POLITECNICO
MILANO 1863

Policy Gradient Algorithms for the Asset Allocation Problem

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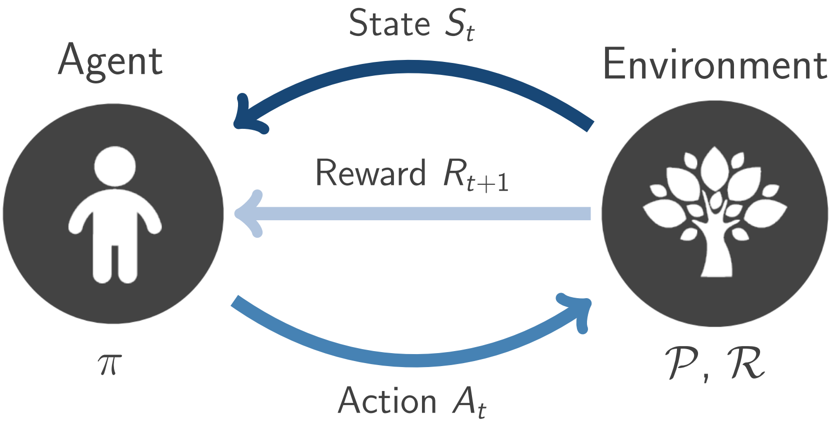
Animal Spirits

The Sound of Silence

Plan

1. Basics of Reinforcement Learning
2. Policy Gradient Algorithms
3. Asset Allocation with Transaction Costs
4. Conclusions



State S_t
Agent
Environment
Reward R_{t+1}
π
$\mathcal{P}, \mathcal{R}$
Action A_t

Mathematical Formulation

State-value function

$$V_\pi(s) = \mathbb{E}_\pi\left[\sum_{t=0}^{\infty} \gamma^t R_{t+1} \middle| S_0 = s\right]$$

Action-value function

$$Q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_{t=0}^{\infty} \gamma^t R_{t+1} \middle| S_0 = s, A_0 = a\right]$$

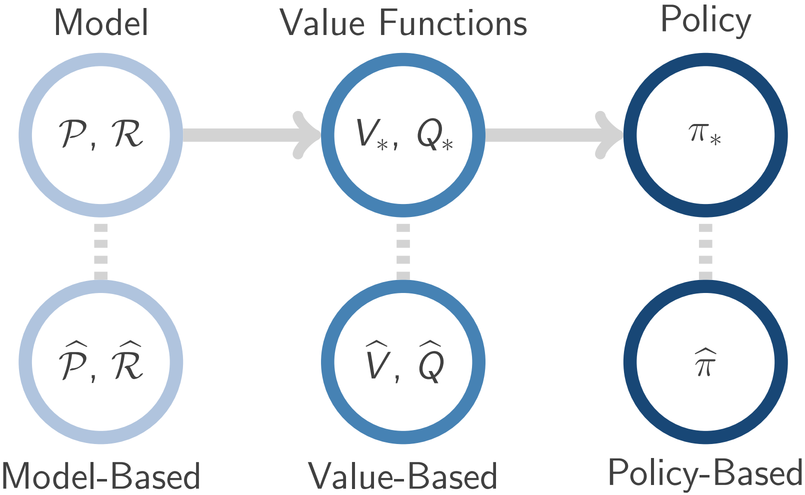
Optimal value functions

$$V_*(s) = \max_\pi V_\pi(s) \qquad Q_*(s,a) = \max_\pi Q_\pi(s,a)$$

Optimal policy

$$\pi_* \text{ s.t. } V_{\pi_*}(s) = V_*(s),\ \forall s \in \mathbb{S}$$

Taxonomy of RL Algorithms



Policy Gradient Algorithms

Key Idea

1. The optimal policy π_* is approximated with a parametric policy π_{θ^*}
2. The parameters θ^* of the policy solve the optimization problem

$$\max_{\theta \in \Theta} J(\theta) = V_{\pi_\theta}(s)$$

3. Which is solved numerically via gradient descent

$$\theta_{k+1} = \theta_k + \alpha_k \, \nabla_\theta J(\theta_k)$$

The Keystone of Policy Gradient Algorithms

Policy Gradient Theorem

$$\nabla_\theta J(\theta) = \mathbb{E}_{\substack{S\sim d^\theta \\ A\sim\pi_\theta}} \left[\nabla_\theta \log \pi_\theta(S, A) Q_\theta(S, A)\right]$$

