EQUILIBRIUM DISTRIBUTIONS AND DEGREE OF RATIONAL APPROXIMATION OF ANALYTIC FUNCTIONS

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ABSTRACT. A theorem is proved on the degree of rational approximation of sequences of analytic functions given by Cauchy-type integrals of the form

$$f_n(z) = \oint_F \Phi_n(t) f(t)(t-z)^{-1}\,dt, \qquad z \in E.$$

The theorem is formulated in terms connected with the equilibrium distribution of the charge on the plates of a capacitor (E, F) under the assumption that an external field $\varphi = \lim_{n\to\infty}(2n)^{-1}\log|\Phi_n|^{-1}$ acts on the plate F, and this plate satisfies a certain symmetry condition in the field φ. The theorem is used to solve the problem of the degree of rational approximation of the function e^{-x} on $[0, +\infty)$.

Bibliography: 44 titles.

In recent years there has been considerable progress in the circle of problems connected with constructive rational approximation of functions. In particular, this has led to the solution of a number of problems on the degree of rational approximation of analytic functions. Here we present a general theorem in this direction (§1, Theorem 1). As an application of the theorem we give a solution of the known problem of the degree of rational approximation of the exponential on the semi-axis; there is a discussion of the corresponding results in §2.

Theorem 1 is formulated in terms connected with the equilibrium distribution of a charge on the plates of a capacitor under the condition that an "external field" acts on one of its plates. The method of proof of the theorem is based on the construction of multipoint Padé approximants. The study of the character of convergence of multipoint Padé approximants leads to questions connected with the limit distribution of the zeros of orthogonal polynomials with variable (depending on the index of the polynomial) weight functions. Potential-theoretic problems on equilibrium in the presence of "external fields" permit us to characterize the corresponding limit distributions. The articles [1]–[6] contain various results in this direction, relating mainly to the "real" case. An important role in the development of this circle of problems for the "complex" case has been played by the results and conjectures of Nuttall relating to local rational approximation; Nuttall's approach is based on the theory of Abelian integrals on compact Riemann surfaces (see the survey article [7], where there is a

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detailed bibliography). The key questions involving the distribution of the zeros of "complex" orthogonal polynomials and convergence of Padé approximants for multivalued analytic functions have been answered recently by Stahl [8]. After these papers it became clear that a potential-theoretic analysis of problems on the limit distribution of the zeros of polynomials satisfying complex orthogonality relations is conveniently based directly on an appropriate "symmetry" property for the corresponding contour. The question of contructing curves having various "symmetry" properties and of computing the parameters of the corresponding potential-theoretic problems is connected with extremal problems in potential theory and the geometric theory of functions, and with trajectories of quadratic differentials on Riemann surfaces (cf. [7] and [8]).

The main role in the present article is played by the "symmetry" property connected with the problem of the equilibrium distribution of the charge in an "external field". The foregoing remarks are illustrated to a certain degree in the material in §§1 and 2; in particular, the potential-theoretic problem to which the question of the degree of approximation of the exponential is reduced admits an explicit solution with the help of elliptic integrals and elliptic functions. The proof of the analogue needed here for Stahl's theorem in the case of arbitrary harmonic "external fields" is presented in the concluding section (§3, Theorem 3). The sphere of applications of this theorem is not confined to the framework of the present article; in this connection the material in §3 is presented in a way essentially independent of the first part of the paper. We remark also that the numbering of the formulas is independent in each section.

The main results in §§1 and 2 were announced in the survey [9] (a report of the first author at the International Congress of Mathematicians in Berkeley, August 3–11, 1986).

§1. Main theorem

1. We first introduce the notation adhered to everywhere below. Let K be a compact set in the extended complex plane $\hat{\mathbf{C}}$. A measure (charge) on K is a positive (respectively, real-valued) Borel measure ν whose support $S(\nu) = \operatorname{supp}(\nu)$ belongs to K. Denote by $M(K)$ the set of all unit measures on K satisfying the condition

$$\int_{|t|\geq 1} \log|t|\,d\nu(t) < +\infty.$$

The logarithmic potential of a charge (in particular, a measure) ν is denoted by V^ν:

$$V^\nu(z) = \int \log\frac{1}{|t-z|}\,d\nu(t);$$

if $\nu \in M(K)$, then $V^\nu(z) > -\infty$ for all $z \in \mathbf{C}$.

Let E and F be disjoint compact sets in the extended plane $\hat{\mathbf{C}}$. The pair (E, F) is called a *capacitor*; E and F are the *plates* of this capacitor. Everywhere below we consider capacitors (E, F) whose plates have *positive* (logarithmic) capacity and whose set F is a compact set in $\mathbf{C}$.

Assume that a continuous function $\varphi : F \to \mathbf{R}$ is given on the plate F of the capacitor (E, F); φ is called an *external field* on F. A capacitor (E, F) with a given field φ on F is denoted by (E, F, φ) and called a *rigged* (оснащенный) capacitor.

Let $M(E, F)$ be the set of all charges of the form $\mu = \mu_F - \mu_E$, where $\mu_E \in M(E)$ and $\mu_F \in M(F)$. The (doubled) energy of the charge μ in the external field φ is denoted by $I_\varphi(\mu)$:

$$I_\varphi = \iint \log \frac{1}{|t-z|}\, d\mu(z)\, d\mu(t) + 2\int \varphi(t)\, d\mu_F(t).$$

Setting $\varphi \equiv 0$ on E, we can write the formula for the energy in the field φ as

$$I_\varphi(\mu) = \int (V^\mu + 2\varphi)\, d\mu. \tag{1}$$

The following proposition can be proved by known methods in potential theory (see, for example, [10], [11], and also §3 below):

There exists a unique charge $\lambda \in M(E,R)$ minimizing the energy I_φ in this class:

$$I_\varphi(\lambda) = \min\{I_\varphi(\mu) \colon \mu \in M(E,F)\}.$$

The charge λ, and only this charge (in the class $M(E,F)$), has the following equilibrium property: there exist constants w_E and w_F such that the relations

$$\begin{gathered} V^\lambda(z) = w_E, \qquad z \in E, \\ (V^\lambda + \varphi)(z)\begin{cases} = w_F, & z \in S(\lambda_F), \\ \geq w_F, & z \in F, \end{cases} \end{gathered} \tag{2}$$

hold approximately everywhere on the indicated sets.

More precisely, the relations (2) are valid at all regular points of E and F (points where E and F are not thin). In particular, if (E,F) is a regular capacitor, then the equilibrium relations can be written in the form

$$\begin{gathered} V^\lambda(z) \equiv w_E, \qquad z \in E, \\ (V^\lambda + \varphi)(z) \equiv \min_F (V^\lambda + \varphi) = w_F, \qquad z \in S(\lambda_F). \end{gathered}$$

The charge $\lambda \in M(E,F)$ satisfying (2) will be called the *equilibrium charge* corresponding to the rigged capacitor (E,F,φ). The assertion that the equilibrium charge $\lambda = \lambda(E,F,\varphi)$ is *unique* (the fact that the equilibrium relations (2) uniquely determine the charge λ and the constants w_E and w_F) plays an important role in our considerations.

Let $w = w(E,F,\varphi) = w_F - w_E$. We remark that the formula

$$w = I_\varphi(\lambda) - \int \varphi\, d\lambda$$

is a consequence of (1) and (2). If the external field is absent ($\varphi \equiv 0$ on F), then $w = 1/c$, where $c = \operatorname{cap}(E,F)$ is the capacity of (E,F).

The problem considered is a very special case of the general problem of equilibrium for vector potentials in the presence of external fields [3]. In the notation adopted in [3] the quantity $\lambda = (\lambda_E, \lambda_F)$ is the vector equilibrium measure corresponding to the compact sets (conductors) $F_1 = E$ and $F_2 = F$, the magnitudes of the measures $\theta_1 = \theta_2 = 1$, the interaction matrix $A = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$, and the external fields $\varphi_1 = 0$ and $\varphi_2 = \varphi$. In this particular case the problem can be reduced to the scalar problem of equilibrium for the Green's potentials of the measures. For a compact set F and a measure $\mu \in M(F)$ we consider the potential

$$V_G^\mu(z) = \int g(z,t)\, d\mu(t), \qquad z \in \hat{\mathbf{C}} \setminus E,$$

where $g(z,t)$ is the Green's function for the complement of E. The Green's potential is expressed in terms of the logarithmic potential by the formula

$$V_G^\mu(z) = V^{\mu-\tilde{\mu}}(z) + \text{const}, \qquad z \in \hat{\mathbf{C}}\setminus E, \tag{3}$$

where $\tilde{\mu}$ is the balayage of the measure μ onto E. We formulate an analogue of the proposition given above.

There exists a unique measure $\lambda' \in M(F)$ minimizing the energy

$$J_\varphi(\mu) = \iint g(z,t)\,d\mu(z)\,d\mu(t) + 2\int \varphi(t)\,d\mu(t).$$

This measure, and only this measure (in the class $M(F)$), has the following equilibrium property: there exists a constant w' such that

$$(V_G^{\lambda'} + \varphi)(z)\begin{cases} = w' & \textit{approximately everywhere on } S(\lambda'), \\ \geq w' & \textit{approximately everywhere on } F. \end{cases} \tag{4}$$

The measure λ' is the equilibrium measure on F in the field φ for the Green's potential with respect to $G = \hat{\mathbf{C}}\setminus E$. It follows from (2) and (3) that

$$\lambda' = \lambda_F, \quad \tilde{\lambda}' = \lambda_E, \quad w' = w, \quad V_G^{\lambda'} = V^\lambda - w_E.$$

The equilibrium problems (2) and (4) are thereby easily reduced to each other.

2. We now define a concept of symmetry connected with the equilibrium problem considered in subsection 1.

Let Γ be a compact set in $\mathbf{C}$ with $\operatorname{cap}(\Gamma) > 0$ (cap is the logarithmic capacity). A point $\zeta \in \Gamma$ is called a *tame* (правильный) *point* of Γ if there exists a neighborhood of this point whose intersection with Γ is a simple (open) analytic arc. The set of all tame points of Γ is denoted by Γ_0. The compact set Γ is called a *tame compact set* if $\operatorname{cap}(\Gamma\setminus\Gamma_0) = 0$.

The open disk of radius $\varepsilon > 0$ about a point ζ is denoted by $U(\zeta, \varepsilon)$. It is clear that if $\zeta \in \Gamma_0$, then for all sufficiently small $\varepsilon > 0$ the intersection $\Gamma \cap \overline{U}$, $U = U(\zeta, \varepsilon)$, is a (closed) analytic arc whose endpoints (and only these points) belong to ∂U.

We say that a rigged capacitor (E, F, φ) has the *S-property* (the *symmetry property*), and we write

$$(E, F, \varphi) \in S,$$

if the following conditions hold:

(i) φ is a harmonic function in some neighborhood Ω of the plate F.

(ii) The support $\Gamma = S(\lambda_F)$ of the F-component of the equilibrium charge $\lambda = \lambda(E, F, \varphi)$ is a tame compact set, and $\Gamma_0 \subset F_0$;

(iii)

$$\frac{\partial(V^\lambda + \varphi)}{\partial n_+}(\zeta) = \frac{\partial(V^\lambda + \varphi)}{\partial n_-}(\zeta), \qquad \zeta \in \Gamma_0,$$

where $\partial/\partial n_\pm$ are the derivatives along the normal to Γ_0 (in opposite directions).

It is easy to see that for a harmonic external field φ the normal derivatives in (iii) exist for all $\zeta \in \Gamma_0$.

It would be more precise to say that under the conditions (i) and (ii) the plate F of the capacitor (E, F, φ) has the S-property; however, in this article we are concerned only with the indicated variant of symmetry for a capacitor, and this refinement is not essential.

The concept of symmetry is discussed in somewhat more detail in §3.

3. Finally, we introduce some notation connected with *holomorphic functions and integrals of Cauchy type*. The class of all holomorphic (single-valued analytic) functions on an open set D will be denoted by $\mathscr{H}(D)$. Let Ω be a neighborhood of the plate F of a capacitor (E, F). We fix a function $f \in \mathscr{H}(\Omega \backslash F)$, and let

$$f_E(z) = \int_\gamma \frac{f(t)\,dt}{t - z}, \qquad z \in E, \tag{5}$$

where γ is an arbitrary contour lying in $\Omega \backslash F$ (we assume that Ω belongs to the complement of E; in general, γ is a complicated contour consisting of finitely many disjoint rectifiable Jordan curves collectively separating F and $\partial\Omega$). The contour γ is positively oriented with respect to the components of the open set $\hat{\mathbf{C}} \backslash \gamma$ having nonempty intersection with E; the union of these components is called the *exterior* of γ.

The function $f_E(z)$, $z \in E$, clearly admits a holomorphic extension to $\hat{\mathbf{C}} \backslash F$, $f_E(\infty) = 0$. Instead of (5) we write

$$f_E(z) = \oint_F \frac{f(t)\,dt}{t - z}, \qquad z \in \hat{\mathbf{C}} \backslash F, \tag{6}$$

understanding the integral $\oint_F$ as $\int_\gamma$, where γ is an arbitrary contour of the indicated form (such that the point z lies in the exterior of γ). In other words, the function f_E is the component of the Laurent expansion of $2\pi i f \in \mathscr{H}(\Omega \backslash F)$ that is holomorphic in the complement of F and normalized by the condition $f_E(\infty) = 0$.

If F is a rectifiable Jordan arc (or a union of disjoint rectifiable arcs and Jordan curves), and the function f has a continuous jump $\chi_f(t)$ on F, then f_E can be represented in the form

$$f_E(z) = \int_F \frac{\chi_f(t)\,dt}{t - z}, \qquad z \in \hat{\mathbf{C}} \backslash F;$$

the choice of sign for the jump χ_f must be coordinated with the orientation of F. Note also that in this case the integral

$$\int_F \frac{\Phi(t)\,dt}{t - z},$$

where $\Phi \in \mathscr{H}(\Omega)$, can be represented in the form (6) by setting

$$f(z) = \frac{\Phi(z)}{2\pi i} \oint_F \frac{dt}{t - z}, \qquad z \in \Omega \backslash F,$$

in particular, if F is a Jordan arc, then

$$f(z) = \Phi(z)(2\pi i)^{-1} \log \frac{z - a}{z - b},$$

where a and b are the endpoints of F.

Suppose that $f \in \mathscr{H}(\Omega \backslash F)$ and $(E, F, \varphi) \in S$; as above, Ω is a neighborhood of F lying in the complement of E. Define $\Gamma = S(\lambda_F)$ and consider a point $\zeta \in \Gamma_0$. For a sufficiently small neighborhood $U(\zeta, \varepsilon) \subset \Omega$ the analytic arc $l = F \cap U$ divides $U \backslash l$ into two regions U_1 and U_2, and $f_j = f|_{U_j} \in \mathscr{H}(U_j)$ (see subsection 2, (ii)). If each of the functions f_j, $j = 1, 2$, admits a continuous extension to the corresponding closed region $\overline{U}_j$, then the continuous jump $\chi_F = f_2 - f_1$ of f is defined on l. We assume that the foregoing holds for all points $\zeta \in \Gamma_0 \backslash e$, where e is a relatively closed subset of Γ_0 having zero capacity; then f *has a continuous jump* χ_f *on* $\Gamma_0 \backslash e$. If, moreover, $\chi_f \neq 0$ on $\Gamma_0 \backslash e$, then we write

$$f \in \mathscr{H}_0(\Omega \backslash F); \tag{7}$$

we emphasize that this condition is connected with a capacitor (E, F, φ) having the S-property.

4. Let g be a function continuous on the compact set $E \subset \hat{\mathbf{C}}$, and let $\mathscr{R}_n$ be the class of all rational functions of z with order at most n. Denote by $\rho_n(g, E)$ the deviation of g from $\mathscr{R}_n$ in the uniform metric on E:

$$\rho_n(g, E) = \inf\{\|g - r\|_E : r \in \mathscr{R}_n\},$$

where $\|\cdot\|_E$ is the sup norm on E.

THEOREM 1. *Suppose that E is a union of finitely many continua in $\hat{\mathbf{C}}$, F is a compact set of positive capacity in $\mathbf{C} \backslash E$, $\{\Phi_n\}$ is a sequence of functions holomorphic in a neighborhood Ω of F, and f is a function holomorphic in $\Omega \backslash F$. Assume the following conditions hold*:

1°. $(1/2n)\log(1/|\Phi_n(z)|) \rightrightarrows \varphi(z), \ z \in \Omega.$

2°. $(E, F, \varphi) \in S.$

3°. $f \in \mathscr{H}_0(\Omega \backslash F).$

Then the sequence of functions

$$f_n(z) = (\Phi_n f)_E(z) = \oint_F \Phi_n(t) \frac{f(t)\,dt}{t - z}, \qquad z \in E, \tag{8}$$

satisfies the relation

$$\lim_{n \to \infty} \rho_n(f_n, E)^{1/n} = e^{-2w}, \tag{9}$$

where $w = w(E, F, \varphi)$.

We define a continuum to be a connected compact set containing more than one point. Note that if F is a union of finitely many rectifiable arcs or Jordan curves and conditions 1° and 2° of the theorem are valid, then relation (9) is valid for the sequence of functions

$$f_n(z) = \int_F \Phi_n(t) \frac{dt}{t - z}$$

(see subsection 3). This is the form of Theorem 1 we use in §2 (F a rectifiable arc). The case $\Phi_n(z) \equiv 1$ is discussed in subsection 6, below.

The scheme of proof of Theorem 1 is based on interpolation by rational functions with *free* poles; the corresponding rational functions are called multipoint Padé approximants. In a simpler situation this scheme was applied in Gonchar's paper [12] (see also [13]) to problems on the degree of rational approximation of analytic functions. The method developed by Stahl (see [8]) enables us to study the asymptotic behavior of multipoint Padé approximants also in the case under consideration here. In our case the "complex" orthogonality relations for the denominators of the Padé approximants contain the additional factor Φ_n, and the limit behavior of these approximants is described in terms of the equilibrium problem for a capacitor in the presence of a harmonic external field φ (see Lemmas 1 and 2 below).

