POLITECNICO
MILANO 1863

Policy Gradient Algorithms for the Asset Allocation Problem

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Animal Spirits

The Sound of Silence

The Computerization of Finance

Plan

1. Basics of Reinforcement Learning
2. Policy Gradient Algorithms
3. Asset Allocation with Transaction Costs
4. Conclusions

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The Reinforcement Learning Framework

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Environment

$\mathcal{P}, \mathcal{R}$

The Reinforcement Learning Framework

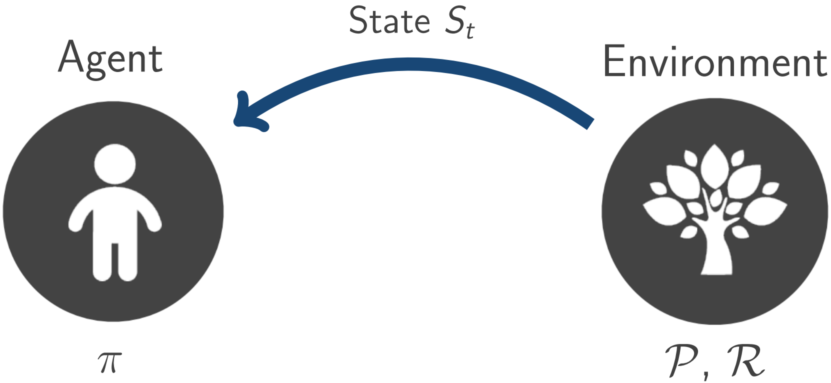
Agent

π

Environment

$\mathcal{P}, \mathcal{R}$

The Reinforcement Learning Framework



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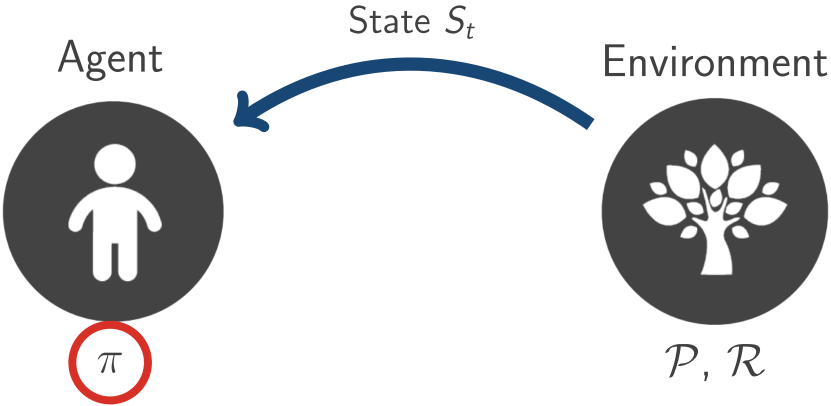

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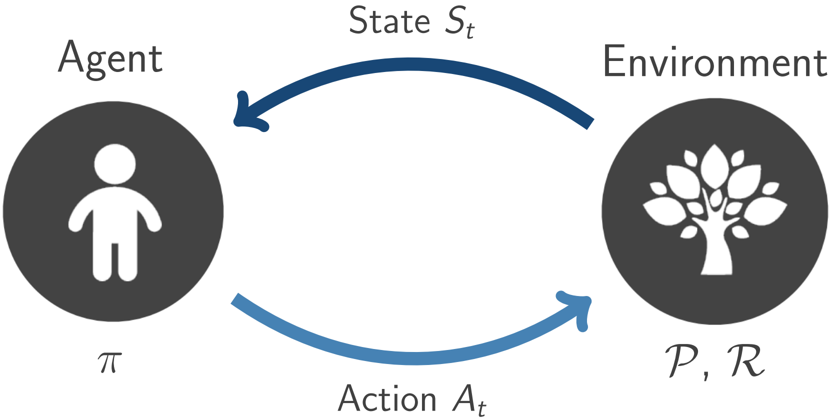