For an episodic environment, the policy gradient can be approximated via Monte-Carlo

$$\nabla_\theta J(\theta_k) \approx \frac{1}{M} \sum_{m=0}^{M} \sum_{u=0}^{T^{(m)}-1} \nabla_\theta \log \pi_{\theta_k}\left(s_u^{(m)}, a_u^{(m)}\right) \sum_{v\geq u}^{T^{(m)}-1} \gamma^{v-u} r_{v+1}^{(m)}$$

However, this estimate is characterized by a large variance. Possible improvements:

1. Optimal baseline
2. Actor-critic methods
3. Natural gradient

Policy Gradient with Parameter-Based Exploration (PGPE)

Key Idea

1. Actions are selected using a deterministic parametric controller F_θ
2. The controller parameters are drawn from a probability distribution p_ξ
3. The search for an optimum is performed in the space of the hyperparameters ξ

More formally, the update scheme becomes

$$\xi_{k+1} = \xi_k + \alpha_k \nabla_\xi J(\xi_k)$$

where the policy gradient is given by

Parameter-Based Policy Gradient Theorem

$$\nabla_\xi J(\xi) = \mathbb{E}_{\substack{S \sim d^\xi \\ \theta \sim p_\xi}} \left[\nabla_\xi \log p_\xi(\theta) Q_\xi\left(S, F_\theta(S)\right)\right]$$

Problem Formulation

Investor's Goal

How to dynamically invest the available capital in a portfolio of different assets in order to maximize the expected total return or another relevant performance measure.

Rewards: portfolio log-return with transaction costs

$$R_{t+1} = \log\left\{1 + \sum_{i=0}^{I}\left[\ a_t^i X_{t+1}^i \ - \ \delta_i \left|a_t^i - \tilde{a}_t^i\right| \ - \ \delta_s (a_t^i)^- \ \right] - \ \delta_f \mathbf{1}_{a_t \neq \tilde{a}_{t-1}} \ \right\}$$

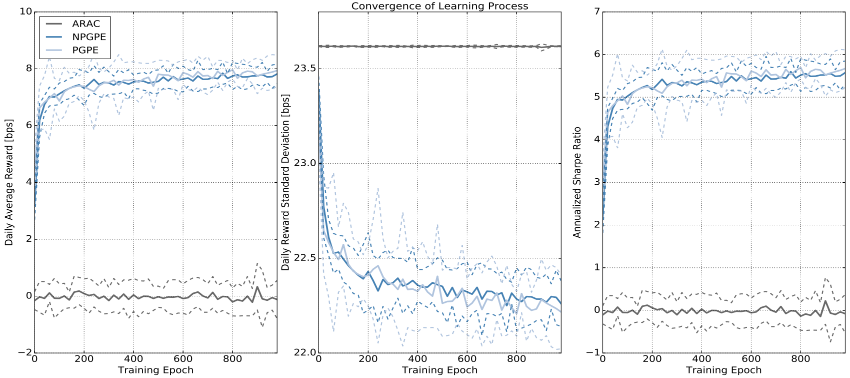
Actions: Portfolio weights

$$\{a_t^i\}_{i=0}^I \quad \text{s.t.} \quad \sum_{i=0}^{I} a_t^i = 1 \qquad \forall t \in \{0, 1, 2, \dots\}$$