5. PROOF OF THEOREM 1. It follows from the formulation of the equilibrium problem in terms of the Green's potential (see (4)) that this problem is invariant with respect to linear fractional transformations $\zeta = L(z)$, $L(F) \subset \mathbf{C}$. The S-property of the capacitor (E, F, φ) is also preserved under such transformations; it suffices to

note that the symmetry condition (iii) in subsection 2 can also be written in terms of the Green's potential of the equilibrium measure λ_F (replace $V^\lambda + \varphi$ by $V_G^{\lambda_F} + \varphi$). Under a linear fractional transformation $\zeta = L(z)$ the integrals (8) are transformed to the form

$$f_n^*(\zeta) = (\zeta - \zeta_0) \oint_{F^*} \Phi_n^*(\tau) \frac{f^*(\tau)\, d\tau}{\tau - \zeta}, \qquad \zeta_0 = L(\infty).$$

It follows from the foregoing that the theorem need be proved only for the case when E is a *bounded* compact set of the form under consideration. In order not to complicate the proof with inessential details we assume also that *the complement of* $E \cup F$ *is connected.*

5.1. On E we fix a table of interpolation nodes of the form $\{\alpha_{n,k}\}$, $k = 1, 2, \dots, 2n+1$, $n = 1, 2, \dots$. Consider the sequence of polynomials

$$\omega_{2n+1}(z) = \prod_{k=1}^{2n+1} (z - \alpha_{n,k}), \qquad n = 1, 2, \dots,$$

and the associated sequence of measures

$$\mu(\omega_{2n+1}) = \sum_{k=1}^{2n+1} \delta_{\alpha_{n,k}}, \qquad n = 1, 2, \dots$$

(δ_α is the Dirac measure at the point α). We choose the table $\{\alpha_{n,k}\}$ so that

$$\frac{1}{2n} \mu(\omega_{2n+1}) \to \lambda_E, \tag{10}$$

where λ_E is the E-component of the equilibrium charge $\lambda = \lambda_F - \lambda_E$ corresponding to the rigged capacitor (E, F, φ). Here and in what follows, the symbol $\to$ denotes weak convergence in connection with sequences of measures (on the Riemann sphere). In other words, the interpolation nodes are chosen to have a limit distribution characterized by the measure λ_E.

Fix a positive integer n. We consider a rational function R_n of the form P_n/Q_n, where P_n and Q_n are arbitrary polynomials of degree at most n ($Q_n \not\equiv 0$) satisfying the condition

$$\frac{Q_n f_n - P_n}{\omega_{2n+1}} \in \mathscr{H}(E).$$

This condition means that $Q_n f_n - P_n = 0$ at the zeros of ω_{2n+1} (together with the appropriate number of derivatives at multiple zeros). The polynomials P_n and Q_n ($Q_n \not\equiv 0$) *exist* for any function f_n; their ratio P_n/Q_n determines a *unique* rational function R_n (to within the standard identification). This rational function is called a *multipoint Padé approximant* of type $[n/n]$ for f_n corresponding to the interpolation nodes $\alpha_{n,1}, \dots, \alpha_{n,2n+1}$. The leading coefficient of the polynomial Q_n is assumed to be 1 in what follows.

We underscore that the rational function R_n need not interpolate f_n at all the zeros of the polynomial ω_{2n+1}. There can be a loss of the interpolation condition (in passing from $Q_n f_n - P_n$ to $f_n - R_n$) only in the case when the polynomials P_n and Q_n have a common zero coinciding with one of the interpolation nodes. Therefore, a loss of d_n interpolation conditions is accompanied by a reduction of the degree of the numerator and the denominator of R_n by the same quantity. More precisely, *if the number of zeros of the polynomial* ω_{2n+1} *at which* $f_n - R_n = 0$ *is equal to* $2n+1-d_n$ *and* $R_n = P_n^*/Q_n^*$, $(P_n^*, Q_n^*) = 1$, *then* $\deg P_n^* \le n - d_n$ *and* $\deg Q_n^* \le n - d_n$ (the multiplicities of the zeros and points of interpolation are always taken into account).

Obviously,

$$\frac{Q_n f_n - P_n}{\omega_{2n+1}}(z) = O\left(\frac{1}{z^{n+1}}\right), \qquad z \to \infty.$$

From this, using the fact that the left-hand side is holomorphic in the open set $D = \hat{\mathbb{C}} \setminus F$, we get that

$$\oint_F \left(\frac{Q_n f_n - P_n}{\omega_{2n+1}}\right)(t) t^j \, dt = 0, \qquad j = 0, 1, \dots, n-1.$$

These relations, with the representation (8) taken into account, lead to *orthogonality relations* for the denominator Q_n:

$$\oint_F Q_n(t) t^j \frac{\Phi_n(t) f(t) \, dt}{\omega_{2n+1}(t)} = 0, \qquad j = 0, 1, \dots, n-1. \tag{11}$$

A formula for the remainder can be derived similarly:

$$(f_n - R_n)(z) = \left(\frac{\omega_{2n+1}}{Q_n Q}\right)(z) \oint_F \left(\frac{Q_n Q \Phi_n}{\omega_{2n+1}}\right)(t) \frac{f(t) \, dt}{t - z}, \qquad z \in D,$$

where Q is an arbitrary polynomial of degree at most n (the variant of Hermite's formula for the interpolation problem under consideration). In particular,

$$(f_n - R_n)(z) = \left(\frac{\omega_{2n+1}}{Q_n^2}\right)(z) \oint_F \left(\frac{Q_n^2 \Phi_n}{\omega_{2n+1}}\right)(t) \frac{f(t) \, dt}{t - z}, \qquad z \in D; \tag{12}$$

here and in what follows $D = \hat{\mathbb{C}} \setminus F$.

5.2. Convergence of a sequence of functions h_n to a function h *with respect to capacity inside* D means that for any compact set $K \subset D$ and any $\varepsilon > 0$

$$\operatorname{cap}\{z \in K : |(h_n - h)(z)| > \varepsilon\} \to 0 \qquad (n \to \infty).$$

For any positive integer n we fix an arbitrary polynomial Q_n satisfying the orthogonality relations (11). Let

$$\mu(Q_n) = \sum_k \delta_{\beta_{n,k}} \qquad (k = 1, \dots, n' \le n),$$

where the sum is taken over all zeros $\beta_{n,k}$ of Q_n.

The following lemma can be proved on the basis of (11).

LEMMA 1. *Suppose that the conditions of Theorem* 1 *hold and the interpolation nodes* (*the polynomials* ω_{2n+1}) *are chosen so that* (10) *holds. Then the following assertions are true*:

(i) $(1/n)\mu(Q_n) \to \lambda_F$, *where* λ_F *is the* F-*component of the equilibrium charge* λ.

(ii) *The polynomials* Q_n *can be normalized in such a way that*

$$\left| \oint_F \left(\frac{Q_n^2 \Phi_n}{\omega_{2n+1}}\right)(t) \frac{f(t) \, dt}{t - z} \right|^{1/n} \to e^{-2w_F}$$

with respect to capacity inside D.

Lemma 1 can be proved by an appropriate modification of Stahl's method [8]; the presence of the factor Φ_n under the integral sign in (11) and (12) has the effect that the external field φ appears in the corresponding capacitor equilibrium problem. In §3 we prove a theorem with Lemma 1 as a corollary; it is in the proof of this theorem that conditions 1°–3° of Theorem 1 are used in an essential way.

Lemma 1 and formula (12) give us

LEMMA 2. *Under the conditions of Lemma* 1,

$$|f_n - R_n|^{1/n} \to \exp 2(V^\lambda - w_F) \tag{13}$$

with respect to capacity inside D.

The fact that the above convergence is with respect to capacity is essential. Although the main part of the poles of the multipoint Padé approximants R_n accumulate at the compact set F (the assertion (i) of Lemma 1 means that the poles of R_n have limit distribution characterized by the measure λ_F with support $S(\lambda_F) \subset F$), outside a fixed neighborhood of F there can be $o(n)$ poles of these rational functions (as $n \to \infty$). Convergence with respect to capacity in Lemma 2 suffices for the proof of (9).

5.3. The subsequent arguments are carried out in a fixed (arbitrarily small) neighborhood of E. Let V^λ be the potential of the equilibrium charge λ, let $G = \hat{\mathbf{C}}\backslash E$, and let $g(z,\zeta)$ be the Green's function of the region G (for $z \in E$ and $\zeta \in G$ let $g(z,\zeta) = 0$). The compact set E is regular in this situation (E is a union of finitely many continua E_j). Consequently, the potential V^λ and the Green's function (for any $\zeta \in G$) are continuous in a neighborhood of E. We can fix an arbitrarily small $\varepsilon > 0$ and then a constant $\theta_0 > 0$ so that the set where $g(z,\infty) < \theta_0$ is a compact subset of D on which

$$|V^\lambda - w_E| < \varepsilon. \tag{14}$$

We also use the notation

$$\gamma(\theta) = \{z : g(z,\infty) = \theta\}, \qquad 0 < \theta \le \theta_0,$$

$$\sigma(\theta', \theta'') = \{z : \theta' < g(z,\infty) < \theta''\}, \qquad 0 \le \theta' < \theta'' \le \theta_0.$$

Let $E = E_1 \cup \cdots \cup E_p$, where the E_j are disjoint continua. We can choose θ_0 small enough that the contour $\gamma(\theta_0)$ splits into p closed analytic curves $\gamma_j(\theta_0)$ encompassing the components E_j of E; then $\sigma(0,\theta_0)$ splits into p annular regions $\sigma_j(0,\theta_0)$. All subsequent arguments are carried out on the set where $g(z,\infty) \le \theta_0$.

The proof of the upper estimate

$$\varlimsup_{n\to\infty} \rho_n(f_n, E)^{1/n} \le e^{-2w} \tag{15}$$

is not hard. Fix any region of the form $\sigma(\theta, \theta_0)$, $\theta > 0$. It follows from Lemma 2 that the inequality

$$\varlimsup_{n\to\infty} \|f_n - R_n\|_{\gamma_n}^{1/n} \le e^{-2w+\varepsilon}, \qquad w = w_F - w_E,$$

is valid for some sequence of curves $\gamma_n = \gamma(\theta_n)$, $\theta < \theta_n < \theta_0$ (a set of sufficiently small capacity cannot intersect all the contours $\gamma(t)$, $\theta < t < \theta_0$); the lengths of the contours γ_n are uniformly bounded. For z lying inside $\gamma(\theta)$ we let

$$r_n(z) = \frac{1}{2\pi i}\int_{\gamma_n} \frac{R_n(t)\,dt}{t - z}.$$

It is clear that $r_n \in \mathscr{R}_n$ (we "removed" the poles of R_n inside γ_n). Representing f_n according to the Cauchy formula, we get that

$$(f_n - r_n)(z) = \frac{1}{2\pi i}\int_{\gamma_n} \frac{(f_n - R_n)(t)\,dt}{t - z}, \qquad z \in E,$$

and

$$\overline{\lim_{n\to\infty}} \|f_n - r_n\|_E^{1/n} \le e^{-2w+\varepsilon};$$

since $\varepsilon > 0$ is arbitrary, this implies (15).

We now prove the corresponding lower estimate

$$\underline{\lim}_{n\to\infty} \rho_n(f_n, E)^{1/n} \ge e^{-2w}. \tag{16}$$

If this inequality does not hold, then there exist an $\eta > 0$ and a sequence of rational functions $r_n = p_n/q_n$, $n \in \Lambda \subset \mathbf{N}$ (p_n and q_n are polynomials of degree at most n) such that

$$\|f_n - r_n\|_E < \exp(-(2w + 3\eta)n), \qquad n \in \Lambda. \tag{17}$$

We show that this assumption leads to a contradiction of (13).

Let us consider the sequence of functions

$$s_n(z) = \log|(f_n - r_n)(z)| - \sum_k g(z, \zeta_{n,k}),$$

where the $\zeta_{n,k}$ are the poles of the rational functions r_n, $k = 1, \dots, n' \le n$, $n \in \Lambda$ (only such n are considered below). The constant $\theta_0 > 0$ is chosen so that (14) holds for $\varepsilon = \eta$. The function s_n is subharmonic in the open set $\sigma(0, \theta_0)$. Relation (17) can be rewritten in the form

$$s_n(z) < -(2w + 3\eta)n, \qquad z \in E. \tag{18}$$

The maximum principle for subharmonic functions implies that

$$\log|r_n(z)| - \sum_k g(z, \zeta_{n,k}) \le \log\|r_n\|_E, \qquad z \in \hat{\mathbf{C}}$$

(the Bernstein-Walsh lemma for rational functions). Using this inequality, we get

$$\begin{aligned} s_n(z) - \log 2 &\le \max(\log|f_n(z)|, \log|r_n(z)|) - \sum_k g(z, \zeta_{n,k}) \\ &\le \max\left(\log|f_n(z)|, \log|r_n(z)| - \sum_k g(z, \zeta_{n,k})\right) \\ &\le \max(\log|f_n(z)|, \|r_n\|_E). \end{aligned}$$

The estimate $\|f\|_{\gamma(\theta_0)} < C_1^n$ follows from the representation (8) and condition 1° of the theorem; using (17), we arrive at the inequality

$$s_n(z) < C_2 n, \qquad z \in \gamma(\theta_0). \tag{19}$$

Let $\omega(z)$, $z \in \sigma_0$, be the harmonic measure of ∂E with respect to $\sigma_0 = \sigma(0, \theta_0)$. The inequality

$$s_n(z) < -(2w + 3\eta)n\omega(z) + C_2 n(1 - \omega(z)), \qquad z \in \sigma_0,$$

follows from (18), (19), and the two constants theorem for subharmonic functions. This gives us that if $\theta'' \in (0, \theta_0)$ is sufficiently small, then

$$s_n(z) < -2(w + \eta)n, \qquad z \in \sigma(0, \theta'').$$

Fix an arbitrary θ'' such that the last inequality holds; we rewrite it in the form

$$|(f_n - r_n)(z)| < \exp\left(- 2(w + \eta)n + \sum_k g(z, \zeta_{n,k})\right), \qquad z \in \sigma(0, \theta''). \tag{20}$$

We now pass to the choice of the parameter $\theta' \in (0, \theta'')$; the choice of this parameter is based on the following lemma. Recall that all considerations are in a neighborhood of E in which the level curves $\gamma(\theta)$, $0 < \theta < \theta_0$, decompose into closed Jordan curves encompassing the components E_j of the compact set E: $\gamma(\theta) = \gamma_1(\theta) \cup \cdots \cup \gamma_p(\theta)$. Let $\sigma_j(\theta, \theta'')$, $j = 1, \dots, p$, denote the annular regions making up $\sigma(\theta, \theta'')$, and let

$$\tilde{\gamma} = \tilde{\gamma}(\theta_1, \dots, \theta_p) = \bigcup_{j=1}^{p} \gamma_j(\theta_j). \tag{21}$$

LEMMA 3. *Let $C > 0$ be an arbitrary (arbitrarily large) number. There exists a $\theta' \in (0, \theta'')$ such that* $\operatorname{cap}(E, K) > C$ *for any compact set K intersecting each contour of the form* (21) *with $\theta_j \in [\theta', \theta'']$, $j = 1, \dots, p$.*

We prove this lemma. Fix an arbitrary $j \in \{1, \dots, p\}$; let $G(z, \infty)$ be the complex Green's function for the complement of E, and let $m_j i$ be the increment of $G(z, \infty)$ on going around E_j in the positive direction. The function

$$\zeta_j(z) = \exp\left(\frac{2\pi}{m_j} G(z, \infty)\right)$$

maps the annular region $\sigma_j(0, \theta'')$ conformally onto the circular annulus $1 < |\zeta| < a_j$ in such a way that the curves $\gamma_j(\theta)$ are carried into the circles $\gamma_j^*(\theta)\colon |\zeta| = a_j(\theta)$. As $\theta \to 0$ we have that $a_j(\theta) \to 1$ and

$$\operatorname{cap}(\overline{U}, [a_j(\theta), a_j]) \to +\infty, \tag{22}$$

where $\overline{U}$ is the closed unit disk. Since all the a_j, $j = 1, \dots, p$, are fixed, it follows from (22) that there exists a $\theta' \in (0, \theta'')$ with

$$\operatorname{cap}(\overline{U}, [a_j(\theta'), a_j]) > C \tag{23}$$

for any $j \in \{1, \dots, p\}$. We show that this θ' works.

If the compact set K satisfies the condition of the lemma, then there exists a $j \in \{1, \dots, p\}$ such that K intersects each curve $\gamma_j(\theta)$, $\theta \in [\theta', \theta'']$. Then the image K_j of the compact set $K_j = K \cap \overline{\sigma}_j(\theta', \theta'')$ intersects each circle $\gamma_j^*(\theta)$. Using the invariance of the capacity of a capacitor under conformal mappings and the symmetrization lemma of Grötzsch, we get that

$$\operatorname{cap}(E_j, K_j) \geq \operatorname{cap}(\overline{U}, [a_j(\theta'), a_j]) > C$$

(see (23)). All the more so, $\operatorname{cap}(E, K) > C$.

The next lemma is easy to get from the definition of the capacity of a capacitor (see, for example, [14]).

LEMMA 4. *If $F_t = \left\{z\colon \sum_k g(z, \zeta_{n,k}) \geq nt\right\}$, $t > 0$, then* $\operatorname{cap}(E, F_t) \leq 1/t$.

