}_{\theta\varphi} &= \frac{1}{2} g^{\sigma\rho}(g_{\rho\theta,\varphi} + g_{\rho\varphi,\theta} - g_{\theta\varphi,\rho}) = \frac{1}{2} g^{\sigma\varphi} g_{\varphi\varphi,\theta} = \frac{1}{2}(\delta^{\sigma\varphi} r^{-2}\sin^{-2}\theta)(r^2\sin 2\theta) \\
&= \delta^{\sigma\varphi}\sin^{-1}\theta\cos\theta = \delta^{\sigma\varphi}\cot\theta\,; \\
\Gamma^\sigma{}_{\varphi\varphi} &= \frac{1}{2} g^{\sigma\rho}(g_{\rho\varphi,\varphi} + g_{\rho\varphi,\varphi} - g_{\varphi\varphi,\rho}) = -\frac{1}{2} g^{\sigma r} g_{\varphi\varphi,r} - \frac{1}{2} g^{\sigma\theta} g_{\varphi\varphi,\theta} \\
&= -\frac{1}{2}(\delta^{\sigma r})(2r\sin^2\theta) - \frac{1}{2}(\delta^{\sigma\theta} r^{-2})(r^2\sin 2\theta) \\
&= -\delta^{\sigma r} r\sin^2\theta - \delta^{\sigma\theta}\sin\theta\cos\theta\,.
\end{aligned}
$$

因此求得 3 维欧氏空间球坐标系的非零克氏符如下：

$$\begin{aligned}
\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} &= r^{-1}\,, \\
\Gamma^{\varphi}{}_{r\varphi} = \Gamma^{\varphi}{}_{\varphi r} &= r^{-1}\,, \\
\Gamma^{r}{}_{\theta\theta} &= -r\,, \\
\Gamma^{\varphi}{}_{\theta\varphi} = \Gamma^{\varphi}{}_{\varphi\theta} &= \cot\theta = \frac{\cos\theta}{\sin\theta}\,, \\
\Gamma^{r}{}_{\varphi\varphi} &= -r\sin^2\theta\,, \\
\Gamma^{\theta}{}_{\varphi\varphi} &= -\sin\theta\cos\theta\,.
\end{aligned}$$

8. 设 I 是 RR 的一个区间， $C: I \to M$ 是 (M, ∇_a) 中的曲线，试证 $\forall s, t \in I$，平移映射 $\psi: V_{C(s)} \to V_{C(t)}$ (见图 3-2) 是同构映射.

证 因 $C(s) \to C(t)$ 是一一到上的线性映射，所以从 $v^a(s)$ 平移到 $\tilde{v}^a(t)$ 也是一一到上的线性映射，故而 $\psi: V_{C(s)} \to V_{C(t)}$ 是同构映射.

~9. 试证定理 3-3-2 、 3-3-3 和 3-3-5.

证 (1) 定理 3-3-2 的证明. 设 T'^a 是重参数化曲线 $\gamma'(t')$ $[=\gamma(t)]$ 的切矢，有关系

$$T'^a = \left(\frac{\partial}{\partial t'}\right)^a = \frac{dt}{dt'}\left(\frac{\partial}{\partial t}\right)^a = \frac{dt}{dt'}T^a\,.$$

要求 $\gamma'(t')$ 为测地线， T'^a 必须满足

$$\begin{aligned}
0 &= T'^b\nabla_b T'^a = \frac{dt}{dt'}T^b\nabla_b\left(\frac{dt}{dt'}T^a\right) \\
&= \frac{dt}{dt'}T^aT^b\nabla_b\left(\frac{dt}{dt'}\right) + \left(\frac{dt}{dt'}\right)^2 T^b\nabla_b T^a \\
&= \frac{dt}{dt'}T^a\left(\frac{\partial}{\partial t}\right)^b\left[d\left(\frac{dt}{dt'}\right)\right]_b + \left(\frac{dt}{dt'}\right)^2\alpha T^a \\
&= \left[\frac{dt}{dt'}\frac{d}{dt}\left(\frac{dt}{dt'}\right) + \alpha\left(\frac{dt}{dt'}\right)^2\right]T^a \\
&= \left[\frac{d^2t}{dt'^2} + \alpha\left(\frac{dt}{dt'}\right)^2\right]T^a\,,
\end{aligned}$$

于是要求

$$\frac{d^2t}{dt'^2} + \alpha\left(\frac{dt}{dt'}\right)^2 = 0\,.$$

这是对 t 的微分方程，可以化为对 t' 的微分方程，因为

$$\begin{aligned}
\frac{d^2t}{dt'^2} &= \frac{d}{dt'}\left(\frac{dt}{dt'}\right) = \frac{dt}{dt'}\frac{d}{dt}\left(\frac{dt'}{dt}\right)^{-1} \\
&= -\left(\frac{dt'}{dt}\right)^{-1}\left(\frac{dt'}{dt}\right)^{-2}\frac{d^2t'}{dt^2} = -\left(\frac{dt'}{dt}\right)^{-3}\frac{d^2t'}{dt^2}\,,
\end{aligned}$$

故有

$$-\left(\frac{dt'}{dt}\right)^{-3}\frac{d^2t'}{dt^2} + \alpha\left(\frac{dt'}{dt}\right)^{-2} = 0\,,$$

即

$$\frac{d^2t'}{dt^2}=\alpha(t)\Big(\frac{dt'}{dt}\Big)\,.$$

这就是 $t'=t'(t)$ 满足的微分方程, 解出 t', 那么就找到了测地线 $\gamma'(t')\ [=\gamma(t)]$.

(2) 定理 3-3-3 的证明. ①必要性：若 t 是测地线 $\gamma(t)$ 的仿射参数，则定理 3-3-2 中的 $\alpha=0$, 这时 t' 满足的方程蜕化为 $\frac{d^2t'}{dt^2}=0$, 其通解必为 $t'=at+b$. 这时 t' 是同一根测地线 $\gamma'(t')$ 的仿射参数. ②充分性：若 $t'=at+b$ 是测地线 $\gamma'(t')$ 的仿射参数，那么定理 3-3-2 中的 $\alpha(t)=(\frac{dt'}{dt})^{-1}\frac{d^2t'}{dt^2}=0$, 于是 $T^b\nabla_bT^a=0$, 即 t 是测地线 $\gamma(t)\ [=\gamma'(t')]$ 的仿射参数.

(3) 定理 3-3-5 的证明. 设 $\gamma(t)$ 为以仿射参数 t 为参数的测地线，沿 $\gamma(t)$ 的切矢为 $T^a(t)\equiv T^a(\gamma(t))$, 其长度 (的平方) 为 $T^2=g(T,T)=T^aT^bg_{ab}$. 因为 T^a 是测地线的切矢，所以满足 $T^c\nabla_cT^a=0$, 另一方面因度规 g_{ab} 与导数算符 ∇_a 相适配，有 $\nabla_cg_{ab}=0$. 于是 $T^c\nabla_cT^2=T^c\nabla_c(T^aT^bg_{ab})=g_{ab}T^bT^c\nabla_cT^a+g_{ab}T^aT^c\nabla_cT^b+T^aT^bT^c\nabla_cg_{ab}=0$, 测地线切矢的长度沿测地线为常数： $|T|=C$. 测地线的线长由式 (2-5-3) 给出： $l=\int_{t_0}^{t}|T(t')|\,dt'=C(t-t_0)$. 那么这同一根测地线也可用重参数化后的 $\gamma'(l)$ 描述， l 是线长参数. 最后根据定理 3-3-3 的结果，如果 t 是 $\gamma(t)$ 的仿射参数，那么 l 必为 $\gamma'(l)\ [=\gamma(t)]$ 的仿射参数.

~10. (a) 写出球面度规 $ds^2=R^2(d\theta^2+\sin^2\theta d\varphi^2)$ (R 为常数) 的测地线方程；(b) 验证任一大圆弧 (配以适当参数) 满足测地线方程. 提示：选球面坐标系 $\{\theta,\varphi\}$ 使所给大圆弧为赤道的一部分，并以 φ 为仿射参数.

证 (a) 球面的度规张量为 $g_{\theta\theta}=R^2$, $g_{\varphi\varphi}=R^2\sin^2\theta$. 利用习题 7 中求得的球坐标系的非零克氏符, 那么现在只有 $\Gamma^\theta{}_{\varphi\varphi}=-\sin\theta\cos\theta$ 和 $\Gamma^\varphi{}_{\varphi\theta}=\Gamma^\varphi{}_{\theta\varphi}=\frac{\cos\theta}{\sin\theta}$. 于是测地线的参数方程 (3-3-1) 为

$$\begin{aligned}0&=\frac{d^2\theta}{dt^2}+\Gamma^\theta{}_{\varphi\varphi}\frac{d\varphi}{dt}\frac{d\varphi}{dt}=\frac{d^2\theta}{dt^2}-\sin\theta\cos\theta\Big(\frac{d\varphi}{dt}\Big)^2\,,\\0&=\frac{d^2\varphi}{dt^2}+\Gamma^\varphi{}_{\varphi\theta}\frac{d\varphi}{dt}\frac{d\theta}{dt}+\Gamma^\varphi{}_{\theta\varphi}\frac{d\theta}{dt}\frac{d\varphi}{dt}=\frac{d^2\varphi}{dt^2}+\frac{2\cos\theta}{\sin\theta}\frac{d\theta}{dt}\frac{d\varphi}{dt}\,,\end{aligned}$$

即测地线方程为

$$\begin{aligned}&\theta_{,tt}-\sin\theta\cos\theta\,\varphi_{,t}^2=0\,,\\&\varphi_{,tt}+2\cot\theta\,\varphi_{,t}\,\theta_{,t}=0\,.\end{aligned}$$

(b) 先做球坐标系的旋转变换. 第一步, 绕 O 系的 x 轴旋转 α 角度得 O' 系, 这两系之间的坐标关系为

$$\begin{cases}x'=x\,,\\y'=y\cos\alpha+z\sin\alpha\,,\\z'=-y\sin\alpha+z\cos\alpha\,.\end{cases}\qquad\qquad\begin{cases}x=x'\,,\\y=y'\cos\alpha-z'\sin\alpha\,,\\z=y'\sin\alpha+z'\cos\alpha\,.\end{cases}$$

然后绕 O' 系的 z' 轴旋转 β 角度得 O'' 系，这两系之间的坐标关系为

$$\begin{cases} x'' = x'\cos\beta + y'\sin\beta\ , \\ y'' = -x'\sin\beta + y'\cos\beta\ , \\ z'' = z'\ . \end{cases} \qquad \begin{cases} x' = x''\cos\beta - y''\sin\beta\ , \\ y' = x''\sin\beta + y''\cos\beta\ , \\ z' = z''\ . \end{cases}$$

由此可得 O 系与 O'' 系的坐标关系：

$$\boldsymbol{x}'' = \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos\beta & \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ -\sin\beta & \cos\alpha\cos\beta & \sin\alpha\cos\beta \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R\boldsymbol{x}\ .$$

$$\begin{aligned} \boldsymbol{x} = R^{-1}\boldsymbol{x}'' = R^T\boldsymbol{x}'' &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \cos\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha \\ \sin\alpha\sin\beta & \sin\alpha\cos\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \\ &= \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \cos\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha \\ \sin\alpha\sin\beta & \sin\alpha\cos\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} R\sin\theta''\cos\varphi'' \\ R\sin\theta''\sin\varphi'' \\ R\cos\theta'' \end{bmatrix} \\ = R\begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} &= R\begin{bmatrix} \sin\theta''\cos(\varphi''+\beta) \\ \sin\theta''\sin(\varphi''+\beta)\cos\alpha - \cos\theta''\sin\alpha \\ \sin\theta''\sin(\varphi''+\beta)\sin\alpha + \cos\theta''\cos\alpha \end{bmatrix}\ , \end{aligned}$$

因此

$$\begin{aligned} \cos\theta &= \sin\theta''\sin(\varphi''+\beta)\sin\alpha + \cos\theta''\cos\alpha\ , \\ \tan\varphi &= \tan(\varphi''+\beta)\cos\alpha - \frac{\cot\theta''\sin\alpha}{\cos(\varphi''+\beta)}\ . \end{aligned}$$

O 系的任何大圆弧 (段) 都可用 O'' 系的 (i) 赤道线 (段) 或 (ii) 经线 (段) 描述. 赤道线 (段) 为 $\theta'' = \frac{\pi}{2}$, $\phi'' = at+b$; 经线 (段) 为 $\theta'' = at+b$, $\phi'' = c$.

(i) 如果用 O'' 的赤道线 (段), $\theta'' = \frac{\pi}{2}$, $\phi'' = at+b$:

$$\begin{aligned} \cos\theta &= \sin(at+b+\beta)\sin\alpha = \sin\alpha\ \sin\phi(t)\ , \\ \tan\varphi &= \tan(at+b+\beta)\cos\alpha = \cos\alpha\ \tan\phi(t)\ , \end{aligned}$$

其中 $\phi(t) = at+b+\beta$, 即

$$\begin{aligned} \theta(t) &= \arccos[\sin\alpha\ \sin\phi(t)]\ , \\ \varphi(t) &= \arctan[\cos\alpha\ \tan\phi(t)]\ . \end{aligned}$$

这时对 t 求导后得

$$
\begin{aligned}
\theta_{,t} &= -a\sin\alpha\,\cos\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-1/2} \\
&= -\frac{a\sin\alpha\,\cos\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{1/2}}\,, \\
\theta_{,tt} &= a^2\sin\alpha\,\sin\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-1/2} \\
&\quad -a\sin\alpha\,\cos\phi\Big\{-\frac{1}{2}(1-\sin^2\alpha\,\sin^2\phi)^{-3/2}[-2a\sin^2\alpha\,\sin\phi\,\cos\phi]\Big\} \\
&= a^2\sin\alpha\,\sin\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-1/2} \\
&\quad -a^2\sin^3\alpha\,\cos^2\phi\,\sin\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-3/2} \\
&= a^2\sin\alpha\,\sin\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-3/2}\Big[(1-\sin^2\alpha\,\sin^2\phi)-\sin^2\alpha\,\cos^2\phi\Big] \\
&= a^2\sin\alpha\,\sin\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-3/2}\Big[1-\sin^2\alpha\Big] \\
&= a^2\cos^2\alpha\,\sin\alpha\,\sin\phi\,(1-\sin^2\alpha\,\sin^2\phi)^{-3/2} \\
&= \frac{a^2\cos^2\alpha\,\sin\alpha\,\sin\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{3/2}}\,; \\
\varphi_{,t} &= a\cos\alpha\,\sec^2\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-1} \\
&= \frac{a\cos\alpha\,\sec^2\phi}{1+\cos^2\alpha\,\tan^2\phi}\,, \\
\varphi_{,tt} &= a\cos\alpha\,2a\sec^2\phi\,\tan\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-1} \\
&\quad -a\cos\alpha\,\sec^2\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-2}\cos^2\alpha\,2a\tan\phi\,\sec^2\phi \\
&= 2a^2\cos\alpha\,\sec^2\phi\,\tan\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-1} \\
&\quad -2a^2\cos^3\alpha\,\sec^4\phi\,\tan\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-2} \\
&= 2a^2\cos\alpha\,\sec^2\phi\,\tan\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-2} \\
&\qquad \times\Big[(1+\cos^2\alpha\,\tan^2\phi)-\cos^2\alpha\,\sec^2\phi\Big] \\
&= 2a^2\cos\alpha\,\sec^2\phi\,\tan\phi\,(1+\cos^2\alpha\,\tan^2\phi)^{-2}\Big[1-\cos^2\alpha\Big] \\
&= \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sec^2\phi\,\tan\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2}\,.
\end{aligned}
$$

测地线方程为：

$$
\begin{aligned}
&\theta_{,tt}-\sin\theta\,\cos\theta\,\varphi_{,t}^2 = 0\,, \\
&\varphi_{,tt}+2\cot\theta\,\varphi_{,t}\,\theta_{,t} = 0\,.
\end{aligned}
$$

代入验证，第一个方程：

$$
\begin{aligned}
&\theta_{,tt}-\sin\theta\cos\theta\,\varphi_{,t}^2 \\
=\;& \frac{a^2\cos^2\alpha\,\sin\alpha\,\sin\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{3/2}}-\sin\theta\cos\theta\left(\frac{a\cos\alpha\,\sec^2\phi}{1+\cos^2\alpha\,\tan^2\phi}\right)^2 \\
=\;& \frac{a^2\cos^2\alpha\,\sin\alpha\,\sin\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{3/2}}-(1-\sin^2\alpha\,\sin^2\phi)^{1/2}\sin\alpha\,\sin\phi\,\frac{a^2\cos^2\alpha\,\sec^4\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2}
\end{aligned}
$$

$$= \frac{a^2\cos^2\alpha\,\sin\alpha\,\sin\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{3/2}} - (1-\sin^2\alpha\,\sin^2\phi)^{1/2}\frac{a^2\cos^2\alpha\,\sin\alpha\,\sin\phi\,\sec^4\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2}$$
$$= \frac{a^2\cos^2\alpha\,\sin\alpha\,\sin\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{3/2}}\left[1-\frac{(1-\sin^2\alpha\,\sin^2\phi)^2\sec^4\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2}\right],$$

其中

$$\frac{(1-\sin^2\alpha\,\sin^2\phi)\sec^2\phi}{(1+\cos^2\alpha\,\tan^2\phi)} = \frac{\sec^2\phi-\sin^2\alpha\,\tan^2\phi}{1+\cos^2\alpha\,\tan^2\phi}$$
$$= \frac{\sec^2\phi-\tan^2\phi+\cos^2\alpha\,\tan^2\phi}{1+\cos^2\alpha\,\tan^2\phi} = 1\,,$$

故第一个方程成立. 第二个方程:

$$\begin{aligned}
&\varphi_{,tt} + 2\cot\theta\,\varphi_{,t}\,\theta_{,t} \\
= & \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sec^2\phi\,\tan\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2} \\
& +2\frac{\cos\theta}{\sin\theta}\left(\frac{a\cos\alpha\,\sec^2\phi}{1+\cos^2\alpha\,\tan^2\phi}\right)\left(-\frac{a\sin\alpha\,\cos\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{1/2}}\right) \\
= & \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sec^2\phi\,\tan\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2} \\
& -\frac{\cos\theta}{\sin\theta}\,\frac{2a^2\cos\alpha\,\sin\alpha\,\sec\phi}{(1+\cos^2\alpha\,\tan^2\phi)(1-\sin^2\alpha\,\sin^2\phi)^{1/2}} \\
= & \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sec^2\phi\,\tan\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2} \\
& -\frac{\sin\alpha\,\sin\phi}{(1-\sin^2\alpha\,\sin^2\phi)^{1/2}}\,\frac{2a^2\cos\alpha\,\sin\alpha\,\sec\phi}{(1+\cos^2\alpha\,\tan^2\phi)(1-\sin^2\alpha\,\sin^2\phi)^{1/2}} \\
= & \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sin\phi\,\sec^3\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2} - \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sin\phi\,\sec\phi}{(1+\cos^2\alpha\,\tan^2\phi)(1-\sin^2\alpha\,\sin^2\phi)} \\
= & \frac{2a^2\cos\alpha\,\sin^2\alpha\,\sin\phi\,\sec^3\phi}{(1+\cos^2\alpha\,\tan^2\phi)^2}\left[1-\frac{(1+\cos^2\alpha\,\tan^2\phi)\cos^2\phi}{(1-\sin^2\alpha\,\sin^2\phi)}\right],
\end{aligned}$$

其中

$$\frac{(1+\cos^2\alpha\,\tan^2\phi)\cos^2\phi}{(1-\sin^2\alpha\,\sin^2\phi)} = \frac{\cos^2\phi+\cos^2\alpha\,\sin^2\phi}{1-\sin^2\alpha\,\sin^2\phi} = 1\,,$$

故第二个方程也成立.

(ii) 如果用 O'' 的经线 (段), $\theta''=at+b$, $\phi''=c$:

$$\begin{aligned}
\cos\theta &= \sin(at+b)\,\sin(c+\beta)\,\sin\alpha + \cos(at+b)\,\cos\alpha\,, \\
\tan\varphi &= \tan(c+\beta)\,\cos\alpha - \frac{\cot(at+b)\,\sin\alpha}{\cos(c+\beta)}\,.
\end{aligned}$$

令 $\phi(t)\equiv at+b$, $A\equiv\sin(c+\beta)$, $B\equiv\sin\alpha$, 则有

$$\begin{aligned}
\cos\theta &= AB\sin\phi(t) + \sqrt{1-B^2}\cos\phi(t)\,, \\
\tan\varphi &= \frac{A\sqrt{1-B^2}}{\sqrt{1-A^2}} - \frac{B}{\sqrt{1-A^2}}\cot\phi(t)\,.
\end{aligned}$$

于是

$$\begin{aligned}\theta(t) &= \arccos\left[AB\sin\phi(t)+\sqrt{1-B^2}\cos\phi(t)\right] , \\ \varphi(t) &= \arctan\left[\frac{A\sqrt{1-B^2}}{\sqrt{1-A^2}}-\frac{B}{\sqrt{1-A^2}}\cot\phi(t)\right] .\end{aligned}$$

可以用 Mathematica 直接验证，这两个参数表达式也满足测地线方程

$$\theta_{,tt}-\sin\theta\,\cos\theta\,\varphi_{,t}^2=0 ,$$
$$\varphi_{,tt}+2\cot\theta\,\varphi_{,t}\,\theta_{,t}=0 .$$

~11. 试证定理 3-4-2. 设 $\omega_c, \omega'_c \in \mathscr{F}(0,1)$ 且 $\omega'_c|_p=\omega_c|_p$, 则

$$[(\nabla_a\nabla_b-\nabla_b\nabla_a)\omega'_c]|_p=[(\nabla_a\nabla_b-\nabla_b\nabla_a)\omega_c]|_p .$$

证 选坐标系 $\{x^\mu\}$ 使其坐标域含 p 点，以该坐标系的对偶基底展开：

$$\omega_c=\omega_\mu(dx^\mu)_c , \qquad \omega'_c=\omega'_\mu(dx^\mu)_c .$$

于是

$$\begin{aligned}[(\nabla_a\nabla_b-\nabla_b\nabla_a)\omega_c]|_p &= \left\{(\nabla_a\nabla_b-\nabla_b\nabla_a)[\omega_\mu(dx^\mu)_c]\right\}\Big|_p \\ &= \left\{\omega_\mu(\nabla_a\nabla_b-\nabla_b\nabla_a)(dx^\mu)_c\right\}\Big|_p \\ &= \omega_\mu|_p\,[(\nabla_a\nabla_b-\nabla_b\nabla_a)(dx^\mu)_c]|_p , \\ [(\nabla_a\nabla_b-\nabla_b\nabla_a)\omega'_c]|_p &= \omega'_\mu|_p\,[(\nabla_a\nabla_b-\nabla_b\nabla_a)(dx^\mu)_c]|_p .\end{aligned}$$

这里用到了定理 3-4-1 [把 ω_μ 和 ω'_μ 看作 f, $(dx^\mu)_c$ 看作 ω_c]. 而 $\omega'_c|_p=\omega_c|_p$ 保证 $\omega'_\mu|_p=\omega_\mu|_p$, 故命题得证.

*12. 试证式 (3-4-10).

证 根据黎曼曲率张量的循环恒等式性质 (3-4-7) 式有

$$\begin{aligned}0=R_{[abc]}{}^e &\overset{(2\text{-}6\text{-}14)}{=} \frac{1}{6}\Big(R_{abc}{}^e+R_{bca}{}^e+R_{cab}{}^e-R_{bac}{}^e-R_{acb}{}^e-R_{cba}{}^e\Big) \\ &\overset{(3\text{-}4\text{-}6)}{=} \frac{1}{3}\Big(R_{abc}{}^e+R_{bca}{}^e+R_{cab}{}^e\Big) .\end{aligned}$$

以 g_{de} 作用上式，由定义 $R_{abcd}=g_{de}R_{abc}{}^e$ 得

$$\frac{1}{3}\Big(R_{abcd}+R_{bcad}+R_{cabd}\Big)=0 .$$

当然这就是 $R_{[abc]d}=0$. 循环这四个指标并相加得

$$0 = 3(R_{[abc]d}+R_{[bcd]a}+R_{[cda]b}+R_{[dab]c})$$

$$\begin{aligned}
&= (R_{abcd} + R_{bcad} + R_{cabd}) + (R_{bcda} + R_{cdba} + R_{dbca}) \\
&\quad +(R_{cdab} + R_{dacb} + R_{acdb}) + (R_{dabc} + R_{abdc} + R_{bdac}) \\
&= (R_{abcd} + R_{bcad} - R_{acbd}) + (-R_{bcad} - R_{cdab} + R_{bdac}) \\
&\quad +(R_{cdab} + R_{adbc} - R_{acbd}) + (-R_{adbc} - R_{abcd} + R_{bdac}) \\
&= -2R_{acbd} + 2R_{bdac} \ ,
\end{aligned}$$

其中用到了性质 (3-4-6) 和 (3-4-9). 因此有 $R_{acbd} = R_{bdac}$, 此即具有对互换对称性 (pair interchange symmetry) 的 (3-4-10) 式

$$R_{abcd} = R_{cdab} \ .$$

~13. 求出球面度规 (见题 10) 的黎曼张量在坐标系 $\{\theta, \varphi\}$ 的全部分量.

解 球面的度规张量为

$$g_{\theta\theta} = R^2 \ , \quad g_{\varphi\varphi} = R^2 \sin^2\theta \ ; \qquad g^{\theta\theta} = R^{-2} \ , \quad g^{\varphi\varphi} = R^{-2}\sin^{-2}\theta \ .$$

利用习题 7 中求得的球坐标系的非零克氏符，那么现在只有

$$\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta \ , \quad \Gamma^\varphi{}_{\varphi\theta} = \Gamma^\varphi{}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta} \ ;$$

以及

$$\begin{aligned}
\Gamma^\theta{}_{\varphi\varphi,\theta} &= \sin^2\theta - \cos^2\theta = -\cos 2\theta \ , \\
\Gamma^\varphi{}_{\varphi\theta,\theta} &= \Gamma^\varphi{}_{\theta\varphi,\theta} = -\sin^{-2}\theta = -\frac{1}{\sin^2\theta} \ .
\end{aligned}$$

利用计算黎曼曲率张量的公式为 (3-4-20'):

$$R_{\mu\nu\sigma}{}^\rho = \Gamma^\rho{}_{\mu\sigma,\nu} - \Gamma^\rho{}_{\nu\sigma,\mu} + \Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\rho{}_{\mu\lambda} \ .$$

因为对前两个指标反对称，所以只须算 $\mu \neq \nu$ 情形，即

$$\begin{aligned}
R_{\theta\varphi\sigma}{}^\rho &= -\Gamma^\rho{}_{\varphi\sigma,\theta} + \Gamma^\lambda{}_{\sigma\theta}\Gamma^\rho{}_{\varphi\lambda} - \Gamma^\lambda{}_{\sigma\varphi}\Gamma^\rho{}_{\theta\lambda} \\
&= -\Gamma^\rho{}_{\varphi\sigma,\theta} + \Gamma^\lambda{}_{\sigma\theta}\Gamma^\rho{}_{\varphi\lambda} - \Gamma^\lambda{}_{\sigma\varphi}\Gamma^\rho{}_{\theta\lambda} \ .
\end{aligned}$$

于是有

$$\begin{aligned}
R_{\theta\varphi\theta}{}^\theta &= -\Gamma^\theta{}_{\varphi\theta,\theta} + \Gamma^\lambda{}_{\theta\theta}\Gamma^\theta{}_{\varphi\lambda} - \Gamma^\lambda{}_{\theta\varphi}\Gamma^\theta{}_{\theta\lambda} \\
&= -0 + 0 - \Gamma^\varphi{}_{\theta\varphi}\Gamma^\theta{}_{\theta\varphi} \\
&= 0 \ , \\
R_{\theta\varphi\theta}{}^\varphi &= -\Gamma^\varphi{}_{\varphi\theta,\theta} + \Gamma^\lambda{}_{\theta\theta}\Gamma^\varphi{}_{\varphi\lambda} - \Gamma^\lambda{}_{\theta\varphi}\Gamma^\varphi{}_{\theta\lambda} \\
&= -\Gamma^\varphi{}_{\varphi\theta,\theta} + 0 - \Gamma^\varphi{}_{\theta\varphi}\Gamma^\varphi{}_{\theta\varphi}
\end{aligned}$$

$$
\begin{aligned}
&= -\Big(-\frac{1}{\sin^2\theta}\Big) - \Big(\frac{\cos\theta}{\sin\theta}\Big)^2 \\
&= 1\,, \\
R_{\theta\varphi\varphi}{}^{\theta} &= -\Gamma^{\theta}{}_{\varphi\varphi,\theta} + \Gamma^{\lambda}{}_{\varphi\theta}\Gamma^{\theta}{}_{\varphi\lambda} - \Gamma^{\lambda}{}_{\varphi\varphi}\Gamma^{\theta}{}_{\theta\lambda} \\
&= -\Gamma^{\theta}{}_{\varphi\varphi,\theta} + \Gamma^{\varphi}{}_{\varphi\theta}\Gamma^{\theta}{}_{\varphi\varphi} - 0 \\
&= -(-\cos 2\theta) + \Big(\frac{\cos\theta}{\sin\theta}\Big)(-\sin\theta\,\cos\theta) \\
&= -\sin^2\theta\,, \\
R_{\theta\varphi\varphi}{}^{\varphi} &= -\Gamma^{\varphi}{}_{\varphi\varphi,\theta} + \Gamma^{\lambda}{}_{\varphi\theta}\Gamma^{\varphi}{}_{\varphi\lambda} - \Gamma^{\lambda}{}_{\varphi\varphi}\Gamma^{\varphi}{}_{\theta\lambda} \\
&= -0 + \Gamma^{\varphi}{}_{\varphi\theta}\Gamma^{\varphi}{}_{\varphi\varphi} - \Gamma^{\theta}{}_{\varphi\varphi}\Gamma^{\varphi}{}_{\theta\theta} \\
&= 0\,.
\end{aligned}
$$

因此求得的非零黎曼曲率张量为

$$
R_{\theta\varphi\theta}{}^{\varphi} = -R_{\varphi\theta\theta}{}^{\varphi} = 1\,, \qquad R_{\theta\varphi\varphi}{}^{\theta} = -R_{\varphi\theta\varphi}{}^{\theta} = -\sin^2\theta\,.
$$

注意到

$$
\begin{aligned}
R_{\theta\varphi\theta}{}^{\varphi} &= g^{\varphi\varphi}R_{\theta\varphi\theta\varphi} \overset{\text{(3-4-9)}}{=} -g^{\varphi\varphi}R_{\theta\varphi\varphi\theta} = -g^{\varphi\varphi}g_{\theta\theta}R_{\theta\varphi\varphi}{}^{\theta} \\
&= -(R^{-2})(\sin^{-2}\theta R^2)R_{\theta\varphi\varphi}{}^{\theta} = -\sin^{-2}\theta\, R_{\theta\varphi\varphi}{}^{\theta}\,,
\end{aligned}
$$

显然上面的结果满足这一关系．事实上，由于

$$
\begin{aligned}
R_{\theta\varphi\theta\varphi} &= g_{\varphi\varphi}R_{\theta\varphi\theta}{}^{\varphi} = (R^2\sin^2\theta)(+1) = R^2\sin^2\theta\,, \\
R_{\theta\varphi\varphi\theta} &= g_{\theta\theta}R_{\theta\varphi\varphi}{}^{\theta} = (R^2)(-\sin^2\theta) = -R^2\sin^2\theta\,,
\end{aligned}
$$

我们有

$$
R_{\theta\varphi\theta\varphi} = -R_{\varphi\theta\theta\varphi} = -R_{\theta\varphi\varphi\theta} = R_{\varphi\theta\varphi\theta} = R^2\sin^2\theta\,,
$$

可以看出它们满足 (3-4-9) 和 (3-4-10) 的关系．现在 $n = \dim M = 2$, 故 $R_{abc}{}^{d}$ 的独立分量的个数为 $N = n^2(n^2-1)/12 = 1$.

14. 求度规 $ds^2 = \Omega^2(t,x)(-dt^2 + dx^2)$ 的黎曼张量在 $\{t,x\}$ 系的全部分量 (在结果中以 $\dot\Omega$ 和 Ω' 分别代表函数 Ω 对 t 和 x 的偏导数).

解 非归一的坐标基底的度规张量分量为

$$
g_{tt} = -\Omega^2\,, \quad g_{xx} = \Omega^2\,; \qquad g^{tt} = -\Omega^{-2}\,, \quad g^{xx} = \Omega^{-2}\,.
$$

于是

$$
g_{tt,t} = -2\Omega\dot\Omega\,, \quad g_{tt,x} = -2\Omega\Omega'\,; \qquad g_{xx,t} = 2\Omega\dot\Omega\,, \quad g_{xx,x} = 2\Omega\Omega'\,.
$$

先利用公式 (3-2-10')

$$
\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})\,,
$$

计算克氏符：

$$
\begin{aligned}
\Gamma^t{}_{tt} &= \frac{1}{2}g^{tt}(g_{tt,t}+g_{tt,t}-g_{tt,t}) = \frac{1}{2}(-\Omega^{-2})(-2\Omega\dot{\Omega}) = \Omega^{-1}\dot{\Omega} \equiv A\ , \\
\Gamma^x{}_{xx} &= \frac{1}{2}g^{xx}(g_{xx,x}+g_{xx,x}-g_{xx,x}) = \frac{1}{2}(\Omega^{-2})(2\Omega\Omega') = \Omega^{-1}\Omega' \equiv B\ , \\
\Gamma^x{}_{tt} &= \frac{1}{2}g^{xx}(g_{xt,t}+g_{xt,t}-g_{tt,x}) = \frac{1}{2}(\Omega^{-2})(2\Omega\Omega') = \Omega^{-1}\Omega' = B\ , \\
\Gamma^t{}_{xx} &= \frac{1}{2}g^{tt}(g_{tx,x}+g_{tx,x}-g_{xx,t}) = \frac{1}{2}(-\Omega^{-2})(-2\Omega\dot{\Omega}) = \Omega^{-1}\dot{\Omega} = A\ ; \\
\Gamma^t{}_{tx} = \Gamma^t{}_{xt} &= \frac{1}{2}g^{tt}(g_{tt,x}+g_{tx,t}-g_{tx,t}) = \frac{1}{2}(-\Omega^{-2})(-2\Omega\Omega') = \Omega^{-1}\Omega' = B\ , \\
\Gamma^x{}_{tx} = \Gamma^x{}_{xt} &= \frac{1}{2}g^{xx}(g_{xt,x}+g_{xx,t}-g_{tx,x}) = \frac{1}{2}(\Omega^{-2})(2\Omega\dot{\Omega}) = \Omega^{-1}\dot{\Omega} = A\ .
\end{aligned}
$$

相应的导数：

$$
\begin{aligned}
\Gamma^t{}_{tt,t} &= A_{,t} = \Omega^{-1}\ddot{\Omega} - \Omega^{-2}\dot{\Omega}^2 \equiv T\ , \\
\Gamma^t{}_{tt,x} &= A_{,x} = \Omega^{-1}\dot{\Omega}' - \Omega^{-2}\dot{\Omega}\Omega' \equiv U\ , \\
\Gamma^x{}_{xx,t} &= B_{,t} = \Omega^{-1}\dot{\Omega}' - \Omega^{-2}\dot{\Omega}\Omega' = U\ , \\
\Gamma^x{}_{xx,x} &= B_{,x} = \Omega^{-1}\Omega'' - \Omega^{-2}\Omega'^2 \equiv X\ ; \\
\Gamma^x{}_{tt,t} &= B_{,t} = U\ , \\
\Gamma^x{}_{tt,x} &= B_{,x} = X\ , \\
\Gamma^t{}_{xx,t} &= A_{,t} = T\ , \\
\Gamma^t{}_{xx,x} &= A_{,x} = U\ ; \\
\Gamma^t{}_{tx,t} = \Gamma^t{}_{xt,t} &= B_{,t} = U\ , \\
\Gamma^t{}_{tx,x} = \Gamma^t{}_{xt,x} &= B_{,x} = X\ , \\
\Gamma^x{}_{tx,t} = \Gamma^x{}_{xt,t} &= A_{,t} = T\ , \\
\Gamma^x{}_{tx,x} = \Gamma^x{}_{xt,x} &= A_{,x} = U\ .
\end{aligned}
$$

然后利用公式 (3-4-20')

$$
R_{\mu\nu\sigma}{}^{\rho} = \Gamma^{\rho}{}_{\mu\sigma,\nu} - \Gamma^{\rho}{}_{\nu\sigma,\mu} + \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\rho}{}_{\mu\lambda}
$$

计算黎曼曲率张量. 因为对前两个指标反对称，所以只须算 $\mu \neq \nu$ 情形，即

$$
R_{tx\sigma}{}^{\rho} = \Gamma^{\rho}{}_{t\sigma,x} - \Gamma^{\rho}{}_{x\sigma,t} + \Gamma^{\lambda}{}_{\sigma t}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{t\lambda}\ .
$$

于是

$$
\begin{aligned}
R_{txt}{}^{t} &= \Gamma^t{}_{tt,x} - \Gamma^t{}_{xt,t} + \Gamma^{\lambda}{}_{tt}\Gamma^t{}_{x\lambda} - \Gamma^{\lambda}{}_{tx}\Gamma^t{}_{t\lambda} \\
&= \Gamma^t{}_{tt,x} - \Gamma^t{}_{xt,t} + \Gamma^t{}_{tt}\Gamma^t{}_{xt} + \Gamma^x{}_{tt}\Gamma^t{}_{xx} - \Gamma^t{}_{tx}\Gamma^t{}_{tt} - \Gamma^x{}_{tx}\Gamma^t{}_{tx} \\
&= U - U + AB + BA - BA - AB
\end{aligned}
$$

$$
\begin{aligned}
&= 0\,, \\
R_{txt}{}^x &= \Gamma^x{}_{tt,x} - \Gamma^x{}_{xt,t} + \Gamma^\lambda{}_{tt}\Gamma^x{}_{x\lambda} - \Gamma^\lambda{}_{tx}\Gamma^x{}_{t\lambda} \\
&= \Gamma^x{}_{tt,x} - \Gamma^x{}_{xt,t} + \Gamma^t{}_{tt}\Gamma^x{}_{xt} + \Gamma^x{}_{tt}\Gamma^x{}_{xx} - \Gamma^t{}_{tx}\Gamma^x{}_{tt} - \Gamma^x{}_{tx}\Gamma^x{}_{tx} \\
&= X - T + AA + BB - BB - AA \\
&= X - T\,, \\
R_{txx}{}^t &= \Gamma^t{}_{tx,x} - \Gamma^t{}_{xx,t} + \Gamma^\lambda{}_{xt}\Gamma^t{}_{x\lambda} - \Gamma^\lambda{}_{xx}\Gamma^t{}_{t\lambda} \\
&= \Gamma^t{}_{tx,x} - \Gamma^t{}_{xx,t} + \Gamma^t{}_{xt}\Gamma^t{}_{xt} + \Gamma^x{}_{xt}\Gamma^t{}_{xx} - \Gamma^t{}_{xx}\Gamma^t{}_{tt} - \Gamma^x{}_{xx}\Gamma^t{}_{tx} \\
&= X - T + BB + AA - AA - BB \\
&= X - T\,, \\
R_{txx}{}^x &= \Gamma^x{}_{tx,x} - \Gamma^x{}_{xx,t} + \Gamma^\lambda{}_{xt}\Gamma^x{}_{x\lambda} - \Gamma^\lambda{}_{xx}\Gamma^x{}_{t\lambda} \\
&= \Gamma^x{}_{tx,x} - \Gamma^x{}_{xx,t} + \Gamma^t{}_{xt}\Gamma^x{}_{xt} + \Gamma^x{}_{xt}\Gamma^x{}_{xx} - \Gamma^t{}_{xx}\Gamma^x{}_{tt} - \Gamma^x{}_{xx}\Gamma^x{}_{tx} \\
&= U - U + BA + AB - AB - BA \\
&= 0\,.
\end{aligned}
$$

因此我们求得非零的黎曼曲率张量:

$$
\begin{aligned}
& R_{txt}{}^x = -R_{xtt}{}^x = R_{txx}{}^t = -R_{xtx}{}^t \\
=\ & X - T = \Omega^{-1}\Omega'' - \Omega^{-2}\Omega'^2 - \Omega^{-1}\ddot{\Omega} + \Omega^{-2}\dot{\Omega}^2 \\
=\ & \frac{\Omega'' - \ddot{\Omega}}{\Omega} + \frac{\dot{\Omega}^2 - \Omega'^2}{\Omega^2}\,.
\end{aligned}
$$

现在 $n = \dim M = 2$, 故 $R_{abc}{}^d$ 的独立分量的个数为 $N = n^2(n^2-1)/12 = 1$. 另外因

$$
\begin{aligned}
R_{txtx} &= g_{xx}R_{txt}{}^x = \Omega^2(X - T)\,, \\
R_{txxt} &= g_{tt}R_{txx}{}^t = -\Omega^2(X - T)\,,
\end{aligned}
$$

我们有

$$
\begin{aligned}
& R_{txtx} = -R_{xttx} = -R_{txxt} = R_{xtxt} \\
=\ & \Omega^2(X - T) = (\Omega'' - \ddot{\Omega})\Omega + \dot{\Omega}^2 - \Omega'^2\,,
\end{aligned}
$$

可见它们满足 (3-4-9) 和 (3-4-10) 的关系.

15. 求度规 $ds^2 = z^{-1/2}(-dt^2 + dz^2) + z(dx^2 + dy^2)$ 的黎曼张量在 $\{t, x, y, z\}$ 系的全部分量.

解 非归一的坐标基底的度规张量分量为

$$
\begin{aligned}
g_{tt} = -g_{zz} = -z^{-1/2}\,, &\qquad g_{xx} = g_{yy} = z\,; \\
g^{tt} = -g^{zz} = -z^{1/2}\,, &\qquad g^{xx} = g^{yy} = z^{-1}\,.
\end{aligned}
$$

于是

$$g_{tt,z} = -g_{zz,z} = z^{-3/2}/2 \,, \quad g_{xx,z} = g_{yy,z} = 1 \,.$$

先利用公式 (3-2-10')

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \,,$$

计算克氏符 (注意 μ, ν, ρ 中必须有 z 才为非零):

$$\begin{aligned}
\Gamma^{t}{}_{\mu\nu} &= \frac{1}{2} g^{tt}(g_{t\mu,\nu} + g_{t\nu,\mu} - g_{\mu\nu,t}) = \frac{1}{2} g^{tt}(g_{t\mu,\nu} + g_{t\nu,\mu}) \quad \Rightarrow \\
\Gamma^{t}{}_{tz} = \Gamma^{t}{}_{zt} &= \frac{1}{2} g^{tt}(g_{tt,z}) = \frac{1}{2}(-z^{1/2})(z^{-3/2}/2) = -\frac{1}{4z} \,; \\
\Gamma^{z}{}_{\mu\nu} &= \frac{1}{2} g^{zz}(g_{z\mu,\nu} + g_{z\nu,\mu} - g_{\mu\nu,z}) \quad \Rightarrow \\
\Gamma^{z}{}_{tt} &= \frac{1}{2} g^{zz}(-g_{tt,z}) = \frac{1}{2}(z^{1/2})(-z^{-3/2}/2) = -\frac{1}{4z} \,, \\
\Gamma^{z}{}_{zz} &= \frac{1}{2} g^{zz}(g_{zz,z}) = \frac{1}{2}(z^{1/2})(-z^{-3/2}/2) = -\frac{1}{4z} \,, \\
\Gamma^{z}{}_{xx} &= \frac{1}{2} g^{zz}(-g_{xx,z}) = \frac{1}{2}(z^{1/2})(-1) = -\frac{z^{1/2}}{2} = \Gamma^{z}{}_{yy} \,; \\
\Gamma^{x}{}_{\mu\nu} &= \frac{1}{2} g^{xx}(g_{x\mu,\nu} + g_{x\nu,\mu} - g_{\mu\nu,x}) = \frac{1}{2} g^{xx}(g_{x\mu,\nu} + g_{x\nu,\mu}) \quad \Rightarrow \\
\Gamma^{x}{}_{xz} = \Gamma^{x}{}_{zx} &= \frac{1}{2} g^{xx}(g_{xx,z}) = \frac{1}{2}(z^{-1})(1) = \frac{1}{2z} = \Gamma^{y}{}_{yz} = \Gamma^{y}{}_{zy} \,.
\end{aligned}$$

总结非零克氏符如下:

$$\begin{aligned}
&\Gamma^{t}{}_{tz} = \Gamma^{t}{}_{zt} = \Gamma^{z}{}_{tt} = \Gamma^{z}{}_{zz} = -\frac{1}{4z} \,, \\
&\Gamma^{z}{}_{xx} = \Gamma^{z}{}_{yy} = -\frac{z^{1/2}}{2} \,, \\
&\Gamma^{x}{}_{xz} = \Gamma^{x}{}_{zx} = \Gamma^{y}{}_{yz} = \Gamma^{y}{}_{zy} = \frac{1}{2z} \,.
\end{aligned}$$

因此求导后有:

$$\begin{aligned}
&\Gamma^{t}{}_{tz,z} = \Gamma^{t}{}_{zt,z} = \Gamma^{z}{}_{tt,z} = \Gamma^{z}{}_{zz,z} = \frac{1}{4z^2} \,, \\
&\Gamma^{z}{}_{xx,z} = \Gamma^{z}{}_{yy,z} = -\frac{1}{4z^{1/2}} \,, \\
&\Gamma^{x}{}_{xz,z} = \Gamma^{x}{}_{zx,z} = \Gamma^{y}{}_{yz,z} = \Gamma^{y}{}_{zy,z} = -\frac{1}{2z^2} \,.
\end{aligned}$$

最后利用公式 (3-4-20)

$$R_{\mu\nu\sigma}{}^{\rho} = \Gamma^{\rho}{}_{\mu\sigma,\nu} - \Gamma^{\rho}{}_{\nu\sigma,\mu} + \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\rho}{}_{\mu\lambda}$$

计算黎曼曲率张量．因为对前两个指标反对称，所以只须计算 $\mu\neq\nu$ 的情形．以下按 t, z, x, y 的次序计算，而且注意 x 和 y 是对称的：

$$
\begin{aligned}
R_{tz\sigma}{}^{\rho} &= \Gamma^{\rho}{}_{t\sigma,z}-\Gamma^{\rho}{}_{z\sigma,t}+\Gamma^{\lambda}{}_{\sigma t}\Gamma^{\rho}{}_{z\lambda}-\Gamma^{\lambda}{}_{\sigma z}\Gamma^{\rho}{}_{t\lambda}\\
&= \Gamma^{\rho}{}_{t\sigma,z}+\Gamma^{\lambda}{}_{\sigma t}\Gamma^{\rho}{}_{z\lambda}-\Gamma^{\lambda}{}_{\sigma z}\Gamma^{\rho}{}_{t\lambda}\ ,\\
R_{tx\sigma}{}^{\rho} &= \Gamma^{\rho}{}_{t\sigma,x}-\Gamma^{\rho}{}_{x\sigma,t}+\Gamma^{\lambda}{}_{\sigma t}\Gamma^{\rho}{}_{x\lambda}-\Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{t\lambda}\\
&= \Gamma^{\lambda}{}_{\sigma t}\Gamma^{\rho}{}_{x\lambda}-\Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{t\lambda}=R_{ty\sigma}{}^{\rho}\ ,\\
R_{zx\sigma}{}^{\rho} &= \Gamma^{\rho}{}_{z\sigma,x}-\Gamma^{\rho}{}_{x\sigma,z}+\Gamma^{\lambda}{}_{\sigma z}\Gamma^{\rho}{}_{x\lambda}-\Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{z\lambda}\\
&= -\Gamma^{\rho}{}_{x\sigma,z}+\Gamma^{\lambda}{}_{\sigma z}\Gamma^{\rho}{}_{x\lambda}-\Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{z\lambda}=R_{zy\sigma}{}^{\rho}\ ,\\
R_{xy\sigma}{}^{\rho} &= \Gamma^{\rho}{}_{x\sigma,y}-\Gamma^{\rho}{}_{y\sigma,x}+\Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{y\lambda}-\Gamma^{\lambda}{}_{\sigma y}\Gamma^{\rho}{}_{x\lambda}\\
&= \Gamma^{\lambda}{}_{\sigma x}\Gamma^{\rho}{}_{y\lambda}-\Gamma^{\lambda}{}_{\sigma y}\Gamma^{\rho}{}_{x\lambda}=-R_{yx\sigma}{}^{\rho}\ .
\end{aligned}
$$

注意虽然 x 和 y 对 t 和 z 来说是对称的，并不意味着没有 $R_{xy\sigma}{}^{\rho}$. 于是

$$
\begin{aligned}
R_{tzt}{}^{\rho} &= \Gamma^{\rho}{}_{tt,z}+\Gamma^{\lambda}{}_{tt}\Gamma^{\rho}{}_{z\lambda}-\Gamma^{\lambda}{}_{tz}\Gamma^{\rho}{}_{t\lambda} \quad\Rightarrow\\
R_{tzt}{}^{t} &= \Gamma^{t}{}_{tt,z}+\Gamma^{\lambda}{}_{tt}\Gamma^{t}{}_{z\lambda}-\Gamma^{\lambda}{}_{tz}\Gamma^{t}{}_{t\lambda}\\
&= 0\ ,\\
R_{tzt}{}^{z} &= \Gamma^{z}{}_{tt,z}+\Gamma^{\lambda}{}_{tt}\Gamma^{z}{}_{z\lambda}-\Gamma^{\lambda}{}_{tz}\Gamma^{z}{}_{t\lambda}\\
&= \Gamma^{z}{}_{tt,z}+\Gamma^{z}{}_{tt}\Gamma^{z}{}_{zz}-\Gamma^{t}{}_{tz}\Gamma^{z}{}_{tt}\\
&= \frac{1}{4z^2}+\Big(-\frac{1}{4z}\Big)\Big(-\frac{1}{4z}\Big)-\Big(-\frac{1}{4z}\Big)\Big(-\frac{1}{4z}\Big)\\
&= \frac{1}{4z^2}\ ,\\
R_{tzt}{}^{x} &= \Gamma^{x}{}_{tt,z}+\Gamma^{\lambda}{}_{tt}\Gamma^{x}{}_{z\lambda}-\Gamma^{\lambda}{}_{tz}\Gamma^{x}{}_{t\lambda}\\
&= 0=R_{tzt}{}^{y}\ ;\\
R_{tzz}{}^{\rho} &= \Gamma^{\rho}{}_{tz,z}+\Gamma^{\lambda}{}_{zt}\Gamma^{\rho}{}_{z\lambda}-\Gamma^{\lambda}{}_{zz}\Gamma^{\rho}{}_{t\lambda} \quad\Rightarrow\\
R_{tzz}{}^{t} &= \Gamma^{t}{}_{tz,z}+\Gamma^{\lambda}{}_{zt}\Gamma^{t}{}_{z\lambda}-\Gamma^{\lambda}{}_{zz}\Gamma^{t}{}_{t\lambda}\\
&= \Gamma^{t}{}_{tz,z}+\Gamma^{t}{}_{zt}\Gamma^{t}{}_{zt}-\Gamma^{z}{}_{zz}\Gamma^{t}{}_{tz}\\
&= \frac{1}{4z^2}+\Big(-\frac{1}{4z}\Big)\Big(-\frac{1}{4z}\Big)-\Big(-\frac{1}{4z}\Big)\Big(-\frac{1}{4z}\Big)\\
&= \frac{1}{4z^2}\ ,\\
R_{tzz}{}^{z} &= \Gamma^{z}{}_{tz,z}+\Gamma^{\lambda}{}_{zt}\Gamma^{z}{}_{z\lambda}-\Gamma^{\lambda}{}_{zz}\Gamma^{z}{}_{t\lambda}\\
&= 0\ ,\\
R_{tzz}{}^{x} &= \Gamma^{x}{}_{tz,z}+\Gamma^{\lambda}{}_{zt}\Gamma^{x}{}_{z\lambda}-\Gamma^{\lambda}{}_{zz}\Gamma^{x}{}_{t\lambda}\\
&= 0=R_{tzz}{}^{y}\ ;\\
R_{tzx}{}^{\rho} &= \Gamma^{\rho}{}_{tx,z}+\Gamma^{\lambda}{}_{xt}\Gamma^{\rho}{}_{z\lambda}-\Gamma^{\lambda}{}_{xz}\Gamma^{\rho}{}_{t\lambda}\\
&= 0=R_{tzy}{}^{\rho}\ .
\end{aligned}
$$

$$
\begin{aligned}
R_{txt}{}^{\rho} &= \Gamma^{\lambda}{}_{tt}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{tx}\Gamma^{\rho}{}_{t\lambda} = \Gamma^{\lambda}{}_{tt}\Gamma^{\rho}{}_{x\lambda} \quad \Rightarrow \\
R_{txt}{}^{t} &= \Gamma^{\lambda}{}_{tt}\Gamma^{t}{}_{x\lambda} \\
&= 0 = R_{tyt}{}^{t} \ , \\
R_{txt}{}^{z} &= \Gamma^{\lambda}{}_{tt}\Gamma^{z}{}_{x\lambda} \\
&= 0 = R_{tyt}{}^{z} \ , \\
R_{txt}{}^{x} &= \Gamma^{\lambda}{}_{tt}\Gamma^{x}{}_{x\lambda} = \Gamma^{z}{}_{tt}\Gamma^{x}{}_{xz} \\
&= \Big(-\frac{1}{4z}\Big)\Big(\frac{1}{2z}\Big) \\
&= -\frac{1}{8z^2} = R_{tyt}{}^{y} \ , \\
R_{txt}{}^{y} &= \Gamma^{\lambda}{}_{tt}\Gamma^{y}{}_{x\lambda} \\
&= 0 = R_{tyt}{}^{x} \ ; \\
R_{txz}{}^{\rho} &= \Gamma^{\lambda}{}_{zt}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{zx}\Gamma^{\rho}{}_{t\lambda} \\
&= 0 = R_{tyz}{}^{\rho} \ ; \\
R_{txx}{}^{\rho} &= \Gamma^{\lambda}{}_{xt}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{xx}\Gamma^{\rho}{}_{t\lambda} = -\Gamma^{\lambda}{}_{xx}\Gamma^{\rho}{}_{t\lambda} = -\Gamma^{z}{}_{xx}\Gamma^{\rho}{}_{tz} \quad \Rightarrow \\
R_{txx}{}^{t} &= -\Gamma^{z}{}_{xx}\Gamma^{t}{}_{tz} \\
&= -\Big(-\frac{z^{1/2}}{2}\Big)\Big(-\frac{1}{4z}\Big) \\
&= -\frac{1}{8z^{1/2}} = R_{tyy}{}^{t} \ ; \\
R_{txy}{}^{\rho} &= \Gamma^{\lambda}{}_{yt}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{yx}\Gamma^{\rho}{}_{t\lambda} \\
&= 0 = R_{tyx}{}^{\rho} \ . \\
R_{zxt}{}^{\rho} &= -\Gamma^{\rho}{}_{xt,z} + \Gamma^{\lambda}{}_{tz}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{tx}\Gamma^{\rho}{}_{z\lambda} \\
&= 0 = R_{zyt}{}^{\rho} \ , \\
R_{zxz}{}^{\rho} &= -\Gamma^{\rho}{}_{xz,z} + \Gamma^{\lambda}{}_{zz}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{zx}\Gamma^{\rho}{}_{z\lambda} \quad \Rightarrow \\
R_{zxz}{}^{t} &= -\Gamma^{t}{}_{xz,z} + \Gamma^{\lambda}{}_{zz}\Gamma^{t}{}_{x\lambda} - \Gamma^{\lambda}{}_{zx}\Gamma^{t}{}_{z\lambda} \\
&= 0 = R_{zyz}{}^{t} \ , \\
R_{zxz}{}^{z} &= -\Gamma^{z}{}_{xz,z} + \Gamma^{\lambda}{}_{zz}\Gamma^{z}{}_{x\lambda} - \Gamma^{\lambda}{}_{zx}\Gamma^{z}{}_{z\lambda} \\
&= \Gamma^{z}{}_{zz}\Gamma^{z}{}_{xz} - \Gamma^{x}{}_{zx}\Gamma^{z}{}_{zx} \\
&= 0 = R_{zyz}{}^{z} \ , \\
R_{zxz}{}^{x} &= -\Gamma^{x}{}_{xz,z} + \Gamma^{\lambda}{}_{zz}\Gamma^{x}{}_{x\lambda} - \Gamma^{\lambda}{}_{zx}\Gamma^{x}{}_{z\lambda} \\
&= -\Gamma^{x}{}_{xz,z} + \Gamma^{z}{}_{zz}\Gamma^{x}{}_{xz} - \Gamma^{x}{}_{zx}\Gamma^{x}{}_{zx} \\
&= -\Big(-\frac{1}{2z^2}\Big) + \Big(-\frac{1}{4z}\Big)\Big(\frac{1}{2z}\Big) - \Big(\frac{1}{2z}\Big)\Big(\frac{1}{2z}\Big) \\
&= \frac{1}{8z^2} = R_{zyz}{}^{y} \ , \\
R_{zxz}{}^{y} &= -\Gamma^{y}{}_{xz,z} + \Gamma^{\lambda}{}_{zz}\Gamma^{y}{}_{x\lambda} - \Gamma^{\lambda}{}_{zx}\Gamma^{y}{}_{z\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= 0 = R_{zyz}{}^{x}\ ; \\
R_{zxx}{}^{\rho} &= -\Gamma^{\rho}{}_{xx,z} + \Gamma^{\lambda}{}_{xz}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{xx}\Gamma^{\rho}{}_{z\lambda} \quad \Rightarrow \\
R_{zxx}{}^{t} &= -\Gamma^{t}{}_{xx,z} + \Gamma^{\lambda}{}_{xz}\Gamma^{t}{}_{x\lambda} - \Gamma^{\lambda}{}_{xx}\Gamma^{t}{}_{z\lambda} \\
&= \Gamma^{x}{}_{xz}\Gamma^{t}{}_{xx} - \Gamma^{z}{}_{xx}\Gamma^{t}{}_{zz} \\
&= 0 = R_{zyy}{}^{t}\ , \\
R_{zxx}{}^{z} &= -\Gamma^{z}{}_{xx,z} + \Gamma^{\lambda}{}_{xz}\Gamma^{z}{}_{x\lambda} - \Gamma^{\lambda}{}_{xx}\Gamma^{z}{}_{z\lambda} \\
&= -\Gamma^{z}{}_{xx,z} + \Gamma^{x}{}_{xz}\Gamma^{z}{}_{xx} - \Gamma^{z}{}_{xx}\Gamma^{z}{}_{zz} \\
&= -\Big(-\frac{1}{4z^{1/2}}\Big) + \Big(\frac{1}{2z}\Big)\Big(-\frac{z^{1/2}}{2}\Big) - \Big(-\frac{z^{1/2}}{2}\Big)\Big(-\frac{1}{4z}\Big) \\
&= -\frac{1}{8z^{1/2}} = R_{zyy}{}^{z}\ , \\
R_{zxx}{}^{x} &= -\Gamma^{x}{}_{xx,z} + \Gamma^{\lambda}{}_{xz}\Gamma^{x}{}_{x\lambda} - \Gamma^{\lambda}{}_{xx}\Gamma^{x}{}_{z\lambda} \\
&= \Gamma^{x}{}_{xz}\Gamma^{x}{}_{xx} - \Gamma^{z}{}_{xx}\Gamma^{x}{}_{zz} \\
&= 0 = R_{zyy}{}^{y}\ , \\
R_{zxx}{}^{y} &= -\Gamma^{y}{}_{xx,z} + \Gamma^{\lambda}{}_{xz}\Gamma^{y}{}_{x\lambda} - \Gamma^{\lambda}{}_{xx}\Gamma^{y}{}_{z\lambda} \\
&= \Gamma^{x}{}_{xz}\Gamma^{y}{}_{xx} - \Gamma^{z}{}_{xx}\Gamma^{y}{}_{zz} \\
&= 0 = R_{zyy}{}^{x}\ ; \\
R_{zxy}{}^{\rho} &= -\Gamma^{\rho}{}_{xy,z} + \Gamma^{\lambda}{}_{yz}\Gamma^{\rho}{}_{x\lambda} - \Gamma^{\lambda}{}_{yx}\Gamma^{\rho}{}_{z\lambda} \\
&= \Gamma^{y}{}_{yz}\Gamma^{\rho}{}_{xy} \\
&= 0 = R_{zyx}{}^{\rho}\ ; \\
R_{xyx}{}^{y} &= \Gamma^{\lambda}{}_{xx}\Gamma^{y}{}_{y\lambda} - \Gamma^{\lambda}{}_{xy}\Gamma^{y}{}_{x\lambda} = \Gamma^{z}{}_{xx}\Gamma^{y}{}_{yz} \\
&= \Big(-\frac{z^{1/2}}{2}\Big)\Big(\frac{1}{2z}\Big) = -\frac{1}{4z^{1/2}} = -R_{yxx}{}^{y}\ , \\
R_{xyy}{}^{x} &= \Gamma^{\lambda}{}_{yx}\Gamma^{x}{}_{y\lambda} - \Gamma^{\lambda}{}_{yy}\Gamma^{x}{}_{x\lambda} = -\Gamma^{z}{}_{yy}\Gamma^{x}{}_{xz} \\
&= -\Big(-\frac{z^{1/2}}{2}\Big)\Big(\frac{1}{2z}\Big) = \frac{1}{4z^{1/2}} = -R_{yxy}{}^{x}\ .
\end{aligned}
$$

我们最终求得非零的黎曼曲率张量为

$$
\begin{aligned}
R_{tzt}{}^{z} &= -R_{ztt}{}^{z} = R_{tzz}{}^{t} = -R_{ztz}{}^{t} = \frac{1}{4z^2}\ , \\
R_{xtt}{}^{x} &= -R_{txt}{}^{x} = R_{ytt}{}^{y} = -R_{tyt}{}^{y} = R_{zxz}{}^{x} = -R_{xzz}{}^{x} = R_{zyz}{}^{y} = -R_{yzz}{}^{y} = \frac{1}{8z^2}\ , \\
R_{xtx}{}^{t} &= -R_{txx}{}^{t} = R_{yty}{}^{t} = -R_{tyy}{}^{t} = R_{xzx}{}^{z} = -R_{zxx}{}^{z} = R_{yzy}{}^{z} = -R_{zyy}{}^{z} = \frac{1}{8z^{1/2}}\ , \\
R_{xyy}{}^{x} &= -R_{yxy}{}^{x} = -R_{xyx}{}^{y} = R_{yxx}{}^{y} = \frac{1}{4z^{1/2}}\ .
\end{aligned}
$$

因为

$$
R_{tztz} = g_{zz}R_{tzt}{}^{z} = z^{-1/2}\frac{1}{4z^2} = \frac{1}{4z^{5/2}}\ ,
$$

$$
\begin{aligned}
R_{tzzt} &= g_{tt}R_{tzz}{}^{t} = -z^{-1/2}\frac{1}{4z^2} = -\frac{1}{4z^{5/2}}\ ;\\
R_{xttx} &= g_{xx}R_{xtt}{}^{x} = z\frac{1}{8z^2} = \frac{1}{8z}\ ,\\
R_{zxzx} &= g_{xx}R_{zxz}{}^{x} = z\frac{1}{8z^2} = \frac{1}{8z}\ ;\\
R_{xtxt} &= g_{tt}R_{xtx}{}^{t} = -z^{-1/2}\frac{1}{8z^{1/2}} = -\frac{1}{8z}\ ,\\
R_{xzxz} &= g_{zz}R_{xzx}{}^{z} = z^{-1/2}\frac{1}{8z^{1/2}} = \frac{1}{8z}\ ;\\
R_{xyyx} &= g_{xx}R_{xyy}{}^{x} = z\frac{1}{4z^{1/2}} = \frac{z^{1/2}}{4}\ .
\end{aligned}
$$

所以黎曼张量又可写为

$$
\begin{aligned}
\frac{1}{4z^{5/2}} &= R_{tztz} = -R_{zttz} = -R_{tzzt} = R_{ztzt}\ ,\\
\frac{1}{8z} &= R_{xttx} = -R_{txtx} = R_{ytty} = -R_{tyty} = R_{zxzx} = -R_{xzzx} = R_{zyzy} = -R_{yzzy}\\
&= -R_{xtxt} = R_{txxt} = -R_{ytyt} = R_{tyyt} = R_{xzxz} = -R_{zxxz} = R_{yzyz} = -R_{zyyz}\ ,\\
\frac{z^{1/2}}{4} &= R_{xyyx} = -R_{yxyx} = -R_{xyxy} = R_{yxxy}\ .
\end{aligned}
$$

很容易看出它们满足 (3-4-9) 和 (3-4-10) 的关系.

16. 设 $\alpha(z)$, $\beta(z)$, $\gamma(z)$ 为任意函数， $h = t + \alpha(z)x + \beta(z)y + \gamma(z)$, 求度规

$$
ds^2 = -dt^2 + dx^2 + dy^2 + h^2dz^2
$$

的黎曼张量在 $\{t, x, y, z\}$ 系的全部分量.

解 度规张量为

$$
\begin{aligned}
&g_{tt} = -g_{xx} = -g_{yy} = -1\ , \qquad g_{zz} = h^2\ ;\\
&g^{tt} = -g^{xx} = -g^{yy} = -1\ , \qquad g^{zz} = h^{-2}\ .
\end{aligned}
$$

于是有

$$
\begin{aligned}
g_{zz,t} &= 2h\ ,\\
g_{zz,z} &= 2h(\alpha' x + \beta' y + \gamma') \equiv 2hh'\ ,\\
g_{zz,x} &= 2h\alpha\ ,\\
g_{zz,y} &= 2h\beta\ .
\end{aligned}
$$

先利用公式 (3-2-10’)

$$
\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})
$$

计算克氏符：

$$
\begin{aligned}
\Gamma^t{}_{\mu\nu} &= \frac{1}{2}g^{tt}(g_{t\mu,\nu}+g_{t\nu,\mu}-g_{\mu\nu,t}) = -\frac{1}{2}g^{tt}g_{\mu\nu,t} \quad \Rightarrow \\
\Gamma^t{}_{zz} &= -\frac{1}{2}g^{tt}g_{zz,t} = -\frac{1}{2}(-1)(2h) = h \ ; \\
\Gamma^z{}_{\mu\nu} &= \frac{1}{2}g^{zz}(g_{z\mu,\nu}+g_{z\nu,\mu}-g_{\mu\nu,z}) \quad \Rightarrow \\
\Gamma^z{}_{zz} &= \frac{1}{2}g^{zz}g_{zz,z} = \frac{1}{2}(h^{-2})(2hh') = h^{-1}h' \ , \\
\Gamma^z{}_{zt} &= \frac{1}{2}g^{zz}(g_{zz,t}+g_{zt,z}-g_{zt,z}) = \frac{1}{2}g^{zz}g_{zz,t} = \frac{1}{2}(h^{-2})(2h) = h^{-1} \ , \\
\Gamma^z{}_{zx} &= \frac{1}{2}g^{zz}(g_{zz,x}+g_{zx,z}-g_{zx,z}) = \frac{1}{2}g^{zz}g_{zz,x} = \frac{1}{2}(h^{-2})(2h\alpha) = h^{-1}\alpha \ , \\
\Gamma^z{}_{zy} &= \frac{1}{2}g^{zz}(g_{zz,y}+g_{zy,z}-g_{zy,z}) = \frac{1}{2}g^{zz}g_{zz,y} = \frac{1}{2}(h^{-2})(2h\beta) = h^{-1}\beta \ ; \\
\Gamma^x{}_{\mu\nu} &= \frac{1}{2}g^{xx}(g_{x\mu,\nu}+g_{x\nu,\mu}-g_{\mu\nu,x}) = -\frac{1}{2}g^{xx}g_{\mu\nu,x} \quad \Rightarrow \\
\Gamma^x{}_{zz} &= -\frac{1}{2}g^{xx}g_{zz,x} = -\frac{1}{2}(1)(2h\alpha) = -h\alpha \ , \\
\Gamma^y{}_{zz} &= -\frac{1}{2}g^{yy}g_{zz,y} = -\frac{1}{2}(1)(2h\beta) = -h\beta \ .
\end{aligned}
$$

总结非零克氏符如下：

$$
\begin{aligned}
\Gamma^t{}_{zz} &= h \ , \\
\Gamma^z{}_{zz} &= h^{-1}(\alpha' x+\beta' y+\gamma') \equiv h^{-1}h' \ , \\
\Gamma^z{}_{zt} = \Gamma^z{}_{tz} &= h^{-1} \ , \\
\Gamma^z{}_{zx} = \Gamma^z{}_{xz} &= h^{-1}\alpha \ , \\
\Gamma^z{}_{zy} = \Gamma^z{}_{yz} &= h^{-1}\beta \ , \\
\Gamma^x{}_{zz} &= -h\alpha \ , \\
\Gamma^y{}_{zz} &= -h\beta \ .
\end{aligned}
$$

因此求导后有：

$$
\begin{aligned}
\Gamma^t{}_{zz,t} &= 1 \ , \\
\Gamma^z{}_{zz,t} &= -h^{-2}h' \ , \\
\Gamma^z{}_{zt,t} = \Gamma^z{}_{tz,t} &= -h^{-2} \ , \\
\Gamma^z{}_{zx,t} = \Gamma^z{}_{xz,t} &= -h^{-2}\alpha \ , \\
\Gamma^z{}_{zy,t} = \Gamma^z{}_{yz,t} &= -h^{-2}\beta \ , \\
\Gamma^x{}_{zz,t} &= -\alpha \ , \\
\Gamma^y{}_{zz,t} &= -\beta \ ; \\
\Gamma^t{}_{zz,z} &= h' \ ,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^z{}_{zz,z} &= -h^{-2}h'^2 + h^{-1}h'' , \\
\Gamma^z{}_{zt,z} = \Gamma^z{}_{tz,z} &= -h^{-2}h' , \\
\Gamma^z{}_{zx,z} = \Gamma^z{}_{xz,z} &= -h^{-2}h'\alpha + h^{-1}\alpha' , \\
\Gamma^z{}_{zy,z} = \Gamma^z{}_{yz,z} &= -h^{-2}h'\beta + h^{-1}\beta' , \\
\Gamma^x{}_{zz,z} &= -h'\alpha - h\alpha' , \\
\Gamma^y{}_{zz,z} &= -h'\beta - h\beta' ; \\
\Gamma^t{}_{zz,x} &= \alpha , \\
\Gamma^z{}_{zz,x} &= -h^{-2}\alpha h' + h^{-1}\alpha' , \\
\Gamma^z{}_{zt,x} = \Gamma^z{}_{tz,x} &= -h^{-2}\alpha , \\
\Gamma^z{}_{zx,x} = \Gamma^z{}_{xz,x} &= -h^{-2}\alpha^2 , \\
\Gamma^z{}_{zy,x} = \Gamma^z{}_{yz,x} &= -h^{-2}\alpha\beta , \\
\Gamma^x{}_{zz,x} &= -\alpha^2 , \\
\Gamma^y{}_{zz,x} &= -\alpha\beta ; \\
\Gamma^t{}_{zz,y} &= \beta , \\
\Gamma^z{}_{zz,y} &= -h^{-2}\beta h' + h^{-1}\beta' , \\
\Gamma^z{}_{zt,y} = \Gamma^z{}_{tz,y} &= -h^{-2}\beta , \\
\Gamma^z{}_{zx,y} = \Gamma^z{}_{xz,y} &= -h^{-2}\alpha\beta , \\
\Gamma^z{}_{zy,y} = \Gamma^z{}_{yz,y} &= -h^{-2}\beta^2 , \\
\Gamma^x{}_{zz,y} &= -\alpha\beta , \\
\Gamma^y{}_{zz,y} &= -\beta^2 .
\end{aligned}
$$

最后利用公式 (3-4-20')

$$
R_{\mu\nu\sigma}{}^{\rho} = \Gamma^{\rho}{}_{\mu\sigma,\nu} - \Gamma^{\rho}{}_{\nu\sigma,\mu} + \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\rho}{}_{\mu\lambda}
$$

计算黎曼曲率张量．因为对前两个指标反对称，所以只须计算 $\mu \neq \nu$ 的情形．以下按 t, z, x, y 的次序计算：

$$
\begin{aligned}
R_{tzt}{}^{t} &= \Gamma^t{}_{tt,z} - \Gamma^t{}_{zt,t} + \Gamma^{\lambda}{}_{tt}\Gamma^t{}_{z\lambda} - \Gamma^{\lambda}{}_{tz}\Gamma^t{}_{t\lambda} = 0 , \\
R_{tzt}{}^{z} &= \Gamma^z{}_{tt,z} - \Gamma^z{}_{zt,t} + \Gamma^{\lambda}{}_{tt}\Gamma^z{}_{z\lambda} - \Gamma^{\lambda}{}_{tz}\Gamma^z{}_{t\lambda} \\
&= -\Gamma^z{}_{zt,t} - \Gamma^z{}_{tz}\Gamma^z{}_{tz} \\
&= -(-h^{-2}) - (h^{-1})^2 \\
&= 0 , \\
R_{tzt}{}^{x} &= \Gamma^x{}_{tt,z} - \Gamma^x{}_{zt,t} + \Gamma^{\lambda}{}_{tt}\Gamma^x{}_{z\lambda} - \Gamma^{\lambda}{}_{tz}\Gamma^x{}_{t\lambda} = 0 = R_{tzt}{}^{y} ; \\
R_{tzz}{}^{t} &= \Gamma^t{}_{tz,z} - \Gamma^t{}_{zz,t} + \Gamma^{\lambda}{}_{zt}\Gamma^t{}_{z\lambda} - \Gamma^{\lambda}{}_{zz}\Gamma^t{}_{t\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= -\Gamma^t{}_{zz,t} + \Gamma^z{}_{zt}\Gamma^t{}_{zz} \\
&= -(1) + (h^{-1})(h) \\
&= 0\,, \\
R_{tzz}{}^z &= \Gamma^z{}_{tz,z} - \Gamma^z{}_{zz,t} + \Gamma^\lambda{}_{zt}\Gamma^z{}_{z\lambda} - \Gamma^\lambda{}_{zz}\Gamma^z{}_{t\lambda} \\
&= \Gamma^z{}_{tz,z} - \Gamma^z{}_{zz,t} + \Gamma^z{}_{zt}\Gamma^z{}_{zz} - \Gamma^z{}_{zz}\Gamma^z{}_{tz} \\
&= (-h^{-2}h') - (-h^{-2}h') \\
&= 0\,, \\
R_{tzz}{}^x &= \Gamma^x{}_{tz,z} - \Gamma^x{}_{zz,t} + \Gamma^\lambda{}_{zt}\Gamma^x{}_{z\lambda} - \Gamma^\lambda{}_{zz}\Gamma^x{}_{t\lambda} \\
&= -\Gamma^x{}_{zz,t} + \Gamma^z{}_{zt}\Gamma^x{}_{zz} \\
&= -(-\alpha) + (h^{-1})(-h\alpha) \\
&= 0 = R_{tzz}{}^y\,.
\end{aligned}
$$

可以用 Mathematica 编程验证黎曼张量的所有分量在该坐标系下均为零！那么根据定理 3-4-9, 这时一定存在 (局域) 平直度规场，即度规场的全部分量为常数！

17. 试证 2 维广义黎曼空间的爱因斯坦张量为零．提示： 2 维广义黎曼空间的黎曼张量只有一个独立分量.