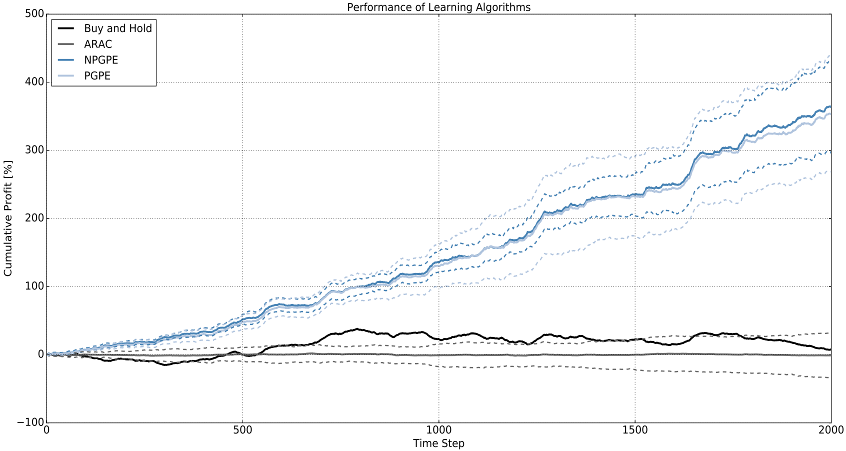
States: assets past returns and current allocation

$$S_t = \{X, X_t, X_{t-1}, \dots, X_{t-P}, \tilde{a}_t\}$$

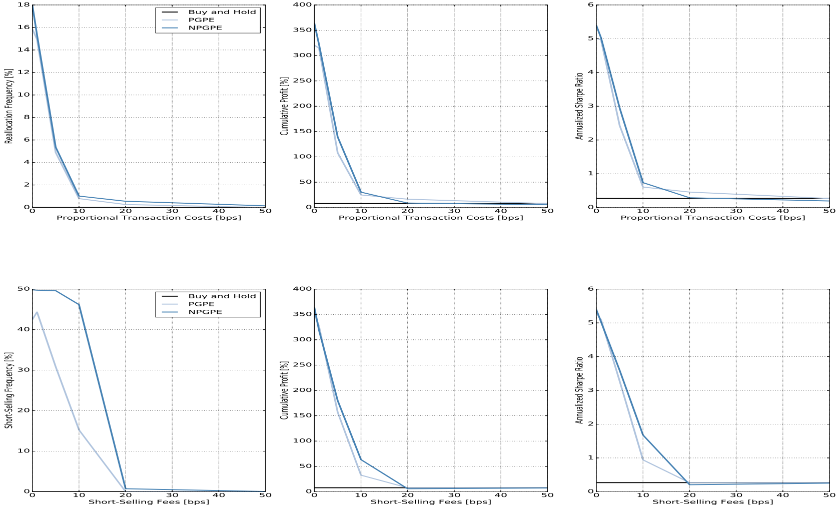
Synthetic Asset: Convergence



Synthetic Asset: Backtest Performance



Synthetic Asset: Impact of Transaction Costs

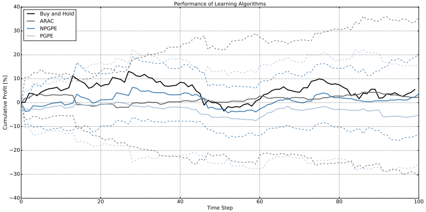


Not So Fast

Insuccess on Historical Data

Successfully applying these RL algorithms to historical data is much more challenging

1. Fail to converge
2. The strategies learned are not profitable



Possible Explanations

1. **Low signal-to-noise ratio**: extremely difficult to find tradable patterns in markets
2. **Quality of data**: unlikely to find patterns in daily prices of liquid stocks
3. **Weak features**: parametric policy must be powerful enough to capture the signal
4. **Non-stationarity of financial time-series**: a signal needs to be persistent

Conclusions

What Has Been Done

1. In-depth bibliographical study of state-of-the-art policy gradient algorithms
2. Innovative contributions to the policy gradient literature
3. Applied these techniques to find a profitable long-short trading strategy

Research Directions

1. Improve the algorithms performance on historical data
2. Develop more complex features for the trading strategy
3. Combine policy gradient algorithms with state-of-the-art deep learning techniques
4. RL framework is versatile and can be applied to other financial decision problems

Thank you for your attention!