We now finish the proof of (16). Fix a $\theta' \in (0, \theta'')$ such that Lemma 3 holds with $C = 2/\eta$ ($\eta \in (0, 1)$ is the constant in (17)). By Lemma 4,

$$\sum_k g(z, \zeta_{n,k}) < n\eta, \qquad z \in \overline{\sigma} \backslash A_n, \tag{24}$$

where $\overline{\sigma}$ is the closure of $\sigma(\theta', \theta'')$ and $\operatorname{cap}(E, A_n) \leq 1/\eta$. Since $\overline{\sigma}$ is a compact set in G, convergence with respect to logarithmic capacity is equivalent on $\overline{\sigma}$ to convergence with respect to Green capacity relative to G, and thereby with respect to capacitor

$(E,\cdot)$-capacity. With this remark taken into account, (13) and (14) with $\varepsilon = \eta$ give us

$$|(f_n - R_n)(z)| \geq e^{-(2w+\eta)n}, \qquad z \in \overline{\sigma}\backslash B_n, \tag{25}$$

and $\operatorname{cap}(E, B_n) < 1$. By the subadditivity property of the capacity, $\operatorname{cap}(E, A_n \cup B_n) < 2/\eta$. Lemma 3 now implies that the compact set $A_n \cup B_n$ cannot intersect all the contours of the form (21) lying in $\overline{\sigma}$. Let $\tilde{\gamma}_n$ be a contour of the form (21) belonging to $\overline{\sigma}\backslash(A_n \cup B_n)$. It follows from (20), (24), and (25) that the inequalities

$$\min_{z\in\tilde{\gamma}_n} |(f_n - R_n)(z)| \geq e^{-(2w+\eta)n}, \tag{26}$$

$$\max_{z\in\tilde{\gamma}_n} |(f_n - r_n)(z)| < e^{-(2w+\eta)n} \tag{27}$$

hold simultaneously on this contour.

Suppose that the rational function R_n interpolates the function f_n at $2n + 1 - d_n$ zeros of the polynomial ω_{2n+1}, and $R_n = P_n^*/Q_n^*$ is an irreducible representation of R_n. We write $R_n - r_n$ in the form

$$R_n - r_n = \frac{P_n^* q_n - p_n Q_n^*}{Q_n^* q_n}.$$

It follows from (26), (27), and Rouché's theorem that the polynomial

$$P_n^* q_n - p_n Q_n^*$$

has at least $2n + 1 - d_n$ zeros; at the same time, the degree of this polynomial does not exceed $2n - d_n$ (see subsection 5.1). Consequently, $r_n \equiv R_n$, which contradicts (26) and (27). This proves (16), and thereby (9).

6. The main goal of this article is to discuss some results on rational approximation connected with equilibrium problems when there is an external field. In this subsection we dwell briefly on the case when all the Φ_n in Theorem 1 are $\equiv 1$, and thus $\varphi \equiv 0$. For $\varphi \equiv 0$ the equilibrium problem in subsection 1 is the classical problem of equilibrium of the distribution of a charge on the plates of a capacitor, λ is the corresponding equilibrium charge, and $w = 1/I_0(\lambda)$ is the *modulus* of the capacitor (E, F); $w = 1/c$, where $c = \operatorname{cap}(E, F)$ is the *capacity* of (E, F). The S-property $(E, F, 0) \in S$ in this case will be written in the form $(E, \underline{F}) \in S$ (this underscores that the plate F of the capacitor (E, F) has the symmetry). The definition of the S-property in this case becomes essentially simpler; $(E, \underline{F}) \in S$ *if* F *is a tame compact set, and*

$$\frac{\partial V^\lambda}{\partial n_+}(\zeta) = \frac{\partial V^\lambda}{\partial n_-}(\zeta), \qquad \zeta \in F_0$$

(*for all* $\zeta \in F_0$). We single out a theorem that is an immediate consequence of Theorem 1 for the case under consideration.

THEOREM 1′. *Suppose that* E *is a union of finitely many capacitors in* $\hat{\mathbf{C}}$, *and* $(E, F) \in S$. *Then for any function* $f \in \mathscr{H}_0(\hat{\mathbf{C}}\backslash F)$

$$\lim_{n\to\infty} \rho_n(f, E)^{1/n} = e^{-2/c}, \tag{28}$$

where $c = \operatorname{cap}(E, F)$ *is the capacity of the capacitor* (E, F).

Theorem 1′ reduces the question of rational approximations on E to the question of the construction of compact sets F having the S-property in the form $(E, \underline{F}) \in S$ (or, what is the same, the construction of open sets D such that $(E, \underline{\partial D}) \in S$). It is well known that many extremal problems in potential theory and geometric function theory (diverse variants of the *modulus problem*) lead to compact sets having a symmetry property of the type considered; the corresponding problems for the cases when E degenerates into a point or a finite set have been especially thoroughly investigated (see [15]–[17], where there is an extensive bibliography). These problems have the following character in the case of interest to us. We fix a compact set $E \subset \hat{\mathbb{C}}$ and a class $\mathscr{F} = \{F\}$ characterized by various conditions on the compact sets F. In this class we pose the problem of the *maximum of the energy* $I_0(\lambda)$ of the equilibrium charge λ, or, what is the same, the problem of the maximum of the modulus w; in papers on rational approximations this problem usually appears as the problem of *minimum capacity* in the class $\mathscr{F}$. It has been proved in a number of important cases that there exists a unique compact set $F^* \in \mathscr{F}$ such that

$$\operatorname{cap}(E, F^*) = \inf\{\operatorname{cap}(E, F) : F \in \mathscr{F}\}, \tag{29}$$

and this extremal compact set has the symmetry property $(E, \underline{F^*}) \in S$. As a rule, this property proves to be a consequence of the fact that for an extremal compact set F^* the set F_0^* consists of trajectories of a quadratic differential connected with the problem under consideration. Let us dwell on the basic examples.

The Chebotarev problem. E is a continuum with connected complement, $A = \{a_1, \dots, a_N\}$ is a finite set of points lying in the complement of E, and $\mathscr{F} = \mathscr{F}(A)$ is the set of all continua $F \subset \hat{\mathbb{C}} \backslash E$ containing A. In the case when the continuum E degenerates into the point $z_0 = \infty$ (V^λ is the equilibrium potential of the continuum F, and $c(\infty, F)$ is its capacity) the problem of a minimum of $c(\infty, F)$ in the class $\mathscr{F}(A)$ is known as *the Chebotarev problem on a continuum of minimal capacity.* In the case of an arbitrary continuum E the problem (29) with $\mathscr{F} = \mathscr{F}(A)$ reduces to the Chebotarev problem for the hyperbolic metric (the case $E\colon |z| \geq 1$). A more general statement is the following: $A = A_1 \cup \dots \cup A_p$, where the A_j are disjoint finite sets (consisting of more than one point), and the class $\mathscr{F} = \mathscr{F}(\{A_j\})$ consists of compact sets F that are unions of finitely many continua, with each of the sets A_j belonging to one of the connected components of F. A degenerate variant of this problem was considered by Nuttall [18] in connection with applications to diagonal Padé approximants. In all the cases indicated an extremal compact set F^* in the problem (29) is a union of finitely many analytic arcs, and $(E, \underline{F^*}) \in S$.

Lavrent′ev's problem. $E = E_1 \cup \dots \cup E_p$, where the E_j are disjoint continua and $\mathscr{F} = \mathscr{F}_E$ is the class of all compact sets $F \subset \hat{\mathbb{C}} \backslash E$ consisting of finitely many continua and separating the components E_j of E. In the degenerate case (the continua $E_1, \dots, E_p$ degenerate into points $z_1, \dots, z_p$, V^λ is the equilibrium potential in the field $\psi(z) = (1/p) \log |(z - z_1) \cdots (z - z_p)|$, and $\operatorname{cap}(\{z_j\}, F)$ is the corresponding capacity; see §3) this problem reduces to the known *Lavrent′ev problem on nonoverlapping regions.* A more general problem of the type (29) is connected with the combination of continua E_j into groups that are separated by the components of the compact sets $F \in \mathscr{F}$. Note that for an extremal compact set F^* in the Lavrent′ev problem all the capacities $\operatorname{cap}(E_j, F)$ coincide; and in this case F^* is a union of finitely many analytic arcs, and $(E, \underline{F^*}) \in S$. We mention a corollary to Theorem 1′ connected with the last problem.

Let $E = E_1 \cup \dots \cup E_p$, *where the* E_j *are disjoint continua in* $\hat{\mathbb{C}}$ *and* f *is a piecewise constant function on* E *taking various (constant) values on the continua* E_j, $j = 1, \dots, p$. *Then*

$$\lim_{n \to \infty} \rho_n(f, E)^{1/n} = e^{-2/c^*}, \tag{30}$$

where $c^* = \operatorname{cap}(E, F^*)$ *is a solution of problem* (29) *for* $\mathscr{F} = \mathscr{F}_E$.

The case $p = 2$ corresponds to the classical Zolotarev problem (see [19]; Zolotarev found the precise values of $\rho_n(\operatorname{sgn} x, E)$, $E = [-1, -k] \cup [k, 1]$ for any $n \in \mathbf{N}$).

Multivalued analytic functions. E is a continuum with connected complement, A is a compact set of zero capacity lying in $\hat{\mathbf{C}} \backslash E$, f is a function holomorphic on E and admitting analytic extension along any path $\hat{\mathbf{C}} \backslash A$ (it is assumed that $f \notin \mathscr{H}(\hat{\mathbf{C}} \backslash A)$, i.e., the corresponding complete analytic function is multivalued), and $\mathscr{F}_f$ is the class of all compact sets F such that f admits a holomorphic (single-valued analytic) extension to $\hat{\mathbf{C}} \backslash F$. The problem of convergence of the diagonal Padé approximants for analytic functions with finitely many branch points reduces to this problem for the case when E degenerates into a point $z_0 = \infty$ and A is a finite set. The problem (29) in the class $\mathscr{F}_f$ and the corresponding problems of convergence of Padé approximants were solved by Stahl in [8] and [20] both for the local case and for the case of a continuum E; he proved that *there exists a unique* (*minimal with respect to inclusion*) *extremal compact set* F^* *in problem* (29) *with* $\mathscr{F} = \mathscr{F}_f$, *and, moreover,* $(E, \underline{F^*}) \in S$ *and the Padé approximants converge to* f *with respect to capacity inside* $\hat{\mathbf{C}} \backslash F^*$ (cf. Lemmas 1 and 2 for $\varphi \equiv 0$). For the f under consideration a formula of type (30) is an immediate consequence of these results and Theorem 1′; cf. [21].

A more general problem of this type can be posed for the case when E is a union of disjoint continua E_j, $j = 1, \dots, p$, f is holomorphic on E, and each element $f_j = f|_{E_j}$ admits an extension along paths lying in $\hat{\mathbf{C}} \backslash A$ (it is required that $f \notin \mathscr{H}(\hat{\mathbf{C}} \backslash A)$; in other respects the class $\mathscr{F}_f$ is defined as above. The conjecture that

$$\lim_{n \to \infty} \rho_n(f, E)^{1/n} = e^{-2/c^*}, \qquad c^* = \inf\{c(E, F) : F \in \mathscr{F}_f\},$$

was discussed in [12] in this general setting; as follows from the foregoing, it is now solved for the particular cases singled out in [12]. We do not know whether problem (29) has been investigated in the general class $\mathscr{F}_f$; this formulation clearly includes all the problems considered above.

The general construction of compact sets F having symmetries of the type $(E, \underline{F}) \in S$ can be described as follows. We fix a compact set $E \subset \hat{\mathbf{C}}$ that is a union of disjoint continua E_j, $j = 1, \dots, p$, and a compact set $A \subset \hat{\mathbf{C}} \backslash E$ of zero capacity, and consider an arbitrary Riemann surface $\mathscr{R}$ that is a two-sheeted (unramified) covering of the region $\hat{\mathbf{C}} \backslash A$ on the Riemann sphere $\hat{\mathbf{C}}$; π is the corresponding projection. Let E^1 and E^2 be disjoint compact sets on $\mathscr{R}$ such that $\pi(E^j) = E$ and $W = 2\omega - 1$, where $\omega(\tilde{z})$, $\tilde{z} \in \Omega \doteq \mathscr{R} \backslash (E^1 \cup E^2)$, is the harmonic measure of ∂E^1 with respect to Ω. Then the compact set

$$F = \pi\{\tilde{z} \in \mathscr{R} : W(\tilde{z}) = 0\}$$

has the property $(E, \underline{F}) \in S$.

The S-compact sets to which this construction leads apparently include a solution of the problem mentioned above in connection with multivalued functions.

7. The condition imposed on the jump of f on F_0 in Theorem 1′ can be weakened; however, relation (28) no longer holds for arbitrary $f \in \mathscr{H}(\hat{\mathbf{C}} \backslash F)$. Moreover, in the class of all $f \in \mathscr{H}(\hat{\mathbf{C}} \backslash F)$ it is not possible to strengthen the universal *Walsh inequality*:

$$\varlimsup_{n \to \infty} \rho_n(f, E)^{1/n} \le e^{-1/c}, \qquad c = \operatorname{cap}(E, F). \tag{31}$$

This inequality is valid for any compact set E and any open set $D = \hat{\mathbf{C}} \backslash F$ ($f \in \mathscr{H}(D)$); it follows from results of Walsh [22] and Bagby [23] relating to interpolation of rational functions with *fixed* poles.

The first of the authors of the present article has conjectured more than once that the inequality

$$\underline{\lim}_{n\to\infty} \rho_n(f,E)^{1/n} \le e^{-2/c}, \qquad c = \operatorname{cap}(E,F), \tag{32}$$

has just as general a character (in particular, see [24]). This conjecture was recently proved by Parfenov for the case when E is a *continuum with connected complement* [25]. The proof is based on the theory of Hankel operators and the Adamyan-Arov-Kreĭn theorem, which permits a characterization of the best rational approximations

$$r_n(f) = \operatorname{dist}_{\mathrm{BMO}}(f,\mathscr{R}_n)$$

of a function $f \in H^\infty$ ($|z| < 1$) as the singular numbers of the Hankel operator corresponding to $f(1/z) - f(0)$ (the infinite Hankel matrix (c_{k+j-1}), $k, j = 1, 2, \dots$, where the c_n are the Taylor coefficients of f at the point $z = 0$); see [26]–[28], where connections with the classical interpolation problems of Carathéodory-Fejér type are also discussed. The inequality

$$\overline{\lim_{n\to\infty}} (r_1(f)\cdots r_n(f))^{1/n^2} \le e^{-1/c} \tag{33}$$

was obtained in [25] for $f \in \mathscr{H}(D)$, where D is an arbitrary region containing the closed unit disk E; in essence, this inequality turns out to be a consequence of general results of Weyl and Horn on singular numbers for products of nonselfadjoint linear operators:

$$\prod_{j=1}^{n} s_j(A\cdot B) \le \prod_{j=1}^{n} s_j(A)s_j(B) \tag{34}$$

(see [29]) and of the Walsh inequality (31). The estimate

$$\underline{\lim}_{n\to\infty} r_n(f)^{1/n} \le e^{-2/c} \tag{35}$$

now follows from (33); it is not hard to pass to the uniform metric and to an arbitrary continuum with connected complement.

See [30] and [31] (also §2.1 below) about other applications of the theory of Hankel operators to questions of rational approximation of analytic functions.

Unfortunately, the proof of (33) and (35) based on the Weyl-Horn inequality (34) is nonconstructive. It would be interesting to get a proof of these inequalities based on *constructive* methods of rational approximation of analytic functions. For example, for the simplest case $E\colon |z| \le 1$ and $D\colon |z| < 1$ inequality (32) would follow from the positive solution of the following conjecture, which is well known in the theory of Padé approximants: *If f is holomorphic in the disk $D\colon |z| < R$, then there exists a subsequence $[n/n]_f$, $n \in \Lambda \subset \mathbf{N}$, of its diagonal Padé approximants that converges to f with respect to capacity inside this disk* (the essentially stronger conjecture that there is a *uniformly* convergent subsequence is usually formulated). An analogous question can be formulated for multipoint Padé approximants (cf. subsection 5.1). We remark that the question of the validity of (32) for *arbitrary* E and $f \in \mathscr{H}(D)$ remains open (it suffices to consider the case when E and $F = \hat{\mathbf{C}}\setminus D$ are unions of finitely many continua).

§2. The degree of rational approximation of the exponential on the semi-axis

1. In this section we consider an example of an application of Theorem 1 he case when there is a nontrivial external field φ. We are concerned with the known problem of the degree of rational approximation of the function e^{-x} on the semi-axis $[0, +\infty)$.

Let $\rho_n = \rho_n(e^{-x}, E)$; here and everywhere in this section $E = [0, +\infty]$ (the closure of $[0, +\infty)$ in $\hat{\mathbf{C}}$). Note that the function e^{-z} has an essential singular point on E.

There is a thorough survey of results of the type

$$0 < c_1 \le \varliminf_{n\to\infty} \rho_n^{1/n}, \qquad \varlimsup_{n\to\infty} \rho_n^{1/n} \le c_2 < 1$$

and of conjectures connected with the (existence and the) value of the limit

$$v = \lim_{n\to\infty} \rho_n^{1/n} \tag{1}$$

in Varga's publications [32] and [33]. The problem arose originally in connection with numerical methods for solving the heat equation (see [34] about this). In [35] Cody, Meinardus, and Varga showed that the relation $\varlimsup \rho_{0,n}^{1/n} < 1/2.298$ holds already for the best approximations $\rho_{0,n} = \rho_{0,n}(e^{-x}, E)$ by means of rational functions of the form $1/q_n$, $\deg q_n \le n$, and thus the sequence $\rho_n = \rho_{n,n} \le \rho_{0,n}$ tends to zero geometrically. Schönhage [36] obtained estimates for $\rho_{0,n}$ which imply the relation $\lim \rho_{0,n}^{1/n} = 1/3$. Apparently on the basis of this result and the first numerical results for ρ_n, $1 \le n \le 14$, Saff and Varga conjectured that the limit (1) exists and $v = 1/9$ (see [37]; it should be mentioned that these first—not very accurate—numerical results showed that $v \simeq 1/9.3$).