证 2 维广义黎曼空间的黎曼张量只有 $\frac{2^2(2^2-1)}{12}=1$ 个独立分量，即有关系

$$R_{1212}=R_{2121}=-R_{1221}=-R_{2112}=a\,.$$

因此里奇张量 $R_{ab}=g^{cd}R_{acbd}$ 的分量式为

$$
\begin{aligned}
R_{11} &= g^{cd}R_{1c1d}=g^{22}R_{1212}=ag^{22}\,, \\
R_{12} &= g^{cd}R_{1c2d}=g^{21}R_{1221}=-ag^{21}\,, \\
R_{21} &= g^{cd}R_{2c1d}=g^{12}R_{2112}=-ag^{12}\,, \\
R_{22} &= g^{cd}R_{2c2d}=g^{11}R_{2121}=ag^{11}\,.
\end{aligned}
$$

标量曲率 $R=g^{ab}R_{ab}$ 的分量式为

$$
\begin{aligned}
R &= g^{11}R_{11}+g^{12}R_{12}+g^{21}R_{21}+g^{22}R_{22} \\
&= g^{11}(ag^{22})+g^{12}(-ag^{21})+g^{21}(-ag^{12})+g^{22}(ag^{11}) \\
&= 2a(g^{11}g^{22}-g^{12}g^{21})\,.
\end{aligned}
$$

于是爱因斯坦张量 $G_{ab}=R_{ab}-\frac{1}{2}Rg_{ab}$ 的分量式为

$$
\begin{aligned}
G_{11} &= R_{11}-\frac{1}{2}Rg_{11}=ag^{22}-\frac{1}{2}2a(g^{11}g^{22}-g^{12}g^{21})g_{11} \\
&= ag^{22}-ag^{11}g_{11}g^{22}+ag^{12}g^{21}g_{11}
\end{aligned}
$$

$$\begin{aligned}
&= ag^{22} - a(\delta^1{}_1 - g^{12}g_{21})g^{22} + ag^{12}g^{21}g_{11} \\
&= ag^{12}g_{21}g^{22} + ag^{12}g^{21}g_{11} = ag^{12}(g^{21}g_{11} + g^{22}g_{21}) \\
&= ag^{12}\delta^2{}_1 = 0\,, \\
G_{22} &= R_{22} - \frac{1}{2}Rg_{22} = ag^{11} - \frac{1}{2}2a(g^{11}g^{22} - g^{12}g^{21})g_{22} \\
&= ag^{11} - ag^{11}g^{22}g_{22} + ag^{12}g^{21}g_{22} \\
&= ag^{11} - ag^{11}(\delta^2{}_2 - g^{21}g_{12}) + ag^{12}g^{21}g_{22} \\
&= ag^{11}g^{21}g_{12} + ag^{12}g^{21}g_{22} = ag^{21}(g^{11}g_{12} + g^{12}g_{22}) \\
&= ag^{21}\delta^1{}_2 = 0\,, \\
G_{12} &= R_{12} - \frac{1}{2}Rg_{12} = -ag^{21} - \frac{1}{2}2a(g^{11}g^{22} - g^{12}g^{21})g_{12} \\
&= -ag^{21} - ag^{11}g_{12}g^{22} + ag^{12}g_{12}g^{21} \\
&= -ag^{21} - ag^{11}g_{12}g^{22} + a(\delta^1{}_1 - g^{11}g_{11})g^{21} \\
&= -ag^{11}g_{12}g^{22} - ag^{11}g_{11}g^{21} = -ag^{11}(g^{22}g_{21} + g^{21}g_{11}) \\
&= -ag^{11}\delta^2{}_1 = 0\,, \\
G_{21} &= R_{21} - \frac{1}{2}Rg_{21} = -ag^{12} - \frac{1}{2}2a(g^{11}g^{22} - g^{12}g^{21})g_{21} \\
&= -ag^{12} - ag^{11}g^{22}g_{21} + ag^{12}g^{21}g_{21} \\
&= -ag^{12} - ag^{11}g^{22}g_{21} + ag^{12}(\delta^2{}_2 - g^{22}g_{22}) \\
&= -ag^{11}g^{22}g_{21} - ag^{12}g_{22}g^{22} = -ag^{22}(g^{11}g_{12} + g^{12}g_{22}) \\
&= -ag^{22}\delta^1{}_2 = 0\,.
\end{aligned}$$

命题得证.

第 4 章 “李导数、 Killing 场和超曲面” 习题

~1. 试证由式 (4-1-1) 定义的 $(\phi_* v)^a$ 满足 §2.2 定义 2 对矢量的两个要求，从而确是 $\phi(p)$ 点的矢量.

证 (a) 线性性：

$$\begin{aligned}
(\phi_* v)(\alpha f + \beta g) &\overset{\text{(4-1-2)}}{=} v[\phi^*(\alpha f + \beta g)] \\
&\overset{\text{(4-1-1)(1)}}{=} v[\alpha(\phi^* f) + \beta(\phi^* g)] \\
&\overset{\text{§2.2定义 2(a)}}{=} \alpha\, v(\phi^* f) + \beta\, v(\phi^* g) \\
&\overset{\text{(4-1-2)}}{=} \alpha\,(\phi_* v)(f) + \beta\,(\phi_* v)(g)\,.
\end{aligned}$$

(b) 莱布尼茨律：

$$(\phi_* v)(fg) \overset{\text{(4-1-2)}}{=} v[\phi^*(fg)]$$

$$
\begin{aligned}
&\overset{\text{(4-1-1)(2)}}{=} \quad v[(\phi^* f)(\phi^* g)] \\
&\overset{\S 2.2\text{定义 }2\text{(b)}}{=} \quad (\phi^* f)|_p\, v(\phi^* g) + (\phi^* g)|_p\, v(\phi^* f) \\
&\overset{\text{(4-1-2)}}{=} \quad (\phi^* f)|_p\, (\phi_* v)(g) + (\phi^* g)|_p\, (\phi_* v)(f) \\
&\overset{\text{定义 }1}{=} \quad f|_{\phi(p)}\, (\phi_* v)(g) + g|_{\phi(p)}\, (\phi_* v)(f)\ .
\end{aligned}
$$

因此 $\phi_* v$ 为点 $\phi(p) \in N$ 的一个矢量， $\phi_* v \in V_{\phi(p)}, \forall f, g \in \mathcal{F}_N$, 满足 §2.2 定义 2 要求的矢量的线性性和莱布尼茨律.

~2. 试证定理 4-1-1 、 4-1-2 和 4-1-3.

证 (a) 定理 4-1-1 的证明. $\forall f \in \mathcal{F}_N$ 有

$$
\begin{aligned}
[\phi_*(\alpha u^a + \beta v^a)](f) &\overset{\text{(4-1-2)}}{=} (\alpha u + \beta v)(\phi^* f) \\
&= \alpha\, u(\phi^* f) + \beta\, v(\phi^* f) \\
&\overset{\text{(4-1-2)}}{=} \alpha\, (\phi_* u^a)(f) + \beta\, (\phi_* v^a)(f)\ .
\end{aligned}
$$

(式中 u 和 v 的矢量上标 a 也可不写.) 因此 $\phi_* : V_p \to V_{\phi(p)}$ 是线性映射，满足

$$
\phi_*(\alpha u^a + \beta v^a) = \alpha\, \phi_* u^a + \beta\, \phi_* v^a\ .
$$

(b) 定理 4-1-2 的证明. 令 $p \equiv C(t_0) \in M$, $\phi(p) \equiv \phi(C(t_0)) \in N$. $\forall f \in \mathcal{F}_N$ 有

$$
\begin{aligned}
&\quad [(\phi_* T^a)(f)]|_{\phi(C(t_0))} = [(\phi_* T^a)(f)]|_{\phi(p)} \\
&\overset{\text{定义 }2}{=} [T(\phi^* f)]|_p = [T(\phi^* f)]|_{C(t_0)} = [T(\phi^* f|_{C(t)})]|_{t=t_0} \\
&\overset{\text{定义 }1}{=} [T(f|_{\phi(C(t))})]|_{t=t_0} = T(f(\phi(C(t)))|_{t=t_0} \\
&= \left.\frac{d(f \circ \phi(C(t)))}{dt}\right|_{t=t_0}\ .
\end{aligned}
$$

根据 §2.2 定义 6 式 (2-2-6), 等式右边定义出曲线 $\phi(C(t))$ 的切矢. 因此 M 上的曲线 $C(t)$ 的切矢 T^a 在 N 上的像 $\phi_* T^a$, 是 M 上的曲线 $C(t)$ 在 N 上的像 $\phi(C(t))$ 的切矢.

(c) 定理 4-1-3 的证明. 在证明此定理前，须先证明式 (4-1-4) 和 (4-1-5). 根据定理 4-1-2: 曲线切矢的推前像等于曲线推前像的切矢. 考虑 M 上 q 点的一个局部坐标系 $\{x'^\mu\}$, 它被 ϕ 映射到 N 上 $\phi(q)$ 点的一个局部坐标系 $\{y^\mu\}$, 即满足 $x'^\mu(q) = y^\mu(\phi(p))$. 因此 $\{x'^\mu\}$ 系的坐标线被映射为 $\{y^\mu\}$ 系的坐标线, 注意到这两组坐标线在 q 和 $\phi(q)$ 点的切矢分别为 $(\partial/\partial x'^\mu)^a|_q$ 和 $(\partial/\partial y^\mu)^a|_{\phi(q)}$, 由定理 4-1-2 知 $\phi_*[(\partial/\partial x'^\mu)^a|_q] = (\partial/\partial y^\mu)^a|_{\phi(q)}$, 此即式 (4-1-4). 另一方面,

$$
\begin{aligned}
\delta^\mu{}_\nu &= \phi_*[\delta^\mu{}_\nu] = \phi_*[(dx'^\mu)_a|_q (\partial/\partial x'^\nu)^a|_q] \overset{\text{(4-1-10)}}{=} \phi_*[(dx'^\mu)_a|_q]\phi_*[(\partial/\partial x'^\nu)^a|_q] \\
&\overset{\text{(4-1-4)}}{=} \phi_*[(dx'^\mu)_a|_q](\partial/\partial y^\nu)^a|_{\phi(q)}\ ,
\end{aligned}
$$

两边作用 $(dy^\nu)_b|_{\phi(q)}$ 得

$$\delta^\mu{}_\nu (dy^\nu)_b|_{\phi(q)} = \phi_*[(dx'^\mu)_a|_q](\partial/\partial y^\nu)^a|_{\phi(q)}(dy^\nu)_b|_{\phi(q)} = \phi_*[(dx'^\mu)_a|_q]\delta^a{}_b \ ,$$

此即式 (4-1-5) $\phi_*[(dx'^\mu)_b|_q] = (dy^\mu)_b|_{\phi(q)}$. 这两个关系也可等价地写成

$$\phi^*[(\partial/\partial y^\mu)^a|_{\phi(q)}] = (\partial/\partial x'^\mu)^a|_q \ , \qquad \phi^*[(dy^\mu)_a|_{\phi(q)}] = (dx'^\mu)_a|_q \ .$$

对于 $p \in M$ 点的 $T \in \mathscr{F}_M(k,l)$, 经过微分同胚的推前映射后变为 $\phi(p) \in N$ 点的 $\phi_* T \in \mathscr{F}_N(k,l)$, 于是用坐标系展开成分量形式:

$$\begin{aligned}
(\phi_*T)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}|_{\phi(p)} &= (\phi_*T)^{a_1\cdots a_k}{}_{b_1\cdots b_l}(dy^{\mu_1})_{a_1}|_{\phi(p)}\cdots(dy^{\mu_k})_{a_k}|_{\phi(p)} \\
&\qquad \Big(\frac{\partial}{\partial y^{\nu_1}}\Big)^{b_1}\Big|_{\phi(p)}\cdots\Big(\frac{\partial}{\partial y^{\nu_l}}\Big)^{b_l}\Big|_{\phi(p)} \\
&= T^{a_1\cdots a_k}{}_{b_1\cdots b_l}\,\phi^*[(dy^{\mu_1})_{a_1}|_{\phi(p)}]\cdots\phi^*[(dy^{\mu_k})_{a_k}|_{\phi(p)}] \\
&\qquad \phi^*\Big[\Big(\frac{\partial}{\partial y^{\nu_1}}\Big)^{b_1}\Big|_{\phi(p)}\Big]\cdots\phi^*\Big[\Big(\frac{\partial}{\partial y^{\nu_l}}\Big)^{b_l}\Big|_{\phi(p)}\Big] \\
&= T^{a_1\cdots a_k}{}_{b_1\cdots b_l}(dx'^{\mu_1})_{a_1}|_p\cdots(dx'^{\mu_k})_{a_k}|_p \\
&\qquad \Big(\frac{\partial}{\partial x'^{\nu_1}}\Big)^{b_1}\Big|_p\cdots\Big(\frac{\partial}{\partial x'^{\nu_l}}\Big)^{b_l}\Big|_p \\
&= T'^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}|_p \ .
\end{aligned}$$

可见式中左边是新点 $\phi(p)$ 的新张量 ϕ_*T 在 (老) 坐标系 $\{y^\mu\}$ 的分量, 右边是老点 p 的老张量 T 在新坐标系 $\{x'^\mu\}$ 的分量.

3. 设 $\phi: M \to N$ 为光滑映射, $p \in M$, $\{y^\mu\}$ 是 $\phi(p)$ 点某邻域上的坐标, 试证

$$(\phi_* v)^a = v(\phi^* y^\mu)(\partial/\partial y^\mu)^a \ , \qquad \forall\, v^a \in V_p \ .$$

证 因 $(\phi_* v)^a \in V_{\phi(p)}$, 以坐标系 $\{y^\mu\}$ 展开有

$$(\phi_* v)^a|_{\phi(p)} = (\phi_* v)^\mu|_{\phi(p)}\Big(\frac{\partial}{\partial y^\mu}\Big)^a\Big|_{\phi(p)} \ .$$

另一方面,

$$\begin{aligned}
(\phi_* v)^\mu|_{\phi(p)} &\overset{(4\text{-}1\text{-}6)}{=} v'^\mu|_p = v^a (dx'^\mu)_a|_p = v^a \phi^*[(dy^\mu)_a|_{\phi(p)}] \\
&= v^a[d(\phi^* y^\mu)]_a = v(\phi^* y^\mu) \ .
\end{aligned}$$

4. 设 M, N 是流形, $\phi: M \to N$ 是微分同胚, $p \in M$, $q \equiv \phi(p)$, 试证推前映射 $\phi_*: V_p \to V_q$ 是同构映射.

证 微分同胚映射是一一到上的映射, 所以推前映射也是一一到上的. 两个矢量空间一一到上的线性映射即是同构映射.

5. 设 M, N, Q 是流形， $\phi: M \to N$ 和 $\psi: N \to Q$ 是光滑映射.

(a) 试证 $(\psi\circ\phi)^* f = (\phi^*\circ\psi^*)f,\ \forall\, f\in\mathscr{F}_Q$.

(b) 试证 $(\psi\circ\phi)_* v^a = \psi_*(\phi_* v^a),\ \forall\, p\in M,\ v^a\in V_p$.

(c) 把 $(\psi\circ\phi)^*$ 和 $\phi^*\circ\psi^*$ 都看作由 $\mathscr{F}_Q(0,l)$ 到 $\mathscr{F}_M(0,l)$ 的映射，试证

$$(\psi\circ\phi)^* = \phi^*\circ\psi^*\ .$$

证 对于复合映射 $\psi\circ\phi : p\mapsto q=\phi(p)\mapsto r=\psi(q)=\psi(\phi(p))$, 其中 $p\in M$, $q\in N$, $r\in Q$.

(a) $\forall\, f\in\mathscr{F}_Q$, 根据定义 1, 拉回映射 $(\psi\circ\phi)^* f|_p = f|_{\psi(\phi(p))}$. 另一方面，$\psi^* f|_q = f|_{\psi(q)} = g|_q$ 和 $\phi^* g|_p = g|_{\phi(p)}$, 所以有 $(\phi^*\circ\psi^*)f|_p = \phi^* g|_p = g|_{\phi(p)} = f|_{\psi(\phi(p))}$. 因此 $(\psi\circ\phi)^* f|_p = (\phi^*\circ\psi^*)f|_p = f|_{\psi(\phi(p))}$.

或者利用关系式 $\phi^* f = f\circ\phi$, 现在有

$$(\psi\circ\phi)^* f = f\circ(\psi\circ\phi) = (f\circ\psi)\circ\phi = (\psi^* f)\circ\phi = \phi^*\circ(\psi^* f) = (\phi^*\circ\psi^*)f\ .$$

(b) 根据定义 2, $\forall\, f\in\mathscr{F}_Q$, 推前映射 $[(\psi\circ\phi)_* v^a](f)|_{\psi(\phi(p))} = v[(\psi\circ\phi)^* f]|_p = v(f)|_{\psi(\phi(p))}$, 最后一步利用了 (a) 的结果. 另一方面，同样根据定义 2:

$$\begin{aligned}
[\psi_*(\phi_* v^a)](f)|_{\psi(\phi(p))} &= (\phi_* v^a)(\psi^* f)|_{\phi(p)} = v(\phi^*(\psi^* f))|_p = v((\phi^*\circ\psi^*)f)|_p \\
&= v(f)|_{\psi(\phi(p))}\ .
\end{aligned}$$

因此有 $[(\psi\circ\phi)_* v^a](f) = [\psi_*(\phi_* v^a)](f)$, 导致 $(\psi\circ\phi)_* v^a = \psi_*(\phi_* v^a)$ 成立.

(c) $\forall\, T\in\mathscr{F}_Q(0,l)$, $\psi^* T\in\mathscr{F}_N(0,l)$, $\phi^*(\psi^* T) = (\phi^*\circ\psi^*)T\in\mathscr{F}_M(0,l)$. 根据定义 3 式 (4-1-3) 有

$$\begin{aligned}
[(\psi\circ\phi)^* T]_{a_1\cdots a_l}|_p (v_1)^{a_1}\cdots(v_l)^{a_l} &= T_{a_1\cdots a_l}|_{\psi(\phi(p))}[(\psi\circ\phi)_* v_1]^{a_1}\cdots[(\psi\circ\phi)_* v_l]^{a_l} \\
&\overset{(b)}{=} T_{a_1\cdots a_l}|_{\psi(\phi(p))}[\psi_*(\phi_* v_1)]^{a_1}\cdots[\psi_*(\phi_* v_l)]^{a_l}\ ,
\end{aligned}$$

其中 $v_1,\cdots,v_l\in V_p$. 另一方面，同样根据定义 3 式 (4-1-3),

$$\begin{aligned}
[(\phi^*\circ\psi^*)T]_{a_1\cdots a_l}|_p (v_1)^{a_1}\cdots(v_l)^{a_l} &= [\phi^*(\psi^* T)]_{a_1\cdots a_l}|_p (v_1)^{a_1}\cdots(v_l)^{a_l} \\
&= (\psi^* T)_{a_1\cdots a_l}|_{\phi(p)}(\phi_* v_1)^{a_1}\cdots(\phi_* v_l)^{a_l} \\
&= T_{a_1\cdots a_l}|_{\psi(\phi(p))}[\psi_*(\phi_* v_1)]^{a_1}\cdots[\psi_*(\phi_* v_l)]^{a_l}\ .
\end{aligned}$$

因此有

$$[(\psi\circ\phi)^* T]_{a_1\cdots a_l}|_p (v_1)^{a_1}\cdots(v_l)^{a_l} = [(\phi^*\circ\psi^*)T]_{a_1\cdots a_l}|_p (v_1)^{a_1}\cdots(v_l)^{a_l}\ ,$$

导致 $(\psi\circ\phi)^* = \phi^*\circ\psi^*$.

6. 设 $\phi : M \to N$ 是微分同胚， v^a, u^a 是 M 上矢量场，试证 $\phi_*([v,u]^a) = [\phi_* v, \phi_* u]^a$, 其中 $[v,u]^a$ 代表对易子.

证 由定义 2

$$\phi_*[v,u](f) = [v,u](\phi^* f) = v(u(\phi^* f)) - u(v(\phi^* f)) \, .$$

而

$$\begin{aligned} [\phi_* v, \phi_* u](f) &= \phi_* v(\phi_* u(f)) - \phi_* u(\phi_* v(f)) \overset{(4\text{-}1\text{-}9)}{=} \phi_*(vu)(f) - \phi_*(uv)(f) \\ &\overset{(4\text{-}1\text{-}2)}{=} (vu)(\phi^* f) - (uv)(\phi^* f) = v(u(\phi^* f)) - u(v(\phi^* f)) \, . \end{aligned}$$

因此 $\phi_*[v,u](f) = [\phi_* v, \phi_* u](f), \forall f \in \mathcal{F}_N$, 给出

$$\phi_*[v,u] = [\phi_* v, \phi_* u] \, .$$

~7. 试证定理 4-2-4.

证 首先，因为李导数满足莱布尼茨律，故有

$$\mathcal{L}_v(\omega_a v^a) = (\mathcal{L}_v \omega_a) v^a + \omega_a (\mathcal{L}_v v^a) = v^a \mathcal{L}_v \omega_a \, ,$$

其中利用了定理 4-2-3 式 (4-2-6) $\mathcal{L}_v v^a = [v,v]^a = 0$. 另一方面，因 $\omega_a v^a$ 是标量场，故由定理 4-2-1 式 (4-2-2) 有

$$\begin{aligned} \mathcal{L}_v(\omega_a v^a) &= v(\omega_a v^a) \overset{\S 3.1\text{定义 } 1(\mathrm{d})}{=} v^b \nabla_b(\omega_a v^a) = v^b[(\nabla_b \omega_a) v^a + \omega_a(\nabla_b v^a)] \\ &= v^a v^b \nabla_b \omega_a + v^b \omega_a \nabla_b v^a = v^a [v^b \nabla_b \omega_a + \omega_b \nabla_a v^b] \, . \end{aligned}$$

比较以上两式，注意到 v^a 的任意性，所以有

$$\mathcal{L}_v \omega_a = v^b \nabla_b \omega_a + \omega_b \nabla_a v^b \, .$$

注意这里的 ∇_a 可以是任一无挠导数算符.

~8. 设 $v^a \in \mathcal{F}_M(1,0)$, $\omega_a \in \mathcal{F}_M(0,1)$, 试证对任一坐标系 $\{x^\mu\}$ 有

$(\mathcal{L}_v \omega)_\mu = v^\nu \partial \omega_\mu / \partial x^\nu + \omega_\nu \partial v^\nu / \partial x^\mu \, .$ 提示：用式 (4-2-7) 并令其 ∇_a 为 ∂_a .

证 以坐标基底展开

$$\begin{aligned} (\mathcal{L}_v \omega)_\mu &= (\partial/\partial x^\mu)^a (\mathcal{L}_v \omega)_a \\ &\overset{(4\text{-}2\text{-}7)}{=} (\partial/\partial x^\mu)^a (v^b \partial_b \omega_a + \omega_b \partial_a v^b) \\ &\overset{(3\text{-}1\text{-}10)}{=} v^b \partial_b [(\partial/\partial x^\mu)^a \omega_a] + \omega_b [(\partial/\partial x^\mu)^a \partial_a] v^b \\ &= v^b \partial_b \omega_\mu + \omega_b \partial_\mu v^b \\ &\overset{(3\text{-}1\text{-}10)}{=} v^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu v^\nu \\ &= v^\nu \partial \omega_\mu / \partial x^\nu + \omega_\nu \partial v^\nu / \partial x^\mu \, . \end{aligned}$$

~9. 设 $u^a,\, v^a \in \mathscr{F}_M(1,0)$, 则下式作用于任意张量场都成立

$$[\mathcal{L}_v,\, \mathcal{L}_u] = \mathcal{L}_{[v,u]} \quad (\text{其中}\, [\mathcal{L}_v,\, \mathcal{L}_u] \equiv \mathcal{L}_v\mathcal{L}_u - \mathcal{L}_u\mathcal{L}_v)\,.$$

试就作用对象为 $f \in \mathscr{F}_M$ 和 $w^a \in \mathscr{F}_M(1,0)$ 的情况给出证明. 提示：当作用对象为 w^a 时可用雅可比恒等式 (第 2 章习题 8).

证 (a) 作用于标量场 f:

$$\begin{aligned}
[\mathcal{L}_v,\, \mathcal{L}_u](f) &= (\mathcal{L}_v\mathcal{L}_u - \mathcal{L}_u\mathcal{L}_v)(f) \\
&= \mathcal{L}_v\mathcal{L}_u(f) - \mathcal{L}_u\mathcal{L}_v(f) \\
&= \mathcal{L}_v(\mathcal{L}_u f) - \mathcal{L}_u(\mathcal{L}_v f) \\
&\overset{(4\text{-}2\text{-}2)}{=} \mathcal{L}_v(u(f)) - \mathcal{L}_u(v(f)) \\
&\overset{(4\text{-}2\text{-}2)}{=} v(u(f)) - u(v(f)) \\
&\overset{(2\text{-}2\text{-}9)}{=} [v,u](f) \\
&\overset{(4\text{-}2\text{-}2)}{=} \mathcal{L}_{[v,u]}(f)\,.
\end{aligned}$$

(b) 作用于矢量场 w^a:

$$\begin{aligned}
[\mathcal{L}_v,\, \mathcal{L}_u](w^a) &= (\mathcal{L}_v\mathcal{L}_u - \mathcal{L}_u\mathcal{L}_v)(w^a) \\
&= \mathcal{L}_v\mathcal{L}_u(w^a) - \mathcal{L}_u\mathcal{L}_v(w^a) \\
&= \mathcal{L}_v(\mathcal{L}_u w^a) - \mathcal{L}_u(\mathcal{L}_v w^a) \\
&\overset{(4\text{-}2\text{-}6)}{=} \mathcal{L}_v([u,w]^a) - \mathcal{L}_u([v,w]^a) \\
&\overset{(4\text{-}2\text{-}6)}{=} [v,[u,w]]^a - [u,[v,w]]^a \\
&= ([v,[u,w]]^a + [u,[w,v]]^a + [w,[v,u]]^a) - [w,[v,u]]^a \\
&= -[w,[v,u]]^a \\
&= [[v,u],w]^a \\
&\overset{(4\text{-}2\text{-}6)}{=} \mathcal{L}_{[v,u]}(w^a)\,,
\end{aligned}$$

其中倒数第四步用了雅可比恒等式.

(c) 作用于对偶矢量场 ω_a. 注意到

$$\begin{aligned}
\mathcal{L}_v\mathcal{L}_u(\omega_a) &= \mathcal{L}_v(\mathcal{L}_u\omega_a) \\
&\overset{(4\text{-}2\text{-}7)}{=} \mathcal{L}_v(u^b\nabla_b\omega_a + \omega_b\nabla_a u^b) \\
&\overset{(4\text{-}2\text{-}7)}{=} v^c\nabla_c(u^b\nabla_b\omega_a + \omega_b\nabla_a u^b) + (u^b\nabla_b\omega_c + \omega_b\nabla_c u^b)\nabla_a v^c \\
&= v^c\nabla_c u^b\nabla_b\omega_a + v^c u^b\nabla_c\nabla_b\omega_a + v^c\nabla_c\omega_b\nabla_a u^b + v^c\omega_b\nabla_c\nabla_a u^b \\
&\quad + u^b\nabla_b\omega_c\nabla_a v^c + \omega_b\nabla_c u^b\nabla_a v^c \\
&= v^c\nabla_c u^b\nabla_b\omega_a + v^c u^b\nabla_c\nabla_b\omega_a + v^c\nabla_a u^b\nabla_c\omega_b + v^c\omega_b\nabla_c\nabla_a u^b
\end{aligned}$$

$$
\begin{aligned}
&+u^b\nabla_a v^c\nabla_b\omega_c+\omega_b\nabla_c u^b\nabla_a v^c \\
=\;& v^c u^b\nabla_c\nabla_b\omega_a+v^c\nabla_c u^b\nabla_b\omega_a+v^c\nabla_a u^b\nabla_c\omega_b+u^b\nabla_a v^c\nabla_b\omega_c \\
&+v^c\omega_b\nabla_c\nabla_a u^b+\omega_b\nabla_c u^b\nabla_a v^c\,,
\end{aligned}
$$

于是

$$
\begin{aligned}
\mathcal{L}_u\mathcal{L}_v(\omega_a) &= u^c v^b\nabla_c\nabla_b\omega_a+u^c\nabla_c v^b\nabla_b\omega_a+u^c\nabla_a v^b\nabla_c\omega_b+v^b\nabla_a u^c\nabla_b\omega_c \\
&\quad +u^c\omega_b\nabla_c\nabla_a v^b+\omega_b\nabla_c v^b\nabla_a u^c \\
&= v^c u^b\nabla_b\nabla_c\omega_a+u^c\nabla_c v^b\nabla_b\omega_a+u^b\nabla_a v^c\nabla_b\omega_c+v^c\nabla_a u^b\nabla_c\omega_b \\
&\quad +u^c\omega_b\nabla_c\nabla_a v^b+\omega_b\nabla_c v^b\nabla_a u^c\,.
\end{aligned}
$$

上两式相减，第一、三、四项相互抵消，得

$$
\begin{aligned}
[\mathcal{L}_v,\,\mathcal{L}_u](\omega_a) &= \mathcal{L}_v\mathcal{L}_u(\omega_a)-\mathcal{L}_u\mathcal{L}_v(\omega_a) \\
&= (v^c\nabla_c u^b-u^c\nabla_c v^b)\nabla_b\omega_a \\
&\quad +\omega_b(v^c\nabla_c\nabla_a u^b-u^c\nabla_c\nabla_a v^b)+\omega_b(\nabla_a v^c\nabla_c u^b-\nabla_a u^c\nabla_c v^b) \\
&= (v^c\nabla_c u^b-u^c\nabla_c v^b)\nabla_b\omega_a+\omega_b\nabla_a(v^c\nabla_c u^b-u^c\nabla_c v^b) \\
&\overset{\text{(4-2-6')}}{=} [v,u]^b\nabla_b\omega_a+\omega_b\nabla_a[v,u]^b \\
&\overset{\text{(4-2-7)}}{=} \mathcal{L}_{[v,u]}(\omega_a)\,.
\end{aligned}
$$

10. 设 F_{ab} 是 4 维闵氏空间上的反对称张量场，其在洛伦兹坐标系 $\{t,\,x,\,y,\,z\}$ 的分量为 $F_{01}=-F_{13}=x\rho^{-1}$, $F_{02}=-F_{23}=y\rho^{-1}$, $F_{03}=F_{12}=0$, 其中 $\rho\equiv(x^2+y^2)^{1/2}$. 试证 F_{ab} 有旋转对称性, 即 $\mathcal{L}_vF_{ab}=0$, 其中 $v^a=-y(\partial/\partial x)^a+x(\partial/\partial y)^a$.

证 根据定理 4-2-5 式 (4-2-8),

$$
\mathcal{L}_vF_{ab}=v^c\nabla_cF_{ab}+F_{cb}\nabla_a v^c+F_{ac}\nabla_b v^c\,.
$$

因为这里的导数算符 ∇_a 可以任意，所以选普通导数 ∂_a, 于是上式的分量式为

$$
\mathcal{L}_vF_{\mu\nu}=v^\sigma\partial_\sigma F_{\mu\nu}+F_{\sigma\nu}\partial_\mu v^\sigma+F_{\mu\sigma}\partial_\nu v^\sigma\,.
$$

于是有

$$
\begin{aligned}
\mathcal{L}_vF_{01} &= v^\sigma\partial_\sigma F_{01}+F_{\sigma 1}\partial_0 v^\sigma+F_{0\sigma}\partial_1 v^\sigma \\
&= (v^1\partial_1F_{01}+v^2\partial_2F_{01})+0+F_{02}\partial_1 v^2 \\
&= (-y)\frac{\partial}{\partial x}(x\rho^{-1})+x\frac{\partial}{\partial y}(x\rho^{-1})+(y\rho^{-1})\frac{\partial}{\partial x}(x) \\
&= -y\rho^{-1}-yx\Big(-\frac{1}{2}\rho^{-3/2}2x\Big)+x^2\Big(-\frac{1}{2}\rho^{-3/2}2y\Big)+y\rho^{-1} \\
&= -y\rho^{-1}+yx^2\rho^{-3/2}-x^2y\rho^{-3/2}+y\rho^{-1} \\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_v F_{02} &= v^\sigma \partial_\sigma F_{02} + F_{\sigma 2}\partial_0 v^\sigma + F_{0\sigma}\partial_2 v^\sigma \\
&= (v^1\partial_1 F_{02} + v^2\partial_2 F_{02}) + 0 + F_{01}\partial_2 v^1 \\
&= (-y)\frac{\partial}{\partial x}(y\rho^{-1}) + x\frac{\partial}{\partial y}(y\rho^{-1}) + (x\rho^{-1})\frac{\partial}{\partial y}(-y) \\
&= -y^2\Big(-\frac{1}{2}\rho^{-3/2}2x\Big) + x\rho^{-1} + xy\Big(-\frac{1}{2}\rho^{-3/2}2y\Big) - x\rho^{-1} \\
&= xy^2\rho^{-3/2} + x\rho^{-1} - xy^2\rho^{-3/2} - x\rho^{-1} \\
&= 0\,, \\
\mathcal{L}_v F_{03} &= \mathcal{L}_v(0) = 0\,, \\
\mathcal{L}_v F_{12} &= \mathcal{L}_v(0) = 0\,, \\
\mathcal{L}_v F_{13} &= v^\sigma \partial_\sigma F_{13} + F_{\sigma 3}\partial_1 v^\sigma + F_{1\sigma}\partial_3 v^\sigma \\
&= (v^1\partial_1 F_{13} + v^2\partial_2 F_{13}) + F_{23}\partial_1 v^2 + 0 \\
&= -(v^1\partial_1 F_{01} + v^2\partial_2 F_{01}) - F_{02}\partial_1 v^2 \\
&= -\mathcal{L}_v F_{01} = 0\,, \\
\mathcal{L}_v F_{23} &= v^\sigma \partial_\sigma F_{23} + F_{\sigma 3}\partial_2 v^\sigma + F_{2\sigma}\partial_3 v^\sigma \\
&= (v^1\partial_1 F_{23} + v^2\partial_2 F_{23}) + F_{13}\partial_2 v^1 + 0 \\
&= -(v^1\partial_1 F_{02} + v^2\partial_2 F_{02}) - F_{01}\partial_2 v^1 \\
&= -\mathcal{L}_v F_{02} = 0\,.
\end{aligned}
$$

因此对 $v^a = -y(\partial/\partial x)^a + x(\partial/\partial y)^a$ 有 $\mathcal{L}_v F_{ab} = 0$.

11. 设 ξ^a 是 $(M,\, g_{ab})$ 中的 Killing 矢量场， ∇_a 与 g_{ab} 适配，试证 $\nabla_a \xi^a = 0$.

证 注意到 $\nabla_a \xi^a = \nabla_a(g^{ab}\xi_b) = \xi_b \nabla_a g^{ab} + g^{ab}\nabla_a \xi_b$. 其中第一项因 ∇_a 与 g_{ab} 的适配性有 $\nabla_a g^{bc} = 0$ (见 §3.2.2 例 1 前的证明), 于是 $\nabla_a g^{ab} = 0$. 第二项 $g^{ab}\nabla_a\xi_b = g^{(ab)}\nabla_a\xi_b \overset{\text{定理 2-6-2(a)}}{=} g^{(ab)}\nabla_{(a}\xi_{b)} \overset{\text{定理 4-3-1}}{=} 0$. 因此 $\nabla_a\xi^a = 0$.

12. 设 ξ^a 是 $(M,\, g_{ab})$ 中的 Killing 矢量场， $\phi: M \to N$【似应为 M 】 是等度规映射，试证 $\phi_*\xi^a$ 也是 $(M,\, g_{ab})$ 中的 Killing 矢量场．提示：利用习题 5(c) 的结论.

证 设 $(M,\, g_{ab})$ 上的 Killing 矢量场 ξ^a 给出的单参微分同胚群为 ψ, 其群元为 ψ_t, t 为 ξ^a 的积分曲线上的参数. 根据 Killing 矢量场的定义 (§4.3 定义 2) 以及李导数的定义 (§4.2 定义 1), 我们有

$$\mathcal{L}_\xi g_{ab} = \lim_{t\to 0}\frac{1}{t}(\psi_t^* g_{ab} - g_{ab}) = 0\,.$$

现在要证的是如果 $\phi: M \to M$ 是等度规映射，即满足 $\phi^* g_{ab} = g_{ab}$ (§4.3 定义 1), 那么有 $\mathcal{L}_{\phi_*\xi} g_{ab} = 0$, 即 $\phi_*\xi^a$ 也是 Killing 矢量场. 首先我们证明矢量场 $\phi_*\xi^a$ 给出的单参微分同胚群为 $\phi\circ\psi_t$. 根据定理 4-2-1 式 (4-2-2), $\mathcal{L}_{\phi_*\xi} = (\phi_*\xi)(f)$,

$\forall f \in \mathcal{F}_M$. 而由推前映射的定义式 (4-1-2) 有 $(\phi_*\xi)(f) = \xi(\phi^* f) = \mathcal{L}_\xi(\phi^* f)$. 将此结果代回李导数的定义式 (4-2-1) 得

$$\begin{aligned}\mathcal{L}_{\phi_*\xi} f &= \mathcal{L}_\xi(\phi^* f) = \lim_{t\to 0}\frac{1}{t}[\psi_t^*(\phi^* f) - f] = \lim_{t\to 0}\frac{1}{t}[(\psi_t^*\circ\phi^*)f - f] \\ &= \lim_{t\to 0}\frac{1}{t}[(\phi\circ\psi_t)^* f - f]\,,\end{aligned}$$

其中最后一步利用了习题 5(c) 的结论 $(\psi_t^*\circ\phi^*) = (\phi\circ\psi_t)^*$. 因为 f 为任意函数，于是看出由 $\phi_*\xi$ 生成的单参微分同胚群元为 $\phi\circ\psi_t$. 最后

$$\begin{aligned}\mathcal{L}_{\phi_*\xi} g_{ab} &= \lim_{t\to 0}\frac{1}{t}[(\phi\circ\psi_t)^* g_{ab} - g_{ab}] = \lim_{t\to 0}\frac{1}{t}[(\psi_t^*\circ\phi^*)g_{ab} - g_{ab}] \\ &= = \lim_{t\to 0}\frac{1}{t}[\psi_t^*(\phi^* g_{ab}) - g_{ab}] = \lim_{t\to 0}\frac{1}{t}[\psi_t^*(g_{ab}) - g_{ab}] \\ &= \mathcal{L}_\xi g_{ab} = 0\,,\end{aligned}$$

其中利用了等度规映射 $\phi^* g_{ab} = g_{ab}$ 和 ξ 的 Killing 性 $\mathcal{L}_\xi g_{ab} = 0$.

13. 设 ξ^a, η^a 是 (M, g_{ab}) 的 Killing 矢量场，试证其对易子 $[\xi,\eta]^a$ 也是 Killing 矢量场. 注：此结论使得 M 上全体 Killing 矢量场的集合不但是矢量空间，而且是李代数 (详见下册附录 G).

证 根据习题 9 的结果，我们有

$$\begin{aligned}\mathcal{L}_{[\xi,\eta]} g_{ab} &= [\mathcal{L}_\xi, \mathcal{L}_\eta] g_{ab} = \mathcal{L}_\xi\mathcal{L}_\eta g_{ab} - \mathcal{L}_\eta\mathcal{L}_\xi g_{ab} \\ &\overset{\text{定义 } 2}{=} \mathcal{L}_\xi(0) - \mathcal{L}_\eta(0) = 0\,,\end{aligned}$$

因此由定义 2 知， $[\xi,\eta]^a$ 也是 Killing 矢量场. 或者直接验证方程 (4-3-1):

$$\begin{aligned}\nabla_a[\xi,\eta]_b + \nabla_b[\xi,\eta]_a &= \nabla_a[\xi^c\nabla_c\eta_b - \eta^c\nabla_c\xi_b] + \nabla_b[\xi^c\nabla_c\eta_a - \eta^c\nabla_c\xi_a] \\ &\overset{(4\text{-}3\text{-}1)}{=} \nabla_a[-\xi^c\nabla_b\eta_c + \eta^c\nabla_b\xi_c] + \nabla_b[-\xi^c\nabla_a\eta_c + \eta^c\nabla_a\xi_c] \\ &= \nabla_a\eta^c\nabla_b\xi_c + \eta^c\nabla_a\nabla_b\xi_c - \nabla_a\xi^c\nabla_b\eta_c - \xi^c\nabla_a\nabla_b\eta_c \\ &\quad +\nabla_b\eta^c\nabla_a\xi_c + \eta^c\nabla_b\nabla_a\xi_c - \nabla_b\xi^c\nabla_a\eta_c - \xi^c\nabla_b\nabla_a\eta_c \\ &= (\nabla_a\eta^c\nabla_b\xi_c - \nabla_b\xi^c\nabla_a\eta_c) + (\nabla_b\eta^c\nabla_a\xi_c - \nabla_a\xi^c\nabla_b\eta_c) \\ &\quad +\eta^c(\nabla_a\nabla_b + \nabla_b\nabla_a)\xi_c - \xi^c(\nabla_a\nabla_b + \nabla_b\nabla_a)\eta_c \\ &= (\nabla_a\eta^c\nabla_b\xi_c - \nabla_b\xi_c\nabla_a\eta^c) + (\nabla_b\eta^c\nabla_a\xi_c - \nabla_a\xi_c\nabla_b\eta^c) \\ &\quad +\eta^c(\nabla_a\nabla_b + \nabla_b\nabla_a)\xi_c - \xi^c(\nabla_a\nabla_b + \nabla_b\nabla_a)\eta_c \\ &= \eta^c(\nabla_a\nabla_b + \nabla_b\nabla_a)\xi_c - \xi^c(\nabla_a\nabla_b + \nabla_b\nabla_a)\eta_c\,.\end{aligned}$$

利用 Killing 性得

$$\begin{aligned}(\nabla_a\nabla_b + \nabla_b\nabla_a)\xi_c &= -\nabla_a\nabla_c\xi_b - \nabla_b\nabla_c\xi_a \\ &= -\nabla_c\nabla_a\xi_b - R_{acbd}\xi^d - \nabla_c\nabla_b\xi_a - R_{bcad}\xi^d\end{aligned}$$

$$
\begin{aligned}
&= -\nabla_c\nabla_a\xi_b - R_{acbd}\xi^d - \nabla_c\nabla_b\xi_a - R_{bcad}\xi^d \\
&= -\nabla_c(\nabla_a\xi_b + \nabla_b\xi_a) - R_{acbd}\xi^d - R_{bcad}\xi^d \\
&= -R_{acbd}\xi^d - R_{bcad}\xi^d\ , \\
(\nabla_a\nabla_b + \nabla_b\nabla_a)\eta_c &= -R_{acbd}\eta^d - R_{bcad}\eta^d\ .
\end{aligned}
$$

于是

$$
\begin{aligned}
\nabla_a[\xi,\eta]_b + \nabla_b[\xi,\eta]_a &= \eta^c(-R_{acbd}\xi^d - R_{bcad}\xi^d) - \xi^c(-R_{acbd}\eta^d - R_{bcad}\eta^d) \\
&= -R_{acbd}\eta^c\xi^d - R_{bcad}\eta^c\xi^d + R_{acbd}\xi^c\eta^d + R_{bcad}\xi^c\eta^d \\
&= -R_{adbc}\eta^d\xi^c - R_{bdac}\eta^d\xi^c + R_{acbd}\xi^c\eta^d + R_{bcad}\xi^c\eta^d \\
&= (R_{bcad} - R_{adbc})\xi^c\eta^d + (R_{acbd} - R_{bdac})\xi^c\eta^d \\
&\overset{(3\text{-}4\text{-}10)}{=} 0\ .
\end{aligned}
$$

14. 设 ξ^a 是广义黎曼空间 (M, g_{ab}) 的 Killing 矢量场， $R_{abc}{}^d$ 是 g_{ab} 的黎曼曲率张量.

(a) 试证 $\nabla_a\nabla_b\xi_c = -R_{bca}{}^d\xi_d$. 注：此式对证明定理 4-3-4 有重要用处. 提示：由 $R_{abc}{}^d$ 的定义以及 Killing 方程 (4-3-1) 可知 $\nabla_a\nabla_b\xi_c + \nabla_b\nabla_c\xi_a = R_{abc}{}^d\xi_d$. 此式称为第一式. 作指标替换 $a\mapsto b$, $b\mapsto c$, $c\mapsto a$ 得第二式，再替换一次得第三式. 以第一、二式之和减第三式并利用式 (3-4-7) 便得证.

(b) 利用 (a) 的结果证明 $\nabla^a\nabla_a\xi_c = -R_{cd}\xi^d$, 其中 R_{cd} 是里奇张量.

证 (a) 根据黎曼曲率张量的定义 (3-4-3) 和 Killing 矢量场满足的方程 (4-3-1) 有

$$
(\nabla_a\nabla_b - \nabla_b\nabla_a)\xi_c = \nabla_a\nabla_b\xi_c - \nabla_b\nabla_a\xi_c = \nabla_a\nabla_b\xi_c + \nabla_b\nabla_c\xi_a = R_{abc}{}^d\xi_d\ .
$$

作指标替换 $a\mapsto b$, $b\mapsto c$, $c\mapsto a$ 得

$$
\nabla_b\nabla_c\xi_a + \nabla_c\nabla_a\xi_b = R_{bca}{}^d\xi_d\ ,
$$

再替换一次得

$$
\nabla_c\nabla_a\xi_b + \nabla_a\nabla_b\xi_c = R_{cab}{}^d\xi_d\ .
$$

第一、二式之和减第三式得

$$
\begin{aligned}
&\nabla_a\nabla_b\xi_c + \nabla_b\nabla_c\xi_a + \nabla_b\nabla_c\xi_a + \nabla_c\nabla_a\xi_b - \nabla_c\nabla_a\xi_b - \nabla_a\nabla_b\xi_c \\
=\ & 2\nabla_b\nabla_c\xi_a \\
=\ & R_{abc}{}^d\xi_d + R_{bca}{}^d\xi_d - R_{cab}{}^d\xi_d = (R_{abc}{}^d + R_{bca}{}^d + R_{cab}{}^d)\xi_d - 2R_{cab}{}^d\xi_d \\
=\ & -2R_{cab}{}^d\xi_d\ ,
\end{aligned}
$$

最后一步用到了黎曼曲率张量的循环恒等式 (3-4-7). 于是 $\nabla_b\nabla_c\xi_a = -R_{cab}{}^d\xi_d$, 此即 $\nabla_a\nabla_b\xi_c = -R_{bca}{}^d\xi_d$.

(b) 上式两边作用 g^{ab}:

$$g^{ab}\nabla_a\nabla_b\xi_c = \nabla^b\nabla_b\xi_c = -g^{ab}R_{bca}{}^d\xi_d = -g^{ab}R_{bcad}\xi^d = -R_{cd}\xi^d\ ,$$

其中最后一步用到了黎曼曲率张量性质 (3-4-6) 、 (3-4-9) 和里奇张量的定义：

$$g^{ab}R_{bcad} = g^{ab}R_{cbda} = R_{cd}\ .$$

因此有 $\nabla^a\nabla_a\xi_c = -R_{cd}\xi^d$.

~15. 验证式 (4-3-3) 中的 $(\partial/\partial\eta)^a$ 的确满足 Killing 方程 (4-3-1).

证 欲证 $\xi^a = (\partial/\partial\eta)^a = t(\partial/\partial x)^a + x(\partial/\partial t)^a$ 满足 Killing 方程 (4-3-1): $\nabla_a\xi_b + \nabla_b\xi_a = 0$, 注意到与闵氏度规 η_{ab} 相适配的导数算符是普通导数 ∂_a, 故只须证 $\partial_a\xi_b + \partial_b\xi_a = 0$. 我们看其相应的分量方程： $\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 0$, 这里 $\mu,\ \nu = 0,\ 1$ 代表 $t,\ x$. 显然 $\xi^0 = x$ 和 $\xi^1 = t$. 于是,

$$\begin{aligned}
(\mu,\ \nu) = (0,\ 0): \quad & \partial_0\xi_0 + \partial_0\xi_0 = 2\partial_0\xi_0 = 2\partial_0(\eta_{0\rho}\xi^\rho) = 2\partial_0(-\xi^0) \\
&= -2\partial_0\xi^0 = -2\frac{\partial}{\partial t}(x) = 0\ , \\
(\mu,\ \nu) = (0,\ 1)\ \text{或}(1,\ 0): \quad & \partial_0\xi_1 + \partial_1\xi_0 = \partial_0(\eta_{1\rho}\xi^\rho) + \partial_1(\eta_{0\rho}\xi^\rho) \\
&= \partial_0(\xi^1) + \partial_1(-\xi^0) = \frac{\partial}{\partial t}(t) - \frac{\partial}{\partial x}(x) = 0\ , \\
(\mu,\ \nu) = (1,\ 1): \quad & \partial_1\xi_1 + \partial_1\xi_1 = 2\partial_1\xi_1 = 2\partial_1(\eta_{1\rho}\xi^\rho) = 2\partial_1(\xi^1) \\
&= 2\frac{\partial}{\partial x}(t) = 0
\end{aligned}$$

故知张量式 $\partial_a\xi_b + \partial_b\xi_a = 0$ 成立.

~16. 找出 2 维欧氏空间中由 $R^a = x(\partial/\partial y)^a - y(\partial/\partial x)^a$ 生出的单参等度规群的任一元素 ϕ_α 诱导的坐标变换.

解 矢量场 $R^a = x(\partial/\partial y)^a - y(\partial/\partial x)^a$ 的积分曲线的参数方程为 $\frac{dx^\mu(t)}{dt} = R^\mu$ $(\mu = 1, 2 = x, y)$, 即

$$\frac{dx^1(t)}{dt} = \frac{dx(t)}{dt} = R^1 = -y(t)\ , \qquad \frac{dx^2(t)}{dt} = \frac{dy(t)}{dt} = R^2 = x(t)\ .$$

$\forall\, p \in RR^2$, 设 $C(t)$ 是满足 $p = C(0)$ 的积分曲线，即 $x(0) = x_p$, $y(0) = y_p$, 则容易看出以上方程的特解 [即该线的参数式] 为

$$x(t) = x_p\cos t - y_p\sin t\ , \qquad y(t) = x_p\sin t + y_p\cos t\ .$$

设 $q \equiv \phi_\alpha(p)$, 则 q 就是 $C(t)$ 上参数值 $t = \alpha$ 的点，即 $q = C(\alpha)$, 故由 ϕ_α 诱导的新坐标 x' 和 y' 满足

$$x'_p \equiv x_q = x_p\cos\alpha - y_p\sin\alpha\ , \qquad y'_p \equiv y_q = x_p\sin\alpha + y_p\cos\alpha\ .$$

因 p 点任意，故可去掉下标 p 而写成

$$x' = x\,\cos\alpha - y\,\sin\alpha\,, \qquad y' = x\,\sin\alpha + y\,\cos\alpha\,.$$

此即熟知的二维平面旋转 (正交) 变换.

*17. 设时空 (M, g_{ab}) 中的超曲面 $\phi[S]$ 上每点都有类光切矢而无类时切矢 (“切矢”指切于 $\phi[S]$), 试证它必为类光超曲面. 提示: ①证明与类时矢量 t^a 正交的矢量必类空 [选正交归一基底 $\{(e_\mu)^a\}$ 使 $(e_0)^a = t^a$]; ②证明类时超曲面上每点都有类时切矢; ③由以上两点证明本命题.

证 ①首先我们证明与类时矢量正交的矢量必类空. 设矢量 t^a 类时, 有 $g_{ab}t^a t^b < 0$. 如果矢量 v^a 与 t^a 正交, 即满足 $g_{ab}v^a t^b = 0$, 则必有 $g_{ab}v^a v^b > 0$. 设 n 维流形 M 的某一正交归一基底为 $\{(e_\mu)^a\}$. 不失一般性可选 $(e_0)^a = t^a$, 于是 $g_{ab}(e_0)^a(e_0)^b = g_{00} < 0$. v^a 在该基底的展开式 $v^a = v^\mu (e_\mu)^a$, 正交性给出

$$g_{ab}v^a t^b = g_{ab}v^\mu (e_\mu)^a (e_0)^b = g_{\mu 0}v^\mu = g_{00}v^0 = 0\,.$$

因此知道 $v^0 = 0$. 而

$$\begin{aligned} g_{ab}v^a v^b &= g_{ab}v^\mu (e_\mu)^a v^\nu (e_\nu)^b = g_{\mu\nu}v^\mu v^\nu \\ &= g_{00}v^0 v^0 + g_{\mu 0}v^\mu v^0 + g_{0\mu}v^0 v^\mu + g_{ij}v^i v^j \\ &= g_{ij}v^i v^j = g_{ii}(v^i)^2 > 0\,, \end{aligned}$$

其中利用了 $n-1$ 维的空间部分度规张量的正定性. 因此得到结论. 由此证明也可以知道与类空矢量正交的矢量未必一定类时, 其原因在于时间只有一维而空间可以高于一维, 此时类空矢量之间可以相互正交.

②其次我们证明类时超曲面上每点都有类时切矢. 所谓类时超曲面, 根据定义 4 是它的法矢处处类空. 设类时超曲面 $\phi[S]$ 上 q 点的切空间为 W_q. 因为它的类空法矢 $n^a \notin W_q$, 故 W_q 的基底中必有一个类时, 它就是超曲面的类时切矢.

③如果超曲面 $\phi[S]$ 上每点都有类光切矢而无类时切矢, 那么它既不可能是类时超曲面也不可能是类空超曲面, 因而只可能是类光超曲面. 如果它是类时超曲面, 那么根据②, 它每点都有类时切矢, 这与题设每点都无类时切矢不符. 如果它是类空超曲面, 那么它每点的法矢都是类时的, 而法矢的性质告诉我们超曲面的切矢都与它正交, 所以从①的结论我们知道这些切矢都是类空的而没有类光的和类时的, 但题设中有类光切矢, 所以也有矛盾. 唯一的可能性就是该超曲面是类光超曲面.

第 5 章“微分形式及其积分”习题

~1. 在定理 5-1-3 证明中补证 $\{(e^1)_a\wedge(e^2)_b,\,(e^2)_a\wedge(e^3)_b,\,(e^3)_a\wedge(e^1)_b\}$ 线性独立.

证 假设它们不线性独立，则必有非零的常数 $a,\,b,\,c$ 满足

$$a\,(e^1)_a\wedge(e^2)_b+b\,(e^2)_a\wedge(e^3)_b+c\,(e^3)_a\wedge(e^1)_b=0\ .$$

以 $(e_1)^a(e_2)^b$ 作用上式，易得

$$(e_1)^a(e_2)^b(e^1)_a\wedge(e^2)_b=(e_1)^a(e_2)^b[(e^1)_a(e^2)_b-(e^2)_a(e^1)_b]=1\ ,$$
$$(e_1)^a(e_2)^b(e^2)_a\wedge(e^3)_b=(e_1)^a(e_2)^b[(e^2)_a(e^3)_b-(e^3)_a(e^2)_b]=0\ ,$$
$$(e_1)^a(e_2)^b(e^3)_a\wedge(e^1)_b=(e_1)^a(e_2)^b[(e^3)_a(e^1)_b-(e^1)_a(e^3)_b]=0\ ,$$

因此上式变为 $a=0$. 同理可知 $b=c=0$. 它们彼此线性独立.

~2. 设 V 为矢量空间， $\{(e^1)_a,\,(e^2)_a,\,(e^3)_a,\,(e^4)_a\}$ 是 V^* 的基底，写出 $\omega_a\in\Lambda(1)$, $\omega_{abc}\in\Lambda(3)$ 和 $\omega_{abcd}\in\Lambda(4)$ 在此基底的展开式，说明展开系数 (如 ω_{12}) 的定义.

解 分别有 $C_4^1=4,\ C_4^3=4,\ C_4^4=1$ 项：

$$\begin{aligned}\omega_a&=\omega_1(e^1)_a+\omega_2(e^2)_a+\omega_3(e^3)_a+\omega_4(e^4)_a\ ,\\ \omega_{abc}&=\omega_{123}(e^1)_a\wedge(e^2)_b\wedge(e^3)_c+\omega_{124}(e^1)_a\wedge(e^2)_b\wedge(e^4)_c\\ &\quad+\omega_{134}(e^1)_a\wedge(e^3)_b\wedge(e^4)_c+\omega_{234}(e^2)_a\wedge(e^3)_b\wedge(e^4)_c\ ,\\ \omega_{abcd}&=\omega_{1234}(e^1)_a\wedge(e^2)_b\wedge(e^3)_c\wedge(e^4)_d\ .\end{aligned}$$

展开系数的定义如

$$\omega_{134}=\omega_{abc}(e_1)^a(e_3)^b(e_4)^c\ .$$

~3. 用数学归纳法证明 $(\omega^1)_{a_1}\wedge\cdots\wedge(\omega^l)_{a_l}=l!(\omega^1)_{[a_1}\cdots(\omega^l)_{a_l]}$, 其中 $(\omega^1)_a,\cdots,(\omega^l)_a$ 为任意对偶矢量.

证 设对 l 成立 $(\omega^1)_{a_1}\wedge\cdots\wedge(\omega^l)_{a_l}=l!(\omega^1)_{[a_1}\cdots(\omega^l)_{a_l]}$, 这是个 l 形式，令它等于 $F^{1\cdots l}_{a_1\cdots a_l}$, 于是根据楔形积的定义 2 式 (5-1-2) 和结合律有

$$\begin{aligned}(\omega^1)_{a_1}\wedge\cdots\wedge(\omega^l)_{a_l}\wedge(\omega^{l+1})_{a_{l+1}}&=F^{1\cdots l}_{a_1\cdots a_l}\wedge(\omega^{l+1})_{a_{l+1}}\\ &\overset{(5\text{-}1\text{-}2)}{=}\frac{(l+1)!}{l!\,1!}F^{1\cdots l}_{[a_1\cdots a_l}(\omega^{l+1})_{a_{l+1}]}\\ &=(l+1)l!(\omega^1)_{[[a_1}\cdots(\omega^l)_{a_l]}(\omega^{l+1})_{a_{l+1}]}\\ &=(l+1)!(\omega^1)_{[a_1}\cdots(\omega^l)_{a_l}(\omega^{l+1})_{a_{l+1}]}\ ,\end{aligned}$$

最后一步用到了定理 2-6-2(b)— 括号内的同种子括号可随意增删. 因此它对 $l+1$ 成立. 事实上此式可由定义 2 式 (5-1-2) 直接写出：

$$(\omega^1)_{a_1}\wedge\cdots\wedge(\omega^l)_{a_l}=\frac{(1+\cdots+1)!}{1!\cdots1!}(\omega^1)_{[a_1}\cdots(\omega^l)_{a_l]}=l!(\omega^1)_{[a_1}\cdots(\omega^l)_{a_l]}\ .$$

~4. 试证定理 5-1-4.

证 因 $\omega_{a_1\cdots a_l}=\sum_C \omega_{\mu_1\cdots\mu_l}(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$, 其中展开系数 $\omega_{\mu_1\cdots\mu_l}$ 为 0 形式，展开基矢 $(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$ 为 l 形式. 由外微分算符定义 3 式 (5-1-11) 知

$$d_b\omega_{a_1\cdots a_l}=(d\omega)_{ba_1\cdots a_l}=(l+1)\nabla_{[b}\omega_{a_1\cdots a_l]}\ .$$

取式中的 ∇_b 为普通导数 ∂_b, 注意到式 (3-1-10) 的结果： $\partial_b(dx^\mu)_a=0$, 有 $\partial_b[(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}]=0$. 于是

$$\begin{aligned}\nabla_b\omega_{a_1\cdots a_l}=\partial_b\omega_{a_1\cdots a_l}\quad&=\quad\sum_C\partial_b\Big[\omega_{\mu_1\cdots\mu_l}(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}\Big]\\&=\quad\sum_C(\partial_b\omega_{\mu_1\cdots\mu_l})(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}\\&\overset{(3\text{-}1\text{-}2)}{=}\quad\sum_C(d\omega_{\mu_1\cdots\mu_l})_b(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}\ .\end{aligned}$$

式中 $(d\omega_{\mu_1\cdots\mu_l})_b$ 为 1 形式而 $(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$ 为 l 形式. 于是根据楔形积的定义 2 式 (5-1-2) 有

$$\begin{aligned}&\sum_C(d\omega_{\mu_1\cdots\mu_l})_b\wedge(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}\\=\ &\sum_C\frac{(1+l)!}{1!l!}(d\omega_{\mu_1\cdots\mu_l})_{[b}(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l]}\\=\ &(l+1)\sum_C(d\omega_{\mu_1\cdots\mu_l})_{[b}(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l]}\\=\ &(l+1)\nabla_{[b}\omega_{a_1\cdots a_l]}\ .\end{aligned}$$

结合外微分算符的定义 3 式 (5-1-11) 即得定理 5-1-4 式 (5-1-12):

$$(d\omega)_{ba_1\cdots a_l}=(l+1)\nabla_{[b}\omega_{a_1\cdots a_l]}=\sum_C(d\omega_{\mu_1\cdots\mu_l})_b\wedge(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}\ .$$

5. 设 $\boldsymbol{\omega}$ 是 1 形式场，u, v 是矢量场, 试证 $d\boldsymbol{\omega}(u,v)=u(\boldsymbol{\omega}(v))-v(\boldsymbol{\omega}(u))-\boldsymbol{\omega}([u,v])$. 等式左边代表 $d\boldsymbol{\omega}$ 对 u, v 的作用结果，即 $(d\boldsymbol{\omega})_{ab}u^av^b$.

证 由定义 3 式 (5-1-11): $(d\boldsymbol{\omega})_{ab}=2\nabla_{[a}\omega_{b]}=\nabla_a\omega_b-\nabla_b\omega_a$. 于是

$$\text{左边}=(\nabla_a\omega_b-\nabla_b\omega_a)u^av^b=u^av^b\nabla_a\omega_b-u^av^b\nabla_b\omega_a\ .$$

右边中

$$\begin{aligned}u(\boldsymbol{\omega}(v))\ &=\ u(\omega_bv^b)=u^a\nabla_a(\omega_bv^b)=u^av^b\nabla_a\omega_b+u^a\omega_b\nabla_av^b\ ,\\v(\boldsymbol{\omega}(u))\ &=\ v(\omega_bu^b)=v^a\nabla_a(\omega_bu^b)=v^au^b\nabla_a\omega_b+v^a\omega_b\nabla_au^b\ ,\\\boldsymbol{\omega}([u,v])\ &=\ \omega_b(u^a\nabla_av^b-v^a\nabla_au^b)=u^a\omega_b\nabla_av^b-v^a\omega_b\nabla_au^b\ .\end{aligned}$$

于是

$$\text{右边}=u^av^b\nabla_a\omega_b-v^au^b\nabla_a\omega_b=u^av^b\nabla_a\omega_b-u^av^b\nabla_b\omega_a=\text{左边}\ .$$

~6. 设 v^b 和 $\omega_{a_1\cdots a_l}$ 分别是流形 M 上的矢量场和 l 形式场，试证

(a) $\mathcal{L}_v\omega_{a_1\cdots a_l} = d_{a_1}(v^b\omega_{ba_2\cdots a_l}) + (d\omega)_{ba_1\cdots a_l}v^b$.

注：令 $\mu_{a_2\cdots a_l} \equiv v^b\omega_{ba_2\cdots a_l}$, 则 $d_{a_1}\mu_{a_2\cdots a_l}$ 是指 $(d\mu)_{a_1a_2\cdots a_l}$.

(b) $\mathcal{L}_v d\boldsymbol{\omega} = d\mathcal{L}_v\boldsymbol{\omega}$ (这本身就是一个很有用的命题).

提示：(1) 证 (a) 时可先证 $l=2$ 的特例，找到感觉后不难推广至一般情况. (2) 利用 (a) 的结果将使 (b) 的证明变得十分简单.

证 (a) 先看 $l=2$ 的特例. 欲证等式的左边根据定理 4-2-5 式 (4-2-8) 为

$$\mathcal{L}_v\omega_{a_1a_2} = v^b\nabla_b\omega_{a_1a_2} + \omega_{ba_2}\nabla_{a_1}v^b + \omega_{a_1b}\nabla_{a_2}v^b\ .$$

右边第一项为

$$\begin{aligned}
d_{a_1}(v^b\omega_{ba_2}) &\overset{(5\text{-}1\text{-}11)}{=} 2\nabla_{[a_1}(v^b\omega_{|b|a_2]}) = \nabla_{a_1}(v^b\omega_{ba_2}) - \nabla_{a_2}(v^b\omega_{ba_1}) \\
&\overset{(5\text{-}1\text{-}1)}{=} \nabla_{a_1}(v^b\omega_{ba_2}) + \nabla_{a_2}(v^b\omega_{a_1b}) \\
&= \omega_{ba_2}\nabla_{a_1}v^b + v^b\nabla_{a_1}\omega_{ba_2} + \omega_{a_1b}\nabla_{a_2}v^b + v^b\nabla_{a_2}\omega_{a_1b} \\
&\overset{(5\text{-}1\text{-}1)}{=} \omega_{ba_2}\nabla_{a_1}v^b + \omega_{a_1b}\nabla_{a_2}v^b + v^b\nabla_{a_1}\omega_{ba_2} - v^b\nabla_{a_2}\omega_{ba_1} \\
&= \omega_{ba_2}\nabla_{a_1}v^b + \omega_{a_1b}\nabla_{a_2}v^b + 2v^b\nabla_{[a_1}\omega_{|b|a_2]}\ ,
\end{aligned}$$

右边第二项为

$$\begin{aligned}
&(d\omega)_{ba_1a_2}v^b \overset{(5\text{-}1\text{-}11)}{=} 3\nabla_{[b}\omega_{a_1a_2]}\,v^b = 3v^b\nabla_{[b}\omega_{a_1a_2]} \\
=\ & 3\frac{1}{3!}v^b\Big[\nabla_b\omega_{a_1a_2} + \nabla_{a_1}\omega_{a_2b} + \nabla_{a_2}\omega_{ba_1} - \nabla_b\omega_{a_2a_1} - \nabla_{a_2}\omega_{a_1b} - \nabla_{a_1}\omega_{ba_2}\Big] \\
\overset{(5\text{-}1\text{-}1)}{=}\ & \frac{1}{2}v^b\Big[2\nabla_b\omega_{a_1a_2} - 2\nabla_{a_1}\omega_{ba_2} + 2\nabla_{a_2}\omega_{ba_1}\Big] \\
=\ & v^b\nabla_b\omega_{a_1a_2} - 2v^b\nabla_{[a_1}\omega_{|b|a_2]}\ .
\end{aligned}$$

因此相加后

$$\text{右边} = \omega_{ba_2}\nabla_{a_1}v^b + \omega_{a_1b}\nabla_{a_2}v^b + v^b\nabla_b\omega_{a_1a_2} = \text{左边}\ .$$

下面看一般 l 情形. 首先左边根据定理 4-2-5 式 (4-2-8) 为

$$\mathcal{L}_v\omega_{a_1\cdots a_l} = v^b\nabla_b\omega_{a_1\cdots a_l} + \sum_{j=1}^{l}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_2}\nabla_{a_j}v^b\ .$$

右边第一项为

$$d_{a_1}(v^b\omega_{ba_2\cdots a_l}) \overset{(5\text{-}1\text{-}11)}{=} l\,\nabla_{[a_1}(v^b\omega_{|b|a_2\cdots a_l]})\ .$$

注意到

$$[a_1a_2\cdots a_l] = \frac{1}{l!}\sum_{\pi}\delta_\pi a_{\pi(1)}a_{\pi(2)}\cdots a_{\pi(l)}$$

$$= \frac{1}{l}\Big\{a_1[a_2a_3\cdots a_l] - a_2[a_1a_3\cdots a_l] + \cdots + (-1)^{l-1}a_l[a_1a_2\cdots a_{l-1}]\Big\}$$
$$= \frac{1}{l}\sum_{j=1}^{l}(-1)^{j-1}a_j[a_1\cdots a_{j-1}a_{j+1}\cdots a_l]\ ,$$

所以有

$$\begin{aligned}
d_{a_1}(v^b\omega_{ba_2\cdots a_l}) &= l\,\nabla_{[a_1}(v^b\omega_{|b|a_2\cdots a_l]}) \\
&= l\frac{1}{l}\sum_{j=1}^{l}(-1)^{j-1}\nabla_{a_j}(v^b\omega_{b[a_1\cdots a_{j-1}a_{j+1}\cdots a_l]}) \\
&= \sum_{j=1}^{l}(-1)^{j-1}\nabla_{a_j}(v^b\omega_{[ba_1\cdots a_{j-1}a_{j+1}\cdots a_l]}) \\
&\overset{(5\text{-}1\text{-}1)}{=} \sum_{j=1}^{l}\nabla_{a_j}(v^b\omega_{[a_1\cdots a_{j-1}ba_{j+1}\cdots a_l]}) \\
&= \sum_{j=1}^{l}\nabla_{a_j}(v^b\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l}) \\
&= \sum_{j=1}^{l}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l}\nabla_{a_j}v^b + v^b\sum_{j=1}^{l}\nabla_{a_j}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l}\ .
\end{aligned}$$

右边第二项为

$$(d\omega)_{ba_1\cdots a_l}v^b \overset{(5\text{-}1\text{-}11)}{=} (l+1)\nabla_{[b}\omega_{a_1a_2\cdots a_l]}\,v^b = v^b(l+1)\nabla_{[b}\omega_{a_1a_2\cdots a_l]}\ .$$

与前面类似，现在注意到

$$\begin{aligned}
&[ba_1a_2\cdots a_l] \\
=\ & \frac{1}{l+1}\Big\{b[a_1a_2\cdots a_l] - a_1[ba_2\cdots a_l] + \cdots + (-1)^l a_l[ba_1a_2\cdots a_{l-1}]\Big\} \\
=\ & \frac{1}{l+1}b[a_1a_2\cdots a_l] - \frac{1}{l+1}\Big\{a_1[ba_2\cdots a_l] - \cdots - (-1)^l a_l[ba_1a_2\cdots a_{l-1}]\Big\} \\
=\ & \frac{1}{l+1}b[a_1a_2\cdots a_l] - \frac{1}{l+1}\sum_{j=1}^{l}(-1)^{j-1}a_j[ba_1\cdots a_{j-1}a_{j+1}\cdots a_l] \\
=\ & \frac{1}{l+1}b[a_1a_2\cdots a_l] - \frac{1}{l+1}\sum_{j=1}^{l}a_j[a_1\cdots a_{j-1}ba_{j+1}\cdots a_l]\ ,
\end{aligned}$$

所以有

$$\begin{aligned}
(d\omega)_{ba_1\cdots a_l}v^b &= v^b(l+1)\nabla_{[b}\omega_{a_1a_2\cdots a_l]} \\
&= v^b\nabla_b\omega_{[a_1a_2\cdots a_l]} - v^b\sum_{j=1}^{l}\nabla_{a_j}\omega_{[a_1\cdots a_{j-1}ba_{j+1}\cdots a_l]} \\
&= v^b\nabla_b\omega_{a_1a_2\cdots a_l} - v^b\sum_{j=1}^{l}\nabla_{a_j}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l}\ .
\end{aligned}$$

因此待证式的

$$
\begin{aligned}
\text{右边} &= d_{a_1}(v^b\omega_{ba_2\cdots a_l}) + (d\omega)_{ba_1\cdots a_l}v^b \\
&= \sum_{j=1}^{l}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l}\nabla_{a_j}v^b + v^b\sum_{j=1}^{l}\nabla_{a_j}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l} \\
&\quad + v^b\nabla_b\omega_{a_1a_2\cdots a_l} - v^b\sum_{j=1}^{l}\nabla_{a_j}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l} \\
&= \sum_{j=1}^{l}\omega_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_l}\nabla_{a_j}v^b + v^b\nabla_b\omega_{a_1a_2\cdots a_l} = \text{左边}\,.
\end{aligned}
$$

(b) 欲证式的左边：

$$
\begin{aligned}
\mathcal{L}_v d\boldsymbol{\omega} &= \mathcal{L}_v(d\omega)_{ba_1\cdots a_l} \\
&\overset{\text{(a)}}{=} d_b[v^c(d\omega)_{ca_1\cdots a_l}] + [d(d\omega)]_{cba_1\cdots a_l}v^c \\
&= d_b[v^c(d\omega)_{ca_1\cdots a_l}]\,,
\end{aligned}
$$

最后一步用了定理 5-1-5 的结果 $d\circ d=0$. 欲证式的右边：

$$
\begin{aligned}
d\mathcal{L}_v\boldsymbol{\omega} &\overset{\text{(a)}}{=} d_b[d_{a_1}(v^c\omega_{ca_2\cdots a_l}) + (d\omega)_{ca_1\cdots a_l}v^c] \\
&= d_b[v^c(d\omega)_{ca_1\cdots a_l}]\,,
\end{aligned}
$$

这里同样用到了 $d\circ d=0$. 因此得

$$
\mathcal{L}_v d\boldsymbol{\omega} = d\mathcal{L}_v\boldsymbol{\omega} = d_b[v^c(d\omega)_{ca_1\cdots a_l}]\,.
$$

7. 设 O 是 n 维流形 M 上坐标系 $\{x^\mu\}$ 的坐标域 (且 O 同胚于 RR^n), ω_a 是 O 上的 1 形式场，试证

$$
\partial\omega_\mu/\partial x^\nu = \partial\omega_\nu/\partial x^\mu\ (\mu,\nu=1,\cdots,n)\ \text{当且仅当存在}\ f:O\to RR\ \text{使}\ \nabla_a f=\omega_a\,.
$$

提示：仿照 §5.1 推论 5-1-6 的证明.