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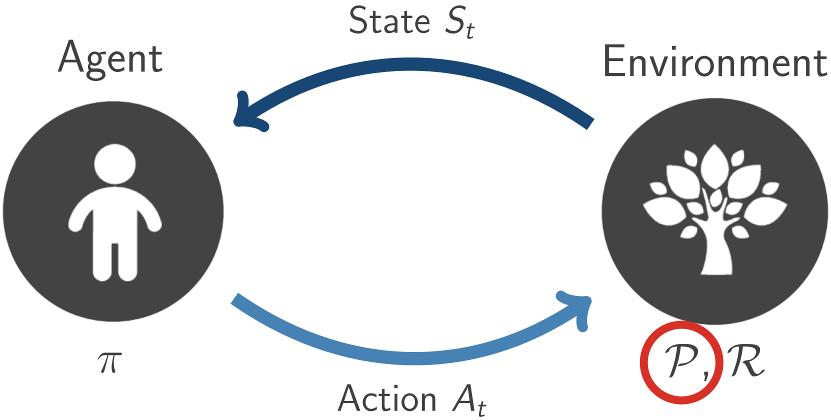

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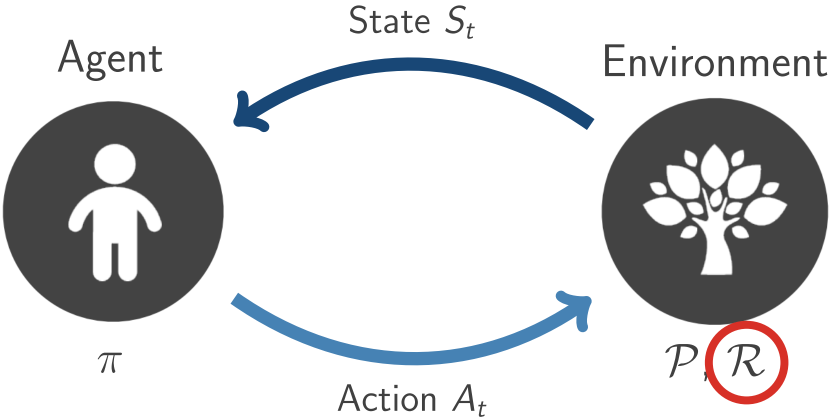

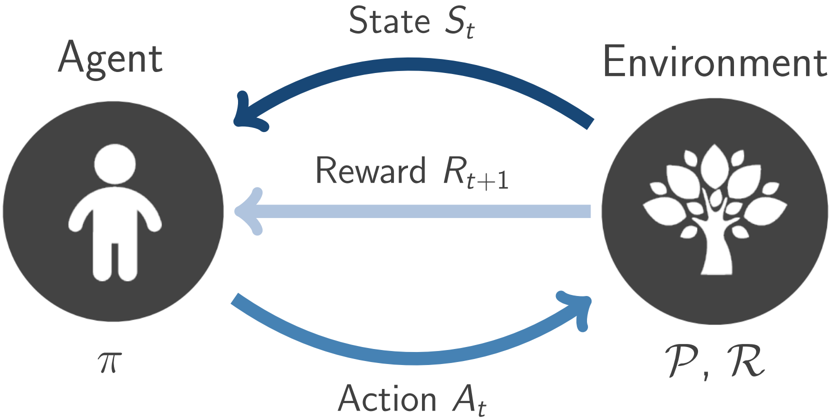
Agent
Environment
State S_t
Reward R_{t+1}
Action A_t
π
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Mathematical Formulation

State-value function

$$V_\pi(s) = \mathbb{E}_\pi \left[\sum_{t=0}^{\infty} \gamma^t R_{t+1} \middle| S_0 = s \right]$$