Without dwelling on the many papers connected with the "1/9 conjecture", we single out the most interesting results of recent years. Approximate values were obtained for the constant v (the question of the existence of the corresponding limit was left open) by different methods of *numerical* analysis of the problem in papers of Trefethen and Gutknecht [38] and of Carpenter, Ruttan, and Varga [39]. The Carathéodory-Fejér method was used in [38]: the authors pass to the function $\exp((x-1)/(x+1))$ on $[-1, 1]$, expand it in a Fourier series of Tchebycheff polynomials, and find approximate values for the singular numbers of the Hankel matrix (c_{j+k-1}), $j, k = 1, 2, \dots$, where the c_n are the coefficients of the indicated series (cf. §1.7; it is clear that the best BMO-approximations of the corresponding power series are being computed). The value

$$v \simeq 1/9.28903$$

was obtained in this way. Very precise computations were used in [39] (the Remez algorithm was used, and the results were processed with the help of Richardson extrapolation) to get the value

$$v \simeq 1/9.289\,025\,491\,920\,81$$

and the coefficients of the numerators and denominators of the rational functions of best approximation were computed approximately for $n \le 30$. Of course, these results had the nature of conjectures (which have now been confirmed); however, numerically they convincingly refuted the 1/9 conjecture. This was formally accomplished in the paper [40] of Opitz and Scherer, where the inequality $c_2 < 1/9.037$ was proved. We also mention a very interesting (with regard to the method used) article of Anderson [41], in which the (very special) problem of an optimal choice of $q \in (-\infty, 0)$ in the approximation of e^{-x} on E by rational functions of the form $p_n(x)/(x - qn)^n$ is investigated; it turns out that $q = 1/\sqrt{2}$ is optimal, and the deviations have order $(1+\sqrt{2})^{-n}$. Our approach to the solution of the general problem has certain points of contact with the method in that paper.

The problem was solved in 1986 by Magnus [42] (see also [33]); by starting from the Carathéodory-Fejér method and analyzing the asymptotic behavior of the singular numbers s_n, he found the correct answer, namely,

$$v = \exp(-\pi K'/K),$$

where K and K' are complete elliptic integrals of first order for the moduli k and $k' = \sqrt{1-k^2}$, and the modulus k is found from the equation

$$K(k) = 2E(k).$$

He used this characterization of the constant v to compute it to thirty significant digits. However, the arguments of Magnus have a heuristic nature at essential points; the result is formulated as the *Magnus conjecture* in Varga's survey [33]. In particular, in passing from n-fold integrals to limit distributions (as $n \to \infty$) Magnus uses a technique of Nuttall connected with the saddle point method (see [7]); this method has not yet been justified for the "complex" case.

We solved the problem on the basis of a construction of multipoint Padé approximants. With the help of Theorem 1 it is possible to *prove* that the limit (1) exists, and to describe the value of v in terms connected with the equilibrium problem for a capacitor of the form (E, F) under the condition that the plate F is in the external field $\varphi(z) = \frac{1}{2}\operatorname{Re} z$, and $(E, F, \varphi) \in S$ (recall that $E = [0, +\infty]$). This equilibrium problem admits an explicit solution with the help of elliptic functions and integrals. Diverse (in form) characterizations of the constant v are possible; in the following theorem we give an answer in the form that seems to us most interesting (other forms of the answer are discussed below).

THEOREM 2. *The limit $v = \lim \rho_n^{1/n}$ exists, and v is the unique positive root of the equation*

$$\sum_{n=1}^{\infty} a_n v^n = 1/8, \quad \textit{where } a_n = \left|\sum_{d|n} (-1)^d d\right|. \tag{2}$$

It is not hard to compute v on the basis of (2).

Before passing to the proof of the theorem we conclude the discussion (it has an interesting ending). The standard notation of the theory of elliptic functions in [43] will be employed below; all the elliptic function formulas used below are in this book (see subsection 4 for more detail). We get (2) from the formula ([43], §20):

$$e_1 = -\frac{\eta}{\omega} + \left(\frac{\pi}{\omega}\right)^2 \left(\frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{h^{2n}}{(1+h^{2n})^2}\right), \qquad h = e^{\pi i \omega'/\omega}. \tag{3}$$

This potential-theoretic equilibrium problem leads to the following value for $v = \exp(-2w)$: $v = -h^2$, where h is such that $e_1 = -\eta/\omega$. It follows from (3) that v is the (positive) root of the equation

$$H(v) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} v^n}{(1+(-v)^n)^2} = \frac{1}{8}. \tag{4}$$

The series on the left-hand side of (2) is the power series expansion of the function $H(v)$. Instead of (3) it is possible to use a formula expressing e_1 in terms of the theta function ϑ_1 and its derivatives; differentiating the second formula in §20 of [43], we get (for $u = \omega$) that

$$e_1 = -\frac{\eta}{\omega} - \frac{1}{\omega}\left(\frac{\vartheta_1'(t)}{\vartheta_1(t)}\right)'\bigg|_{t=1/2} \qquad (t = u/2\omega).$$

With the foregoing taken into account, this implies that h satisfies $\vartheta_1''(h) = 0$, and thus

$$\sum_{n=0}^{\infty}(2n+1)^2(-v)^{n(n+1)/2} = 0. \tag{5}$$

We announced Theorem 2 in reports at the International Congress of Mathematicians in Berkeley (August 1986) and at a conference in Segovia (September 1986). At the end of the same year we received a letter from Magnus which discussed, in particular, our approach to the solution of the problem and various equations for v. He had discovered that equation (5), as well as the equation

$$\sum_{n=1}^{\infty}\frac{nv^n}{1-(-v)^n} = \frac{1}{8},$$

which is equivalent to (2) and (4), are in Halphen's book [44] (1886!). Halphen arrived at (5) (and computed the value of v to six significant digits) in connection with a study of variations of the theta functions. It is interesting that the problem of the degree of rational approximation of the exponential on the semi-axis proved to be connected with this problem, and the *Halphen constant* gives its solution.

The following subsections 2–4 are devoted to a proof of Theorem 2.

2. The reduction of Theorem 2 to Theorem 1 (more precisely, to the corresponding potential-theoretic problem) is based on the fact that the quantity $\rho_n = \rho_n(e^{-x}, E)$ is invariant under linear transformations of the form $x \to cx$, where c is an arbitrary positive number; in particular,

$$\rho_n = \rho_n(e^{-x}, E) = \rho_n(e^{-nx}, E). \tag{6}$$

We fix a point of the form $b = 3 + i\beta$, $\beta < 0$ (the parameter β will be given later). Let F be an arbitrary rectifiable arc lying in the intersection of the region $G = \mathbf{C}\backslash E$ with the half-plane $\operatorname{Re} z \le 3$ and joining the points b and $\overline{b}$; the class of all such arcs is denoted by $\mathscr{F}$. Further, let γ be an unbounded contour consisting of F and (for example) two rays parallel to the real line $\mathbf{R}$ and joining b and $\overline{b}$ with the point at infinity (the rays lie in the half-plane $\operatorname{Re} z \ge 3$). For a suitable orientation of γ we have that

$$e^{-nx} = \frac{1}{2\pi i}\int_\gamma \frac{e^{-nt}\,dt}{t-x} = f_n(x) + \Delta_n(x), \qquad x \in E,$$

where

$$f_n(x) = \frac{1}{2\pi i}\int_F \frac{e^{-nt}\,dt}{t-x}, \qquad \Delta_n(x) = \frac{1}{2\pi i}\int_{\gamma\backslash F} \frac{e^{-nt}\,dt}{t-x}. \tag{7}$$

Obviously,

$$\varlimsup_{n\to\infty} \|\Delta_n\|_E^{1/n} \le e^{-3}.$$

By (6), this implies that *the assertion of Theorem 2 is equivalent to the existence of the limit*

$$\lim_{n\to\infty} \rho_n(f_n, E)^{1/n} = v,$$

where v is the solution of (2) (the fact is used that $e^{-3} < v$, where v is the root of (2); the choice of the number 3 is connected only with this inequality).

To use Theorem 1 we must construct, for a given E and the given external field $\varphi(z) = \frac{1}{2}\operatorname{Re} z$, an $F^* \in \mathscr{F}$ such that

$$(E, F^*, \varphi) \in S, \tag{8}$$

and find the corresponding constant $v = \exp(-2w)$, $w = w(E, F^*, \varphi)$ (more precisely, we must prove that this constant is the root of (2)). We formulate the assertions and formulas connected with the solution of this problem in Lemma 5 below. Before proceeding to the lemma we make some remarks.

Condition (8) does not determine the contour $F^* \in \mathscr{F}$ uniquely. The equilibrium charge λ is uniquely determined; hence so is the arc $\Gamma = S(\lambda_{F^*})$; the endpoints a and $\overline{a}$ of this arc are the basic parameters of the problem (clearly Γ is symmetric with respect to $\mathbf{R}$). The remaining part of F^* can be chosen arbitrarily in the region $\{z: (V^\lambda + \varphi)(z) > w\}$. We use the segments $[a, b]$ and $[\overline{a}, \overline{b}]$, setting $\beta = \operatorname{Im} a$; the contour F^* is thereby uniquely associated with the point a.

Assume that problem (8) has been solved, λ is the corresponding equilibrium charge, $\Gamma = S(\lambda_{F^*})$, and a and $\overline{a}$ are the endpoints of Γ. The complex logarithmic potential of the charge λ will be denoted by $\mathscr{V}^\lambda$:

$$\mathscr{V}^\lambda(z) = V^\lambda(z) + i\tilde{V}^\lambda(z) = \int \log \frac{1}{t-z} \, d\lambda(t).$$

The potential $\mathscr{V}^\lambda$ is a multivalued analytic function on $G \backslash \Gamma$ with single-valued real part V^λ and single-valued (holomorphic) derivative

$$\frac{d\mathscr{V}^\lambda(z)}{dz} = \int \frac{d\lambda(t)}{t-z}, \qquad z \in G\backslash\Gamma. \tag{9}$$

The complex field corresponding to the field φ under consideration is $z/2$. By taking into account all the given information (the equilibrium, the S-property and the symmetry with respect to $\mathbf{R}$, the linearity of the field) as well as the possible behavior of the density of the equilibrium charge near the endpoints of the plates E and Γ, it is not hard to conclude that the second derivative of the complex potential of λ must have the representation

$$\frac{d^2\mathscr{V}^\lambda(z)}{dz^2} = \frac{C}{\sqrt{z^3(z-a)(z-\overline{a})}},$$

where C is a constant. Further, the first derivative

$$g(z) = C \int_\infty^z \frac{dt}{\sqrt{t^3(t-z)(t-\overline{a})}}, \qquad z \in G\backslash\Gamma,$$

must be a single-valued analytic function in the region $G\backslash\Gamma$. This is the key condition of the problem; it is equivalent to the relation

$$\int_a^{\overline{a}} \frac{dt}{\sqrt{t^3(t-a)(t-\overline{a})}} = 0$$

(the integral is over any arc in $G = \mathbf{C}\backslash E$). From this, passing to elliptic functions, we can at once arrive at equation (4), and hence (2) (without finding the rest of the problem parameters). It turns out that the solution v of these equations coincides with the quantity $\exp(-2w)$ of interest to us. By using the other conditions of the problem (λ_E and λ_{F^*} are unit measures, and the coefficient of the field is $1/2$) we can find its complete solution explicitly. It remains to verify that the explicit formulas really do give a solution of problem (8). In essence, this scheme is realized in Lemma 5.

3. As above, let $G = \mathbf{C}\backslash E$. We set $P(z, a) = 4z(z-a)(z-\overline{a})$ for any $a \in G$ with $\operatorname{Im} a < 0$. The branch of the square root $\sqrt{P(z,a)}$ in any region of the form $G\backslash L$

(L an arc lying in G and joining the points a and $\overline{a}$) is chosen so that $\sqrt{P(z,a)} > 0$ on the upper side E^+ of the cut E. Let

$$\omega(a) = \int_0^{+\infty} \frac{dt}{\sqrt{P(t,a)}} \quad (>0),$$

$$\omega'(a) = \int_\infty^a \frac{dt}{\sqrt{P(t,a)}} \qquad (\operatorname{Im}\omega'(a) > 0)$$

(the second integral is taken, for example, along an arc parallel to $\mathbf{R}$).

LEMMA 5. (i) *There exists a unique a with $\operatorname{Im} a < 0$ such that*

$$\frac{\pi i}{\omega(a)} \int_\infty^a \frac{dt}{t\sqrt{P(t,a)}} = -\frac{1}{2}. \tag{10}$$

Let $v \in (0,1)$ be the solution of (2); then the indicated a can be found from the formula

$$a = -4\vartheta_0^{-4}, \quad \vartheta_0 = 1 + 2\sum_{n=1}^{\infty}(-h)^{n^2}, \quad h = i\sqrt{v}.$$

All the remaining assertions of the lemma are formulated for this a; for this choice of a let $P(z,a) = P(z)$, $\omega(a) = \omega$, and $\omega'(a) = \omega'$.

(ii) *The function*

$$R(z) = \left(\frac{\pi i}{\omega}\int_a^z \frac{dt}{t\sqrt{P(t)}}\right)^2$$

is holomorphic in G, has simple zeros at the points a and $\overline{a}$, and does not have other zeros in G. Among the trajectories of the quadratic differential $-R(z)(dz)^2$ there is one that is compact in G and joins a and $\overline{a}$:

$$\Gamma_0\colon R(z)(dz)^2 < 0, \qquad a, \overline{a} \in \overline{\Gamma}_0 = \Gamma \subset G.$$

(iii) *Let $F^* = \Gamma \cup \{x + iy\colon \operatorname{Re} a \le x \le 3,\ y = \pm\operatorname{Im} a\}$. Then*

$$(E, F^*, \varphi) \in S, \qquad \varphi(z) = \tfrac{1}{2}\operatorname{Re} z;$$

the equilibrium charge $\lambda = \lambda_{F^*} - \lambda_E$ of the capacitor (E, F^*, φ) is determined by the relations

$$d\lambda_E(x) = \frac{1}{\omega}\int_x^\infty \frac{dt}{t\sqrt{P(t)}}dx, \qquad x \in E,$$

$$d\lambda_{F^*}(z) = \frac{1}{\omega}\int_a^z \frac{dt}{t\sqrt{P(t)}}dz, \qquad z \in S(\lambda_{F^*}) = \Gamma.$$

The equilibrium potential is found from the formula

$$V^\lambda(z) = \operatorname{Re}\left(\frac{\pi i}{\omega}\int_\infty^z dt \int_\infty^t \frac{d\tau}{\tau\sqrt{P(\tau)}}\right), \qquad z \in G\backslash\Gamma.$$

Finally, for $w = w(E, F^*, \varphi)$

$$w = w_{F^*} = -\operatorname{Re}\left(\frac{\pi i\omega'}{\omega}\right) = -\pi i\left(\frac{\omega'}{\omega} - \frac{1}{2}\right)$$

and

$$e^{-2w} = -h^2 = v, \qquad h = e^{\pi i\omega'/\omega}$$

(v is the root of (2)).

4. PROOF OF THE LEMMA. (i) Let $\gamma \subset G$ be a contour symmetric with respect to $\mathbf{R}$ and joining the points $(\infty, a, \overline{a}, \infty)$ (for example, as in subsection 1). Using the symmetry of γ and the fact that $\sqrt{P(\overline{z})} = -\overline{\sqrt{P(z)}}$, $z \in G\setminus L$ (L is the part of γ joining a and $\overline{a}$), we get that

$$0 = \int_\gamma \frac{dt}{t\sqrt{P(t,a)}} = \int_a^{\overline{a}} \frac{dt}{t\sqrt{P(t,a)}} + 2\operatorname{Re}\int_\infty^a \frac{dt}{t\sqrt{P(t,a)}}. \tag{11}$$

Note that, in connection with the integral of $P(t,a)^{-1/2}$, the same arguments lead to the equality

$$\omega(a) = 2\operatorname{Re}\omega'(a). \tag{12}$$

It follows from (11) that the relation

$$\int_a^{\overline{a}} \frac{dt}{t\sqrt{P(t,a)}} = 0 \tag{13}$$

is equivalent to the left-hand side of (10) being a real number; in particular, (13) *follows from* (10).

We show first that (13) *holds for precisely those a such that the quantity*

$$v = -h^2, \quad h = e^{\pi i\omega'/\omega}, \quad \omega = \omega(a), \quad \omega' = \omega'(a),$$

is the root of (2) (since $\operatorname{Im}\omega' > 0$, it follows from (12) that $|h| < 1$, $h^2 < 0$, and hence $v \in (0,1)$).

Let $\wp(u)$ be the Weierstrass function corresponding to the periods 2ω and $2\omega'$; the polynomial

$$Q(u) = 4u^3 - g_2u - g_3$$

associated with $\wp$ has real parameters g_2 and g_3, one real root e_1, and the pair of conjugate roots e_2 and e_3:

$$Q(u) = P(u - e_1, a), \quad \overline{a} = e_2 - e_1, \quad a = e_3 - e_1.$$

Relation (13) can be rewritten as

$$\int_{\omega'}^{\omega'+\omega} (\wp(u) - e_1)^{-1}\,du = 0.$$

Using the formula (see [43], Table VI)

$$\wp(u+\omega) - e_1 = \frac{(e_2 - e_1)(e_3 - e_1)}{\wp(u) - e_1},$$

we get that

$$0 = \int_{\omega'}^{\omega'+\omega} (e_1 - \wp(u+\omega))\,du = e_1\omega + \zeta(\omega' + 2\omega) - \zeta(\omega' + \omega) = e_1\omega + \eta,$$

where $\zeta(u)$ is the Weierstrass zeta function, and $\eta = \zeta(\omega)$. Hence, (13) is equivalent to

$$e_1 = -\eta/\omega. \tag{14}$$

Using (3), we now get that

$$\sum_{n=1}^\infty \frac{h^{2n}}{(1+h^{2n})^2} = -\frac{1}{8},$$

and, passing to $v = -h^2$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} v^n}{(1+(-v)^n)^2} = \frac{1}{8}. \tag{15}$$

As mentioned in subsection 1, the left-hand side of (2) is obtained by expanding the function on the left-hand side of (14) in a power series. Since (2) has a unique solution $v \in (0, 1)$, our assertion is proved (all the steps have the nature of an equivalence).