证 (A) [充分性] 如果 1 形式 $\omega_a=\nabla_a f=d_a f$ 是恰当的，那么根据定理 5-1-5 它必是闭的，有 $0=d_b\omega_a=2\nabla_{[b}\omega_{a]}=\nabla_b\omega_a-\nabla_a\omega_b$, 即 $\nabla_b\omega_a=\nabla_a\omega_b$. 取 ∇_a 为普通导数 ∂_a, 则有 $\partial_b\omega_a=\partial_a\omega_b$. 用坐标系的分量表示就是 $\partial_\nu\omega_\mu=\partial_\mu\omega_\nu$, 即 $\partial\omega_\mu/\partial x^\nu=\partial\omega_\nu/\partial x^\mu$.

(B) [必要性] 如果 $\partial_\nu\omega_\mu=\partial_\mu\omega_\nu$, 于是有 $\partial_b\omega_a=\partial_a\omega_b$ 和 $d_b\omega_a=\nabla_b\omega_a-\nabla_a\omega_b=0$, 即 ω_a 是闭的. 对 RR^n 流形定理 5-1-5 的逆定理成立，所以它必是恰当的，即可表示为 $\omega_a=d_a f=\nabla_a f$.

8. 设 $\{x,y,z\}$ 和 $\{r,\theta,\varphi\}$ 分别为 3 维欧氏空间的笛卡尔坐标系和球坐标系，写出 $dr\wedge d\theta\wedge d\varphi$ 用 $dx\wedge dy\wedge dz$ 的表达式.

解 由 $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, 得

$$
\begin{aligned}
dx &= \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi\ , \\
dy &= \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi\ , \\
dz &= \cos\theta\, dr - r\sin\theta\, d\theta\ .
\end{aligned}
$$

于是

$$
\begin{aligned}
dx\wedge dy &= \sin\theta\cos\varphi\, dr\wedge[r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi] \\
&\quad + r\cos\theta\cos\varphi\, d\theta\wedge[\sin\theta\sin\varphi\, dr + r\sin\theta\cos\varphi\, d\varphi] \\
&\quad - r\sin\theta\sin\varphi\, d\varphi\wedge[\sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta] \\
&= r\sin\theta\cos\theta\sin\varphi\cos\varphi\, dr\wedge d\theta + r\sin^2\theta\cos^2\varphi\, dr\wedge d\varphi \\
&\quad + r\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\wedge dr + r^2\sin\theta\cos\theta\cos^2\varphi\, d\theta\wedge d\varphi \\
&\quad - r\sin^2\theta\sin^2\varphi\, d\varphi\wedge dr - r^2\sin\theta\cos\theta\sin^2\varphi\, d\varphi\wedge d\theta \\
&= r\sin^2\theta\, dr\wedge d\varphi + r^2\sin\theta\cos\theta\, d\theta\wedge d\varphi\ ,
\end{aligned}
$$

以及

$$
\begin{aligned}
dx\wedge dy\wedge dz &= [r\sin^2\theta\, dr\wedge d\varphi + r^2\sin\theta\cos\theta\, d\theta\wedge d\varphi]\wedge[\cos\theta\, dr - r\sin\theta\, d\theta] \\
&= -r^2\sin^3\theta\, dr\wedge d\varphi\wedge d\theta + r^2\sin\theta\cos^2\theta\, d\theta\wedge d\varphi\wedge dr \\
&= r^2\sin\theta\, dr\wedge d\theta\wedge d\varphi\ .
\end{aligned}
$$

反过来写就是

$$
\begin{aligned}
dr\wedge d\theta\wedge d\varphi &= \frac{1}{r^2\sin\theta}\, dx\wedge dy\wedge dz \\
&= \frac{1}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\, dx\wedge dy\wedge dz\ .
\end{aligned}
$$

~9. 连通流形 M 配以洛伦兹号差的度规场 g_{ab} 叫 **时空** (spacetime). 设 F_{ab} 是任意 4 维时空的 2 形式场 (第 6 章将看到电磁场张量 F_{ab} 就是一个 2 形式场), 试证

$$
\frac{1}{2}(F_{ac}F_b{}^c + {}^*F_{ac}\,{}^*F_b{}^c) = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\ ,
$$

其中 ${}^*F_{ac} \equiv ({}^*F)_{ac}$, ${}^*F_b{}^c = g^{ac}\,{}^*F_{ba}$ (此式对研究电磁场有帮助).

证 首先, 对于 4 维流形 (闵氏时空), 2 形式场 F_{ab} 的微分对偶形式仍是 2 形式场:

$$
{}^*F_{ac} \overset{(5\text{-}6\text{-}1)}{=} \frac{1}{2}\, F^{de}\varepsilon_{deac}\ ,
$$

于是有 ${}^*F^{fc} = \frac{1}{2}\, F_{gh}\varepsilon^{ghfc}$ 和

$$
{}^*F_b{}^c = g_{bf}\,{}^*F^{fc} = \frac{1}{2}\, g_{bf}F_{gh}\varepsilon^{ghfc}\ .
$$

计算等式左边的第二项:

$$\begin{aligned}
{}^*F_{ac}\,{}^*F_b{}^c &= \Big(\frac{1}{2}F^{de}\varepsilon_{deac}\Big)\Big(\frac{1}{2}g_{bf}F_{gh}\varepsilon^{ghfc}\Big) \\
&= \frac{1}{4}g_{bf}F^{de}F_{gh}\,[\varepsilon^{ghfc}\varepsilon_{deac}] \\
&\overset{(5\text{-}4\text{-}10)}{=} \frac{1}{4}g_{bf}F^{de}F_{gh}\,[(-1)^1(4-3)!3!\,\delta^{[g}{}_d\delta^h{}_e\delta^{f]}{}_a] \\
&= -\frac{3}{2}F^{de}F_{gh}g_{fb}\,\delta^{[g}{}_d\delta^h{}_e\delta^{f]}{}_a \\
&\overset{(2\text{-}6\text{-}19)}{=} -\frac{3}{2}F^{de}F_{[gh}g_{f]b}\,\delta^g{}_d\delta^h{}_e\delta^f{}_a \\
&= -\frac{3}{2}F^{de}F_{[de}g_{a]b} \\
&= -\frac{3}{2}F^{de}\frac{1}{3}(F_{[de]}g_{ab}+F_{[ad]}g_{eb}+F_{[ea]}g_{db}) \\
&\overset{(2\text{-}6\text{-}20)}{=} -\frac{1}{2}F^{de}(F_{de}g_{ab}+F_{ad}g_{eb}+F_{ea}g_{db}) \\
&= -\frac{1}{2}g_{ab}F_{de}F^{de}-\frac{1}{2}F_{ad}F^d{}_b-\frac{1}{2}F_{ea}F_b{}^e \\
&= -\frac{1}{2}g_{ab}F_{de}F^{de}+\frac{1}{2}F_{ad}F_b{}^d+\frac{1}{2}F_{ae}F_b{}^e \\
&= -\frac{1}{2}g_{ab}F_{cd}F^{cd}+F_{ac}F_b{}^c\,.
\end{aligned}$$

将此结果代入待证等式的左边:

$$\frac{1}{2}\Big(F_{ac}F_b{}^c-\frac{1}{2}g_{ab}F_{cd}F^{cd}+F_{ac}F_b{}^c\Big)=F_{ac}F_b{}^c-\frac{1}{4}g_{ab}F_{cd}F^{cd}\,.$$

等式成立.

*10. 试证 $\hat{\varepsilon}_{a_1\cdots a_{n-1}}\equiv\pm n^b\hat{\varepsilon}_{ba_1\cdots a_{n-1}}$【应为 $\varepsilon_{ba_1\cdots a_{n-1}}$】是 ∂N 上与诱导度规场 h_{ab} 相适配的体元.

证 即要证明 $\hat{\varepsilon}_{a_1\cdots a_{n-1}}=\pm n^b\varepsilon_{ba_1\cdots a_{n-1}}$ 满足式 (5-5-5):

$$\hat{\varepsilon}^{a_1\cdots a_{n-1}}\hat{\varepsilon}_{a_1\cdots a_{n-1}}=h^{a_1b_1}\cdots h^{a_{n-1}b_{n-1}}\hat{\varepsilon}_{b_1\cdots b_{n-1}}\hat{\varepsilon}_{a_1\cdots a_{n-1}}=(-1)^{\hat{s}}(n-1)!\,,$$

其中 $\hat{s}$ 为 N 上的度规 g_{ab} 在超曲面 ∂N 上的诱导度规 $h_{ab}=g_{ab}\mp n_an_b$ 的对角元的负数的个数. 首先

$$\begin{aligned}
&h^{a_1b_1}\cdots h^{a_{n-1}b_{n-1}}\hat{\varepsilon}_{b_1\cdots b_{n-1}}\hat{\varepsilon}_{a_1\cdots a_{n-1}} \\
=\ &h^{a_1b_1}\cdots h^{a_{n-1}b_{n-1}}(\pm n^d\varepsilon_{db_1\cdots b_{n-1}})(\pm n^c\varepsilon_{ca_1\cdots a_{n-1}}) \\
=\ &n^cn^dh^{a_1b_1}\cdots h^{a_{n-1}b_{n-1}}\varepsilon_{db_1\cdots b_{n-1}}\varepsilon_{ca_1\cdots a_{n-1}} \\
=\ &n^cn^d(g^{a_1b_1}\mp n^{a_1}n^{b_1})\cdots(g^{a_{n-1}b_{n-1}}\mp n^{a_{n-1}}n^{b_{n-1}})\varepsilon_{db_1\cdots b_{n-1}}\varepsilon_{ca_1\cdots a_{n-1}}\,.
\end{aligned}$$

很容易看出上式中的 $(g^{a_jb_j}\mp n^{a_j}n^{b_j})$ 中的 $n^{a_j}n^{b_j}$ 没有任何贡献, 因为乘开来后必有因子

$$n^cn^dn^{a_j}n^{b_j}\varepsilon_{db_1\cdots b_{n-1}}\varepsilon_{ca_1\cdots a_{n-1}}\quad=\quad n^cn^{a_j}\varepsilon_{ca_j\cdots}\;n^dn^{b_j}\varepsilon_{db_j\cdots}$$

$$\begin{aligned}
&= n^{(c}n^{a_j)}\varepsilon_{[ca_j\cdots a_{n-1}]}\, n^{(d}n^{b_j)}\varepsilon_{[db_j\cdots b_{n-1}]} \\
&\overset{(2\text{-}6\text{-}19)}{=} n^c n^{a_j}\varepsilon_{[(ca_j)\cdots a_{n-1}]}\, n^d n^{b_j}\varepsilon_{[(db_j)\cdots b_{n-1}]} \\
&\overset{(2\text{-}6\text{-}21)}{=} 0\,.
\end{aligned}$$

于是

$$\begin{aligned}
& h^{a_1b_1}\cdots h^{a_{n-1}b_{n-1}}\hat{\varepsilon}_{b_1\cdots b_{n-1}}\hat{\varepsilon}_{a_1\cdots a_{n-1}} \\
= \;& n^c n^d g^{a_1b_1}\cdots g^{a_{n-1}b_{n-1}}\varepsilon_{db_1\cdots b_{n-1}}\varepsilon_{ca_1\cdots a_{n-1}} \\
= \;& n^c n_d \varepsilon^{da_1\cdots a_{n-1}}\varepsilon_{ca_1\cdots a_{n-1}} \\
\overset{(5\text{-}4\text{-}10)}{=} \;& n^c n_d(-1)^s(n-1)!\delta^d{}_c = n^c n_c(-1)^s(n-1)!\,.
\end{aligned}$$

令 $(-1)^{\hat{s}} \equiv n^c n_c(-1)^s$. 如果 ∂N 是类时超曲面， $n^c n_c = +1$, 这时 $\hat{s} = s$; 如果 ∂N 是类空超曲面， $n^c n_c = -1$, 这时 $\hat{s} = s-1$ (见 §4.4 注 3 后的例子).

因此得到 $\hat{\varepsilon}^{a_1\cdots a_{n-1}}\hat{\varepsilon}_{a_1\cdots a_{n-1}} = (-1)^{\hat{s}}(n-1)!$, 超曲面上的体元与超曲面上的诱导度规相适配.

11. 试证定理 5-6-1 和 5-6-2.

证 定理 5-6-1 的证明. 由 §5.6 对偶微分形式的定义 1 式 (5-6-1), 对 l 形式 $\boldsymbol{\omega} = \omega_{a_1\cdots a_l}$, 有

$$\begin{aligned}
{}^{**}\omega_{a_1\cdots a_l} &= \frac{1}{(n-l)!}\,{}^*\omega^{b_1\cdots b_{n-l}}\varepsilon_{b_1\cdots b_{n-l}a_1\cdots a_l} \\
&= \frac{1}{(n-l)!}\left[\frac{1}{l!}\omega_{c_1\cdots c_l}\varepsilon^{c_1\cdots c_l b_1\cdots b_{n-l}}\right]\varepsilon_{b_1\cdots b_{n-l}a_1\cdots a_l} \\
&= \frac{1}{(n-l)!\,l!}\,[\varepsilon^{c_1\cdots c_l b_1\cdots b_{n-l}}\varepsilon_{b_1\cdots b_{n-l}a_1\cdots a_l}]\omega_{c_1\cdots c_l} \\
&= \frac{1}{(n-l)!\,l!}\,(-1)^{l(n-l)}[\varepsilon^{b_1\cdots b_{n-l}c_1\cdots c_l}\varepsilon_{b_1\cdots b_{n-l}a_1\cdots a_l}]\omega_{c_1\cdots c_l} \\
&\overset{(5\text{-}4\text{-}10)}{=} \frac{1}{(n-l)!\,l!}\,(-1)^{l(n-l)}[(-1)^s l!(n-l)!\delta^{[c_1}{}_{a_1}\cdots\delta^{c_l]}{}_{a_l}]\omega_{c_1\cdots c_l} \\
&= (-1)^{s+l(n-l)}\,\delta^{[c_1}{}_{a_1}\cdots\delta^{c_l]}{}_{a_l}\omega_{c_1\cdots c_l} \\
&\overset{(2\text{-}6\text{-}19)}{=} (-1)^{s+l(n-l)}\,\delta^{c_1}{}_{a_1}\cdots\delta^{c_l}{}_{a_l}\omega_{[c_1\cdots c_l]} \\
&= (-1)^{s+l(n-l)}\,\omega_{a_1\cdots a_l}\,,
\end{aligned}$$

此即定理 5-6-1 式 (5-6-2)

$$^{**}\boldsymbol{\omega} = (-1)^{s+l(n-l)}\boldsymbol{\omega}\,.$$

定理 5-6-2 的证明. 设 f 和 $\vec{A}$ 是 3 维欧氏空间的函数和矢量场，则

$$\text{(a) grad}\, f = df\,, \qquad \text{(b) curl}\,\vec{A} = {}^*d\boldsymbol{A}\,, \qquad \text{(c) div}\,\vec{A} = {}^*d({}^*\boldsymbol{A})\,.$$

(a) f 为 0 形式场， df 为 1 形式场，有

$$(df)_a = \nabla_a f = \partial_a f = (\vec{\nabla} f)_a = (\text{grad}\, f)_a\ .$$

(b) $\boldsymbol{A}$ 为 1 形式场， $d\boldsymbol{A}$ 为 2 形式场，故 ${}^*d\boldsymbol{A}$ 为 1 形式场．由外微分定义式 (5-1-11)

$$(dA)_{ba} = d_b A_a = 2\nabla_{[b} A_{a]} = 2\partial_{[b} A_{a]}\ ,$$

以及对偶微分形式定义式 (5-6-1)

$$\begin{aligned}
{}^*(dA)_c &= \frac{1}{2!}\,(dA)^{ba}\,\varepsilon_{bac} = \frac{1}{2}\,2\partial^{[b}A^{a]}\,\varepsilon_{bac} \\
&= \varepsilon_{abc}\,\partial^{[a}A^{b]} = \varepsilon_{[abc]}\,\partial^{[a}A^{b]} \\
&\overset{(2\text{-}6\text{-}19)}{=} \varepsilon_{[[ab]c]}\,\partial^a A^b \overset{(2\text{-}6\text{-}20)}{=} \varepsilon_{[abc]}\,\partial^a A^b \\
&= \varepsilon_{abc}\,\partial^a A^b \overset{(5\text{-}6\text{-}5)(\text{c})}{=} (\vec{\nabla}\times\vec{A})_c \\
&= (\text{curl}\,\vec{A})_c\ .
\end{aligned}$$

(c) $\boldsymbol{A}$ 为 1 形式场， ${}^*\boldsymbol{A}$ 为 2 形式场， $d({}^*\boldsymbol{A})$ 为 3 形式场， ${}^*d({}^*\boldsymbol{A})$ 为 0 形式场 (标量场)．首先 A_a 的对偶微分形式

$${}^*A_{bc} \overset{(5\text{-}6\text{-}1)}{=} A^a \varepsilon_{abc}\ ,$$

它的外微分为

$$d({}^*A)_{dbc} \overset{(5\text{-}1\text{-}11)}{=} 3\partial_{[d}\,{}^*A_{bc]} = 3\partial_{[d}A^a\varepsilon_{|a|bc]}\ ,$$

再取对偶微分形式

$$\begin{aligned}
{}^*d({}^*A) &\overset{(5\text{-}6\text{-}1)}{=} \frac{1}{3!}\,d({}^*A)^{dbc}\,\varepsilon_{dbc} = \frac{1}{3!}\left(3\partial^{[d}A_a\varepsilon^{|a|bc]}\right)\varepsilon_{dbc} \\
&= \frac{1}{2}\left(\partial^{[d}A_a\varepsilon^{|a|bc]}\right)\varepsilon_{dbc} \overset{(5\text{-}4\text{-}6)}{=} \frac{1}{2}\,\partial^{[d}\left(A_a\varepsilon^{|a|bc]}\varepsilon_{dbc}\right) \\
&\overset{(2\text{-}6\text{-}19)}{=} \frac{1}{2}\,\partial^d\left(A_a\varepsilon^{abc}\varepsilon_{[dbc]}\right) = \frac{1}{2}\,\partial^d\left(A_a\varepsilon^{abc}\varepsilon_{dbc}\right) \\
&\overset{(5\text{-}4\text{-}10)}{=} \frac{1}{2}\,\partial^d\left(A_a 2\delta^a{}_d\right) = \partial^a A_a = \partial_a A^a \\
&\overset{(5\text{-}6\text{-}5)(\text{b})}{=} \vec{\nabla}\cdot\vec{A} = \text{div}\,\vec{A}\ .
\end{aligned}$$

~12. 设 x, y, z 是 3 维欧氏空间的笛卡尔坐标，试证

(a) ${}^*dx = dy\wedge dz$;

(b) ${}^*(dx\wedge dy\wedge dz) = 1$.

证 (a) 1 形式 $(dx)_a$ 的对偶微分形式

$${}^*(dx)_{bc} \overset{(5\text{-}6\text{-}1)}{=} (\partial/\partial x)^a\varepsilon_{abc}\ ,$$

而 3 维笛卡尔坐标系的适配体元 (右手) 为

$$\varepsilon_{abc} \overset{(5\text{-}4\text{-}4)}{=} (dx)_a \wedge (dy)_b \wedge (dz)_c = (dx \wedge dy \wedge dz)_{abc} ,$$

于是

$$^*(dx)_{bc} = (\partial/\partial x)^a (dx \wedge dy \wedge dz)_{abc} \overset{(5\text{-}1\text{-}2)}{=} (\partial/\partial x)^a 3!(dx)_{[a}(dy)_b(dz)_{c]} .$$

而

$$[abc] = \frac{1}{3}(a[bc] + b[ca] + c[ab]) ,$$

利用 $(\partial/\partial x)^a(dx)_a = 1$ 和 $(\partial/\partial x)^a(dy)_a = (\partial/\partial x)^a(dz)_a = 0$ 得

$$\begin{aligned} ^*(dx)_{bc} &= 2(\partial/\partial x)^a\Big[(dx)_a(dy)_{[b}(dz)_{c]} + (dx)_b(dy)_{[c}(dz)_{a]} + (dx)_c(dy)_{[a}(dz)_{b]}\Big] \\ &= 2(dy)_{[b}(dz)_{c]} \overset{(5\text{-}1\text{-}2)}{=} (dy)_b \wedge (dz)_c = (dy \wedge dz)_{bc} , \end{aligned}$$

此即 $^*dx = dy \wedge dz$.

(b) 3 形式 $(dx \wedge dy \wedge dz)_{abc} = (dx)_a \wedge (dy)_b \wedge (dz)_c$, 其实它就是 3 维笛卡尔坐标系的适配右手体元 ε_{abc} [见 (5-4-4) 式]. 其对偶微分形式为 0 形式:

$$\begin{aligned} ^*(dx \wedge dy \wedge dz) &\overset{(5\text{-}6\text{-}1)}{=} \frac{1}{3!}(\partial/\partial x)^a \wedge (\partial/\partial y)^b \wedge (\partial/\partial z)^c \varepsilon_{abc} \\ &\overset{(5\text{-}4\text{-}4)}{=} \frac{1}{3!}[(\partial/\partial x)^a \wedge (\partial/\partial y)^b \wedge (\partial/\partial z)^c][(dx)_a \wedge (dy)_b \wedge (dz)_c] \\ &\overset{(5\text{-}1\text{-}2)}{=} 3![(\partial/\partial x)^{[a}(\partial/\partial y)^b(\partial/\partial z)^{c]}][(dx)_{[a}(dy)_b(dz)_{c]}] \\ &\overset{(2\text{-}6\text{-}19)}{=} 3![(\partial/\partial x)^a(\partial/\partial y)^b(\partial/\partial z)^c][(dx)_{[a}(dy)_b(dz)_{c]}] \\ &= 3![(\partial/\partial y)^b(\partial/\partial z)^c]\frac{1}{3}[(dy)_{[b}(dz)_{c]}] \\ &= 2[(\partial/\partial z)^c]\frac{1}{2}[(dz)_c] \\ &= 1 . \end{aligned}$$

其实可以利用 (5-4-3) 式的结果直接得到，因为对于正交归一的笛卡尔系 $(\partial/\partial x)^a \wedge (\partial/\partial y)^b \wedge (\partial/\partial z)^c = \varepsilon^{abc}$,

$$^*(dx \wedge dy \wedge dz) = \frac{1}{3!}\varepsilon^{abc}\varepsilon_{abc} \overset{(5\text{-}4\text{-}3)}{=} \frac{1}{3!}3! = 1 .$$

13. 设 $\{r, \theta, \varphi\}$ 是 3 维欧氏空间的球坐标系，试证 $^*dr = (r^2\sin\theta)\, d\theta \wedge d\varphi$.

证 首先，根据式 (5-4-4), 3 维欧氏空间球坐标系的右手适配体元为 (因 $g = \det(g_{\mu\nu}) = r^4\sin^2\theta$)

$$\varepsilon_{abc} = r^2\sin\theta\,(dr)_a \wedge (d\theta)_b \wedge (d\varphi)_c = r^2\sin\theta\,(dr \wedge d\theta \wedge d\varphi)_{abc} .$$

形式 1 $(dr)_a$ 的对偶微分形式为形式 2:

$$\begin{aligned} {}^*(dr)_{bc} &\overset{(5\text{-}6\text{-}1)}{=} (\partial/\partial r)^a\,\varepsilon_{abc} = (\partial/\partial r)^a\, r^2\sin\theta\,(dr)_a\wedge(d\theta)_b\wedge(d\varphi)_c \\ &= r^2\sin\theta\,(d\theta)_b\wedge(d\varphi)_c = r^2\sin\theta\,(d\theta\wedge d\varphi)_{bc}\ , \end{aligned}$$

即为 ${}^*dr = (r^2\sin\theta)\,d\theta\wedge d\varphi$.

14. 设 $\vec{A}, \vec{B}$ 为 RR^3 上的矢量场, $\vec{\nabla}$ 为 RR^3 上与欧氏度规相适配的导数算符, 试证

$$\vec{\nabla}\times(\vec{A}\times\vec{B}\,) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{\nabla}\cdot\vec{B})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} - (\vec{\nabla}\cdot\vec{A})\vec{B}\ .$$

证 利用 $(\vec{A}\times\vec{B})^c = \varepsilon^{cab}A_aB_b$ 和 $(\vec{\nabla}\times\vec{B})^c = \varepsilon^{cab}\partial_aB_b$, 上式左边为

$$\begin{aligned} [\vec{\nabla}\times(\vec{A}\times\vec{B}\,)]^c &= \varepsilon^{cab}\partial_a(\vec{A}\times\vec{B}\,)_b = \varepsilon^{cab}\partial_a(\varepsilon_{bde}A^dB^e) \\ &\overset{(5\text{-}4\text{-}6)}{=} \varepsilon^{cab}\varepsilon_{bde}\partial_a(A^dB^e) = \varepsilon^{bca}\varepsilon_{bde}\partial_a(A^dB^e) \\ &\overset{(5\text{-}4\text{-}10)}{=} 2\delta^{[c}{}_d\delta^{a]}{}_e\partial_a(A^dB^e) \\ &= (\delta^c{}_d\delta^a{}_e - \delta^a{}_d\delta^c{}_e)\partial_a(A^dB^e) \\ &= \partial_a(A^cB^a) - \partial_a(A^aB^c) \\ &= B^a\partial_aA^c + (\partial_aB^a)A^c - A^a\partial_aB^c - (\partial_aA^a)B^c \\ &= (\vec{B}\cdot\vec{\nabla})A^c + (\vec{\nabla}\cdot\vec{B})A^c - (\vec{A}\cdot\vec{\nabla})B^c - (\vec{\nabla}\cdot\vec{A})B^c \\ &= [(\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{\nabla}\cdot\vec{B})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} - (\vec{\nabla}\cdot\vec{A})\vec{B}]^c \end{aligned}$$

即为上式右边.

15. 用微分形式证明 3 维欧氏空间场论中并不易证的下列熟知命题:

(1) 无旋矢量场必可表为梯度;

(2) 无散矢量场必可表为旋度 (见 §5.6 末).

证 (1) 由定理 5-6-2 的 $\operatorname{curl}\vec{A} = {}^*d\boldsymbol{A}$ 知, 如果 $\operatorname{curl}\vec{A} = 0$, 则 ${}^*d\boldsymbol{A} = 0$, 有 $d\boldsymbol{A} = 0$, 即 1 形式场 $\boldsymbol{A}$ 是闭的. 而对于平凡流形 RR^3, 闭形式场必恰当 (即定理 5-1-5 的逆成立), 因此 $\boldsymbol{A}$ 必为某 0 形式场 (标量场) ϕ 的外微分, 即有

$$A^a = A_a = d_a\phi \overset{(5\text{-}1\text{-}11)}{=} \nabla_a\phi = \partial_a\phi\ ,$$

此即 $\vec{A} = \vec{\nabla}\phi$.

(2) 由定理 5-6-2 的 $\operatorname{div}\vec{A} = {}^*d({}^*\boldsymbol{A})$ 知, 如果 $\operatorname{div}\vec{A} = 0$, 则 ${}^*d({}^*\boldsymbol{A}) = 0$, 有 $d({}^*\boldsymbol{A}) = 0$, 即 2 形式场 ${}^*\boldsymbol{A}$ 是闭的. 而对于平凡流形 RR^3, 闭形式场必恰当 (即定理 5-1-5 的逆成立), 因此 ${}^*\boldsymbol{A}$ 必为某 1 形式场 B_a 的外微分, 即有

$$^*A_{bc} = d_bB_c \overset{(5\text{-}1\text{-}11)}{=} 2\nabla_{[b}B_{c]} = 2\partial_{[b}B_{c]}\ ,$$

于是

$$
\begin{aligned}
{}^{**}A^a &\overset{(5\text{-}6\text{-}1)}{=} \frac{1}{2}\,{}^*A_{bc}\varepsilon^{bca} = \partial_{[b}B_{c]}\varepsilon^{bca} \overset{(2\text{-}6\text{-}19)}{=} \partial_b B_c \varepsilon^{[bc]a} \\
&\overset{(2\text{-}6\text{-}20)}{=} \partial_b B_c \varepsilon^{bca} \overset{(5\text{-}6\text{-}5)(\mathrm{c})}{=} (\vec{\nabla}\times\vec{B}\,)^a \,.
\end{aligned}
$$

最后，由定理 5-6-1 式 (5-6-2) 知

$$
{}^{**}A^a = (-1)^{0+1(3-1)}A^a = A^a \,,
$$

因此总有 $\vec{A} = \vec{\nabla}\times\vec{B}$.

16. 设 ∇_a 是广义黎曼空间 (M, g_{ab}) 上的适配导数算符 (即 $\nabla_a g_{bc} = 0$), $\boldsymbol{\varepsilon}$ 是适配体元 (即 $\nabla_a \varepsilon_{b_1\cdots b_n} = 0$), v^a 是 M 上是矢量场， $v_a \equiv g_{ab}v^b$ 是 v^a 相应的 1 形式场， ${}^*\boldsymbol{v}$ 是 v_a 的对偶形式场，试证 $(\nabla_a v^a)\boldsymbol{\varepsilon} = d{}^*\boldsymbol{v}$.

注：这个结论可做如下推广：设 $F_{a_1\cdots a_k}$ 是 k 形式场 $(k\le n)$, 简记作 $\boldsymbol{F}$, 把 $k-1$ 形式场 $\nabla^{a_k}F_{a_1\cdots a_k}$ 记作 $\operatorname{div}\boldsymbol{F}$, 则 ${}^*(\operatorname{div}\boldsymbol{F}) = d{}^*\boldsymbol{F}$. 电磁场的麦氏方程 [式 (12-6-2)] 就是一例.

证 1 形式场 v_a 的对偶微分形式 $[(n-1)$ 形式] 为

$$
{}^*v_{b_1\cdots b_{n-1}} \overset{(5\text{-}6\text{-}1)}{=} v^c \varepsilon_{cb_1\cdots b_{n-1}} \,.
$$

它的外微分 (n 形式) 为

$$
d_a{}^*v_{b_1\cdots b_{n-1}} = (d{}^*v)_{ab_1\cdots b_{n-1}} \overset{(5\text{-}1\text{-}11)}{=} n\nabla_{[a}{}^*v_{b_1\cdots b_{n-1}]} = n\nabla_{[a}(v^c\varepsilon_{|c|b_1\cdots b_{n-1}]}) \,.
$$

因为 n 维流形的 n 形式的集合是 1 维矢量空间，故它与 $\varepsilon_{ab_1\cdots b_{n-1}}$ 应该只差一个因子 (设为 h), 即有

$$
n\nabla_{[a}(v^c\varepsilon_{|c|b_1\cdots b_{n-1}]}) = h\varepsilon_{ab_1\cdots b_{n-1}} \,.
$$

用 $\varepsilon^{ab_1\cdots b_{n-1}}$ 缩并此式，右边得 $h(-1)^s n!$ [适配体元的性质式 (5-4-3)], 而左边为

$$
\begin{aligned}
n\varepsilon^{ab_1\cdots b_{n-1}}\nabla_{[a}(v^c\varepsilon_{|c|b_1\cdots b_{n-1}]}) &\overset{(2\text{-}6\text{-}19)}{=} n\varepsilon^{[ab_1\cdots b_{n-1}]}\nabla_a(v^c\varepsilon_{cb_1\cdots b_{n-1}}) \\
&= n\varepsilon^{ab_1\cdots b_{n-1}}\nabla_a(v^c\varepsilon_{cb_1\cdots b_{n-1}}) \\
&\overset{(5\text{-}4\text{-}6)}{=} n\varepsilon^{ab_1\cdots b_{n-1}}\varepsilon_{cb_1\cdots b_{n-1}}\nabla_a v^c \\
&\overset{(5\text{-}4\text{-}10)}{=} n[(-1)^s(n-1)!\delta^a{}_c]\nabla_a v^c \\
&= (-1)^s n!\nabla_c v^c \,.
\end{aligned}
$$

因此知 $h = \nabla_c v^c$. 代入前面的结果有

$$
d_a{}^*v_{b_1\cdots b_{n-1}} = (\nabla_c v^c)\varepsilon_{ab_1\cdots b_{n-1}} \,,
$$

可简记作 $d^*\boldsymbol{v}=(\nabla_a v^a)\boldsymbol{\varepsilon}$.

推广到 k 形式场 $\boldsymbol{F}=F_{a_1\cdots a_k}$, 其对偶微分 $(n-k)$ 形式为

$$^*F_{b_1\cdots b_{n-k}} \overset{(5\text{-}6\text{-}1)}{=} \frac{1}{k!}F^{c_1\cdots c_k}\varepsilon_{c_1\cdots c_k b_1\cdots b_{n-k}}\ .$$

取其外微分 $(n-k+1)$ 形式

$$\begin{aligned}(d^*\boldsymbol{F})_{ab_1\cdots b_{n-k}}=(d^*F)_{ab_1\cdots b_{n-k}} &= d_a{}^*F_{b_1\cdots b_{n-k}} \overset{(5\text{-}1\text{-}11)}{=} (n-k+1)\nabla_{[a}{}^*F_{b_1\cdots b_{n-k}]}\\ &= (n-k+1)\nabla_{[a}\Big(\frac{1}{k!}F^{c_1\cdots c_k}\varepsilon_{|c_1\cdots c_k|b_1\cdots b_{n-k}]}\Big)\\ &= \frac{(n-k+1)}{k!}\nabla_{[a}(F^{c_1\cdots c_k}\varepsilon_{|c_1\cdots c_k|b_1\cdots b_{n-k}]})\ .\end{aligned}$$

另一方面, $(k-1)$ 形式场 $\nabla^{c_k}F_{c_1\cdots c_{k-1}c_k}=\operatorname{div}\boldsymbol{F}$ 的对偶微分形式为 $(n-k+1)$ 形式:

$$^*(\operatorname{div}\boldsymbol{F})_{ab_1\cdots b_{n-k}} \overset{(5\text{-}6\text{-}1)}{=} \frac{1}{(k-1)!}(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})\varepsilon_{c_1\cdots c_{k-1}ab_1\cdots b_{n-k}}\ .$$

欲证 $^*(\operatorname{div}\boldsymbol{F})=d^*\boldsymbol{F}$ 即证

$$\begin{aligned}&\frac{(n-k+1)}{k!}\nabla_{[a}(F^{c_1\cdots c_k}\varepsilon_{|c_1\cdots c_k|b_1\cdots b_{n-k}]})\\ =&\ \frac{1}{(k-1)!}(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})\varepsilon_{c_1\cdots c_{k-1}ab_1\cdots b_{n-k}}\ ,\end{aligned}$$

即

$$\begin{aligned}&(n-k+1)\nabla_{[a}(F^{c_1\cdots c_k}\varepsilon_{|c_1\cdots c_k|b_1\cdots b_{n-k}]})\\ =&\ k(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})\varepsilon_{c_1\cdots c_{k-1}ab_1\cdots b_{n-k}}\ .\end{aligned}$$

亦即

$$\begin{aligned}&(n-k+1)\nabla_{[a}(F^{c_1\cdots c_k}\varepsilon_{b_1\cdots b_{n-k}]c_1\cdots c_k})\\ =&\ k(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_{k-1}a}\ .\end{aligned}$$

两边作用 $\varepsilon^{ab_1\cdots b_{n-k}d_1\cdots d_{k-1}}$

$$\begin{aligned}&(n-k+1)\nabla_{[a}(F^{c_1\cdots c_k}\varepsilon_{b_1\cdots b_{n-k}]c_1\cdots c_k}\varepsilon^{ab_1\cdots b_{n-k}d_1\cdots d_{k-1}})\\ =&\ (n-k+1)\nabla_a(F^{c_1\cdots c_k}\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_k}\varepsilon^{[ab_1\cdots b_{n-k}]d_1\cdots d_{k-1}})\\ =&\ (n-k+1)\nabla_a(F^{c_1\cdots c_k}\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_k}\varepsilon^{ab_1\cdots b_{n-k}d_1\cdots d_{k-1}})\\ =&\ k(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_{k-1}a}\varepsilon^{ab_1\cdots b_{n-k}d_1\cdots d_{k-1}}\ .\end{aligned}$$

即

$$\begin{aligned}&(n-k+1)\nabla_a(F^{c_1\cdots c_k}\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_k}\varepsilon^{b_1\cdots b_{n-k}d_1\cdots d_{k-1}a})\\ =&\ k(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_{k-1}a}\varepsilon^{b_1\cdots b_{n-k}d_1\cdots d_{k-1}a}\ .\end{aligned}$$

利用式 (5-4-10)

$$\begin{aligned}
\varepsilon^{b_1\cdots b_{n-k}d_1\cdots d_{k-1}a}\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_k} &= (-1)^s k!(n-k)!\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a]}{}_{c_k}\ , \\
\varepsilon^{b_1\cdots b_{n-k}d_1\cdots d_{k-1}a}\varepsilon_{b_1\cdots b_{n-k}c_1\cdots c_{k-1}a} &= \varepsilon^{ab_1\cdots b_{n-k}d_1\cdots d_{k-1}}\varepsilon_{ab_1\cdots b_{n-k}c_1\cdots c_{k-1}} \\
&= (-1)^s(k-1)!(n-k+1)!\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}]}{}_{c_{k-1}}\ .
\end{aligned}$$

待证式左边为

$$\begin{aligned}
&(n-k+1)\nabla_a\left(F^{c_1\cdots c_k}(-1)^s k!(n-k)!\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a]}{}_{c_k}\right) \\
=\ &(-1)^s k!(n-k+1)!\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a]}{}_{c_k}\nabla_a F^{c_1\cdots c_k}\ ,
\end{aligned}$$

待证式右边为

$$\begin{aligned}
&k(\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k})(-1)^s(k-1)!(n-k+1)!\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}]}{}_{c_{k-1}} \\
=\ &(-1)^s k!(n-k+1)!\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}]}{}_{c_{k-1}}\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k}\ .
\end{aligned}$$

于是待证式变为

$$\delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a]}{}_{c_k}\nabla_a F^{c_1\cdots c_k} = \delta^{[d_1}{}_{c_1}\cdots\delta^{d_{k-1}]}{}_{c_{k-1}}\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k}\ .$$

这个等式是显然的，因为

$$\begin{aligned}
\text{左边} &= \delta^{d_1}{}_{[c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a}{}_{c_k]}\nabla_a F^{c_1\cdots c_k} \\
&\overset{(2\text{-}6\text{-}19)}{=} \delta^{d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a}{}_{c_k}\nabla_a F^{[c_1\cdots c_k]} \\
&= \delta^{d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\delta^{a}{}_{c_k}\nabla_a F^{c_1\cdots c_k} = \nabla_a F^{d_1\cdots d_{k-1}a}\ , \\
\text{右边} &= \delta^{d_1}{}_{[c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}]}\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k} \\
&\overset{(2\text{-}6\text{-}19)}{=} \delta^{d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\nabla_{c_k}F^{[c_1\cdots c_{k-1}]c_k} \\
&= \delta^{d_1}{}_{c_1}\cdots\delta^{d_{k-1}}{}_{c_{k-1}}\nabla_{c_k}F^{c_1\cdots c_{k-1}c_k} = \nabla_{c_k}F^{d_1\cdots d_{k-1}c_k}\ .
\end{aligned}$$

命题得证.

17. 试证由式 (5-7-2) 定义的 $\Gamma^\sigma{}_{\mu\tau}$ 正是 §3.1 定义的克氏符 $\Gamma^c{}_{ab}$ 在式 (5-7-2) 涉及的坐标基底的分量.

证 克氏符的定义为 §3.1 定义 2: $\nabla_a\omega_b = \partial_b\omega_a - \Gamma^c{}_{ab}\omega_c$ [见 (3-1-6)]. 取这里的 ω_a 为对偶坐标基矢 $(dx^\nu)_a$, 则有

$$\nabla_a(dx^\nu)_b = \partial_b(dx^\nu)_a - \Gamma^c{}_{ab}(dx^\nu)_c\ .$$

以基矢 $(\partial/\partial x^\mu)^a(\partial/\partial x^\nu)^d$ 左作用:

$$\begin{aligned}
&(\partial/\partial x^\mu)^a(\partial/\partial x^\nu)^d\nabla_a(dx^\nu)_b \\
=\ &(\partial/\partial x^\mu)^a(\partial/\partial x^\nu)^d\partial_b(dx^\nu)_a - (\partial/\partial x^\mu)^a(\partial/\partial x^\nu)^d\Gamma^c{}_{ab}(dx^\nu)_c \\
\overset{(3\text{-}1\text{-}10)}{=}\ &(\partial/\partial x^\mu)^a\partial_b[(\partial/\partial x^\nu)^d(dx^\nu)_a] - (\partial/\partial x^\mu)^a[(\partial/\partial x^\nu)^d(dx^\nu)_c]\Gamma^c{}_{ab} \\
=\ &(\partial/\partial x^\mu)^a\partial_b\delta^d{}_a - (\partial/\partial x^\mu)^a\delta^d{}_c\Gamma^c{}_{ab} = -(\partial/\partial x^\mu)^a\Gamma^d{}_{ab}\ ,
\end{aligned}$$

得到 [其实根据式 (3-1-10), 直接有 $\partial_b(dx^\nu)_a=0$]

$$(\partial/\partial x^\mu)^a\Gamma^d{}_{ab}=-(\partial/\partial x^\mu)^a(\partial/\partial x^\nu)^d\nabla_a(dx^\nu)_b\ .$$

注意到

$$0=\nabla_a\delta^d{}_b=\nabla_a[(\partial/\partial x^\nu)^d(dx^\nu)_b]=(\partial/\partial x^\nu)^d\nabla_a(dx^\nu)_b+(dx^\nu)_b\nabla_a(\partial/\partial x^\nu)^d\ ,$$

所以

$$(\partial/\partial x^\mu)^a\Gamma^d{}_{ab}=(\partial/\partial x^\mu)^a(dx^\nu)_b\nabla_a(\partial/\partial x^\nu)^d\ .$$

克氏符的坐标分量

$$\begin{aligned}\Gamma^\sigma{}_{\mu\tau} &= (\partial/\partial x^\tau)^b(dx^\sigma)_d(\partial/\partial x^\mu)^a\Gamma^d{}_{ab}\\ &= (\partial/\partial x^\tau)^b(dx^\sigma)_d(\partial/\partial x^\mu)^a(dx^\nu)_b\nabla_a(\partial/\partial x^\nu)^d\\ &= (dx^\sigma)_d(\partial/\partial x^\mu)^a\delta^\nu{}_\tau\nabla_a(\partial/\partial x^\nu)^d\\ &= (dx^\sigma)_d(\partial/\partial x^\mu)^a\nabla_a(\partial/\partial x^\tau)^d\ .\end{aligned}$$

左作用 $(\partial/\partial x^\sigma)^b$:

$$\begin{aligned}(\partial/\partial x^\sigma)^b\Gamma^\sigma{}_{\mu\tau} &= (\partial/\partial x^\sigma)^b(dx^\sigma)_d(\partial/\partial x^\mu)^a\nabla_a(\partial/\partial x^\tau)^d\\ &= \delta^b{}_d(\partial/\partial x^\mu)^a\nabla_a(\partial/\partial x^\tau)^d\\ &= (\partial/\partial x^\mu)^a\nabla_a(\partial/\partial x^\tau)^b\ .\end{aligned}$$

此即式 (5-7-2) 的定义 (因 $\Gamma^\sigma{}_{\mu\tau}=\Gamma^\sigma{}_{\tau\mu}$). 又见第 3 章习题 4.

*18. 用正交归一标架分别求第 3 章习题 14~16 所给度规的曲率张量的全部标架分量,并验证所得结果与用坐标基底法求得的曲率张量相同. 为与 $R_{abc}{}^d$ 的坐标分量 $R_{\mu\nu\sigma}{}^\rho$ 区别, 在求得 $R_{abc}{}^d$ 的全部标架分量后宜改用符号 $R_{(\mu)(\nu)(\sigma)}{}^{(\rho)}$ 代表标架分量.

解 (A) 习题 14.

(a) 选正交归一标架. 线元 $ds^2=\Omega^2(t,x)(-dt^2+dx^2)$, 故非归一坐标基底的度规分量为

$$g_{tt}=-\Omega^2(t,x)\ ,\quad g_{xx}=\Omega^2(t,x)\ ;\qquad g^{tt}=-\Omega^{-2}(t,x)\ ,\quad g^{xx}=\Omega^{-2}(t,x)\ .$$

度规张量场为

$$\begin{aligned}g_{ab} &= g_{tt}(dt)_a(dt)_b+g_{xx}(dx)_a(dx)_b\\ &= \eta_{00}(e^0)_a(e^0)_b+\eta_{11}(e^1)_a(e^1)_b\ ,\\ g^{ab} &= g^{tt}(\partial_t)^a(\partial_t)^b+g^{xx}(\partial_x)^a(\partial_x)^b\\ &= \eta^{00}(e_0)^a(e_0)^b+\eta^{11}(e_1)^a(e_1)^b\ ,\end{aligned}$$

其中 $\{(e_\mu)^a\}$ 和 $\{(e^\mu)_a\}$ $(\mu = 0, 1)$ 为正交归一的基底和对偶基底，即度规分量为 $-\eta_{00} = -\eta^{00} = \eta_{11} = \eta^{11} = 1$. 比较得

$$(e_0)^a = \Omega^{-1}\,(\partial_t)^a\,, \qquad (e_1)^a = \Omega^{-1}\,(\partial_x)^a\,;$$
$$(e^0)_a = \Omega\,(dt)_a\,, \qquad (e^1)_a = \Omega\,(dx)_a\,.$$

用 g_{ab} 降 $(e_\mu)^b$ 或用 $\eta_{\mu\nu}$ 降 $(e^\nu)_a$, 如： $(e_0)_a = g_{ab}(e_0)^b = \eta_{00}(e^0)_a(e^0)_b(e_0)^b = -(e^0)_a = -\Omega\,(dt)_a$, 或 $(e_0)_a = \eta_{0\nu}(e^\nu)_a = -(e^0)_a = -\Omega\,(dt)_a$. 因此

$$(e_0)_a = -\Omega\,(dt)_a\,, \qquad (e_1)_a = \Omega\,(dx)_a\,.$$

(b) 用式 (5-7-19) 计算 $\Lambda_{\mu\nu\rho}$ 和式 (5-7-20) 计算 $\omega_{\mu\nu\rho}$. 注意到反称关系 $\Lambda_{\mu\nu\rho} = -\Lambda_{\rho\nu\mu}$, 现在只有 2 个独立： $\Lambda_{001} = -\Lambda_{100}$ 和 $\Lambda_{011} = -\Lambda_{110}$. 因为

$$\begin{aligned}
(e_0)_\lambda &= (e_0)_a(\partial_\lambda)^a = -\Omega\,(dt)_a(\partial_\lambda)^a = -\Omega\,\delta^0{}_\lambda\,, \\
(e_1)_\lambda &= (e_1)_a(\partial_\lambda)^a = \Omega\,(dx)_a(\partial_\lambda)^a = \Omega\,\delta^1{}_\lambda\,,
\end{aligned}$$

有

$$\begin{aligned}
(e_0)_{\lambda,\tau} &= \partial_\tau(-\Omega\,\delta^0{}_\lambda) = -\delta^0{}_\lambda\delta^0{}_\tau\dot\Omega - \delta^0{}_\lambda\delta^1{}_\tau\Omega'\,, \\
(e_1)_{\lambda,\tau} &= \partial_\tau(\Omega\,\delta^1{}_\lambda) = \delta^1{}_\lambda\delta^0{}_\tau\dot\Omega + \delta^1{}_\lambda\delta^1{}_\tau\Omega'\,.
\end{aligned}$$

代入式 (5-7-19) $\Lambda_{\mu\nu\rho} = [(e_\nu)_{\lambda,\tau} - (e_\nu)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau$:

$$\begin{aligned}
\Lambda_{\mu0\rho} &= [(e_0)_{\lambda,\tau} - (e_0)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau \\
&= [-\delta^0{}_\lambda\delta^0{}_\tau\dot\Omega - \delta^0{}_\lambda\delta^1{}_\tau\Omega' + \delta^0{}_\tau\delta^0{}_\lambda\dot\Omega + \delta^0{}_\tau\delta^1{}_\lambda\Omega'](e_\mu)^\lambda(e_\rho)^\tau \\
&= -\dot\Omega\,(e_\mu)^0(e_\rho)^0 - \Omega'\,(e_\mu)^0(e_\rho)^1 + \dot\Omega\,(e_\mu)^0(e_\rho)^0 + \Omega'\,(e_\mu)^1(e_\rho)^0 \\
&= -\Omega'\,(e_\mu)^0(e_\rho)^1 + \Omega'\,(e_\mu)^1(e_\rho)^0\,, \\
\Lambda_{\mu1\rho} &= [(e_1)_{\lambda,\tau} - (e_1)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau \\
&= [\delta^1{}_\lambda\delta^0{}_\tau\dot\Omega + \delta^1{}_\lambda\delta^1{}_\tau\Omega' - \delta^1{}_\tau\delta^0{}_\lambda\dot\Omega - \delta^1{}_\tau\delta^1{}_\lambda\Omega'](e_\mu)^\lambda(e_\rho)^\tau \\
&= \dot\Omega(e_\mu)^1(e_\rho)^0 + \Omega'(e_\mu)^1(e_\rho)^1 - \dot\Omega(e_\mu)^0(e_\rho)^1 - \Omega'(e_\mu)^1(e_\rho)^1 \\
&= \dot\Omega(e_\mu)^1(e_\rho)^0 - \dot\Omega(e_\mu)^0(e_\rho)^1\,.
\end{aligned}$$

得到非零的 $\Lambda_{\mu\nu\rho}$

$$\begin{aligned}
\Lambda_{001} &= -\Lambda_{100} = -\Omega'\,(e_0)^0(e_1)^1 = -\Omega'\,\Omega^{-1}\Omega^{-1} = -\Omega'\,\Omega^{-2}\,, \\
\Lambda_{011} &= -\Lambda_{110} = -\dot\Omega\,(e_0)^0(e_1)^1 = -\dot\Omega\,\Omega^{-1}\Omega^{-1} = -\dot\Omega\,\Omega^{-2}\,.
\end{aligned}$$

代入式 (5-7-20) $\omega_{\mu\nu\rho} = \frac{1}{2}(\Lambda_{\mu\nu\rho} + \Lambda_{\rho\mu\nu} - \Lambda_{\nu\rho\mu})$ 求得非零的 $\omega_{\mu\nu\rho}$ (注意反称关系，非零时 $\mu \neq \nu$):

$$\begin{aligned}
\omega_{010} &= \frac{1}{2}(\Lambda_{010} + \Lambda_{001} - \Lambda_{100}) = \Lambda_{001} = -\Omega'\,\Omega^{-2} = -\omega_{100}\,, \\
\omega_{011} &= \frac{1}{2}(\Lambda_{011} + \Lambda_{101} - \Lambda_{110}) = \Lambda_{011} = -\dot\Omega\,\Omega^{-2} = -\omega_{101}\,.
\end{aligned}$$

联络 1 形式为 $\boldsymbol{\omega}_{\mu\nu} = \omega_{\mu\nu a} = \omega_{\mu\nu\lambda}(e^\lambda)_a = \omega_{\mu\nu\lambda}\boldsymbol{e}^\lambda$:

$$\begin{aligned}\boldsymbol{\omega}_{01} &= \omega_{010}\boldsymbol{e}^0 + \omega_{011}\boldsymbol{e}^1 = -\Omega'\,\Omega^{-2}\boldsymbol{e}^0 - \dot\Omega\,\Omega^{-2}\boldsymbol{e}^1 \\ &= -\Omega'\,\Omega^{-2}\Omega\,dt - \dot\Omega\,\Omega^{-2}\Omega\,dx = -\Omega'\,\Omega^{-1}\,dt - \dot\Omega\,\Omega^{-1}\,dx\,,\end{aligned}$$

只有一个非零的独立元素.

(c) 用嘉当第二结构方程求曲率 2 形式. 由 $\boldsymbol{\omega}_\mu{}^\nu = \eta^{\nu\sigma}\boldsymbol{\omega}_{\mu\sigma}$ 知 $\boldsymbol{\omega}_0{}^1 = \boldsymbol{\omega}_{01}$. 代入式 (5-7-8): $\boldsymbol{R}_\mu{}^\nu = d\boldsymbol{\omega}_\mu{}^\nu + \boldsymbol{\omega}_\mu{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^\nu$, 得黎曼曲率 2 形式

$$\begin{aligned}\boldsymbol{R}_0{}^1 &= d\boldsymbol{\omega}_0{}^1 + \boldsymbol{\omega}_0{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^1 = d\boldsymbol{\omega}_{01} + 0 \\ &= d(-\Omega'\,\Omega^{-1}\,dt - \dot\Omega\,\Omega^{-1}\,dx) \\ &\overset{(5\text{-}1\text{-}12)}{=} \Big(-\frac{\dot\Omega'\Omega - \Omega'\dot\Omega}{\Omega^2}\,dt - \frac{\Omega''\Omega - \Omega'^2}{\Omega^2}\,dx\Big)\wedge dt \\ &\quad +\Big(-\frac{\ddot\Omega\Omega - \dot\Omega^2}{\Omega^2}\,dt - \frac{\dot\Omega'\Omega - \dot\Omega\Omega'}{\Omega^2}\,dx\Big)\wedge dx \\ &= -\frac{\Omega''\Omega - \Omega'^2}{\Omega^2}\,dx\wedge dt - \frac{\ddot\Omega\Omega - \dot\Omega^2}{\Omega^2}\,dt\wedge dx \\ &= \frac{\Omega''\Omega - \Omega'^2 - \ddot\Omega\Omega + \dot\Omega^2}{\Omega^2}\,dt\wedge dx \\ &= \frac{\Omega''\Omega - \Omega'^2 - \ddot\Omega\Omega + \dot\Omega^2}{\Omega^4}\,(\Omega\,dt)\wedge(\Omega\,dx) \\ &= \Big(\frac{\Omega'' - \ddot\Omega}{\Omega^3} + \frac{\dot\Omega^2 - \Omega'^2}{\Omega^4}\Big)\,\boldsymbol{e}^0\wedge\boldsymbol{e}^1 \\ &\equiv R\,\boldsymbol{e}^0\wedge\boldsymbol{e}^1\,.\end{aligned}$$

此式即为

$$(R_0{}^1)_{ab} = R\,(e^0)_a\wedge(e^1)_b = 2R\,(e^0)_{[a}(e^1)_{b]}\,.$$

因此黎曼曲率在正交归一标架基底的分量为

$$\begin{aligned}R_{\mu\nu0}{}^1 &= (R_0{}^1)_{ab}(e_\mu)^a(e_\nu)^b \\ &= 2R\,(e^0)_{[a}(e^1)_{b]}(e_\mu)^a(e_\nu)^b \\ &= 2R\,\delta^0{}_{[\mu}\delta^1{}_{\nu]} \\ &= R\,(\delta^0{}_\mu\delta^1{}_\nu - \delta^0{}_\nu\delta^1{}_\mu)\,,\end{aligned}$$

因此求得黎曼曲率张量

$$R_{010}{}^1 = -R_{100}{}^1 = R = \frac{\Omega'' - \ddot\Omega}{\Omega^3} + \frac{\dot\Omega^2 - \Omega'^2}{\Omega^4}\,.$$

它与第 3 章习题 14 的结果 $R_{txt}{}^x$ 的关系为

$$\begin{aligned}R_{txt}{}^x &= R_{abc}{}^d(\partial_t)^a(\partial_x)^b(\partial_t)^c(dx)_d \\ &= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_t)^a(\partial_x)^b(\partial_t)^c(dx)_d\end{aligned}$$

$$
\begin{aligned}
&= R_{\mu\nu\sigma}{}^{\tau}(e^{\mu})_a(e^{\nu})_b(e^{\sigma})_c(e_{\tau})^d\Omega(e_0)^a\Omega(e_1)^b\Omega(e_0)^c\Omega^{-1}(e^1)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}\Omega^2(e^{\mu})_a(e_0)^a(e^{\nu})_b(e_1)^b(e^{\sigma})_c(e_0)^c(e_{\tau})^d(e^1)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}\Omega^2\delta^{\mu}{}_0\delta^{\nu}{}_1\delta^{\sigma}{}_0\delta^1{}_{\tau} \\
&= R_{010}{}^1\Omega^2 = \frac{\Omega''-\ddot{\Omega}}{\Omega}+\frac{\dot{\Omega}^2-\Omega'^2}{\Omega^2}\ ,
\end{aligned}
$$

即是前面通过坐标基底场的度规张量计算的结果.

(B) 习题 15.

(a) 选正交归一标架. 线元 $ds^2 = z^{-1/2}(-dt^2+dz^2)+z(dx^2+dy^2)$, 故非归一坐标基底的度规分量为

$$
\begin{aligned}
&g_{tt} = -z^{-1/2}\ , \quad g_{zz} = z^{-1/2}\ , \quad g_{xx} = z\ , \quad g_{yy} = z\ ; \\
&g^{tt} = -z^{1/2}\ , \quad g^{zz} = z^{1/2}\ , \quad g^{xx} = z^{-1}\ , \quad g^{yy} = z^{-1}\ .
\end{aligned}
$$

度规张量场为

$$
\begin{aligned}
g_{ab} &= g_{tt}(dt)_a(dt)_b + g_{zz}(dz)_a(dz)_b + g_{xx}(dx)_a(dx)_b + g_{yy}(dy)_a(dy)_b \\
&= \eta_{00}(e^0)_a(e^0)_b + \eta_{33}(e^3)_a(e^3)_b + \eta_{11}(e^1)_a(e^1)_b + \eta_{22}(e^2)_a(e^2)_b\ , \\
g^{ab} &= g^{tt}(\partial_t)^a(\partial_t)^b + g^{zz}(\partial_z)^a(\partial_z)^b + g^{xx}(\partial_x)^a(\partial_x)^b + g^{yy}(\partial_y)^a(\partial_y)^b \\
&= \eta^{00}(e_0)^a(e_0)^b + \eta^{33}(e_3)^a(e_3)^b + \eta^{11}(e_1)^a(e_1)^b + \eta^{22}(e_2)^a(e_2)^b\ ,
\end{aligned}
$$

其中 $\{(e_{\mu})^a\}$ 和 $\{(e^{\mu})_a\}$ $(\mu = 0, 1, 2, 3)$ 为正交归一的基底和对偶基底, 即度规分量为洛伦兹度规 $-\eta_{00} = -\eta^{00} = \eta_{11} = \eta^{11} = \eta_{22} = \eta^{22} = \eta_{33} = \eta^{33} = 1$. 比较得

$$
\begin{aligned}
&(e_0)^a = z^{1/4}\,(\partial_t)^a\ ,\ (e_3)^a = z^{1/4}\,(\partial_z)^a\ ,\ (e_1)^a = z^{-1/2}\,(\partial_x)^a\ ,\ (e_2)^a = z^{-1/2}\,(\partial_y)^a\ ; \\
&(e^0)_a = z^{-1/4}\,(dt)_a\ ,\ (e^3)_a = z^{-1/4}\,(dz)_a\ ,\ (e^1)_a = z^{1/2}\,(dx)^a\ ,\ (e^2)_a = z^{1/2}\,(dy)_a\ .
\end{aligned}
$$

用 g_{ab} 降 $(e_{\mu})^b$ 或用 $\eta_{\mu\nu}$ 降 $(e^{\nu})_a$, 如: $(e_0)_a = g_{ab}(e_0)^b = \eta_{00}(e^0)_a(e^0)_b(e_0)^b = -(e^0)_a = -z^{-1/4}\,(dt)_a$, 或 $(e_0)_a = \eta_{0\nu}(e^{\nu})_a = -(e^0)_a = -z^{-1/4}\,(dt)_a$. 因此

$$
(e_0)_a = -z^{-1/4}\,(dt)_a,\ (e_3)_a = z^{-1/4}\,(dz)_a,\ (e_1)_a = z^{1/2}\,(dx)^a,\ (e_2)_a = z^{1/2}\,(dy)_a\ .
$$

(b) 用式 (5-7-19) 计算 $\Lambda_{\mu\nu\rho}$ 和式 (5-7-20) 计算 $\omega_{\mu\nu\rho}$. 注意到反称关系 $\Lambda_{\mu\nu\rho} = -\Lambda_{\rho\nu\mu}$, 只须计算 $\mu \neq \rho$ 情形. 因为

$$
\begin{aligned}
(e_0)_{\lambda} &= (e_0)_a(\partial_{\lambda})^a = -z^{-1/4}\,(dt)_a(\partial_{\lambda})^a = -z^{-1/4}\,\delta^0{}_{\lambda}\ , \\
(e_3)_{\lambda} &= (e_3)_a(\partial_{\lambda})^a = z^{-1/4}\,(dz)_a(\partial_{\lambda})^a = z^{-1/4}\,\delta^3{}_{\lambda}\ , \\
(e_1)_{\lambda} &= (e_1)_a(\partial_{\lambda})^a = z^{1/2}\,(dx)_a(\partial_{\lambda})^a = z^{1/2}\,\delta^1{}_{\lambda}\ , \\
(e_2)_{\lambda} &= (e_2)_a(\partial_{\lambda})^a = z^{1/2}\,(dy)_a(\partial_{\lambda})^a = z^{1/2}\,\delta^2{}_{\lambda}\ ,
\end{aligned}
$$

有

$$
\begin{aligned}
(e_0)_{\lambda,\tau} &= \partial_\tau(-z^{-1/4}\,\delta^0{}_\lambda) = \delta^0{}_\lambda\delta^3{}_\tau\frac{1}{4}z^{-5/4}\ , \\
(e_3)_{\lambda,\tau} &= \partial_\tau(z^{-1/4}\,\delta^3{}_\lambda) = -\delta^3{}_\lambda\delta^3{}_\tau\frac{1}{4}z^{-5/4}\ , \\
(e_1)_{\lambda,\tau} &= \partial_\tau(z^{1/2}\,\delta^1{}_\lambda) = \delta^1{}_\lambda\delta^3{}_\tau\frac{1}{2}z^{-1/2}\ , \\
(e_2)_{\lambda,\tau} &= \partial_\tau(z^{1/2}\,\delta^2{}_\lambda) = \delta^2{}_\lambda\delta^3{}_\tau\frac{1}{2}z^{-1/2}\ .
\end{aligned}
$$

代入式 (5-7-19) $\Lambda_{\mu\nu\rho} = [(e_\nu)_{\lambda,\tau} - (e_\nu)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau$:

$$
\begin{aligned}
\Lambda_{\mu 0\rho} &= [(e_0)_{\lambda,\tau} - (e_0)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{4}z^{-5/4}[\delta^0{}_\lambda\delta^3{}_\tau - \delta^0{}_\tau\delta^3{}_\lambda](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{4}z^{-5/4}[(e_\mu)^0(e_\rho)^3 - (e_\mu)^3(e_\rho)^0]\ , \\
\Lambda_{\mu 3\rho} &= [(e_3)_{\lambda,\tau} - (e_3)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{4}z^{-5/4}[-\delta^3{}_\lambda\delta^3{}_\tau + \delta^3{}_\tau\delta^3{}_\lambda](e_\mu)^\lambda(e_\rho)^\tau \\
&= 0\ , \\
\Lambda_{\mu 1\rho} &= [(e_1)_{\lambda,\tau} - (e_1)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{2}z^{-1/2}[\delta^1{}_\lambda\delta^3{}_\tau - \delta^1{}_\tau\delta^3{}_\lambda](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{2}z^{-1/2}[(e_\mu)^1(e_\rho)^3 - (e_\mu)^3(e_\rho)^1]\ , \\
\Lambda_{\mu 2\rho} &= [(e_2)_{\lambda,\tau} - (e_2)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{2}z^{-1/2}[\delta^2{}_\lambda\delta^3{}_\tau - \delta^2{}_\tau\delta^3{}_\lambda](e_\mu)^\lambda(e_\rho)^\tau \\
&= \frac{1}{2}z^{-1/2}[(e_\mu)^2(e_\rho)^3 - (e_\mu)^3(e_\rho)^2]\ .
\end{aligned}
$$

得到非零的 $\Lambda_{\mu\nu\rho}$

$$
\begin{aligned}
\Lambda_{003} &= -\Lambda_{300} = \frac{1}{4}z^{-5/4}\,(e_0)^0(e_3)^3 = \frac{1}{4}z^{-5/4}\,z^{1/4}z^{1/4} = \frac{1}{4}z^{-3/4}\ , \\
\Lambda_{\mu 3\rho} &= 0\ , \\
\Lambda_{113} &= -\Lambda_{311} = \frac{1}{2}z^{-1/2}\,(e_1)^1(e_3)^3 = \frac{1}{2}z^{-1/2}\,z^{-1/2}z^{1/4} = \frac{1}{2}z^{-3/4} \\
\Lambda_{223} &= -\Lambda_{322} = \frac{1}{2}z^{-1/2}\,(e_2)^2(e_3)^3 = \frac{1}{2}z^{-1/2}\,z^{-1/2}z^{1/4} = \frac{1}{2}z^{-3/4}\ .
\end{aligned}
$$

代入式 (5-7-20) $\omega_{\mu\nu\rho} = \frac{1}{2}(\Lambda_{\mu\nu\rho} + \Lambda_{\rho\mu\nu} - \Lambda_{\nu\rho\mu})$ 求得非零的 $\omega_{\mu\nu\rho}$ (注意反称关系，非零时 $\mu \neq \nu$). 容易看出 $(\mu,\nu) = (0,1)$ 时没有， $(0,2)$ 时没有， $(0,3)$ 时 可以有, $(1,2)$ 时没有， $(1,3)$ 时 可以有, $(2,3)$ 时 可以有:

$$
\omega_{030} = \frac{1}{2}(\Lambda_{030} + \Lambda_{003} - \Lambda_{300}) = \Lambda_{003} = \frac{1}{4}z^{-3/4} = -\omega_{300}\ ,
$$

$$\begin{aligned}
\omega_{131} &= \frac{1}{2}(\Lambda_{131}+\Lambda_{113}-\Lambda_{311}) = \Lambda_{113} = \frac{1}{2}z^{-3/4} = -\omega_{311} \\
\omega_{232} &= \frac{1}{2}(\Lambda_{232}+\Lambda_{223}-\Lambda_{322}) = \Lambda_{223} = \frac{1}{2}z^{-3/4} = -\omega_{322} \ .
\end{aligned}$$

联络 1 形式为 $\boldsymbol{\omega}_{\mu\nu} = \omega_{\mu\nu a} = \omega_{\mu\nu\lambda}(e^\lambda)_a = \omega_{\mu\nu\lambda}\boldsymbol{e}^\lambda$:

$$\begin{aligned}
\boldsymbol{\omega}_{03} &= \omega_{030}\,\boldsymbol{e}^0 = \frac{1}{4}z^{-3/4}\,\boldsymbol{e}^0 \\
&= \frac{1}{4}z^{-3/4}\,z^{-1/4}\,dt = \frac{1}{4}z^{-1}\,dt \ , \\
\boldsymbol{\omega}_{13} &= \omega_{131}\,\boldsymbol{e}^1 = \frac{1}{2}z^{-3/4}\,\boldsymbol{e}^1 \\
&= \frac{1}{2}z^{-3/4}\,z^{1/2}\,dx = \frac{1}{2}z^{-1/4}\,dx \ , \\
\boldsymbol{\omega}_{23} &= \omega_{232}\,\boldsymbol{e}^2 = \frac{1}{2}z^{-3/4}\,\boldsymbol{e}^2 \\
&= \frac{1}{2}z^{-3/4}\,z^{1/2}\,dy = \frac{1}{2}z^{-1/4}\,dy \ ,
\end{aligned}$$

有 3 个非零的独立元素.