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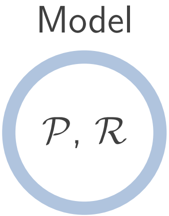
Optimal policy

$$\pi_* \text{ s.t. } V_{\pi_*}(s) = V_*(s), \ \forall s \in \mathbb{S}$$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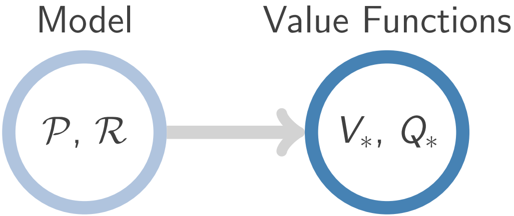
Model
Value Functions
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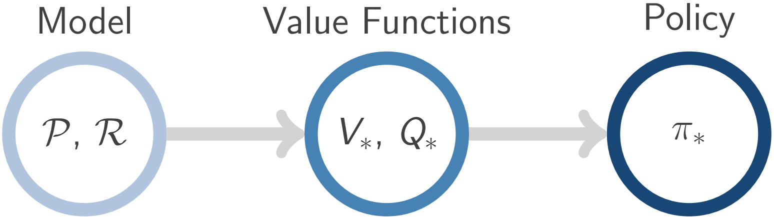
Taxonomy of RL Algorithms



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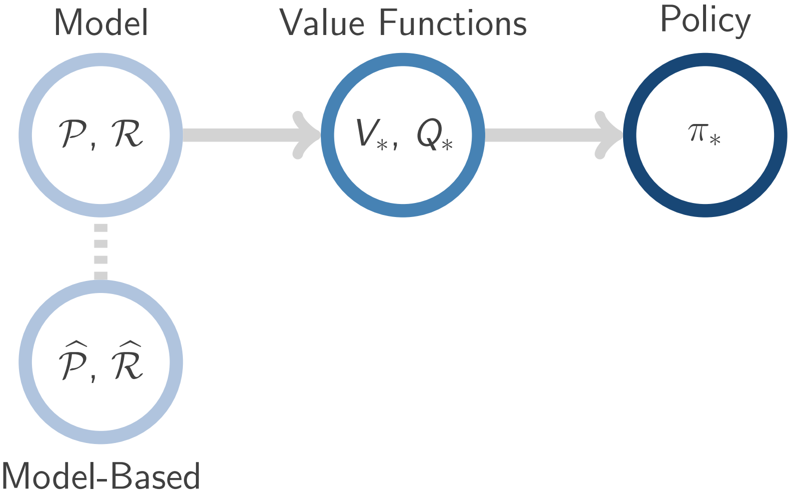

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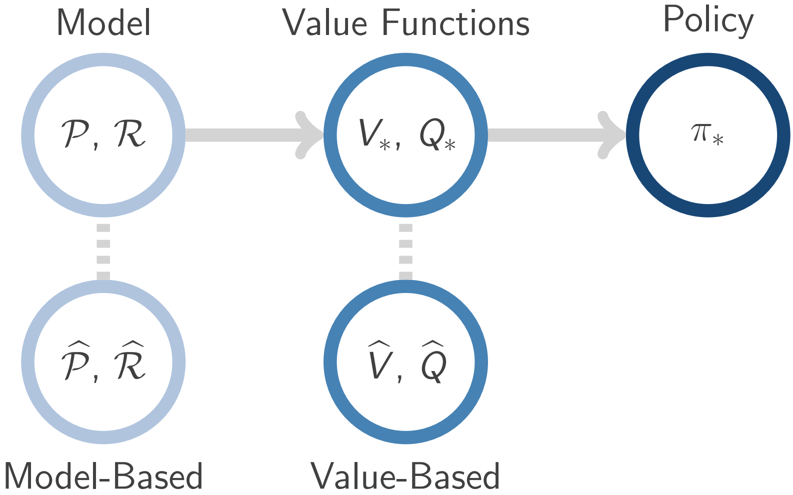