We derived an equation for $h = \sqrt{-v}$ from (13); using (10), we can find a. Again passing to elliptic functions, we write (10) in the form

$$-\frac{\pi i}{\omega} \int_0^{\omega'} \frac{\wp(u+\omega) - e_1}{(e_2 - e_1)(e_3 - e_1)} du = \frac{\pi i}{\omega} \frac{\zeta(\omega' + \omega) - \zeta(\omega) + e_1 \omega'}{(e_2 - e_1)(e_3 - e_1)}$$
$$= \frac{\pi i}{\omega} \frac{\eta' + e\omega'}{(e_2 - e_1)(e_3 - e_1)} = \frac{1}{2}.$$

Using (14) and the Legendre formula

$$\eta \omega' - \eta' \omega = \pi i/2,$$

we get

$$(e_2 - e_1)(e_3 - e_1) = \pi^2/\omega^2,$$

which is equivalent to (10). Finally, after using the formula (see [43], §21)

$$e_2 - e_1 = -\pi^2 \vartheta_0^4 / 4\omega^2,$$

we express a in terms of h:

$$a = e_3 - e_1 = -4\vartheta_0^{-4}.$$

The proof of (i) is complete.

(ii) Relation (13) holds for our fixed a; this means that the function

$$g(z) = \frac{\pi i}{\omega} \int_{\infty}^{z} \frac{dt}{t\sqrt{P(t)}}$$

is holomorphic in any region of the form $G \setminus L$, where L is an arbitrary arc in G joining a and $\overline{a}$. The function

$$g(z) - g(a) = \frac{\pi i}{\omega} \int_a^z \frac{dt}{t\sqrt{P(t)}}$$

vanishes at a and $\overline{a}$ and changes sign in going around these points in $G \setminus \{a, \overline{a}\}$. This implies that the function

$$R(z) = (g(z) - g(a))^2, \qquad z \in G,$$

is holomorphic in G and has simple zeros at a and $\overline{a}$; it is not hard to see that R is univalent in both the upper and lower half-planes, and $R(z) \neq 0$, $z \in G \setminus \{a, \overline{a}\}$.

We now study the structure of the critical trajectories of the quadratic differential $-R(z)dz^2$. Let

$$\mathscr{V}(z) = \int_{\infty}^{z} g(t)\, dt, \qquad \mathscr{W}(z) = \int_a^z (g(t) - g(a))\, dt.$$

Integration by parts establishes that

$$\int_{\infty}^{a} g(t)\,dt = \frac{\pi i}{\omega}\int_{\infty}^{a} dt \int_{\infty}^{t} \frac{d\tau}{\tau\sqrt{P(\tau)}} = ag(a) - \frac{\pi i \omega'}{\omega}, \tag{16}$$

$$\int_{0}^{\infty} g(t)\,dt = \frac{\pi i}{\omega}\int_{0}^{\infty} dt \int_{\infty}^{t} \frac{d\tau}{\tau\sqrt{P(\tau)}} = -\pi i. \tag{17}$$

Using the fact that $g(a) = -1/2$ (see (10) and (16)), we get

$$\mathscr{W}(z) = \mathscr{V}(z) + z/2 + \pi i \omega'/\omega. \tag{18}$$

Here $\mathscr{V}$ and $\mathscr{W}$ are multivalued analytic functions in regions of the form $G\backslash L$; for their periods in going around L and E (in the positive directions with respect to $G\backslash L$) we have that

$$\Delta_L \mathscr{W} = -\Delta_E \mathscr{V} = 2\pi i \tag{19}$$

(see (17)). Let $V = \operatorname{Re}\mathscr{V}$ and $W = \operatorname{Re}\mathscr{W}$; we have that (see (18))

$$W(z) = V(z) + \operatorname{Re} z/2 - w, \tag{20}$$

where

$$w = -\operatorname{Re}(\pi i \omega'/\omega) > 0. \tag{21}$$

It follows from (19) that V and W are (single-valued) harmonic functions in any region of the form $G\backslash L$. It is immediately clear from the construction of $\mathscr{V}$ that it takes pure imaginary values on E^+ and E^-; consequently,

$$V(z) \equiv 0, \qquad z \in E. \tag{22}$$

We now construct an arc Γ on which W has the same property. Since $W(a) = 0$ and $W(\overline{a}) = \frac{1}{2}\Delta_L W = 0$, the formula for $\mathscr{W}$ implies that $W(z)$ changes sign in going around the points a and $\overline{a}$ in $G\backslash\{a, \overline{a}\}$. Therefore, W extends to a (single-valued) harmonic function on the two-sheeted Riemann surface $\tilde{G}$ over G with branch points a and $\overline{a}$. Let $p\colon \tilde{G} \to G$ be the corresponding projection and

$$p^{-1}(\partial G) = E^1 \cup E^2 = \partial\tilde{G}.$$

Assuming that we have extended W "from the first sheet to the second" and taking (20) and (22) into account, we get that

$$W(\tilde{x}) - \begin{cases} \tilde{x}/2 - w, & \tilde{x} \in E^1, \\ -\tilde{x}/2 + w, & \tilde{x} \in E^2. \end{cases} \tag{23}$$

Let us consider the set $\tilde{\mathscr{N}} \in \{\tilde{z} \in \tilde{G} \colon W(\tilde{z}) = 0\}$ and its projection $\mathscr{N} = p(\tilde{\mathscr{N}}) \subset G$. The derivative $\mathscr{W}'(\tilde{z}) = \sqrt{R(\tilde{z})}$ is nonzero in $\tilde{G}\backslash\{a, \overline{a}\}$, and $\tilde{\mathscr{N}} = p^{-1}(\mathscr{N})$; consequently, the set $\mathscr{N}$ consists of disjoint analytic arcs with endpoints at a and $\overline{a}$ and on the boundary of G.

On the other hand, the arcs forming $\mathscr{N}$ are trajectories of the quadratic differential $-R(z)(dz)^2$. It follows from a theorem on the local structure of the trajectories (see [16]) that each of a and $\overline{a}$ is an endpoint of three arcs in $\mathscr{N}$ (which come together at equal angles $2\pi/3$). It follows from (23) that a pair of such arcs goes to E^+ and E^- at the point $x = 2w$, a second pair goes to ∞ with the vertical asymptote $\operatorname{Re} z = 2w$, and no other arcs in $\mathscr{N}$ can go to the boundary. The third pair of arcs emanating from a and $\overline{a}$ thereby closes and forms an analytic arc (trajectory) joining a and $\overline{a}$. This trajectory is denoted by Γ_0; $\Gamma = \overline{\Gamma}_0 \ni a, \overline{a}$.

The function $W(z)$, $z \in G\backslash\Gamma$ (the branch $W(\tilde{z})$, $\tilde{z} \in \tilde{G}$, on the "first sheet") has boundary values $W(x) = x/2 - w$, $x \in E$. It follows from the definition of Γ that $W \equiv 0$ on Γ; thus (see (20)),

$$V(z) + \tfrac{1}{2}\operatorname{Re} z \equiv w, \quad z \in \Gamma. \tag{24}$$

The set $\mathscr{N}$ partitions G into four regions, and Γ is the intersection of the boundaries of the regions in which $W(z) < 0$. Since $W(z) \equiv 0$, $z \in \Gamma$, we have that

$$\frac{\partial W}{\partial n_\pm}(z) = -|\mathscr{W}'(z)| = -|R(z)|^{1/2}, \qquad z \in \Gamma_0;$$

$\partial/\partial n_\pm$ are the derivatives along the normal to Γ_0 that are inward with respect to $G\backslash\Gamma$; therefore, $\partial W/\partial n_+ = \partial W/\partial n_-$ on Γ_0. This means that

$$\frac{\partial(V+\varphi)}{\partial n_+}(z) = \frac{\partial(V+\varphi)}{\partial n_-}(z), \qquad z \in \Gamma_0,\ \varphi(z) = \frac{1}{2}\operatorname{Re} z. \tag{25}$$

(iii) It remains to represent V as the potential of an admissible charge λ. By the Cauchy formula,

$$g(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{g(t)\,dt}{t-z} + \frac{1}{2\pi i}\oint_E \frac{g(t)\,dt}{t-z}, \qquad z \in G\backslash\Gamma;$$

let us pass to the integration of the jump $\chi_g(t)$ of this function on Γ and E. Denoting by g^+ the boundary values of g on Γ^+ and E^+ (Γ^+ is the side of Γ on which the orientation "from a to $\overline{a}$" corresponds to the positive orientation of the boundary $\partial(G\backslash\Gamma)$, and E^+ is the upper side of E, oriented "from 0 to ∞"), we get that

$$\begin{aligned} g(z) &= \frac{1}{\pi i}\int_{\Gamma^+} \frac{(g^+(t) - g(a))\,dt}{t-z} + \frac{1}{\pi i}\int_{E^+} \frac{g^+(t)\,dt}{t-z} \\ &= \int_{\Gamma^+} \frac{\left(\frac{1}{\omega}\int_a^t \frac{d\tau}{\tau\sqrt{P(\tau)}}\right)dt}{t-z} - \int_{E^+} \frac{\left(\frac{1}{\omega}\int_t^\infty \frac{d\tau}{\tau\sqrt{P(\tau)}}\right)dt}{t-z}, \qquad z \in G\backslash\Gamma. \end{aligned} \tag{26}$$

Let

$$d\lambda_\Gamma(t) = \frac{1}{\omega}\int_a^t \frac{d\tau}{\tau\sqrt{P(\tau)}}dt, \qquad t \in \Gamma,$$

$$d\lambda_E(t) = \frac{1}{\omega}\int_t^\infty \frac{d\tau}{\tau\sqrt{P(\tau)}}dt, \qquad t \in E$$

(the integration is along Γ^+ and E^+). It follows from the construction of Γ and (19) that λ_Γ and λ_E are unit measures, $\lambda_\Gamma \in M(\Gamma)$ and $\lambda_E \in M(E)$. By (26),

$$g(z) = \int \frac{d\lambda(t)}{t-z}, \qquad z \in G\backslash\Gamma,$$

where $\lambda = \lambda_\Gamma - \lambda_E$. This gives us that $\mathscr{V} = \mathscr{V}^\lambda$ (see (9)); thus,

$$V^\lambda(z) = V(z) = \operatorname{Re}\int_\infty^z g(t)\,dt.$$

It now follows from (22) and (24) that the charge λ solves the equilibrium problem for (E, Γ, φ), $\varphi(z) = \frac{1}{2}\operatorname{Re} z$; relation (25) means that $(E, \Gamma, \varphi) \in S$. The constant $w = w(E, \Gamma, \varphi)$ coincides with the constant w in (20); formula (21) holds for it. Therefore (see (12)),

$$e^{-2w} = e^{2\pi i\omega'/\omega - \pi i/2} = -h^2 = v,$$

where v is the root of (2). It remains to observe that the same solution is valid for the equilibrium problem with the field $\varphi(z) = (\operatorname{Re} z)/2$ for any capacitor (E, F, φ) with $F \in \mathcal{F}$ an arbitrary compact set in G such that

$$\Gamma \subset F, \qquad F \backslash \Gamma \subset \left\{ z\colon\ W(z) = (V^{\lambda} + \varphi)(z) > w \right\}.$$

In particular, this applies to the capacitor in (iii).

Theorem 2 is a consequence of Theorem 1 and Lemma 5; apply Theorem 1 to a sequence of functions f_n of the form (7), where F is chosen to be the arc F^* in (iii) of Lemma 5.

It is not hard to pass from the expression for v in terms of $h = \exp(\pi i \omega'/\omega)$ given above to the answer in the form given by Magnus.

5. Theorem 1 enables us also to investigate other problems connected with the approximation of entire functions, in particular, problems on the degree of rational approximation of the function e^{-z} on $E_\theta = \{z\colon |\arg z| \le \theta < \pi/2\}$, of the function $e^{-p(x)}$ on $E = [0, +\infty]$, where $p(x)$ is an arbitrary polynomial with positive leading coefficient, and so on. The answers are given in potential-theoretic terms; in some cases it is possible to write an equation for v (unfortunately, it is not as simple as in the problem we have considered). We present the result for e^{-z}, $z \in E_\theta$.

The limit

$$\lim_{n\to\infty} \rho_n(e^{-z}, E_\theta)^{1/n} \doteq v_\theta \in (0, 1)$$

exists, where $v_\theta = -h^2$ and $h = \exp(\pi i \omega'/\omega)$ satisfies the equation

$$\int_0^{1/2} \left(\frac{\vartheta_0(t)}{\vartheta_3(t)} \right)^{2\alpha} dt = 0, \qquad \alpha = 1 - \frac{\theta}{\pi} \tag{27}$$

(ϑ_0 and ϑ_3 are the theta functions corresponding to $\tau = \omega'/\omega$).

Setting $x = 2\pi t$, we can write this equation in the form

$$\int_0^{\pi} \left(\frac{1 + 2\sum_{n=1}^{\infty} (-h)^{n^2} \cos nx}{1 + 2\sum_{n=1}^{\infty} h^{n^2} \cos nx} \right)^{2\alpha} dx = 0.$$

For example, $v_\theta = 1/4.42$ for $\theta = \pi/4$.

§3. Equilibrium and symmetry in a harmonic external field. The generalized Stahl theorem

1. This section is devoted to a proof of a general theorem on the asymptotic behavior of "complex" orthogonal polynomials; we are concerned with the weak (logarithmic) asymptotics. In essence, we are interested in the limit distribution of the zeros of polynomials satisfying complex orthogonality relations with holomorphic weight functions depending on the index of the polynomial. This limit distribution is characterized by the equilibrium problem on a compact set (conductor) F in the presence of an external field ψ; further, F must have the property of symmetry with respect to the harmonic field ψ. For the case when F belongs to the real line the corresponding theorem on the limit distribution of the zeros of the orthogonal (and somewhat more general) polynomials was obtained in our article [1]; in the same paper there are some applications of this theorem to problems in which the external field plays an essential role (see [1] about other papers in this direction). Our concern here is essentially to carry over the theorem to the complex case; this problem has turned out to be fairly complicated. In [8] Stahl developed a method leading to proofs of the corresponding theorems, based directly on the symmetry property of

the compact set F in the field ψ. We emphasize that the polynomials considered satisfy complex orthogonality relations of the form

$$\int_F Q_n(t)t^j\Psi_n(t)\,dt = 0, \qquad j = 0, 1, \dots, n-1,$$

where F is a system of curves in $\mathbf{C}$, and Ψ_n is a sequence of holomorphic functions in a fixed neighborhood Ω of F (as above, a somewhat more general situation connected with the replacement of dt by $f(t)\,dt$ is considered). It is essential that in this case the curves F admit deformations in Ω that preserve the orthogonality relations, and the polynomials Q_n are thereby associated not with a given compact set but with the class $\mathscr{F}$ of all $F \subset \Omega$ for which the same orthogonality relations hold. For example, if F is a rectifiable arc with endpoints at a and b, then $\mathscr{F}$ is the class of all rectifiable arcs joining a and b and lying in Ω. Further, the zeros of the orthogonal polynomials "select" the compact set F^* in $\mathscr{F}$ that has a definite symmetry property with respect to the field ψ, which is characterized by the asymptotic behavior of the sequence Ψ_n (if such a "selection" is possible). The limit distribution of the zeros of the polynomials Q_n on this compact set F^* is the same as for the polynomials P_n satisfying the corresponding Hermitian orthogonality relations

$$\int_F P_n(t)\bar{t}^j|\Psi_n(t)\,dt| = 0, \qquad j = 0, 1, \dots, n-1,$$

for $F = F^*$. For polynomials P_n satisfying these relations the theorems on the limit distribution of the zeros do not, in essence, have a "complex" specific nature; they are formulated and proved just as simply as in [1].

We present a theorem for the case corresponding to the equilibrium problem for a conductor in an arbitrary harmonic external field. In this form the theorem is applicable to a broad circle of problems. The case of the capacitor considered above can serve as one example; thus, Lemma 1 is the special case of Theorem 3 corresponding to

$$\Psi_n = \frac{\Phi_n}{\omega_{2n+1}}, \qquad \Psi = \varphi - V^{\lambda_E}.$$

In the process of proving Theorem 3 we use a number of concepts and devices from Stahl's paper [8]. However, many arguments require essential modification in our case, and direct references to certain propositions in [8] would make it much more difficult for the reader of the present article; therefore, we give a presentation here independent of [8]. In essence, it does not depend on the material in §§1 and 2; the needed concepts are introduced in subsections 2–4 below.

2. Let F be a compact set in $\mathbf{C}$ having *positive* (logarithmic) capacity. As above, $M(F)$ is the set of all unit measures on F. If μ is a measure (on the Riemann sphere $\hat{\mathbf{C}}$), then $S(\mu)$ is its support, $|\mu| = \mu(\hat{\mathbf{C}})$, and V^μ is the logarithmic potential of μ; logarithmic potentials are considered only for measures μ satisfying the condition

$$\int_{|t|\geq 1} \log|t|\,d\mu(t) < +\infty. \tag{1}$$

Let $\psi: F \to \mathbf{R}$ be a given continuous function (*external field*) on the compact set F. The pair (F, ψ) is the original data of the equilibrium problem of interest to us.

LEMMA 6. *There exists a unique measure $\lambda \in M(F)$ with the following equilibrium property: for some constant w the relations*

$$(V^\lambda + \psi)(z)\begin{cases} = w, & z \in S(\lambda), \\ \geq w, & z \in F, \end{cases} \tag{2}$$

hold approximately everywhere on $S(\lambda)$ and F.