(c) 用嘉当第二结构方程求曲率 2 形式. 由 $\boldsymbol{\omega}_\mu{}^\nu = \eta^{\nu\sigma}\boldsymbol{\omega}_{\mu\sigma}$ 知 $\boldsymbol{\omega}_0{}^i = \boldsymbol{\omega}_{0i}$, $\boldsymbol{\omega}_i{}^0 = -\boldsymbol{\omega}_{i0} = \boldsymbol{\omega}_{0i}$ 以及 $\boldsymbol{\omega}_i{}^j = \boldsymbol{\omega}_{ij}$. 代入式 (5-7-8): $\boldsymbol{R}_\mu{}^\nu = d\boldsymbol{\omega}_\mu{}^\nu + \boldsymbol{\omega}_\mu{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^\nu$, 得黎曼曲率 2 形式. 显然有定理 3-4-6 性质 (4) 式 (3-4-9): $\boldsymbol{R}_{\mu\nu} = -\boldsymbol{R}_{\nu\mu}$. 证明如下: $\boldsymbol{R}_{\mu\nu} = d\boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_\mu{}^\lambda \wedge \boldsymbol{\omega}_{\lambda\nu} = -d\boldsymbol{\omega}_{\nu\mu} - \boldsymbol{\omega}_{\lambda\nu} \wedge \boldsymbol{\omega}_\mu{}^\lambda = -d\boldsymbol{\omega}_{\nu\mu} + \boldsymbol{\omega}_{\nu\lambda} \wedge \boldsymbol{\omega}_\mu{}^\lambda = -d\boldsymbol{\omega}_{\nu\mu} + \boldsymbol{\omega}_\nu{}^\lambda \wedge \boldsymbol{\omega}_{\mu\lambda} = -d\boldsymbol{\omega}_{\nu\mu} - \boldsymbol{\omega}_\nu{}^\lambda \wedge \boldsymbol{\omega}_{\lambda\mu} = -\boldsymbol{R}_{\nu\mu}$.

$$\begin{aligned}
\boldsymbol{R}_0{}^1 &= d\boldsymbol{\omega}_0{}^1 + \boldsymbol{\omega}_0{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^1 = 0 + \boldsymbol{\omega}_{03} \wedge \boldsymbol{\omega}_{31} \\
&= \Big(\frac{1}{4}z^{-3/4}\,\boldsymbol{e}^0\Big) \wedge \Big(-\frac{1}{2}z^{-3/4}\,\boldsymbol{e}^1\Big) \\
&= -\frac{1}{8}z^{-3/2}\,\boldsymbol{e}^0 \wedge \boldsymbol{e}^1 \ , \\
\boldsymbol{R}_0{}^2 &= d\boldsymbol{\omega}_0{}^2 + \boldsymbol{\omega}_0{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^2 = 0 + \boldsymbol{\omega}_{03} \wedge \boldsymbol{\omega}_{32} \\
&= \Big(\frac{1}{4}z^{-3/4}\,\boldsymbol{e}^0\Big) \wedge \Big(-\frac{1}{2}z^{-3/4}\,\boldsymbol{e}^2\Big) \\
&= -\frac{1}{8}z^{-3/2}\,\boldsymbol{e}^0 \wedge \boldsymbol{e}^2 \ , \\
\boldsymbol{R}_0{}^3 &= d\boldsymbol{\omega}_0{}^3 + \boldsymbol{\omega}_0{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^3 = d\boldsymbol{\omega}_{03} + 0 \\
&= d\Big(\frac{1}{4}z^{-1}\,dt\Big) \\
&\overset{(5\text{-}1\text{-}12)}{=} -\frac{1}{4}z^{-2}\,dz \wedge dt = \frac{1}{4}z^{-2}\,dt \wedge dz \\
&= \frac{1}{4}z^{-2}z^{1/4}z^{1/4}\,(z^{-1/4}\,dt) \wedge (z^{-1/4}\,dz) \\
&= \frac{1}{4}z^{-3/2}\,\boldsymbol{e}^0 \wedge \boldsymbol{e}^3 \ , \\
\boldsymbol{R}_1{}^0 &= d\boldsymbol{\omega}_1{}^0 + \boldsymbol{\omega}_1{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^0 = 0 + \boldsymbol{\omega}_{13} \wedge \boldsymbol{\omega}_{03} \\
&= \Big(\frac{1}{2}z^{-3/4}\,\boldsymbol{e}^1\Big) \wedge \Big(\frac{1}{4}z^{-3/4}\,\boldsymbol{e}^0\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8} z^{-3/2}\, \boldsymbol{e}^1 \wedge \boldsymbol{e}^0 \\
&= -\frac{1}{8} z^{-3/2}\, \boldsymbol{e}^0 \wedge \boldsymbol{e}^1 = \boldsymbol{R}_0{}^1\ , \\
\boldsymbol{R}_1{}^2 &= d\boldsymbol{\omega}_1{}^2 + \boldsymbol{\omega}_1{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^2 = 0 + \boldsymbol{\omega}_{13} \wedge \boldsymbol{\omega}_{32} \\
&= \Big(\frac{1}{2} z^{-3/4}\, \boldsymbol{e}^1\Big) \wedge \Big(-\frac{1}{2} z^{-3/4}\, \boldsymbol{e}^2\Big) \\
&= -\frac{1}{4} z^{-3/2}\, \boldsymbol{e}^1 \wedge \boldsymbol{e}^2\ , \\
\boldsymbol{R}_1{}^3 &= d\boldsymbol{\omega}_1{}^3 + \boldsymbol{\omega}_1{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^3 = d\boldsymbol{\omega}_{13} + 0 \\
&= d\Big(\frac{1}{2} z^{-1/4}\, dx\Big) \\
&\overset{(5\text{-}1\text{-}12)}{=} -\frac{1}{8} z^{-5/4}\, dz \wedge dx = \frac{1}{8} z^{-5/4}\, dx \wedge dz \\
&= \frac{1}{8} z^{-5/4} z^{-1/2} z^{1/4}\, (z^{1/2}\, dx) \wedge (z^{-1/4}\, dz) \\
&= \frac{1}{8} z^{-3/2}\, \boldsymbol{e}^1 \wedge \boldsymbol{e}^3\ , \\
\boldsymbol{R}_2{}^0 &= d\boldsymbol{\omega}_2{}^0 + \boldsymbol{\omega}_2{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^0 = 0 + \boldsymbol{\omega}_{23} \wedge \boldsymbol{\omega}_{03} \\
&= \Big(\frac{1}{2} z^{-3/4}\, \boldsymbol{e}^2\Big) \wedge \Big(\frac{1}{4} z^{-3/4}\, \boldsymbol{e}^0\Big) \\
&= \frac{1}{8} z^{-3/2}\, \boldsymbol{e}^2 \wedge \boldsymbol{e}^0 = -\frac{1}{8} z^{-3/2}\, \boldsymbol{e}^0 \wedge \boldsymbol{e}^2 = \boldsymbol{R}_0{}^2\ , \\
\boldsymbol{R}_2{}^1 &= d\boldsymbol{\omega}_2{}^1 + \boldsymbol{\omega}_2{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^1 = 0 + \boldsymbol{\omega}_{23} \wedge \boldsymbol{\omega}_{31} \\
&= \Big(\frac{1}{2} z^{-3/4}\, \boldsymbol{e}^2\Big) \wedge \Big(-\frac{1}{2} z^{-3/4}\, \boldsymbol{e}^1\Big) \\
&= -\frac{1}{4} z^{-3/2}\, \boldsymbol{e}^2 \wedge \boldsymbol{e}^1\ , \\
\boldsymbol{R}_2{}^3 &= d\boldsymbol{\omega}_2{}^3 + \boldsymbol{\omega}_2{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^3 = d\boldsymbol{\omega}_{23} + 0 \\
&= d\Big(\frac{1}{2} z^{-1/4}\, dy\Big) \\
&\overset{(5\text{-}1\text{-}12)}{=} -\frac{1}{8} z^{-5/4}\, dz \wedge dy = \frac{1}{8} z^{-5/4}\, dy \wedge dz \\
&= \frac{1}{8} z^{-5/4} z^{-1/2} z^{1/4}\, (z^{1/2}\, dy) \wedge (z^{-1/4}\, dz) \\
&= \frac{1}{8} z^{-3/2}\, \boldsymbol{e}^2 \wedge \boldsymbol{e}^3 \\
\boldsymbol{R}_3{}^0 &= d\boldsymbol{\omega}_3{}^0 + \boldsymbol{\omega}_3{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^0 = d\boldsymbol{\omega}_{03} + 0 \\
&= \boldsymbol{R}_0{}^3\ , \\
\boldsymbol{R}_3{}^1 &= -\boldsymbol{R}_1{}^3\ , \\
\boldsymbol{R}_3{}^2 &= -\boldsymbol{R}_2{}^3\ .
\end{aligned}
$$

这些式子可写为

$$(R_\sigma{}^\tau)_{ab} = R(\sigma,\tau)\,(e^\sigma)_a \wedge (e^\tau)_b = 2R(\sigma,\tau)\,(e^\sigma)_{[a}(e^\tau)_{b]}\ .$$

因此黎曼曲率在正交归一标架基底的分量为

$$R_{\mu\nu\sigma}{}^\tau \ = \ (R_\sigma{}^\tau)_{ab}(e_\mu)^a(e_\nu)^b$$

$$
\begin{aligned}
&= 2R(\sigma,\tau)\,(e^\sigma)_{[a}(e^\tau)_{b]}(e_\mu)^a(e_\nu)^b \\
&= 2R(\sigma,\tau)\,\delta^\sigma{}_{[\mu}\delta^\tau{}_{\nu]} \\
&= R(\sigma,\tau)\,(\delta^\sigma{}_\mu\delta^\tau{}_\nu - \delta^\sigma{}_\nu\delta^\tau{}_\mu)\ ,
\end{aligned}
$$

于是求得非零黎曼曲率张量

$$
\begin{aligned}
R_{010}{}^1 &= -R_{100}{}^1 = R(0,1) = -\frac{1}{8}z^{-3/2}\ , \\
R_{020}{}^2 &= -R_{200}{}^2 = R(0,2) = -\frac{1}{8}z^{-3/2}\ , \\
R_{030}{}^3 &= -R_{300}{}^3 = R(0,3) = \frac{1}{4}z^{-3/2}\ , \\
R_{121}{}^2 &= -R_{211}{}^2 = R(1,2) = -\frac{1}{4}z^{-3/2}\ , \\
R_{131}{}^3 &= -R_{311}{}^3 = R(1,3) = \frac{1}{8}z^{-3/2}\ , \\
R_{232}{}^3 &= -R_{322}{}^3 = R(2,3) = \frac{1}{8}z^{-3/2}\ .
\end{aligned}
$$

它们与第 3 章习题 15 的结果 $R_{(\mu)(\nu)(\sigma)}{}^{(\tau)}$ 的关系为

$$
\begin{aligned}
R_{txt}{}^x &= R_{abc}{}^d(\partial_t)^a(\partial_x)^b(\partial_t)^c(dx)_d \\
&= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_t)^a(\partial_x)^b(\partial_t)^c(dx)_d \\
&= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d z^{-1/4}(e_0)^a z^{1/2}(e_1)^b z^{-1/4}(e_0)^c z^{-1/2}(e^1)_d \\
&= R_{\mu\nu\sigma}{}^\tau z^{-1/2}(e^\mu)_a(e_0)^a(e^\nu)_b(e_1)^b(e^\sigma)_c(e_0)^c(e_\tau)^d(e^1)_d \\
&= R_{\mu\nu\sigma}{}^\tau z^{-1/2}\delta^\mu{}_0\delta^\nu{}_1\delta^\sigma{}_0\delta^1{}_\tau \\
&= R_{010}{}^1 z^{-1/2} = -\frac{1}{8}z^{-3/2}z^{-1/2} = -\frac{1}{8z^2}\ , \\
R_{tyt}{}^y &= R_{abc}{}^d(\partial_t)^a(\partial_y)^b(\partial_t)^c(dy)_d \\
&= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_t)^a(\partial_y)^b(\partial_t)^c(dy)_d \\
&= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d z^{-1/4}(e_0)^a z^{1/2}(e_2)^b z^{-1/4}(e_0)^c z^{-1/2}(e^2)_d \\
&= R_{\mu\nu\sigma}{}^\tau z^{-1/2}(e^\mu)_a(e_0)^a(e^\nu)_b(e_2)^b(e^\sigma)_c(e_0)^c(e_\tau)^d(e^2)_d \\
&= R_{\mu\nu\sigma}{}^\tau z^{-1/2}\delta^\mu{}_0\delta^\nu{}_2\delta^\sigma{}_0\delta^2{}_\tau \\
&= R_{020}{}^2 z^{-1/2} = -\frac{1}{8}z^{-3/2}z^{-1/2} = -\frac{1}{8z^2}\ , \\
R_{tzt}{}^z &= R_{abc}{}^d(\partial_t)^a(\partial_z)^b(\partial_t)^c(dz)_d \\
&= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_t)^a(\partial_z)^b(\partial_t)^c(dz)_d \\
&= R_{\mu\nu\sigma}{}^\tau(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d z^{-1/4}(e_0)^a z^{-1/4}(e_3)^b z^{-1/4}(e_0)^c z^{1/4}(e^3)_d \\
&= R_{\mu\nu\sigma}{}^\tau z^{-1/2}(e^\mu)_a(e_0)^a(e^\nu)_b(e_3)^b(e^\sigma)_c(e_0)^c(e_\tau)^d(e^3)_d \\
&= R_{\mu\nu\sigma}{}^\tau z^{-1/2}\delta^\mu{}_0\delta^\nu{}_3\delta^\sigma{}_0\delta^3{}_\tau \\
&= R_{030}{}^3 z^{-1/2} = \frac{1}{4}z^{-3/2}z^{-1/2} = \frac{1}{4z^2}\ ,
\end{aligned}
$$

$$
\begin{aligned}
R_{xyx}{}^{y} &= R_{abc}{}^{d}(\partial_x)^a(\partial_y)^b(\partial_x)^c(dy)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_x)^a(\partial_y)^b(\partial_x)^c(dy)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d z^{1/2}(e_1)^a z^{1/2}(e_2)^b z^{1/2}(e_1)^c z^{-1/2}(e^2)_d \\
&= R_{\mu\nu\sigma}{}^{\tau} z(e^\mu)_a(e_1)^a(e^\nu)_b(e_2)^b(e^\sigma)_c(e_1)^c(e_\tau)^d(e^2)_d \\
&= R_{\mu\nu\sigma}{}^{\tau} z\delta^\mu{}_1\delta^\nu{}_2\delta^\sigma{}_1\delta^2{}_\tau \\
&= R_{121}{}^{2} z = -\frac{1}{4}z^{-3/2}z = -\frac{1}{4z^{1/2}}\ , \\
R_{xzx}{}^{z} &= R_{abc}{}^{d}(\partial_x)^a(\partial_z)^b(\partial_x)^c(dz)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_x)^a(\partial_z)^b(\partial_x)^c(dz)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d z^{1/2}(e_1)^a z^{-1/4}(e_3)^b z^{1/2}(e_1)^c z^{1/4}(e^3)_d \\
&= R_{\mu\nu\sigma}{}^{\tau} z(e^\mu)_a(e_1)^a(e^\nu)_b(e_3)^b(e^\sigma)_c(e_1)^c(e_\tau)^d(e^3)_d \\
&= R_{\mu\nu\sigma}{}^{\tau} z\delta^\mu{}_1\delta^\nu{}_3\delta^\sigma{}_1\delta^3{}_\tau \\
&= R_{131}{}^{3} z = \frac{1}{8}z^{-3/2}z = \frac{1}{8z^{1/2}}\ , \\
R_{yzy}{}^{z} &= R_{abc}{}^{d}(\partial_y)^a(\partial_z)^b(\partial_y)^c(dz)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d(\partial_y)^a(\partial_z)^b(\partial_y)^c(dz)_d \\
&= R_{\mu\nu\sigma}{}^{\tau}(e^\mu)_a(e^\nu)_b(e^\sigma)_c(e_\tau)^d z^{1/2}(e_2)^a z^{-1/4}(e_3)^b z^{1/2}(e_2)^c z^{1/4}(e^3)_d \\
&= R_{\mu\nu\sigma}{}^{\tau} z(e^\mu)_a(e_2)^a(e^\nu)_b(e_3)^b(e^\sigma)_c(e_2)^c(e_\tau)^d(e^3)_d \\
&= R_{\mu\nu\sigma}{}^{\tau} z\delta^\mu{}_2\delta^\nu{}_3\delta^\sigma{}_2\delta^3{}_\tau \\
&= R_{232}{}^{3} z = \frac{1}{8}z^{-3/2}z = \frac{1}{8z^{1/2}}\ .
\end{aligned}
$$

这正是前面通过坐标基底场的度规张量计算的结果.

(C) 习题 16.

(a) 选正交归一标架. 线元 $ds^2 = -dt^2 + dx^2 + dy^2 + h^2dz^2$, 其中 $h = t + \alpha(z)x + \beta(z)y + \gamma(z)$, 故非归一坐标基底的度规分量为

$$
\begin{aligned}
&g_{tt} = -1\ , \quad g_{zz} = h^2\ , \quad g_{xx} = 1\ , \quad g_{yy} = 1\ ; \\
&g^{tt} = -1\ , \quad g^{zz} = h^{-2}\ , \quad g^{xx} = 1\ , \quad g^{yy} = 1\ .
\end{aligned}
$$

度规张量场为

$$
\begin{aligned}
g_{ab} &= g_{tt}(dt)_a(dt)_b + g_{zz}(dz)_a(dz)_b + g_{xx}(dx)_a(dx)_b + g_{yy}(dy)_a(dy)_b \\
&= \eta_{00}(e^0)_a(e^0)_b + \eta_{33}(e^3)_a(e^3)_b + \eta_{11}(e^1)_a(e^1)_b + \eta_{22}(e^2)_a(e^2)_b\ , \\
g^{ab} &= g^{tt}(\partial_t)^a(\partial_t)^b + g^{zz}(\partial_z)^a(\partial_z)^b + g^{xx}(\partial_x)^a(\partial_x)^b + g^{yy}(\partial_y)^a(\partial_y)^b \\
&= \eta^{00}(e_0)^a(e_0)^b + \eta^{33}(e_3)^a(e_3)^b + \eta^{11}(e_1)^a(e_1)^b + \eta^{22}(e_2)^a(e_2)^b\ ,
\end{aligned}
$$

其中 $\{(e_\mu)^a\}$ 和 $\{(e^\mu)_a\}$ ($\mu = 0, 1, 2, 3$) 为正交归一的基底和对偶基底，即度规分量为洛伦兹度规 $-\eta_{00} = -\eta^{00} = \eta_{11} = \eta^{11} = \eta_{22} = \eta^{22} = \eta_{33} = \eta^{33} = 1$.

比较得

$$(e_0)^a = (\partial_t)^a\ ,\ (e_3)^a = h^{-1}\,(\partial_z)^a\ ,\ (e_1)^a = (\partial_x)^a\ ,\ (e_2)^a = (\partial_y)^a\ ;$$
$$(e^0)_a = (dt)_a\ ,\ (e^3)_a = h\,(dz)_a\ ,\ (e^1)_a = (dx)^a\ ,\ (e^2)_a = (dy)_a\ .$$

用 g_{ab} 降 $(e_\mu)^b$ 或用 $\eta_{\mu\nu}$ 降 $(e^\nu)_a$, 如: $(e_0)_a = g_{ab}(e_0)^b = \eta_{00}(e^0)_a(e^0)_b(e_0)^b = -(e^0)_a = -(dt)_a$, 或 $(e_0)_a = \eta_{0\nu}(e^\nu)_a = -(e^0)_a = -(dt)_a$. 因此

$$(e_0)_a = -(dt)_a,\ (e_3)_a = h\,(dz)_a,\ (e_1)_a = (dx)^a,\ (e_2)_a = (dy)_a\ .$$

(b) 用式 (5-7-19) 计算 $\Lambda_{\mu\nu\rho}$ 和式 (5-7-20) 计算 $\omega_{\mu\nu\rho}$. 注意到反称关系 $\Lambda_{\mu\nu\rho} = -\Lambda_{\rho\nu\mu}$, 只须计算 $\mu \neq \rho$ 情形. 因为

$$\begin{aligned}
(e_0)_\lambda &= (e_0)_a(\partial_\lambda)^a = -(dt)_a(\partial_\lambda)^a = -\delta^0{}_\lambda\ ,\\
(e_3)_\lambda &= (e_3)_a(\partial_\lambda)^a = h\,(dz)_a(\partial_\lambda)^a = h\,\delta^3{}_\lambda\ ,\\
(e_1)_\lambda &= (e_1)_a(\partial_\lambda)^a = (dx)_a(\partial_\lambda)^a = \delta^1{}_\lambda\ ,\\
(e_2)_\lambda &= (e_2)_a(\partial_\lambda)^a = (dy)_a(\partial_\lambda)^a = \delta^2{}_\lambda\ ,
\end{aligned}$$

有

$$\begin{aligned}
(e_0)_{\lambda,\tau} &= \partial_\tau(\delta^0{}_\lambda) = 0\ ,\\
(e_3)_{\lambda,\tau} &= \partial_\tau(h\,\delta^3{}_\lambda)\\
&= \delta^3{}_\lambda(\delta^0{}_\tau\,h_t + \delta^3{}_\tau\,h_z + \delta^1{}_\tau\,h_x + \delta^2{}_\tau\,h_y)\\
&= h_\sigma\,\delta^3{}_\lambda\delta^\sigma{}_\tau\ ,\\
(e_1)_{\lambda,\tau} &= \partial_\tau(\delta^1{}_\lambda) = 0\ ,\\
(e_2)_{\lambda,\tau} &= \partial_\tau(\delta^2{}_\lambda) = 0\ ,
\end{aligned}$$

其中因 $h = t + \alpha(z)x + \beta(z)y + \gamma(z)$,

$$\begin{aligned}
h_t &\equiv h_0 = 1\ ,\\
h_z &\equiv h_3 = \alpha'(z)x + \beta'(z)y + \gamma'(z)\ ,\\
h_x &\equiv h_1 = \alpha(z)\ ,\\
h_y &\equiv h_2 = \beta(z)\ .
\end{aligned}$$

代入式 (5-7-19) $\Lambda_{\mu\nu\rho} = [(e_\nu)_{\lambda,\tau} - (e_\nu)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau$:

$$\begin{aligned}
\Lambda_{\mu0\rho} &= [(e_0)_{\lambda,\tau} - (e_0)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau = 0\ ,\\
\Lambda_{\mu3\rho} &= [(e_3)_{\lambda,\tau} - (e_3)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau\\
&= h_\sigma\,[\delta^3{}_\lambda\delta^\sigma{}_\tau - \delta^3{}_\tau\delta^\sigma{}_\lambda](e_\mu)^\lambda(e_\rho)^\tau\\
&= h_\sigma\,[(e_\mu)^3(e_\rho)^\sigma - (e_\mu)^\sigma(e_\rho)^3]\ ,\\
\Lambda_{\mu1\rho} &= [(e_1)_{\lambda,\tau} - (e_1)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau = 0\ ,\\
\Lambda_{\mu2\rho} &= [(e_2)_{\lambda,\tau} - (e_2)_{\tau,\lambda}](e_\mu)^\lambda(e_\rho)^\tau = 0\ .
\end{aligned}$$

因此只有 $\Lambda_{\mu 3\rho}$ 非零：

$$\begin{aligned}\Lambda_{33\sigma}=-\Lambda_{\sigma 33} &= h_\sigma\,(e_3)^3(e_\sigma)^\sigma = h_\sigma\,h^{-1}(e_\sigma)^\sigma \\ &= h^{-1}h_\sigma\,(e_\sigma)^\sigma\;, \qquad (\sigma\text{不求和},\ \sigma\neq 3)\end{aligned}$$

即

$$\begin{aligned}\Lambda_{330}=-\Lambda_{033} &= h^{-1}h_0\,(e_0)^0 = h^{-1}h_t\;,\\ \Lambda_{331}=-\Lambda_{133} &= h^{-1}h_1\,(e_1)^1 = h^{-1}h_x\;,\\ \Lambda_{332}=-\Lambda_{133} &= h^{-1}h_2\,(e_2)^2 = h^{-1}h_y\;.\end{aligned}$$

代入式 (5-7-20) $\omega_{\mu\nu\rho}=\frac{1}{2}(\Lambda_{\mu\nu\rho}+\Lambda_{\rho\mu\nu}-\Lambda_{\nu\rho\mu})$ 求得非零的 $\omega_{\mu\nu\rho}$ (注意反称关系，非零时 $\mu\neq\nu$). 容易看出 $(\mu,\nu)=(0,1)$ 时没有， $(0,2)$ 时没有， $\underline{(0,3)}$ 时可以有， $(1,2)$ 时没有， $\underline{(1,3)}$ 时可以有， $\underline{(2,3)}$ 时可以有：

$$\begin{aligned}\omega_{033} &= \frac{1}{2}(\Lambda_{033}+\Lambda_{303}-\Lambda_{330}) = \Lambda_{033} = -h^{-1}h_t = -\omega_{330}\;,\\ \omega_{133} &= \frac{1}{2}(\Lambda_{133}+\Lambda_{313}-\Lambda_{331}) = \Lambda_{133} = -h^{-1}h_x = -\omega_{331}\\ \omega_{233} &= \frac{1}{2}(\Lambda_{233}+\Lambda_{323}-\Lambda_{332}) = \Lambda_{233} = -h^{-1}h_y = -\omega_{322}\;.\end{aligned}$$

联络 1 形式为 $\boldsymbol{\omega}_{\mu\nu}=\omega_{\mu\nu a}=\omega_{\mu\nu\lambda}(e^\lambda)_a=\omega_{\mu\nu\lambda}\boldsymbol{e}^\lambda$:

$$\begin{aligned}\boldsymbol{\omega}_{03} &= \omega_{033}\,\boldsymbol{e}^3 = -h^{-1}h_t\,\boldsymbol{e}^3\\ &= -h^{-1}h_t\,h\,dz = -dz\;,\\ \boldsymbol{\omega}_{13} &= \omega_{133}\,\boldsymbol{e}^3 = -h^{-1}h_x\,\boldsymbol{e}^3\\ &= -h^{-1}h_x\,h\,dz = -\alpha(z)\,dz\;,\\ \boldsymbol{\omega}_{23} &= \omega_{233}\,\boldsymbol{e}^3 = -h^{-1}h_y\,\boldsymbol{e}^3\\ &= -h^{-1}h_y\,h\,dz = -\beta(z)\,dz\;,\end{aligned}$$

有 3 个非零的独立元素.

(c) 用嘉当第二结构方程求曲率 2 形式. 由 $\boldsymbol{\omega}_\mu{}^\nu=\eta^{\nu\sigma}\boldsymbol{\omega}_{\mu\sigma}$ 知 $\boldsymbol{\omega}_0{}^i=\boldsymbol{\omega}_{0i}$, $\boldsymbol{\omega}_i{}^0=-\boldsymbol{\omega}_{i0}=\boldsymbol{\omega}_{0i}$ 以及 $\boldsymbol{\omega}_i{}^j=\boldsymbol{\omega}_{ij}$. 代入式 (5-7-8): $\boldsymbol{R}_\mu{}^\nu=d\boldsymbol{\omega}_\mu{}^\nu+\boldsymbol{\omega}_\mu{}^\lambda\wedge\boldsymbol{\omega}_\lambda{}^\nu$, 得黎曼曲率 2 形式. 显然有定理 3-4-6 性质 (4) 式 (3-4-9): $\boldsymbol{R}_{\mu\nu}=-\boldsymbol{R}_{\nu\mu}$. 证明如下： $\boldsymbol{R}_{\mu\nu}=d\boldsymbol{\omega}_{\mu\nu}+\boldsymbol{\omega}_\mu{}^\lambda\wedge\boldsymbol{\omega}_{\lambda\nu}=-d\boldsymbol{\omega}_{\nu\mu}-\boldsymbol{\omega}_{\lambda\nu}\wedge\boldsymbol{\omega}_\mu{}^\lambda=-d\boldsymbol{\omega}_{\nu\mu}+\boldsymbol{\omega}_{\nu\lambda}\wedge\boldsymbol{\omega}_\mu{}^\lambda=-d\boldsymbol{\omega}_{\nu\mu}+\boldsymbol{\omega}_\nu{}^\lambda\wedge\boldsymbol{\omega}_{\mu\lambda}=-d\boldsymbol{\omega}_{\nu\mu}-\boldsymbol{\omega}_\nu{}^\lambda\wedge\boldsymbol{\omega}_{\lambda\mu}=-\boldsymbol{R}_{\nu\mu}$.

$$\begin{aligned}\boldsymbol{R}_0{}^1 &= d\boldsymbol{\omega}_0{}^1+\boldsymbol{\omega}_0{}^\lambda\wedge\boldsymbol{\omega}_\lambda{}^1 = 0+\boldsymbol{\omega}_{03}\wedge\boldsymbol{\omega}_{31}\\ &= (-dz)\wedge(\alpha\,dz)\\ &= 0\;,\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{R}_0{}^2 &= d\boldsymbol{\omega}_0{}^2 + \boldsymbol{\omega}_0{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^2 = 0 + \boldsymbol{\omega}_{03} \wedge \boldsymbol{\omega}_{32} \\
&= (-dz) \wedge (\beta\, dz) \\
&= 0\,, \\
\boldsymbol{R}_0{}^3 &= d\boldsymbol{\omega}_0{}^3 + \boldsymbol{\omega}_0{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^3 = d\boldsymbol{\omega}_{03} + 0 \\
&= d(-dz) \\
&= 0\,, \qquad\qquad (\text{定理 5-1-5}) \\
\boldsymbol{R}_1{}^2 &= d\boldsymbol{\omega}_1{}^2 + \boldsymbol{\omega}_1{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^2 = 0 + \boldsymbol{\omega}_{13} \wedge \boldsymbol{\omega}_{32} \\
&= (-\alpha\, dz) \wedge (\beta\, dz) \\
&= 0\,, \\
\boldsymbol{R}_1{}^3 &= d\boldsymbol{\omega}_1{}^3 + \boldsymbol{\omega}_1{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^3 = d\boldsymbol{\omega}_{13} + 0 \\
&= d(-\alpha\, dz) \\
&\overset{(5\text{-}1\text{-}12)}{=} -\alpha'\, dz \wedge dz \\
&= 0\,, \\
\boldsymbol{R}_2{}^3 &= d\boldsymbol{\omega}_2{}^3 + \boldsymbol{\omega}_2{}^\lambda \wedge \boldsymbol{\omega}_\lambda{}^3 = d\boldsymbol{\omega}_{23} + 0 \\
&= d(-\beta\, dz) \\
&\overset{(5\text{-}1\text{-}12)}{=} -\beta'\, dz \wedge dz \\
&= 0\,.
\end{aligned}
$$

因此知道曲率张量恒为零！与前面通过坐标基底场的度规张量计算的结果相同.

第 6 章“狭义相对论”习题

~1. 惯性观者 G 和 G' 相对速率为 $u = 0.6c$, 相遇时把钟读数都调为零. 用时空图讨论: (a) 在 G 所属的惯性参考系看来 (以其同时观判断), 当 G 钟读数为 5μs 时, G' 钟的读数是多少? (b) 当 G 钟读数为 5μs 时, 他实际看见 G' 钟的读数是多少?

解 $\gamma = (1-u^2)^{-1/2} = 1.25$. 由图中的几何关系知:

(a) $l_{ob} = \sqrt{l_{oa}^2 - l_{ab}^2} = \sqrt{l_{oa}^2 - (ul_{oa})^2} = \sqrt{1-u^2}\, l_{oa} = \gamma^{-1} l_{oa} = 5/1.25 = 4\mu\text{s}$. 当 G 钟读数为 5μs 时, G' 钟的读数是 4μs.

(b) $l_{oc} = \gamma^{-1} l_{od} = \gamma^{-1}(l_{oa} - l_{ad}) = \gamma^{-1}(l_{oa} - l_{cd}) = \gamma^{-1}(l_{oa} - ul_{od}) = \gamma^{-1}(l_{oa} - u\gamma l_{oc})$. 解得 $l_{oc} = (1+u)^{-1}\gamma^{-1} l_{oa} = \sqrt{\frac{1-u}{1+u}}\, l_{oa} = \sqrt{\frac{1-0.6}{1+0.6}} \times 5 = 2.5\mu\text{s}$. 当 G 钟读数为 5μs 时, 他实际看见 G' 钟的读数是 2.5μs.

~2. 远方星体以 0.8c 的速率 (匀速直线地) 离开我们，我们测得它辐射来的闪光按 5 昼夜的周期变化．用时空图求星上观者测得的闪光周期．

解 根据上题 (b) 的结果我们知道 $l_{ob} = (1+v)^{-1}\gamma^{-1}l_{oa}$, $l_{od} = (1+v)^{-1}\gamma^{-1}l_{ob}$, 两式相减即得 $\Delta t' = l_{bd} = l_{od} - l_{ob} = (1+v)^{-1}\gamma^{-1}(l_{oc} - l_{oa}) = (1+v)^{-1}\gamma^{-1}l_{ac} = (1+v)^{-1}\gamma^{-1}\Delta t = \sqrt{\frac{1-u}{1+u}}\,\Delta t = \sqrt{\frac{1-0.8}{1+0.8}} \times 5 = \frac{5}{3}$ 昼夜.

这里的因子 $(1+u)^{-1}\gamma^{-1} = \sqrt{\frac{1-u}{1+u}}$ 也可以通过洛伦兹变换如下求得：设时空原点两参考系 (我们和星体) 重合．当我们的时间为 t 时，星体距离我们为 vt, 星体上的钟走过 $\gamma^{-1}t$. 但是我们看到这一刻度必定在 $vt/c = vt$ 时间之后．因此我们的钟走过 $t + vt = (1+v)t$ 时 "看到" 星体的钟走过 $\gamma^{-1}t$. 换句话说当我们的钟走过 t 时我们 "看到" 星体上的钟走过 $(1+u)^{-1}\gamma^{-1}t$.

~3. 用图 6-20 的 oa 段和 oe 段线长分别记作 τ 和 τ'. (a) 用两钟的相对速率 u 表出 τ'/τ; (b) 在 $u = 0.6c$ 和 $u = 0.8c$ 两种情况下求出 τ'/τ 的数值.

解 (a) 因 $l_{oa} = l_{ob} - l_{ab} = l_{ob} - l_{be} = l_{ob} - ul_{ob} = (1-u)l_{ob} = (1-u)\gamma l_{oe} = (1+u)^{-1/2}l_{oe}$, 即 $\tau = (1+u)^{-1/2}\tau'$, 故 $\tau'/\tau = (1+u)^{1/2}$.

(b) 当 $u = 0.6c$ 时， $\tau'/\tau = (1+0.6)^{1/2} = 1.265$; 当 $u = 0.8c$ 时， $\tau'/\tau = (1+0.8)^{1/2} = 1.342$. 注意这个比值有个极限 $\sqrt{2} = 1.414$.

4. 惯性质点 A, B, C 排成一直线并沿此线相对运动 (见图 6-42), 相对速率 $u_{BA} = 0.6c$, $u_{CA} = 0.8c$, A, B 所在惯性系各为 $\mathcal{R}_A$ 和 $\mathcal{R}_B$. 设 $\mathcal{R}_B$ 系认为 (测得) C 走了 60m, 画出时空图并求 $\mathcal{R}_A$ 认为 (测得) 这一过程的时间.

解 解法 1. 转化成图中的几何语言，待解的问题是：已知 l_{oa}, 求出 $l_{od} = l_{fb}$. 令 $u_B \equiv u_{BA}$, $u_C \equiv u_{CA}$. 首先由关系 $l_{of} = l_{fb}u_C$, 即 $l_{og} + l_{gf} = (l_{fe} + l_{eb})u_C$, 知

$$l_{oa}\gamma_B + l_{ab}\gamma_B u_B = (l_{oa}\gamma_B u_B + l_{ab}\gamma_B)u_C \ ,$$

解得关系

$$\frac{l_{oa}}{l_{ab}} = \frac{u_C - u_B}{1 - u_C u_B} \ .$$

这其实就是相对论速度迭加公式．于是

$$\begin{aligned} l_{od} &= l_{fb} = l_{fe} + l_{eb} = l_{oa}\gamma_B u_B + l_{ab}\gamma_B = \gamma_B(l_{oa}u_B + l_{ab}) \\ &= \gamma_B\left(l_{oa}u_B + l_{oa}\frac{1 - u_C u_B}{u_C - u_B}\right) = l_{oa}\gamma_B\frac{1 - u_B^2}{u_C - u_B} \\ &= l_{oa}\frac{\sqrt{1 - u_B^2}}{u_C - u_B} \ . \end{aligned}$$

解法 2. A 和 B 之间的相对速率为 $v_A = u_{BA} = u_B$, 而 B 和 C 之间的相对速率为 $v_C = \frac{u_{CA} - u_{BA}}{1 - u_{CA}u_{BA}} = \frac{u_C - u_B}{1 - u_C u_B}$. 待解的问题仍然是：已知 l_{oa}, 求出 l_{od}. 首先

注意关系 $l_{df}=l_{of}v_A=l_{od}\gamma_A v_A$, 而

$$\begin{aligned} l_{of} &= l_{oc}+l_{ce}+l_{ef}=l_{ab}+l_{cb}v_A+l_{df}v_A \\ &= l_{oa}/v_C+l_{oa}v_A+(l_{od}\gamma_A v_A)v_A=l_{od}\gamma_A \ , \end{aligned}$$

解得

$$l_{od}=l_{oa}\gamma_A\left(\frac{1}{v_C}+v_A\right)=l_{oa}\gamma_B\left(\frac{1-u_Cu_B}{u_C-u_B}+u_B\right)=l_{oa}\frac{\sqrt{1-u_B^2}}{u_C-u_B}\ .$$

与解法 1 的结果相同.

因此，最后的答案是 $\mathcal{R}_A$ 测得这一过程的时间为 $60\times\frac{\sqrt{1-0.6^2}}{0.8-0.6}=240\mathrm{m}/c$.

˜5. A, B 是同一惯性系的两个惯性观者，他们互相发射中子，每一中子以相对速率 $0.6c$ 离开中子枪. 设 B 测得 B 枪的中子发射率为 $10^4\mathrm{s}^{-1}$ (即每秒发 10^4 个), 求 A 所发中子 (根据中子自己的标准钟) 测得的 B 枪的中子发射率 (要求画时空图求解).

解 从时空图可以找出 l_{oa} 和 l_{ob} 之间的关系.

$$\begin{aligned} l_{oc} &= l_{oa}\gamma_A \ , \\ l_{oe} &= l_{ob}\gamma_B \ ; \\ l_{dc} &= l_{ca}v_A=(l_{oc}v_A)v_A=l_{oa}\gamma_A v_A^2 \ , \\ l_{ed} &= l_{be}v_A=(l_{oe}v_B)v_A=l_{ob}\gamma_B v_A v_B \ . \end{aligned}$$

而

$$l_{oc}=l_{oe}+l_{ed}+l_{dc}=l_{ob}\gamma_B+l_{ob}\gamma_B v_A v_B+l_{oa}\gamma_A v_A^2=l_{oa}\gamma_A \ ,$$

解得

$$l_{oa}=l_{ob}\frac{\gamma_B(1+v_Av_B)}{\gamma_A(1-v_A^2)}=l_{ob}\gamma_A\gamma_B(1+v_Av_B)\ .$$

对于本题 $v_A=v_B=v=0.6c$, $\gamma_A=\gamma_B=\gamma=1.25$, $l_{oa}=l_{ob}\frac{1+v^2}{1-v^2}$. 故 A 所发中子测得的 B 枪的中子发射率为 $10^4\times\frac{1-0.6^2}{1+0.6^2}=4.71\times10^3\,\mathrm{s}^{-1}$. 从所得结果的对称形式可以知道， B 所发中子测得的 A 枪的中子发射率也是 $4.71\times10^3\,\mathrm{s}^{-1}$.

˜6. 静止 μ 子的平均寿命为 $\tau_0=2\times10^{-6}$s. 宇宙线产生的 μ 子相对于地球以 $0.995c$ 的速率匀速直线下落，用时空图求地球观者测得的 (a) μ 子的平均寿命； (b) μ 子在其平均寿命内所走过的距离.

解 $l_{ob}=l_{oa}\gamma$, $l_{oc}=l_{ca}v=l_{ob}v=l_{oa}\gamma v$. 因此 (a) μ 子的平均寿命 $\tau=l_{ob}=l_{oa}\gamma=\tau_0\gamma=2\times10^{-6}\times(1-0.995^2)^{-1/2}=2.00\times10^{-5}$s. (b) μ 子在其平均寿命内所走过的距离 $l_{oc}=l_{oa}\gamma v=\tau v=1.99\times10^{-5}\mathrm{s}\ \times c$.

7. 从惯性系 $\mathcal{R}$ 看来 (认为，测得), 位于某地 A 的两标准钟甲、乙指零时开始以速率 $v = 0.6c$ 一同做匀速直线运动. 两钟指 1s 时到达某地 B. 甲钟在到达 B 时立即以速率 v 向 A 地匀速返回，乙钟在 B 地停留 1s (按他的钟) 后以速率 v 向 A 地匀速返回. 另有丙钟一直呆在 A 地，且当甲、乙离 A 地时也指零， (a) 画出甲、乙、丙的世界线； (b) 求乙钟返回 A 地时三钟的读数 $\tau_{甲}, \tau_{乙}$ 和 $\tau_{丙}$.

解 $\gamma = 1.25$. 乙钟的读数 $\tau_{乙} = l_{oa} + l_{ac} + l_{cd} = 1 + 1 + 1 = 3\text{s}$. 甲钟的读数为 $\tau_{甲} = l_{oa} + l_{ab} + l_{bd} = l_{oa} + l_{ab} + l_{ac} = 1 + 1 + 1 = 3\text{s}$. 而丙钟的读数为

$$\begin{aligned} \tau_{丙} &= l_{od} = l_{oe} + l_{ef} + l_{fd} = l_{oa}\gamma + l_{ac} + l_{cd}\gamma \\ &= 2 \times 1.25 + 1 = 3.5 \text{ s} . \end{aligned}$$

~8. (单选题) 双子 A, B 静止于某惯性系 $\mathcal{R}$ 中的同一空间点上. A 从某时刻 (此时 A, B 年龄相等) 开始向东以速率 u 相对于 $\mathcal{R}$ 系做惯性运动，一段时间后 B 以速率 $v > u$ 向东追上 A, 则相遇时 A 的年龄

(1) 比 B 大， (2) 比 B 小， (3) 与 B 等.

解 (1) 比 B 大. A 流逝的时间为 l_{oa}, B 流逝的时间为 $l_{ob} + l_{ba}$. 因为类时世界线以测地线 (直线) 为最长，故 $l_{oa} > l_{ob} + l_{ba}$. 下面我们证明这一不等关系. 注意到 $l_{ca} = l_{oc}u = l_{oa}\gamma_u u$, 有 $l_{ba} = l_{bc}/\gamma_v = l_{ca}/v\gamma_v = l_{oa}\gamma_u u/v\gamma_v$. 另外 $l_{ob} = l_{oc} - l_{bc} = l_{oa}\gamma_u - l_{ca}/v = l_{oa}\gamma_u - l_{oa}\gamma_u u/v$. 于是

$$l_{ob} + l_{ba} = l_{oa}\gamma_u - l_{oa}\gamma_u u/v + l_{oa}\gamma_u u/v\gamma_v = l_{oa}\gamma_u(1 - u/v + u/v\gamma_v)$$

可以证明当 $v > u$ 时 $\gamma_u(1 - u/v + u/v\gamma_v) < 1$. 即

$$1 - \frac{u}{v}(1 - \sqrt{1 - v^2}\,) < \sqrt{1 - u^2}\,.$$

因左边恒为正，可以平方得

$$1 + \frac{u^2}{v^2}(1 - \sqrt{1 - v^2}\,)^2 - 2\frac{u}{v}(1 - \sqrt{1 - v^2}\,) < 1 - u^2\,,$$

即

$$\begin{aligned} & 1 + \frac{u^2}{v^2}(2 - v^2 - 2\sqrt{1 - v^2}) - 2\frac{u}{v}(1 - \sqrt{1 - v^2}\,) \\ =\ & 1 + \frac{2u^2}{v^2} - u^2 - \frac{2u^2}{v^2}\sqrt{1 - v^2} - \frac{2u}{v} + \frac{2u}{v}\sqrt{1 - v^2} \\ =\ & (1 - u^2) - \frac{2u}{v}\left(1 - \frac{u}{v}\right) - \frac{2u}{v}\left(1 - \frac{u}{v}\right)\sqrt{1 - v^2} < (1 - u^2)\,. \end{aligned}$$

可见该不等式在 $v > u$ 时成立 ($v = u$ 时变为等式).

~9. 标准钟 A, B 静止于某惯性系中的同一空间点上. A 钟从某时刻开始以速率 $u=0.6c$ 匀速直线飞出，2s(根据 A 钟) 后以 $u=0.6c$ 匀速直线返航. 已知分手时两钟皆指零. (1) 求重逢时两钟的读数； (2) 当 A 钟指 3s 时 A 看见 B 钟指多少？

解 $\gamma=1.25$. (1) 因 $l_{ob}=l_{oc}+l_{cb}=l_{oa}\gamma+l_{ab}\gamma$, 故重逢时 A 钟的读数为 $l_{oa}+l_{ab}=2+2=4\,\text{s}$, B 钟的读数为 $l_{ob}=2\times1.25+2\times1.25=5\,\text{s}$.

(2) A 钟在 3s 时 (d 点) 看到 B 钟指向的时刻为 l_{of}, 可以求出

$$\begin{aligned}
l_{of} &= l_{ob}-l_{eb}-l_{fe}=l_{ob}-l_{db}\gamma-l_{ed}=l_{ob}-l_{eb}-l_{ed}\\
&= l_{ob}-l_{db}\gamma-l_{db}\gamma u=(l_{oa}+l_{ab})\gamma-\frac{1}{2}l_{ab}\gamma(1+u)\\
&= 4\times1.25-\frac{1}{2}\times2\times1.25\times1.6=3\,\text{s}\,.
\end{aligned}$$

因此当 A 钟指 3s 时 A 看见 B 钟也刚好指 3s.

~10. 地球自转线速率在赤道之值约为每小时 1600km. 甲、乙为赤道上的一对孪生子. 甲乘飞机以每小时 1600km 的速率向西绕赤道飞行一圈后回家与乙重逢 (忽略地球和太阳引力场的影响. 由第 7 章可知引力的存在对应于时空的弯曲.). (a) 画出地球表面的世界面和甲、乙的世界线 (甲相对于地面的运动抵消了地球自转的效应，所以甲是惯性观者.); (b) 甲与乙中谁更年轻？ (c) 两者年龄差多少？ (答：约为 10^{-7}s.) 注：本实验已于 1971 年完成，当然不是对人而是对铯原子钟. 见 Hafele and Keating (1972).

解 乙更年轻，因为甲是惯性系，世界线是测地线，为竖直的时间线，而乙的世界线为螺旋线，其线元为 $ds=dt/\gamma$. 所以 $\tau_{乙}=\tau_{甲}/\gamma$. 两者的年龄差为 $\tau_{甲}(1-1/\gamma)$. 因为

$$\frac{v}{c}=\frac{1.6\times10^6/(60\times60)}{3\times10^8}=1.48\times10^{-6}\ll1\,,$$

所以

$$1-1/\gamma=1-[1-(v/c)^2]^{1/2}\approx\frac{v^2}{2c^2}=1.097\times10^{-12}\,.$$

得年龄差

$$\tau_{甲}\frac{v^2}{2c^2}\approx24\times60\times60\times1.097\times10^{-12}=9.48\times10^{-8}\,\text{s}\,.$$

因此乙 (静止于赤道上) 比甲 (绕赤道反向飞行) 年轻约 10^{-7} s.

~11. 静长 $l=5$m 的汽车以 $u=0.6c$ 的速率匀速进库，库有坚硬后墙. 为简化问题，假定车头撞墙的信息以光速传播，车身任一点接到信息立即停下. (a) 设司库测得在车头撞墙的同时车尾的钟 C_W 指零，求车尾 “获悉” 车头撞墙

这一信息时 C_W 的读数； (b) 求车完全停下后的静长 $\hat{l}$; (c) 用 u 表出新旧静长比 $\hat{l}/l$.

解 见文中图 6-23. (a) 司库看车头撞墙时车尾在时空点 c, 车上的钟 C_W 指零. 然后车尾的世界线为 l_{cf}, 直到在时空点 f 收到车头在撞墙时发出的信号. 因为 $l_{gc} = l_{gf}u = l_{cf}\gamma u$ 和 $l_{og} = l_{gf} = l_{cf}\gamma$, 而 $l_{og} + l_{gc} = l_{oc} = l/\gamma$, 有 $l_{cf}\gamma + l_{cf}\gamma u = l_{cf}\gamma(1+u) = l/\gamma$. 解得车尾“获悉”车头撞墙这一信息时 C_W 的读数

$$l_{cf} = l(1+u)^{-1}\gamma^{-2} = l(1-u) = \frac{5\times(1-0.6)}{3\times 10^8} = 6.67\times 10^{-9}\,\mathrm{s}\,.$$

(b) 车完全停下后的静长为 $l_{fh} = l_{og} = l_{gf} = l_{cf}\gamma$, 即

$$\hat{l} = l(1-u)\gamma = l\sqrt{\frac{1-u}{1+u}}\,.$$

(c) 车的新旧静长比为

$$\frac{\hat{l}}{l} = \sqrt{\frac{1-u}{1+u}} = \sqrt{\frac{1-0.6}{1+0.6}} = 0.5\,.$$

12. 试证命题 6-3-4.

证 由 $0 = U^b\partial_b(-1) = U^b\partial_b(U^aU_a) = U^bU^a\partial_bU_a + U^bU_a\partial_bU^a$, 其中 $U^bU^a\partial_bU_a = U^bU^a\partial_b(\eta_{ac}U^c) = U^bU^a\eta_{ac}\partial_bU^c = U^bU_c\partial_bU^c = U^bU_b\partial_bU^b$, 这里利用了 ∂_a 与 η_{ab} 的适配性 $\partial_b\eta_{ac} = 0$. 于是 $0 = 2U^bU_a\partial_bU^a = 2U_aA^a$. 命题得证.

事实上，这一结果也可从 U^a 和 A^a 的 $3+1$ 分解后的分量式 (6-3-30) 和 (6-3-37) 看出来. 因为

$$\begin{aligned} U^\mu &= (\gamma,\, \gamma\vec{u})\,, \\ A^\mu &= (\gamma^4\vec{u}\cdot\vec{a},\, \gamma^2\vec{a} + \gamma^4(\vec{u}\cdot\vec{a})\vec{u})\,, \end{aligned}$$

故有

$$\begin{aligned} U^aA_a = U^\mu A_\mu &= -\gamma\,\gamma^4\vec{u}\cdot\vec{a} + \gamma\vec{u}\cdot[\gamma^2\vec{a} + \gamma^4(\vec{u}\cdot\vec{a})\vec{u}] \\ &= -\gamma^5\vec{u}\cdot\vec{a} + \gamma^3\vec{u}\cdot\vec{a} + u^2\gamma^5(\vec{u}\cdot\vec{a}) \\ &= [-\gamma^5(1-u^2) + \gamma^3]\vec{u}\cdot\vec{a} \\ &= [-\gamma^3 + \gamma^3]\vec{u}\cdot\vec{a} = 0\,. \end{aligned}$$

~13. 设观者世界线为 $t\sim x$ 面内的双曲线 G (见图 6-43), 图中 K 值为已知， A^a 为观者的 4 加速，求 A^aA_a (结论是 A^aA_a 为常数，因此 G 称为匀加速运动观者. 请注意这指的是 4 加速.)

解 双曲线方程为 $x^2 - t^2 = K^2$. 对 t 求导 $2xu - 2t = 0$ 得 3 速 $u = t/x$. 再求导得 3 加速

$$a = \frac{x - tu}{x^2} = \frac{x - t^2/x}{x^2} = \frac{x^2 - t^2}{x^3} = \frac{K^2}{x^3} \ .$$

有 $\gamma = (1-u^2)^{-1/2} = (1-t^2/x^2)^{-1/2} = x/K$. 故由式 (6-3-37) 得 4 加速在该惯性洛伦兹参考系上的分量为

$$\begin{aligned} A^0 &= \Big(\frac{x}{K}\Big)^4\Big(\frac{t}{x}\Big)\Big(\frac{K^2}{x^3}\Big) = \frac{t}{K^2} \ , \\ A^1 &= \Big(\frac{x}{K}\Big)^2\Big(\frac{K^2}{x^3}\Big) + \Big(\frac{x}{K}\Big)^4\Big(\frac{t}{x}\Big)\Big(\frac{K^2}{x^3}\Big)\Big(\frac{t}{x}\Big) \\ &= \frac{1}{x} + \frac{t^2}{K^2 x} = \frac{1}{x} + \frac{x^2 - K^2}{K^2 x} = \frac{x}{K^2} \ , \\ A^2 &= A^3 = 0 \ . \end{aligned}$$

因此有

$$A^a A_a = A^\mu A_\mu = -(A^0)^2 + (A^1)^2 = -\Big(\frac{t}{K^2}\Big)^2 + \Big(\frac{x}{K^2}\Big)^2 = \frac{x^2 - t^2}{K^4} = \frac{1}{K^2} \ .$$

该世界线的观者的 4 加速是个常数. (此世界线又称为 Rindler 世界线，观者为 Rindler 观者.)

~14. 试证命题 6-6-2.

证 电磁场满足张量变换关系而非矢量变换关系！

$$\begin{aligned} E'_1 &= F'_{10} = F_{ab}(e'_1)^a(e'_0)^b = F_{\mu\nu}(e^\mu)_a(e^\nu)_b(e'_1)^a(e'_0)^b \\ &= F_{\mu\nu}[(dx^\mu)_a(\partial/\partial x')^a][(dx^\nu)_b(\partial/\partial t')^b] = F_{\mu\nu}(\partial x^\mu/\partial x')(\partial x^\nu/\partial t') \\ &= F_{01}(\partial t/\partial x')(\partial x/\partial t') + F_{10}(\partial x/\partial x')(\partial t/\partial t') = F_{01}(\gamma v)(\gamma v) + F_{10}(\gamma)(\gamma) \\ &= F_{10}\gamma^2(1-v^2) = F_{10} = E_1 \ , \\ E'_2 &= F'_{20} = F_{ab}(e'_2)^a(e'_0)^b = F_{\mu\nu}(e^\mu)_a(e^\nu)_b(e'_2)^a(e'_0)^b \\ &= F_{\mu\nu}[(dx^\mu)_a(\partial/\partial y')^a][(dx^\nu)_b(\partial/\partial t')^b] = F_{\mu\nu}(\partial x^\mu/\partial y')(\partial x^\nu/\partial t') \\ &= F_{20}(\partial y/\partial y')(\partial t/\partial t') + F_{21}(\partial y/\partial y')(\partial x/\partial t') = F_{20}(1)(\gamma) + F_{21}(1)(\gamma v) \\ &= \gamma F_{20} - \gamma v F_{12} = \gamma(E_2 - vB_3) \ , \\ E'_3 &= F'_{30} = F_{ab}(e'_3)^a(e'_0)^b = F_{\mu\nu}(e^\mu)_a(e^\nu)_b(e'_3)^a(e'_0)^b \\ &= F_{\mu\nu}[(dx^\mu)_a(\partial/\partial z')^a][(dx^\nu)_b(\partial/\partial t')^b] = F_{\mu\nu}(\partial x^\mu/\partial z')(\partial x^\nu/\partial t') \\ &= F_{30}(\partial z/\partial z')(\partial t/\partial t') + F_{31}(\partial z/\partial z')(\partial x/\partial t') = F_{30}(1)(\gamma) + F_{31}(1)(\gamma v) \\ &= \gamma F_{30} + \gamma v F_{31} = \gamma(E_3 + vB_2) \ . \end{aligned}$$

也可利用选读 6-6-1 中的关系式 (6-6-6): $(e'_0)^a = \gamma(e_0)^a + \gamma v(e_1)^a$, 和 (6-6-7): $(e'_1)^a = \gamma v(e_0)^a + \gamma(e_1)^a$.

$$B'_1 = F'_{23} = F_{ab}(e'_2)^a(e'_3)^b = F_{ab}(e_2)^a(e_3)^b = F_{23} = B_1 \ ,$$

$$\begin{aligned}
B_2' &= F_{31}' = F_{ab}(e_3')^a(e_1')^b = F_{ab}(e_3)^a[\gamma v(e_0)^b + \gamma(e_1)^b] \\
&= \gamma v F_{ab}(e_3)^a(e_0)^b + \gamma F_{ab}(e_3)^a(e_1)^b = \gamma v F_{30} + \gamma F_{31} \\
&= \gamma(B_2 + vE_3)\ , \\
B_3' &= F_{12}' = F_{ab}(e_1')^a(e_2')^b = F_{ab}[\gamma v(e_0)^a + \gamma(e_1)^a](e_2)^b \\
&= \gamma v F_{ab}(e_0)^a(e_2)^b + \gamma F_{ab}(e_1)^a(e_2)^b = \gamma v F_{02} + \gamma F_{12} \\
&= \gamma(B_3 - vE_2)\ .
\end{aligned}$$

*15. 设瞬时观者测 F_{ab} 所得电场和磁场分别为 E^a 和 B^a (也记作 $\vec{E}$ 和 $\vec{B}$), 试证:

(a) $F_{ab}F^{ab} = 2(B^2 - E^2)\,$,

(b) $F_{ab}\,{}^*F^{ab} = 4\vec{E}\cdot\vec{B}$. 提示：可用惯性坐标基底把 $F_{ab}\,{}^*F^{ab}$ 写成分量表达式.

注：本题表明，虽然 $\vec{E}$ 和 $\vec{B}$ 都是观者依赖的， $B^2 - E^2$ 和 $\vec{E}\cdot\vec{B}$ 却同观者无关. 事实上，由 F_{ab} 能构造的独立的不变量只有这两个.

证 电磁场和电磁场张量 (2 形式场) 在惯性坐标基底的分量关系为:

$$\begin{aligned}
E_i &= E^i = F_{i0} = -F_{0i} = -F^{i0} = F^{0i} = \frac{1}{2}\hat{\varepsilon}_{ijk}\,{}^*F^{jk} = \frac{1}{2}\hat{\varepsilon}^{ijk}\,{}^*F_{jk}\ , \\
B_i &= B^i = -\,{}^*F_{i0} = {}^*F_{0i} = {}^*F^{i0} = -\,{}^*F^{0i} = \frac{1}{2}\hat{\varepsilon}_{ijk}F^{jk} = \frac{1}{2}\hat{\varepsilon}^{ijk}F_{jk}\ ,
\end{aligned}$$

反转形式为

$${}^*F^{ij} = \hat{\varepsilon}^{ijk}E_k\ , \quad {}^*F_{ij} = \hat{\varepsilon}_{ijk}E^k\ , \qquad F^{ij} = \hat{\varepsilon}^{ijk}B_k\ , \quad F_{ij} = \hat{\varepsilon}_{ijk}B^k\ .$$

于是有:

(a)
$$\begin{aligned}
F_{ab}F^{ab} &= F_{\mu\nu}F^{\mu\nu} = F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} \\
&= (-E_i)(E^i) + (E_i)(-E^i) + (\hat{\varepsilon}_{ijk}B^k)(\hat{\varepsilon}^{ijl}B_l) \\
&\overset{(5\text{-}4\text{-}10)}{=} -2E_iE^i + (-1)^0(3-2)!2!\delta^l{}_kB^kB_l \\
&= -2E_iE^i + 2B_iB^i = 2(B^2 - E^2)\ .
\end{aligned}$$

(b)
$$\begin{aligned}
F_{ab}\,{}^*F^{ab} &= F_{\mu\nu}\,{}^*F^{\mu\nu} = F_{0i}\,{}^*F^{0i} + F_{i0}\,{}^*F^{i0} + F_{ij}\,{}^*F^{ij} \\
&= (-E_i)(-B^i) + (E_i)(B^i) + (\hat{\varepsilon}_{ijk}B^k)(\hat{\varepsilon}^{ijl}E_l) \\
&= 2E_iB^i + (-1)^0(3-2)!2!\delta^l{}_kB^kE_l \\
&= 2E_iB^i + 2E_iB^i = 4\vec{E}\cdot\vec{B}\ .
\end{aligned}$$

(c)
$$\begin{aligned}
{}^*F_{ab}\,{}^*F^{ab} &= {}^*F_{\mu\nu}\,{}^*F^{\mu\nu} = {}^*F_{0i}\,{}^*F^{0i} + {}^*F_{i0}\,{}^*F^{i0} + {}^*F_{ij}\,{}^*F^{ij} \\
&= (-B_i)(-B^i) + (-B_i)(B^i) + (\hat{\varepsilon}_{ijk}E^k)(\hat{\varepsilon}^{ijl}E_l) \\
&= -2B_iB^i + (-1)^0(3-2)!2!\delta^l{}_kE^kE_l \\
&= -2B_iB^i + 2E_iE^i = 2(E^2 - B^2) = -F_{ab}F^{ab}\ .
\end{aligned}$$

负号是由于对偶场的 E, B 互换.

~16. 试证命题 6-6-5 (只须证后两个麦氏方程).

证 以 δ_{ab} 代表所选惯性系的等 t 面上的 (诱导) 欧氏度规，$\hat{\partial}_a$ 和 ∂_a 分别代表与 δ_{ab} 和 η_{ab} 适配的导数算符，令 $Z^a \equiv (\partial/\partial t)^a$. 麦氏方程 (6-6-12) 中的 (c): 注意到空间矢量 B^a 满足 $B_0 = 0$, 便有

$$\begin{aligned}\vec{\nabla}\cdot\vec{B} &= \hat{\partial}^a B_a = \partial B_i/\partial x_i = \partial^a B_a = \partial^a(-{}^*F_{ab}Z^b) = -Z^b\partial^a\,{}^*F_{ab} \\ &= -Z^b\partial^a\Big(\frac{1}{2}\varepsilon_{abcd}F^{cd}\Big) = -\frac{1}{2}Z^b\varepsilon_{abcd}\partial^a F^{cd} = \frac{1}{2}\hat{\varepsilon}_{acd}\partial^a F^{cd} \\ &= \frac{1}{2}\hat{\varepsilon}_{[acd]}\partial^a F^{cd} \overset{(2\text{-}6\text{-}19)}{=} \frac{1}{2}\hat{\varepsilon}_{acd}\partial^{[a}F^{cd]} \overset{(6\text{-}6\text{-}11)}{=} 0\ ,\end{aligned}$$

其中 $\hat{\varepsilon}_{acd} \equiv Z^b\varepsilon_{bacd}$ 是等 t 面上与 δ_{ab} 适配的体元.

麦氏方程 (6-6-12) 中的 (d): 由式 (5-6-5c) 知

$$(\vec{\nabla}\times\vec{B}\,)^c = \hat{\varepsilon}^{abc}\,\hat{\partial}_a B_b\ ,$$

其中的 $\hat{\partial}_a B_b$ 可表为 [据式 (3-1-9)]

$$\hat{\partial}_a B_b = (dx^i)_a(dx^j)_b\,\hat{\partial}_i B_j = (dx^i)_a(dx^j)_b\,\partial_i B_j\ ,$$

而 $B_0 = 0$ 导致

$$\partial_a B_b = (dx^\mu)_a(dx^j)_b\partial_\mu B_j = (dx^0)_a(dx^j)_b\partial_0 B_j + (dx^i)_a(dx^j)_b\partial_i B_j\ ,$$

将上式投影到等 t 面，注意到 $(dx^0)_a$ 的投影为零，$(dx^i)_a$ 的投影等于自身，与前式比较得

$$\hat{\partial}_a B_b = h_a{}^d h_b{}^e\,\partial_d B_e\ .$$

注意到 $\hat{\varepsilon}^{abc} = Z_d\varepsilon^{dabc}$ 是空间张量，其投影等于自身：

$$\begin{aligned}h_a{}^d h_b{}^e\hat{\varepsilon}^{abc} &= (\delta_a{}^d + Z_aZ^d)(\delta_b{}^e + Z_bZ^e)Z_f\varepsilon^{fabc} \\ &= (\delta_a{}^d + Z_aZ^d)Z_f\varepsilon^{faec} = Z_f\varepsilon^{fdec} = \hat{\varepsilon}^{dec}\ ,\end{aligned}$$

代入 $(\vec{\nabla}\times\vec{B}\,)^c$, 便得

$$(\vec{\nabla}\times\vec{B}\,)^c = \hat{\varepsilon}^{abc}\,h_a{}^d h_b{}^e\,\partial_d B_e = \hat{\varepsilon}^{dec}\,\partial_d B_e\ ,$$

于是

$$\begin{aligned}(\vec{\nabla}\times\vec{B}\,)^c &= \hat{\varepsilon}^{abc}\,\partial_a B_b = \hat{\varepsilon}^{abc}\,\partial_a(-{}^*F_{bd}Z^d) = -Z^d\hat{\varepsilon}^{abc}\,\partial_a\,{}^*F_{bd} \\ &= -Z^d\hat{\varepsilon}^{abc}\,\partial_a\Big(\frac{1}{2}\varepsilon_{bdef}F^{ef}\Big) = -\frac{1}{2}Z^d\varepsilon_{bdef}\hat{\varepsilon}^{abc}\,\partial_a F^{ef} \\ &= \frac{1}{2}\hat{\varepsilon}_{bef}\hat{\varepsilon}^{abc}\,\partial_a F^{ef} = -\frac{1}{2}\hat{\varepsilon}_{bef}\hat{\varepsilon}^{bac}\,\partial_a F^{ef} \\ &\overset{(5\text{-}4\text{-}10)}{=} -\frac{1}{2}(-1)^0(3-1)!1!\delta^{[a}{}_e\delta^{c]}{}_f\,\partial_a F^{ef} \\ &= -\partial_a F^{ac} \overset{(6\text{-}6\text{-}10)}{=} 4\pi J^c\ .\end{aligned}$$

注意这个结果是错的！正确的做法是

$$\begin{aligned}
(\vec{\nabla}\times\vec{B})^c &= -\frac{1}{2}Z^d\varepsilon_{bdef}\hat{\varepsilon}^{abc}\,\partial_a F^{ef} = -\frac{1}{2}Z^d\varepsilon_{bdef}Z_g\varepsilon^{gabc}\,\partial_a F^{ef}\\
&= -\frac{1}{2}Z^d Z_g\varepsilon_{bdef}\varepsilon^{bgac}\,\partial_a F^{ef}\\
&\overset{(5\text{-}4\text{-}10)}{=} -\frac{1}{2}Z^d Z_g(-1)^1(4-1)!1!\delta^{[g}{}_d\delta^a{}_e\delta^{c]}{}_f\,\partial_a F^{ef}\\
&= 3Z_g\delta^g{}_{[d}\delta^a{}_e\delta^c{}_{f]}\,\partial_a(Z^dF^{ef}) \overset{(2\text{-}6\text{-}19)}{=} 3Z_g\delta^g{}_d\delta^a{}_e\delta^c{}_f\,\partial_a(Z^{[d}F^{ef]})\\
&= 3Z_g\partial_a(Z^{[g}F^{ac]}) = Z_g\partial_a(Z^gF^{ac}+Z^cF^{ga}+Z^aF^{cg})\\
&= Z_gZ^g\partial_aF^{ac} - Z^c\partial_a(F^{ag}Z_g) + Z^a\partial_a(F^{cg}Z_g)\\
&= -\partial_aF^{ac} - Z^c\partial_aE^a + Z^a\partial_aE^c\\
&\overset{(6\text{-}6\text{-}10)}{=} 4\pi J^c - Z^c\partial_aE^a + Z^a\partial_aE^c\,.
\end{aligned}$$

于是有

$$\begin{aligned}
(\vec{\nabla}\times\vec{B})^i &= (dx^i)_c(\vec{\nabla}\times\vec{B})^c\\
&= (dx^i)_c(4\pi J^c - Z^c\partial_aE^a + Z^a\partial_aE^c)\\
&= 4\pi J^i - 0 + Z^a\partial_aE^i = 4\pi j^i + \left(\frac{\partial}{\partial t}\right)^a\partial_aE^i\\
&= 4\pi j^i + \frac{\partial E^i}{\partial t}\,,
\end{aligned}$$

其中利用了 $(dx^i)_cZ^c = (dx^i)_c(\partial/\partial t)^c = \partial x^i/\partial t = 0$. 这就是麦氏方程 (6-6-12) 中的 (d).

为什么不能用

$$Z^d\varepsilon_{bdef}\hat{\varepsilon}^{abc} = -\hat{\varepsilon}_{bef}\hat{\varepsilon}^{abc} = \hat{\varepsilon}_{bef}\hat{\varepsilon}^{bac} \overset{(5\text{-}4\text{-}10)}{=} (-1)^0(3-1)!1!\delta^{[a}{}_e\delta^{c]}{}_f = 2\delta^{[a}{}_e\delta^{c]}{}_f\ ?$$

左边按定义为

$$\begin{aligned}
Z^d\varepsilon_{bdef}Z_g\varepsilon^{gabc} &= Z^dZ_g\varepsilon_{bdef}\varepsilon^{bgac} \overset{(5\text{-}4\text{-}10)}{=} Z^dZ_g(-1)^1(4-1)!1!\delta^{[g}{}_d\delta^a{}_e\delta^{c]}{}_f\\
&= -6Z^dZ_g\delta^{[g}{}_d\delta^a{}_e\delta^{c]}{}_f\\
&= -2Z^dZ_g(\delta^g{}_d\delta^{[a}{}_e\delta^{c]}{}_f + \delta^c{}_d\delta^{[g}{}_e\delta^{a]}{}_f + \delta^a{}_d\delta^{[c}{}_e\delta^{g]}{}_f)\\
&= -2(Z^dZ_d\delta^{[a}{}_e\delta^{c]}{}_f + Z^cZ_g\delta^g{}_{[e}\delta^a{}_{f]} + Z^aZ_g\delta^c{}_{[e}\delta^g{}_{f]})\\
&= -2(-\delta^{[a}{}_e\delta^{c]}{}_f + Z^cZ_{[e}\delta^a{}_{f]} + Z^a\delta^c{}_{[e}Z_{f]})\\
&= 2\delta^{[a}{}_e\delta^{c]}{}_f - 2(Z^cZ_{[e}\delta^a{}_{f]} + Z^a\delta^c{}_{[e}Z_{f]})\,.
\end{aligned}$$

显然前面用 $\hat{\varepsilon}_{bef}\hat{\varepsilon}^{bac}$ 计算时，少了后面两项！只有当指标限制在 3 维空间时， $\hat{\varepsilon}_{bef}\hat{\varepsilon}^{bac}$ 才能得到正确结果.

~17. 试证瞬时观者测得的电磁场能量密度和 3 动量密度分别为 $T_{00} = (E^2 + B^2)/8\pi$ 和 $w_i = -T_{i0} = (\vec{E}\times\vec{B})_i/4\pi$, $i = 1, 2, 3$. 提示：用 F_{ab} 的对称表达式 (6-6-28') 可简化 T_{00} 的计算.

证 电磁场和电磁场张量 (2 形式场) 在惯性坐标基底的分量关系为:

$$\begin{aligned} E_i &= E^i = F_{i0} = -F_{0i} = -F^{i0} = F^{0i} = \frac{1}{2}\hat{\varepsilon}_{ijk}\,{}^*F^{jk} = \frac{1}{2}\hat{\varepsilon}^{ijk}\,{}^*F_{jk}\ , \\ B_i &= B^i = -{}^*F_{i0} = {}^*F_{0i} = {}^*F^{i0} = -{}^*F^{0i} = \frac{1}{2}\hat{\varepsilon}_{ijk}F^{jk} = \frac{1}{2}\hat{\varepsilon}^{ijk}F_{jk}\ , \end{aligned}$$

反转形式为

$${}^*F^{ij} = \hat{\varepsilon}^{ijk}E_k\ ,\quad {}^*F_{ij} = \hat{\varepsilon}_{ijk}E^k\ ,\qquad F^{ij} = \hat{\varepsilon}^{ijk}B_k\ ,\quad F_{ij} = \hat{\varepsilon}_{ijk}B^k\ .$$

利用电磁场能动张量的表达式 (6-6-28') $T_{ab} = \frac{1}{8\pi}(F_{ac}F_b{}^c + {}^*F_{ac}{}^*F_b{}^c)$ 可知电磁场能量密度为

$$\begin{aligned} T_{00} &= \frac{1}{8\pi}(F_{0i}F_0{}^i + {}^*F_{0i}{}^*F_0{}^i) = -\frac{1}{8\pi}(F_{0i}F^{0i} + {}^*F_{0i}{}^*F^{0i}) \\ &= -\frac{1}{8\pi}[(-E_i)(E^i) + (B_i)(-B^i)] = \frac{1}{8\pi}(E^2 + B^2)\ . \end{aligned}$$

也可利用表达式 (6-6-28) $T_{ab} = \frac{1}{4\pi}(F_{ac}F_b{}^c - \frac{1}{4}\eta_{ab}F_{cd}F^{cd})$,

$$\begin{aligned} T_{00} &= \frac{1}{4\pi}\Big(F_{0i}F_0{}^i - \frac{1}{4}\eta_{00}F_{cd}F^{cd}\Big) = \frac{1}{4\pi}\Big(-F_{0i}F^{0i} + \frac{1}{4}F_{cd}F^{cd}\Big) \\ &= \frac{1}{4\pi}\Big[-(-E_i)(E^i) + \frac{1}{4}2(B^2 - E^2)\Big] = \frac{1}{8\pi}(E^2 + B^2)\ . \end{aligned}$$

利用式 (6-6-28) 计算电磁场的能流密度 (坡印廷矢量) 和动量密度 ($c = 1$ 时它们相等) $w_i = -T_{i0}$:

$$\begin{aligned} w_i = -T_{i0} &= -\frac{1}{4\pi}\Big(F_{ij}F_0{}^j - \frac{1}{4}\eta_{i0}F_{cd}F^{cd}\Big) = \frac{1}{4\pi}F_{ij}F^{0j} \\ &= \frac{1}{4\pi}\hat{\varepsilon}_{ijk}B^kE^j = \frac{1}{4\pi}(\vec{E}\times\vec{B})_i\ . \end{aligned}$$

【最后我们补充计算动量流密度 (3 维) 张量, 也即选读 6-4-1 中讨论过的 3 应力张量 $T^{ij} = T_{ij} = \frac{1}{4\pi}(F_{ic}F_j{}^c - \frac{1}{4}\eta_{ij}F_{cd}F^{cd})$, 其中 $\eta_{ij} = \delta_{ij}$, $F_{cd}F^{cd} = 2(B^2 - E^2)$, 而

$$\begin{aligned} F_{ic}F_j{}^c &= F_{i\mu}F_j{}^\mu = F_{i0}F_j{}^0 + F_{ik}F_j{}^k = F_{i0}F^{j0} + F_{ik}F^{jk} \\ &= (E_i)(-E^j) + (\hat{\varepsilon}_{ikl}B^l)(\hat{\varepsilon}^{jkm}B_m) = -E_iE_j + 2\delta^{[j}{}_i\delta^{m]}{}_lB^lB_m \\ &= -E_iE_j + (\delta^j{}_i\delta^m{}_l - \delta^m{}_i\delta^j{}_l)B^lB_m = -E_iE_j + \delta^j{}_iB^lB_l - B^jB_i \\ &= -E_iE_j + \delta_{ij}B^2 - B_iB_j\ . \end{aligned}$$

代回上式得

$$\begin{aligned} T_{ij} &= \frac{1}{4\pi}\Big[-E_iE_j + \delta_{ij}B^2 - B_iB_j - \frac{1}{4}\delta_{ij}2(B^2 - E^2)\Big] \\ &= \frac{1}{4\pi}\Big[-E_iE_j - B_iB_j + \delta_{ij}\frac{1}{2}(E^2 + B^2)\Big] \\ &= -\frac{1}{4\pi}(E_iE_j + B_iB_j) + \delta_{ij}\frac{1}{8\pi}(E^2 + B^2)\ . \end{aligned}$$

这正是电磁场动量流密度 (3 维) 张量 [见郭硕鸿 (1995) 书 220 页式 (7.5) 的 $\overleftrightarrow{T}$]. 】

18. (a) 试证 4 电流密度为 J^a 的电磁场 F_{ab} 的能动张量 T_{ab} 满足 $\partial^a T_{ab} = -F_{bc}J^c$ (由此可知当 $J^a = 0$ 时有 $\partial^a T_{ab} = 0$); *(b) 试证上式在惯性坐标系中的时间分量反映能量守恒，即郭硕鸿 (1995) 40 页式 (6.2); 空间分量反映 3 动量守恒，即郭书 220 页式 (7.6). 提示：用 4 洛伦兹力表达式 (6-6-20) 把 $F_{bc}J^c$ 改写为洛伦兹力密度.