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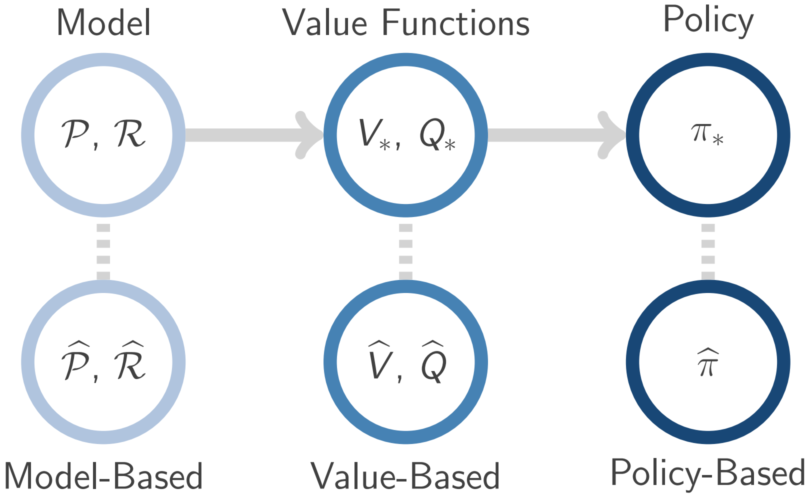

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Key Idea

1. The optimal policy π_* is approximated with a parametric policy π_{θ^*}

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$$\max_{\theta \in \Theta} J(\theta) = V_{\pi_\theta}(s)$$

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$$\theta_{k+1} = \theta_k + \alpha_k + \ \nabla_\theta J\left(\theta_k\right)$$

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The Keystone of Policy Gradient Algorithms

Policy Gradient Theorem

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For an episodic environment, the policy gradient can be approximated via Monte Carlo

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However, this estimate is characterized by a large variance. Possible improvements:

1. Optimal baseline
2. Actor-critic methods
3. Natural gradient

Policy Gradient with Parameter-Based Exploration (PGPE)

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3. The search for an optimum is performed in the space of the hyperparameters ξ

More formally, the update scheme becomes

$$\xi_{k+1} = \xi_k + \alpha_k \nabla_\xi J(\xi_k)$$

where the gradient is given by

$$\nabla_\xi J(\xi) = \mathbb{E}_{\substack{S \sim d^\xi \\ \theta \sim p_\xi}} \left[\nabla_\xi \log p_\xi(\theta) Q_\xi \left(S, F_\theta(S) \right) \right]$$

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Investor's Goal

How to dynamically invest the available capital in a portfolio of different assets in order to maximize the expected total return or another relevant performance measure.

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$$R_{t+1} = \log \left\{ 1 + \sum_{i=0}^{I} \left[a_t^i X_{t+1}^i - \delta_i \left| a_t^i - \tilde{a}_t^i \right| - \delta_s (a_t^i)^- \right] - \delta_f \mathbf{1}_{a_t \neq \tilde{a}_{t-1}} \right\}$$

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Actions: Portfolio weights

$$\{a_t^i\}_{i=0}^{I} \quad \text{s.t.} \quad \sum_{i=0}^{I} a_t^i = 1 \qquad \forall t \in \{0, 1, 2, \ldots\}$$