The measure $\lambda = \lambda(F, \psi)$ is called the *equilibrium measure* of the compact set F in the field ψ; $w = w(F, \psi)$ is the corresponding *equilibrium constant.* We underscore that the equilibrium relations (2) uniquely determine λ and w. At regular points of F we have equality (on $S(\lambda)$) and the inequality $\geq$ (on F) in (2). In particular, if F is a regular compact set, then (2) can be written in the form

$$(V^\lambda + \psi)(z) \equiv \min_F (V^\lambda + \psi) = w, \qquad z \in S(\lambda).$$

Lemma 6 will be proved here (as noted in connection with the corresponding assertions in §1.1, it is proved by standard methods in potential theory; cf. [10] and [11], the case $\psi \equiv 0$). Let

$$I(\mu) = \iint \log \frac{1}{|t-z|} d\mu(z)\, d\mu(t),$$
$$I_\psi(\mu) = I(\mu) + 2 \int \psi(t)\, d\mu(t)$$

be the (doubled) characteristic energy of the measure μ and the total energy when the interaction with the external field is taken into account. Let $M^0(F)$ denote the set of measures $\mu \in M(F)$ having finite energy $I(\mu)$. It follows from the weak compactness of $M(F)$ and the reduction principle for $I_\psi(\mu)$ that there exists a *minimizing measure* λ:

$$I_\psi(\lambda) = \min \{I_\psi(\mu)\colon\ \mu \in M(F)\}. \tag{3}$$

The relations (for measures in $M^0(F)$)

$$I_\psi\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{2}(I_\psi(\mu) + I_\psi(\nu)),$$
$$I_\psi(\mu - \nu) = 2I_\psi(\mu) + 2I_\psi(\nu) - 4I_\psi\left(\frac{\mu+\nu}{2}\right)$$

imply that the minimizing measure λ is unique. Further, the identity

$$I_\psi(\varepsilon\sigma + (1-\varepsilon)\lambda) - I_\psi(\lambda) = 2\varepsilon \int (V^\lambda + \psi) d(\sigma - \lambda) + \varepsilon^2 I(\sigma - \lambda) \tag{4}$$

can be directly verified for any $\varepsilon > 0$ and any measure $\sigma \in M^0(F)$. This gives us that *the minimizing measure λ is the unique measure in $M^0(F)$ satisfying the condition*

$$\int (V^\lambda + \psi) d(\sigma - \lambda) \geq 0, \quad \forall \sigma \in M^0(F). \tag{5}$$

Indeed, (5) follows directly from (3) and (4) ($\varepsilon \to 0$). On the other hand, the fact that the energy $I(\nu)$ is positive on neutral charges implies that $I(\sigma - \lambda) \geq 0$ for any measure $\sigma \in M^0(F)$; using (4) with $\varepsilon = 1$, we conclude that every measure $\lambda \in M^0(F)$ satisfying (5) minimizes the energy integral I_ψ.

If λ satisfies (5), then the equilibrium relation (2) holds for it with

$$w = \int (V^\lambda + \psi)\, d\lambda.$$

Indeed, if $V^\lambda + \psi < w$ on a closed set $e \subset F$ with $\operatorname{cap}(e) > 0$, then there exists a $\sigma \in M^0(e)$ such that $\int (V^\lambda + \psi)\, d\sigma < w$, and hence (5) is violated. Consequently, $V^\lambda + \psi \geq w$ approximately everywhere on F. If $V^\lambda + \psi > w$ on a nonempty set

$e \subset S(\lambda)$, then the inequality $\int (V^\lambda + \psi)\, d\lambda > w$ follows from the lower semicontinuity of the function $V^\lambda + \psi$ (a contradiction to the definition of w).

If λ is an equilibrium measure, then $V^\lambda + \psi \le w$ everywhere on $S(\lambda)$; therefore, $\lambda \in M^0(F)$. Since sets of zero inner capacity do not play a role in integration with respect to measures in $M^0(F)$, we get (5).

We have shown that *the minimizing measure λ and only this measure satisfies the equilibrium relation* (2). The lemma is proved.

It is possible to determine a number of other characteristic properties of the equilibrium measure $\lambda = \lambda(F, \psi)$. For example, the following *extremal property* characterizes the equilibrium measure λ for regular compact sets F without interior points:

$$w = \min_F (V^\lambda + \psi) = \max_{\mu \in M(F)} \min_F (V^\mu + \psi).$$

This property connects λ and w with extremal problems for polynomials and, in particular, with orthogonal polynomials on the line $\mathbf{R}$ and with polynomials satisfying Hermitian orthogonal relations on $\mathbf{C}$.

We mention that the support of λ is not known in advance, which makes an analysis of the problem of equilibrium in a field essentially more difficult in many cases; see [1] (also §2 above) about some examples in which λ and w can be found explicitly, and applications of them. There is not this difficulty, in particular, in the important problem of equilibrium on F in the field $\psi = V^{-\sigma} = -V^\sigma$ of a unit negative charge with support outside F; in this case $S(\lambda) = \partial F$, and λ is the result of balayage of σ from $\hat{\mathbf{C}} \setminus F$ onto F. This is the case we had in §1.1—with respect to the component λ_E of the equilibrium charge $\lambda = \lambda_F - \lambda_E$; the external field was given only on the plate F of the capacitor (E, F), and the measure λ_E smoothed out the potential V^{λ_E} on E under the effect of the external field $-V^{\lambda_F}$.

3. We now define the S-property for a compact set F in a harmonic external field ψ.

As above, we define a *tame point* ζ of a compact set Γ to be a point in Γ such that the intersection of Γ with a small neighborhood of ζ is a simple (open) analytic arc; Γ_0 denotes the set of all tame points of Γ. A compact set Γ is said to be *tame* if $\operatorname{cap}(\Gamma) > 0$ and $\operatorname{cap}(\Gamma \setminus \Gamma_0) = 0$.

We say that a compact set F in a field ψ *has the S-property*, and write $(F, \psi) \in S$, if the following conditions hold:

(i) ψ is a harmonic function in some neighborhood Ω of F.

(ii) The support $\Gamma = S(\lambda)$ of the equilibrium measure $\lambda = \lambda(F, \psi)$ is a tame compact set, and $\Gamma_0 \subset F_0$.

(iii) The equality

$$\frac{\partial (V^\lambda + \psi)}{\partial n_+}(\zeta) = \frac{\partial (V^\lambda + \psi)}{\partial n_-}(\zeta)$$

holds at each point $\zeta \in \Gamma_0$, where the $n_\pm$ are the normals to Γ_0, directed in opposite directions.

The results in §§1 and 2 show that the problem of constructing compact sets that are symmetric with respect to a field is closely connected with problems of rational approximation of functions. We are not now formally connected with this problem; the S-property is given as a *condition* in Theorem 3 (as in Theorem 1). We confine ourselves here to some remarks on this.

If $(F, \psi) \in S$, then the analytic arcs forming Γ_0, $\Gamma = S(\lambda)$, are *trajectories of quadratic differentials in Ω*. Indeed, let $\mathscr{V}^\lambda$ and Ψ be multivalued analytic functions in $\Omega\backslash\Gamma$ for which $V^\lambda = \operatorname{Re}\mathscr{V}^\lambda$ and $\psi = \operatorname{Re}\Psi$. The derivative of the complex potential (with the field taken into account)

$$r(z) = \frac{d}{dz}(\mathscr{V}^\lambda + \Psi)(z), \qquad z \in \Omega\backslash\Gamma,$$

is single-valued (and holomorphic) in $\Omega\backslash\Gamma$. It follows from the equilibrium condition that this derivative extends continuously to both sides of any arc $L \subset \Gamma_0$. It follows from the equilibrium and symmetry conditions that its boundary values on the different sides of these arcs differ only by a sign. The function $R = r^2$ thereby extends to a holomorphic function in the open set $\Omega\backslash A$, where $A = \Gamma\backslash\Gamma_0$. More interesting is the case when $A = \{a_1, \dots, a_N\}$ is a finite set. It can be shown that in this case each point a_j is a simple pole or a zero (not necessarily simple) of R, and hence

$$R(z) = \frac{B(z)}{A(z)}, \quad B(z) \in \mathscr{H}(\Omega), \quad A(z) = \prod_{j=1}^{N}(z - a_j),$$

is a meromorphic function in Ω with poles among the points a_j. The potential with the field taken into account has the representation

$$(V^\lambda + \psi)(z) = \operatorname{Re}\int^z R(t)^{1/2}\,dt, \qquad z \in \Omega\backslash\Gamma,$$

and the set Γ_0 turns out to be a collection of trajectories of the quadratic differential $-R(z)(dz)^2$. In a number of concrete cases it is possible to find the functions $B(z)$ and $A(z)$ from the conditions of the problem and to construct the necessary trajectories of $-R(z)(dz)^2$ by inverse arguments; the symmetry condition holds for them. This approach was used in proving Theorem 2.

Another general approach to the problem of constructing symmetric compact sets is based on the problem of the *maximum of the equilibrium energy* in various classes $\mathscr{F} = \{F\}$ and the problem of the *stationarity property* of the corresponding equilibrium measures with respect to this functional. For an example we consider a problem of Chebotarev type. Let ψ be a given harmonic field in the region Ω, and fix a finite set $A = \{a_1, \dots, a_N\}$. Denote by $\mathscr{F} = \mathscr{F}(A)$ the class of all continua $F \subset \Omega$ containing A. It is not hard to show by classical variational methods that if there exists a continuum $F^* \subset \mathscr{F}(A)$ maximizing the equilibrium energy in the field ψ, namely,

$$I_\psi(\lambda(F^*, \psi)) = \max\{I_\psi(\lambda(F, \psi)) : F \subset \mathscr{F}(A)\}, \tag{6}$$

then $(F^*, \psi) \in S$. Relation (6) means that the equilibrium measure $\lambda^* = \lambda(F^*, \psi)$ is a stationary point of the energy functional with respect to "physical" (i.e., connected with small displacements) variations of the equilibrium measure (including those that go beyond the bounds of the conductor F^*). In other words, the measure λ^* remains in an (unstable) equilibrium position if we fasten the points $\{a_j\}$ and then make the whole region Ω a conductor.

Problem (6) is a natural generalization of the problem of a continuum of minimal capacity, which relates to the case $\psi \equiv 0$. The question of the existence of a continuum F^* maximizing the equilibrium energy $I_\psi(\lambda)$ turns out to be more complicated because of the presence of the field ψ. A positive solution of it requires additional assumptions about the properties of the external field; in the general case a maximizing sequence of continua $F_n \in \mathscr{F}$ can go to the boundary of Ω, and then there can fail to be a solution of (6). We remark also that in the presence of a field the condition (6) does not in general determine the continuum F^* but only the support Γ of the equilibrium measure λ^* (cf. §2.2).

4. In connection with sequences of measures the symbol $\to$ denotes *weak convergence on the Riemann sphere*:

$$\mu_n \to \mu \Leftrightarrow \int f\, d\mu_n \to \int f\, d\mu \quad \forall f \in C(\hat{\mathbf{C}}).$$

The set $M(\hat{\mathbf{C}})$ is compact in the sense of this convergence.

It is convenient to use spherical normalization of the logarithmic potential in connection with the concept of weak convergence on the sphere. For any measure μ, $S(\mu) \subset \hat{\mathbf{C}}$, we define the potential by the formula

$$V_*^\mu(z) = \int_{|t|\le 1} \log \frac{1}{|t-z|}\, d\mu(t) + \int_{|t|>1} \log \frac{1}{|1-z/t|}\, d\mu(t);$$

such potentials are said to be *spherically normalized* or simply *spherical*. With this modification the definition of a potential extends to measures μ for which condition (1) is not satisfied. For measures with this condition the usual potential differs from the spherical potential by a constant term; for measures with support in the disk $|z| \le 1$ the spherical potential coincides with the usual one.

We use the symbol $\xrightarrow{\text{cap}}$ for denoting *convergence with respect to capacity*; the expression $h_n \xrightarrow{\text{cap}} h$, $z \in D$, means that the sequence of functions $h_n(z)$ converges to the function $h(z)$ *with respect to capacity on compact subsets of* D; further, the capacity of sets lying in a fixed neighborhood of the point at infinity (for example, in the disk $|z| > 2$) can be understood as the Green capacity with respect to the disk $|z| > 1$. Weak convergence of measures is equivalent to convergence of their spherical potentials with respect to capacity:

$$\mu_n \to \mu \Leftrightarrow V^{\mu_n} \xrightarrow{\text{cap}} V^\mu, \qquad z \in \hat{\mathbf{C}}.$$

An analogous normalization is introduced for polynomials. A polynomial P_* is said to be *spherically normalized* if

$$P_*(z) = C\prod_j (z-\zeta_j) = \prod_{|\zeta_j|\le 1}(z-\zeta_j)\prod_{|\zeta_j|>1}\left(1-\frac{z}{\zeta_j}\right).$$

Let $\mu(P)$ denote the *measure associated with a polynomial* P:

$$\mu(P) = \sum_j \delta_{\zeta_j},$$

where δ_ζ is the Dirac measure with support at the point ζ, and the summation is over all zeros of P (counting multiplicity). We underscore that in the spherically normalized expression for the polynomial P_* and in the definition of $\mu(P_*)$ we do not allow the value $\zeta_j = \infty$, and hence $\deg P = |\mu(P)|$.

For a spherically normalized polynomial $P = P_*$

$$\log|P| = -V_*^{\mu(P)}.$$

If a sequence of such polynomials satisfies the condition $(1/n)\mu(P_n) \to \mu$, then

$$|P_n|^{1/n} \xrightarrow{\text{cap}} \exp(-V_*^\mu), \qquad z \in \hat{\mathbf{C}}.$$

The notation introduced in §1.3 (associated with integrals of Cauchy type) is used below. We dwell on the definition of the class $\mathscr{H}_0(\Omega\setminus F)$; this definition is connected with a compact set F and a field ψ such that $(F, \psi) \in S$. We write $f \in \mathscr{H}_0(\Omega\setminus F)$ to indicate that *the function* f *is holomorphic in the open set* $\Omega\setminus F$, *on arcs in* $\Gamma_0\setminus e$

(Γ_0 is the set of tame points of $\Gamma = S(\lambda)$, and e is a compact set of zero capacity) *it has continuous boundary values (from two sides), and its jump χ_f does not have zeros on $\Gamma_0 \backslash e$.* The exceptional set e includes a (relatively closed) subset of Γ_0 consisting of possible zeros and singular points of χ_f, and of its limit set, which belongs to $\Gamma \backslash \Gamma_0$. Since $\operatorname{cap}(\Gamma \backslash \Gamma_0) = 0$, we require in essence that the first of these sets have zero capacity.

5. We now formulate the main result of this section.

THEOREM 3. *Suppose that F is a compact set of positive capacity belonging to the unit disk $U_1 \colon |z| < 1$, and let Ω be a neighborhood of F. Assume that the sequence of functions Ψ_n and the function f satisfy the conditions:*

1°. *The Ψ_n are holomorphic functions in Ω, and*

$$\psi_n(z) = \frac{1}{2n} \log \frac{1}{|\Psi_n(z)|} \rightrightarrows \psi(z), \quad z \in \Omega \quad (n \to \infty); \tag{7}$$

2°. $(F, \psi) \in S$, *$\mathbf{C} \backslash \Gamma$ is connected.*

3°. $f \in \mathscr{H}_0(\Omega \backslash F)$.

If the polynomials Q_n, $\deg Q_n \leq n$ ($Q_n \not\equiv 0$), satisfy the orthogonality relations

$$\oint_F Q_n(t) t^j \Psi_n(t) f(t)\, dt = 0, \qquad j = 0, 1, \dots, n-1 \ (n = 1, 2, \dots), \tag{8}$$

then the following assertions hold for them:

(i) $(1/n)\mu(Q_n) \to \lambda = \lambda(F, \psi)$ $(n \to \infty)$.

(ii) *If the polynomials Q_n are spherically normalized, then*

$$\left| \oint_F Q_n^2(t) \frac{\Psi_n(t) f(t)\, dt}{t - z} \right|^{1/n} \xrightarrow{\text{cap}} e^{-2w}, \qquad z \in D,$$

where $w = w(F, \psi)$ and $D = \hat{\mathbf{C}} \backslash F$.

The condition $F \subset U_1$ in the above formulation is connected with the fact that assertion (ii) depends on the normalization of the polynomials Q_n. Assertion (i) does not depend on the normalization (nor do the orthogonality relations (8)); it thus holds for compact sets in $\mathbf{C}$. Assertion (ii) also remains in force for compact sets $F \subset \mathbf{C}$ if the constant w in (ii) depends on the constant w_* determined by the equilibrium relations (2) for the normalized potentials V_*^λ. We note that, in the use of Theorem 3 in questions involving approximation of analytic functions, the asymptotic behavior of functions of the form

$$\Delta_n(z) = \frac{1}{Q_n^2(z)} \oint_F Q_n^2(t) \frac{\Psi_n(t) f(t)\, dt}{t - z}$$

is analyzed. These functions do not in general depend on the normalization of the polynomials Q_n, and we can always assert that

$$|\Delta_n(z)|^{1/n} \xrightarrow{\text{cap}} e^{-2(w - V^\lambda(z))}, \qquad z \in D$$

(cf. §1.5).

Everywhere below we assume that the neighborhood Ω of F lies in the disk $U_{1/2} \colon |z| < 1/2$ (it clearly suffices to prove the theorem for this case). *All the potentials and polynomials in what follows are assumed to be spherically normalized.* The spherical normalization coincides with the usual for potentials of measures with supports in Ω; further, they are all positive in Ω. The leading coefficient of the polynomials with zeros in Ω is equal to 1.

In subsections 6 and 7 we single out the lemmas forming the basis of the proof of Theorem 3. The proof of the theorem is concluded in subsection 8.

6. As an initial result we single out the following simple statement about the asymptotic behavior of integrals of positive functions.

LEMMA 7. *Suppose that L is a union of finitely many rectifiable curves in $\mathbf{C}$, the functions Ψ_n are continuous on L and satisfy the condition*

$$\psi_n = \log\frac{1}{n}\cdot\frac{1}{|\Psi_n|} \rightrightarrows \psi, \qquad z \in L$$

(it is assumed that $\Psi_n \neq 0$ on L), and the function χ is such that $|\chi|$ is integrable and positive almost everywhere on L. If the sequence of polynomials P_n satisfies the condition $\nu_n = (1/n)\mu(P_n) \to \nu$, then

$$\lim_{n\to\infty}\left(\int_L |(P_n\Psi_n\chi)(t)\,dt|\right)^{1/n} = e^{-m}, \qquad m = \min_L(V^\nu + \psi).$$

PROOF. Since L is a regular set, the conditions $\nu_n \to \nu$ and $\psi_n \rightrightarrows \psi$ imply that

$$\min_L(V^{\nu_n} + \psi_n) \to \min_L(V^\nu + \psi).$$

From this we get

$$||P_n\Psi_n||_L^{1/n} \to e^{-m},$$

and hence the upper estimate

$$\overline{\lim_{n\to\infty}}\left(\int_L |P_n\Psi_n\chi\,dt|\right)^{1/n} \leq e^{-m}.$$

We prove the corresponding lower estimate. The potential V^λ is continuous in the weak topology, and thus the function $V^\nu + \psi$ is approximately continuous with respect to Lebesgue measure (length) on L. This gives us that for any $\varepsilon > 0$ the set

$$e = \{z \in L\colon\ (V^\nu + \psi)(z) < m + \varepsilon\}$$

has positive measure. It follows from the conditions of the lemma that $V^{\nu_n} + \psi_n \to V^\nu + \psi$ in measure on L, and the measure of the set $e_n = \{z \in e\colon\ (V^{\nu_n} + \psi_n)(z) < m + \varepsilon\}$ thus tends to the measure of e (as $n \to \infty$). Consequently,

$$\underline{\lim_{n\to\infty}}\left(\int_L |P_n\Psi_n\chi\,dt|\right)^{1/n} \geq e^{-(m+\varepsilon)}\lim_{n\to\infty}\left(\int_{e_n}|\chi\,dt|\right)^{1/n} = e^{-(m+\varepsilon)};$$

since ε is arbitrary, the same estimate is true with $\varepsilon = 0$. The lemma is proved.