Deng, Y., Bao, F., Kong, Y., Ren, Z., and Dai, Q. (2016).
Deep direct reinforcement learning for financial signal representation and trading.
IEEE Transactions on Neural Networks and Learning Systems.

Peters, J. and Schaal, S. (2008).
Reinforcement learning of motor skills with policy gradients.
Neural networks, 21(4):682–697.

Sehnke, F. (2012).
Parameter exploring policy gradients and their implications.
PhD thesis, Technische Universität München.

Sutton, R. S. and Barto, A. G. (1998).
Introduction to reinforcement learning, volume 135.
MIT Press Cambridge.

Markov Decision Processes

Reinforcement Learning

General class of algorithms that allow an agent to learn how to behave in a stochastic and possibly unknown environment by trial-and-error.

Markov Decision Process (MDP)

stochastic dynamical system specified by $< \mathbb{S}, \mathbb{A}, \mathcal{P}, \mathcal{R}, \gamma >$

1. $(\mathbb{S}, \mathcal{S})$ is a measurable state space
2. $(\mathbb{A}, \mathcal{A})$ is a measurable action space
3. $\mathcal{P} : \mathbb{S} \times \mathbb{A} \times \mathcal{S} \to \mathbb{R}$ is a Markov transition kernel
4. $\mathcal{R} : \mathbb{S} \times \mathbb{A} \to \mathbb{R}$ is a reward function
5. $0 < \gamma < 1$ is the discount factor.

Policy Gradient Theorem: Statement and Proof

Policy Gradient Theorem

Let π_θ be a differentiable policy. For the gradient of the average reward is

$$\nabla_\theta \rho(\theta) = \mathbb{E}_{\substack{S\sim d^\theta \\ A\sim\pi_\theta}} \left[\nabla_\theta \log \pi_\theta(S, A) Q_\theta(S, A)\right]$$

where d^θ is the stationary distribution of the Markov chain induced by π_θ.

Proof

$$\nabla_\theta V_\theta(s) = \nabla_\theta \int_{\mathbb{A}} \pi_\theta(s,a) Q_\theta(s,a) da = \int_{\mathbb{A}} \left[\nabla_\theta \pi_\theta(s,a) Q_\theta(s,a) + \pi_\theta(s,a) \nabla_\theta Q_\theta(s,a)\right] da$$

$$\nabla_\theta Q_\theta(s,a) = \nabla_\theta \left[\mathcal{R}(s,a) - \rho_\theta + \int_{\mathbb{S}} \mathcal{P}(s,a,s') V_\theta(s') ds'\right] = -\nabla_\theta \rho_\theta + \int_{\mathbb{S}} \mathcal{P}(s,a,s') \nabla_\theta V_\theta(s') ds'$$

$$\nabla_\theta V_\theta(s) = \int_{\mathbb{A}} \nabla_\theta \pi_\theta(s,a) Q_\theta(s,a) da - \nabla_\theta \rho_\theta + \int_{\mathbb{A}} \pi_\theta(s,a) \int_{\mathbb{S}} \mathcal{P}(s,a,s') \nabla_\theta V_\theta(s') ds'$$

$$\int_{\mathbb{S}} d^\theta(s) \int_{\mathbb{A}} \pi(s,a) \int_{\mathbb{S}} \mathcal{P}(s,a,s') \nabla_\theta V(s') ds' da ds = \int_{\mathbb{S}} d^\theta(s) \nabla_\theta V_\theta(s) ds$$

$$\nabla_\theta \rho_\theta = \int_{\mathbb{S}} d^\theta(s) \int_{\mathbb{A}} \nabla_\theta \pi_\theta(s,a) Q_\theta(s,a) da ds = \int_{\mathbb{S}} d^\theta(s) \int_{\mathbb{A}} \pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a) Q_\theta(s,a) da ds$$

Monte-Carlo Policy Gradient: Pseudocode

Input: Stochastic policy π_θ, Initial parameters θ_0, learning rate $\{\alpha_k\}$
Output: Approximation of the optimal policy $\pi_{\theta^*} \approx \pi_*$