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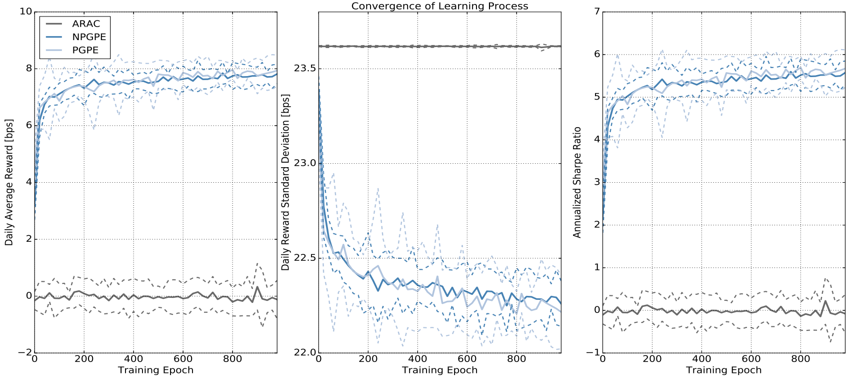
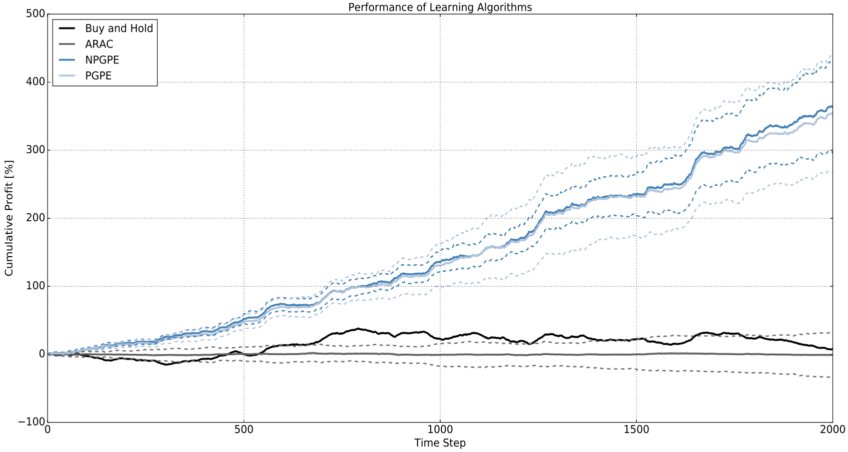
States: assets past returns and current allocation

$$S_t = \{X, X_t, X_{t-1}, \ldots, X_{t-P}, \tilde{a}_t\}$$

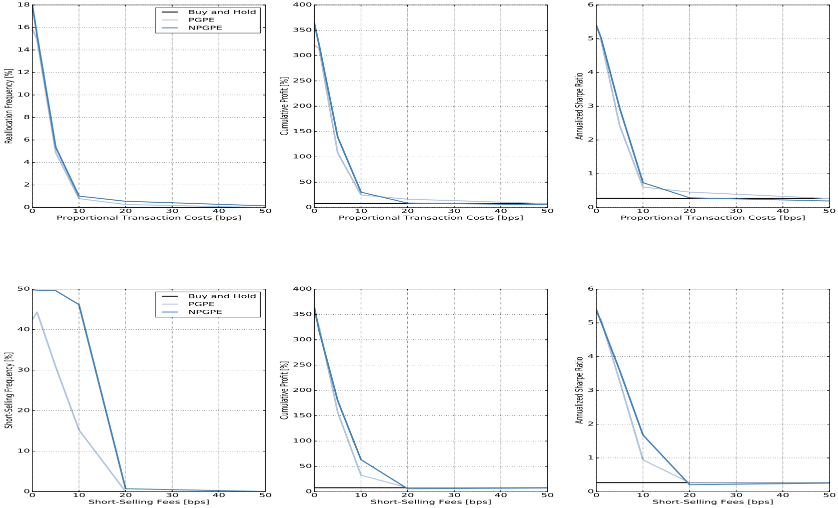
Synthetic Asset: Convergence



Synthetic Asset: Backtest Performance



Synthetic Asset: Impact of Transaction Costs

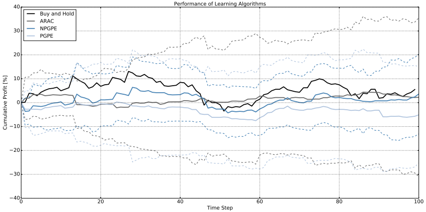


Not So Fast

Insuccess on Historical Data

Successfully applying these RL algorithms to historical data is much more challenging

1. Fail to converge
2. The strategies learned are not profitable



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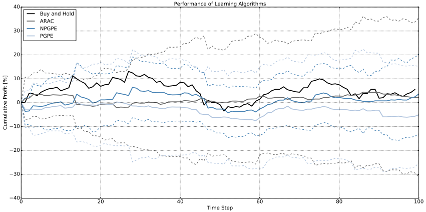