Note that in Lemma 7 (and in Lemma 8 below) the meaning of the functions ψ_n and ψ is not quite the same as in condition (7) (there is a doubling relative to (7)).

Our next goal is to formulate additional conditions on P_n, Ψ_n, and L needed for the assertion of Lemma 7 to remain in force on replacement of the integral $\int_L |(\cdot)\,dt|$ by $|\int_L (\cdot)\,dt|$. Here we must pass from arbitrary rectifiable curves to analytic arcs L and impose a certain symmetry condition on $P_n\Psi_n$ (with respect to L) in a neighborhood of a minimum point of the limit potential $V^\nu + \psi$. The following definitions enable us to formulate the corresponding symmetry requirement.

Let L be a simple analytic arc and z_0 an interior point of L. There exist a neighborhood $\mathscr{U}$ of z_0 and a conformal mapping $\varphi\colon \mathscr{U} \to U_1$ (U_1 is the open unit disk) such that

$$\varphi(\mathscr{U} \cap L) = (-1, 1), \qquad \varphi(z_0) = 0.$$

This enables us to define an anticonformal mapping of $\mathscr{U}$ onto itself (an analogue of complex conjugation): $z \to z^* = \varphi^{-1}(\overline{\varphi(z)})$ (as usual, the bar denotes complex conjugation). A neighborhood $\mathscr{U}$ of this type will be called *symmetric*. The mapping $z \to z^*$ allows us in the standard way to introduce the mapping $\mu \to \mu^*$ of measures on $\mathscr{U}$: $d\mu^*(z) = d\mu(z^*)$.

LEMMA 8. *Suppose that the functions Ψ_n are holomorphic in a neighborhood $\mathscr{U}$ of the analytic arc L,*

$$\psi_n = \frac{1}{n}\log\frac{1}{|\Psi_n|} \rightrightarrows \psi, \qquad z \in U, \tag{9}$$

χ is a function that is continuous and nonzero on L, P_n is a sequence of polynomials satisfying the condition

$$\nu_n = \frac{1}{n}\mu(P_n) \to \nu, \tag{10}$$

and z_0 is an interior point of L. Assume the following conditions in some *-symmetric neighborhood $\mathscr{U}$ of z_0:

(i) $(\nu_n|_{\mathscr{U}})^* = \nu_n|_{\mathscr{U}}$, *and all the zeros of P_n on $\mathscr{L} = L \cap \mathscr{U}$ have even multiplicity.*

(ii) $(V^\nu + \psi)(z^*) = (V^\nu + \psi)(z)$, $z \in \mathscr{U}$.

(iii) $(V^\nu + \psi)(z_0) < (V^\nu + \psi)(z)$, $z \in L\setminus\{z_0\}$.

Then there exists a sequence of polynomials q_n with zeros in $\mathscr{U}$, $k_n = \deg q_n = o(n)$ $(n \to \infty)$, such that

$$\lim_{n\to\infty}\left|\int_L (q_n P_n \Psi_n \chi)(t)\,dt\right|^{1/n} = e^{-m_0}, \qquad m_0 = (V^\nu + \psi)(z_0).$$

PROOF. Let

$$W_n(z) = (V^{\nu_n} + \psi_n)(z), \qquad z \in U.$$

Denote by $\tilde{u}$ a function conjugate to the harmonic function u. All equalities involving arguments of functions are understood at the points where the argument is defined (outside the set of zeros of P_n) and hold modulo 2π. We have

$$\begin{aligned}\operatorname{Arg}(P_n(z)\Psi_n(z)) &= -n\tilde{W}_n(z)\\ &= -\frac{n}{2}(\tilde{W}_n(z) + \tilde{W}_n(z^*)) + \frac{n}{2}(\tilde{W}_n(z^*) - \tilde{W}_n(z)), \qquad z \in \mathscr{U}.\end{aligned} \tag{11}$$

It is clear that the equalities

$$\frac{\partial}{\partial n}(W_n(z) + W_n(z^*)) = 0, \qquad \frac{\partial}{\partial \tau}(\tilde{W}_n(z) + \tilde{W}_n(z^*)) = 0$$

are valid at each point $z \in \mathscr{L}$ at which $P_n(z) \neq 0$, where $\partial/\partial n$ and $\partial/\partial\tau$ are the derivatives with respect to the normal and with respect to the tangent to L, respectively (the first equality is a consequence of the symmetry of the functions $W_n(z) + W_n(z^*)$ with respect to $\mathscr{L}$, and the second follows from the first in view of the Cauchy-Riemann conditions). Since the zeros of P_n on $\mathscr{L}$ have even multiplicity, the increments of the function

$$\frac{n}{2}(\tilde{W}_n(z) + \tilde{W}_n(z^*))$$

in passing through these zeros (in moving along $\mathscr{L}$) are multiplies of 2π; from this,

$$-\frac{n}{2}(\tilde{W}_n(z) + \tilde{W}_n(z^*)) = \text{const } [\operatorname{mod} 2\pi] \tag{12}$$

on the arcs belonging to $\mathscr{L}\setminus\{z\colon P_n(z) = 0\}$. Further, it follows from (9) and (10) that

$$W_n \xrightarrow{\text{cap}} W = V^\nu + \psi, \qquad z \in U;$$

thus,

$$W_n(z^*) - W_n(z) \xrightarrow{\text{cap}} W(z^*) - W(z) = 0, \qquad z \in \mathscr{U}$$

(see condition (ii) of the lemma). Condition (i) of the lemma gives us that the functions $W_n(z^*) - W_n(z)$ are harmonic in $\mathscr{U}$; for such functions convergence with respect to capacity implies uniform convergence, namely,

$$h_n(z) = W_n(z^*) - W_n(z) \rightrightarrows 0, \qquad z \in \mathscr{U}.$$

The disk r-neighborhood of z_0 is denoted by U_r below; fix a $\delta > 0$ such that $\bar{U}_\delta \subset \mathscr{U}$. Then

$$m_n = \max_{z \in \gamma} \left| \frac{\partial}{\partial n} h_n(z) \right| \to 0, \qquad \gamma = \partial U_\delta, \tag{13}$$

where $\partial/\partial n$ is the derivative with respect to the outward normal to γ. On the circle γ we consider the absolutely continuous (with respect to Lebesgue measure) measure η_n with the density

$$\frac{d\eta_n(\zeta)}{|d\zeta|} = n\left(m_n + \frac{\partial h_n}{\partial n}(\zeta)\right) + r_n/2\pi\delta,$$

where $r_n \in (0, 1)$ is chosen so that the quantity

$$k_n = 2\pi n \delta m_n + r_n$$

is an integer. It follows from (13) that $k_n = o(n)$ $(n \to \infty)$. Using the fact that for any function h harmonic in $\bar{U}_\delta$ the logarithmic potential of the measure $(\partial h/\partial n)|d\zeta|$ on γ differs from h by a constant quantity, we get that

$$V^{\eta_n}(z) = nh_n(z) + \alpha_n, \qquad z \in \gamma. \tag{14}$$

Let

$$\kappa_n = \sum_{j=1}^{k_n} \delta_{\zeta_j}$$

be a discrete measure on γ whose points of concentration $\zeta_j = \zeta_{n,j}$ are uniformly distributed with respect to the measure η_n (the η_n-measure of each arc (ζ_{j-1}, ζ_j) is equal to 1). This construction has the following property: for any function g on γ

$$\left| \int g d(\eta_n - \kappa_n) \right| \le \operatorname{Var}(g)$$

(this is verified by decomposing a function of bounded variation as a difference of monotone functions). Applying this inequality to the functions $g(\zeta) = \log|\zeta - z|^{-1}$, $z \in U_{\delta/3}$, we get

$$|(V^{\eta_n} - V^{\kappa_n})(z)| \le \log 2, \qquad z \in U_{\delta/3}. \tag{15}$$

Let

$$q_n(z) = \prod_{j=1}^{k_n} (z - \zeta_{n,j}) \qquad (\deg q_n = k_n = o(n)).$$

Since $\log|q_n| = -V^{\kappa_n}$, we get from (14) and (15) that

$$|nh_n(z) + \log|q_n(z)| + \alpha_n| \le \log 2, \qquad z \in U_{\delta/3}. \tag{16}$$

For any function u harmonic in $\overline{U}_R$,

$$\|\tilde{u} - \tilde{u}(z_0)\|_{U_r} \le \frac{\sqrt{2}Rr}{R^2 - r^2} \|u\|_{U_R}, \qquad r < R.$$

Applying this estimate with $R = \delta/3$ and $r = \delta/6$ to the function inside the modulus bars in (16), we get that

$$|n\tilde{h}_n(z) + \operatorname{Arg} q_n(z) - \beta_n| < \pi/3, \qquad z \in U_{\delta/6}$$

(β_n is a sequence of real numbers). Together with (11) and (12), this leads to

$$|\operatorname{Arg}(q_n P_n \Psi_n)(z) e^{-i\beta_n}| < \pi/3, \qquad z \in \mathscr{L}_{\delta/6}$$

(we see $\mathscr{L}_r = \mathscr{L} \cap U_r$). Let

$$\varphi_n = q_n P_n \Psi_n \chi, \quad M_n = |\varphi_n|, \quad \theta_n(z) = \operatorname{Arg}(\varphi_n(z)dz/|dz|).$$

Using the fact that the function $\operatorname{Arg}(\chi(z)dz)$ is continuous in a neighborhood of z_0 on L, we conclude that, for some $\varepsilon \in (0, \delta/6)$ and some sequence of real numbers γ_n,

$$|\theta_n(z) - \gamma_n| < \pi/3, \qquad z \in \mathscr{L}_\varepsilon.$$

For such z we have that $\cos(\theta_n(z) - \gamma_n) > 1/2$; consequently,

$$\begin{aligned}\int_{\mathscr{L}_\varepsilon} M_n(z)|dz| &\geq \left|\int_{\mathscr{L}_\varepsilon} M_n(z) e^{i\theta_n(z)}|dz|\right| \\ &\geq \operatorname{Re}\int_{\mathscr{L}_\varepsilon} M_n(z) e^{i(\theta_n(z)-\gamma_n)}|dz| \geq \frac{1}{2}\int_{\mathscr{L}_\varepsilon} M_n(z)|dz|.\end{aligned}$$

Since $k_n = \deg q_n = o(n)$, we have that $(1/n)\mu(q_n P_n) \to \nu$. Lemma 7 is applicable to the integrals of $M_n(z)|dz|$, and leads to the relation

$$\lim_{n\to\infty} \left|\int_{\mathscr{L}_\varepsilon} \varphi_n(t)\,dt\right|^{1/n} = e^{-m_0}. \tag{17}$$

By Lemma 7,

$$\overline{\lim_{n\to\infty}} \left|\int_{L\setminus\mathscr{L}_\varepsilon} \varphi_n(t)\,dt\right|^{1/n} \leq e^{-m}, \qquad m = \min_{L\setminus\mathscr{L}_\varepsilon}(V^\nu + \psi).$$

Since z_0 is a strict minimum point for the function $V^\lambda + \psi$ on L, it follows that $m > m_0$. Consequently, (17) remains in force when the integral over $\mathscr{L}_\varepsilon$ is replaced by the integral over L. The lemma is proved.

7. Having in view the proof of the main assertion (i) of Theorem 3, we can now argue as follows. If this assertion is not true, then $(1/n)\mu(Q_n) \to \mu \neq \lambda$ for some subsequence of measures associated with the polynomials Q_n. Using this assumption, we must contradict the orthogonality conditions

$$\oint_F Q_n p_n \Psi_n f\,dt = 0, \quad \forall p_n, \ \deg p_n < n.$$

This goal will be achieved if it is possible to construct polynomials p_n, $(1/n)\deg p_n \leq \theta < 1$ (leaving a "reserve" for polynomials of degrees $k_n = o(n)$) such that the $P_n = Q_n p_n$ satisfy the conditions of Lemma 8 in a neighborhood of some point $z_0 \in \Gamma_0$. The convergence condition (10) and condition (i) of this lemma are easy to satisfy. The matter is more complicated with the main conditions (ii) and (iii) of Lemma 8. This question is solved on the basis of Lemma 9 below; the S-property of the compact set F in the field ψ is used only in this lemma (in the same form as in [8]). Namely, we use the symmetry property of the potential $V^\lambda + \psi$ (in neighborhoods of points in Γ_0) with respect to the corresponding mappings *, which follows from the S-property.

More precisely, *let* $(F, \psi) \in S$, *and let* $\Gamma = S(\lambda)$; *then for each point* $z_0 \in \Gamma_0$ *there exists a* **-symmetric neighborhood* $\mathscr{U}$ *such that*

$$(V^\lambda + \psi)(z) = (V^\lambda + \psi)(z^*), \qquad z \in \mathscr{U}. \tag{18}$$

Indeed, let φ: $\mathscr{U} \to U_1$ be the mapping in the definition of * (for the given point z_0; see subsection 6). Let

$$g(\zeta) = (V^\lambda + \psi - w)(\varphi^{-1}(\zeta)), \qquad \zeta \in U_1,$$
$$h(\zeta) = \begin{cases} g(\zeta), & \operatorname{Im}\zeta \geq 0, \\ -g(\zeta), & \operatorname{Im}\zeta < 0, \end{cases}$$

where $w = w(F, \psi)$. Since $V^\lambda + \psi - w \equiv 0$ on $\Gamma_0 \cap \mathscr{U}$, the function h is equal to zero on $(-1, 1)$ and is continuous in the disk U_1. It follows from the S-property that h is a smooth function on U_1; h is thereby harmonic in U_1. By the symmetry principle for harmonic functions, $h(\zeta) = -h(\overline{\zeta})$; returning to $\mathscr{U}$, we get (18).

We make one more remark relating to exceptional sets. First, the conditions of the lemma contain an arbitrary compact set e of zero capacity; in the proof of the theorem the role of e will be played by the closure of the set of zeros and singularities of the jump χ_f on Γ_0. Second, the subset of F where $V^\lambda + \psi < w$ (the equilibrium condition (2) is violated) is an exceptional set in applications to the integral estimates. In the general case it can have a complicated nature, but it has a sufficiently simple structure in the case $(F, \psi) \in S$. Namely, part of this set can belong to the compact set $A = \Gamma \setminus \Gamma_0$ of zero capacity. Let

$$A_1 = \left\{ z \in F \setminus \Gamma \colon\ (V^\lambda + \psi)(z) < w \right\};$$

it follows from the continuity of $V^\lambda + \psi$ on $F \setminus \Gamma$ that A_1 is a relatively open subset of $F \setminus \Gamma$; thus, $\operatorname{cap}(A_1) = 0$. Note that the closure of A_1 belongs to $F \setminus \Gamma_0$ (the set $\overline{A}_1$ can have positive capacity, contain curves, and so on). The meaning of the symbols A and A_1 in Lemma 9 is the same as above.

LEMMA 9. *Suppose that* $(F, \psi) \in S$, *the measure* μ, $|\mu| \leq 1$, *is different from the equilibrium measure* $\lambda = \lambda(F, \psi)$, *and* e *is a fixed compact set of zero capacity. Then there exist a point* $z_0 \in \Gamma_0 \setminus e$, *a* **-symmetric neighborhood* $\mathscr{U}$ *of it, and a measure* σ *such that the following conditions hold*:

(i) $|\sigma| < 1$, $S(\sigma) \subset (F \cup \mathscr{U})$.
(ii) $\sigma|_{\mathscr{U}} = (\mu|_{\mathscr{U}})^*$.
(iii) $V^\sigma(z) \equiv +\infty$, $z \in B = e \cup A \cup A_1$.
(iv) $(V^{\mu+\sigma} + 2\psi)(z^*) = (V^{\mu+\sigma} + 2\psi)(z)$, $z \in \mathscr{U}$.
(v) $(V^{\mu+\sigma} + 2\psi)(z_0) < (V^{\mu+\sigma} + 2\psi)(z)$, $z \in F \setminus \{z_0\}$.

PROOF. Let $t \in [0, 1/2)$, and let $\eta \in M(\overline{U})$, where U is a neighborhood of F that is compact in Ω. Let

$$\begin{aligned} W(z) = W(z; t, \eta) = V^{\mu + t\eta}(z) &+ (1 + 2t)\psi(z) \\ &- (1 - 2t)(V^\lambda + \psi)(z), \qquad z \in \Omega. \end{aligned}$$

Since $V^\lambda + \psi \leq w$ everywhere on $\Gamma = S(\lambda)$, the potential V^λ is bounded on Γ and in Ω, and the function W is well-defined (as a function with values in $(-\infty, +\infty]$). Let

$$w_0 = w_0(t, \eta) = \inf_\Gamma W(z; t, \eta), \qquad w_1 = w_1(t, \eta) = \min_{\partial U} W(z; t, \eta).$$

For $t = 0$ the function W coincides with $V^{\mu} - V^{\lambda}$ and is thereby superharmonic and not identically zero in $\hat{\mathbf{C}}\backslash\Gamma$; by the minimum principle for superharmonic functions, we can assert that $w_0(0, \eta) < w_1(0, \eta)$ (these quantities do not depnd on η). On the other hand, it is not hard to see that

$$\lim_{t\to 0} w_j(t, \eta) = w_j(0, \eta), \qquad j = 0, 1,$$

uniformly with respect to $\eta \in M(\overline{U})$. This implies that there exists a t such that

$$w_0(\tau, \eta) < w_1(\tau, \eta), \quad \forall\tau \in [0, t], \ \forall\eta \in M(\overline{U}). \tag{19}$$

Fix a $t \in (0, 1/2)$ for which (19) holds.