1: **repeat**
2: Sample M trajectories $h^{(m)} = \{(s_t^{(m)}, a_t^{(m)}, r_{t+1}^{(m)})\}_{t=0}^{T^{(m)}}$ under policy π_{θ_k}
3: Approximate policy gradient

$$\nabla_\theta J(\theta_k) \approx \frac{1}{M} \sum_{m=0}^{M} \sum_{u=0}^{T^{(m)}-1} \nabla_\theta \log \pi_{\theta_k} \left(s_u^{(m)}, a_u^{(m)}\right) \sum_{v \geq u}^{T^{(m)}-1} \gamma^{v-u} r_{v+1}^{(m)}$$

4: Update parameters using gradient ascent $\theta_{k+1} = \theta_k + \alpha_k \nabla_\theta J(\theta_k)$
5: $k \leftarrow k + 1$
6: **until** converged

Episodic PGPE Algorithm: Pseudocode

Input: Controller F_θ, hyper-distribution p_ξ, initial guess ξ_0, learning rate $\{\alpha_k\}$
Output: Approximation of the optimal policy $F_{\xi^*} \approx \pi_*$

repeat
  for $m = 1, \dots, M$ **do**
    Sample controller parameters $\theta^{(m)} \sim p_{\xi_k}$
    Sample trajectory $h^{(m)} = \{(s_t^{(m)}, a_t^{(m)}, r_{t+1}^{(m)})\}_{t=0}^{T^{(m)}}$ under policy $F_{\theta^{(m)}}$
  end for
  Approximate policy gradient

$$\nabla_\xi J(\xi_k) \approx \frac{1}{M} \sum_{m=1}^{M} \nabla_\xi \log p_\xi \left(\theta^{(m)}\right) \left[G\left(h^{(m)}\right) - b\right]$$

  Update hyperparameters using gradient ascent $\xi_{k+1} = \xi_k + \alpha_k \nabla_\xi J(\xi_k)$
  $k \leftarrow k + 1$
until converged

Natural PGPE Algorithm: Pseudocode

Input: Controller F_θ, hyper-distribution p_ξ, initial guess ξ_0, learning rate $\{\alpha_k\}$

Output: Approximation of the optimal policy $F_{\xi^*} \approx \pi_*$

repeat

  Observe current state s_k

  Draw $\zeta_k \sim \mathcal{N}(0, I_n)$

  Compute controller parameters $\theta_k = \mu_k + \Gamma^T \zeta_k$

  Perform action $a_k = F_{\theta_k}(s_k)$ and receive reward r_{k+1}

  Update average reward estimate $\widehat{\rho}_{k+1} = \widehat{\rho}_k + \alpha_k(r_{k+1} - \widehat{\rho}_k)$

  Compute natural policy gradients

$$\widetilde{\nabla}_\mu \log p_{\xi_k}(\theta_k) = \theta_k - \mu_k \qquad \widetilde{\nabla}_\Gamma \log p_{\xi_k}(\theta_k) = \left(\operatorname{triu}(\zeta_k\zeta_k^T) - \frac{1}{2}\operatorname{diag}(\zeta_k\zeta_k^T) - \frac{1}{2}I\right)\Gamma$$

  Update eligibility trace $e_k = \lambda e_{k-1} + \nabla_\xi \log p_{\xi_k}(\theta_k)$

  Update hyper-parameters $\xi_{k+1} = \xi_k + \alpha_k(r_{k+1} - \widehat{\rho}_k)e_k$

  $k \leftarrow k + 1$

until converged

Experiment on Synthetic Asset

The synthetic asset price is given by

$$Z_t = \exp\left(\frac{z_t}{\max_t z_t - \min_t z_t}\right)$$

where $\{z_t\}$ is a random walk with autoregressive trend $\{\beta_t\}$

$$\begin{aligned} z_t &= z_{t-1} + \beta_{t-1} + \kappa\epsilon_t \\ \beta_t &= \alpha\beta_{t-1} + \nu_t \end{aligned}$$