Possible Explanations

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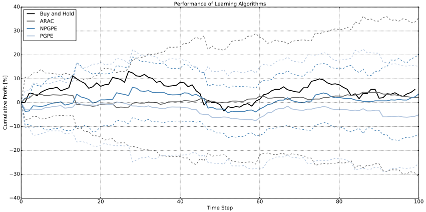

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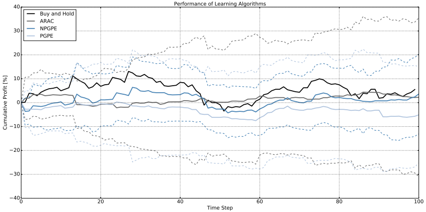

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1. **Low signal-to-noise ratio**: extremely difficult to find tradable patterns in markets
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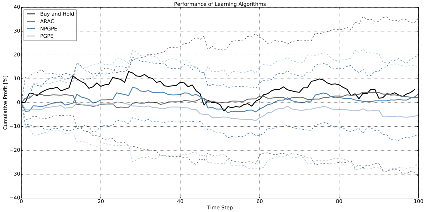

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4. **Non-stationarity of financial time-series**: a signal needs to be persistent

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What Has Been Done

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2. Develop more complex features for the trading strategy
3. Combine policy gradient algorithms with state-of-the-art deep learning techniques
4. RL framework is versatile and can be applied to other financial decision problems

Thank you for your attention!

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Markov Decision Processes

Reinforcement Learning

General class of algorithms that allow an agent to learn how to behave in a stochastic and possibly unknown environment by trial-and-error.

Markov Decision Process (MDP)

stochastic dynamical system specified by $< \mathbb{S}, \mathbb{A}, \mathcal{P}, \mathcal{R}, \gamma >$

1. $(\mathbb{S}, \mathcal{S})$ is a measurable state space
2. $(\mathbb{A}, \mathcal{A})$ is a measurable action space
3. $\mathcal{P} : \mathbb{S} \times \mathbb{A} \times \mathcal{S} \to \mathbb{R}$ is a Markov transition kernel
4. $\mathcal{R} : \mathbb{S} \times \mathbb{A} \to \mathbb{R}$ is a reward function
5. $0 < \gamma < 1$ is the discount factor.

Policy Gradient Theorem: Statement and Proof

Policy Gradient Theorem

Let π_θ be a differentiable policy. The policy gradient is given by

$$\nabla_\theta J(\theta) = \mathbb{E}_{\substack{S\sim d^\theta \\ A\sim\pi_\theta}} \left[\nabla_\theta \log \pi_\theta(S, A) Q_\theta(S, A)\right]$$

where d^θ is the stationary distribution of the Markov chain induced by π_θ.

Proof

$$\nabla_\theta V_\theta(s) = \nabla_\theta \int_{\mathbb{A}} \pi_\theta(s,a) Q_\theta(s,a) da = \int_{\mathbb{A}} \left[\nabla_\theta \pi_\theta(s,a) Q_\theta(s,a) + \pi_\theta(s,a) \nabla_\theta Q_\theta(s,a)\right] da$$

$$\nabla_\theta Q_\theta(s,a) = \nabla_\theta \left[\mathcal{R}(s,a) - \rho_\theta + \int_{\mathbb{S}} \mathcal{P}(s,a,s') V_\theta(s') ds'\right] = -\nabla_\theta \rho_\theta + \int_{\mathbb{S}} \mathcal{P}(s,a,s') \nabla_\theta V_\theta(s') ds'$$

$$\nabla_\theta V_\theta(s) = \int_{\mathbb{A}} \nabla_\theta \pi_\theta(s,a) Q_\theta(s,a) da - \nabla_\theta \rho_\theta + \int_{\mathbb{A}} \pi_\theta(s,a) \int_{\mathbb{S}} \mathcal{P}(s,a,s') \nabla_\theta V_\theta(s') ds'$$