证 (a) 能动张量 $T_{ab} = \frac{1}{4\pi}(F_{ac}F_b{}^c - \frac{1}{4}\eta_{ab}F_{cd}F^{cd})$. 因

$$
\begin{aligned}
\partial^a(F_{ac}F_b{}^c) &= F_b{}^c\partial^a F_{ac} + F_{ac}\partial^a F_b{}^c \\
&\overset{(6\text{-}6\text{-}10)}{=} F_b{}^c(-4\pi J_c) + F^{ac}\partial_a F_{bc} \\
&= -4\pi F_{bc}J^c + F^{ac}\partial_a F_{bc}\ , \\
\partial^a(\eta_{ab}F_{cd}F^{cd}) &= \partial_b(F_{cd}F^{cd}) = F^{cd}\partial_b F_{cd} + F_{cd}\partial_b F^{cd} \\
&= 2F^{cd}\partial_b F_{cd} \overset{(6\text{-}6\text{-}11)}{=} 2F^{cd}(-\partial_d F_{bc} - \partial_c F_{db}) \\
&= 2F^{cd}(\partial_d F_{cb} - \partial_c F_{db}) = 4F^{cd}\partial_{[d}F_{c]b} \\
&\overset{(2\text{-}6\text{-}19)}{=} -4F^{[dc]}\partial_d F_{cb} = -4F^{dc}\partial_d F_{cb} \\
&= 4F^{ac}\partial_a F_{bc}\ .
\end{aligned}
$$

代入能动张量的表达式得

$$
\partial^a T_{ab} = \frac{1}{4\pi}\Big([-4\pi F_{bc}J^c + F^{ac}\partial_a F_{bc}] - \frac{1}{4}[4F^{ac}\partial_a F_{bc}]\Big) = -F_{bc}J^c\ .
$$

当无源 $J^c = 0$ 时，有 $\partial^a T_{ab} = 0$.

(b) 洛伦兹 4 力密度的定义为

$$
\tilde{f}^a = F^a{}_b J^b\ , \qquad \text{或} \qquad \tilde{f}_a = F_{ab}J^b\ .
$$

将 4 电流密度做 $3+1$ 分解 $J^b = \rho Z^b + j^b = \rho Z^b + \rho u^b$, 有

$$
\tilde{f}_a = F_{ab}(\rho Z^b + j^b) = \rho F_{ab}Z^b + F_{ab}j^b = \rho E_a + F_{ab}j^b\ .
$$

于是 (注意 E_a 和 j^b 没有时间分量)

$$
\begin{aligned}
\tilde{f}_0 &= F_{0i}j^i = -E_i j^i = -\vec{E}\cdot\vec{j}\ , \\
\tilde{f}_i &= \rho E_i + F_{ij}j^j = \rho E_i + \hat{\varepsilon}_{ijk}B^k j^j = \rho E_i + (\vec{j}\times\vec{B}\,)_i\ .
\end{aligned}
$$

第二式即为 $\vec{\tilde{f}} = \rho\vec{E} + \vec{j}\times\vec{B}$. 由于 $\vec{j} = \rho\vec{u}$, 有 $\vec{\tilde{f}}\cdot\vec{u} = \rho\vec{u}\cdot\vec{E} = \vec{j}\cdot\vec{E}$, 因此 $\tilde{f}_0 = -\vec{\tilde{f}}\cdot\vec{u}$ 或 $\tilde{f}^0 = \vec{\tilde{f}}\cdot\vec{u}$.

也可仿照式 (6-6-27), 我们计算

$$\begin{aligned}
(\vec{j}\times\vec{B})_c &= \hat{\varepsilon}_{cab}j^a B^b = \hat{\varepsilon}_{cab}j^a(-{}^*F^{bd}Z_d) = \hat{\varepsilon}_{cab}j^a\Big(-\frac{1}{2}\varepsilon^{bdef}F_{ef}Z_d\Big)\\
&= -\frac{1}{2}Z_d\hat{\varepsilon}_{cab}\varepsilon^{bdef}j^aF_{ef} = -\frac{1}{2}Z_dZ^g\varepsilon_{gcab}\varepsilon^{bdef}j^aF_{ef}\\
&= \frac{1}{2}Z_dZ^g\varepsilon_{bgca}\varepsilon^{bdef}j^aF_{ef} \overset{(5\text{-}4\text{-}10)}{=} \frac{1}{2}Z_dZ^g(-1)^1(4-1)!1!\delta^{[d}{}_g\delta^e{}_c\delta^{f]}{}_a j^aF_{ef}\\
&= -3Z^g\delta^{[d}{}_g\delta^e{}_c\delta^{f]}{}_aZ_dF_{ef}j^a \overset{(2\text{-}6\text{-}19)}{=} -3Z^g\delta^d{}_g\delta^e{}_c\delta^f{}_aZ_{[d}F_{ef]}j^a\\
&= -3Z^gZ_{[g}F_{ca]}j^a = -Z^g(Z_gF_{ca}+Z_aF_{gc}+Z_cF_{ag})j^a\\
&= -(Z^gZ_gF_{ca}+Z_cF_{ag}Z^g)j^a = -(-F_{ca}+Z_cE_a)j^a\\
&= F_{ca}j^a - Z_cE_aj^a\ ,
\end{aligned}$$

其中用到了 j^a 只有空间分量，故 $Z_aj^a=0$. 因此得

$$\begin{aligned}
\tilde{f}_a &= F_{ab}J^b = F_{ab}(\rho Z^b + j^b) = \rho F_{ab}Z^b + F_{ab}j^b\\
&= \rho E_a + Z_aE_bj^b + (\vec{j}\times\vec{B})_a\ .
\end{aligned}$$

所以同样有

$$\begin{aligned}
\tilde{f}_0 &= (e_0)^a\tilde{f}_a = Z^a\tilde{f}_a = \rho Z^aE_a + Z^aZ_aE_bj^b + Z^a(\vec{j}\times\vec{B})_a\\
&= 0 - E_bj^b + 0 = -\vec{E}\cdot\vec{j}\ ,\\
\tilde{f}_i &= (e_i)^a\tilde{f}_a = (\partial/\partial x^i)^a\tilde{f}_a = \rho E_i + 0 + (\vec{j}\times\vec{B})_i\\
&= \rho E_i + (\vec{j}\times\vec{B})_i\ .
\end{aligned}$$

洛伦兹 4 力密度 $\tilde{f}_a = F_{ab}J^b$ 总结如下：

$$\text{空间分量}\quad \vec{\tilde{f}} = \rho\vec{E} + \vec{j}\times\vec{B}\ ,\qquad \text{时间分量}\quad \tilde{f}^0 = \vec{j}\cdot\vec{E} = \vec{\tilde{f}}\cdot\vec{u}\ .$$

有了这些结果下面我们看能量守恒和动量守恒，利用 $\partial^aT_{ab} = -F_{bc}J^c = -\tilde{f}_b$, 即 $\partial_aT^{ab} = -F^{bc}J_c = -\tilde{f}^b$ 的分量式 $\partial_\mu T^{\mu\nu} = -\tilde{f}^\nu$.

(i) 能量守恒. 取 $\nu=0$, $\partial_\mu T^{\mu 0} = \partial_0T^{00} + \partial_iT^{i0} = -\tilde{f}^0$. 利用习题 17 的结果, 其中 $\partial_0T^{00} = \partial_0T_{00} = \frac{\partial}{\partial t}w$, 引入 $w\equiv T_{00} = \frac{1}{8\pi}(E^2+B^2)$ 就是郭书的电磁场能量密度； $\partial_iT^{i0} = \partial_i(-T_{i0}) = \partial_iw_i = \partial_iw^i = \vec{\nabla}\cdot\vec{w} = \vec{\nabla}\cdot\vec{S}$, 引入 $\vec{S}\equiv\vec{w} = \frac{1}{4\pi}\vec{E}\times\vec{B}$ 就是郭书的电磁场能流密度 (坡印廷矢量). 因此我们得到

$$\frac{\partial w}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{j}\cdot\vec{E} = -\vec{\tilde{f}}\cdot\vec{u}\ .$$

这正是代表电磁场能量守恒性质的郭硕鸿书 (1995) 40 页的式 (6.2).

(ii) 动量守恒. 取 $\nu=i$, $\partial_\mu T^{\mu i} = \partial_0T^{0i} + \partial_jT^{ji} = -\tilde{f}^i$. 利用习题 17 的结果, 其中 $\partial_0T^{0i} = \partial_0(-T_{0i}) = \partial_0(-T_{i0}) = \frac{\partial}{\partial t}w_i = \frac{\partial}{\partial t}S_i = \frac{\partial}{\partial t}g_i = \frac{\partial g^i}{\partial t}$, 引入 3 维动量

密度 g^i, 它等于能流密度 ($c=1$ 时); $\partial_j T^{ji} = \partial_j T_{ji} = (\nabla\cdot \overleftrightarrow{T})_i = (\nabla\cdot \overleftrightarrow{T})^i$, 其中 T_{ji} 为动量流密度 (3 维) 张量 (见习题 17 的补充计算). 因此我们得到

$$\frac{\partial g^i}{\partial t} + (\nabla\cdot \overleftrightarrow{T})^i = -\tilde{f}^i \ ,$$

此即代表电磁场动量守恒性质的郭硕鸿书 (1995) 220 页的 3 维式 (7.6):

$$\vec{\tilde{f}} + \frac{\partial \vec{g}}{\partial t} = -\nabla\cdot \overleftrightarrow{T} \ .$$

19. 试证式 (6-6-29) 中的 a^a 和 ϕ 满足 $\vec{B} = \vec{\nabla}\times\vec{a}$ 和 $\vec{E} = -\vec{\nabla}\phi - \partial\vec{a}/\partial t$, 因而的确是电动力学中的 3 矢势和标势.

证 利用 4 势 A^a 在任意惯性系 $\{t, x^i\}$ 的分解，式 (6-6-29):

$$A^a = \phi Z^a + a^a = \phi(\partial/\partial t)^a + a^a \ , \qquad \text{或} \qquad A_a = \phi Z_a + a_a = -\phi(dt)_a + a_a \ ,$$

我们有

$$\begin{aligned}
(\vec{\nabla}\times\vec{a})_c &\overset{\text{(5-6-5c)}}{=} \hat{\varepsilon}_{cab}\partial^a a^b = \hat{\varepsilon}_{cab}\partial^a(A^b - \phi Z^b) \\
&= \hat{\varepsilon}_{cab}\partial^a A^b = \hat{\varepsilon}_{c[ab]}\partial^a A^b \overset{\text{(2-6-19)}}{=} \hat{\varepsilon}_{cab}\partial^{[a}A^{b]} \\
&= \frac{1}{2}\hat{\varepsilon}_{cab}F^{ab} = \frac{1}{2}Z^d\varepsilon_{dcab}F^{ab} = Z^d\,{}^*F_{dc} \\
&= -{}^*F_{cd}Z^d = B_c \ ,
\end{aligned}$$

其中利用了 $\hat{\varepsilon}_{cab}$ 的空间性，有： $\hat{\varepsilon}_{cab}Z^b = Z^d\varepsilon_{dcab}Z^b = Z^{(d}Z^{b)}\varepsilon_{[d|ca|b]} = 0$. 此即关系式 $\vec{B} = \vec{\nabla}\times\vec{a}$.

另一方面，因为

$$\begin{aligned}
F^{ab} &= \partial^a A^b - \partial^b A^a = \partial^a(\phi Z^b + a^b) - \partial^b(\phi Z^a + a^a) \\
&\overset{\text{(3-1-10)}}{=} Z^b\partial^a\phi + \partial^a a^b - Z^a\partial^b\phi - \partial^b a^a \ ,
\end{aligned}$$

得

$$\begin{aligned}
E^a &= F^{ab}Z_b = Z_b(Z^b\partial^a\phi + \partial^a a^b - Z^a\partial^b\phi - \partial^b a^a) \\
&= Z_b Z^b\partial^a\phi + \partial^a(Z_b a^b) - Z^a Z_b\partial^b\phi - Z_b\partial^b a^a \\
&= -\partial^a\phi - Z^a Z_b\partial^b\phi - Z_b\partial^b a^a \ ,
\end{aligned}$$

最后一步用到了 a^b 的空间性： $Z_b a^b = 0$. 取上式的空间分量：

$$\begin{aligned}
E^i &= (dx^i)_a E^a = (dx^i)_a(-\partial^a\phi - Z^a Z_b\partial^b\phi - Z_b\partial^b a^a) \\
&= -(dx^i)_a\partial^a\phi - (dx^i)_a Z^a Z_b\partial^b\phi - Z_b\partial^b[(dx^i)_a a^a] \\
&= -\partial^i\phi - 0 - Z^b\partial_b a^i = -\partial_i\phi - (\partial/\partial t)^b\partial_b a^i \\
&= -\frac{\partial\phi}{\partial x^i} - \frac{\partial a^i}{\partial t} \ ,
\end{aligned}$$

其中用到了 Z^a 的时间性： $(dx^i)_a Z^a = (dx^i)_a(\partial/\partial t)^a = \partial x^i/\partial t = 0$. 此即关系式 $\vec{E} = -\vec{\nabla}\phi - \partial\vec{a}/\partial t$. 最后可以验证上面的 E^a 的形式的确没有时间分量：

$$\begin{aligned}
E^0 &= (dt)_a E^a = -Z_a(-\partial^a\phi - Z^a Z_b \partial^b\phi - Z_b\partial^b a^a) \\
&= -Z_a\partial^a\phi - Z_a Z^a Z_b\partial^b\phi - Z_b\partial^b(Z_a a^a) \\
&= -Z_a\partial^a\phi + Z_b\partial^b\phi - Z_b\partial^b(0) \\
&= 0\,.
\end{aligned}$$

20. 在选读 6-1-1 中， (a) 试证 $\nabla_a(dt)_b = 0$, 其中 t 为绝对时间， ∇_a 为牛顿时空的导数算符 [提示：从式 (5-7-2) 出发.]; (b) 设 w^a 为空间矢量 (即切于绝对同时面的矢量), v^a 为任一 4 维矢量，试证 $v^a\nabla_a w^b$ 仍为空间矢量 [提示：注意 $\nabla_a t$ 是绝对同时面的法余矢.].

证 (a) 根据式 (5-7-2):

$$\Big(\frac{\partial}{\partial x^\nu}\Big)^a \nabla_a \Big(\frac{\partial}{\partial x^\mu}\Big)^b = \Gamma^\sigma{}_{\mu\nu}\Big(\frac{\partial}{\partial x^\sigma}\Big)^b\,.$$

而牛顿时空唯一的非零克氏符为 $\Gamma^i{}_{00}$, 故上式变为

$$\Big(\frac{\partial}{\partial t}\Big)^a \nabla_a \Big(\frac{\partial}{\partial t}\Big)^b = \Gamma^i{}_{00}\Big(\frac{\partial}{\partial x^i}\Big)^b\,.$$

两边作用 $(dt)_b$, 右边为零，而左边等于

$$\begin{aligned}
(dt)_b\Big(\frac{\partial}{\partial t}\Big)^a \nabla_a \Big(\frac{\partial}{\partial t}\Big)^b &= \Big(\frac{\partial}{\partial t}\Big)^a \nabla_a\Big[(dt)_b\Big(\frac{\partial}{\partial t}\Big)^b\Big] - \Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial t}\Big)^b\nabla_a(dt)_b \\
&= \Big(\frac{\partial}{\partial t}\Big)^a \nabla_a[1] - \Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial t}\Big)^b\nabla_a(dt)_b \\
&= -\Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial t}\Big)^b\nabla_a(dt)_b\,.
\end{aligned}$$

因此有 $\nabla_a(dt)_b = 0$.

(b) 因为 $\nabla_a t \overset{(3\text{-}1\text{-}2)}{=} (dt)_a$ 是绝对同时面的法余矢，所以它与空间矢量 w^a 正交： $(\nabla_a t)w^a = (dt)_a w^a = 0$. 为了证明 $v^a\nabla_a w^b$ 仍为空间矢量，只须证明它也与 $\nabla_b t = (dt)_b$ 正交. 利用 (a) 的结果，我们有

$$(dt)_b v^a\nabla_a w^b = v^a\nabla_a[(dt)_b w^b] = v^a\nabla_a[0] = 0\,.$$

因此对任意 4 矢 v^a, $v^a\nabla_a w^b$ 仍是空间矢量.

附. 试推导任意绝对 4 矢 F^a 在任意两个惯性系的分量之间的洛伦兹变换关系式.

解 设有两个惯性系 $\mathcal{R}$ 和 $\mathcal{R}'$, 它们的 4 速分别为 $U^a = (\frac{\partial}{\partial t})^a$ 和 $U^a = (\frac{\partial}{\partial t'})^a$, 选相应的坐标系为 $\{x^\mu\}$ 和 $\{x'^\mu\}$. 我们要找出任意 4 矢 F^a 的分量在这两个坐标系之间的变换关系. 首先

$$\Big(\frac{\partial}{\partial x'^\mu}\Big)^a = \frac{\partial x^\nu}{\partial x'^\mu}\Big(\frac{\partial}{\partial x^\nu}\Big)^a\,.$$

因闵氏度规为

$$\eta_{ab} = \eta_{\mu\nu}(dx^\mu)_a(dx^\nu)_b = \eta_{\mu\nu}(dx'^\mu)_a(dx'^\nu)_b \ ,$$

于是有

$$\begin{aligned}\eta_{\mu\nu} &= \eta_{ab}\Big(\frac{\partial}{\partial x'^\mu}\Big)^a\Big(\frac{\partial}{\partial x'^\nu}\Big)^b \\ &= \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\eta_{ab}\Big(\frac{\partial}{\partial x^\lambda}\Big)^a\Big(\frac{\partial}{\partial x^\rho}\Big)^b = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\eta_{\lambda\rho} \ .\end{aligned}$$

当然，这其实就是定理 2-4-2 的张量变换律．令

$$\begin{aligned}\frac{\partial x^0}{\partial x'^0} &= \frac{\partial t}{\partial t'} \equiv \gamma \ , \\ \frac{\partial x^i}{\partial x'^0} &= \frac{\partial x^i}{\partial t'} = \frac{\partial t}{\partial t'}\frac{dx^i}{dt} \equiv \gamma u^i \ , \\ \frac{\partial x^0}{\partial x'^i} &= \frac{\partial t}{\partial x'^i} \equiv \alpha_i \ , \\ \frac{\partial x^i}{\partial x'^j} &= \beta^i{}_j \ ,\end{aligned}$$

其中 u^i 为 $\mathcal{R}'$ 系相对于 $\mathcal{R}$ 的运动 3 速在坐标系 $\{x^\mu\}$ 上的分量．于是有

$$\begin{aligned}-1 &= \eta_{00} = \frac{\partial x^0}{\partial x'^0}\frac{\partial x^0}{\partial x'^0}\eta_{00} + \frac{\partial x^i}{\partial x'^0}\frac{\partial x^i}{\partial x'^0}\eta_{ii} \\ &= (\gamma)(\gamma)(-1) + (\gamma u^i)(\gamma u^i)(\delta_{ii}) = -\gamma^2(1-u^2) \ , \\ 0 &= \eta_{0i} = \frac{\partial x^0}{\partial x'^0}\frac{\partial x^0}{\partial x'^i}\eta_{00} + \frac{\partial x^j}{\partial x'^0}\frac{\partial x^j}{\partial x'^i}\eta_{jj} \\ &= (\gamma)(\alpha_i)(-1) + (\gamma u^j)(\beta^j{}_i)(\delta_{ii}) = -\gamma\alpha_i + \gamma u_j\beta_{ji} \ , \\ \delta_{ij} &= \eta_{ij} = \frac{\partial x^0}{\partial x'^i}\frac{\partial x^0}{\partial x'^j}\eta_{00} + \frac{\partial x^k}{\partial x'^i}\frac{\partial x^k}{\partial x'^j}\eta_{kk} \\ &= (\alpha_i)(\alpha_j)(-1) + (\beta^k{}_i)(\beta^k{}_j)(\delta_{kk}) = -\alpha_i\alpha_j + \beta_{ki}\beta_{kj} \ .\end{aligned}$$

由第一个方程得 $\gamma = (1-u^2)^{-1/2}$, 第二个方程得关系式

$$\alpha_i = u_j\beta_{ji} = \beta_{ij}u_j \ ,$$

注意到这里的 $\beta_{ij} = \beta_{ji}$ 是对称的．代入第三个方程得

$$\delta_{ij} = -\beta_{ik}u_k u_l\beta_{lj} + \beta_{ik}\beta_{kj} \ .$$

下面将 α_i 和 u_i 看成列矢， β_{ij} 看成矩阵，则上两式分别为

$$\begin{aligned}\boldsymbol{\alpha}^T &= \boldsymbol{u}^T\boldsymbol{\beta} \ , \qquad \text{或} \qquad \boldsymbol{\alpha} = \boldsymbol{\beta}\boldsymbol{u} \ , \\ \boldsymbol{I} &= -\boldsymbol{\beta}\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{\beta} + \boldsymbol{\beta}^2 \ .\end{aligned}$$

猜 $\boldsymbol{\beta}$ 的解的形式为 $\boldsymbol{I}+A\boldsymbol{u}\boldsymbol{u}^T$, 其中 A 为待定常数，代入第二式有

$$\begin{aligned}
\boldsymbol{I} &= -(\boldsymbol{I}+A\boldsymbol{u}\boldsymbol{u}^T)\boldsymbol{u}\boldsymbol{u}^T(\boldsymbol{I}+A\boldsymbol{u}\boldsymbol{u}^T)+(\boldsymbol{I}+A\boldsymbol{u}\boldsymbol{u}^T)^2 \\
&= -(\boldsymbol{u}\boldsymbol{u}^T+2A\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T+A^2\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T)+(\boldsymbol{I}+2A\boldsymbol{u}\boldsymbol{u}^T+A^2\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T) \\
&= -\boldsymbol{u}\boldsymbol{u}^T-2Au^2\boldsymbol{u}\boldsymbol{u}^T-A^2u^4\boldsymbol{u}\boldsymbol{u}^T+\boldsymbol{I}+2A\boldsymbol{u}\boldsymbol{u}^T+A^2u^2\boldsymbol{u}\boldsymbol{u}^T \\
&= \boldsymbol{I}-\boldsymbol{u}\boldsymbol{u}^T+2A(1-u^2)\boldsymbol{u}\boldsymbol{u}^T+A^2u^2(1-u^2)\boldsymbol{u}\boldsymbol{u}^T \\
&= \boldsymbol{I}-\boldsymbol{u}\boldsymbol{u}^T+2A\gamma^{-2}\boldsymbol{u}\boldsymbol{u}^T+A^2u^2\gamma^{-2}\boldsymbol{u}\boldsymbol{u}^T \\
&= \boldsymbol{I}+\gamma^{-2}(A^2u^2+2A-\gamma^2)\boldsymbol{u}\boldsymbol{u}^T\ ,
\end{aligned}$$

其中利用了 $\boldsymbol{u}^T\boldsymbol{u}=u^2$. 因此 A 满足

$$u^2A^2+2A-\gamma^2=0\ .$$

其解为

$$A_\pm=\frac{-2\pm\sqrt{4+4u^2\gamma^2}}{2u^2}=\frac{-1\pm\gamma}{u^2}=\pm(\gamma\mp1)u^{-2}\ .$$

取其正的解【凭什么？凭 $\boldsymbol{\alpha}$ 必须是 $\gamma\boldsymbol{u}$, 如果是 $-\gamma\boldsymbol{u}$ 的话时间分量的变换就已经错了.】，于是得到

$$\begin{aligned}
\boldsymbol{\beta} &= \boldsymbol{I}+(\gamma-1)u^{-2}\boldsymbol{u}\boldsymbol{u}^T\ , \\
\boldsymbol{\alpha} &= \boldsymbol{\beta}\boldsymbol{u}=[\boldsymbol{I}+(\gamma-1)u^{-2}\boldsymbol{u}\boldsymbol{u}^T]\boldsymbol{u}=\boldsymbol{u}+(\gamma-1)u^{-2}\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u} \\
&= \boldsymbol{u}+(\gamma-1)u^{-2}u^2\boldsymbol{u}=\gamma\boldsymbol{u}\ .
\end{aligned}$$

写回分量形式就是：

$$\begin{aligned}
\alpha_i &= \gamma u_i\ , \\
\beta^i{}_j &= \delta^i{}_j+(\gamma-1)u^{-2}u^iu_j\ .
\end{aligned}$$

设 F^a 是任意的 4 矢，是个绝对量，它既可以在 $\mathcal{R}$ 系展开 (用 $\{x^\mu\}$ 坐标) 也可在 $\mathcal{R}'$ 系展开 (用 $\{x'^\mu\}$ 坐标):

$$F^a=f^0\Big(\frac{\partial}{\partial t}\Big)^a+f^i\Big(\frac{\partial}{\partial x^i}\Big)^a=f'^0\Big(\frac{\partial}{\partial t'}\Big)^a+f'^i\Big(\frac{\partial}{\partial x'^i}\Big)^a\ ,$$

而

$$\begin{aligned}
\Big(\frac{\partial}{\partial t'}\Big)^a &= \frac{\partial t}{\partial t'}\Big(\frac{\partial}{\partial t}\Big)^a+\frac{\partial x^i}{\partial t'}\Big(\frac{\partial}{\partial x^i}\Big)^a=\gamma\Big(\frac{\partial}{\partial t}\Big)^a+\gamma u^i\Big(\frac{\partial}{\partial x^i}\Big)^a\ , \\
\Big(\frac{\partial}{\partial x'^i}\Big)^a &= \frac{\partial t}{\partial x'^i}\Big(\frac{\partial}{\partial t}\Big)^a+\frac{\partial x^j}{\partial x'^i}\Big(\frac{\partial}{\partial x^j}\Big)^a=\alpha_i\Big(\frac{\partial}{\partial t}\Big)^a+\beta^j{}_i\Big(\frac{\partial}{\partial x^j}\Big)^a\ ,
\end{aligned}$$

所以有

$$\begin{aligned}
f'^0\Big(\frac{\partial}{\partial t'}\Big)^a+f'^i\Big(\frac{\partial}{\partial x'^i}\Big)^a &= f'^0\Big[\gamma\Big(\frac{\partial}{\partial t}\Big)^a+\gamma u^i\Big(\frac{\partial}{\partial x^i}\Big)^a\Big]+f'^i\Big[\alpha_i\Big(\frac{\partial}{\partial t}\Big)^a+\beta^j{}_i\Big(\frac{\partial}{\partial x^j}\Big)^a\Big] \\
&= (f'^0\gamma+f'^i\alpha_i)\Big(\frac{\partial}{\partial t}\Big)^a+(f'^0\gamma u^i+f'^j\beta^i{}_j)\Big(\frac{\partial}{\partial x^i}\Big)^a \\
&= f^0\Big(\frac{\partial}{\partial t}\Big)^a+f^i\Big(\frac{\partial}{\partial x^i}\Big)^a\ ,
\end{aligned}$$

得到变换关系

$$\begin{aligned}
f^0 &= f'^0\gamma + f'^i\alpha_i = f'^0\gamma + f'^i\gamma u_i = \gamma(f'^0 + f'^i u_i)\ ,\\
f^i &= f'^0\gamma u^i + f'^j\beta^i{}_j = f'^0\gamma u^i + f'^j[\delta^i{}_j + (\gamma-1)u^{-2}u^i u_j]\\
&= f'^0\gamma u^i + f'^i + (\gamma-1)u^{-2}u^i f'^j u_j\ ,
\end{aligned}$$

即

$$\begin{aligned}
f^0 &= \gamma(f'^0 + \vec{f}'\cdot\vec{u})\ ,\\
\vec{f} &= \vec{f}' + \gamma\vec{u}f'^0 + (\gamma-1)u^{-2}\vec{u}(\vec{f}'\cdot\vec{u})\ .
\end{aligned}$$

这其实是逆变换，其正变换为

$$\begin{aligned}
f'^0 &= \gamma(f^0 - \vec{f}\cdot\vec{u})\ ,\\
\vec{f}' &= \vec{f} - \gamma\vec{u}f^0 + (\gamma-1)u^{-2}\vec{u}(\vec{f}\cdot\vec{u})\ .
\end{aligned}$$

注意这里的 $\vec{u}$ 是惯性系 $\mathcal{R}'$ 相对于惯性系 $\mathcal{R}$ 的 3 速度，即用 $\{x^\mu\}$ 坐标描述的 $\mathcal{R}'$ 的速度.

第 7 章“广义相对论基础”习题

~1. 试证弯曲时空麦氏方程 $\nabla^a F_{ab} = -4\pi J_b$ 蕴含电荷守恒定律，即 $\nabla_a J^a = 0$. 注： $\nabla^a F_{ab} = -4\pi J_b$ 等价于式 (7-2-8) 而非式 (7-2-9), 故本题表明式 (7-2-8) 而非式 (7-2-9) 可推出电荷守恒.

证 由方程 $\nabla_a F^{ab} = -4\pi J^b$ 知

$$\begin{aligned}
-4\pi\nabla_b J^b &= \nabla_b\nabla_a F^{ab} \overset{(3\text{-}4\text{-}5)}{=} \nabla_a\nabla_b F^{ab} + R_{abc}{}^a F^{cb} + R_{abc}{}^b F^{ac}\\
&= \nabla_a\nabla_b F^{ab} - R_{bc}F^{cb} + R_{ac}F^{ac} = \nabla_a\nabla_b F^{ab} - R_{cb}F^{cb} + R_{ac}F^{ac}\\
&= \nabla_a\nabla_b F^{ab}\ ,
\end{aligned}$$

其中利用了里奇张量的对称性 $R_{ac} = R_{ca} = R_{(ac)}$. 因此有

$$-4\pi\nabla_b J^b = \nabla_{(b}\nabla_{a)}F^{ab} = \nabla_{(b}\nabla_{a)}F^{[ab]} = 0\ .$$

命题得证.

也可用加了洛伦兹规范条件的 (7-2-8) 式 $\nabla_a\nabla^a A^b - R^{bd}A_d = -4\pi J^b$ 得到电荷守恒律：

$$-4\pi\nabla_b J^b \quad = \quad \nabla_b\nabla_a\nabla^a A^b - \nabla_b(R^{bd}A_d)$$

$$\begin{aligned}
&\overset{(3\text{-}4\text{-}5)}{=} \nabla_a\nabla_b\nabla^a A^b + R_{abc}{}^a\nabla^c A^b + R_{abc}{}^b\nabla^a A^c - \nabla^b(R_{bd}A^d) \\
&= \nabla_a\nabla_b\nabla^a A^b - R_{cb}\nabla^c A^b + R_{ac}\nabla^a A^c - \nabla^b(R_{bd}A^d) \\
&= \nabla_a\nabla_b\nabla^a A^b - \nabla^b(R_{bd}A^d) \\
&= \nabla^a\nabla_b\nabla_a A^b - \nabla^b(R_{bd}A^d) \\
&= \nabla^a(\nabla_a\nabla_b A^b + R_{ad}A^d) - \nabla^b(R_{bd}A^d) \\
&= 0\,,
\end{aligned}$$

最后一步用到了洛伦兹条件 $\nabla_b A^b = 0$. 这是必须的，因为得到 (7-2-8) 式时已经用过这一条件. 于是可以看出从 (7-2-9) 式推不出电荷守恒.

~2. 试证 $\frac{D_F\omega_a}{d\tau} = \frac{D\omega_a}{d\tau} + (A_a \wedge Z_b)\omega^b \ \ \forall \omega_a \in \mathscr{F}_G(0,1)$.

证 利用 $\frac{D_F g_{ab}}{d\tau} = 0$ 和 $\frac{Dg_{ab}}{d\tau} = 0$, 我们有

$$\begin{aligned}
\frac{D_F\omega_a}{d\tau} &= \frac{D_F(g_{ab}\omega^b)}{d\tau} = g_{ab}\frac{D_F\omega^b}{d\tau} = g_{ab}\Big[\frac{D\omega^b}{d\tau} + (A^b \wedge Z^c)\omega_c\Big] \\
&= \frac{D(g_{ab}\omega^b)}{d\tau} + g_{ab}(A^b \wedge Z^c)\omega_c = \frac{D\omega_a}{d\tau} + (A_a \wedge Z_c)\omega^c\,.
\end{aligned}$$

~3. 试证费米导数性质 3.

证 若 w^a 是 $G(\tau)$ 上的空间矢量场，则有 $w^a Z_a = 0$. 这时根据费米导数的定义

$$\frac{D_F w^a}{d\tau} = \frac{Dw^a}{d\tau} + (A^a Z^b - Z^a A^b)w_b = \frac{Dw^a}{d\tau} - Z^a A^b w_b\,.$$

另一方面，由于投影映射 $h^a{}_b = \delta^a{}_b + Z^a Z_b$, 故有

$$\begin{aligned}
h^a{}_b\frac{Dw^b}{d\tau} &= (\delta^a{}_b + Z^a Z_b)\frac{Dw^b}{d\tau} = \frac{Dw^a}{d\tau} + Z^a Z_b Z^c\nabla_c w^b \\
&= \frac{Dw^a}{d\tau} + Z^a Z^c(Z_b\nabla_c w^b) = \frac{Dw^a}{d\tau} - Z^a Z^c(w^b\nabla_c Z_b) \\
&= \frac{Dw^a}{d\tau} - Z^a(Z^c\nabla_c Z_b)w^b = \frac{Dw^a}{d\tau} - Z^a A_b w^b \\
&= \frac{Dw^a}{d\tau} - Z^a A^b w_b\,,
\end{aligned}$$

其中用到了 $0 = \nabla_c(0) = \nabla_c(w^b Z_b) = Z_b\nabla_c w^b + w^b\nabla_c Z_b$. 因此有

$$\frac{D_F w^a}{d\tau} = h^a{}_b\frac{Dw^b}{d\tau}\,.$$

4. 试证类时线 $G(\tau)$ 上长度不变 (且非零) 的矢量场 v^a 必经受时空转动. 提示：令 $u^a \equiv Dv^a/d\tau$, 则 $u_a v^a = 0$. 先证：无论 $v_a v^a$ 为零与否，总有 $G(\tau)$ 上矢量场 v'^a 使 $v'_a v^a = 1$. 再验证 v^a 经受以 $\Omega_{ab} \equiv 2v'_{[a}u_{b]}$ 为角速度 2 形式的时空转动.

证 因为 v^a 沿 $G(\tau)$ 长度不变，故有 $0=\frac{D(v^a v_a)}{d\tau}=2v^a\frac{Dv_a}{d\tau}=2v^a u_a$, 其中令 $u_a\equiv\frac{Dv_a}{d\tau}$. 总可以找到 $G(\tau)$ 上的矢量场 v'^a 满足 $v'_a v^a=1$. 于是有关系式

$$\begin{aligned}\frac{Dv^a}{d\tau} &= u^a=u^a(1)-v'^a(0)=u^a(v'^b bv_b)-v'^a(u^b v_b)\\ &= -(v'^a u^b-u^a v'^b)v_b=-2v'^{[a}u^{b]}v_b=-\Omega^{ab}v_b\ ,\end{aligned}$$

其中 $\Omega^{ab}\equiv 2v'^{[a}u^{b]}$, 它的角速度 2 形式为 $\Omega_{ab}\equiv 2v'_{[a}u_{b]}=v'_a\wedge u_b$. 可见矢量场 v^a 沿类时线 $G(\tau)$ 以角速度 Ω^{ab} 做时空转动.

5. 设 $\{T,\,X,\,Y,\,Z\}$ 为闵氏时空的洛伦兹坐标系，曲线 $G(\tau)$ 的参数表达式为

$$T=A^{-1}\sinh A\tau\ ,\qquad X=A^{-1}\cosh A\tau\ ,\qquad Y=Z=0\ ,\qquad (\text{其中 } A \text{ 为常数})$$

(a) 试证 $G(\tau)$ 是类时双曲线 (即图 6-43 的 G), τ 是固有时， A 是 $G(\tau)$ 的 4 加速 A^a 的长度.

*(b) 试证从 $\{T,\,X,\,Y,\,Z\}$ 系原点 o 出发的与 $G(\tau)$ 有交的任一半直线 $\mu(s)$ 都与 $G(\tau)$ 正交.

*(c) 设 (b) 中的 $\mu(s)$ 的参数 s 是 μ 的线长，随着 $\mu(s)$ 取遍所有从 o 出发并与 $G(\tau)$ 有交的半直线，便得 $G(\tau)$ 上的一个空间矢量场 $w^a\equiv(\partial/\partial s)^a$, 试证 w^a 沿 $G(\tau)$ 费移.

*(d) 令 $Z^a\equiv(\partial/\partial\tau)^a$, 选 $\{Z^a,w^a,(\partial/\partial Y)^a,(\partial/\partial Z)^a\}$ 为 $G(\tau)$ 上的正交归一 4 标架场，求出 $G(\tau)$ 的固有坐标系 $\{t,x,y,z\}$ 并指出其坐标域.

答： $T=(A^{-1}+x)\sinh At\,,\quad X=(A^{-1}+x)\cosh At\,,\quad Y=y\quad Z=z\,.$

(e) 写出闵氏度规在上述固有坐标系中的线元表达式. 计算闵氏度规在该系的克氏符，验证它满足引理 7-4-3, 即式 (7-4-10).

解 (a) 双曲线 $G(\tau)$ 由惯性洛伦兹系的参数坐标 $(A^{-1}\sinh A\tau,\ A^{-1}\cosh A\tau,\ 0,\ 0)$ 即 $X^2-T^2=A^{-2}$ 描述，它的切矢 (4 速) 为 $Z^a=(\frac{\partial}{\partial\tau})^a$, 在洛伦兹系的参数式为

$$\begin{aligned}Z^\mu(\tau) &= Z^a(dX^\mu)_a=\Big(\frac{\partial}{\partial\tau}\Big)^a(dX^\mu)_a=\frac{\partial X^\mu}{\partial\tau}\\ &= \frac{\partial}{\partial\tau}(T,X,Y,Z)=(\cosh A\tau,\sinh A\tau,0,0)\ .\end{aligned}$$

因为

$$Z^aZ_a=Z^\mu(\tau)Z_\mu(\tau)=\eta_{\mu\nu}Z^\mu(\tau)Z^\nu(\tau)=-\cosh^2A\tau+\sinh^2A\tau=-1\ ,$$

所以它是类时双曲线. $G(\tau)$ 的 4 加速为 $A^a=Z^b\nabla_bZ^a$, 与闵氏时空的度规 η_{ab} 相适配的导数算符为普通导数 ∂_a, 故 $A^a=Z^b\partial_bZ^a$, 其在洛伦兹坐标系的

分量为

$$\begin{aligned} A^\mu(\tau) &= (dX^\mu)_a Z^b \partial_b Z^a = Z^b \partial_b [(dX^\mu)_a Z^a] \\ &= \frac{\partial}{\partial\tau} Z^\mu(\tau) = \frac{\partial}{\partial\tau}(\cosh A\tau, \sinh A\tau, 0, 0) \\ &= (A\sinh A\tau, A\cosh A\tau, 0, 0)\ . \end{aligned}$$

因此 4 加速 A^a 的长度 (平方) 为

$$A^a A_a = A^\mu A_\mu = \eta_{\mu\nu} A^\mu A^\nu = -(A\sinh A\tau)^2 + (A\cosh A\tau)^2 = A^2\ ,$$

即 A 是 A^a 的长度，所以 G 做匀加速运动 (见第 6 章习题 13).

(b) 从原点 o 出发的与 $G(\tau)$ 有交的任一半直线 $\mu(s)$ 都是闵氏时空的类空测地线, 其洛伦兹坐标的参数表达式为 $(sA^{-1}\sinh A\tau,\ sA^{-1}\cosh A\tau,\ 0,\ 0)$, 其中 s 为仿射参数. 如果要求 s 就是线长, 可取坐标参数为 $(s\sinh A\tau,\ s\cosh A\tau,\ 0,\ 0)$. $\mu(s)$ 的切矢为 $w^a = (\frac{\partial}{\partial s})^a$, 在洛伦兹系的参数式为

$$\begin{aligned} w^\mu(s) &= w^a (dX^\mu)_a = \left(\frac{\partial}{\partial s}\right)^a (dX^\mu)_a = \frac{\partial X^\mu}{\partial s} \\ &= \frac{\partial}{\partial s}(s\sinh A\tau,\ s\cosh A\tau,\ 0,\ 0) = (\sinh A\tau, \cosh A\tau, 0, 0)\ . \end{aligned}$$

可见

$$w^a w_a = w^\mu w_\mu = -\sinh^2 A\tau + \cosh^2 A\tau = 1\ ,$$

w^a 是类空单位矢. 另外，显然有

$$w^a Z_a = w^\mu Z_\mu = -\sinh A\tau\cosh A\tau + \cosh A\tau\sinh A\tau = 0\ ,$$

即 w^a 与 Z^a 正交，也就是 $\mu(s)$ 与 $G(\tau)$ 正交，交点 p 的洛伦兹坐标为

$$(s\sinh A\tau,\ s\cosh A\tau,\ 0,\ 0) = (A^{-1}\sinh A\tau,\ A^{-1}\cosh A\tau,\ 0,\ 0)$$

即仿射参数 (从 o 到交点的线长) 为 $s_p = A^{-1}$.

(c) w^a 沿 $G(\tau)$ 的费米导数为

$$\frac{D_F w^a}{d\tau} = \frac{Dw^a}{d\tau} + (A^a Z^b - Z^a A^b) w_b = \frac{Dw^a}{d\tau} - Z^a A^b w_b\ ,$$

相应的洛伦兹系的分量式为

$$\frac{D_F w^\mu}{d\tau} = \frac{Dw^\mu}{d\tau} - Z^\mu A^\nu w_\nu\ ,$$

其中

$$\begin{aligned} \frac{Dw^\mu}{d\tau} &= \frac{dw^\mu}{d\tau} = \frac{d}{d\tau}(\sinh A\tau, \cosh A\tau, 0, 0) = (A\cosh A\tau, A\sinh A\tau, 0, 0)\ , \\ A^\nu w_\nu &= -(A\sinh A\tau)\sinh A\tau + (A\cosh A\tau)\cosh A\tau = A\ , \\ Z^\mu A^\nu w_\nu &= (\cosh A\tau, \sinh A\tau, 0, 0)A = (A\cosh A\tau, A\sinh A\tau, 0, 0)\ . \end{aligned}$$

因此有

$$\frac{D_F w^\mu}{d\tau} = 0 \,,$$

即 w^a 沿 $G(\tau)$ 费移.

(d) 选 $\{Z^a, w^a, (\partial/\partial Y)^a, (\partial/\partial Z)^a\}$ 为 $G(\tau)$ 上的正交归一 4 标架场. 设 $\mu(s)$ 与 $G(\tau)$ 交于 p, 则 $\mu(s)$ 上任意一点 q 在此标架上的分量为 $(t(q), x(q), y(q), z(q))$. 根据式 (7-4-1) 的定义，$t(q) = \tau_p$, 而 τ_p 等于 $G(\tau)$ 线的 p 点到到任意一点 (如 p 与 X 轴交点) 的线长；$x(q)$ 等于 $\mu(s)$ 上的直线段 pq 的线长,即 $x(q) = s_q - s_p = s_q - A^{-1}$. 注意到 q 点的洛伦兹坐标分量为 $(s_q \sinh A\tau_q, s_q \cosh A\tau_q, 0, 0)$, 故得到关系

$$\begin{aligned}
T(q) &= s_q \sinh A\tau_q = [A^{-1} + x(q)] \sinh At(q) \,, \\
X(q) &= s_q \cosh A\tau_q = [A^{-1} + x(q)] \cosh At(q) \,, \\
Y(q) &= Z(q) = 0 \,.
\end{aligned}$$

p 可以是 $G(\tau)$ 上的任意点， q 可以是 $\mu(s)$ 上的任意点，于是我们找到 $G(\tau)$ 的固有坐标系 $\{t, x, y, z\}$ 和洛伦兹坐标系 $\{T, X, Y, Z\}$ 的关系：

$$\begin{aligned}
T &= (A^{-1} + x) \sinh At \,, \\
X &= (A^{-1} + x) \cosh At \,, \\
Y &= y \,, \\
Z &= z \,.
\end{aligned}$$

可以看出 t, y, z 都可从负无穷到正无穷，但因 $X \geq 0$, 所以 x 的坐标域为 $[-A^{-1}, +\infty)$

(e) 因为

$$\begin{aligned}
dT &= (1 + Ax) \cosh At\, dt + \sinh At\, dx \,, \\
dX &= (1 + Ax) \sinh At\, dt + \cosh At\, dx \,, \\
dY &= dy \,, \\
dZ &= dz \,,
\end{aligned}$$

我们得闵氏度规的线元为

$$\begin{aligned}
ds^2 &= -dT^2 + dX^2 + dY^2 + dZ^2 \\
&= -[(1 + Ax) \cosh At\, dt + \sinh At\, dx]^2 \\
&\quad +[(1 + Ax) \sinh At\, dt + \cosh At\, dx]^2 + dy^2 + dz^2 \\
&= -(1 + Ax)^2 dt^2 + dx^2 + dy^2 + dz^2 \,.
\end{aligned}$$

因此闵氏度规在 G 的固有坐标系的分量为

$$g_{00} = -(1 + Ax)^2 \ , \qquad g_{11} = g_{22} = g_{33} = 1 \ ,$$

或

$$g^{00} = -(1 + Ax)^{-2} \ , \qquad g^{11} = g^{22} = g^{33} = 1 \ .$$

注意坐标基底虽然正交却不归一. 为了得到正交归一基底，由度规张量场

$$g_{ab} = g_{\mu\nu}(dx^\mu)_a(dx^\nu)_b = \eta_{\mu\nu}(e^\mu)_a(e^\nu)_b$$

对比得出对偶基底为

$$(e^0)_a = (1 + Ax)(dt)_a \ , \quad (e^1)_a = (dx)_a \ , \quad (e^2)_a = (dy)_a \ , \quad (e^3)_a = (dz)_a \ .$$

由

$$(e_\mu)^a = \eta_{\mu\nu}g^{ab}(e^\nu)_b = \eta_{\mu\nu}g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(e^\nu)_b$$

知基底为

$$\begin{aligned}
(e_0)^a &= \eta_{00}g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(e^0)_b = -g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(1 + Ax)(dt)_b \\
&= -g^{00}(\partial/\partial t)^a(\partial/\partial t)^b(1 + Ax)(dt)_b = (1 + Ax)^{-1}(\partial/\partial t)^a \ , \\
(e_1)^a &= \eta_{11}g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(e^1)_b = g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(dx)_b \\
&= g^{11}(\partial/\partial x)^a(\partial/\partial x)^b(dx)_b = (\partial/\partial x)^a \ , \\
(e_2)^a &= \eta_{22}g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(e^2)_b = g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(dy)_b \\
&= g^{22}(\partial/\partial y)^a(\partial/\partial y)^b(dy)_b = (\partial/\partial y)^a \ , \\
(e_3)^a &= \eta_{33}g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(e^3)_b = g^{\sigma\rho}(\partial/\partial x^\sigma)^a(\partial/\partial x^\rho)^b(dz)_b \\
&= g^{33}(\partial/\partial z)^a(\partial/\partial z)^b(dz)_b = (\partial/\partial z)^a \ ,
\end{aligned}$$

即

$$(e_0)^a = (1 + Ax)^{-1}(\partial_t)^a \ , \quad (e_1)^a = (\partial_x)^a \ , \quad (e_2)^a = (\partial_y)^a \ , \quad (e_3)^a = (\partial_z)^a \ .$$

这一关系也可由度规张量场

$$g^{ab} = g^{\mu\nu}(\partial/\partial x^\mu)_a(\partial/\partial x^\nu)_b = \eta^{\mu\nu}(e_\mu)^a(e_\nu)^b$$

直接对比得出. 另外还有 [如 $(e_0)_a = \eta_{0\nu}(e^\nu)_a = \eta_{00}(e^0)_a = -(e^0)_a$]

$$(e_0)_a = -(1 + Ax)(dt)_a \ , \quad (e_1)_a = (dx)_a \ , \quad (e_2)_a = (dy)_a \ , \quad (e_3)_a = (dz)_a \ .$$

下面利用式 (5-7-19) 计算 $\Lambda_{\alpha\beta\gamma}$ 和式 (5-7-20) 计算 $\omega_{\alpha\beta\gamma}$. 注意到反称关系 $\Lambda_{\alpha\beta\gamma} = -\Lambda_{\gamma\beta\alpha}$, 只须计算 $\alpha \neq \gamma$ 情形. 因为

$$\begin{aligned}
(e_0)_\lambda &= (e_0)_a(\partial/\partial x^\lambda)^a = -(1 + Ax)(dt)_a(\partial/\partial x^\lambda)^a = -(1 + Ax)\delta^0{}_\lambda \ , \\
(e_1)_\lambda &= (e_1)_a(\partial/\partial x^\lambda)^a = (dx)_a(\partial/\partial x^\lambda)^a = \delta^1{}_\lambda \ , \\
(e_2)_\lambda &= (e_2)_a(\partial/\partial x^\lambda)^a = (dy)_a(\partial/\partial x^\lambda)^a = \delta^2{}_\lambda \ , \\
(e_3)_\lambda &= (e_3)_a(\partial/\partial x^\lambda)^a = (dz)_a(\partial/\partial x^\lambda)^a = \delta^3{}_\lambda \ .
\end{aligned}$$

有

$$\begin{aligned}(e_1)_{\lambda,\tau} &= (e_2)_{\lambda,\tau} = (e_3)_{\lambda,\tau} = 0\ ,\\ (e_0)_{\lambda,\tau} &= \frac{\partial}{\partial x^\tau}[-(1+Ax)\delta^0{}_\lambda] = -A\delta^1{}_\tau\delta^0{}_\lambda\ .\end{aligned}$$

代入式 (5-7-19) $\Lambda_{\alpha\beta\gamma} = [(e_\beta)_{\lambda,\tau} - (e_\beta)_{\tau,\lambda}](e_\alpha)^\lambda(e_\gamma)^\tau$ (注意 这里的 α, β 为标架指标 0, 1, 2, 3, 而 λ, τ 为坐标系指标 0, 1, 2, 3 !):

$$\begin{aligned}\Lambda_{\alpha0\gamma} &= [(e_0)_{\lambda,\tau} - (e_0)_{\tau,\lambda}](e_\alpha)^\lambda(e_\gamma)^\tau\\ &= [-A\delta^1{}_\tau\delta^0{}_\lambda + A\delta^1{}_\lambda\delta^0{}_\tau](e_\alpha)^\lambda(e_\gamma)^\tau\\ &= -A(e_\alpha)^0(e_\gamma)^1 + A(e_\alpha)^1(e_\gamma)^0\\ &= -A(1+Ax)^{-1}\delta^0{}_\alpha\delta^1{}_\gamma + A(1+Ax)^{-1}\delta^1{}_\alpha\delta^0{}_\gamma\ .\end{aligned}$$

因此得到非零的 $\Lambda_{\alpha\beta\gamma}$:

$$\Lambda_{001} = -\Lambda_{100} = -A(1+Ax)^{-1}\ .$$

代入式 (5-7-20) $\omega_{\alpha\beta\gamma} = \frac{1}{2}(\Lambda_{\alpha\beta\gamma} + \Lambda_{\gamma\alpha\beta} - \Lambda_{\beta\gamma\alpha})$ 求得非零的 $\omega_{\alpha\beta\gamma}$ (注意反称关系，非零时 $\alpha \neq \beta$):

$$\omega_{010} = \frac{1}{2}(\Lambda_{010} + \Lambda_{001} - \Lambda_{100}) = \Lambda_{001} = -A(1+Ax)^{-1} = -\omega_{100}\ ,$$

联络 1 形式为 $\boldsymbol{\omega}_{\alpha\beta} = \omega_{\alpha\beta a} = \omega_{\alpha\beta\gamma}(e^\gamma)_a = \omega_{\alpha\beta\gamma}\boldsymbol{e}^\gamma$:

$$-\boldsymbol{\omega}_{10} = \boldsymbol{\omega}_{01} = \omega_{010}\boldsymbol{e}^0 = -A(1+Ax)^{-1}\boldsymbol{e}^0 = -A(dt)_a\ ,$$

即有

$$\boldsymbol{\omega}_1{}^0 = \boldsymbol{\omega}_0{}^1 = -A(1+Ax)^{-1}\boldsymbol{e}^0 = -A(dt)_a\ .$$

由嘉当第二结构方程式 (5-7-8) 很容易看出现在的黎曼张量为零: $\boldsymbol{R}_1{}^0 = d\boldsymbol{\omega}_1{}^0 + \boldsymbol{\omega}_1{}^\gamma\wedge\boldsymbol{\omega}_\gamma{}^0 = -Ad(dt) = 0$, 因为闵氏时空的平直性.

在此正交归一标架上的联络由式 (5-7-1) 给出:

$$(e_\beta)^b\nabla_b(e_\alpha)^a = \gamma^\gamma{}_{\alpha\beta}(e_\gamma)^a\ ,$$

即

$$\gamma^\gamma{}_{\alpha\beta} = (e^\gamma)_a(e_\beta)^b\nabla_b(e_\alpha)^a = -(e^\gamma)_a(e_\beta)^b\omega_\alpha{}^\delta{}_b(e_\delta)^a = -\omega_\alpha{}^\gamma{}_b(e_\beta)^b\ ,$$

当然，这就是式 (5-7-4). 于是非零的联络只有

$$\begin{aligned}\gamma^1{}_{0\beta} &= -\omega_0{}^1{}_b(e_\beta)^b = A(1+Ax)^{-1}(e^0)_b(e_\beta)^b = A(1+Ax)^{-1}\delta^0{}_\beta\ ,\\ \gamma^0{}_{1\beta} &= -\omega_1{}^0{}_b(e_\beta)^b = A(1+Ax)^{-1}(e^0)_b(e_\beta)^b = A(1+Ax)^{-1}\delta^0{}_\beta\ ,\end{aligned}$$

即

$$\gamma^0{}_{01} = \gamma^0{}_{10} = \gamma^1{}_{00} = A(1 + Ax)^{-1} \ .$$

也可以按如下方式求得. 度规在 $G(\tau)$ 的固有坐标系的克氏符由式 (3-2-10')

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$$

给出，为此先计算

$$g_{00,1} = -2A(1 + Ax) \ .$$

首先容易看出，只有当 μ, ν 中至少有一个为 0 时，克氏符才不为零，于是 $\Gamma^\sigma{}_{ij} = 0$. 显然

$$\Gamma^0{}_{00} = \frac{1}{2} g^{0\rho}(g_{\rho 0,0} + g_{\rho 0,0} - g_{00,\rho}) = \frac{1}{2} g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) = 0 \ .$$

而

$$\begin{aligned}
\Gamma^0{}_{0i} = \Gamma^0{}_{i0} &= \frac{1}{2} g^{0\rho}(g_{\rho 0,i} + g_{\rho i,0} - g_{0i,\rho}) = \frac{1}{2} g^{00}(g_{00,i} + g_{0i,0} - g_{0i,0}) \\
&= \frac{1}{2} g^{00}(g_{00,i}) = \frac{1}{2}[-(1 + Ax)^{-2}][-2A(1 + Ax)]\delta_{i1} \\
&= A(1 + Ax)^{-1}\delta_{i1} \ , \\
\Gamma^i{}_{00} &= \frac{1}{2} g^{i\rho}(g_{\rho 0,0} + g_{\rho 0,0} - g_{00,\rho}) = \frac{1}{2} g^{ii}(g_{i0,0} + g_{i0,0} - g_{00,i}) \\
&= -\frac{1}{2} g^{ii} g_{00,i} = -\frac{1}{2}[1][-2A(1 + Ax)]\delta_{i1} \\
&= A(1 + Ax)\delta_{i1} \ , \\
\Gamma^i{}_{0j} = \Gamma^i{}_{j0} &= \frac{1}{2} g^{i\rho}(g_{\rho 0,j} + g_{\rho j,0} - g_{0j,\rho}) = \frac{1}{2} g^{ii}(g_{i0,j} + g_{ij,0} - g_{0j,i}) \\
&= 0 \ .
\end{aligned}$$

转到正交归一标架上的克氏符 (联络):

$$\begin{aligned}
\gamma^0{}_{0i} &= \gamma^c{}_{ab}(e^0)_c(e_0)^a(e_i)^b = \gamma^c{}_{ab}[(1 + Ax)(dt)_c][(1 + Ax)^{-1}(\partial_t)^a](\partial_i)^b \\
&= \gamma^c{}_{ab}(dt)_c(\partial_t)^a(\partial_i)^b = \Gamma^0{}_{0i} = A(1 + Ax)^{-1}\delta_{i1} \ , \\
\gamma^i{}_{00} &= \gamma^c{}_{ab}(e^i)_c(e_0)^a(e_0)^b = \gamma^c{}_{ab}(dx^i)_c[(1 + Ax)^{-1}(\partial_t)^a][(1 + Ax)^{-1}(\partial_t)^b] \\
&= (1 + Ax)^{-2}\gamma^c{}_{ab}(dx^i)_c(\partial_t)^a(\partial_t)^b = (1 + Ax)^{-2}\Gamma^i{}_{00} = A(1 + Ax)^{-1}\delta_{i1} \ .
\end{aligned}$$

因此正交归一标架上的非零克氏符 (应为联络，因为是非坐标基底) 为

$$\gamma^0{}_{01} = \gamma^0{}_{10} = \gamma^1{}_{00} = A(1 + Ax)^{-1} \ .$$

与前面的结果相同.

最后我们看 G 的 4 加速 A^a 在 G 的固有坐标系 $\{x^\mu\}$ 上的分量表达式，令它为 $\hat{A}^\mu$, 即有

$$\hat{A}^\mu = (dx^\mu)_a A^a = (dx^\mu)_a A^\nu \Big(\frac{\partial}{\partial X^\nu}\Big)^a = \frac{\partial x^\mu}{\partial X^\nu} A^\nu \ ,$$

其中 $A^\nu = (A\sinh At, A\cosh At, 0, 0)$ 为它在洛伦兹系的分量式，已在 (a) 中得到．下面我们先求矩阵 $\frac{\partial X^\nu}{\partial x^\mu}$, 它的逆矩阵即为 $\frac{\partial x^\mu}{\partial X^\nu}$. 很容易看出

$$\left[\frac{\partial X^\nu}{\partial x^\mu}\right] = \begin{bmatrix} (1+Ax)\cosh At & \sinh At & 0 & 0 \\ (1+Ax)\sinh At & \cosh At & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

因此得

$$\left[\frac{\partial x^\mu}{\partial X^\nu}\right] = \left[\frac{\partial X^\nu}{\partial x^\mu}\right]^{-1} = \begin{bmatrix} (1+Ax)^{-1}\cosh At & -(1+Ax)^{-1}\sinh At & 0 & 0 \\ -\sinh At & \cosh At & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

最后我们得

$$\begin{aligned}\hat{A}^\mu &= \begin{bmatrix} \hat{A}^0 \\ \hat{A}^1 \\ \hat{A}^2 \\ \hat{A}^3 \end{bmatrix} = \begin{bmatrix} (1+Ax)^{-1}\cosh At & -(1+Ax)^{-1}\sinh At & 0 & 0 \\ -\sinh At & \cosh At & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A\sinh At \\ A\cosh At \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ A \\ 0 \\ 0 \end{bmatrix}.\end{aligned}$$

因此 G 的 4 加速 A^a 在 G 的固有坐标系上只有在 x 方向上有一个分量，大小为 4 加速的大小 A, 即 $\hat{A}^1 = A$. 因为引理 7-4-3 仅对 G 线上 $(x=0)$ 成立，显然现在

$$\Gamma^0{}_{01}|_{x=0} = \Gamma^0{}_{10}|_{x=0} = \Gamma^1{}_{00}|_{x=0} = \hat{A}^1 .$$

6. 设 G 是质点 L 在点 $p \in L$ 的瞬时静止 自由下落 观者 (即 G 的 4 速 Z^a 与 L 的 4 速 U^a 在 p 点相切), A^a 是 L 在 p 点的 4 加速， a^a 是 L 在 p 点相对于 G 的 3 加速 [由式 (7-4-3) 定义], 试证 $a^a = A^a$. 注：本命题可视为命题 6-3-6 在弯曲时空的推广.

证 我们考虑一个更加一般的情形：设 $G(t)$ 是任意 (不一定自由下落而且可以有自转) 观者，其 4 速为 $Z^a = (\frac{\partial}{\partial t})^a$, 在它的固有坐标系 $\{t, x^i\}$ 的 (某段) 坐标域内有个任意运动的质点 $L(\tau_L)$, 其 4 速为 $U^a = (\frac{\partial}{\partial \tau_L})^a$. 质点 L 的运动可由用 G 的固有坐标的参数式表达的世界线 $\{x^\mu(t)\} = \{t, x^i(t)\}$ 描述，那么质点 L 相对于观者 G 的 3 速为

$$u^a = \frac{dx^i(t)}{dt}\Big(\frac{\partial}{\partial x^i}\Big)^a ,$$

3 加速为

$$a^a=\frac{d^2x^i(t)}{dt^2}\Big(\frac{\partial}{\partial x^i}\Big)^a\ .$$

因为 G 和 L 都是任意运动，所以它们都可以有 4 加速，分别为 $\hat{A}=Z^b\nabla_bZ^a$ 和 $A=U^b\nabla_bU^a$. 观者 G 还可以有自转的空间转动角速度 ω^a. 题目中要证明的是：当观者 G 无自转 ($\omega^a=0$) 且自由下落 ($\hat{A}^a=0$) 时，若在某一时空点 p 正好和质点 L 相对静止 (它们的世界线在 p 点相切且 4 速相等), 即 $Z|_p=U|_p$, 那么质点 L 的 4 加速 A^a 等于观者 G 看到的质点的 3 加速 a^a (即质点相对于观者的 3 加速).

首先对质点 L 的 4 速 U^a 做 $3+1$ 分解

$$\begin{aligned}U^a &= \Big(\frac{\partial}{\partial\tau_L}\Big)^a=\frac{dt}{d\tau_L}\Big(\frac{\partial}{\partial t}\Big)^a+\frac{dx^i}{d\tau_L}\Big(\frac{\partial}{\partial x^i}\Big)^a=\gamma\Big(\frac{\partial}{\partial t}\Big)^a+\gamma\frac{dx^i}{dt}\Big(\frac{\partial}{\partial x^i}\Big)^a\\ &= \gamma(e_0)^a+\gamma u^i(e_i)^a=\gamma Z^a+\gamma u^a\ ,\end{aligned}$$

其中令 $\gamma\equiv\frac{dt}{d\tau_L}$, $(e_\mu)^a\equiv(\frac{\partial}{\partial x^\mu})^a$ 为固有坐标基矢. 质点 L 的 4 加速 A^a 也可做 $3+1$ 分解

$$\begin{aligned}A^a &= U^b\nabla_bU^a=U^b\nabla_b[\gamma(e_0)^a+\gamma u^i(e_i)^a]\\
&= U^b\Big[(e_0)^a\nabla_b\gamma+\gamma\nabla_b(e_0)^a+(e_i)^a\nabla_b(\gamma u^i)+\gamma u^i\nabla_b(e_i)^a\Big]\\
&= (e_0)^a\frac{d\gamma}{d\tau_L}+\gamma[\gamma(e_0)^b+\gamma u^j(e_j)^b]\nabla_b(e_0)^a\\
&\quad +(e_i)^a\frac{d(\gamma u^i)}{d\tau_L}+\gamma u^i[\gamma(e_0)^b+\gamma u^j(e_j)^b]\nabla_b(e_i)^a\\
&= (e_0)^a\gamma\frac{d\gamma}{dt}+\gamma^2[(e_0)^b\nabla_b(e_0)^a+u^j(e_j)^b\nabla_b(e_0)^a]\\
&\quad +(e_i)^a\gamma\frac{d(\gamma u^i)}{dt}+\gamma^2u^i[(e_0)^b\nabla_b(e_i)^a+u^j(e_j)^b\nabla_b(e_i)^a]\\
&\overset{(7\text{-}4\text{-}13)}{=} (e_0)^a\gamma\frac{d\gamma}{dt}+\gamma^2[\Gamma^\sigma{}_{00}(e_\sigma)^a+u^j\Gamma^\sigma{}_{0j}(e_\sigma)^a]\\
&\quad +(e_i)^a\gamma\frac{d(\gamma u^i)}{dt}+\gamma^2u^i[\Gamma^\sigma{}_{i0}(e_\sigma)^a+u^j\Gamma^\sigma{}_{ij}(e_\sigma)^a]\\
&\overset{(7\text{-}4\text{-}10)}{=} (e_0)^a\gamma\frac{d\gamma}{dt}+\gamma^2[\Gamma^i{}_{00}(e_i)^a+u^j\Gamma^0{}_{0j}(e_0)^a+u^j\Gamma^i{}_{0j}(e_i)^a]\\
&\quad +(e_i)^a\gamma\frac{d(\gamma u^i)}{dt}+\gamma^2u^i[\Gamma^0{}_{i0}(e_0)^a+\Gamma^j{}_{i0}(e_j)^a+0]\\
&\overset{(7\text{-}4\text{-}10)}{=} (e_0)^a\gamma\frac{d\gamma}{dt}+\gamma^2[\hat{A}^i(e_i)^a+u^j\hat{A}_j(e_0)^a+u^j(-\omega_k\varepsilon^{ki}{}_j)(e_i)^a]\\
&\quad +(e_i)^a\gamma\frac{d(\gamma u^i)}{dt}+\gamma^2u^i[\hat{A}_i(e_0)^a+(-\omega_k\varepsilon^{kj}{}_i)(e_j)^a]\\
&= (e_0)^a\gamma\frac{d\gamma}{dt}+(e_i)^a\gamma\frac{d(\gamma u^i)}{dt}+\gamma^2\hat{A}^i(e_i)^a+2\gamma^2(u^j\hat{A}_j)(e_0)^a+2\gamma^2\varepsilon^i{}_{kj}\omega^ku^j(e_i)^a\\
&= (e_0)^a\Big[\gamma\frac{d\gamma}{dt}+2\gamma^2(u^j\hat{A}_j)\Big]+(e_i)^a\Big[\gamma\frac{d(\gamma u^i)}{dt}+\gamma^2\hat{A}^i+2\gamma^2\varepsilon^i{}_{kj}\omega^ku^j\Big]\end{aligned}$$

$$= \quad (e_0)^a\left[\gamma\frac{d\gamma}{dt}+2\gamma^2(u^j\hat{A}_j)\right]+(e_i)^a\left[\gamma^2a^i+u^i\gamma\frac{d\gamma}{dt}+\gamma^2\hat{A}^i+2\gamma^2\varepsilon^i{}_{kj}\omega^ku^j\right].$$

[注意引理 7-4-3 式 (7-4-10) 仅对 G 线上的点成立.] 我们先看几种特殊情况: ①如果 G 是无自转自由观者 ($\hat{A}^a=0$, $\omega^a=0$), 那么

$$A^a=(e_0)^a\left[\gamma\frac{d\gamma}{dt}\right]+(e_i)^a\left[\gamma^2a^i+u^i\gamma\frac{d\gamma}{dt}\right].$$

这一式子在 G 为惯性系 (平直的闵氏时空) 时就退回到命题 6-3-5 的结论 [式 (6-3-37)], 因为这时有 $\frac{d\gamma}{dt}=\gamma^3u^ia_i=\gamma^3\vec{u}\cdot\vec{a}$. ②如果 L 是自由下落质点，而观者 G 的运动任意，这时 $L(\tau_L)$ 为类时测地线，质点的 4 加速 $A^a=0$, 于是有关系式

$$\gamma\frac{d\gamma}{dt}+2\gamma^2(u^j\hat{A}_j)=0\,,$$

这正是命题 7-4-2 的证明过程中用到的一个等式 (见选读 7-4-1 最后一个式子), 以及

$$\gamma^2a^i+u^i\gamma\frac{d\gamma}{dt}+\gamma^2\hat{A}^i+2\gamma^2\varepsilon^i{}_{kj}\omega^ku^j=0\,,$$

利用上面的等式得

$$a^i-2u^i(u^j\hat{A}_j)+\hat{A}^i+2\varepsilon^i{}_{kj}\omega^ku^j=0\,,$$

因为式中都是空间量，没有时间分量，改写为抽象指标即为

$$a^a-2u^a(u^b\hat{A}_b)+\hat{A}^a+2\varepsilon^a{}_{bc}\omega^bu^c=0\,,$$

这正是命题 7-4-2 式 (7-4-7) 的结果.

最后我们回到习题本身. 如果 G 是无自转观者，这时

$$A^a=(e_0)^a\left[\gamma\frac{d\gamma}{dt}+2\gamma^2(u^j\hat{A}_j)\right]+(e_i)^a\left[\gamma^2a^i+u^i\gamma\frac{d\gamma}{dt}+\gamma^2\hat{A}^i\right].$$

我们证明当 $Z|_p=U|_p$ 时 $u^i|_p=0$ (即 $u^a|_p=0$) 以及 $\gamma|_p=1$, $\frac{d\gamma}{dt}|_p=0$. 首先因 $U^b=\gamma Z^b+\gamma u^b$, 有 $g_{ab}Z^aU^b=g_{ab}Z^a(\gamma Z^b+\gamma u^b)$. 于是在 p 点上左边为 $g_{ab}|_pZ^a|_pU^b|_p=\eta_{ab}Z^a|_pZ^b|_p=-1$, 右边为 $g_{ab}|_pZ^a|_p(\gamma|_pZ^b|_p+\gamma|_pu^b|_p)=\gamma|_pg_{ab}|_pZ^a|_pZ^b|_p=-\gamma$, 得 $\gamma|_p=1$. 这时 $U^a|_p=\gamma|_pZ^a|_p+\gamma|_pu^a|_p=Z^a|_p+u^a|_p$, 故有 $u^a|_p=U^a|_p-Z^a|_p=0$. 【如何证明 $\frac{d\gamma}{dt}|_p=0$? 】 将这些结果代回上式，我们有

$$A^a|_p=(e_i)^a\left[a^i|_p+\hat{A}^i|_p\right]=a^a|_p+\hat{A}^a|_p\,.$$

当观者 G 自由下落时有 $A^a|_p=a^a|_p$.

~7. 度规 g_{ab} 叫 **里奇平直** 的，若 g_{ab} 的里奇张量为零. 试证 g_{ab} 是真空爱因斯坦方程的解的充要条件为 g_{ab} 是里奇平直的.

证 真空爱因斯坦方程为 $G_{ab}=R_{ab}-\frac{1}{2}g_{ab}R=0$. 如果 g_{ab} 是里奇平直的，则里奇张量 $R_{ab}=0$, 于是标量曲率 $R=g^{ab}R_{ab}=0$, 显然这是爱因斯坦方程的解. 如果 R_{ab} 是爱因斯坦方程的解，满足 $R_{ab}-\frac{1}{2}g_{ab}R=0$, 以度规 g^{ab} 作用，有 $g^{ab}R_{ab}-\frac{1}{2}g^{ab}g_{ab}R=R-\frac{1}{2}\delta^a{}_aR=R-2R=-R=0$, 代回方程即有 $R_{ab}=0$, 所以里奇平直.

~8. 设 (M,g_{ab}) 为里奇平直时空 (定义见上题), ξ^a 是其中的一个 Killing 矢量场，试证 $F_{ab}:=(d\xi)_{ab}$ 满足 (M,g_{ab}) 的无源 $(J_a=0)$ 麦氏方程. 提示: 利用 Killing 场 ξ^a 满足的 $\nabla_a\xi^a=0$ (第 4 章习题 11 的结果).

证 无源麦氏方程为 $\nabla^aF_{ab}=0$ 和 $\nabla_{[a}F_{bc]}=0$. 现在

$$F_{ab}=(d\xi)_{ab}\overset{(5\text{-}1\text{-}11)}{=}2\nabla_{[a}\xi_{b]}=\nabla_a\xi_b-\nabla_b\xi_a\ .$$

第一个方程需证 $\nabla^a\nabla_a\xi_b-\nabla^a\nabla_b\xi_a=0$. 因 ξ_a 是 Killing 场, 满足 $\nabla_a\xi_b=-\nabla_b\xi_a$, 于是上式变为 $-2\nabla^a\nabla_b\xi_a=0$. 由于度规与导数算符适配, 即要证 $\nabla_a\nabla_b\xi^a=0$. 注意到

$$\begin{aligned}\nabla_a\nabla_b\xi^a &\overset{(3\text{-}4\text{-}4)}{=} \nabla_b\nabla_a\xi^a-R_{abc}{}^a\xi^c\overset{(3\text{-}4\text{-}6)}{=}\nabla_b\nabla_a\xi^a+R_{bac}{}^a\xi^c\\ &= \nabla_b\nabla_a\xi^a+R_{bc}\xi^c\ .\end{aligned}$$

因为里奇平直，所以 $R_{bc}=0$; 又由第 4 章习题 11 的结果知对 Killing 场有 $\nabla_a\xi^a=0$, 因此 $\nabla_a\nabla_b\xi^a=0$, 即第一个麦氏方程 $\nabla^aF_{ab}=0$ 成立. 又由于

$$\nabla_a\nabla_b\xi_c\overset{(3\text{-}4\text{-}3)}{=}\nabla_b\nabla_a\xi_c+R_{abc}{}^d\xi_d\ ,$$

对第二个方程有

$$\begin{aligned}\nabla_{[a}F_{bc]} &= 2\nabla_{[a}\nabla_{[b}\xi_{c]]}\overset{(2\text{-}6\text{-}20)}{=}2\nabla_{[a}\nabla_b\xi_{c]}\\ &= 2\nabla_{[b}\nabla_a\xi_{c]}+2R_{[abc]}{}^d\xi_d\\ &\overset{(3\text{-}4\text{-}7)}{=} 2\nabla_{[b}\nabla_a\xi_{c]}=-2\nabla_{[a}\nabla_b\xi_{c]}\ ,\end{aligned}$$

因此 $\nabla_{[a}F_{bc]}=0$, 第二个方程也成立.