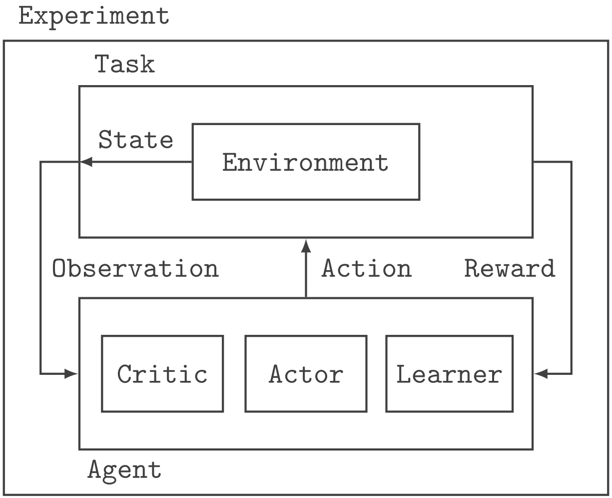
The policy used for the PGPE and the NPGPE algorithms is

$$F_\theta(s) = \mathrm{sign}(\theta \cdot s)$$

where

$$\theta \sim \mathcal{N}(\mu, \mathrm{diag}(\sigma))$$

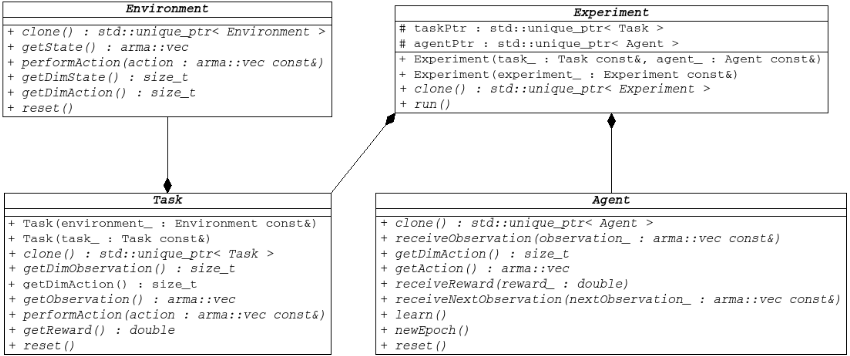
PyBrain's Architecture for a RL Problem



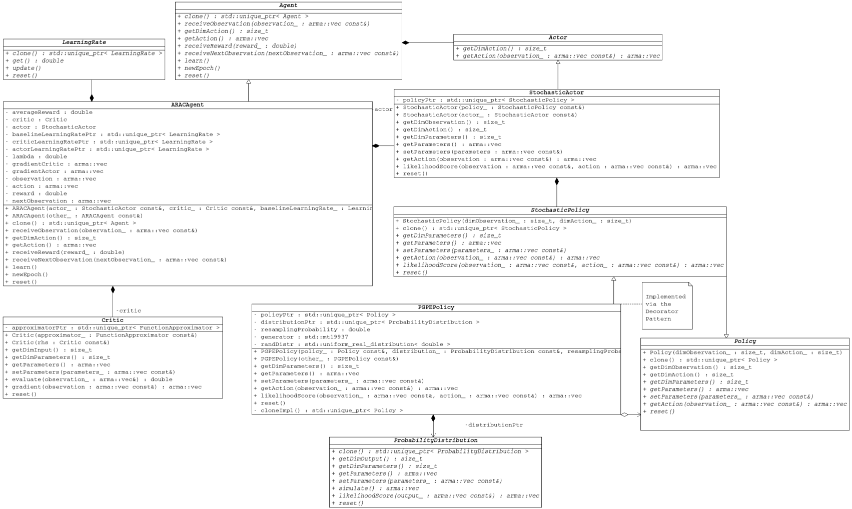
Agent-Environment Interaction in C++

Adapting PyBrain's Architecture

1. Defined standard interfaces via pure abstract classes
2. Achieved modularity via polymorphic composition



Agent's Architecture in C++



Execution Pipeline

experiment_launcher.py

1. Program execution is handled by a Python script
2. Responsible for analyzing the output of the C++ engine