$$\int_{\mathbb{S}} d^\theta(s) \int_{\mathbb{A}} \pi(s,a) \int_{\mathbb{S}} \mathcal{P}(s,a,s') \nabla_\theta V(s') ds' da ds = \int_{\mathbb{S}} d^\theta(s) \nabla_\theta V_\theta(s) ds$$

$$\nabla_\theta \rho_\theta = \int_{\mathbb{S}} d^\theta(s) \int_{\mathbb{A}} \nabla_\theta \pi_\theta(s,a) Q_\theta(s,a) da ds = \int_{\mathbb{S}} d^\theta(s) \int_{\mathbb{A}} \pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a) Q_\theta(s,a) da ds$$

Episodic PGPE Algorithm: Pseudocode

Input: Deterministic controller F_θ, Initial hyper-parameters ξ^0, learning rate $\{\alpha_k\}$
Output: Approximation of the optimal policy $F_{\xi^*} \approx \pi_*$

repeat
  for $m = 1, \dots, M$ **do**
    Sample controller parameters $\theta^{(m)} \sim p_{\xi_k}$
    Sample trajectory $h^{(m)} = \{(s_t^{(m)}, a_t^{(m)})\}_{t \geq 0}$ under policy $F_{\theta^{(m)}}$
  end for
  Approximate policy gradient

$$\nabla_\xi J(\xi_k) \approx \frac{1}{M} \sum_{m=1}^{M} \nabla_\xi \log p_\xi \left(\theta^{(m)}\right) \left[G\left(h^{(m)}\right) - b\right]$$

  Update hyperparameters using gradient ascent $\xi_{k+1} = \xi_k + \alpha_k \nabla_\xi J(\xi_k)$
  $k \leftarrow k + 1$
until converged

Natural PGPE Algorithm: Pseudocode

Input: Deterministic controller F_θ, Initial hyper-parameters ξ^0, learning rate $\{\alpha_k\}$
Output: Approximation of the optimal hyper-parameters ξ_*

repeat
    Observe current state s_k
    Draw $\zeta_k \sim \mathcal{N}(0, I_n)$
    Compute controller parameters $\theta_k = \mu_k + \Gamma^T \zeta_k$
    Perform action $a_k = F_{\theta_k}(s_k)$ and receive reward r_{k+1}
    Update average reward estimate $\widehat{\rho}_{k+1} = \widehat{\rho}_k + \alpha_k(r_{k+1} - \widehat{\rho}_k)$
    Compute natural policy gradients

$$\widetilde{\nabla}_\mu \log p_{\xi_k}(\theta_k) = \theta_k - \mu_k \qquad \widetilde{\nabla}_\Gamma \log p_{\xi_k}(\theta_k) = \left(\mathrm{triu}(\zeta_k \zeta_k^T) - \frac{1}{2}\mathrm{diag}(\zeta_k \zeta_k^T) - \frac{1}{2} I\right)\Gamma$$

    Update eligibility trace $e_k = \lambda e_{k-1} + \nabla_\xi \log p_{\xi_k}(\theta_k)$
    Update hyper-parameters $\xi_{k+1} = \xi_k + \alpha_k(r_{k+1} - \widehat{\rho}_k) e_k$
    $k \leftarrow k + 1$
until converged

Experiment on Synthetic Asset

The synthetic asset price is given by

$$Z_t = \exp\left(\frac{z_t}{\max_t z_t - \min_t z_t}\right)$$

where $\{z_t\}$ is a random walk with autoregressive trend $\{\beta_t\}$

$$\begin{aligned} z_t &= z_{t-1} + \beta_{t-1} + \kappa\epsilon_t \\ \beta_t &= \alpha\beta_{t-1} + \nu_t \end{aligned}$$