9. 设 ξ_μ $(\mu=0,1,2,3)$ 为方程 $\partial^b\partial_b\xi_\mu=0$ 在初始条件式 (7-9-10)~(7-9-13) 下的解, 试证由 $\xi_a=\xi_\mu(dx^\mu)_a$ 及 γ_{ab} 按式 (7-9-8) 构造的 γ'_{ab} 在无源区既满足洛伦兹规范条件 $\partial^a\bar\gamma'_{ab}=0$ 又满足 $\gamma'=0$ 和 $\gamma'_{0i}=0$ $(i=1,2,3)$. 提示: (1) 根据解的唯一性定理，只须证明 $\gamma'=0$ 和 $\gamma'_{0i}=0$ 分别是方程 $\partial^c\partial_c\gamma'=0$ 和 $\partial^c\partial_c\gamma'_{0i}=0$ 的满足初始条件 $\gamma'|_{\Sigma_0}=0$, $\partial\gamma'/\partial t|_{\Sigma_0}=0$, $\gamma'_{0i}|_{\Sigma_0}=0$ 和 $\partial\gamma'_{0i}/\partial t|_{\Sigma_0}=0$ 的解. (2) 由 $\partial^b\partial_b\xi_\mu=0$ 可得 $\partial^2\xi_\mu/\partial t^2=\nabla^2\xi_\mu$.

证 首先，如果 γ_{ab} (即 $\bar\gamma_{ab}$) 是洛伦兹规范条件 $(\partial^a\bar\gamma_{ab}=0)$ 下的线性爱因斯坦方程 $(\partial^c\partial_c\bar\gamma_{ab}=-16\pi T_{ab})$ 的解，那么 (再一次) 通过规范变换式 (7-9-8), 并满足条件 $\partial^b\partial_b\xi_a=0$, 得到的

$$\gamma'_{ab}=\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a\ ,$$

也是这一方程的解，即满足 $\partial^a\bar{\gamma}'_{ab}=0$. 这是显然的，因为规范变换不会改变黎曼张量，故不会改变方程，或可直接验证．注意到

$$\begin{aligned}\gamma' &= \eta^{ab}\gamma'_{ab}=\eta^{ab}(\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a)=\gamma+2\partial^c\xi_c\ ,\\ \bar{\gamma}'_{ab} &= \gamma'_{ab}-\frac{1}{2}\eta_{ab}\gamma'=(\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a)-\frac{1}{2}\eta_{ab}(\gamma+2\partial^c\xi_c)\\ &= \bar{\gamma}_{ab}+\partial_a\xi_b+\partial_b\xi_a-\eta_{ab}\partial^c\xi_c\ ,\end{aligned}$$

如果 $\partial^c\partial_c\xi_a=0$, 显然有 $\partial^c\partial_c\bar{\gamma}'_{ab}=\partial^c\partial_c\bar{\gamma}_{ab}=-16\pi T_{ab}$, 可见 $\bar{\gamma}'_{ab}$ 也满足 (洛伦兹规范条件下的) 线性爱因斯坦方程，所以需要验证相应的洛伦兹规范条件 $\partial^a\bar{\gamma}'_{ab}=0$ 是否满足．显然有

$$\begin{aligned}\partial^a\bar{\gamma}'_{ab} &= \partial^a(\bar{\gamma}_{ab}+\partial_a\xi_b+\partial_b\xi_a-\eta_{ab}\partial^c\xi_c)\\ &= \partial^a\bar{\gamma}_{ab}+\partial^a\partial_a\xi_b+\partial_b\partial^a\xi_a-\partial_b\partial^c\xi_c\\ &= \partial^a\bar{\gamma}_{ab}+\partial^a\partial_a\xi_b=0\ ,\end{aligned}$$

其中用到了洛伦兹条件 $\partial^a\bar{\gamma}_{ab}=0$ 和 $\partial^a\partial_a\xi_b=0$.

下面我们从满足洛伦兹条件的 γ_{ab} (即 $\bar{\gamma}_{ab}$) 出发，通过这一规范变换使得在<u>无源区</u> $\gamma'=0$, $\gamma'_{0i}=0$. 首先由于 ξ_a 满足方程 $\partial^c\partial_c\xi_a=0$, 在坐标分量下为 $\partial^\nu\partial_\nu\xi_\mu=0$, 即要求 ξ_μ 满足

$$\Big(-\frac{\partial^2}{\partial t^2}+\nabla^2\Big)\xi_\mu=0\ ,\qquad \text{即}\quad \frac{\partial^2\xi_\mu}{\partial t^2}=\nabla^2\xi_\mu\ .$$

因为在无源区， $\bar{\gamma}'_{ab}$ 满足 $\partial^c\partial_c\bar{\gamma}'_{ab}=\partial^c\partial_c(\gamma'_{ab}-\frac{1}{2}\eta_{ab}\gamma')=0$, 以 η^{ab} 作用得 $\partial^c\partial_c(\gamma'-\frac{1}{2}4\gamma')=-\partial^c\partial_c\gamma'=0$. 设 Σ_0 是 $t=t_0$ 时刻的超曲面，如果 $\gamma|_{\Sigma_0}$ 满足式 (7-9-10) 以及 $\frac{\partial\gamma}{\partial t}|_{\Sigma_0}$ 满足式 (7-9-11), 则因 $\gamma'=\gamma+2\partial^c\xi_c=\gamma+2\partial^\mu\xi_\mu=\gamma-2\partial_0\xi_0+2\partial_i\xi_i=\gamma+2(\vec{\nabla}\cdot\vec{\xi}-\frac{\partial\xi_0}{\partial t})$ 有

$$\begin{aligned}\gamma'|_{\Sigma_0} &= \gamma|_{\Sigma_0}+2\Big[\vec{\nabla}\cdot\vec{\xi}-\frac{\partial\xi_0}{\partial t}\Big]_{\Sigma_0}\overset{(7\text{-}9\text{-}10)}{=}0\ ,\\ \frac{\partial\gamma'}{\partial t}\Big|_{\Sigma_0} &= \frac{\partial\gamma}{\partial t}\Big|_{\Sigma_0}+2\Big[\vec{\nabla}\cdot\Big(\frac{\partial\vec{\xi}}{\partial t}\Big)-\frac{\partial^2\xi_0}{\partial t^2}\Big]_{\Sigma_0}\\ &= \frac{\partial\gamma}{\partial t}\Big|_{\Sigma_0}+2\Big[\vec{\nabla}\cdot\Big(\frac{\partial\vec{\xi}}{\partial t}\Big)-\nabla^2\xi_0\Big]_{\Sigma_0}\overset{(7\text{-}9\text{-}11)}{=}0\ ,\end{aligned}$$

其中利用了 $\frac{\partial^2\xi_0}{\partial t^2}=\nabla^2\xi_0$. 方程 $\partial^c\partial_c\gamma'=0$ 加上这两个初值条件，决定唯一解 $\gamma'=0$. 于是现在真空线性爱因斯坦方程变为 $\partial^c\partial_c\bar{\gamma}'_{ab}=\partial^c\partial_c(\gamma'_{ab}-\frac{1}{2}\eta_{ab}\gamma')=\partial^c\partial_c\gamma'_{ab}=0$. 下面看 γ'_{0i}. 因 $\gamma'_{0i}=\gamma_{0i}+\partial_0\xi_i+\partial_i\xi_0=\gamma_{0i}+\frac{\partial\xi_i}{\partial t}+\frac{\partial\xi_0}{\partial x^i}$, 则因式 (7-9-12) 和 (7-9-13) 有

$$\begin{aligned}\gamma'_{0i}|_{\Sigma_0} &= \gamma_{0i}|_{\Sigma_0}+\Big[\frac{\partial\xi_i}{\partial t}+\frac{\partial\xi_0}{\partial x^i}\Big]_{\Sigma_0}\overset{(7\text{-}9\text{-}12)}{=}0\ ,\\ \frac{\partial\gamma'_{0i}}{\partial t}\Big|_{\Sigma_0} &= \frac{\partial\gamma_{0i}}{\partial t}\Big|_{\Sigma_0}+\Big[\frac{\partial^2\xi_i}{\partial t^2}+\frac{\partial}{\partial t}\Big(\frac{\partial\xi_0}{\partial x^i}\Big)\Big]_{\Sigma_0}\\ &= \frac{\partial\gamma_{0i}}{\partial t}\Big|_{\Sigma_0}+\Big[\nabla^2\xi_i+\frac{\partial}{\partial x^i}\Big(\frac{\partial\xi_0}{\partial t}\Big)\Big]_{\Sigma_0}\overset{(7\text{-}9\text{-}13)}{=}0\ ,\end{aligned}$$

其中利用了 $\frac{\partial^2\xi_i}{\partial t^2}=\nabla^2\xi_i$. 方程 $\partial^c\partial_c\gamma'_{0i}=0$ 加上这两个初值条件，决定唯一解 $\gamma'_{0i}=0$.

与电磁场的情况比较有点不同，那里 [方程 (7-9-6) 和 (7-9-7)] 只用到了初始值 $A_0|_{\Sigma_0}$ 和 $\vec{a}|_{\Sigma_0}$，并没有用到 $\frac{\partial A_0}{\partial t}|_{\Sigma_0}$，因为它可以通过洛伦兹条件变为 $\vec{\nabla}\cdot\vec{a}|_{\Sigma_0}$，所以现在也应该利用洛伦兹条件 $\partial^a\bar{\gamma}_{ab}=0$ 化掉式 (7-9-11) 和 (7-9-13) 右边的 $\frac{\partial\gamma}{\partial t}|_{\Sigma_0}$ 和 $\frac{\partial\gamma_{0i}}{\partial t}|_{\Sigma_0}$，这似乎只有在 $\gamma_{00}=0$ 时才能做到！因为这时由洛伦兹条件

$$0=\partial^\mu\bar{\gamma}_{\mu\nu}=\partial^\mu\Big(\gamma_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\gamma\Big)=\partial^\mu\gamma_{\mu\nu}-\frac{1}{2}\partial_\nu\gamma\ ,$$

有

$$\frac{\partial\gamma}{\partial t}\bigg|_{\Sigma_0}=\partial_0\gamma|_{\Sigma_0}=2\partial^\mu\gamma_{\mu 0}|_{\Sigma_0}=2\partial^i\gamma_{i0}|_{\Sigma_0}=2\frac{\partial\gamma_{0i}}{\partial x^i}\bigg|_{\Sigma_0}\ ,$$

所以只需知道 $\gamma_{0i}|_{\Sigma_0}$，就有 $\frac{\partial\gamma_{0i}}{\partial x^i}|_{\Sigma_0}$ (因为不需要对时间求导，所以不需要知道 Σ_0 外的行为) 和 $\frac{\partial\gamma}{\partial t}|_{\Sigma_0}$. 最后

$$\begin{aligned}\frac{\partial\gamma_{0i}}{\partial t}\bigg|_{\Sigma_0}&=\partial_0\gamma_{0i}|_{\Sigma_0}=-\partial^0\gamma_{0i}|_{\Sigma_0}=\Big[\partial^j\gamma_{ji}-\frac{1}{2}\partial_i\gamma\Big]_{\Sigma_0}\\&=\bigg[\frac{\partial\gamma_{ij}}{\partial x^j}-\frac{1}{2}\frac{\partial\gamma}{\partial x^i}\bigg]_{\Sigma_0}\ ,\end{aligned}$$

这样只要知道 $\gamma_{0i}|_{\Sigma_0}$ 和 $\gamma_{ij}|_{\Sigma_0}$，就有了 $\frac{\partial\gamma_{0i}}{\partial t}|_{\Sigma_0}$.

10. 设 γ_{ab} 满足 (a) $\partial^a\bar{\gamma}_{ab}=0$; (b) $\gamma=0$; (c) $\gamma_{0i}=0\ (i=1,2,3)$; (d) $\gamma_{00}=$ 常数. 试找出一个“无限小”矢量场 ξ^a 使 $\tilde{\gamma}_{ab}\equiv\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a$ 满足

(a) $\partial^a\bar{\tilde{\gamma}}_{ab}=0$; (b) $\tilde{\gamma}=0$; (c) $\tilde{\gamma}_{0i}=0\ (i=1,2,3)$; (d) $\tilde{\gamma}_{00}=0$.

解 (a) 由前题知道在规范变换下仍保持洛伦兹规范条件，要求 $\partial^b\partial_b\xi_a=0$, 即

$$\Big(-\frac{\partial^2}{\partial t^2}+\nabla^2\Big)\xi_\mu=0\ .$$

(b) 由 $\tilde{\gamma}=\eta^{ab}\tilde{\gamma}_{ab}=\gamma+2\partial^a\xi_a=2\partial^a\xi_a=0$, 要求 $\partial_0\xi_0=\partial_i\xi_i$, 即

$$\frac{\partial\xi_0}{\partial t}=\frac{\partial\xi_i}{\partial x^i}=\vec{\nabla}\cdot\vec{\xi}\,.$$

(c) 要求 $\tilde{\gamma}_{0i}=\gamma_{0i}+\partial_0\xi_i+\partial_i\xi_0=\partial_0\xi_i+\partial_i\xi_0=0$, 即

$$\frac{\partial\xi_i}{\partial t}+\frac{\partial\xi_0}{\partial x^i}=0\ .$$

(d) 要求 $\tilde{\gamma}_{00}=\gamma_{00}+2\partial_0\xi_0=0$, 因 γ_{00} 是常数，即要求

$$\frac{\partial\xi_0}{\partial t}=-\frac{\gamma_{00}}{2}\ .$$

满足以上 (a)–(d) 的一个显而易见的解为

$$\xi_0 = -\frac{\gamma_{00}}{2}t = -\frac{\gamma_{00}}{2}x^0\ , \qquad \xi_i = -\frac{\gamma_{00}}{6}x^i \quad (\text{即}\ \vec{\xi} = -\frac{\gamma_{00}}{6}\vec{x})\ .$$

验证如下： (a) 因为解对时空参数是线性依赖，其二阶导数都为零. (b) $\frac{\partial \xi_0}{\partial t} = \frac{\partial}{\partial t}(-\frac{\gamma_{00}}{2}t) = -\frac{\gamma_{00}}{2}$, $\vec{\nabla}\cdot\vec{\xi} = \vec{\nabla}\cdot(-\frac{\gamma_{00}}{6}\vec{x}) = -\frac{\gamma_{00}}{6}\times 3 = -\frac{\gamma_{00}}{2}$. (c) $\frac{\partial \xi_i}{\partial t} = \frac{\partial}{\partial t}(-\frac{\gamma_{00}}{6}x^i) = 0$, $\frac{\partial \xi_0}{\partial x^i} = \frac{\partial}{\partial x^i}(-\frac{\gamma_{00}}{2}t) = 0$. (d) 易见.

11. 试证命题 7-9-2.

证 已经求出黎曼张量式 (7-9-32)

$$\begin{aligned} R_{abc}{}^{d} &= [f(e^1)_a\wedge(e^4)_b + g(e^2)_a\wedge(e^4)_b][(e^1)_c(e_3)^d + (e^4)_c(e_1)^d] \\ &\quad +[g(e^1)_a\wedge(e^4)_b - f(e^2)_a\wedge(e^4)_b][(e^2)_c(e_3)^d + (e^4)_c(e_2)^d]\ , \end{aligned}$$

所以有

$$\begin{aligned} &R_{abcd} = g_{de}R_{abc}{}^{e} \\ =\ &[f(e^1)_a\wedge(e^4)_b + g(e^2)_a\wedge(e^4)_b][(e^1)_c g_{de}(e_3)^e + (e^4)_c g_{de}(e_1)^e] \\ &+[g(e^1)_a\wedge(e^4)_b - f(e^2)_a\wedge(e^4)_b][(e^2)_c g_{de}(e_3)^e + (e^4)_c g_{de}(e_2)^e]\ , \end{aligned}$$

其中

$$\begin{aligned} g_{de}(e_1)^e &= (e_1)_d = g_{1\beta}(e^\beta)_d = g_{11}(e^1)_d = (e^1)_d\ , \\ g_{de}(e_2)^e &= (e_2)_d = g_{2\beta}(e^\beta)_d = g_{22}(e^2)_d = (e^1)_d\ , \\ g_{de}(e_3)^e &= (e_3)_d = g_{3\beta}(e^\beta)_d = g_{34}(e^4)_d = -(e^4)_d\ . \end{aligned}$$

当然这些关系也可用度规场 g_{ab} 的式 (7-9-23) 硬算，如：

$$\begin{aligned} g_{de}(e_3)^e &= \Big(\eta_{de} + 2P[(dt)_d - (dz)_d][(dt)_e - (dz)_e]\Big)[(\partial/\partial t)^e + (\partial/\partial z)^e] \\ &= -(dt)_d + (dz)_d + 2P[(dt)_d - (dz)_d][1-1] = -(du)_d = -(e^4)_d\ . \end{aligned}$$

因此得

$$\begin{aligned} R_{abcd} &= [f(e^1)_a\wedge(e^4)_b + g(e^2)_a\wedge(e^4)_b][-(e^1)_c(e^4)_d + (e^4)_c(e^1)_d] \\ &\quad +[g(e^1)_a\wedge(e^4)_b - f(e^2)_a\wedge(e^4)_b][-(e^2)_c(e^4)_d + (e^4)_c(e^2)_d] \\ &= [f(e^1)_a\wedge(e^4)_b + g(e^2)_a\wedge(e^4)_b](e^4)_c\wedge(e^1)_d \\ &\quad +[g(e^1)_a\wedge(e^4)_b - f(e^2)_a\wedge(e^4)_b](e^4)_c\wedge(e^2)_d\ , \end{aligned}$$

即命题 7-9-2 的结果.

12. 验证式 (7-9-41) 后的 (1)∼(3).

证 ① $\{(E_i)^a\}$ 的正交归一性．由式 (7-9-41)

$$\begin{aligned}
(E_1)^a &= (\partial/\partial x)^a + E^{-1}Z_1K^a = (e_1)^a + E^{-1}Z_1(e_3)^a \ , \\
(E_2)^a &= (\partial/\partial y)^a + E^{-1}Z_2K^a = (e_2)^a + E^{-1}Z_2(e_3)^a \ , \\
(E_3)^a &= E^{-1}K^a - Z^a = E^{-1}(e_3)^a - Z^a \ ,
\end{aligned}$$

其中

$$\begin{aligned}
E &= -g_{ab}Z^aK^b = -g_{ab}Z^a(e_3)^b \ , \\
Z_1 &= g_{ab}Z^a(\partial/\partial x)^b = g_{ab}Z^a(e_1)^b \ , \\
Z_2 &= g_{ab}Z^a(\partial/\partial y)^b = g_{ab}Z^a(e_2)^b \ .
\end{aligned}$$

显然归一：

$$\begin{aligned}
g_{ab}(E_1)^a(E_1)^b &= g_{ab}[(e_1)^a + E^{-1}Z_1(e_3)^a][(e_1)^b + E^{-1}Z_1(e_3)^b] \\
&= g_{11} + g_{13}E^{-1}Z_1 + g_{31}E^{-1}Z_1 + g_{33}E^{-2}Z_1^2 \\
&= 1 + 0 + 0 + 0 = 1 \ , \\
g_{ab}(E_2)^a(E_2)^b &= g_{ab}[(e_2)^a + E^{-1}Z_2(e_3)^a][(e_2)^b + E^{-1}Z_2(e_3)^b] \\
&= g_{22} + g_{23}E^{-1}Z_2 + g_{32}E^{-1}Z_2 + g_{33}E^{-2}Z_1^2 \\
&= 1 + 0 + 0 + 0 = 1 \ , \\
g_{ab}(E_3)^a(E_3)^b &= g_{ab}[E^{-1}(e_3)^a - Z^a][E^{-1}(e_3)^b - Z^b] \\
&= g_{33}E^{-2} - g_{ab}(e_3)^aZ^bE^{-1} - g_{ab}Z^a(e_3)^bE^{-1} + g_{ab}Z^aZ^b \\
&= 0 - (-E)E^{-1} - (-E)E^{-1} + (-1) = 1 \ ,
\end{aligned}$$

而且正交：

$$\begin{aligned}
g_{ab}(E_1)^a(E_2)^b &= g_{ab}[(e_1)^a + E^{-1}Z_1(e_3)^a][(e_2)^b + E^{-1}Z_2(e_3)^b] \\
&= g_{12} + g_{13}E^{-1}Z_2 + g_{32}E^{-1}Z_1 + g_{33}E^{-2}Z_1^2 \\
&= 0 + 0 + 0 + 0 = 0 \ , \\
g_{ab}(E_1)^a(E_3)^b &= g_{ab}[(e_1)^a + E^{-1}Z_1(e_3)^a][E^{-1}(e_3)^b - Z^b] \\
&= g_{13}E^{-1} - g_{ab}(e_1)^aZ^b + g_{ab}(e_3)^a(e_3)^bE^{-2}Z_1 - g_{ab}(e_3)^aZ^bE^{-1}Z_1 \\
&= 0 - Z_1 + 0 - (-E)E^{-1}Z_1 = 0 \ , \\
g_{ab}(E_2)^a(E_3)^b &= g_{ab}[(e_2)^a + E^{-1}Z_2(e_3)^a][E^{-1}(e_3)^b - Z^b] \\
&= g_{23}E^{-1} - g_{ab}(e_2)^aZ^b + g_{ab}(e_3)^a(e_3)^bE^{-2}Z_2 - g_{ab}(e_3)^aZ^bE^{-1}Z_2 \\
&= 0 - Z_2 + 0 - (-E)E^{-1}Z_2 = 0 \ .
\end{aligned}$$

②与 Z^a 正交的投影算符 $h^a{}_b = \delta^a{}_b + Z^aZ_b$ 可将 p 点的任意 4 矢投影到 W_p 3 维空间子空间，于是 K^a 在 W_p 上的投影为

$$h^a{}_bK^b = (\delta^a{}_b + Z^aZ_b)K^b = K^a + Z^aZ_bZ^b = K^a + Z^a(-E) = K^a - EZ^a \ ,$$

它的“长度”的平方为

$$\begin{aligned} & g_{ab}(K^a - EZ^a)(K^b - EZ^b) \\ = \; & g_{ab}K^aK^b - g_{ab}K^aZ^bE - g_{ab}E^aK^bE + g_{ab}Z^aZ^bE^2 \\ = \; & 0 - (-E)E - (-E)E + (-1)E^2 = E^2 \,, \end{aligned}$$

因此把该投影归一化后

$$E^{-1}(K^a - EZ^a) = E^{-1}K^a - Z^a = (E_3)^a \,.$$

③首先证明 $(E_3)^a$ 沿测地线 $\gamma(\tau)$ 平移，即 $Z^c\nabla_c(E_3)^a = 0$. 注意到

$$Z^c\nabla_c E = Z^c\nabla_c(-g_{ab}Z^aK^b) = -g_{ab}K^bZ^c\nabla_c Z^a - g_{ab}Z^aZ^c\nabla_c K^b = 0 \,,$$

其中利用了 $\gamma(\tau)$ 的测地性 $Z^c\nabla_c Z^a = 0$ 和 K^a 的 Killing 矢量性 $\nabla_c K^b = 0$.
于是有

$$\begin{aligned} Z^c\nabla_c(E_3)^a &= Z^c\nabla_c(E^{-1}K^a - Z^a) = Z^c\nabla_c(E^{-1}K^a) - Z^c\nabla_c Z^a \\ &= K^aZ^c\nabla_c E^{-1} + E^{-1}Z^c\nabla_c K^a - Z^c\nabla_c Z^a \\ &= 0 \,. \end{aligned}$$

为了证明 $(E_1)^a$ 沿测地线平移,利用式 (5-7-5): $\omega_\alpha{}^\beta{}_a = (e_\alpha)^c\nabla_a(e^\beta)_c = -(e^\beta)_c\nabla_a(e_\alpha)^c$, 两边作用 $(e_\beta)^b$: $(e_\beta)^b\omega_\alpha{}^\beta{}_a = -(e_\beta)^b(e^\beta)_c\nabla_a(e_\alpha)^c = -\nabla_a(e_\alpha)^b$, 即 $\nabla_a(e_\alpha)^b = -\omega_\alpha{}^\beta{}_a(e_\beta)^b$. 现在

$$\nabla_a(e_1)^b = -\omega_1{}^\beta{}_a(e_\beta)^b \overset{\text{(7-9-30)}}{=} -\omega_1{}^3{}_a(e_3)^b \,,$$

其中 $\omega_1{}^3{}_a$ 根据 (7-9-30) 为 $\omega_1{}^3{}_a = (fx+gy)(du)_a$. 于是有

$$\begin{aligned} Z^a\nabla_a(E_1)^b &= Z^a\nabla_a[(e_1)^b + E^{-1}Z_1(e_3)^b] \\ &= Z^a\nabla_a(e_1)^b + Z_1(e_3)^bZ^a\nabla_a E^{-1} + E^{-1}(e_3)^bZ^a\nabla_a Z_1 + E^{-1}Z_1Z^a\nabla_a(e_3)^b \\ &= Z^a\nabla_a(e_1)^b + 0 + E^{-1}(e_3)^bZ^a\nabla_a[g_{cd}Z^c(e_1)^d] + 0 \\ &= Z^a\nabla_a(e_1)^b + E^{-1}g_{cd}Z^c(e_3)^bZ^a\nabla_a(e_1)^d \\ &= Z^a[-\omega_1{}^3{}_a(e_3)^b] + E^{-1}g_{cd}Z^c(e_3)^bZ^a[-\omega_1{}^3{}_a(e_3)^d] \\ &= -\omega_1{}^3{}_aZ^a(e_3)^b - \omega_1{}^3{}_aE^{-1}[g_{cd}Z^c(e_3)^d]Z^a(e_3)^b \\ &= -\omega_1{}^3{}_aZ^a(e_3)^b - \omega_1{}^3{}_aE^{-1}[-E]Z^a(e_3)^b \\ &= 0 \,. \end{aligned}$$

同样可证 $(E_2)^a$ 沿测地线平移，即 $Z^a\nabla_a(E_2)^b = 0$.

13. 试证式 (7-9-43).

证 把式 (7-9-33) 代入式 (7-9-42), 我们有

$$\begin{aligned}
\psi^i{}_j &= -R_{abcd}Z^a(E_j)^bZ^c(E_i)^d \\
&= -[f(e^1)_a\wedge(e^4)_b+g(e^2)_a\wedge(e^4)_b](e^4)_c\wedge(e^1)_dZ^a(E_j)^bZ^c(E_i)^d \\
&\quad -[g(e^1)_a\wedge(e^4)_b-g(e^2)_a\wedge(e^4)_b](e^4)_c\wedge(e^2)_dZ^a(E_j)^bZ^c(E_i)^d \\
&= -[f(e^1)_a\wedge(e^4)_bZ^a(E_j)^b+g(e^2)_a\wedge(e^4)_bZ^a(E_j)^b](e^4)_c\wedge(e^1)_dZ^c(E_i)^d \\
&\quad -[g(e^1)_a\wedge(e^4)_bZ^a(E_j)^b-f(e^2)_a\wedge(e^4)_bZ^a(E_j)^b](e^4)_c\wedge(e^2)_dZ^c(E_i)^d ,
\end{aligned}$$

其中

$$\begin{aligned}
(e^1)_a\wedge(e^4)_bZ^a(E_j)^b &= [(e^1)_a(e^4)_b-(e^4)_a(e^1)_b]Z^a(E_j)^b \\
&= [(e^1)_aZ^a][(e^4)_b(E_j)^b]-[(e^4)_aZ^a][(e^1)_b(E_j)^b] , \\
(e^2)_a\wedge(e^4)_bZ^a(E_j)^b &= [(e^2)_a(e^4)_b-(e^4)_a(e^2)_b]Z^a(E_j)^b \\
&= [(e^2)_aZ^a][(e^4)_b(E_j)^b]-[(e^4)_aZ^a][(e^2)_b(E_j)^b] , \\
(e^4)_c\wedge(e^1)_dZ^c(E_i)^d &= [(e^4)_c(e^1)_d-(e^1)_c(e^4)_d]Z^c(E_i)^d \\
&= [(e^4)_cZ^c][(e^1)_d(E_i)^d]-[(e^1)_cZ^c][(e^4)_d(E_i)^d] , \\
(e^4)_c\wedge(e^2)_dZ^c(E_i)^d &= [(e^4)_c(e^2)_d-(e^2)_c(e^4)_d]Z^c(E_i)^d \\
&= [(e^4)_cZ^c][(e^2)_d(E_i)^d]-[(e^2)_cZ^c][(e^4)_d(E_i)^d] .
\end{aligned}$$

因为

$$\begin{aligned}
(e^1)_aZ^a &= g^{11}(e_1)_aZ^a=g_{ab}(e_1)^aZ^b=Z_1 , \\
(e^2)_aZ^a &= g^{22}(e_2)_aZ^a=g_{ab}(e_2)^aZ^b=Z_2 , \\
(e^4)_aZ^a &= g^{43}(e_3)_aZ^a=-g_{ab}(e_3)^aZ^b=E , \\
(e^1)_a(E_i)^a &= g^{11}(e_1)_a\Big\{\delta^1{}_i[(e_1)^a+E^{-1}Z_1(e_3)^a]+\delta^2{}_i[(e_2)^a+E^{-1}Z_2(e_3)^a] \\
&\qquad +\delta^3{}_i[E^{-1}(e_3)^a-Z^a]\Big\} \\
&= \delta^1{}_i[g_{11}+g_{13}E^{-1}Z_1]+\delta^2{}_i[g_{12}+g_{13}E^{-1}Z_2] \\
&\quad +\delta^3{}_i[g_{13}E^{-1}-(e_1)_aZ^a] \\
&= \delta^1{}_i-\delta^3{}_iZ_1 , \\
(e^2)_a(E_i)^a &= g^{22}(e_2)_a\Big\{\delta^1{}_i[(e_1)^a+E^{-1}Z_1(e_3)^a]+\delta^2{}_i[(e_2)^a+E^{-1}Z_2(e_3)^a] \\
&\qquad +\delta^3{}_i[E^{-1}(e_3)^a-Z^a]\Big\} \\
&= \delta^1{}_i[g_{21}+g_{23}E^{-1}Z_1]+\delta^2{}_i[g_{22}+g_{23}E^{-1}Z_2] \\
&\quad +\delta^3{}_i[g_{23}E^{-1}-(e_2)_aZ^a] \\
&= \delta^2{}_i-\delta^3{}_iZ_2 , \\
(e^4)_a(E_i)^a &= g^{43}(e_3)_a\Big\{\delta^1{}_i[(e_1)^a+E^{-1}Z_1(e_3)^a]+\delta^2{}_i[(e_2)^a+E^{-1}Z_2(e_3)^a]
\end{aligned}$$

$$
\begin{aligned}
& +\delta^3{}_i[E^{-1}(e_3)^a - Z^a]\Big\} \\
= & -\delta^1{}_i[g_{31}+g_{33}E^{-1}Z_1] - \delta^2{}_i[g_{32}+g_{33}E^{-1}Z_2] \\
& -\delta^3{}_i[g_{33}E^{-1} - (e_3)_a Z^a] \\
= & \ \delta^3{}_i(e_3)_a Z^a = -\delta^3{}_i E\ .
\end{aligned}
$$

于是

$$
\begin{aligned}
(e^1)_a \wedge (e^4)_b Z^a (E_j)^b &= [Z_1][-\delta^3{}_j E] - [E][\delta^1{}_j - \delta^3{}_j Z_1] = -\delta^1{}_j E\ , \\
(e^2)_a \wedge (e^4)_b Z^a (E_j)^b &= [Z_2][-\delta^3{}_j E] - [E][\delta^2{}_j - \delta^3{}_j Z_2] = -\delta^2{}_j E\ , \\
(e^4)_c \wedge (e^1)_d Z^c (E_i)^d &= [E][\delta^1{}_i - \delta^3{}_i Z_1] - [Z_1][-\delta^3{}_i E] = \delta^1{}_i E\ , \\
(e^4)_c \wedge (e^2)_d Z^c (E_i)^d &= [E][\delta^2{}_i - \delta^3{}_i Z_2] - [Z_2][-\delta^3{}_i E] = \delta^2{}_i E\ .
\end{aligned}
$$

代入 $\psi^i{}_j$ 的表达式

$$
\begin{aligned}
\psi^i{}_j &= -[f(-\delta^1{}_j E) + g(-\delta^2{}_j E)](\delta^1{}_i E) - [g(-\delta^1{}_j E) - f(-\delta^2{}_j E)](\delta^2{}_i E) \\
&= E^2 f \delta^1{}_i \delta^1{}_j + E^2 g \delta^1{}_i \delta^2{}_j + E^2 g \delta^2{}_i \delta^1{}_j - E^2 f \delta^2{}_i \delta^2{}_j \\
&= \alpha \delta^1{}_i \delta^1{}_j + \beta \delta^1{}_i \delta^2{}_j + \beta \delta^2{}_i \delta^1{}_j - \alpha \delta^2{}_i \delta^2{}_j\ ,
\end{aligned}
$$

其中 $\alpha = E^2 f$, $\beta = E^2 g$. 写成矩阵形式

$$
(\psi^i{}_j) = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}\ ,
$$

即式 (7-9-43).

14. 试证式 (7-9-36), 即 $\nabla^a \nabla_a P = (\partial^2 P/\partial x^2) + (\partial^2 P/\partial y^2)$.

证 首先因

$$
\begin{aligned}
\nabla_a P &= \nabla_a P(x,y,u) \overset{(3\text{-}1\text{-}2)}{=} (dP)_a = P_x (dx)_a + P_y (dy)_a + P_u (du)_a \\
&\overset{(7\text{-}9\text{-}28)}{=} P_x (e^1)_a + P_y (e^2)_a + P_u (e^4)_a\ ,
\end{aligned}
$$

其中记 $P_x \equiv \partial P/\partial x$ 等. 故

$$
\begin{aligned}
\nabla^a P &= P_x (e^1)^a + P_y (e^2)^a + P_u (e^4)^a \\
&= P_x g^{11} (e_1)^a + P_y g^{22} (e_2)^a + P_u g^{43} (e_3)^a \\
&= P_x (e_1)^a + P_y (e_2)^a - P_u (e_3)^a\ .
\end{aligned}
$$

于是

$$
\nabla_a \nabla^a P = \nabla_a [P_x (e_1)^a] + \nabla_a [P_y (e_2)^a] - \nabla_a [P_u (e_3)^a]
$$

$$
\begin{aligned}
&= (e_1)^a\nabla_a P_x + P_x\nabla_a(e_1)^a \\
&\quad +(e_2)^a\nabla_a P_y + P_y\nabla_a(e_2)^a \\
&\quad -(e_3)^a\nabla_a P_u - P_u\nabla_a(e_3)^a \\
&= (e_1)^a[P_{xx}(dx)_a + P_{xy}(dy)_a + P_{xu}(du)_a] + P_x\nabla_a(e_1)^a \\
&\quad +(e_2)^a[P_{yx}(dx)_a + P_{yy}(dy)_a + P_{yu}(du)_a] + P_y\nabla_a(e_2)^a \\
&\quad -(e_3)^a[P_{ux}(dx)_a + P_{uy}(dy)_a + P_{uu}(du)_a] - P_u\nabla_a(e_3)^a \\
&= (e_1)^a[P_{xx}(e_1)_a + P_{xy}(e_2)_a - P_{xu}(e_3)_a] + P_x\nabla_a(e_1)^a \\
&\quad +(e_2)^a[P_{yx}(e_1)_a + P_{yy}(e_2)_a + P_{yu}(e_3)_a] + P_y\nabla_a(e_2)^a \\
&\quad -(e_3)^a[P_{ux}(e_1)_a + P_{uy}(e_2)_a + P_{uu}(e_3)_a] - P_u\nabla_a(e_3)^a \\
&= g_{11}P_{xx} + g_{12}P_{xy} - g_{13}P_{xu} + P_x\nabla_a(e_1)^a \\
&\quad +g_{21}P_{yx} + g_{22}P_{yy} + g_{23}P_{yu} + P_y\nabla_a(e_2)^a \\
&\quad -g_{31}P_{ux} - g_{32}P_{uy} - g_{33}P_{uu} - P_u\nabla_a(e_3)^a \\
&= P_{xx} + P_{yy} + P_x\nabla_a(e_1)^a + P_y\nabla_a(e_2)^a - P_u\nabla_a(e_3)^a\ .
\end{aligned}
$$

利用 $\nabla_a(e_\alpha)^b = -\omega_\alpha{}^\beta{}_a(e_\beta)^b$ 并注意式 (7-9-30), 我们有

$$
\begin{aligned}
\nabla_a(e_1)^a &= -\omega_1{}^\beta{}_a(e_\beta)^a = -\omega_1{}^3{}_a(e_3)^a = -(fx+gy)(du)_a(e_3)^a \\
&= (fx+gy)(e_3)_a(e_3)^a = g_{33}(fx+gy) = 0\ , \\
\nabla_a(e_2)^a &= -\omega_2{}^\beta{}_a(e_\beta)^a = -\omega_2{}^3{}_a(e_3)^a = -(gx-fy)(du)_a(e_3)^a \\
&= (gx-fy)(e_3)_a(e_3)^a = g_{33}(gx-fy) = 0\ , \\
\nabla_a(e_3)^a &= -\omega_3{}^\beta{}_a(e_\beta)^a = 0\ .
\end{aligned}
$$

最后得到

$$
\nabla_a\nabla^a P = P_{xx} + P_{yy} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\ .
$$

第 8 章 “爱因斯坦方程的求解” 习题

~1. 试证命题 8-1-1.

证 由 Killing 矢量场 $\xi^a = (\partial/\partial t)^a$ 决定的单参等度规群元为 ϕ_t, 它将处处与 ξ^a 正交的超曲面 $\Sigma_0 = \{p \in M|\, t(p) = 0\}$ 上的点 p 映射到超曲面 $\Sigma_{t_1} = \{q \in M|\, t(q) = t_1\}$ 上的 $q = \phi_{t_1}(p)$ 点 [有 $p = \phi_{t_1}^{-1}(p) = \phi_{-t_1}(p)$]. 设 q 点的切空间为 V_q, 与超曲面 Σ_{t_1} 相切的子空间为 W_q (即 “躺在” Σ_{t_1} 面内的矢量构成的矢量空间). 下面我们要证明属于 W_q 的矢量 $w^a|_q$ 都与 $\xi^a|_q$ 正交，即

$$
(g_{ab}\xi^a w^b)|_q = g_{ab}|_q \xi^a|_q w^b|_q = 0\ .
$$

因为 ϕ_t 为等度规映射,根据 §4.3 注 1, $\phi_t^{-1}=\phi_{-t}$ 也为等度规映射,有 $\phi^*_{-t_1}g_{ab}=g_{ab}$. 所以

$$\begin{aligned}(g_{ab}\xi^a w^b)|_q &= (\phi^*_{-t_1}g_{ab})|_q\xi^a|_q w^b|_q \overset{(4\text{-}1\text{-}3)}{=} g_{ab}|_{\phi_{-t_1}(q)}(\phi_{-t_1*}\xi)^a|_{\phi_{-t_1}(q)}(\phi_{-t_1*}w)^b|_{\phi_{-t_1}(q)}\\ &= g_{ab}|_p(\phi_{-t_1*}\xi)^a|_p(\phi_{-t_1*}w)^b|_p = g_{ab}|_p(\phi^*_{t_1}\xi)^a|_p(\phi^*_{t_1}w)^b|_p\ .\end{aligned}$$

下面我们证明 $\phi^*_{t_1}(\xi^a|_q)=(\phi^*_{t_1}\xi)^a|_p=\xi^a|_p$, 而 $\phi^*_{t_1}(w^b|_q)=(\phi^*_{t_1}w)^b|_p$ 为超曲面 Σ_0 上 p 点切于超曲面 (“躺在” 超曲面 Σ_0 内) 的矢量，即 $(\phi^*_{t_1}w)^b|_p\in W_p$.

设 f 是任一光滑函数，则 Σ_0 上 p 点的矢量 $(\phi^*_{t_1}\xi)^a|_p$ 对 f 的作用为

$$(\phi^*_{t_1}\xi)|_p(f) \overset{(4\text{-}1\text{-}2)}{=} \xi(\phi_{t_1*}f)=\left.\frac{\partial}{\partial t}\right|_{t=t_1}(\phi_{t_1*}f)=\lim_{\Delta t\to 0}[(\phi_{t_1*}f)|_r-(\phi_{t_1*}f)|_q]\ .$$

其中 r 为 ξ^a 的积分曲线 $C(t)$ 上的一点： $r=C(t_1+\Delta t)$ $[p=C(0),\ q=C(t_1)]$. 于是有

$$(\phi^*_{t_1}\xi)|_p(f)=\lim_{\Delta t\to 0}[f|_s-f|_p]=\xi|_p(f)\ ,$$

其中 s 也是 $C(t)$ 上的一点： $s=\phi_{-t_1}(r)=C(\Delta t)$. f 的任意性给出 $(\phi^*_{t_1}\xi)^a|_p=\xi^a|_p$.

最后，设 $\mu(s)$ 是躺在 (切于) Σ_{t_1} 面内过 q 点并由 $w^a|_q$ 决定的测地线， r' 是该线上的一点： $r'=\mu(\Delta s)$ $[q=\mu(0)]$, 则

$$\begin{aligned}(\phi^*_{t_1}w)|_p(f) &\overset{(4\text{-}1\text{-}2)}{=} w(\phi_{t_1*}f)=\left.\frac{\partial}{\partial s}\right|_{\Delta s=0}(\phi_{t_1*}f)=\lim_{\Delta s\to 0}[(\phi_{t_1*}f)|_{r'}-(\phi_{t_1*}f)|_q]\\ &= \lim_{\Delta s\to 0}[f|_{s'}-f|_p]\equiv\bar{w}|_p(f)\ ,\end{aligned}$$

注意其中 $s'=\phi_{-t_1}(r')$ 是 Σ_0 面上的点，因此 $(\phi^*_{t_1}w)^a|_p=\bar{w}^a|_p$ 与超曲面 Σ_0 相切，即 $(\phi^*_{t_1}w)^a|_p=\bar{w}^a|_p\in W_p$.

于是我们得到

$$(g_{ab}\xi^a w^b)|_q=g_{ab}|_p(\phi^*_{t_1}\xi)^a|_p(\phi^*_{t_1}w)^b|_p=g_{ab}|_p\xi^a|_p\bar{w}^b|_p=(g_{ab}\xi^a\bar{w}^b)|_p=0\ ,$$

最后一个等式是由于 Σ_0 与 ξ^a 正交. 既然 q 是超曲面 Σ_{t_1} 上的任意一点，因此超曲面本身与 ξ^a 处处正交.

~2. 设 $\gamma(r)$ 是图 8-6 中 Σ_t 上从 p_1 到 p_2 的、θ 和 φ 都为常数的曲线 (以径向坐标 r 为曲线参数), 试证 $\gamma(r)$ 是 (非仿射参数化的) 测地线. 提示：用式 (5-7-2).

证 令曲线 $\gamma(r)$ 的切矢 $T^a=(\frac{\partial}{\partial r})^a$, 因

$$\begin{aligned}T^b\nabla_b T^a &= \Big(\frac{\partial}{\partial r}\Big)^b\nabla_b\Big(\frac{\partial}{\partial r}\Big)^a \overset{(5\text{-}7\text{-}2)}{=} \Gamma^\sigma{}_{11}\Big(\frac{\partial}{\partial x^\sigma}\Big)^a \overset{(8\text{-}3\text{-}20)}{=} \Gamma^1{}_{11}(r)\Big(\frac{\partial}{\partial r}\Big)^a\\ &= \alpha(r)T^a\ ,\end{aligned}$$

其中 $\alpha(r)=\Gamma^1{}_{11}(r)=-\frac{M}{r^2}(1-\frac{2M}{r})^{-1}$. 由定理 3-3-2, 知道 $\gamma(r)$ 为非仿射参数化的测地线. 令重参数化 $\gamma'(r')=\gamma(r)$ 可获得仿射参数化的测地线 $\gamma'(r')$. 利

用第 3 章习题 9 (定理 3-3-2 的证明) 的结果，函数关系 $r' = r'(r)$ 满足常微分方程

$$\frac{d^2r'(r)}{dr^2} = \alpha(r)\frac{dr'(r)}{dr} \ .$$

$\gamma'(r')$ 的切矢 $T'^a = (\frac{\partial}{\partial r'})^a$ 满足测地线方程 $T'^b\nabla_b T'^a = 0$, r' 是仿射参数.

~3. 设 ξ^a 是稳态时空的类时 Killing 矢量场， $\chi \equiv (-g_{ab}\xi^a\xi^b)^{1/2}$.

(a) 试证 χ 在 ξ^a 的积分曲线上为常数;

(b) 试证稳态观者的 4 加速 $A^a = \nabla^a(\ln\chi)$. 提示：利用 Killing 方程 $\nabla^{(a}\xi^{b)} = 0$ 和 (a) 的结果.

证 令 $\chi \equiv (-\xi_a\xi^a)^{1/2} = (-g_{ab}\xi^a\xi^b)^{1/2} = (-g_{00})^{1/2}$.

(a) χ 在 ξ^a 的积分曲线上的变化为

$$\begin{aligned}\xi^b\nabla_b\chi &= \xi^b\nabla_b(-\xi_a\xi^a)^{1/2} = \frac{1}{2}(-\xi_a\xi^a)^{-1/2}\Big[-\xi^b\xi_a\nabla_b\xi^a - \xi^b\xi^a\nabla_b\xi_a\Big] \\ &= -\chi^{-1}\xi^b\xi^a\nabla_b\xi_a \overset{(4\text{-}3\text{-}1)}{=} -\chi^{-1}\xi^{(b}\xi^{a)}\nabla_{[b}\xi_{a]} = 0 \ ,\end{aligned}$$

其中利用了 ξ^a 的 Killing 性. 这一结果说明在 ξ^a 的积分曲线上矢量 ξ^a 的 “长度” 不变.

(b) 设 τ 是稳态观者的固有时，其世界线与 ξ^a 的积分曲线重合. 稳态观者的 4 速为 $Z^a = (\frac{\partial}{\partial\tau})^a$, 因

$$\begin{aligned}-1 &= Z_aZ^a = g_{ab}Z^aZ^b = g_{ab}\Big(\frac{\partial}{\partial\tau}\Big)^a\Big(\frac{\partial}{\partial\tau}\Big)^b \\ &= g_{ab}\Big(\frac{dt}{d\tau}\Big)^2\Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial t}\Big)^b = g_{00}\Big(\frac{dt}{d\tau}\Big)^2 \ ,\end{aligned}$$

得

$$\frac{d\tau}{dt} = (-g_{00})^{1/2} = \chi \ .$$

于是有关系 $Z^a = \chi^{-1}\xi^a$. 稳态观者的 4 加速按定义为

$$A^a = Z^b\nabla_b Z^a = \chi^{-1}\xi^b\nabla_b(\chi^{-1}\xi^a) = \chi^{-2}\xi^b\nabla_b\xi^a \ ,$$

最后一步利用了 (a) 的结果. 另一方面,

$$\begin{aligned}\nabla^a\ln\chi &= \chi^{-1}\nabla^a\chi = \chi^{-1}\nabla^a(-\xi_b\xi^b)^{1/2} = \chi^{-1}\frac{1}{2}(-\xi_b\xi^b)^{-1/2}\Big[-2\xi_b\nabla^a\xi^b\Big] \\ &= -\chi^{-2}\xi_b\nabla^a\xi^b \overset{(4\text{-}3\text{-}1)}{=} \chi^{-2}\xi_b\nabla^b\xi^a = \chi^{-2}\xi^b\nabla_b\xi^a \ .\end{aligned}$$

因此有 $A^a = \nabla^a\ln\chi$.

~4. 试证： (a) 电磁场能动张量的迹为零，即 $T \equiv g^{ab}T_{ab} = 0$; (b) 电磁真空时空的标量曲率 $R = 0$.

证 (a) 电磁场的能动张量为式 (8-4-1)

$$T_{ab} = \frac{1}{4\pi}\Big(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\Big) \ ,$$

它的迹为

$$g^{ab}T_{ab} = T_a{}^a = \frac{1}{4\pi}\Big(F_{ac}F^{ac} - \frac{1}{4}g_a{}^aF_{cd}F^{cd}\Big) = \frac{1}{4\pi}\Big(F_{ac}F^{ac} - F_{cd}F^{cd}\Big) = 0 \ ,$$

其中利用了 $g_a{}^a = \delta_a{}^a = 4$.

(b) 因为电磁真空时空满足的爱因斯坦方程为 $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$, 求迹后有

$$R_a{}^a - \frac{1}{2}Rg_a{}^a = R - 2R = T_a{}^a = 0 \ ,$$

于是标量曲率 $R = 0$.

~5. 试证式 (8-4-7) 和 (8-4-28).

证 式 (8-4-7) 的证明：

由式 (8-4-5) 的定义 $\Sigma_{ab} = F_{ab} + i\,{}^*F_{ab}$, 知 $\Sigma^{ab} = F^{ab} + i\,{}^*F^{ab}$, 于是

$$\begin{aligned}
\Sigma_{ab}\Sigma^{ab} &= (F_{ab} + i\,{}^*F_{ab})(F^{ab} + i\,{}^*F^{ab}) \\
&= F_{ab}F^{ab} - {}^*F_{ab}\,{}^*F^{ab} + iF_{ab}\,{}^*F^{ab} + i\,{}^*F_{ab}F^{ab} \\
&= F_{ab}F^{ab} - {}^*F_{ab}\,{}^*F^{ab} + 2iF_{ab}\,{}^*F^{ab} \ ,
\end{aligned}$$

其中

$$\begin{aligned}
{}^*F_{ab}\,{}^*F^{ab} &\overset{(5\text{-}6\text{-}1)}{=} \frac{1}{2}F^{cd}\varepsilon_{cdab}\frac{1}{2}F_{ef}\varepsilon^{efab} = \frac{1}{4}F^{cd}F_{ef}\varepsilon_{cdab}\varepsilon^{efab} \\
&\overset{(5\text{-}4\text{-}10)}{=} \frac{1}{4}F^{cd}F_{ef}(-1)^1 2!2!\delta^{[e}{}_c\delta^{f]}{}_d = -F^{cd}F_{[ef]}\delta^e{}_c\delta^f{}_d = -F^{cd}F_{cd} \ .
\end{aligned}$$

由此得

$$\Sigma_{ab}\Sigma^{ab} = 2(F_{ab}F^{ab} + iF_{ab}\,{}^*F^{ab}) \ .$$

此即式 (8-4-7).

式 (8-4-28) 的证明：

与第 3 题 (b) 类似，根据线元式 (8-4-23), 静态观者 G 的 4 速 Z^a 与 Killing 场 ξ^a 的关系为

$$Z^a = \Big(\frac{\partial}{\partial\tau}\Big)^a = (-g_{00})^{-1/2}\Big(\frac{\partial}{\partial t}\Big)^a = (-g_{00})^{-1/2}\xi^a \ .$$

现在

$$g_{00} = -\left(1 + \frac{Q^2}{r^2} + \frac{C}{r}\right) \equiv -f\ ,$$

有 $Z^a = f^{-1/2}\xi^a = f^{-1/2}(\partial/\partial t)^a$. G 的正交归一 4 标架的对偶基底可从线元式 (8-4-23) 看出：

$$(e^0)_a = f^{1/2}(dt)_a\ , \quad (e^1)_a = f^{-1/2}(dr)_a\ , \quad (e^2)_a = r(d\theta)_a\ , \quad (e^3)_a = r\sin\theta(d\varphi)_a\ ,$$

相应的基底为：

$$(e_0)^a = f^{-1/2}(\partial_t)_a,\ (e_1)^a = f^{1/2}(\partial_r)^a,\ (e_2)^a = r^{-1}(\partial_\theta)_a,\ (e_3)^a = (r\sin\theta)^{-1}(\partial_\varphi)_a.$$

静态观者 G 测得的电场和磁场分别为 $E_a = F_{ab}Z^b$ 和 $B_a = -{}^*F_{ab}Z^b$, 其中 $Z^b = (e_0)^b$. 注意到式 (8-4-27)

$$\begin{aligned} F_{ab} &= -\frac{Q}{r^2}[f^{-1/2}(e^0)_a] \wedge [f^{1/2}(e^1)_b] = -\frac{Q}{r^2}(e^0)_a \wedge (e^1)_b \\ &= -\frac{Q}{r^2}[(e^0)_a(e^1)_b - (e^0)_b(e^1)_a]\ , \end{aligned}$$

故

$$F^{ab} = -\frac{Q}{r^2}[(e^0)^a(e^1)^b - (e^0)^b(e^1)^a] = \frac{Q}{r^2}[(e_0)^a(e_1)^b - (e_0)^b(e_1)^a]\ .$$

于是有

$$\begin{aligned} E_a &= F_{ab}Z^b = -\frac{Q}{r^2}[(e^0)_a(e^1)_b - (e^0)_b(e^1)_a](e_0)^b = \frac{Q}{r^2}(e^1)_a\ , \\ B_a &= -{}^*F_{ab}Z^b = -\frac{1}{2}F^{cd}\varepsilon_{cdab}Z^b \\ &= -\frac{1}{2}\frac{Q}{r^2}[(e_0)^c(e_1)^d - (e_0)^d(e_1)^c]\varepsilon_{cdab}(e_0)^b \\ &= -\frac{Q}{r^2}(e_0)^c(e_1)^d\varepsilon_{cdab}(e_0)^b = \frac{Q}{r^2}(e_0)^b(e_0)^c(e_1)^d\varepsilon_{bcda} \\ &= \frac{Q}{r^2}(e_0)^{(b}(e_0)^{c)}(e_1)^d\varepsilon_{[bc]da} = 0\ , \end{aligned}$$

和

$$E^a = \frac{Q}{r^2}(e_1)^a\ , \quad B^a = 0\ , \quad [\text{其中}\ (e_1)^a = f^{1/2}(\partial_r)^a = f^{1/2}(\partial/\partial r)^a\,].$$

此即式 (8-4-28).

6. 设 F_{ab} 是任意时空中的 2 形式场， ${}^*F_{ab}$ 是 F_{ab} 的对偶 2 形式场， $\alpha \in [0, 2\pi]$ 为常实数，则 $F'_{ab} \equiv F_{ab}\cos\alpha - {}^*F_{ab}\sin\alpha$ 称为 F_{ab} 的、角度为 α 的一个 **对偶转动** (duality rotation).

(a) 试证 F_{ab} 为无源电磁场当且仅当 F'_{ab} 为无源电磁场 [证明很易. 若用麦氏方程的外微分表达式 (7-2-4') 和 (7-2-5') 甚至一望便知.].

(b) 试证电磁场 F_{ab} 和 F'_{ab} 有相同能动张量. 提示：用 T_{ab} 的对称表示式 (6-6-28') 可简化证明.

(c) 令 $M \equiv 2F_{ab}F^{ab}$, $N \equiv 2F_{ab}{}^*F^{ab}$, $M' \equiv 2F'_{ab}F'^{ab}$, $N' \equiv 2F'_{ab}{}^*F'^{ab}$, 试证

$$M' = M\cos 2\alpha - N\sin 2\alpha\ , \qquad N' = M\sin 2\alpha + N\cos 2\alpha\ .$$

(d) 令 $\Sigma_{ab} \equiv F_{ab} + i\,{}^*F_{ab}$, $\Sigma'_{ab} \equiv F'_{ab} + i\,{}^*F'_{ab}$, 则 $K \equiv \Sigma_{ab}\Sigma^{ab}$ 和 $K' \equiv \Sigma'_{ab}\Sigma'^{ab}$ 为复标量场，故在每一时空点的 K 和 K' 相当于复平面上的两个矢量. 试用 (c) 的结果证明矢量 K' 是矢量 K 逆时针转 2α 角的结果 (即 $|K| = |K'|$, K' 与 K 的辐角差为 2α.).

(e) 设 $(\vec{E}, \vec{B})$ 和 $(\vec{E}', \vec{B}')$ 是瞬时观者分别测 F_{ab} 和 F'_{ab} 所得的电场和磁场，试证

$$\vec{E}' = \vec{E}\cos\alpha + \vec{B}\sin\alpha\ , \qquad \vec{B}' = -\vec{E}\sin\alpha + \vec{B}\cos\alpha\ ,.$$

注：对偶转动的进一步物理意义见本书下册及 Jackson (1975).

证 (a) 首先由于

$$^{**}F_{ab} \overset{(5\text{-}6\text{-}2)}{=} (-1)^{1+2(4-2)}F_{ab} = -F_{ab}\ ,$$

根据 $F'_{ab} \equiv F_{ab}\cos\alpha - {}^*F_{ab}\sin\alpha$, 有

$$^*F'_{ab} = {}^*F_{ab}\cos\alpha - {}^{**}F_{ab}\sin\alpha = F_{ab}\sin\alpha + {}^*F_{ab}\cos\alpha\ ,$$

其反变换为

$$\begin{cases} F_{ab} = F'_{ab}\cos\alpha + {}^*F'_{ab}\sin\alpha\ . \\ ^*F_{ab} = -F'_{ab}\sin\alpha + {}^*F'_{ab}\cos\alpha\ . \end{cases}$$

麦氏方程的外微分表达式由式 (7-2-4') 和 (7-2-5') 给出：

$$\begin{aligned} d\,{}^*\boldsymbol{F} &= 4\pi\,{}^*\boldsymbol{J}\ , \\ d\boldsymbol{F} &= 0\ , \end{aligned}$$

在无源时为齐次. 因此从上面的线性变换关系立即知道 F_{ab} 为无源电磁场当且仅当 F'_{ab} 为无源电磁场.

(b) 电磁场能动张量 T_{ab} 的对称表达式为 (6-6-28'):

$$T_{ab} = \frac{1}{8\pi}(F_{ac}F_b{}^c + {}^*F_{ac}{}^*F_b{}^c)\ .$$

显然有

$$\begin{aligned} T'_{ab} &= \frac{1}{8\pi}(F'_{ac}F'^{\,c}_b + {}^*F'_{ac}{}^*F'^{\,c}_b) \\ &= \frac{1}{8\pi}\Big[(F_{ac}\cos\alpha - {}^*F_{ac}\sin\alpha)(F_b{}^c\cos\alpha - {}^*F_b{}^c\sin\alpha) \\ &\qquad + (F_{ac}\sin\alpha + {}^*F_{ac}\cos\alpha)(F_b{}^c\sin\alpha + {}^*F_b{}^c\cos\alpha)\Big] \\ &= \frac{1}{8\pi}(F_{ac}F_b{}^c + {}^*F_{ac}{}^*F_b{}^c) = T_{ab}\ . \end{aligned}$$

(c) 注意第 5 题第一问的证明时得到的一个关系式 ${}^*F_{ab}{}^*F^{ab} = -F_{ab}F^{ab}$. 于是有

$$\begin{aligned} M' &= 2F'_{ab}F'^{ab} = 2(F_{ab}\cos\alpha - {}^*F_{ab}\sin\alpha)(F^{ab}\cos\alpha - {}^*F^{ab}\sin\alpha) \\ &= 2F_{ab}F^{ab}\cos^2\alpha - 2F_{ab}{}^*F^{ab}\sin\alpha\cos\alpha \\ &\quad -2{}^*F_{ab}F^{ab}\sin\alpha\cos\alpha + 2{}^*F_{ab}{}^*F^{ab}\sin^2\alpha \\ &= 2F_{ab}F^{ab}(\cos^2\alpha - \sin^2\alpha) - 4F_{ab}{}^*F^{ab}\sin\alpha\cos\alpha \\ &= M\cos 2\alpha - N\sin 2\alpha\ ; \\ N' &= 2F'_{ab}{}^*F'^{ab} = 2(F_{ab}\cos\alpha - {}^*F_{ab}\sin\alpha)(F^{ab}\sin\alpha + {}^*F^{ab}\cos\alpha) \\ &= 2F_{ab}F^{ab}\sin\alpha\cos\alpha + 2F_{ab}{}^*F^{ab}\cos^2\alpha \\ &\quad -2{}^*F_{ab}F^{ab}\sin^2\alpha - 2{}^*F_{ab}{}^*F^{ab}\sin\alpha\cos\alpha \\ &= 4F_{ab}F^{ab}\sin\alpha\cos\alpha + 2F_{ab}{}^*F^{ab}(\cos^2\alpha - \sin^2\alpha) \\ &= M\sin 2\alpha + N\cos 2\alpha\ . \end{aligned}$$

(d) 根据定义我们有

$$\begin{aligned} K' &= \Sigma'_{ab}\Sigma'^{ab} = (F'_{ab} + i\,{}^*F'_{ab})(F'^{ab} + i\,{}^*F'^{ab}) \\ &= F'_{ab}F'^{ab} + iF'_{ab}{}^*F'^{ab} + i\,{}^*F'_{ab}F'^{ab} - {}^*F'_{ab}{}^*F'^{ab} \\ &= 2F'_{ab}F'^{ab} + 2iF'_{ab}{}^*F'^{ab} = M' + iN'\ . \end{aligned}$$

因此由 (c) 的结果知复平面上的矢量 K' 是矢量 K 逆时针转 2α 角的结果.

(e) 瞬时静态观者测得的电场和磁场分别为 $E_a = F_{ab}Z^b$ 和 $B_a = -{}^*F_{ab}Z^b$, 因此有

$$\begin{aligned} E'_a &= F'_{ab}Z^b = (F_{ab}\cos\alpha - {}^*F_{ab}\sin\alpha)Z^b \\ &= F_{ab}Z^b\cos\alpha - {}^*F_{ab}Z^b\sin\alpha \\ &= E_a\cos\alpha + B_a\sin\alpha\ ; \\ B'_a &= -{}^*F'_{ab}Z^b = -(F_{ab}\sin\alpha + {}^*F_{ab}\cos\alpha)Z^b \\ &= -F_{ab}Z^b\sin\alpha - {}^*F_{ab}Z^b\cos\alpha \\ &= -E_a\sin\alpha + B_a\cos\alpha\ . \end{aligned}$$

此即

$$\vec{E}' = \vec{E}\cos\alpha + \vec{B}\sin\alpha\ , \qquad \vec{B}' = -\vec{E}\sin\alpha + \vec{B}\cos\alpha\ ,.$$

7. n 维时空称为 **爱因斯坦时空**, 若 $R_{ab} = Rg_{ab}/2$, 其中 g_{ab}, R_{ab} 和 R 分别为度规、里奇张量和标量曲率. 试证电磁真空时空 (其中电磁场非零) 不是爱因斯坦时空. 注：由第 3 章习题 17 可知任意 2 维时空必为爱因斯坦时空.

证 爱因斯坦时空即为爱因斯坦张量 $G_{ab}=0$ 的时空，而电磁真空时空的爱因斯坦方程为

$$G_{ab}=R_{ab}-\frac{1}{2}Rg_{ab}=8\pi T_{ab}\ ,$$

其中 T_{ab} 为电磁场的能动张量. 因为在有电磁场时 T_{ab} 一般不为零，故 G_{ab} 一般也不为零，所以电磁真空时空不是爱因斯坦时空.

但是第 3 章习题 17 的结论告诉我们：2 维空间或 2 维时空的爱因斯坦张量都为零, 因此 2 维时空必为爱因斯坦时空, 反过来也就是说 2 维时空的电磁场能动张量必须为零. 但是对于 1+1 维时空, 情况并非如此. 电磁场的能动张量为 $T_{ab}=\frac{1}{4\pi}(F_{ac}F_b{}^c-\frac{1}{4}g_{ab}F_{cd}F^{cd})$, 其分量式为 $T_{\mu\nu}=\frac{1}{4\pi}(F_{\mu\sigma}F_\nu{}^\sigma-\frac{1}{4}g_{\mu\nu}F_{\sigma\rho}F^{\sigma\rho})$. 利用场强张量的反称性, 现在只有一个独立分量 $F_{01}=-F_{10}$. 注意到 $F_{\sigma\rho}F^{\sigma\rho}=F_{01}F^{01}+F_{10}F^{10}=2F_{01}F^{01}$, 于是有

$$\begin{aligned}
T_{00} &= \frac{1}{4\pi}\Big(F_{0\sigma}F_0{}^\sigma-\frac{1}{4}g_{00}2F_{01}F^{01}\Big)=\frac{1}{4\pi}\Big(F_{01}F_0{}^1-\frac{1}{2}g_{00}F_{01}F^{01}\Big)\\
&= \frac{1}{4\pi}\Big(g_{0\sigma}F_{01}F^{\sigma 1}-\frac{1}{2}g_{00}F_{01}F^{01}\Big)=\frac{1}{8\pi}g_{00}F_{01}F^{01}\ ,\\
T_{01} &= \frac{1}{4\pi}\Big(F_{0\sigma}F_1{}^\sigma-\frac{1}{4}g_{01}2F_{01}F^{01}\Big)=\frac{1}{4\pi}\Big(F_{01}F_1{}^1-\frac{1}{2}g_{01}F_{01}F^{01}\Big)\\
&= \frac{1}{4\pi}\Big(g_{1\sigma}F_{01}F^{\sigma 1}-\frac{1}{2}g_{01}F_{01}F^{01}\Big)=\frac{1}{8\pi}g_{01}F_{01}F^{01}\ ,\\
T_{11} &= \frac{1}{4\pi}\Big(F_{1\sigma}F_1{}^\sigma-\frac{1}{4}g_{11}2F_{01}F^{01}\Big)=\frac{1}{4\pi}\Big(F_{10}F_1{}^0-\frac{1}{2}g_{11}F_{01}F^{01}\Big)\\
&= \frac{1}{4\pi}\Big(g_{1\sigma}F_{10}F^{\sigma 0}-\frac{1}{2}g_{11}F_{01}F^{01}\Big)=\frac{1}{8\pi}g_{11}F_{01}F^{01}\ .
\end{aligned}$$

可见 $T_{\mu\nu}\neq 0$, 与爱因斯坦张量 $G_{\mu\nu}=0$ 不相容！因此 $1+3$ 维形式的电磁场能动张量不适用于 $1+1$ 维情形.

8. 考虑 Taub 的平面对称真空解 (8-6-1’).

(a) 写出静态观者的 4 速用坐标基矢的表达式;

(b) 设两静态观者的空间坐标分别为 (x,y,z_1) 和 (x,y,z_2), 求他们间的空间距离.

解 (a) 由 Taub 平面对称真空解

$$ds^2=z^{-1/2}(-dt^2+dz^2)+z(dx^2+dy^2)$$

知 $g_{00}=-z^{-1/2}$. 设静态观者的固有时为 τ, 因 $d\tau=\sqrt{-g_{00}}\,dt$, 于是静态观者的 4 速为

$$Z^a=\Big(\frac{\partial}{\partial\tau}\Big)^a=(-g_{00})^{-1/2}\Big(\frac{\partial}{\partial t}\Big)^a=z^{1/4}\Big(\frac{\partial}{\partial t}\Big)^a=z^{1/4}\xi^a\ .$$

(b) 位于 (x,y,z_1) 和 (x,y,z_2) 的两静态观者的空间距离为 (设 $z_2>z_1>0$)

$$l=\int_{z_1}^{z_2}\sqrt{z^{-1/2}dz^2}=\int_{z_1}^{z_2}z^{-1/4}dz=\frac{4}{3}(z_2^{3/4}-z_1^{3/4})\ .$$

设他们的坐标距离为 $l_c = z_2 - z_1$. 当 $z_1 < 1$ 时，对于小的 l_c 有 $l > l_c$, 对于大的 l_c 有 $l < l_c$; 相等时的 l_c 由方程

$$\frac{4}{3}[(l_c + z_1)^{3/4} - z_1^{3/4}] = l_c$$

决定．而当 $z_1 > 1$ 时，总有 $l < l_c$.

9. 试证式 (eq8-6-5) 的 F_{ab} 有平面对称性，即 $\mathcal{L}_{\xi_i} F_{ab} = 0$ $(i = 1, 2, 3)$, 其中 $\xi_1^a \equiv (\partial/\partial x)^a$, $\xi_2^a \equiv (\partial/\partial y)^a$, $\xi_3^a \equiv -y(\partial/\partial x)^a + x(\partial/\partial y)^a$ 是反映度规 (8-6-3) 平面对称性的 Killing 场.

证 这是显而易见的,因为 ξ_1^a 和 ξ_2^a 的积分曲线分别为 x 和 y 坐标线,而 ξ_3^a 的积分曲线为 φ 坐标线，其中 φ 满足 $\cos\varphi = x/\sqrt{x^2+y^2}$ (或 $\sin\varphi = y/\sqrt{x^2+y^2}$). 注意到 F_{ab} 的分量 [式 (8-6-5)] 都只是 z 的函数，故根据定理 4-2-2 式 (4-2-3) 有

$$(\mathcal{L}_{\xi_1} F)_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x} = 0\ , \quad (\mathcal{L}_{\xi_2} F)_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial y} = 0\ , \quad (\mathcal{L}_{\xi_3} F)_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial \varphi} = 0\ ,$$

亦即 $\mathcal{L}_{\xi_i} F_{ab} = 0$.