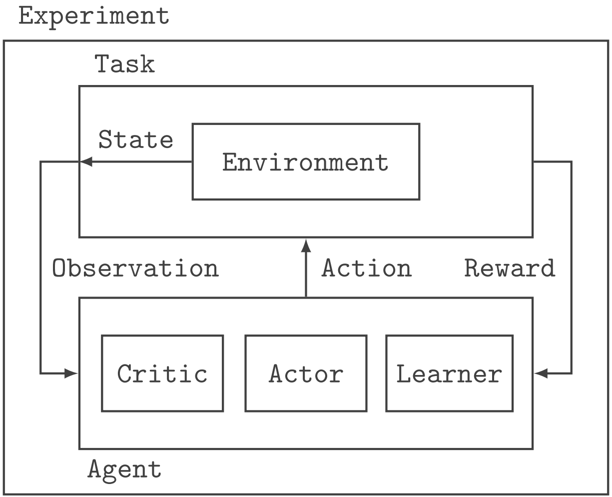
The policy used for the PGPE and the NPGPE algorithms is

$$F_\theta(s) = \text{sign}(\theta \cdot s)$$

where

$$\theta \sim \mathcal{N}(\mu, \text{diag}(\sigma))$$

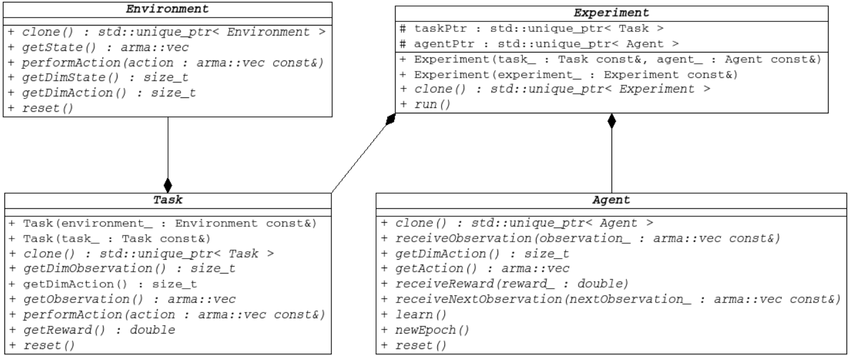
PyBrain's Architecture for a RL Problem



Agent-Environment Interaction in C++

Adapting PyBrain's Architecture

1. Defined standard interfaces via pure abstract classes
2. Achieved modularity via polymorphic composition



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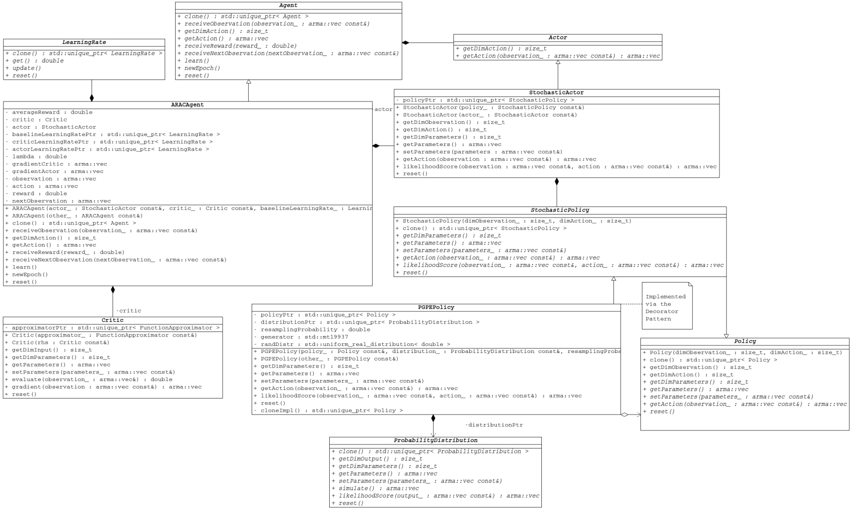

Execution Pipeline

experiment_launcher.py

1. Program execution is handled by a Python script
2. Responsible for analyzing the output of the C++ engine