或者利用定理 4-2-5 的公式 (4-2-8) 来计算：

$$\mathcal{L}_{\xi_i} F_{ab} = \xi_i^c \partial_c F_{ab} + F_{cb} \partial_a \xi_i^c + F_{ac} \partial_b \xi_i^c\ .$$

右边的第一项为

$$\begin{aligned}
\xi_i^c \partial_c F_{ab} &= (dx^\mu)_a (dx^\nu)_b \xi_i^\sigma \partial_\sigma F_{\mu\nu}(z) = (dx^\mu)_a (dx^\nu)_b \xi_i^\sigma \delta^3{}_\sigma \frac{\partial}{\partial z} F_{\mu\nu}(z) \\
&= (dx^\mu)_a (dx^\nu)_b \xi_i^3 \frac{\partial}{\partial z} F_{\mu\nu}(z) = 0\ .
\end{aligned}$$

最后一步是因为 $\xi_i^3 = 0$ (非零的 Killing 场分量只有 $\xi_1^1 = 1$, $\xi_2^2 = 1$, $\xi_3^1 = -y$ 和 $\xi_3^2 = x$). 右边的后两项中的 $\partial_a \xi_i^c$ 根据式 (3-1-10) 有

$$\partial_a \xi_1^c = \partial_a (\partial/\partial x)^c = \partial_a \xi_2^c = \partial_a (\partial/\partial y)^c = 0\ ,$$

显然对 ξ_1^a 和 ξ_2^a 结论成立．而对 $\partial_a \xi_3^c$ 有

$$\begin{aligned}
\partial_a \xi_3^c &= (dx^\mu)\partial_\mu \Big[-y\Big(\frac{\partial}{\partial x}\Big)^c + x\Big(\frac{\partial}{\partial y}\Big)^c \Big] = (dy)_a \Big[-\Big(\frac{\partial}{\partial x}\Big)^c \Big] + (dx)_a \Big[\Big(\frac{\partial}{\partial y}\Big)^c \Big] \\
&= -(dx^2)_a \Big(\frac{\partial}{\partial x^1}\Big)^c + (dx^1)_a \Big(\frac{\partial}{\partial x^2}\Big)^c\ ,
\end{aligned}$$

导致

$$\begin{aligned}
& F_{cb} \partial_a \xi_3^c + F_{ac} \partial_b \xi_3^c \\
= \quad & F_{cb} \Big[-(dx^2)_a \Big(\frac{\partial}{\partial x^1}\Big)^c + (dx^1)_a \Big(\frac{\partial}{\partial x^2}\Big)^c \Big]
\end{aligned}$$

$$
\begin{aligned}
& +F_{ac}\Big[-(dx^2)_b\Big(\frac{\partial}{\partial x^1}\Big)^c+(dx^1)_b\Big(\frac{\partial}{\partial x^2}\Big)^c\Big] \\
= & -(dx^2)_aF_{1b}+(dx^1)_aF_{2b}-(dx^2)_bF_{a1}+(dx^1)_bF_{a2} \\
= & -(dx^2)_a(dx^\mu)_bF_{1\mu}+(dx^1)_a(dx^\mu)_bF_{2\mu}-(dx^2)_b(dx^\mu)_aF_{\mu1}+(dx^1)_b(dx^\mu)_aF_{\mu2} \\
\overset{(8\text{-}6\text{-}5)}{=} & -(dx^2)_a(dx^2)_bF_{12}+(dx^1)_a(dx^1)_bF_{21}-(dx^2)_b(dx^2)_aF_{21}+(dx^1)_b(dx^1)_aF_{12} \\
= & \ 0\ .
\end{aligned}
$$

对 ξ_3^a 结论也成立．故命题得证．

*10. 推出有源麦氏方程在 NP 形式中的表达式．答案：在式 (8-8-3a)–(8-8-3d) 的每式右边各加一项,依次为 $-4\pi J_4$, $-4\pi J_2$, $-4\pi J_1$, $-4\pi J_3$【似应为 2π!】(J_1, J_2, J_3, J_4 是 J_a 在类光标架的分量)．

解 可仿照式 (8-8-3a) 的推导．第一个方程：

$$
\begin{aligned}
2D\Phi_1 &= k^c\nabla_c[F_{ab}(k^al^b+\bar m^am^b)] = F_{ab}k^ak^c\nabla_cl^b+F_{ab}l^bk^c\nabla_ck^a+k^al^bk^c\nabla_cF_{ab} \\
&\quad +F_{ab}\bar m^ak^c\nabla_cm^b+F_{ab}m^bk^c\nabla_c\bar m^a+\bar m^am^bk^c\nabla_cF_{ab}\ ,
\end{aligned}
$$

其中

$$
\begin{aligned}
F_{ab}k^ak^c\nabla_cl^b &= F_{4\nu}(\varepsilon^\nu)_b(\varepsilon_4)^c\nabla_c(\varepsilon_3)^b = F_{4\nu}g^{\nu\mu}\omega_{\mu34} \\
&= F_{41}g^{12}\omega_{234}+F_{42}g^{21}\omega_{134}+F_{43}g^{34}\omega_{434} \\
&= F_{41}\omega_{234}+F_{42}\omega_{134}+F_{43}\omega_{344}\ , \\
F_{ab}l^bk^c\nabla_ck^a &= F_{\nu3}(\varepsilon^\nu)_a(\varepsilon_4)^c\nabla_c(\varepsilon_4)^a = F_{\nu3}g^{\nu\mu}\omega_{\mu44} \\
&= F_{13}g^{12}\omega_{244}+F_{23}g^{21}\omega_{144}+F_{43}g^{43}\omega_{344} \\
&= F_{13}\omega_{244}+F_{23}\omega_{144}-F_{43}\omega_{344}\ , \\
F_{ab}\bar m^ak^c\nabla_cm^b &= F_{2\nu}(\varepsilon^\nu)_b(\varepsilon_4)^c\nabla_c(\varepsilon_1)^b = F_{2\nu}g^{\nu\mu}\omega_{\mu14} \\
&= F_{21}g^{12}\omega_{214}+F_{23}g^{34}\omega_{414}+F_{24}g^{43}\omega_{314} \\
&= -F_{21}\omega_{124}+F_{23}\omega_{144}-F_{42}\omega_{134}\ , \\
F_{ab}m^bk^c\nabla_c\bar m^a &= F_{\nu1}(\varepsilon^\nu)_a(\varepsilon_4)^c\nabla_c(\varepsilon_2)^a = F_{\nu1}g^{\nu\mu}\omega_{\mu24} \\
&= F_{21}g^{21}\omega_{124}+F_{31}g^{34}\omega_{424}+F_{41}g^{43}\omega_{324} \\
&= F_{21}\omega_{124}-F_{13}\omega_{244}+F_{41}\omega_{234}\ ,
\end{aligned}
$$

即

$$
\begin{aligned}
& F_{ab}k^ak^c\nabla_cl^b+F_{ab}l^bk^c\nabla_ck^a+F_{ab}\bar m^ak^c\nabla_cm^b+F_{ab}m^bk^c\nabla_c\bar m^a \\
= & \ 2F_{41}\omega_{234}+2F_{23}\omega_{144} = 2(\pi\Phi_0-\kappa\Phi_2)\ .
\end{aligned}
$$

得

$$
2D\Phi_1 = 2(\pi\Phi_0-\kappa\Phi_2)+k^al^bk^c\nabla_cF_{ab}+\bar m^am^bk^c\nabla_cF_{ab}\ .
$$

类似地,

$$\bar{\delta}\Phi_0 = \bar{m}^c\nabla_c[F_{ab}k^a m^b] = F_{ab}k^a\bar{m}^c\nabla_c m^b + F_{ab}m^b\bar{m}^c\nabla_c k^a + k^a m^b\bar{m}^c\nabla_c F_{ab} ,$$

其中

$$\begin{aligned} F_{ab}k^a\bar{m}^c\nabla_c m^b &= F_{4\nu}(\varepsilon^\nu)_b(\varepsilon_2)^c\nabla_c(\varepsilon_1)^b = F_{4\nu}g^{\nu\mu}\omega_{\mu 12} \\ &= F_{41}g^{12}\omega_{212} + F_{42}g^{21}\omega_{112} + F_{43}g^{34}\omega_{412} \\ &= -F_{41}\omega_{122} + F_{43}\omega_{142} , \\ F_{ab}m^b\bar{m}^c\nabla_c k^a &= F_{\nu 1}(\varepsilon^\nu)_a(\varepsilon_2)^c\nabla_c(\varepsilon_4)^a = F_{\nu 1}g^{\nu\mu}\omega_{\mu 42} \\ &= F_{21}g^{21}\omega_{142} + F_{31}g^{34}\omega_{442} + F_{41}g^{43}\omega_{342} \\ &= F_{21}\omega_{142} - F_{41}\omega_{342} , \end{aligned}$$

即

$$\begin{aligned} & F_{ab}k^a\bar{m}^c\nabla_c m^b + F_{ab}m^b\bar{m}^c\nabla_c k^a \\ = & -F_{41}(\omega_{122} + \omega_{342}) + (F_{43} + F_{21})\omega_{142} \\ = & -\Phi_0(-2\alpha) + 2\Phi_1(-\rho) = 2(\alpha\Phi_0 - \rho\Phi_1) . \end{aligned}$$

得

$$\bar{\delta}\Phi_0 = 2(\alpha\Phi_0 - \rho\Phi_1) + k^a m^b\bar{m}^c\nabla_c F_{ab} .$$

于是有

$$\begin{aligned} & D\Phi_1 - \bar{\delta}\Phi_0 \\ = & (\pi\Phi_0 - \kappa\Phi_2) + \frac{1}{2}(k^a l^b k^c + \bar{m}^a m^b k^c)\nabla_c F_{ab} - 2(\alpha\Phi_0 - \rho\Phi_1) - k^a m^b\bar{m}^c\nabla_c F_{ab} \\ = & (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 + \frac{1}{2}(k^a l^b k^c + \bar{m}^a m^b k^c - 2k^a m^b\bar{m}^c)\nabla_c F_{ab} . \end{aligned}$$

第二个方程:

$$D\Phi_2 = k^c\nabla_c[F_{ab}\bar{m}^a l^b] = F_{ab}\bar{m}^a k^c\nabla_c l^b + F_{ab}l^b k^c\nabla_c\bar{m}^a + \bar{m}^a l^b k^c\nabla_c F_{ab} ,$$

其中

$$\begin{aligned} F_{ab}\bar{m}^a k^c\nabla_c l^b &= F_{2\nu}(\varepsilon^\nu)_b(\varepsilon_4)^c\nabla_c(\varepsilon_3)^b = F_{2\nu}g^{\nu\mu}\omega_{\mu 34} \\ &= F_{21}g^{12}\omega_{234} + F_{23}g^{34}\omega_{434} + F_{24}g^{43}\omega_{334} \\ &= F_{21}\omega_{234} + F_{23}\omega_{344} , \\ F_{ab}l^b k^c\nabla_c\bar{m}^a &= F_{\nu 3}(\varepsilon^\nu)_a(\varepsilon_4)^c\nabla_c(\varepsilon_2)^a = F_{\nu 3}g^{\nu\mu}\omega_{\mu 23} \\ &= F_{13}g^{12}\omega_{224} + F_{23}g^{21}\omega_{124} + F_{43}g^{43}\omega_{324} \\ &= F_{23}\omega_{124} + F_{43}\omega_{234} , \end{aligned}$$

即

$$\begin{aligned}
&F_{ab}\bar{m}^a k^c\nabla_c l^b + F_{ab}l^b k^c\nabla_c\bar{m}^a \\
= \ & F_{23}(\omega_{124}+\omega_{344}) + (F_{21}+F_{43})\omega_{234} \\
= \ & \Phi_2(-2\varepsilon) + 2\Phi_1\pi = 2(\pi\Phi_1 - \varepsilon\Phi_2)\ .
\end{aligned}$$

得

$$D\Phi_2 = 2(\pi\Phi_1 - \varepsilon\Phi_2) + \bar{m}^a l^b k^c\nabla_c F_{ab}\ .$$

类似地，

$$\begin{aligned}
2\bar{\delta}\Phi_1 \ &= \ \bar{m}^c\nabla_c[F_{ab}(k^a l^b + \bar{m}^a m^b)] = F_{ab}k^a\bar{m}^c\nabla_c l^b + F_{ab}l^b\bar{m}^c\nabla_c k^a + k^a l^b\bar{m}^c\nabla_c F_{ab} \\
&\quad + F_{ab}\bar{m}^a\bar{m}^c\nabla_c m^b + F_{ab}m^b\bar{m}^c\nabla_c\bar{m}^a + \bar{m}^a m^b\bar{m}^c\nabla_c F_{ab}\ ,
\end{aligned}$$

其中

$$\begin{aligned}
F_{ab}k^a\bar{m}^c\nabla_c l^b \ &= \ F_{4\nu}(\varepsilon^\nu)_b(\varepsilon_2)^c\nabla_c(\varepsilon_3)^b = F_{4\nu}g^{\nu\mu}\omega_{\mu 32} \\
&= \ F_{41}g^{12}\omega_{232} + F_{42}g^{21}\omega_{132} + F_{43}g^{34}\omega_{432} \\
&= \ F_{41}\omega_{232} + F_{42}\omega_{132} + F_{43}\omega_{342}\ , \\
F_{ab}l^b\bar{m}^c\nabla_c k^a \ &= \ F_{\nu 3}(\varepsilon^\nu)_a(\varepsilon_2)^c\nabla_c(\varepsilon_4)^a = F_{\nu 3}g^{\nu\mu}\omega_{\mu 42} \\
&= \ F_{13}g^{12}\omega_{242} + F_{23}g^{21}\omega_{142} + F_{43}g^{43}\omega_{342} \\
&= \ F_{13}\omega_{242} + F_{23}\omega_{142} - F_{43}\omega_{342}\ , \\
F_{ab}\bar{m}^a\bar{m}^c\nabla_c m^b \ &= \ F_{2\nu}(\varepsilon^\nu)_b(\varepsilon_2)^c\nabla_c(\varepsilon_1)^b = F_{2\nu}g^{\nu\mu}\omega_{\mu 14} \\
&= \ F_{21}g^{12}\omega_{212} + F_{23}g^{34}\omega_{412} + F_{24}g^{43}\omega_{312} \\
&= \ -F_{21}\omega_{122} + F_{23}\omega_{142} - F_{42}\omega_{132}\ , \\
F_{ab}m^b\bar{m}^c\nabla_c\bar{m}^a \ &= \ F_{\nu 1}(\varepsilon^\nu)_a(\varepsilon_2)^c\nabla_c(\varepsilon_2)^a = F_{\nu 1}g^{\nu\mu}\omega_{\mu 22} \\
&= \ F_{21}g^{21}\omega_{122} + F_{31}g^{34}\omega_{422} + F_{41}g^{43}\omega_{322} \\
&= \ F_{21}\omega_{122} - F_{13}\omega_{242} + F_{41}\omega_{232}\ ,
\end{aligned}$$

即

$$\begin{aligned}
&F_{ab}k^a\bar{m}^c\nabla_c l^b + F_{ab}l^b\bar{m}^c\nabla_c k^a + F_{ab}\bar{m}^a\bar{m}^c\nabla_c m^b + F_{ab}m^b\bar{m}^c\nabla_c\bar{m}^a \\
= \ & 2F_{41}\omega_{232} + 2F_{23}\omega_{142} = 2(\lambda\Phi_0 - \rho\Phi_2)\ .
\end{aligned}$$

得

$$2\bar{\delta}\Phi_1 = 2(\lambda\Phi_0 - \rho\Phi_2) + k^a l^b\bar{m}^c\nabla_c F_{ab} + \bar{m}^a m^b\bar{m}^c\nabla_c F_{ab}\ .$$

于是有

$$\begin{aligned}
&D\Phi_2 - \bar{\delta}\Phi_1 \\
= \ & 2(\pi\Phi_1 - \varepsilon\Phi_2) + \bar{m}^a l^b k^c\nabla_c F_{ab} - (\lambda\Phi_0 - \rho\Phi_2) - \frac{1}{2}(k^a l^b\bar{m}^c + \bar{m}^a m^b\bar{m}^c)\nabla_c F_{ab} \\
= \ & -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\varepsilon)\Phi_2 + \frac{1}{2}(-k^a l^b\bar{m}^c - \bar{m}^a m^b\bar{m}^c + 2\bar{m}^a l^b k^c)\nabla_c F_{ab}\ .
\end{aligned}$$

第三个方程:

$$
\begin{aligned}
2\delta\Phi_1 &= m^c\nabla_c[F_{ab}(k^a l^b + \bar{m}^a m^b)] = F_{ab}k^a m^c\nabla_c l^b + F_{ab}l^b m^c\nabla_c k^a + k^a l^b m^c\nabla_c F_{ab} \\
&\quad + F_{ab}\bar{m}^a m^c\nabla_c m^b + F_{ab}m^b m^c\nabla_c\bar{m}^a + \bar{m}^a m^b m^c\nabla_c F_{ab} ,
\end{aligned}
$$

其中

$$
\begin{aligned}
F_{ab}k^a m^c\nabla_c l^b &= F_{4\nu}(\varepsilon^\nu)_b(\varepsilon_1)^c\nabla_c(\varepsilon_3)^b = F_{4\nu}g^{\nu\mu}\omega_{\mu 31} \\
&= F_{41}g^{12}\omega_{231} + F_{42}g^{21}\omega_{131} + F_{43}g^{34}\omega_{431} \\
&= F_{41}\omega_{231} + F_{42}\omega_{131} + F_{43}\omega_{341} , \\
F_{ab}l^b m^c\nabla_c k^a &= F_{\nu 3}(\varepsilon^\nu)_a(\varepsilon_1)^c\nabla_c(\varepsilon_4)^a = F_{\nu 3}g^{\nu\mu}\omega_{\mu 41} \\
&= F_{13}g^{12}\omega_{241} + F_{23}g^{21}\omega_{141} + F_{43}g^{43}\omega_{341} \\
&= F_{13}\omega_{241} + F_{23}\omega_{141} - F_{43}\omega_{341} , \\
F_{ab}\bar{m}^a m^c\nabla_c m^b &= F_{2\nu}(\varepsilon^\nu)_b(\varepsilon_1)^c\nabla_c(\varepsilon_1)^b = F_{2\nu}g^{\nu\mu}\omega_{\mu 11} \\
&= F_{21}g^{12}\omega_{211} + F_{23}g^{34}\omega_{411} + F_{24}g^{43}\omega_{311} \\
&= -F_{21}\omega_{121} + F_{23}\omega_{141} - F_{42}\omega_{131} , \\
F_{ab}m^b m^c\nabla_c\bar{m}^a &= F_{\nu 1}(\varepsilon^\nu)_a(\varepsilon_1)^c\nabla_c(\varepsilon_2)^a = F_{\nu 1}g^{\nu\mu}\omega_{\mu 21} \\
&= F_{21}g^{21}\omega_{121} + F_{31}g^{34}\omega_{421} + F_{41}g^{43}\omega_{321} \\
&= F_{21}\omega_{121} - F_{13}\omega_{241} + F_{41}\omega_{231} ,
\end{aligned}
$$

即

$$
\begin{aligned}
& F_{ab}k^a m^c\nabla_c l^b + F_{ab}l^b m^c\nabla_c k^a + F_{ab}\bar{m}^a m^c\nabla_c m^b + F_{ab}m^b m^c\nabla_c\bar{m}^a \\
= \; & 2F_{41}\omega_{231} + 2F_{23}\omega_{141} = 2(\mu\Phi_0 - \sigma\Phi_2) .
\end{aligned}
$$

得

$$
2\delta\Phi_1 = 2(\mu\Phi_0 - \sigma\Phi_2) + k^a l^b m^c\nabla_c F_{ab} + \bar{m}^a m^b m^c\nabla_c F_{ab} .
$$

类似地,

$$
\Delta\Phi_0 = l^c\nabla_c[F_{ab}k^a m^b] = F_{ab}k^a l^c\nabla_c m^b + F_{ab}m^b l^c\nabla_c k^a + k^a m^b l^c\nabla_c F_{ab} ,
$$

其中

$$
\begin{aligned}
F_{ab}k^a l^c\nabla_c m^b &= F_{4\nu}(\varepsilon^\nu)_b(\varepsilon_3)^c\nabla_c(\varepsilon_1)^b = F_{4\nu}g^{\nu\mu}\omega_{\mu 13} \\
&= F_{41}g^{12}\omega_{213} + F_{42}g^{21}\omega_{113} + F_{43}g^{34}\omega_{413} \\
&= -F_{41}\omega_{123} + F_{43}\omega_{143} , \\
F_{ab}m^b l^c\nabla_c k^a &= F_{\nu 1}(\varepsilon^\nu)_a(\varepsilon_3)^c\nabla_c(\varepsilon_4)^a = F_{\nu 1}g^{\nu\mu}\omega_{\mu 43} \\
&= F_{21}g^{21}\omega_{143} + F_{31}g^{34}\omega_{443} + F_{41}g^{43}\omega_{343} \\
&= F_{21}\omega_{143} - F_{41}\omega_{343} ,
\end{aligned}
$$

即

$$
\begin{aligned}
& F_{ab}k^a l^c \nabla_c m^b + F_{ab} m^b l^c \nabla_c k^a \\
= & -F_{41}(\omega_{123}+\omega_{343}) + (F_{43}+F_{21})\omega_{143} \\
= & -\Phi_0(-2\gamma) + 2\Phi_1(-\tau) = 2(\gamma\Phi_0 - \tau\Phi_1) \ .
\end{aligned}
$$

得

$$
\Delta\Phi_0 = 2(\gamma\Phi_0 - \tau\Phi_1) + k^a m^b l^c \nabla_c F_{ab} \ .
$$

于是有

$$
\begin{aligned}
& \delta\Phi_1 - \Delta\Phi_0 \\
= & (\mu\Phi_0 - \sigma\Phi_2) + \frac{1}{2}(k^a l^b m^c + \bar{m}^a m^b m^c)\nabla_c F_{ab} - 2(\gamma\Phi_0 - \tau\Phi_1) - k^a m^b l^c \nabla_c F_{ab} \\
= & (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 + \frac{1}{2}(k^a l^b m^c + \bar{m}^a m^b m^c - 2k^a m^b l^c)\nabla_c F_{ab} \ .
\end{aligned}
$$

第四个方程：

$$
\delta\Phi_2 = m^c \nabla_c [F_{ab}\bar{m}^a l^b] = F_{ab}\bar{m}^a m^c \nabla_c l^b + F_{ab} l^b m^c \nabla_c \bar{m}^a + \bar{m}^a l^b m^c \nabla_c F_{ab} \ ,
$$

其中

$$
\begin{aligned}
F_{ab}\bar{m}^a m^c \nabla_c l^b &= F_{2\nu}(\varepsilon^\nu)_b(\varepsilon_1)^c \nabla_c(\varepsilon_3)^b = F_{2\nu} g^{\nu\mu}\omega_{\mu 31} \\
&= F_{21} g^{12}\omega_{231} + F_{23} g^{34}\omega_{431} + F_{24} g^{43}\omega_{331} \\
&= F_{21}\omega_{231} + F_{23}\omega_{341} \ , \\
F_{ab} l^b m^c \nabla_c \bar{m}^a &= F_{\nu 3}(\varepsilon^\nu)_a(\varepsilon_1)^c \nabla_c(\varepsilon_2)^a = F_{\nu 3} g^{\nu\mu}\omega_{\mu 21} \\
&= F_{13} g^{12}\omega_{221} + F_{23} g^{21}\omega_{121} + F_{43} g^{43}\omega_{321} \\
&= F_{23}\omega_{121} + F_{43}\omega_{231} \ ,
\end{aligned}
$$

即

$$
\begin{aligned}
& F_{ab}\bar{m}^a m^c \nabla_c l^b + F_{ab} l^b m^c \nabla_c \bar{m}^a \\
= & F_{23}(\omega_{121}+\omega_{341}) + (F_{21}+F_{43})\omega_{231} \\
= & \Phi_2(-2\beta) + 2\Phi_1\mu = 2(\mu\Phi_1 - \beta\Phi_2) \ .
\end{aligned}
$$

得

$$
\delta\Phi_2 = 2(\mu\Phi_1 - \beta\Phi_2) + \bar{m}^a l^b m^c \nabla_c F_{ab} \ .
$$

类似地，

$$
\begin{aligned}
2\Delta\Phi_1 = & l^c \nabla_c [F_{ab}(k^a l^b + \bar{m}^a m^b)] = F_{ab} k^a l^c \nabla_c l^b + F_{ab} l^b l^c \nabla_c k^a + k^a l^b l^c \nabla_c F_{ab} \\
& + F_{ab}\bar{m}^a l^c \nabla_c m^b + F_{ab} m^b l^c \nabla_c \bar{m}^a + \bar{m}^a m^b l^c \nabla_c F_{ab} \ ,
\end{aligned}
$$

其中

$$\begin{aligned}
F_{ab}k^a l^c \nabla_c l^b &= F_{4\nu}(\varepsilon^\nu)_b(\varepsilon_3)^c\nabla_c(\varepsilon_3)^b = F_{4\nu}g^{\nu\mu}\omega_{\mu 33} \\
&= F_{41}g^{12}\omega_{233} + F_{42}g^{21}\omega_{133} + F_{43}g^{34}\omega_{433} \\
&= F_{41}\omega_{233} + F_{42}\omega_{133} + F_{43}\omega_{343}\ , \\
F_{ab}l^b l^c \nabla_c k^a &= F_{\nu 3}(\varepsilon^\nu)_a(\varepsilon_3)^c\nabla_c(\varepsilon_4)^a = F_{\nu 3}g^{\nu\mu}\omega_{\mu 43} \\
&= F_{13}g^{12}\omega_{243} + F_{23}g^{21}\omega_{143} + F_{43}g^{43}\omega_{343} \\
&= F_{13}\omega_{243} + F_{23}\omega_{143} - F_{43}\omega_{343}\ , \\
F_{ab}\bar{m}^a l^c \nabla_c m^b &= F_{2\nu}(\varepsilon^\nu)_b(\varepsilon_3)^c\nabla_c(\varepsilon_1)^b = F_{2\nu}g^{\nu\mu}\omega_{\mu 13} \\
&= F_{21}g^{12}\omega_{213} + F_{23}g^{34}\omega_{413} + F_{24}g^{43}\omega_{313} \\
&= -F_{21}\omega_{123} + F_{23}\omega_{143} - F_{42}\omega_{133}\ , \\
F_{ab}m^b l^c \nabla_c \bar{m}^a &= F_{\nu 1}(\varepsilon^\nu)_a(\varepsilon_3)^c\nabla_c(\varepsilon_2)^a = F_{\nu 1}g^{\nu\mu}\omega_{\mu 23} \\
&= F_{21}g^{21}\omega_{123} + F_{31}g^{34}\omega_{423} + F_{41}g^{43}\omega_{323} \\
&= F_{21}\omega_{123} - F_{13}\omega_{243} + F_{41}\omega_{233}\ ,
\end{aligned}$$

即

$$\begin{aligned}
& F_{ab}k^a l^c \nabla_c l^b + F_{ab}l^b l^c \nabla_c k^a + F_{ab}\bar{m}^a l^c \nabla_c m^b + F_{ab}m^b l^c \nabla_c \bar{m}^a \\
=\ & 2F_{41}\omega_{233} + 2F_{23}\omega_{143} = 2(\nu\Phi_0 - \tau\Phi_2)\ .
\end{aligned}$$

得

$$2\Delta\Phi_1 = 2(\nu\Phi_0 - \tau\Phi_2) + k^a l^b l^c \nabla_c F_{ab} + \bar{m}^a m^b l^c \nabla_c F_{ab}\ .$$

于是有

$$\begin{aligned}
& \delta\Phi_2 - \Delta\Phi_1 \\
=\ & 2(\mu\Phi_1 - \beta\Phi_2) + \bar{m}^a l^b m^c \nabla_c F_{ab} - (\nu\Phi_0 - \tau\Phi_2) - \frac{1}{2}(k^a l^b l^c + \bar{m}^a m^b l^c)\nabla_c F_{ab} \\
=\ & -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 + \frac{1}{2}(-k^a l^b l^c - \bar{m}^a m^b l^c + 2\bar{m}^a l^b m^c)\nabla_c F_{ab}\ .
\end{aligned}$$

令

$$\begin{aligned}
G_1 &= \frac{1}{2}(k^a l^b k^c + \bar{m}^a m^b k^c - 2k^a m^b \bar{m}^c)\nabla_c F_{ab}\ , \\
G_2 &= \frac{1}{2}(-k^a l^b \bar{m}^c - \bar{m}^a m^b \bar{m}^c + 2\bar{m}^a l^b k^c)\nabla_c F_{ab}\ , \\
G_3 &= \frac{1}{2}(k^a l^b m^c + \bar{m}^a m^b m^c - 2k^a m^b l^c)\nabla_c F_{ab}\ , \\
G_4 &= \frac{1}{2}(-k^a l^b l^c - \bar{m}^a m^b l^c + 2\bar{m}^a l^b m^c)\nabla_c F_{ab}\ ,
\end{aligned}$$

四个方程变为

$$\begin{aligned}
D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 + G_1 \ , \\
D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\varepsilon)\Phi_2 + G_2 \ , \\
\delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 + G_3 \ , \\
\delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 + G_4 \ .
\end{aligned}$$

由式 (8-7-3) 可知

$$g^{ac} = m^a\bar{m}^c + \bar{m}^a m^c - l^a k^c - k^a l^c \ ,$$

故有源麦氏方程 $\nabla^a F_{ab} = -4\pi J_b$ 可表为

$$(m^a\bar{m}^c + \bar{m}^a m^c - l^a k^c - k^a l^c)\nabla_c F_{ab} = -4\pi J_b \ .$$

与 k^b 缩并得

$$\begin{aligned}
-4\pi J_4 &= (m^a k^b \bar{m}^c + \bar{m}^a k^b m^c - l^a k^b k^c - k^a k^b l^c)\nabla_c F_{ab} \\
&= [m^a k^b \bar{m}^c + \bar{m}^a k^b m^c - (l^a k^b k^c + k^a k^b l^c)]\nabla_c F_{ab} \\
&= [m^a k^b \bar{m}^c - (m^a \bar{m}^b k^c + k^a m^b \bar{m}^c) + k^a l^b k^c]\nabla_c F_{ab} \\
&= [-m^b k^a \bar{m}^c - (-m^b \bar{m}^a k^c + k^a m^b \bar{m}^c) + k^a l^b k^c)]\nabla_c F_{ab} \\
&= (k^a l^b k^c + \bar{m}^a m^b k^c - 2k^a m^b \bar{m}^c)\nabla_c F_{ab} \\
&= 2G_1 \ ,
\end{aligned}$$

其中第三步是因为 $\nabla_{[c}F_{ab]} = 0$ 导致 $\bar{m}^{[a}k^b m^{c]}\nabla_c F_{ab} = 0$ 和 $l^{[a}k^b k^{c]}\nabla_c F_{ab} = 0.$
与 $\bar{m}^b$ 缩并得

$$\begin{aligned}
-4\pi J_2 &= (m^a \bar{m}^b \bar{m}^c + \bar{m}^a \bar{m}^b m^c - l^a \bar{m}^b k^c - k^a \bar{m}^b l^c)\nabla_c F_{ab} \\
&= [(m^a \bar{m}^b \bar{m}^c + \bar{m}^a \bar{m}^b m^c) - l^a \bar{m}^b k^c - k^a \bar{m}^b l^c]\nabla_c F_{ab} \\
&= [-\bar{m}^a m^b \bar{m}^c - l^a \bar{m}^b k^c + (\bar{m}^a l^b k^c + l^a k^b \bar{m}^c)]\nabla_c F_{ab} \\
&= [-\bar{m}^a m^b \bar{m}^c + l^b \bar{m}^a k^c + (\bar{m}^a l^b k^c - l^b k^a \bar{m}^c)]\nabla_c F_{ab} \\
&= (k^a l^b m^c + \bar{m}^a m^b m^c - 2k^a m^b l^c)\nabla_c F_{ab} \\
&= 2G_3 \ ,
\end{aligned}$$

其中第三步是因为 $\nabla_{[c}F_{ab]} = 0$ 导致 $m^{[a}\bar{m}^b \bar{m}^{c]}\nabla_c F_{ab} = 0$ 和 $k^{[a}\bar{m}^b l^{c]}\nabla_c F_{ab} = 0.$
与 m^b 缩并得

$$\begin{aligned}
-4\pi J_1 &= (m^a m^b \bar{m}^c + \bar{m}^a m^b m^c - l^a m^b k^c - k^a m^b l^c)\nabla_c F_{ab} \\
&= [(m^a m^b \bar{m}^c + \bar{m}^a m^b m^c) - l^a m^b k^c - k^a m^b l^c]\nabla_c F_{ab} \\
&= [-m^a \bar{m}^b m^c + (k^a l^b m^c + m^a k^b l^c) - k^a m^b l^c]\nabla_c F_{ab} \\
&= [m^b \bar{m}^a m^c + (k^a l^b m^c - m^b k^a l^c) - k^a m^b l^c]\nabla_c F_{ab} \\
&= (k^a l^b m^c + \bar{m}^a m^b m^c - 2k^a m^b l^c)\nabla_c F_{ab} \\
&= 2G_3 \ ,
\end{aligned}$$

其中第三步是因为 $\nabla_{[c}F_{ab]}=0$ 导致 $m^{[a}m^b\bar{m}^{c]}\nabla_c F_{ab}=0$ 和 $l^{[a}m^b k^{c]}\nabla_c F_{ab}=0$. 与 l^b 缩并得

$$\begin{aligned}
-4\pi J_3 &= (m^a l^b \bar{m}^c + \bar{m}^a l^b m^c - l^a l^b k^c - k^a l^b l^c)\nabla_c F_{ab} \\
&= [m^a l^b \bar{m}^c + \bar{m}^a l^b m^c - (l^a l^b k^c + k^a l^b l^c)]\nabla_c F_{ab} \\
&= [-(\bar{m}^a m^b l^c + l^a \bar{m}^b m^c) + \bar{m}^a l^b m^c + l^a k^b l^c]\nabla_c F_{ab} \\
&= [-(\bar{m}^a m^b l^c - l^b \bar{m}^a m^c) + \bar{m}^a l^b m^c - l^b k^a l^c]\nabla_c F_{ab} \\
&= (-k^a l^b l^c - \bar{m}^a m^b l^c + 2\bar{m}^a l^b m^c)\nabla_c F_{ab} \\
&= 2G_4\ .
\end{aligned}$$

其中第三步是因为 $\nabla_{[c}F_{ab]}=0$ 导致 $m^{[a}l^b\bar{m}^{c]}\nabla_c F_{ab}=0$ 和 $l^{[a}l^b k^{c]}\nabla_c F_{ab}=0$.

结合前面的结果，我们最终推得有源麦氏方程的 NP 形式为

$$\begin{aligned}
D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi-2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 - 2\pi J_4\ , \\
D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho-2\varepsilon)\Phi_2 - 2\pi J_2\ , \\
\delta\Phi_1 - \Delta\Phi_0 &= (\mu-2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 - 2\pi J_1\ , \\
\delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau-2\beta)\Phi_2 - 2\pi J_3\ .
\end{aligned}$$

这四个方程是方程 (8-8-3a)–(8-8-3d) 在有源时的推广.

*11. 试证式 (8-8-7) 和 (8-8-10).

证 式 (8-8-7) 的证明. 电磁场的能动张量为式 (7-2-6)

$$T_{\mu\nu} = \frac{1}{4\pi}\Big(F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big)\ .$$

首先

$$\begin{aligned}
F_{\rho\sigma}F^{\rho\sigma} &= 2(F_{43}F^{43} + F_{42}F^{42} + F_{41}F^{41} + F_{32}F^{32} + F_{31}F^{31} + F_{21}F^{21}) \\
&= 2(F_{43}F_{34} - F_{42}F_{31} - F_{41}F_{32} - F_{32}F_{41} - F_{31}F_{42} + F_{21}F_{12}) \\
&= 2(-F_{43}^2 + F_{42}F_{13} + F_{41}F_{23} + F_{23}F_{41} + F_{13}F_{42} - F_{21}^2) \\
&= 2(-F_{43}^2 + 2F_{42}F_{13} + 2F_{41}F_{23} - F_{21}^2) \\
&= -2(F_{43}^2 + F_{21}^2) + 4(F_{41}F_{23} + F_{42}F_{13}) \\
&= -2(F_{43}^2 + F_{21}^2) + 4(F_{41}F_{23} + \bar{F}_{41}\bar{F}_{23})\ .
\end{aligned}$$

利用式 (8-8-1b), $\Phi_1 = \frac{1}{2}(F_{43}+F_{21})$, 有 $\bar{\Phi}_1 = \frac{1}{2}(F_{43}+F_{12}) = \frac{1}{2}(F_{43}-F_{21})$, 故

$$\Phi_1^2 + \bar{\Phi}_1^2 = \frac{1}{2}(F_{43}^2 + F_{21}^2)\ , \qquad \Phi_1^2 - \bar{\Phi}_1^2 = F_{43}F_{21}\ , \qquad \Phi_1\bar{\Phi}_1 = \frac{1}{4}(F_{43}^2 - F_{21}^2)\ .$$

于是结合式 (8-8-1a) 和 (8-8-1c) 得

$$F_{\rho\sigma}F^{\rho\sigma} = -4(\Phi_1^2 + \bar{\Phi}_1^2) + 4(\Phi_0\Phi_2 + \bar{\Phi}_0\bar{\Phi}_2)\ .$$

电磁场的能动张量在类光标架的分量为

$$
\begin{aligned}
T_{11} &= \frac{1}{4\pi}\Big(F_{1\sigma}F_1{}^{\sigma} - \frac{1}{4}g_{11}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_1{}^2 + F_{13}F_1{}^3 + F_{14}F_1{}^4 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_{11} - F_{13}F_{14} - F_{14}F_{13}\Big) \\
&= \frac{1}{2\pi}F_{41}F_{13} = \frac{1}{2\pi}\Phi_0\bar{\Phi}_2 \ , \\
T_{12} &= \frac{1}{4\pi}\Big(F_{1\sigma}F_2{}^{\sigma} - \frac{1}{4}g_{12}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_2{}^2 + F_{13}F_2{}^3 + F_{14}F_2{}^4 - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_{21} - F_{13}F_{24} - F_{14}F_{23} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(-F_{21}^2 + F_{13}F_{42} + F_{41}F_{23} - \frac{1}{4}[-2(F_{43}^2 + F_{21}^2) + 4(F_{41}F_{23} + \bar{F}_{41}\bar{F}_{23})]\Big) \\
&= \frac{1}{4\pi}\Big(-F_{21}^2 + \bar{F}_{41}\bar{F}_{23} + F_{41}F_{23} + \frac{1}{2}(F_{43}^2 + F_{21}^2) - (F_{41}F_{23} + \bar{F}_{41}\bar{F}_{23})\Big) \\
&= \frac{1}{8\pi}(F_{43}^2 - F_{21}^2) = \frac{1}{2\pi}\Phi_1\bar{\Phi}_1 \ , \\
T_{13} &= \frac{1}{4\pi}\Big(F_{1\sigma}F_3{}^{\sigma} - \frac{1}{4}g_{13}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_3{}^2 + F_{13}F_1{}^3 + F_{14}F_3{}^4 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_{31} - F_{13}F_{34} - F_{14}F_{33}\Big) \\
&= \frac{1}{4\pi}F_{13}(F_{21} + F_{43}) = \frac{1}{2\pi}\bar{\Phi}_2\Phi_1 \ , \\
T_{14} &= \frac{1}{4\pi}\Big(F_{1\sigma}F_4{}^{\sigma} - \frac{1}{4}g_{14}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_4{}^2 + F_{13}F_4{}^3 + F_{14}F_4{}^4 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{12}F_{41} - F_{13}F_{44} - F_{14}F_{43}\Big) \\
&= \frac{1}{4\pi}F_{41}(F_{43} - F_{21}) = \frac{1}{2\pi}\Phi_0\bar{\Phi}_1 \ , \\
T_{22} &= \frac{1}{4\pi}\Big(F_{2\sigma}F_2{}^{\sigma} - \frac{1}{4}g_{22}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{21}F_2{}^1 + F_{23}F_2{}^3 + F_{24}F_2{}^4 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{21}F_{22} - F_{23}F_{24} - F_{24}F_{23}\Big) \\
&= \frac{1}{2\pi}F_{23}F_{42} = \frac{1}{2\pi}\Phi_2\bar{\Phi}_0 \ , \\
T_{23} &= \frac{1}{4\pi}\Big(F_{2\sigma}F_3{}^{\sigma} - \frac{1}{4}g_{23}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{21}F_3{}^1 + F_{23}F_3{}^3 + F_{24}F_3{}^4 - 0\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\Big(F_{21}F_{32} - F_{23}F_{34} - F_{24}F_{33}\Big) \\
&= \frac{1}{4\pi}F_{23}(F_{43} - F_{21}) = \frac{1}{2\pi}\Phi_2\bar{\Phi}_1 \ , \\
T_{24} &= \frac{1}{4\pi}\Big(F_{2\sigma}F_4{}^{\sigma} - \frac{1}{4}g_{24}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{21}F_4{}^1 + F_{23}F_4{}^3 + F_{24}F_4{}^4 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{21}F_{42} - F_{23}F_{44} - F_{24}F_{43}\Big) \\
&= \frac{1}{4\pi}F_{42}(F_{43} + F_{21}) = \frac{1}{2\pi}\bar{\Phi}_0\Phi_1 \ , \\
T_{33} &= \frac{1}{4\pi}\Big(F_{3\sigma}F_3{}^{\sigma} - \frac{1}{4}g_{33}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{31}F_3{}^1 + F_{32}F_3{}^2 + F_{34}F_3{}^4 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{31}F_{32} + F_{32}F_{31} - F_{34}F_{33}\Big) \\
&= \frac{1}{2\pi}F_{23}F_{13} = \frac{1}{2\pi}\Phi_2\bar{\Phi}_2 \ , \\
T_{34} &= \frac{1}{4\pi}\Big(F_{3\sigma}F_4{}^{\sigma} - \frac{1}{4}g_{34}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{31}F_4{}^1 + F_{32}F_4{}^2 + F_{34}F_4{}^4 + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{31}F_{42} + F_{32}F_{41} - F_{34}F_{43} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(- F_{13}F_{42} - F_{23}F_{41} + F_{43}^2 + \frac{1}{4}[-2(F_{43}^2 + F_{21}^2) + 4(F_{41}F_{23} + \bar{F}_{41}\bar{F}_{23})]\Big) \\
&= \frac{1}{4\pi}\Big(- \bar{F}_{23}\bar{F}_{41} - F_{23}F_{41} + F_{43}^2 - \frac{1}{2}(F_{43}^2 + F_{21}^2) + (F_{41}F_{23} + \bar{F}_{41}\bar{F}_{23})\Big) \\
&= \frac{1}{8\pi}(F_{43}^2 - F_{21}^2) = \frac{1}{2\pi}\Phi_1\bar{\Phi}_1 \ , \\
T_{44} &= \frac{1}{4\pi}\Big(F_{4\sigma}F_4{}^{\sigma} - \frac{1}{4}g_{44}F_{\rho\sigma}F^{\rho\sigma}\Big) \\
&= \frac{1}{4\pi}\Big(F_{41}F_4{}^1 + F_{42}F_4{}^2 + F_{43}F_4{}^3 - 0\Big) \\
&= \frac{1}{4\pi}\Big(F_{41}F_{42} + F_{42}F_{41} - F_{43}F_{44}\Big) \\
&= \frac{1}{2\pi}F_{41}F_{42} = \frac{1}{2\pi}\Phi_0\bar{\Phi}_0 \ .
\end{aligned}
$$

此即 (8-8-7) 中诸式.

式 (8-8-10) 的证明. 由式 (8-4-7) 知

$$\Sigma_{ab}\Sigma^{ab} = 2(F_{ab}F^{ab} + iF_{ab}\,{}^*F^{ab}) \ .$$

上面我们已经证明了

$$F_{ab}F^{ab} = F_{\mu\nu}F^{\mu\nu} = -4(\Phi_1^2 + \bar{\Phi}_1^2) + 4(\Phi_0\Phi_2 + \bar{\Phi}_0\bar{\Phi}_2) \ .$$

下面我们求 $F_{ab}{}^*F^{ab}$. 根据对偶微分形式的定义

$$
\begin{aligned}
F_{ab}{}^*F^{ab} &= F_{ab}\frac{1}{2}F_{cd}\varepsilon^{cdab} = \frac{1}{2}F^{ab}F^{cd}\varepsilon_{cdab} = \frac{1}{2}F^{\mu\nu}F^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} \\
&= F^{12}F^{\rho\sigma}\varepsilon_{12\rho\sigma} + F^{13}F^{\rho\sigma}\varepsilon_{13\rho\sigma} + F^{14}F^{\rho\sigma}\varepsilon_{14\rho\sigma} \\
&\quad + F^{23}F^{\rho\sigma}\varepsilon_{23\rho\sigma} + F^{24}F^{\rho\sigma}\varepsilon_{24\rho\sigma} + F^{34}F^{\rho\sigma}\varepsilon_{34\rho\sigma} \\
&= 2F^{12}F^{34}\varepsilon_{1234} + 2F^{13}F^{24}\varepsilon_{1324} + 2F^{14}F^{23}\varepsilon_{1423} \\
&\quad + 2F^{23}F^{14}\varepsilon_{2314} + 2F^{24}F^{13}\varepsilon_{2413} + 2F^{34}F^{12}\varepsilon_{3412}) \\
&= \varepsilon_{1234}(2F^{12}F^{34} - 2F^{13}F^{24} + 2F^{14}F^{23} + 2F^{23}F^{14} - 2F^{24}F^{13} + 2F^{34}F^{12}) \\
&= \varepsilon_{1234}(4F^{12}F^{34} - 4F^{13}F^{24} + 4F^{14}F^{23}) \\
&= 4\varepsilon_{1234}(F_{21}F_{43} - F_{24}F_{13} + F_{23}F_{14}) \\
&= 4\varepsilon_{1234}(F_{43}F_{21} + F_{42}F_{13} - F_{41}F_{23}) \\
&= 4\varepsilon_{1234}(\Phi_1^2 - \bar{\Phi}_1^2 + \bar{\Phi}_0\bar{\Phi}_2 - \Phi_0\Phi_2)\ .
\end{aligned}
$$

现在计算类光标架中的体元 ε_{1234}, 应该转成正交归一标架中的相应量:

$$
\begin{aligned}
\varepsilon_{1234} &= \varepsilon_{abcd}(\varepsilon_1)^a(\varepsilon_2)^b(\varepsilon_3)^c(\varepsilon_4)^d \\
&\overset{(8\text{-}7\text{-}1)}{=} \varepsilon_{abcd}\frac{1}{4}[(e_1)^a - i(e_2)^a][(e_1)^b + i(e_2)^b][(e_0)^c - (e_3)^c][(e_0)^d + (e_3)^d] \\
&= \frac{1}{4}\varepsilon_{abcd}\Big\{i[(e_1)^a(e_2)^b - (e_2)^a(e_1)^b][(e_0)^c(e_3)^d - (e_3)^c(e_0)^d]\Big\} \\
&= i\varepsilon_{abcd}(e_1)^a(e_2)^b(e_0)^c(e_3)^d \\
&= i\varepsilon_{1203} = i\varepsilon_{0123} = i\ .
\end{aligned}
$$

因此

$$
F_{ab}{}^*F^{ab} = F_{\mu\nu}{}^*F^{\mu\nu} = 4i(\Phi_1^2 - \bar{\Phi}_1^2 + \bar{\Phi}_0\bar{\Phi}_2 - \Phi_0\Phi_2)\ .
$$

最后得

$$
\begin{aligned}
&\Sigma_{ab}\Sigma^{ab} = \Sigma_{\mu\nu}\Sigma^{\mu\nu} \\
&= 2(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}{}^*F^{\mu\nu}) \\
&= 2\Big[-4(\Phi_1^2 + \bar{\Phi}_1^2) + 4(\Phi_0\Phi_2 + \bar{\Phi}_0\bar{\Phi}_2) - 4(\Phi_1^2 - \bar{\Phi}_1^2 + \bar{\Phi}_0\bar{\Phi}_2 - \Phi_0\Phi_2)\Big] \\
&= 16(\Phi_0\Phi_2 - \Phi_1^2)\ .
\end{aligned}
$$

此即式 (8-8-10).

第 9 章 “施瓦西时空” 习题

~1. 考虑 Taub 的平面对称静态时空，其线元为式 (8-6-1’), 试借助 Killing 矢量场写出类时测地线 $\gamma(\tau)$ 的参数表达式 $t(\tau)$, $x(\tau)$, $y(\tau)$, $z(\tau)$ 所满足的解耦方程 (参考 §9.1).

解 Taub 平面对称静态时空的线元为

$$ds^2 = z^{-1/2}(-dt^2 + dz^2) + z(dx^2 + dy^2) ,$$

从线元式很容易看出度规分量为

$$g_{00} = -g_{33} = -z^{-1/2} , \qquad g_{11} = g_{22} = z ,$$

其相应的克氏符已在第 3 章习题 15 中求得:

$$\begin{aligned}
&\Gamma^0{}_{03} = \Gamma^0{}_{30} = \Gamma^3{}_{00} = \Gamma^3{}_{33} = -\frac{1}{4z} ,\\
&\Gamma^3{}_{11} = \Gamma^3{}_{22} = -\frac{z^{1/2}}{2} ,\\
&\Gamma^1{}_{13} = \Gamma^1{}_{31} = \Gamma^2{}_{23} = \Gamma^2{}_{23} = \frac{1}{2z} .
\end{aligned}$$

从度规只是 z 的函数知平面对称静态时空的独立 Killing 场有 4 个 — 1 个反映时间平移对称性的类时 Killing 场: $\xi_0^a = (\frac{\partial}{\partial t})^a$; 2 个反映空间平移对称性的类空 Killing 场 $\xi_1^a = (\frac{\partial}{\partial x})^a$ 和 $\xi_2^a = (\frac{\partial}{\partial y})^a$; 1 个反映空间转动对称性的类空 Killing 场 $\xi_3^a = -y(\frac{\partial}{\partial x})^a + x(\frac{\partial}{\partial y})^a$.

利用定理 4-3-3 定义测地线 $\gamma(\tau)$ 上的 3 个常量:

$$\begin{aligned}
E &= -g_{ab}\Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial \tau}\Big)^b = -g_{00}(dt)_b\Big(\frac{\partial}{\partial \tau}\Big)^b = -g_{00}\frac{dt}{d\tau} = z^{-1/2}\frac{dt}{d\tau} ,\\
P_x &= g_{ab}\Big(\frac{\partial}{\partial x}\Big)^a\Big(\frac{\partial}{\partial \tau}\Big)^b = g_{11}(dx)_b\Big(\frac{\partial}{\partial \tau}\Big)^b = g_{11}\frac{dx}{d\tau} = z\frac{dx}{d\tau} ,\\
P_y &= g_{ab}\Big(\frac{\partial}{\partial y}\Big)^a\Big(\frac{\partial}{\partial \tau}\Big)^b = g_{22}(dy)_b\Big(\frac{\partial}{\partial \tau}\Big)^b = g_{22}\frac{dx}{d\tau} = z\frac{dy}{d\tau} ,
\end{aligned}$$

另外根据测地线的类时性 ($\kappa = 1$) 或类光性 ($\kappa = 0$) 定义:

$$\begin{aligned}
\kappa &:= -g_{ab}U^aU^b = -g_{ab}\Big(\frac{\partial}{\partial \tau}\Big)^a\Big(\frac{\partial}{\partial \tau}\Big)^b\\
&= -g_{00}\Big(\frac{dt}{d\tau}\Big)^2 - g_{33}\Big(\frac{dz}{d\tau}\Big)^2 - g_{11}\Big(\frac{dx}{d\tau}\Big)^2 - g_{22}\Big(\frac{dy}{d\tau}\Big)^2\\
&= z^{-1/2}\Big(\frac{dt}{d\tau}\Big)^2 - z^{-1/2}\Big(\frac{dz}{d\tau}\Big)^2 - z\Big(\frac{dx}{d\tau}\Big)^2 - z\Big(\frac{dy}{d\tau}\Big)^2 .
\end{aligned}$$

以 3 个常量代入得:

$$\kappa = z^{1/2}E^2 - z^{-1/2}\Big(\frac{dz}{d\tau}\Big)^2 - z^{-1}P_x^2 - z^{-1}P_y^2 ,$$

即

$$\Big(\frac{dz}{d\tau}\Big)^2 = E^2 z - (P_x^2 + P_y^2)z^{-1/2} - \kappa z^{1/2} = E^2 z - P^2 z^{-1/2} - \kappa z^{1/2} ,$$

其中 $P^2 \equiv P_x^2 + P_y^2$. 先从

$$\frac{dz}{d\tau} = \pm\sqrt{E^2 z - P^2 z^{-1/2} - \kappa z^{1/2}} ,$$

解出 $z=z(\tau)$, 然后代入另外 3 个微分方程

$$\frac{dt}{d\tau}=Ez^{1/2}\ ,\qquad \frac{dx}{d\tau}=P_x z^{-1}\ ,\qquad \frac{dy}{d\tau}=P_y z^{-1}\ ,$$

即可求得测地线 $\gamma(\tau)$ 的参数表达式 $x^\mu=x^\mu(\tau)$.

下面我们验证以上 4 个方程的确与测地线方程组

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}=0\ ,\qquad \mu=0,\,1,\,2,\,3$$

一致：

$$\begin{aligned}
\mu=0: \qquad & \frac{d^2t}{d\tau^2}+2\Big(-\frac{1}{4z}\Big)\Big(\frac{dt}{d\tau}\Big)\Big(\frac{dz}{d\tau}\Big)=\frac{1}{2}Ez^{-1/2}\frac{dz}{d\tau}-\frac{1}{2z}\Big(\frac{dt}{d\tau}\Big)\Big(\frac{dz}{d\tau}\Big)\\
=&\ \frac{1}{2}Ez^{-1/2}\frac{dz}{d\tau}-\frac{1}{2z}(Ez^{1/2})\frac{dz}{d\tau}=0\ ,\\
\mu=1: \qquad & \frac{d^2x}{d\tau^2}+2\Big(\frac{1}{2z}\Big)\Big(\frac{dx}{d\tau}\Big)\Big(\frac{dz}{d\tau}\Big)=-P_xz^{-2}\frac{dz}{d\tau}+\frac{1}{z}\Big(\frac{dx}{d\tau}\Big)\Big(\frac{dz}{d\tau}\Big)\\
=&\ -P_xz^{-2}\frac{dz}{d\tau}+\frac{1}{z}(P_xz^{-1})\frac{dz}{d\tau}=0\ ,\\
\mu=2: \qquad & \frac{d^2y}{d\tau^2}+2\Big(\frac{1}{2z}\Big)\Big(\frac{dy}{d\tau}\Big)\Big(\frac{dz}{d\tau}\Big)=-P_yz^{-2}\frac{dz}{d\tau}+\frac{1}{z}\Big(\frac{dy}{d\tau}\Big)\Big(\frac{dz}{d\tau}\Big)\\
=&\ -P_yz^{-2}\frac{dz}{d\tau}+\frac{1}{z}(P_yz^{-1})\frac{dz}{d\tau}=0\ ,\\
\mu=3: \qquad & \frac{d^2z}{d\tau^2}+\Big(-\frac{1}{4z}\Big)\Big[\Big(\frac{dt}{d\tau}\Big)^2+\Big(\frac{dz}{d\tau}\Big)^2\Big]+\Big(-\frac{z^{1/2}}{2}\Big)\Big[\Big(\frac{dx}{d\tau}\Big)^2+\Big(\frac{dy}{d\tau}\Big)^2\Big]\\
=&\ \frac{d^2z}{d\tau^2}-\frac{1}{4z}(E^2z+E^2z-P^2z^{-1/2}-\kappa z^{1/2})-\frac{z^{1/2}}{2}(P_x^2z^{-2}+P_y^2z^{-2})\\
=&\ \frac{d^2z}{d\tau^2}-\frac{1}{2}E^2+\frac{1}{4}P^2z^{-3/2}+\frac{1}{4}\kappa z^{-1/2}-\frac{1}{2}P^2z^{-3/2}\\
=&\ \frac{d^2z}{d\tau^2}-\frac{1}{2}E^2-\frac{1}{4}P^2z^{-3/2}+\frac{1}{4}\kappa z^{-1/2}=0\ ,
\end{aligned}$$

最后一步是由于 $(\frac{dz}{d\tau})^2=E^2z-P^2z^{-1/2}-\kappa z^{1/2}$, 两边对 z 求导得

$$2\Big(\frac{dz}{d\tau}\Big)\frac{d^2z}{d\tau^2}=\Big[E^2+\frac{1}{2}P^2z^{-3/2}-\frac{1}{2}\kappa z^{-1/2}\Big]\Big(\frac{dz}{d\tau}\Big)\ ,$$

于是有

$$\frac{d^2z}{d\tau^2}=\frac{1}{2}E^2+\frac{1}{4}P^2z^{-3/2}-\frac{1}{4}\kappa z^{-1/2}\ .$$

可见前面的 4 个退耦方程与测地线方程的 4 个方程一致.

最后我们看 3 个沿测地线的常量 E 、 P_x 和 P_y 的物理意义：

$$\begin{aligned}
E\ &=\ z^{-1/2}\frac{dt}{d\tau}=z^{-1/2}\gamma=\frac{z^{-1/2}}{m}\gamma m=\frac{z^{-1/2}}{m}E_{\text{当}}\ ,\\
P_x\ &=\ z\frac{dx}{d\tau}=z\gamma\frac{dx}{dt}=z\gamma v^x_{\text{当}}=\frac{z}{m}\gamma m v^x_{\text{当}}=\frac{z}{m}p^x_{\text{当}}\ ,\\
P_y\ &=\ z\frac{dy}{d\tau}=z\gamma\frac{dy}{dt}=z\gamma v^y_{\text{当}}=\frac{z}{m}\gamma m v^y_{\text{当}}=\frac{z}{m}p^y_{\text{当}}\ ,
\end{aligned}$$

这里等式右边的 $E_{当}$、$p^x_{当}$ 和 $p^y_{当}$ 分别为当时当地测量的质点的能量和沿着 x 和 y 方向的动量，而等式左边为相应的总量.

2. 用牛顿引力论借图 9-8 直接推出式 (9-3-18).

解 由图 9-8 知，在空间体元 $dV = drdS$ 内的质量为 $\rho dV = \rho drdS$, 它受到的向内的引力大小为 $\frac{m(r)(\rho drdS)}{r^2}$, 其中 $m(r)$ 是半径 r 内的星体的质量，由式 (9-3-8) 给出．此外，因为存在压强梯度，该体元还受到向外的压力，大小为 $[p-(p+dp)]dS = -dpdS$, 两者平衡得关系式 $\frac{m(r)(\rho drdS)}{r^2} = -dpdS$, 于是即有方程 (9-3-18): $\frac{dp}{dr} = -\frac{\rho m(r)}{r^2}$.

~3. 试证 OV 流体静力学平衡方程 (9-3-17) 可改写为

$$\left[1-\frac{2m(r)}{r}\right]^{1/2}\frac{dp}{dr} = -(\rho+p)g\ , \tag{9-4-60}$$

其中 g 代表流体质点的 4 加速 $U^b\nabla_b U^a$ 的大小.

注 在牛顿近似下 $[1-2m(r)/r]^{1/2}\cong 1$, $p\cong 0$, 式 (9-4-60) 成为 $dp/dr\cong -\rho g$. 而 $g\cong m(r)/r^2$, 故得式 (9-3-18), 即 $dp/dr\cong -\rho m(r)/r^2$.

证 利用第 8 章习题 3 的结论流体质点的 4 加速 $A^a = U^b\nabla_b U^a = \nabla^a \ln\chi$, 其中 $\chi = (-\xi^a\xi_a)^{1/2} = (-g_{00})^{1/2} = [e^{2A(r)}]^{1/2} = e^{A(r)}$. 于是 4 加速的大小 (的平方)

$$\begin{aligned}
g^2 &= A^a A_a = g^{ab}(\nabla_a\ln\chi)(\nabla_b\ln\chi) = \chi^{-2}g^{ab}(\nabla_a\chi)(\nabla_b\chi)\\
&= e^{-2A(r)}\left[e^{A(r)}\frac{dA(r)}{dr}\right]^2 g^{ab}(dr)_a(dr)_b\\
&\overset{(9\text{-}3\text{-}11)}{=} \left[\frac{m(r)+4\pi pr^3}{r[r-2m(r)]}\right]^2 g^{11}\\
&= \left[\frac{m(r)+4\pi pr^3}{r[r-2m(r)]}\right]^2\left[1-\frac{2m(r)}{r}\right]\ .
\end{aligned}$$

在牛顿近似下， $g\cong\frac{m(r)}{r^2}$, 即重力加速度．式 (9-4-60) 可写为

$$\begin{aligned}
\frac{dp}{dr} &= -(\rho+p)\left[1-\frac{2m(r)}{r}\right]^{-1/2} g\\
&= -(\rho+p)\left[1-\frac{2m(r)}{r}\right]^{-1/2}\left[\frac{m(r)+4\pi pr^3}{r[r-2m(r)]}\right]\left[1-\frac{2m(r)}{r}\right]^{1/2}\\
&= -(\rho+p)\frac{m(r)+4\pi pr^3}{r[r-2m(r)]}\ .
\end{aligned}$$

此即 OV 流体静力学平衡方程 (9-3-17).

~4. 试证当 $R\gg M$ 时式 (9-3-26) 近似回到牛顿引力论的式 (9-3-23).

证 均匀密度星的施瓦西内解为式 (9-3-25):

$$p(r)=\rho\frac{(1-2M/R)^{1/2}-(1-2Mr^2/R^3)^{1/2}}{(1-2Mr^2/R^3)^{1/2}-3(1-2M/R)^{1/2}}\ ,$$

中心压强为式 (9-3-26):

$$p_0=\rho\frac{1-(1-2M/R)^{1/2}}{3(1-2M/R)^{1/2}-1}\ .$$

当 $R\gg M$ 时，它们分别回到牛顿引力论的式 (9-3-24) 和 (9-3-23):

$$\begin{aligned}
p(r) &\approx \rho\frac{(1-M/R)-(1-Mr^2/R^3)}{(1-Mr^2/R^3)-3(1-M/R)}\\
&\approx \rho\frac{Mr^2/R^3-M/R}{-2}=\rho\frac{M}{2R^3}(R^2-r^2)\\
&= \rho\frac{4\pi R^3\rho/3}{2R^3}(R^2-r^2)=\frac{2}{3}\pi\rho^2(R^2-r^2)\ ,\\
p_0 &\approx \rho\frac{1-(1-M/R)}{3(1-M/R)-1}\approx\rho\frac{M/R}{2}\\
&= \rho\frac{4\pi R^3/3\rho}{2R}=\frac{2}{3}\pi\rho^2R^2\ .
\end{aligned}$$

~5. 求闵氏时空中 Rindler 坐标 t, x 与洛伦兹坐标 T, X 的关系.

解 由关系式 (9-4-16) 、(9-4-11) 、(9-4-12) 、(9-4-6) 可得洛伦兹坐标与 Rindler 坐标的关系:

$$\begin{aligned}
T &= \frac{1}{2}(V+U)=\frac{1}{2}(e^v-e^{-u})=\frac{1}{2}(e^{\ln x+t}-e^{\ln x-t})=x\sinh t\ ,\\
X &= \frac{1}{2}(V-U)=\frac{1}{2}(e^v+e^{-u})=\frac{1}{2}(e^{\ln x+t}+e^{\ln x-t})=x\cosh t\ .
\end{aligned}$$

于是有

$$\begin{aligned}
dT &= x\cosh t\,dt+\sinh t\,dx\ ,\\
dX &= x\sinh t\,dt+\cosh t\,dx\ ,
\end{aligned}$$

线元为

$$\begin{aligned}
ds^2 &= -dT^2+dX^2=-(x\cosh t\,dt+\sinh t\,dx)^2+(x\sinh t\,dt+\cosh t\,dx)^2\\
&= -x^2dt^2+dx^2\ .
\end{aligned}$$

~6. Rindler 时空的类时 Killing 矢量场 $(\partial/\partial t)^a$ 是闵氏时空的哪个 Killing 矢量场？

解 由上题的结果 $T=x\sinh t$, $X=x\cosh t$ 知

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^a &= \left(\frac{\partial}{\partial T}\right)^a\frac{\partial T}{\partial t}+\left(\frac{\partial}{\partial X}\right)^a\frac{\partial X}{\partial t}\\ &= \left(\frac{\partial}{\partial T}\right)^a x\cosh t+\left(\frac{\partial}{\partial X}\right)^a x\sinh t\\ &= X\left(\frac{\partial}{\partial T}\right)^a+T\left(\frac{\partial}{\partial X}\right)^a,\end{aligned}$$

代表 2 维闵氏时空伪转动的 Killing 矢量场.

~7. 求施瓦西时空中静态观者的 4 加速的长度 $A\equiv(A^aA_a)^{1/2}$. 提示：可借用第 8 章习题 3 的结论，即 $A_a=\nabla_a\ln\chi$.

解 根据第 8 章习题 3 的结论，设 $\xi^a=(\frac{\partial}{\partial t})^a$ 为施瓦西时空的类时 Killing 矢量场，则静态观者的 4 加速为

$$\begin{aligned}A_a &= \nabla_a\ln\chi=\nabla_a\ln(-\xi^a\xi_a)^{1/2}=\nabla_a\ln(-g_{00})^{1/2}\\ &= \nabla_a\ln\left(1-\frac{2M}{r}\right)^{1/2}=\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}(dr)_a\,,\end{aligned}$$

故得

$$\begin{aligned}A &= (A^aA_a)^{1/2}=(g^{ab}A_aA_b)^{1/2}=(g^{11}A_1A_1)^{1/2}\\ &= \left\{\left(1-\frac{2M}{r}\right)\left[\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\right]^2\right\}^{1/2}\\ &= \left(1-\frac{2M}{r}\right)^{-1/2}\frac{M}{r^2}\,.\end{aligned}$$

在牛顿近似下 $A\cong\frac{M}{r^2}=g$, 即为重力加速度.

~8. 把图 9-13(a) 的 N_1 (或 N_2) 所代表的径向类光测地线简称为 N_1 (或 N_2), 试证：(1) 坐标 V (或 U) 是类光测地线 N_1 (或 N_2) 的仿射参数；(2) 坐标 r 是除 N_1 和 N_2 外的径向类光测地线的仿射参数.

证 设 $\eta(\lambda)$ 为任一径向类光测地线，其参数式为 $t=t(\lambda)$, $r=r(\lambda)$, $\theta=$ 常数，$\varphi=$ 常数. 而其切矢

$$\left(\frac{\partial}{\partial\lambda}\right)^a=\left(\frac{\partial}{\partial t}\right)^a\frac{dt(\lambda)}{d\lambda}+\left(\frac{\partial}{\partial r}\right)^a\frac{dr(\lambda)}{d\lambda}$$

满足

$$\begin{aligned}0 &= g_{ab}\left(\frac{\partial}{\partial\lambda}\right)^a\left(\frac{\partial}{\partial\lambda}\right)^b=g_{00}\left(\frac{dt(\lambda)}{d\lambda}\right)^2+g_{11}\left(\frac{dr(\lambda)}{d\lambda}\right)^2\\ &= -\left(1-\frac{2M}{r}\right)\left(\frac{dt(\lambda)}{d\lambda}\right)^2+\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr(\lambda)}{d\lambda}\right)^2,\end{aligned}$$

即

$$dt=\pm\left(1-\frac{2M}{r}\right)^{-1}dr=\pm dr_*\,.$$

于是径向类光测地线对应于 $v=t+r_*=0$ 或 $u=t-r_*=0$.

因为 $\xi^a=(\frac{\partial}{\partial t})^a$ 是 Killing 矢量场，根据定理 4-3-3 下面定义的 E 沿测地线 $\eta(\lambda)$ 为常数

$$E:=-g_{ab}\Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial\lambda}\Big)^b=-g_{00}(dt)_b\Big(\frac{\partial}{\partial\lambda}\Big)^b=-g_{00}\frac{dt}{d\lambda}=\Big(1-\frac{2M}{r}\Big)\frac{dt}{d\lambda}\,.$$

由以上关系 $(1-\frac{2M}{r})dt=\pm dr$, 得径向测地线上有关系 $d\lambda=\pm\frac{dr}{E}$, 即

$$\lambda=\pm\frac{r}{E}+c\,,\qquad c=\text{常数}\,.$$

因为 λ 是类光测地线的仿射参数，根据定理 3-3-3, r 也是这一测地线的仿射参数.

但是以上结果不适用于 N_1 或 N_2 所代表的类光测地线，因为在 N_1 或 N_2 上， $r=2M$ 而 $t=\pm\infty$, 故以上关系不成立. 但根据式 (9-4-26)–(9-4-28),

$$d\hat{s}^2=\frac{32M^3}{r}e^{-r/2M}(-dT^2+dX^2)=-\frac{32M^3}{r}e^{-r/2M}dV\,dU\,,$$

可知 N_1 由 $U=T-X=0$ 描述，它是 V 坐标线，切矢为 $(\frac{\partial}{\partial V})^a$, 所以它的仿射参数就是 V; 类似地， N_2 由 $V=T+X=0$ 描述，它是 U 坐标线，切矢为 $(\frac{\partial}{\partial U})^a$, 所以它的仿射参数就是 U. 它们都是类光测地线，满足 $g_{ab}(\frac{\partial}{\partial V})^a(\frac{\partial}{\partial V})^b=g_{ab}(\frac{\partial}{\partial U})^a(\frac{\partial}{\partial U})^b=0$.

下面我们讨论径向类时测地线 $\gamma(\tau)$, 其参数式为 $t=t(\tau)$, $r=r(\tau)$, $\theta=$ 常数， $\varphi=$ 常数. 由方程 (9-1-6) ($\kappa=1$, $L=0$) 知

$$-1=-\Big(1-\frac{2M}{r}\Big)^{-1}E^2+\Big(1-\frac{2M}{r}\Big)^{-1}\Big(\frac{dr}{d\tau}\Big)^2\,,$$

即有

$$\frac{dr}{d\tau}=\pm\sqrt{E^2-(1-2M/r)}\,,$$

积分后可得函数关系 $\tau=\tau(r)$. 因为 r 终止于 $r=0$, 故 τ 也终止于 $\tau(0)$, 故径向类时测地线 $\gamma(\tau)$ 也是不完备的. 从图 9-13a 来看 $\gamma(\tau)$ 必然与锯齿线有交.

~9. 引入与 Kruskal 坐标类似的坐标消除下列线元的坐标奇性 $r=R$:

$$ds^2=-(1-r^2/R^2)dt^2+(1-r^2/R^2)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\,,\quad R=\text{常数}\,.$$

解 仿照得到 Kruskal 坐标的过程，令

$$\begin{aligned}d\hat{s}^2&=-(1-r^2/R^2)dt^2+(1-r^2/R^2)^{-1}dr^2\\&=(1-r^2/R^2)[-dt^2+(1-r^2/R^2)^{-2}dr^2]\\&=(1-r^2/R^2)(-dt^2+dr_*^2)\,,\end{aligned}$$

其中

$$dr_* = (1-r^2/R^2)^{-1}dr = d[R\,\mathrm{arctanh}(r/R)] = d\left[\frac{R}{2}\ln\frac{1+r/R}{1-r/R}\right] ,$$

因此对 $0<r<R$, 可取

$$r_* := \frac{R}{2}\ln\frac{1+r/R}{1-r/R} .$$

再令

$$v := t + r_* , \quad u := t - r_* \quad \text{即} \;\; t = (v+u)/2 , \quad r_* = (v-u)/2 ,$$

则 v 和 u 的取值范围是

$$-\infty < v, u < \infty .$$

因 $-dt^2 + dr_*^2 = -dvdu$, 得

$$d\hat{s}^2 = -(1-r^2/R^2)dvdu .$$

令

$$V := e^{\beta v} , \qquad U := -e^{-\beta u} \quad (\beta\,\text{为待定常数}) ,$$

则 V 和 U 的取值范围是

$$0 < V < \infty , \qquad -\infty < U < 0 ,$$

且

$$dvdu = \beta^{-2}e^{\beta(u-v)}dVdU ,$$

故

$$\begin{aligned} d\hat{s}^2 &= -\beta^{-2}(1-r^2/R^2)e^{\beta(u-v)}dVdU \\ &= -\beta^{-2}(1-r^2/R^2)e^{-2\beta r_*}dVdU \\ &= -\beta^{-2}(1-r^2/R^2)e^{-\beta R\ln\frac{1+r/R}{1-r/R}}dVdU \\ &= -\beta^{-2}(1-r/R)(1+r/R)\left(\frac{1-r/R}{1+r/R}\right)^{\beta R}dVdU . \end{aligned}$$

为了消除上式在 $r=R$ 处的奇性，可选 $\beta R = -1$, 即

$$\beta = -1/R .$$

于是

$$d\hat{s}^2 = -R^2(1+r/R)^2dVdU = -(r+R)^2dVdU .$$

上式表明度规分量在 $r=R$ 处不再奇异，故可把 V, U 的取值范围延拓至 $V\le 0$ 和 $U \ge 0$ 的区域. 因为 $r=0$ 仍可能是 (后两个指标 θ 和 φ 的) 奇点，所以对 V 和 U 的取值的限制是必须满足 $r>0$ 的条件. 再令

$$T := \frac{1}{2}(V+U) , \qquad X := \frac{1}{2}(V-U) ,$$

并补上后两维，便得新坐标系 $\{T,\, X,\, \theta,\, \varphi\}$ 中的线元表达式为

$$ds^2 = (r+R)^2(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\ .$$

与 Kruskal 对施瓦西时空所做的延拓类似，现在也可分为 4 个区 (借用了 Kruskal 的标记).

A 区 ($X > |T|$, $0 < r < R$, $r_* = \frac{R}{2}\ln\frac{1+r/R}{1-r/R}$):

$$\begin{aligned}
V &= e^{-v/R} = e^{-(t+r_*)/R} = \left(\frac{1-r/R}{1+r/R}\right)^{1/2} e^{-t/R}\ ,\\
U &= -e^{u/R} = -e^{(t-r_*)/R} = -\left(\frac{1-r/R}{1+r/R}\right)^{1/2} e^{t/R}\ ,\\
T &= \frac{1}{2}(V+U) = -\left(\frac{1-r/R}{1+r/R}\right)^{1/2} \sinh(t/R)\ ,\\
X &= \frac{1}{2}(V-U) = \left(\frac{1-r/R}{1+r/R}\right)^{1/2} \cosh(t/R)\ ;
\end{aligned}$$

B 区 ($T > |X|$, $r > R$, $r_* = \frac{R}{2}\ln\frac{r/R+1}{r/R-1}$):

$$\begin{aligned}
V &= e^{-v/R} = e^{-(t+r_*)/R} = \left(\frac{r/R-1}{r/R+1}\right)^{1/2} e^{-t/R}\ ,\\
U &= e^{u/R} = e^{(t-r_*)/R} = \left(\frac{r/R-1}{r/R+1}\right)^{1/2} e^{t/R}\ ,\\
T &= \frac{1}{2}(V+U) = \left(\frac{r/R-1}{r/R+1}\right)^{1/2} \cosh(t/R)\ ,\\
X &= \frac{1}{2}(V-U) = -\left(\frac{r/R-1}{r/R+1}\right)^{1/2} \sinh(t/R)\ ;
\end{aligned}$$

W 区 ($T < -|X|$, $r > R$, $r_* = \frac{R}{2}\ln\frac{r/R+1}{r/R-1}$):

$$\begin{aligned}
V &= -e^{-v/R} = -e^{-(t+r_*)/R} = -\left(\frac{r/R-1}{r/R+1}\right)^{1/2} e^{-t/R}\ ,\\
U &= -e^{u/R} = -e^{(t-r_*)/R} = -\left(\frac{r/R-1}{r/R+1}\right)^{1/2} e^{t/R}\ ,\\
T &= \frac{1}{2}(V+U) = -\left(\frac{r/R-1}{r/R+1}\right)^{1/2} \cosh(t/R)\ ,\\
X &= \frac{1}{2}(V-U) = \left(\frac{r/R-1}{r/R+1}\right)^{1/2} \sinh(t/R)\ ;
\end{aligned}$$

A’ 区 ($X < -|T|$, $0 < r < R$, $r_* = \frac{R}{2}\ln\frac{1+r/R}{1-r/R}$):

$$\begin{aligned}
V &= -e^{-v/R} = -e^{-(t+r_*)/R} = -\left(\frac{1-r/R}{1+r/R}\right)^{1/2} e^{-t/R}\ ,\\
U &= e^{u/R} = e^{(t-r_*)/R} = \left(\frac{1-r/R}{1+r/R}\right)^{1/2} e^{t/R}\ ,
\end{aligned}$$

$$
\begin{aligned}
T &= \frac{1}{2}(V+U) = \left(\frac{1-r/R}{1+r/R}\right)^{1/2}\sinh(t/R)\ , \\
X &= \frac{1}{2}(V-U) = -\left(\frac{1-r/R}{1+r/R}\right)^{1/2}\cosh(t/R)\ .
\end{aligned}
$$

逆变换为

A, B, W, A' 区	$\dfrac{1-r/R}{1+r/R} = X^2 - T^2\ ,$
A, A' 区	$t/R = -\,\mathrm{arctanh}(T/X)\ ,$
B, W 区	$t/R = -\,\mathrm{arctanh}(X/T)\ .$

在任何区域都有

$$
(r+R)^2(-dT^2+dX^2) = -(1-r^2/R^2)dt^2 + (1-r^2/R^2)^{-1}dr^2\ ,
$$

其中 $r+R$ 可表为 $2R(X^2-T^2+1)^{-1}$. 要注意的是现在的 $r=0$ 对应的是 $X^2-T^2=1$, 位于 A 区和 A' 区，而不是 B 区和 W 区.

为了看出 $r=0$ 处是否存在奇性，可以将线元改写为

$$
ds^2 = (r+R)^2(-dT^2+dX^2) - dr^2 + [dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]\ ,
$$

其中方括号中为 3 维平直欧氏空间的线元，所以没有奇性. 而由 $\frac{1-r/R}{1+r/R} = X^2-T^2$ 得 $dr = R^{-1}(r+R)^2(TdT - XdX)$, 故

$$
dr^2 = R^{-2}(r+R)^4(T^2dT^2 + X^2dX^2 - 2TXdTdX)\ ,
$$

由此可以看出 $r=0$ 也只是坐标系 $\{T,\, X,\, \theta,\, \varphi\}$ 的坐标奇性.

r 在 R 和 0 处的坐标奇性也可通过计算曲率的标量多项式 (s.p.) 而获得有用的信息. 为此先求得 (借助 Mathematica) 与初始坐标相应的克氏符如下：

$$
\begin{aligned}
\Gamma^0{}_{01} = \Gamma^0{}_{10} &= -(1-r^2/R^2)^{-1}r/R^2\ , \\
\Gamma^1{}_{00} &= -(1-r^2/R^2)r/R^2\ , \\
\Gamma^1{}_{11} &= (1-r^2/R^2)^{-1}r/R^2\ , \\
\Gamma^1{}_{22} &= -(1-r^2/R^2)r\ , \\
\Gamma^1{}_{33} &= -(1-r^2/R^2)r\sin^2\theta\ , \\
\Gamma^2{}_{12} = \Gamma^2{}_{21} &= 1/r\ , \\
\Gamma^2{}_{33} &= -\sin\theta\cos\theta\ , \\
\Gamma^3{}_{13} = \Gamma^3{}_{31} &= 1/r\ , \\
\Gamma^3{}_{23} = \Gamma^3{}_{32} &= \cot\theta\ ,
\end{aligned}
$$

然后可得黎曼张量在初始坐标下的非零分量如下：

$$
\begin{aligned}
R_{0101} &= -1/R^2 \ , \\
R_{0202} &= (1-r^2/R^2)r^4/R^4 \ , \\
R_{0303} &= (1-r^2/R^2)(r^4/R^4)\sin^2\theta \ , \\
R_{1212} &= (1-r^2/R^2)^{-1}r^2/R^2 \ , \\
R_{1313} &= (1-r^2/R^2)^{-1}(r^2/R^2)\sin^2\theta \ , \\
R_{2323} &= (r^4/R^2)\sin^2\theta \ .
\end{aligned}
$$

于是可求得里奇张量的非零分量为：

$$
\begin{aligned}
R_{00} &= -3R^{-2}(1-r^2/R^2) \ , \\
R_{11} &= 3R^{-2}(1-r^2/R^2)^{-1} \ , \\
R_{22} &= 3R^{-2}r^2 \ , \\
R_{33} &= 3R^{-2}r^2\sin^2\theta \ .
\end{aligned}
$$

下面我们计算 s.p.:
①标量曲率 R:

$$
R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = 12R^{-2} \ .
$$

② $R_{ab}R^{ab}$

$$
R_{ab}R^{ab} = g^{\mu\mu'}g^{\nu\nu'}R_{\mu\nu}R_{\mu'\nu'} = 36R^{-4} \ .
$$

③ $R_{abcd}R^{abcd}$

$$
R_{abcd}R^{abcd} = g^{\mu\mu'}g^{\nu\nu'}g^{\sigma\sigma'}g^{\rho\rho'}R_{\mu\nu\sigma\rho}R_{\mu'\nu'\sigma'\rho'} = 24R^{-4} \ .
$$

至少这些 s.p. 都没有任何奇性.

10. 试证最大延拓施瓦西时空有 s.p. 曲率奇性．提示：利用式 (8-3-21).

证 利用施瓦西时空的黎曼张量表达式 (8-3-21), 注意黎曼张量的性质式 (3-4-6) 、 (3-4-9) 和 (3-4-10), 我们有 (也可借助 Mathematica 简单算得):

$$
\begin{aligned}
R_{abcd}R^{abcd} &= g^{\mu\mu'}g^{\nu\nu'}g^{\sigma\sigma'}g^{\rho\rho'}R_{\mu\nu\sigma\rho}R_{\mu'\nu'\sigma'\rho'} \\
&= 4[(g^{00}g^{11}R_{0101})^2 + (g^{00}g^{22}R_{0202})^2 + (g^{00}g^{33}R_{0303})^2 \\
&\quad +(g^{11}g^{22}R_{1212})^2 + (g^{11}g^{33}R_{1313})^2 + (g^{22}g^{33}R_{2323})^2] \\
&= 4\Big\{\Big[-\Big(1-\frac{2M}{r}\Big)^{-1}\Big(1-\frac{2M}{r}\Big)\Big(-\frac{2M}{r^3}\Big)\Big]^2 \\
&\quad +\Big[-\Big(1-\frac{2M}{r}\Big)^{-1}\Big(\frac{1}{r^2}\Big)\frac{M}{r}\Big(1-\frac{2M}{r}\Big)\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-\Big(1-\frac{2M}{r}\Big)^{-1}\Big(\frac{1}{r^2\sin^2\theta}\Big)\frac{M}{r}\Big(1-\frac{2M}{r}\Big)\sin^2\theta\Big]^2 \\
&+\Big[\Big(1-\frac{2M}{r}\Big)\Big(\frac{1}{r^2}\Big)\Big(-\frac{M}{r}\Big)\Big(1-\frac{2M}{r}\Big)^{-1}\Big]^2 \\
&+\Big[\Big(1-\frac{2M}{r}\Big)\Big(\frac{1}{r^2\sin^2\theta}\Big)\Big(-\frac{M}{r}\Big)\Big(1-\frac{2M}{r}\Big)^{-1}\sin^2\theta\Big]^2 \\
&+\Big[\Big(\frac{1}{r^2}\Big)\Big(\frac{1}{r^2\sin^2\theta}\Big)2Mr\sin^2\theta\Big]^2\Big\} \\
=\;& 4\Big\{\frac{4M^2}{r^6}+\frac{M^2}{r^6}+\frac{M^2}{r^6}+\frac{M^2}{r^6}+\frac{M^2}{r^6}+\frac{4M^2}{r^6}\Big\} \\
=\;& \frac{48M^2}{r^6}\,.
\end{aligned}
$$

故 $r\to 2M$ 有限而 $r\to 0$ 发散，可见 $r\to 0$ 有 s.p. 曲率奇性．注意施瓦西时空的里奇张量 R_{ab} 为零，因而标量曲率 R 和 $R_{ab}R^{ab}$ 都为零.

11. 试证图 9-13(a) 的 N_1 是类光超曲面．提示：只须证明其法矢 n^a 类光．请注意 N_1 的方程为 $U=0$, 其法余矢为 $n_a=\nabla_a U$.

证 超曲面 N_1 由方程 $U=0$ 决定，其法余矢为 $n_a=\nabla_a U=(dU)_a$. 而从线元式 (9-4-28) 和 (9-4-26) 知

$$
\begin{aligned}
ds^2 &= \frac{32M^3}{r}e^{-r/2M}(-dT^2+dX^2)+r^2(d\theta^2+\sin^2\theta\,d\varphi^2) \\
&= -\frac{32M^3}{r}e^{-r/2M}dV\,dU+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\,,
\end{aligned}
$$

于是有

$$
n_a n^a = g^{ab}n_a n_b = g^{ab}(dU)_a(dU)_b = g^{UU} = 0\,.
$$

因此 N_1 为类光超曲面.

12. 试由式 (9-4-50) 推出式 (9-4-51), 再推出式 (9-4-54).

解 由式 (9-4-50)

$$
d\hat{s}^2 = -(1-2M/r)dt^2+(1-2M/r)^{-1}dr^2\,,
$$

定义 $v:=t+r_*$ 并利用 $(1-2M/r)^{-1}dr=dr_*$, 有

$$
\begin{aligned}
d\hat{s}^2 &= (1-2M/r)[-dt^2+(1-2M/r)^{-2}dr^2] \\
&= (1-2M/r)[-dt^2+dr_*^2] \\
&= (1-2M/r)d(r_*+t)d(r_*-t) \\
&= (1-2M/r)dv\,d(2r_*-v) \\
&= 2(1-2M/r)dv\,dr_*-(1-2M/r)dv^2 \\
&= -(1-2M/r)dv^2+2dv\,dr\,,
\end{aligned}
$$

此即式 (9-4-51). 再定义 $\tilde{t} := v - r$,

$$\begin{aligned} dv^2 &= (d\tilde{t} + dr)^2 = d\tilde{t}^2 + dr^2 + 2d\tilde{t}dr\ , \\ dvdr &= (d\tilde{t} + dr)dr = d\tilde{t}dr + dr^2\ , \end{aligned}$$

于是

$$\begin{aligned} d\hat{s}^2 &= -(1-2M/r)(d\tilde{t}^2 + dr^2 + 2d\tilde{t}dr) + 2(d\tilde{t}dr + dr^2) \\ &= -(1-2M/r)d\tilde{t}^2 + (1+2M/r)dr^2 + (4M/r)d\tilde{t}dr\ , \end{aligned}$$

此即式 (9-4-54).

~13. 写出施瓦西度规在外向 Eddington 坐标系 $\{u,\, r,\, \theta,\, \varphi\}$ $(u \equiv t - r_*)$ 的线元表达式.

解 与上题类似,

$$\begin{aligned} d\hat{s}^2 &= (1-2M/r)[-dt^2 + (1-2M/r)^{-2}dr^2] \\ &= (1-2M/r)[-dt^2 + dr_*^2] \\ &= (1-2M/r)d(r_* + t)d(r_* - t) \\ &= (1-2M/r)d(2r_* + u)(-du) \\ &= -2(1-2M/r)dudr_* - (1-2M/r)du^2 \\ &= -(1-2M/r)du^2 - 2dudr\ , \end{aligned}$$

这就是施瓦西度规在外向 Eddington 坐标系 $\{u,\, r,\, \theta,\, \varphi\}$ 的线元表达式.

*14. 试证用 $(\partial/\partial V)^a$ 和 $(\partial/\partial U)^a$ 定义的 ξ^a [见式 (9-4-40)] 在 N_1 和 N_2 上是类光 Killing 矢量场.

证 由式 (9-4-26) 知 $g_{VU} = -\frac{16M^3}{r}e^{-r/2M}$, 故有

$$\begin{aligned} g_{ab}\xi^a\xi^b &= g_{ab}\frac{1}{(4M)^2}\Big[V\Big(\frac{\partial}{\partial V}\Big)^a - U\Big(\frac{\partial}{\partial U}\Big)^a\Big]\Big[V\Big(\frac{\partial}{\partial V}\Big)^b - U\Big(\frac{\partial}{\partial U}\Big)^b\Big] \\ &= -\frac{VU}{(4M)^2}\Big[g_{ab}\Big(\frac{\partial}{\partial V}\Big)^a\Big(\frac{\partial}{\partial U}\Big)^b + g_{ab}\Big(\frac{\partial}{\partial U}\Big)^a\Big(\frac{\partial}{\partial V}\Big)^b\Big] \\ &= -\frac{VU}{(4M)^2}2g_{VU} = VU\frac{2M}{r}e^{-r/2M}\ . \end{aligned}$$

在 N_1 上 $U=0$, 在 N_2 上 $V=0$, 所以在 N_1 和 N_2 上 ξ^a 是类光 Killing 矢量场.

*15. 把图 9-21 改画为图 9-23. 试通过计算图中的 $\Delta\tau'/\Delta\tau$ 给出式 (9-4-58) 的另一推导. 提示: (1) $U \equiv -e^{(r_*-t)/4M}$ 在每条外向类光测地线上为常数. 先后沿外部静态观者世界线和星面自由下落观者世界线求得同一 dU 的两个表

达式 (分别含 $d\tau'$ 和 $d\tau$), 在两式之间画等号便得式 (9-4-58). (2) 在写出用 $d\tau$ 表出 dU 的式子时要用到以能量 E 表达 $dt/d\tau$ 和 $dr/d\tau$ 的公式，这可借 §9.1 的手法求得.

解 首先注意到以下的讨论都限于径向运动，不涉及 θ 和 φ (即 $d\theta = d\varphi = 0$), 故只须考虑前两维. 如果 $G(\tau)$ 为星体外静态观者的世界线 (在 $T\sim X$ 图中表现为 A 区中的双曲线族), 它的 4 速为 $Z^a = (\frac{\partial}{\partial\tau})^a$. 因为它与类时 Killing 矢量场 $\xi^a = (\frac{\partial}{\partial t})^a$ 的积分曲线重合，故由 4 速归一性得

$$-1 = Z_a Z^a = g_{ab}\Big(\frac{\partial}{\partial\tau}\Big)^a\Big(\frac{\partial}{\partial\tau}\Big)^b = g_{00}(dt)_a(dt)_b\Big(\frac{\partial}{\partial\tau}\Big)^a\Big(\frac{\partial}{\partial\tau}\Big)^b = g_{00}\Big(\frac{dt}{d\tau}\Big)^2 ,$$

得

$$\frac{dt}{d\tau} = (-g_{00})^{-1/2} = \Big(1 - \frac{2M}{r}\Big)^{-1/2} \equiv \chi^{-1}(r) ,$$

此即关系 $\xi^a = \chi Z^a$. 如果 p 和 p' 是由类光测地线 (即等 U 线) 联系的两个星外静态观者世界线上的两点，那么就有关系

$$\frac{\omega'}{\omega} = \frac{\chi}{\chi'} \qquad \text{或} \qquad \frac{\lambda'}{\lambda} = \frac{\chi'}{\chi} ,$$

此即代表引力红移的式 (9-2-2). 令 $\Delta\tau$ 和 $\Delta\tau'$ 分别为由两根等 U 线在 $G(\tau)$ (p 点) 和 $G'(\tau')$ (p' 点) 截得的线长 (固有时线段，见图), 那么有关系

$$\frac{\Delta\tau'}{\Delta\tau} = \frac{\chi'}{\chi} = \frac{\chi(r(p'))}{\chi(r(p))} = \frac{[1 - 2M/r(p')]^{1/2}}{[1 - 2M/r(p)]^{1/2}} .$$

下面可把 p 看成星体表面静态观者 $G(\tilde\tau)$ 的点而把 p' 看成星外静态观者的点 (见图), 上式中的 $\Delta\tau$ 换成 $\Delta\tilde\tau$:

$$\frac{\Delta\tau'}{\Delta\tilde\tau} = \frac{\chi(r(p'))}{\chi(r(p))} = \frac{\chi'}{\chi} .$$

这其实就是式 (9-4-55).

对于无压强 (尘埃) 球对称恒星，星体表面每点的坍缩世界线为径向 (内向) 类时测地线 $\gamma(\tau)$, 其 4 速为 $(\frac{\partial}{\partial\tau})^a$. 由定理 4-3-3 知该测地线的能量为

$$\begin{aligned} E &= -g_{ab}\Big(\frac{\partial}{\partial t}\Big)^a\Big(\frac{\partial}{\partial\tau}\Big)^b = -g_{00}(dt)_b\Big(\frac{\partial}{\partial\tau}\Big)^b = -g_{00}\frac{dt}{d\tau} \\ &= \Big(1 - \frac{2M}{r}\Big)\frac{dt}{d\tau} = \chi^2(r)\frac{dt}{d\tau} , \end{aligned}$$

即有

$$\frac{dt}{d\tau} = \frac{E}{\chi^2(r)} ,$$

其中 $r=r(p)$, 为星体表面在某一刻的半径 [此即式 (9-1-4)]. 另外由 4 速的归一条件得

$$\begin{aligned} -1 &= g_{ab}\Big(\frac{\partial}{\partial\tau}\Big)^a\Big(\frac{\partial}{\partial\tau}\Big)^b = \Big[g_{00}(dt)_a(dt)_b + g_{11}(dr)_a(dr)_b\Big]\Big(\frac{\partial}{\partial\tau}\Big)^a\Big(\frac{\partial}{\partial\tau}\Big)^b \\ &= g_{00}\Big(\frac{dt}{d\tau}\Big)^2 + g_{11}\Big(\frac{dr}{d\tau}\Big)^2 = -\chi^2(\chi^{-2}E)^2 + \chi^{-2}\Big(\frac{dr}{d\tau}\Big)^2 , \end{aligned}$$

即

$$\Big(\frac{dr}{d\tau}\Big)^2 = E^2 - \chi^2(r) .$$

由于 $\gamma(\tau)$ 内向，故

$$\frac{dr}{d\tau} = -\sqrt{E^2-\chi^2} .$$

于是对于星面测地线有关系，

$$\frac{dr}{dt} = -\frac{\sqrt{E^2-\chi^2}}{\chi^{-2}E} = -\frac{\chi^2\sqrt{E^2-\chi^2}}{E} .$$

最后注意到 $U=-e^{(r_*-t)/4M}$, 知

$$\begin{aligned} dU &= U\frac{1}{4M}(dr_*-dt) = \frac{U}{4M}\Big[\Big(1-\frac{2M}{r}\Big)^{-1}dr - dt\Big] = \frac{U}{4M}[\chi^{-2}dr - dt] \\ &= \frac{U}{4M}\Big[\chi^{-2}\Big(\frac{dr}{dt}\Big)-1\Big]dt = \frac{U}{4M}\Big[-\frac{\sqrt{E^2-\chi^2}}{E}-1\Big]dt \\ &= -\frac{U}{4M}\frac{\sqrt{E^2-\chi^2}+E}{E}\,dt \\ &= -\frac{U}{4M}\frac{\sqrt{E^2-\chi^2(r(p))}+E}{E}\Big[\frac{E}{\chi^2(r(p))}d\tau\Big] \\ &= -\frac{U}{4M}\frac{\sqrt{E^2-\chi^2(r(p))}+E}{\chi^2(r(p))}\,d\tau . \end{aligned}$$

另一方面，对于星外静态观者的世界线 $G'(\tau')$, 由于 $r=$ 常数，故有

$$dU = U\frac{1}{4M}(-dt) = -\frac{U}{4M}[\chi^{-1}(r(p'))d\tau'] .$$

与前式相等得

$$\frac{d\tau'}{d\tau} = \frac{\chi(r(p'))(\sqrt{E^2-\chi^2(r(p))}+E)}{\chi^2(r(p))} = \frac{\chi'(\sqrt{E^2-\chi^2}+E)}{\chi^2} ,$$

此即式 (9-4-59), 其中 $\chi=[1-2M/r(p)]^{1/2}$, $\chi'=[1-2M/r(p')]^{1/2}$.

附. 对尘埃星估算星体表面自由坍缩观者从越过事件视界 $(r=2M)$ 后到跌入奇点 $(r=0)$ 的固有时流逝 (对 $M=3M_\odot$ 的黑洞).

解 由上题的求解过程知道，星体表面自由坍缩观者的世界线为径向测地线 $\gamma(\tau)$, 并且有关系

$$\frac{dr}{d\tau} = -\sqrt{E^2-\chi^2(r)} ,$$

其中 E 表征该测地线的能量，是个常数，而 $\chi(r)=(1-2M/r)^{1/2}$. 积分上式得

$$\begin{aligned}\Delta\tau &= -\int_{r=2M}^{0}\frac{dr}{\sqrt{E^2-1+2M/r}}\\ &= \frac{M}{(E^2-1)^{3/2}}\left\{2E(E^2-1)^{1/2}-\ln\left[2E(E^2-1)^{1/2}+2E^2-1\right]\right\}.\end{aligned}$$

因此如果知道能量值 E, 就可知固有时的流逝 $\Delta\tau$. 作为估算，可设 E 取允许的最小值 1 (对应落入事件视界时的坍缩速率最小). 于是有

$$\begin{aligned}\Delta\tau &= \lim_{E\to 1}\frac{M}{(E^2-1)^{3/2}}\left\{2E(E^2-1)^{1/2}-\ln\left[2E(E^2-1)^{1/2}+2E^2-1\right]\right\}\\ &= \frac{4M}{3}.\end{aligned}$$

如果取 $M=3M_\odot$, 则 $\Delta\tau=4M_\odot$. 恢复国际单位制，我们有

$$\Delta\tau=\frac{4GM_\odot}{c^2}\Big/c=\frac{4\times 6.67\times 10^{-11}\times 1.99\times 10^{30}}{(3\times 10^8)^3}=1.97\times 10^{-5}\,\mathrm{s}\,.$$

此即 §9.4.6 小节第一段末的结论！

第 10 章 “宇宙论” 习题

~1. 试验证度规 (10-1-12) 的曲率张量 ${}^{(3)}R_{abc}{}^{d}$ 满足 ${}^{(3)}R_{ab}{}^{cd}=2\bar{R}^{-2}\delta_a{}^{[c}\delta_b{}^{d]}$.

证 由 3 维球面线元

$$dl^2=\bar{R}^2[d\psi^2+\sin^2\psi(d\theta^2+\sin^2\theta\,d\varphi^2)]$$

知度规为

$$g_{11}\equiv g_{\psi\psi}=\bar{R}^2\,,\quad g_{22}\equiv g_{\theta\theta}=(\bar{R}\sin\psi)^2\,,\quad g_{33}\equiv g_{\varphi\varphi}=(\bar{R}\sin\psi\sin\theta)^2\,;$$

以及

$$g^{11}=\bar{R}^{-2}\,,\qquad g^{22}=(\bar{R}\sin\psi)^{-2}\,,\qquad g^{33}=(\bar{R}\sin\psi\sin\theta)^{-2}\,.$$

于是,

$$g_{22,1}=\bar{R}^2\sin 2\psi\,,\quad g_{33,1}=\bar{R}^2\sin 2\psi\sin^2\theta\,,\quad g_{33,2}=\bar{R}^2\sin^2\psi\sin 2\theta\,.$$

代入式 (3-4-19) 得非零克氏符 (Mathematica):

$$\begin{aligned}\Gamma^1{}_{22} &= -\sin\psi\cos\psi\,, & \Gamma^1{}_{33} &= -\sin\psi\cos\psi\sin^2\theta\,,\\ \Gamma^2{}_{12} &= \Gamma^2{}_{21}=\cot\psi\,, & \Gamma^2{}_{33} &= -\sin\theta\cos\theta\,,\\ \Gamma^3{}_{13} &= \Gamma^3{}_{31}=\cot\psi\,, & \Gamma^3{}_{23} &= \Gamma^3{}_{32}=\cot\theta\,.\end{aligned}$$

代入式 (3-4-20') 得非零黎曼张量 (Mathematica):

$$
\begin{aligned}
R_{121}{}^{2} &= 1\,,\\
R_{122}{}^{1} &= -\sin^2\psi\,,\\
R_{131}{}^{3} &= 1\,,\\
R_{133}{}^{1} &= -\sin^2\psi\sin^2\theta\,,\\
R_{211}{}^{2} &= -1\,,\\
R_{212}{}^{1} &= \sin^2\psi\,,\\
R_{232}{}^{3} &= \sin^2\psi\,,\\
R_{233}{}^{2} &= -\sin^2\psi\sin^2\theta\,,\\
R_{311}{}^{3} &= -1\,,\\
R_{313}{}^{1} &= \sin^2\psi\sin^2\theta\,,\\
R_{322}{}^{3} &= -\sin^2\psi\,,\\
R_{323}{}^{2} &= \sin^2\psi\sin^2\theta\,,
\end{aligned}
$$

其对称形式为:

$$
\begin{aligned}
R_{1212} &= -R_{1221} = -R_{2112} = R_{2121} = \bar{R}^2\sin^2\psi\,,\\
R_{1313} &= -R_{1331} = -R_{3113} = R_{3131} = \bar{R}^2\sin^2\psi\sin^2\theta\,,\\
R_{2323} &= -R_{2332} = -R_{3223} = R_{3232} = \bar{R}^2\sin^4\psi\sin^2\theta\,.
\end{aligned}
$$

于是有

$$
\begin{aligned}
R_{12}{}^{12} &= g^{11}R_{121}{}^{2} = \bar{R}^{-2} = \bar{R}^{-2}(\delta_1{}^1\delta_2{}^2 - \delta_1{}^2\delta_2{}^1)\,,\\
R_{12}{}^{21} &= g^{22}R_{122}{}^{1} = -\bar{R}^{-2} = \bar{R}^{-2}(\delta_1{}^2\delta_2{}^1 - \delta_1{}^1\delta_2{}^2)\,,\\
R_{13}{}^{13} &= g^{11}R_{131}{}^{3} = \bar{R}^{-2} = \bar{R}^{-2}(\delta_1{}^1\delta_3{}^3 - \delta_1{}^3\delta_3{}^1)\,,\\
R_{13}{}^{31} &= g^{33}R_{133}{}^{1} = -\bar{R}^{-2} = \bar{R}^{-2}(\delta_1{}^3\delta_3{}^1 - \delta_1{}^1\delta_3{}^3)\,,\\
R_{21}{}^{12} &= g^{11}R_{211}{}^{2} = -\bar{R}^{-2} = \bar{R}^{-2}(\delta_2{}^1\delta_1{}^2 - \delta_2{}^2\delta_1{}^1)\,,\\
R_{21}{}^{21} &= g^{22}R_{212}{}^{1} = \bar{R}^{-2} = \bar{R}^{-2}(\delta_2{}^2\delta_1{}^1 - \delta_2{}^1\delta_1{}^2)\,,\\
R_{23}{}^{23} &= g^{22}R_{232}{}^{3} = \bar{R}^{-2} = \bar{R}^{-2}(\delta_2{}^2\delta_3{}^3 - \delta_2{}^3\delta_3{}^2)\,,\\
R_{23}{}^{32} &= g^{33}R_{233}{}^{2} = -\bar{R}^{-2} = \bar{R}^{-2}(\delta_2{}^3\delta_3{}^2 - \delta_2{}^2\delta_3{}^3)\,,\\
R_{31}{}^{13} &= g^{11}R_{311}{}^{3} = -\bar{R}^{-2} = \bar{R}^{-2}(\delta_3{}^1\delta_1{}^3 - \delta_3{}^3\delta_1{}^1)\,,\\
R_{31}{}^{31} &= g^{33}R_{313}{}^{1} = \bar{R}^{-2} = \bar{R}^{-2}(\delta_3{}^3\delta_1{}^1 - \delta_3{}^1\delta_1{}^3)\,,\\
R_{32}{}^{23} &= g^{22}R_{322}{}^{3} = -\bar{R}^{-2} = \bar{R}^{-2}(\delta_3{}^2\delta_2{}^3 - \delta_3{}^3\delta_2{}^2)\,,\\
R_{32}{}^{32} &= g^{33}R_{323}{}^{2} = \bar{R}^{-2} = \bar{R}^{-2}(\delta_3{}^3\delta_2{}^2 - \delta_3{}^2\delta_2{}^3)\,.
\end{aligned}
$$

此即 $R_{ab}{}^{cd} = 2\bar{R}^{-2}\delta_a{}^{[c}\delta_b{}^{d]}$, 为常曲率 3 维空间, $K = \bar{R}^{-2} > 0$.

由 3 维双曲面线元

$$dl^2 = \bar{\xi}^2[d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2)]$$

知度规为

$$g_{11} \equiv g_{\psi\psi} = \bar{\xi}^2\ , \quad g_{22} \equiv g_{\theta\theta} = (\bar{\xi}\sinh\psi)^2\ , \quad g_{33} \equiv g_{\varphi\varphi} = (\bar{\xi}\sinh\psi\sin\theta)^2\ ;$$

以及

$$g^{11} = \bar{\xi}^{-2}\ , \qquad g^{22} = (\bar{\xi}\sinh\psi)^{-2}\ , \qquad g^{33} = (\bar{\xi}\sinh\psi\sin\theta)^{-2}\ .$$

于是，

$$g_{22,1} = \bar{\xi}^2\sinh 2\psi\ , \quad g_{33,1} = \bar{\xi}^2\sinh 2\psi\sin^2\theta\ , \quad g_{33,2} = \bar{\xi}^2\sinh^2\psi\sin 2\theta\ .$$

代入式 (3-4-19) 得非零克氏符 (Mathematica):

$$\begin{aligned}
\Gamma^1{}_{22} &= -\sinh\psi\cosh\psi\ , & \Gamma^1{}_{33} &= -\sinh\psi\cosh\psi\sin^2\theta\ ,\\
\Gamma^2{}_{12} &= \Gamma^2{}_{21} = \coth\psi\ , & \Gamma^2{}_{33} &= -\sin\theta\cos\theta\ ,\\
\Gamma^3{}_{13} &= \Gamma^3{}_{31} = \coth\psi\ , & \Gamma^3{}_{23} &= \Gamma^3{}_{32} = \cot\theta\ .
\end{aligned}$$

代入式 (3-4-20') 得非零黎曼张量 (Mathematica):

$$\begin{aligned}
R_{121}{}^2 &= -1\ ,\\
R_{122}{}^1 &= \sinh^2\psi\ ,\\
R_{131}{}^3 &= -1\ ,\\
R_{133}{}^1 &= \sinh^2\psi\sin^2\theta\ ,\\
R_{211}{}^2 &= 1\ ,\\
R_{212}{}^1 &= -\sinh^2\psi\ ,\\
R_{232}{}^3 &= -\sinh^2\psi\ ,\\
R_{233}{}^2 &= \sinh^2\psi\sin^2\theta\ ,\\
R_{311}{}^3 &= 1\ ,\\
R_{313}{}^1 &= -\sinh^2\psi\sin^2\theta\ ,\\
R_{322}{}^3 &= \sinh^2\psi\ ,\\
R_{323}{}^2 &= -\sinh^2\psi\sin^2\theta\ ,
\end{aligned}$$

其对称形式为：

$$\begin{aligned}
R_{1212} &= -R_{1221} = -R_{2112} = R_{2121} = -\bar{\xi}^2\sinh^2\psi\ ,\\
R_{1313} &= -R_{1331} = -R_{3113} = R_{3131} = -\bar{\xi}^2\sinh^2\psi\sin^2\theta\ ,\\
R_{2323} &= -R_{2332} = -R_{3223} = R_{3232} = -\bar{\xi}^2\sinh^4\psi\sin^2\theta\ .
\end{aligned}$$

于是有

$$
\begin{aligned}
R_{12}{}^{12} &= g^{11}R_{121}{}^{2} = -\bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_1{}^1\delta_2{}^2 - \delta_1{}^2\delta_2{}^1)\ , \\
R_{12}{}^{21} &= g^{22}R_{122}{}^{1} = \bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_1{}^2\delta_2{}^1 - \delta_1{}^1\delta_2{}^2)\ , \\
R_{13}{}^{13} &= g^{11}R_{131}{}^{3} = -\bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_1{}^1\delta_3{}^3 - \delta_1{}^3\delta_3{}^1)\ , \\
R_{13}{}^{31} &= g^{33}R_{133}{}^{1} = \bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_1{}^3\delta_3{}^1 - \delta_1{}^1\delta_3{}^3)\ , \\
R_{21}{}^{12} &= g^{11}R_{211}{}^{2} = \bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_2{}^1\delta_1{}^2 - \delta_2{}^2\delta_1{}^1)\ , \\
R_{21}{}^{21} &= g^{22}R_{212}{}^{1} = -\bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_2{}^2\delta_1{}^1 - \delta_2{}^1\delta_1{}^2)\ , \\
R_{23}{}^{23} &= g^{22}R_{232}{}^{3} = -\bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_2{}^2\delta_3{}^3 - \delta_2{}^3\delta_3{}^2)\ , \\
R_{23}{}^{32} &= g^{33}R_{233}{}^{2} = \bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_2{}^3\delta_3{}^2 - \delta_2{}^2\delta_3{}^3)\ , \\
R_{31}{}^{13} &= g^{11}R_{311}{}^{3} = \bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_3{}^1\delta_1{}^3 - \delta_3{}^3\delta_1{}^1)\ , \\
R_{31}{}^{31} &= g^{33}R_{313}{}^{1} = -\bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_3{}^3\delta_1{}^1 - \delta_3{}^1\delta_1{}^3)\ , \\
R_{32}{}^{23} &= g^{22}R_{322}{}^{3} = \bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_3{}^2\delta_2{}^3 - \delta_3{}^3\delta_2{}^2)\ , \\
R_{32}{}^{32} &= g^{33}R_{323}{}^{2} = -\bar{\xi}^{-2} = -\bar{\xi}^{-2}(\delta_3{}^3\delta_2{}^2 - \delta_3{}^2\delta_2{}^3)\ .
\end{aligned}
$$

此即 $R_{ab}{}^{cd} = -2\bar{\xi}^{-2}\delta_a{}^{[c}\delta_b{}^{d]}$, 也为常曲率 3 维空间， $K = -\bar{\xi}^{-2} < 0$.

2. 试证各向同性观者的世界线是测地线. 提示：利用式 (10-2-5) 后的克氏符表达式及式 (5-7-2) 几乎一望而知.

证 各向同性观者的世界线与时间坐标线重合, 其切矢 (4 速) 为 $Z^a = (\partial/\partial t)^a$. 根据 §3.3 定义 1, 对测地线应满足 $Z^b\nabla_b Z^a = 0$. 而根据式 (5-7-2):

$$\Big(\frac{\partial}{\partial x^\nu}\Big)^b \nabla_b \Big(\frac{\partial}{\partial x^\mu}\Big)^a = \Gamma^\sigma{}_{\mu\nu}\Big(\frac{\partial}{\partial x^\sigma}\Big)^a\ ,$$

我们有

$$Z^b\nabla_b Z^a = \Big(\frac{\partial}{\partial t}\Big)^b \nabla_b \Big(\frac{\partial}{\partial t}\Big)^a = \Gamma^\sigma{}_{00}\Big(\frac{\partial}{\partial x^\sigma}\Big)^a = 0\ ,$$

最后一步是因为由式 (10-2-5) 后的克氏符知 $\Gamma^\sigma{}_{00} = 0$. 因此各向同性观者的世界线是测地线.

3. 试用如下步骤导出宇宙学红移公式 (10-2-8):

(a) 证明沿任一类光测地线 $\eta(\beta)$ (β 为仿射参数) 有 $d\omega/d\beta = -K^aK^b\nabla_a Z_b$, 其中

$$K^a \equiv (\partial/\partial\beta)^a\ , \qquad Z^a \equiv (\partial/\partial t)^a\ , \qquad \omega \equiv -g_{ab}Z^aK^b\ .$$

(b) 证明 $\nabla_a Z_b = (\dot{a}/a)h_{ab}$, 其中 h_{ab} 是由 g_{ab} 在均匀面上的诱导度规， $\dot{a} \equiv da/dt$.

提示：先证明 $\nabla_a Z_b$ 是空间张量场，即 $Z^a\nabla_a Z_b = 0 = Z^b\nabla_a Z_b$, 再证明待证等式两边作用于 $(\partial/\partial x^i)^a(\partial/\partial x^j)^b$ $(i, j = 1, 2, 3)$ 得相同结果.

(c) 利用 (a) 、 (b) 的结果推出 $d\omega/\omega=-da/a$, 从而得式 (10-2-8).

解 (a) 因 ω 为标量函数,

$$\begin{aligned}\frac{d\omega}{d\beta} &= K^c\nabla_c\omega=-K^c\nabla_c(g_{ab}Z^aK^b)=-K^c\nabla_c(Z_bK^b)\\ &= -K^cK^b\nabla_cZ_b-K^cZ_b\nabla_cK^b=-K^aK^b\nabla_aZ_b\ ,\end{aligned}$$

最后一步利用了类光测地线的性质 $K^c\nabla_cK^b=0$.

(b) 利用上一题的结果知道各向同性观者的世界线为测地线，故它的切矢满足 $Z^a\nabla_aZ^c=0$ (见上题的证明). 以适配度规 g_{bc} 作用得 $g_{bc}Z^a\nabla_aZ^c=Z^a\nabla_aZ_b=0$. 另一方面，由 4 速 Z^a 的归一性 $Z_aZ^a=-1$ 自然有 $0=\nabla_a(Z_bZ^b)=Z^b\nabla_aZ_b+Z_b\nabla_aZ^b=2Z^b\nabla_aZ_b$ (用到了度规与微分算符的适配性). 既然 $Z^a\nabla_aZ_b=0=Z^b\nabla_aZ_b$, 可知张量 ∇_aZ_b 与 Z^a 正交，是空间张量，即它的分量只有空间指标.

利用关系式

$$\nabla_aZ_b=\partial_aZ_b-\Gamma^c{}_{ab}Z_c\ ,$$

故得

$$\begin{aligned}\Big(\frac{\partial}{\partial x^i}\Big)^a\Big(\frac{\partial}{\partial x^j}\Big)^b\nabla_aZ_b &= \Big(\frac{\partial}{\partial x^i}\Big)^a\Big(\frac{\partial}{\partial x^j}\Big)^b\partial_aZ_b-\Big(\frac{\partial}{\partial x^i}\Big)^a\Big(\frac{\partial}{\partial x^j}\Big)^b\Gamma^c{}_{ab}Z_c\\ &= \partial_iZ_j-\Gamma^\sigma{}_{ij}Z_\sigma\ .\end{aligned}$$

由于 $Z_\sigma=(\frac{\partial}{\partial x^\sigma})^aZ_a=(\frac{\partial}{\partial x^\sigma})^a[-(dt)_a]=-\delta^0{}_\sigma$, 故 $Z_j=0$, 得

$$\begin{aligned}\Big(\frac{\partial}{\partial x^i}\Big)^a\Big(\frac{\partial}{\partial x^j}\Big)^b\nabla_aZ_b &= \Gamma^\sigma{}_{ij}\delta^0{}_\sigma=\Gamma^0{}_{ij}\\ &= a\dot a(1-kr^2)^{-1}\delta^1{}_i\delta^1{}_j+a\dot ar^2\delta^2{}_i\delta^2{}_j+a\dot ar^2\sin^2\theta\delta^3{}_i\delta^3{}_j\\ &= a\dot a[a^{-2}g_{11}\delta^1{}_i\delta^1{}_j+a^{-2}g_{22}\delta^2{}_i\delta^2{}_j+a^{-2}g_{33}\delta^3{}_i\delta^3{}_j]\\ &= a^{-1}\dot ag_{ij}[\delta^1{}_i\delta^1{}_j+\delta^2{}_i\delta^2{}_j+\delta^3{}_i\delta^3{}_j]\\ &= a^{-1}\dot ah_{ij}[\delta^1{}_i\delta^1{}_j+\delta^2{}_i\delta^2{}_j+\delta^3{}_i\delta^3{}_j]\\ &= a^{-1}\dot ah_{ab}\Big(\frac{\partial}{\partial x^i}\Big)^a\Big(\frac{\partial}{\partial x^j}\Big)^b\ ,\end{aligned}$$

故有 $\nabla_aZ_b=a^{-1}\dot ah_{ab}$.

(c) 结合 (a) 和 (b) 的结果，我们有 (设 K^a 沿径向类光测地线)

$$\begin{aligned}\frac{d\omega}{d\beta} &= -K^aK^b\nabla_aZ_b=-K^aK^b(a^{-1}\dot ah_{ab})\\ &= -a^{-1}\dot ah_{ab}\Big[\Big(\frac{\partial}{\partial t}\Big)^a\frac{dt}{d\beta}+\Big(\frac{\partial}{\partial r}\Big)^a\frac{dr}{d\beta}\Big]\Big[\Big(\frac{\partial}{\partial t}\Big)^b\frac{dt}{d\beta}+\Big(\frac{\partial}{\partial r}\Big)^b\frac{dr}{d\beta}\Big]\\ &= -a^{-1}\dot a\Big[h_{00}\Big(\frac{dt}{d\beta}\Big)^2+h_{11}\Big(\frac{dr}{d\beta}\Big)^2\Big] \qquad (h_{00}=0)\\ &= -a^{-1}\dot ah_{11}\Big(\frac{dr}{d\beta}\Big)^2=-a^{-1}\dot ag_{11}\Big(\frac{dr}{d\beta}\Big)^2\end{aligned}$$

$$= \quad -\frac{a\dot{a}}{1-kr^2}\Big(\frac{dr}{d\beta}\Big)^2$$
$$\overset{(10\text{-}2\text{-}7)}{=} \quad -\frac{\dot{a}}{a}\Big(\frac{dt}{d\beta}\Big)^2 = -\frac{1}{a}\frac{da}{dt}\frac{dt}{d\beta}\,\omega = -\frac{\omega}{a}\frac{da}{d\beta}\ ,$$

即有 $d\omega/\omega = -da/a$, 从而得 $\omega = \omega_0 a^{-1}$.

4. 宇宙当今年龄是宇宙从 $a = 0$ 演化至 $a_0 \equiv a(t_0)$ 所需的时间．给定任一 a 值都可谈及宇宙的尺度因子演化至该值所需的时间，称为该 a 值相应的宇宙年龄，因此年龄 t 可看作 a 的函数.

(a) 从式 (10-2-29a)–(10-2-29c) 和 (10-2-25) 出发证明 $\Lambda = 0$ 的物质宇宙的年龄函数由以下三式给出：

对 $\Omega_0 = 1$,
$$t = \frac{2}{3}H_0^{-1}\Big(\frac{a}{a_0}\Big)^{3/2}\ ,$$

对 $\Omega_0 > 1$,
$$t = H_0^{-1}\left\{\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\cos^{-1}\left[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\right]-\frac{1}{\Omega_0-1}\left[\Omega_0\frac{a}{a_0}-(\Omega_0-1)\Big(\frac{a}{a_0}\Big)^2\right]^{1/2}\right\},$$

对 $\Omega_0 < 1$,
$$t = H_0^{-1}\left\{\frac{-\Omega_0}{2(1-\Omega_0)^{3/2}}\cosh^{-1}\left[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\right]+\frac{1}{1-\Omega_0}\left[\Omega_0\frac{a}{a_0}+(1-\Omega_0)\Big(\frac{a}{a_0}\Big)^2\right]^{1/2}\right\}.$$

(b) 由以上三式导出 $\Omega_0 = 1$, $\Omega_0 > 1$ 和 $\Omega_0 < 1$ 三种情况下当今宇宙年龄 t_0 的表达式.

解 (a) 引入哈勃参数 $H(t) = \dot{a}(t)/a(t)$ 和密度参数 $\Omega(t)$, 式 (10-3-8) 和 (10-3-12) 给出
$$H^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}\ , \qquad \Omega = \frac{8\pi\rho}{3H^2}\ .$$

由此可解出 ρ 和 a: 当 $k = 0$ 时， $\Omega = 1$,
$$\rho = \frac{3H^2}{8\pi}\ ;$$

当 $k = \pm 1$ 时， $\Omega \gtrless 1$,
$$\rho = \frac{3H^2\Omega}{8\pi}\ , \qquad a = \frac{1}{|\Omega-1|^{1/2}|H|}\ .$$

另外由式 (10-2-25) 知对物质 (尘埃) 宇宙 $8\pi\rho a^3/3 = A$ 为常数，于是对 $k = 0$ $(\Omega = 1)$:
$$A = \frac{8\pi}{3}\frac{3H^2}{8\pi}a^3 = H^2a^3 = H_0^2a_0^3\ ;$$

对 $k=\pm 1$ ($\Omega \gtrless 1$):

$$\begin{aligned} A &= \frac{8\pi}{3}\frac{3H^2\Omega}{8\pi}\frac{1}{|\Omega-1|^{3/2}|H|^3} = \frac{\Omega}{|\Omega-1|^{3/2}|H|} = \frac{\Omega_0}{|\Omega_0-1|^{3/2}|H_0|} \\ &= \frac{\Omega_0 a_0}{|\Omega_0-1|}\,, \end{aligned}$$

最后一步利用了 $a_0 = \frac{1}{|\Omega_0-1|^{1/2}|H_0|}$. 注意到当今是膨胀宇宙，故 $H_0>0$, $|H_0|=H_0$.

把以上关系代入物质宇宙的解，式 (10-2-29a)–(10-2-29c). 对 $k=0$ ($\Omega_0=1$), 式 (10-2-29b) 给出：

$$\begin{aligned} t &= \Big(\frac{4}{9A}\Big)^{1/2} a^{3/2} = \frac{2}{3A^{1/2}}a^{3/2} = \frac{2}{3(H_0^2 a_0^3)^{1/2}}a^{3/2} \\ &= \frac{2}{3}H_0^{-1}\Big(\frac{a}{a_0}\Big)^{3/2}\,. \end{aligned}$$

对 $k=+1$ ($\Omega_0>1$), 式 (10-2-29a) 第一式给出：

$$\begin{aligned} \hat t &= \arccos\Big(1-\frac{2a}{A}\Big) = \arccos\Big[1-\frac{2a}{\Omega_0 a_0/(\Omega_0-1)}\Big] \\ &= \arccos\Big[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\Big]\,, \end{aligned}$$

有

$$\begin{aligned} \sin\hat t &= (1-\cos^2\hat t)^{1/2} = \Big\{1-\Big[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\Big]^2\Big\}^{1/2} \\ &= \Big\{4(1-\Omega_0^{-1})\frac{a}{a_0} - 4(1-\Omega_0^{-1})^2\Big(\frac{a}{a_0}\Big)^2\Big\}^{1/2} \\ &= \frac{2(\Omega_0-1)^{1/2}}{\Omega_0}\Big[\Omega_0\frac{a}{a_0}-(\Omega_0-1)\Big(\frac{a}{a_0}\Big)^2\Big]^{1/2}\,, \end{aligned}$$

代入第二式得

$$\begin{aligned} t &= \frac{\Omega_0}{2(\Omega_0-1)^{3/2}H_0}\Big\{\arccos\Big[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\Big] \\ &\qquad -\frac{2(\Omega_0-1)^{1/2}}{\Omega_0}\Big[\Omega_0\frac{a}{a_0}-(\Omega_0-1)\Big(\frac{a}{a_0}\Big)^2\Big]^{1/2}\Big\} \\ &= H_0^{-1}\Big\{\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\arccos\Big[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\Big] \\ &\qquad -\frac{1}{\Omega_0-1}\Big[\Omega_0\frac{a}{a_0}-(\Omega_0-1)\Big(\frac{a}{a_0}\Big)^2\Big]^{1/2}\Big\}\,. \end{aligned}$$

对 $k=-1$ ($\Omega_0<1$), 式 (10-2-29c) 第一式给出：

$$\begin{aligned} \hat t &= \operatorname{arccosh}\Big(1+\frac{2a}{A}\Big) = \operatorname{arccosh}\Big[1+\frac{2a}{\Omega_0 a_0/(1-\Omega_0)}\Big] \\ &= \operatorname{arccosh}\Big[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\Big]\,, \end{aligned}$$

有

$$\begin{aligned}\sinh\hat{t} &= (\cosh^2\hat{t}-1)^{1/2} = \left\{\left[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\right]^2-1\right\}^{1/2}\\ &= \left\{-4(1-\Omega_0^{-1})\frac{a}{a_0}+4(1-\Omega_0^{-1})^2\left(\frac{a}{a_0}\right)^2\right\}^{1/2}\\ &= \frac{2(1-\Omega_0)^{1/2}}{\Omega_0}\left[\Omega_0\frac{a}{a_0}+(1-\Omega_0)\left(\frac{a}{a_0}\right)^2\right]^{1/2},\end{aligned}$$

代入第二式得

$$\begin{aligned}t &= \frac{\Omega_0}{2(1-\Omega_0)^{3/2}H_0}\left\{-\operatorname{arccosh}\left[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\right]\right.\\ &\qquad\left.+\frac{2(1-\Omega_0)^{1/2}}{\Omega_0}\left[\Omega_0\frac{a}{a_0}+(1-\Omega_0)\left(\frac{a}{a_0}\right)^2\right]^{1/2}\right\}\\ &= H_0^{-1}\left\{-\frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\operatorname{arccosh}\left[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\right]\right.\\ &\qquad\left.+\frac{1}{1-\Omega_0}\left[\Omega_0\frac{a}{a_0}+(1-\Omega_0)\left(\frac{a}{a_0}\right)^2\right]^{1/2}\right\}\\ &= H_0^{-1}\left\{\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\operatorname{arccosh}\left[1-2(1-\Omega_0^{-1})\frac{a}{a_0}\right]\right.\\ &\qquad\left.-\frac{1}{\Omega_0-1}\left[\Omega_0\frac{a}{a_0}-(\Omega_0-1)\left(\frac{a}{a_0}\right)^2\right]^{1/2}\right\},\end{aligned}$$

可见只须把前一种情形的 arccos 换成 arccosh, 其他不变.

(b) 对于当今宇宙年龄，取上式中 $a=a_0$, 故得：

若 $\Omega_0=1$,

$$t_0=\frac{2}{3}H_0^{-1};$$

若 $\Omega_0>1$,

$$\begin{aligned}t_0 &= H_0^{-1}\left\{\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\arccos\left[1-2(1-\Omega_0^{-1})\right]-\frac{1}{\Omega_0-1}\left[\Omega_0-(\Omega_0-1)\right]^{1/2}\right\}\\ &= H_0^{-1}\left[\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\arccos(2\Omega_0^{-1}-1)-\frac{1}{\Omega_0-1}\right];\end{aligned}$$

若 $\Omega_0<1$,

$$\begin{aligned}t_0 &= H_0^{-1}\left\{-\frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\operatorname{arccosh}\left[1-2(1-\Omega_0^{-1})\right]+\frac{1}{1-\Omega_0}\left[\Omega_0+(1-\Omega_0)\right]^{1/2}\right\}\\ &= H_0^{-1}\left[-\frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\operatorname{arccosh}(2\Omega_0^{-1}-1)+\frac{1}{1-\Omega_0}\right].\end{aligned}$$

可以证明：对于 $\Omega_0\gtrless 1$ 情形，如果取极限 $\Omega_0\to 1$, 它们都回到 $t_0=\frac{2}{3}H_0^{-1}$. 因此如果把 t_0 表示成

$$t_0=H_0^{-1}f(\Omega_0),$$

其中

$$f(x)=\begin{cases}\frac{x}{2(x-1)^{3/2}}\arccos(2x^{-1}-1)-\frac{1}{x-1}\,, & x<1\,,\\ \frac{2}{3}\,, & x=1\,,\\ \frac{x}{2(x-1)^{3/2}}\operatorname{arccosh}(2x^{-1}-1)-\frac{1}{x-1}\,, & x>1\,.\end{cases}$$

$f(x)$ 为单调递减函数， $f(0)=1$ 而 $f(2)=\frac{\pi}{2}-1=0.5708$.

~5. 试证含 Λ 项的爱因斯坦方程即使无物质场 ($T_{ab}=0$) 也不允许平直度规解.
提示：从含 Λ 项的爱因斯坦方程出发求得 R 与 T 的关系，以此消去方程中的 R, 便发现 $T_{ab}=0$ 时 R_{ab} 不能为零.

证 含 Λ 项的爱因斯坦方程为

$$G_{ab}+\Lambda g_{ab}=R_{ab}-\frac{1}{2}Rg_{ab}+\Lambda g_{ab}=8\pi T_{ab}\,.$$

两边作用 (求迹) g^{ab} 得：

$$R-\frac{1}{2}R\delta^a{}_a+\Lambda\delta^a{}_a=8\pi T^a{}_a=8\pi T\,,$$

其中 $T\equiv T^a{}_a$ 为物质场能动张量的迹. 因 4 维时空的 $\delta^a{}_a=4$, 故有

$$4\Lambda-R=8\pi T\,,$$

即 $R=4\Lambda-8\pi T$. 代回爱因斯坦方程得：

$$R_{ab}-\frac{1}{2}(4\Lambda-8\pi T)g_{ab}+\Lambda g_{ab}=R_{ab}-\Lambda g_{ab}+4\pi Tg_{ab}=8\pi T_{ab}\,,$$

于是知里奇张量满足

$$R_{ab}=\Lambda g_{ab}+8\pi T_{ab}-4\pi Tg_{ab}\,.$$

可见即使无物质场 ($T_{ab}=0$, $T=0$), $R_{ab}=\Lambda g_{ab}\neq 0$, 时空也非平直.

6. 试证 $k=-1$ 和 $k=+1$ 的 RW 度规也是 (局部) 共形平直的.
提示：用式 (10-4-2) 定义 $\hat{t}$, 把式 (10-1-23a) 和 (10-1-23c) 的线元改用坐标 $\hat{t}$, ψ, θ, φ 表出，再分别对 $k=-1$ 和 $k=+1$ 的情况做如下坐标变换 $(\hat{t},\psi)\mapsto(\tilde{t},\tilde{r})$:

对 $k=-1$, 令

$$\tilde{t}=e^{\hat{t}}\cosh\psi\,,\qquad \tilde{r}=e^{\hat{t}}\sinh\psi\,,$$

对 $k=+1$, 令

$$\tilde{t}=\tan\frac{1}{2}(\hat{t}+\psi)+\tan\frac{1}{2}(\hat{t}-\psi)\,,\qquad \tilde{r}=\tan\frac{1}{2}(\hat{t}+\psi)-\tan\frac{1}{2}(\hat{t}-\psi)\,,$$

则线元分别取如下的明显共形平直形式：

对 $k=-1$,

$$ds^2 = a^2(t(\hat{t}))e^{-2\hat{t}}[-d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]\ ,$$

对 $k=+1$,

$$ds^2 = \frac{a^2(t(\hat{t}))}{4}(\cos\hat{t} + \cos\psi)^2[-d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]\ .$$

证 引入新坐标

$$\hat{t}(t) \equiv \int_0^t dt'/a(t')\ ,$$

有 $d\hat{t} = dt/a(t)$ 或 $a(\hat{t})d\hat{t} = dt$, 故 $a^2(\hat{t})d\hat{t}^2 = dt^2$. 于是线元 (10-1-23a) 和 (10-1-23c) 分别为

$$\begin{aligned} ds^2 &= a^2(\hat{t})[-d\hat{t}^2 + d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2)]\ , \qquad [\text{对}\, k=+1\,]\ , \\ ds^2 &= a^2(\hat{t})[-d\hat{t}^2 + d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2)]\ , \qquad [\text{对}\, k=-1\,]\ . \end{aligned}$$

对 $k=+1$, 令

$$\tilde{t} = \tan\frac{1}{2}(\hat{t}+\psi) + \tan\frac{1}{2}(\hat{t}-\psi)\ , \qquad \tilde{r} = \tan\frac{1}{2}(\hat{t}+\psi) - \tan\frac{1}{2}(\hat{t}-\psi)\ ,$$

其逆变换为

$$\hat{t} = \arctan\frac{1}{2}(\tilde{t}+\tilde{r}) + \arctan\frac{1}{2}(\tilde{t}-\tilde{r})\ , \qquad \psi = \arctan\frac{1}{2}(\tilde{t}+\tilde{r}) - \arctan\frac{1}{2}(\tilde{t}-\tilde{r})\ .$$

于是

$$\begin{aligned} d\hat{t} &= \frac{(d\tilde{t}+d\tilde{r})/2}{1+[\frac{1}{2}(\tilde{t}+\tilde{r})]^2} + \frac{(d\tilde{t}-d\tilde{r})/2}{1+[\frac{1}{2}(\tilde{t}-\tilde{r})]^2} \\ &= \frac{(d\tilde{t}+d\tilde{r})/2}{1+\tan^2\frac{1}{2}(\hat{t}+\psi)} + \frac{(d\tilde{t}-d\tilde{r})/2}{1+\tan^2\frac{1}{2}(\hat{t}-\psi)} \\ &= \frac{\cos^2\frac{1}{2}(\hat{t}+\psi)}{2}(d\tilde{t}+d\tilde{r}) + \frac{\cos^2\frac{1}{2}(\hat{t}-\psi)}{2}(d\tilde{t}-d\tilde{r})\ , \\ d\psi &= \frac{(d\tilde{t}+d\tilde{r})/2}{1+[\frac{1}{2}(\tilde{t}+\tilde{r})]^2} - \frac{(d\tilde{t}-d\tilde{r})/2}{1+[\frac{1}{2}(\tilde{t}-\tilde{r})]^2} \\ &= \frac{(d\tilde{t}+d\tilde{r})/2}{1+\tan^2\frac{1}{2}(\hat{t}+\psi)} - \frac{(d\tilde{t}-d\tilde{r})/2}{1+\tan^2\frac{1}{2}(\hat{t}-\psi)} \\ &= \frac{\cos^2\frac{1}{2}(\hat{t}+\psi)}{2}(d\tilde{t}+d\tilde{r}) - \frac{\cos^2\frac{1}{2}(\hat{t}-\psi)}{2}(d\tilde{t}-d\tilde{r})\ , \end{aligned}$$

得

$$\begin{aligned} -d\hat{t}^2 + d\psi^2 &= -(d\hat{t}+d\psi)(d\hat{t}-d\psi) \\ &= -\cos^2[(\hat{t}+\psi)/2](d\tilde{t}+d\tilde{r})\cos^2[(\hat{t}-\psi)/2](d\tilde{t}-d\tilde{r}) \\ &= \{\cos[(\hat{t}+\psi)/2]\cos[(\hat{t}-\psi)/2]\}^2(-d\tilde{t}^2+d\tilde{r}^2) \\ &= \frac{1}{4}(\cos\hat{t}+\cos\psi)^2(-d\tilde{t}^2+d\tilde{r}^2)\ . \end{aligned}$$

另一方面，因 $\sin^2\psi = \frac{\tan^2\psi}{1+\tan^2\psi}$, 而

$$\begin{aligned}\tan\psi &= \tan\left[\arctan\frac{1}{2}(\tilde{t}+\tilde{r}) - \arctan\frac{1}{2}(\tilde{t}-\tilde{r})\right] \\ &= \frac{\frac{1}{2}(\tilde{t}+\tilde{r}) - \frac{1}{2}(\tilde{t}-\tilde{r})}{1+\frac{1}{4}(\tilde{t}+\tilde{r})(\tilde{t}-\tilde{r})} = \frac{\tilde{r}}{1+\frac{1}{4}(\tilde{t}+\tilde{r})(\tilde{t}-\tilde{r})}\ ,\end{aligned}$$

得

$$\sin^2\psi = \frac{\tilde{r}^2}{\tilde{r}^2 + [1+\frac{1}{4}(\tilde{t}+\tilde{r})(\tilde{t}-\tilde{r})]^2}\ ,$$

其中分母

$$\begin{aligned}&\tilde{r}^2 + \left[1+\frac{1}{4}(\tilde{t}+\tilde{r})(\tilde{t}-\tilde{r})\right]^2 \\ =&\ \left[\tan\frac{1}{2}(\hat{t}+\psi) - \tan\frac{1}{2}(\hat{t}-\psi)\right]^2 + \left[1+\tan\frac{1}{2}(\hat{t}+\psi)\tan\frac{1}{2}(\hat{t}-\psi)\right]^2 \\ =&\ \frac{4}{(\cos\hat{t}+\cos\psi)^2}\ .\end{aligned}$$

因此有

$$\sin^2\psi = \frac{1}{4}(\cos\hat{t}+\cos\psi)^2\,\tilde{r}^2\ .$$

将以上结果代回 $k=+1$ 的线元表达式：

$$\begin{aligned}ds^2 &= a^2\left[\frac{1}{4}(\cos\hat{t}+\cos\psi)^2(-d\tilde{t}^2+d\tilde{r}^2) + \frac{1}{4}(\cos\hat{t}+\cos\psi)^2\,\tilde{r}^2(d\theta^2+\sin^2\theta\,d\varphi^2)\right] \\ &= \frac{a^2}{4}(\cos\hat{t}+\cos\psi)^2[-d\tilde{t}^2+d\tilde{r}^2+\tilde{r}^2(d\theta^2+\sin^2\theta\,d\varphi^2)]\ .\end{aligned}$$

对 $k=-1$, 令

$$\tilde{t} = e^{\hat{t}}\cosh\psi\ , \qquad \tilde{r} = e^{\hat{t}}\sinh\psi\ ,$$

其逆变换为

$$\hat{t} = \frac{1}{2}\ln(\tilde{t}^2-\tilde{r}^2)\ , \qquad \psi = \operatorname{arctanh}(\tilde{r}/\tilde{t})\ .$$

于是

$$\begin{aligned}d\hat{t} &= \frac{\tilde{t}d\tilde{t}-\tilde{r}d\tilde{r}}{\tilde{t}^2-\tilde{r}^2} = e^{-2\hat{t}}(\tilde{t}d\tilde{t}-\tilde{r}d\tilde{r})\ , \\ d\psi &= \frac{1}{1-(\tilde{r}/\tilde{t})^2}\frac{\tilde{t}d\tilde{r}-\tilde{r}d\tilde{t}}{\tilde{t}^2} = \frac{\tilde{t}d\tilde{r}-\tilde{r}d\tilde{t}}{\tilde{t}^2-\tilde{r}^2} = e^{-2\hat{t}}(\tilde{t}d\tilde{r}-\tilde{r}d\tilde{t})\ ,\end{aligned}$$

得

$$\begin{aligned}-d\hat{t}^2+d\psi^2 &= -e^{-4\hat{t}}(\tilde{t}d\tilde{t}-\tilde{r}d\tilde{r})^2 + e^{-4\hat{t}}(\tilde{t}d\tilde{r}-\tilde{r}d\tilde{t})^2 \\ &= e^{-4\hat{t}}(\tilde{t}^2-\tilde{r}^2)(-d\tilde{t}^2+d\tilde{r}^2) \\ &= e^{-2\hat{t}}(-d\tilde{t}^2+d\tilde{r}^2)\end{aligned}$$

另一方面，因 $\sinh^2\psi = \frac{\tanh^2\psi}{1-\tanh^2\psi}$, 而 $\tanh\psi = \tilde{r}/\tilde{t}$, 得

$$\sinh^2\psi = \frac{\tilde{r}^2}{\tilde{t}^2 - \tilde{r}^2} = e^{-2\hat{t}}\,\tilde{r}^2\ .$$

将以上结果代回 $k=-1$ 的线元表达式：

$$\begin{aligned} ds^2 &= a^2[e^{-2\hat{t}}(-d\tilde{t}^2 + d\tilde{r}^2) + e^{-2\hat{t}}\,\tilde{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] \\ &= a^2 e^{-2\hat{t}}[-d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)]\ . \end{aligned}$$

综上结论，对 $k=+1$ 和 $k=-1$, RW 度规也是 (局部) 共形平直的，共形联系的正定函数分别为 $\frac{a^2}{4}(\cos\hat{t} + \cos\psi)^2$ 和 $a^2 e^{-2\hat{t}}$.

~7. 设 p 为各向同性观者 G 世界线上的一点，试证 G 在 t_p 时刻的视界距离满足式 (10-4-5). 提示：利用式 (10-1-28) 和 (10-2-7).

证 等时面上两点的距离由式 (10-4-5) 给出：

$$D_{AB}(t) = a(t)\int_{r_A}^{r_B} \frac{dr}{\sqrt{1-kr^2}}\ .$$

取 A 为各向同性观者 G 世界线上的一点 p, B 为视界边界，径向坐标为 r_B, 则

$$D_H(t_p) = a(t_p)\int_0^{r_B} \frac{dr}{\sqrt{1-kr^2}}\ .$$

另一方面，任一径向类光测地线的参数式 $\{t(\beta), r(\beta)\}$ 满足方程 (10-2-7):

$$\left(\frac{dt}{d\beta}\right)^2 = \frac{a^2}{1-kr^2}\left(\frac{dr}{d\beta}\right)^2,$$

即对内向类光测地线有关系：

$$\frac{dr}{dt} = -\frac{\sqrt{1-kr^2}}{a(t)}\ .$$

由视界距离的定义 (参见图 10-16) 知类光测地线从 $\{0, r_B\}$ 传到 $\{t_p, 0\}$, 因此，

$$\begin{aligned} D_H(t_p) &= a(t_p)\int_{t_p}^{0} \frac{1}{\sqrt{1-kr^2}}\left[-\frac{\sqrt{1-kr^2}}{a(t)}\right]dt \\ &= a(t_p)\int_0^{t_p} \frac{dt}{a(t)}\ . \end{aligned}$$

此即式 (10-4-5).

*~8. (a) 设 $\eta(\beta)$ 是径向 $(d\theta/d\beta = d\varphi/d\beta = 0)$ 类光测地线， $p_1 = (t_1, \psi_1, \theta, \varphi)$ 和 $p_2 = (t_2, \psi_2, \theta, \varphi)$ 是 η 上任意两点，试证对 $k = 1,\ 0,\ -1$ 三种情况都有

$$\psi_2 - \psi_1 = \int_{t_1}^{t_2} dt/a(t)\ .$$

(b) 对 $k = 1$ 的宇宙，从大爆炸奇点发出的任一径向光线在膨胀着的 3 球面上沿大圆弧前进. 试证： (b1) 对物质宇宙, 该光线在 3 球面膨胀至最大时刚走完半个大圆，在 3 球面又缩为一点 (大挤压) 时刚走完一个大圆. 因此，在球面膨胀至最大时任一各向同性观者只要向各个方向看去，总能看到任一各向同性粒子发来的光，表明他的粒子视界从膨胀至最大时开始消失 [参见 Wald (1984) P.106]. (b2) 对辐射宇宙，该光线在 3 球面又缩为一点 (大挤压) 时刚刚走完半个大圆. 因此，任一各向同性观者的任一时刻都存在粒子视界.

证 (a) 考虑到类光测地线满足的条件式 (10-2-7), 它的外向形式为：

$$\frac{dr}{dt} = \frac{\sqrt{1 - kr^2}}{a(t)}\ .$$

再利用关系式 (10-1-24), 对 $k = +1$,

$$\psi_2 - \psi_1 = \arcsin r_2 - \arcsin r_1 = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - r^2}} = \int_{t_1}^{t_2} \frac{dt}{a(t)}\ ;$$

对 $k = 0$,

$$\psi_2 - \psi_1 = r_2 - r_1 = \int_{r_1}^{r_2} dr = \int_{t_1}^{t_2} \frac{dt}{a(t)}\ ;$$

对 $k = -1$,

$$\psi_2 - \psi_1 = \operatorname{arcsinh} r_2 - \operatorname{arcsinh} r_1 = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 + r^2}} = \int_{t_1}^{t_2} \frac{dt}{a(t)}\ .$$

因此，综合有

$$\psi_2 - \psi_1 = \int_{t_1}^{t_2} \frac{dt}{a(t)}\ .$$

(b1) 对 $k = 1$ 的物质宇宙，尺度因子的解为式 (10-2-29a):

$$a = A(1 - \cos\hat{t})/2\ , \qquad t = A(\hat{t} - \sin\hat{t})/2\ .$$

由此可知当 3 球面膨胀至最大时 $\hat{t} = \pi$, $t = A\pi/2$. 从坐标 $(0, 0, \theta, \varphi)$ 出发的光子这时走到 $(A\pi/2, \psi, \theta, \varphi)$, 其中 ψ 坐标根据 (a) 的结果为

$$\psi = \int_0^{A\pi/2} \frac{dt}{a(t)} = \int_0^{\pi} d\hat{t} = \pi\ ,$$

正好为 3 球面的半个大圆弧. 其后当 3 球面收缩 (大挤压) 到一点时 $\hat{t}=2\pi$, $t=A\pi$, 这时光子的 ψ 坐标从 π 缩为零, 到大挤压的一点时它的坐标为 $(A\pi,\,0,\,\theta,\,\varphi)$, 刚好走完 3 球面的一个大圆弧.

另外从 3 球面的体积公式 (10-1-29) 可以知道, 当 $\psi<\pi$ 时, 与之对应的 3 球面的体积占全体积的

$$\frac{\int_0^{\psi}\sin^2\psi' d\psi'}{\int_0^{\pi}\sin^2\psi' d\psi'}=\frac{1}{\pi}(\psi-\sin\psi\cos\psi)\ .$$

于是不难想象在宇宙膨胀至最大之前, 即当 $\hat{t}<\pi$ $(t<A\pi/2)$ 时, 由 $\psi=\hat{t}$ 知, 粒子视界占全空间的体积随时间的变化关系为

$$\eta(t)=\frac{1}{\pi}(\hat{t}-\sin\hat{t}\cos\hat{t})\ ,\qquad t=A(\hat{t}-\sin\hat{t})/2\ .$$

它从 0 开始增长到 1, 当宇宙开始收缩, 粒子视界仍是全空间. 而且容易算得, 当 t (即 $\hat{t}$) 很小时,

$$\begin{aligned}\hat{t}&=\left(\frac{12}{A}\right)^{1/3}t^{1/3}+\frac{1}{5A}t+O(t^{5/2})\ ,\\ \eta(t)&=\frac{8}{A\pi}t-\frac{31104^{1/3}}{5A^{5/3}\pi}t^{5/3}+O(t^{7/3})\ ,\end{aligned}$$

粒子视界初始随时间线性增长. 当 t 接近 $A\pi/2$ (即 $\hat{t}$ 接近 π) 时, 令 $\bar{\hat{t}}\equiv\pi-\hat{t}$, $\bar{t}\equiv\frac{A\pi}{2}-t$, 则以上的关系变为

$$\eta(\bar{t})=1-\frac{1}{\pi}(\bar{\hat{t}}-\sin\bar{\hat{t}}\cos\bar{\hat{t}})\ ,\qquad \bar{t}=A(\bar{\hat{t}}+\sin\bar{\hat{t}})/2\ .$$

容易算得

$$\begin{aligned}\bar{\hat{t}}&=\frac{1}{A}\bar{t}+\frac{1}{12A^3}\bar{t}^3+O(\bar{t}^5)\ ,\\ \eta(\bar{t})&=1-\frac{2}{3A^3\pi}\bar{t}^3-\frac{1}{30A^5\pi}\bar{t}^5+O(\bar{t}^7)\ .\end{aligned}$$

因此当宇宙膨胀至最大前, $\eta(t)$ 以以下方式趋于饱和:

$$\eta(t)=1-\frac{2}{3A^3\pi}\Big(\frac{A\pi}{2}-t\Big)^3-\frac{1}{30A^5\pi}\Big(\frac{A\pi}{2}-t\Big)^5+\cdots\ .$$

(b2) 对 $k=1$ 的辐射宇宙, 尺度因子的解为式 (10-2-24a):

$$a=\sqrt{2Bt-t^2}\ .$$

由此可知当 3 球面膨胀至最大时 $t=B$. 从坐标 $(0,\,0,\,\theta,\,\varphi)$ 出发的光子这时走到 $(B,\,\psi,\,\theta,\,\varphi)$, 其中 ψ 坐标根据 (a) 的结果为

$$\psi=\int_0^B\frac{dt}{a(t)}=\int_0^B\frac{dt}{\sqrt{2Bt-t^2}}=\frac{\pi}{2}\ ,$$

正好为 3 球面的 1/4 个大圆弧. 其后当 3 球面收缩 (大挤压) 到一点时 $t=2B$, 这时光子的 ψ 坐标从 $\pi/2$ 增至 π, 到大挤压的一点时它的坐标为 $(2B,\,\pi,\,\theta,\,\varphi)$, 刚好走完 3 球面的半个大圆弧.

类似地也可求出辐射宇宙从大爆炸到大挤压的整个历史中粒子视界占整个空间的比例函数 $\eta(t)$ 的演化. 因

$$\psi(t)=\int_0^t\frac{dt'}{\sqrt{2Bt'-t'^2}}=2\arctan\sqrt{\frac{t}{2B-t}}\ ,$$

故得

$$\begin{aligned}\eta(t)&=\frac{1}{\pi}(\psi-\sin\psi\cos\psi)\\&=\frac{1}{\pi}\left[2\arctan\sqrt{\frac{t}{2B-t}}-\frac{(B-t)\sqrt{2Bt-t^2}}{B^2}\right]\ .\end{aligned}$$

它在大爆炸 $(t=0)$ 、最大宇宙 $(t=B)$ 和大挤压 $(t=2B)$ 附近的行为分别为:

$$\begin{aligned}\eta(t)&=\frac{4\sqrt{2}}{3B^{3/2}\pi}t^{3/2}-\frac{\sqrt{2}}{5B^{5/2}\pi}t^{5/2}+O(t^{7/2})\ ,\\\eta(t)&=\frac{1}{2}+\frac{2}{B\pi}(t-B)-\frac{1}{3B^3\pi}(t-B)^3+O((t-B)^5)\ ,\\\eta(t)&=1-\frac{4\sqrt{2}}{3B^{3/2}\pi}(2B-t)^{3/2}-\frac{\sqrt{2}}{5B^{5/2}\pi}(2B-t)^{5/2}+O((2B-t)^{7/2})\ .\end{aligned}$$

显然大爆炸和大挤压两点是对称的, 演化的对称中心就是最大宇宙, 因为成立关系

$$\eta(t)+\eta(2B-t)=1\ .$$

{**(Dis)claimer:** **Since I thank no one for helping me in solving these problems, all errors are definitely my own. — 639**}