It follows from the properties of the exceptional set B that there exists a measure $\eta_1 \in M(B)$ such that $V^{\eta_1}(z) \equiv +\infty$, $z \in B$. Clearly, the minimum of the function $W(z; t, \eta_1/2)$ on Γ is attained at points of the set Γ_0; fix any point $z_0 \in \Gamma_0$ at which the minimum of this function is attained. Finally, fix an arbitrary measure $\eta_2 \in M(F)$ such that

$$z_0 \notin S(\eta_2), \quad V^{\eta_2}(z_0) < V^{\eta_2}(z), \quad z \in F\backslash\{z_0\}$$

(for example, let D be an open set containing z_0, and η_2 the Robin distribution for $F\backslash D$; by varying the set D it is not hard to ensure that η_2 has the necessasry properties). Let $\eta = \frac{1}{2}(\eta_1 + \eta_2)$.

The function $W(z) = W(z; t, \eta)$ has a strict minimum on Γ at z_0, and, by (19),

$$W(z_0) < W(z), \qquad z \in \overline{U}\backslash\{z_0\}. \tag{20}$$

Let $\varphi\colon \mathscr{U} \to U_1$ be an anticonformal mapping for which $\varphi(F \cap \mathscr{U}) = (-1, 1)$ and $\varphi(z_0) = 0$. The notation $\mathscr{U}_r = \{z: |\varphi(z)| < r\}$ is used below. Fix an annular region

$$\mathscr{K} = \mathscr{U}_{\varepsilon_1}\backslash\overline{\mathscr{U}}_{\varepsilon_2}$$

such that $0 < \varepsilon_2 < \varepsilon_1$ and $\overline{\mathscr{U}}_{\varepsilon_1} \cap S(\eta) = \varnothing$. By (20), there exists a $\delta > 0$ such that $W(z) > W(z_0) + \delta$, $z \in \mathscr{K}$, and hence

$$W(z^*) > W(z_0) + \delta, \qquad z \in \mathscr{K}. \tag{21}$$

For $\varepsilon \in (0, \varepsilon_2)$ let $\nu(\varepsilon) = (\mu|_{\mathscr{U}_\varepsilon})^*$; it is clear that $\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$. We fix an $\varepsilon_3 \in (0, \varepsilon_2)$ such that for $\nu = \nu(\varepsilon_3)$

$$|\nu| < t/2, \quad 0 < V^{\nu}(z) < \delta/2, \quad z \in F \cup \overline{\mathscr{U}}_{\varepsilon_1}.$$

By the last inequality, (21) gives us that

$$W(z^*) - V^{\nu}(z) > W(z_0) + \delta/2, \qquad z \in \mathscr{K}. \tag{22}$$

The function on the left-hand side of this inequality is continuous on $\mathscr{U}_{\varepsilon_3}$, and its value at $z_0 = z_0^*$ is less than $W(z_0)$. Consequently, there exists an $\varepsilon \in (0, \varepsilon_3)$ such that

$$W(z^*) - V^{\nu}(z) < W(z_0) + \delta/2, \qquad z \in \mathscr{U} = \mathscr{U}_\varepsilon. \tag{23}$$

Now consider the two superharmonic functions ($z \in \Omega$):

$$W_1(z) = V^{\mu+t\eta}(z) + (1 + 2t)\psi(z),$$
$$W_2(z) = (1 - 2t)(V^{\lambda} + \psi)(z) + W(z_0) + V^{\nu}(z) + \delta/2.$$

Using the symmetry property (18), we can rewrite (22) and (23) in the form

$$W_1(z^*) > W_2(z), \qquad z \in \mathscr{K}, \tag{24}$$
$$W_1(z^*) < W_2(z), \qquad z \in \mathscr{U}. \tag{25}$$

We let

$$W_3(z) = \begin{cases} \min(W_1(z^*), W_2(z)), & z \in \mathscr{U}_{\varepsilon_1}, \\ W_2(z), & z \in U \backslash \mathscr{U}_{\varepsilon_1} \end{cases}$$

and show that this function can be represented in the form

$$W_3(z) = V^{\sigma_1}(z) + (1-2t)\psi(z) + C, \qquad z \in U, \tag{26}$$

where C is a real constant and σ_1 is a measure on $F \cup \mathscr{U}$ such that $|\sigma_1| = 1 - \frac{3}{2}t$. Indeed, W_3 is superharmonic both in the region $\mathscr{U}_{\varepsilon_1}$ (as a pointwise minimum of superharmonic functions) and in $U \backslash \mathscr{U}_{\varepsilon_1}$. It follows from (24) that $W_3(z) = W_2(z)$ for $z \in U \backslash \mathscr{U}_{\varepsilon_2}$, i.e., W_3 is superharmonic in a neighborhood of the junction curve $\partial \mathscr{U}_{\varepsilon_1}$. Thus, W_3 is superharmonic on U. Further, by the definition of W_2,

$$W_3(z) - (1-2t)\psi(z) = (1-2t)V^{\lambda}(z) + V^{\nu}(z) + C_1, \qquad z \in U \backslash \mathscr{U}_{\varepsilon_2}.$$

This formula determines a superharmonic (and harmonic in a neighborhood of $z = \infty$) extension of its left-hand side to the whole complex plane, with the asymptotic expression $(1 - \frac{3}{2}t)\log(1/|z|) + O(1)$ (as $z \to \infty$). This implies the existence of a measure σ_1 with the indicated properties.

We can now conclude the proof; we show that all the assertions of the lemma are true for the measure $\sigma = \sigma_1 + t\eta$, $|\sigma| = 1 - t/2$. Assertions (i) and (iii) follow immediately from the construction of σ. According to (25) and (26),

$$V^{\sigma_1}(z) = W_3(z) - (1-2t)\psi(z) - C = W_1(z^*) - (1-2t)\psi(z) - C$$

for $z \in \mathscr{U}$. Applying the Laplace operator to both sides of the equality and using the fact that $\sigma|_{\mathscr{U}} = \sigma_1|_{\mathscr{U}}$, we get (ii). Let

$$V(z) = (V^{\mu+\sigma} + 2\psi)(z) = W_1(z) + W_3(z) - C, \qquad z \in U$$

(the second equality follows from (26) and the definition of W_1), and let

$$E_1 = \{z \in U\colon\ W_3(z) = W_1(z^*)\}, \qquad E_2 = \{z \in U\colon\ W_3(z) = W_2(z)\}.$$

For $z \in E_1$, in particular, the function $V(z)$ coincides with $W_1(z) + W_1(z^*)$ to within a constant when $z \in \mathscr{U}$. This implies (iv) and (v) for $z \in E_1 \cap (F \backslash \{z_0\})$. It remains to verify (v) for $z \in (E_2 \cap F) \backslash A_1$; for such z

$$\begin{aligned} V(z) - V(z_0) &= W_1(z) + W_2(z) - 2W_1(z_0) \\ &= (W(z) - W(z_0)) + 2(1-2t)((V^{\lambda} + \psi)(z) - (V^{\lambda} + \psi)(z_0)) \\ &\quad + (V^{\nu}(z) + \delta/2) > 0. \end{aligned}$$

The lemma is proved.

8. Proof of Theorem 3. (i) Assume that $\mu(Q_n)/n \not\to \lambda$. Then, taking a convergent subsequence, we get that $\mu(Q_n)/n \to \mu \neq \lambda$, $n \in \Lambda \subset \mathbf{N}$. In this situation Lemma 9 is applicable. It follows from condition 3° in the theorem that there exists a compact set e of zero capacity such that on $\Gamma_0 \backslash e$ the function f has a continuous and nonvanishing jump χ_f. Using Lemma 9 with this e, we find a point $z_0 \in \Gamma_0 \backslash e$, a *-symmetric neighborhood $\mathscr{U}$ of it, and a measure σ such that (i)–(v) of the lemma are valid. The neighborhood $\mathscr{U}$ is taken to be small enough that the intersection $F \cap \overline{\mathscr{U}}$ is a simple analytic arc L belonging to Γ_0.

We deform the contour of integration γ representing the integral in (8) in such a way that two to its arcs coincide (from different sides) with the arc L. Let U_ε be a neighborhood of z_0, $\overline{U_\varepsilon} \subset \mathscr{U}$. It follows from Lemma 9(v) that

$$\min_{F \backslash U_\varepsilon}(V^{\mu+\sigma} + 2\psi) > (V^{\mu+\sigma} + 2\psi)(z_0)$$

for the compact set $F\backslash U_\varepsilon$. Since the potential is lower semicontinous, the same inequality is valid in a neighborhood of this compact set. It is now clear that the remaining part γ_1 of the contour γ can be chosen in such a way that

$$\min_{\gamma_1}(V^{\mu+\sigma}+2\psi) > (V^{\mu+\sigma}+2\psi)(z_0). \tag{27}$$

We construct a sequence of polynomials p_n, $\deg p_n \le n|\sigma|$, such that

$$\frac{1}{n}\mu(p_n) \to \sigma, \qquad \mu(p_n)|_{\mathscr{U}} = (\mu(Q_n)|_{\mathscr{U}})^*.$$

The second of these conditions means that points *-symmetric to zeros of Q_n in $\mathscr{U}$ are chosen as the zeros of p_n in U; such a choice is possible in view of assertion (ii) of Lemma 9. The zeros of the polynomial $Q_n p_n$ on $\Gamma_0 \cap \mathscr{U}$ will have even multiplicity. All the conditions of Lemma 8 are thereby satisfied for

$$P_n = Q_n p_n, \quad \nu = \mu+\sigma, \quad L = F\cap\overline{\mathscr{U}}, \quad \chi = \chi_f$$

and for 2ψ in place of ψ. This lemma implies that there exists a sequence of polynomials q_n, $k_n = \deg q_n = o(n)$, such that

$$\lim_{n\to\infty}\left|\int_L (Q_n p_n q_n \Psi_n \chi_f)(t)\,dt\right|^{1/n} = e^{-m_0}, \tag{28}$$

where

$$m_0 = (V^{\mu+\sigma}+2\psi)(z_0).$$

On the other hand, Lemma 7 and (27) give us

$$\overline{\lim_{n\to\infty}}\left|\int_{\gamma_1} (Q_n p_n q_n \Psi_n)(t)\,dt\right|^{1/n} < e^{-m_0}. \tag{29}$$

The sum of the integrals in (28) and (29) forms the integral $\oint_F$ of the same product of functions as in (29). Since $\deg(p_n q_n) < n$ for sufficiently large n, it follows from the orthogonality conditions that this integral is equal to zero. This contradicts (28) and (29).

(ii) Let

$$A_n(z) = \oint_F Q_n^2(t)\frac{\Psi_n(t)f(t)\,dt}{t-z}, \qquad z\in D. \tag{30}$$

The orthogonality conditions (8) imply

$$A_n(z) = \frac{Q_n(z)}{p_n(z)}\oint_F (Q_n p_n)(t)\frac{\Psi_n(t)f(t)\,dt}{t-z}, \qquad z\in D, \tag{30'}$$

where p_n is an *arbitrary* polynomial of degree at most n.

We must prove that for the sequence

$$\varphi_n(z) = -\frac{1}{2n}\log|A_n(z)|$$

we have

$$\varphi_n(z) \xrightarrow{\text{cap}} w, \qquad z\in D. \tag{31}$$

Let us prove first that for all $z\in D$

$$\underline{\lim_{n\to\infty}}\ \varphi_n(z) \ge w. \tag{32}$$

If F is a regular compact set, then for any $\varepsilon>0$ the inequality $\underline{\lim}\,\varphi_n > w-\varepsilon$ is obtained at once by estimating the modulus of the integral (30) in terms of the integral of the modulus of the integrand along a contour γ sufficiently close to F, and then using Lemma 7. In the general case it is necessary to exclude the influence of points of F at which $V^\Gamma + \psi < w$ (the equilibrium condition (2) is violated). For the case $(F, \psi) \in S$ this subset of F consists of two parts; one belongs to $A = \Gamma \backslash \Gamma_0$ (a *compact* set of zero capacity), and the other, $A_1 \subset F \backslash \Gamma$, is a relatively open subset of F with $\operatorname{cap}(A_1) = 0$ (cf. subsection 7). Let us turn to the representation (30′) and use the arbitrariness of the choice of the polynomial p_n in this representation. We fix a point $z \in D$ and construct a polynomial p_n with zeros in F, $\deg p_n \le n$, and a contour γ representing the contour integral along F such that

$$\varlimsup_{n\to\infty} \left| \oint_F (Q_n p_n)(t) \frac{\Psi_n(t) f(t)\, dt}{t - z} \right|^{1/2n} < e^{-w+\varepsilon}. \tag{33}$$

Since A_1 is not a compact set (and $\overline{A}_1$ can have positive capacity), we construct the corresponding limit measure in two steps. Consider the set $F_1 = F \backslash A_1$; note that for the compact set F_1 the equilibrium problem (2) has the same solution (the measure λ and the constant w) as for F. Let the measure $\eta_1 \in M(A)$ be such that $V^{\eta_1} \equiv +\infty$ on A. It is clear that on any contour $\gamma_1 \subset \Omega \backslash F$ sufficiently close to F_1

$$V^{\nu_1}(z) > w - \varepsilon/2, \qquad \nu_1 = (1 - 2\delta)\lambda + \delta\eta_1$$

for any sufficiently small $\delta > 0$. We fix any such contour γ_1 separating F_1 from $\partial\Omega$ (this is possible, because $\operatorname{cap} A_1 = 0$). On a *compact* set $A_2 \subset A_1$ not separated from $\partial\Omega$ by the contour γ_1 we now construct a measure $\eta_2 \in M(A_2)$ such that $V^{\eta_2} \equiv +\infty$ on A_2. Let $\nu = \nu_1 + \delta\eta_2$. We can now fix a contour $\gamma_2 \subset \Omega \backslash F$ separating A_2 from $\partial\Omega$ and a $\delta > 0$ small enough that

$$\tfrac{1}{2} V^{\nu+\lambda}(z) > w - \varepsilon, \quad z \in \gamma, \quad \nu = (1 - 2\delta)\lambda + \delta(\eta_1 + \eta_2), \quad |\nu| = 1,$$

on $\gamma = \gamma_1 \cup \gamma_2$. Let p_n be a sequence of polynomials satisfying the following conditions: the zeros of p_n belong to $\Gamma \cup A_2 \subset F$, $\deg p_n \le n$, and $\mu(p_n)/n \to \nu$. Since $\mu(Q_n)/n \to \lambda$, the estimate (33) follows from Lemma 7.

Further,

$$\log |Q_n/p_n|^{1/n} \xrightarrow{\text{cap}} V^{\nu-\lambda} = \delta V^{\eta_1+\eta_2-2\lambda}. \tag{34}$$

It is clear that (33) holds uniformly on compact subsets of D. Using (33) and (34), we get from the representation (30′) that for any compact set $K \subset D$

$$\varlimsup_{n\to\infty} \varphi_n(z) \ge w - 2\varepsilon \tag{35}$$

with respect to capacity on K. Since the functions φ_n are superharmonic on D, this implies that (35) is valid at any point $z \in D$; this proves (32), which also holds uniformly on compact subsets of D.

With the help of the two constants theorem it is now not hard to show that *if relation* (31) *does not hold, then for any compact set* $K \subset D$ *there exists a constant* $w' = w'(K) > w$ *such that*

$$\varlimsup_{n\to\infty} \varphi_n(z) \ge w' \quad \textit{uniformly on } K. \tag{36}$$

We fix an arbitrary measure $\mu \in M(F)$ different from the equilibrium measure $\lambda = \lambda(F, \psi)$; suppose that the point $z_0 \in \lambda_0$, the neighborhood $\mathscr{U}$ of it, and the measure σ are constructed from the measure μ in such a way that (i)–(v) of Lemma 9 are valid (the set e is the same as in the proof of (i) of the theorem). Let $\nu = \mu + \sigma$.

Take $\varepsilon > 0$; we construct a sequence of polynomials s_n such that

$$\mu(s_n)/n \to \varepsilon\nu, \qquad \mu(s_n)|_{\mathscr{U}} = (\mu(s_n)|_{\mathscr{U}})^*,$$

and all the zeros of s_n of $F \cap \mathscr{U}$ have even multiplicity. We choose the sequence p_n such that the zeros of p_n lie on $F \cup \mathscr{U}$ and

$$\mu(p_n)/n \to \lambda, \quad \mu(p_n)|_{\mathscr{U}} = (\mu(Q_n)|_{\mathscr{U}})^*, \quad \deg p_n \leq n.$$

The polynomials $P_n = Q_n p_n s_n$ now satisfy the conditions of Lemma 8 with $L = \Gamma_0 \cap \overline{\mathscr{U}}$, $\nu = 2\lambda + \varepsilon\nu$, and $\Psi_n^{1+\varepsilon}$ instead of Ψ_n; let q_n be the sequence of polynomials whose existence is asserted in that lemma.

Let

$$B_n(z) = \oint_F (Q_n, p_n)(t) \frac{\Psi_n(t) f(t)\, dt}{t - z} = \left(\frac{Q_n}{p_n} A_n\right)(z).$$

We fix a contour γ_1 lying in $\Omega \backslash F$ and separating F from $\partial\Omega$. Since

$$\varlimsup_{n \to \infty} \left| \frac{Q_n(z)}{p_n(z)} \right|^{1/n} \leq 1$$

uniformly on γ_1, it follows from (36) that for some constant $w_1 = w'(\gamma_1) > w$

$$\varlimsup_{n \to \infty} |B_+ n(z)|^{1/2n} \leq e^{-w_1}, \qquad z \in \gamma_1. \tag{37}$$

We now consider the sequence

$$I_n = \left| \int_{\gamma_1} (s_n q_n B_n \Psi_n^{\varepsilon})(t)\, dt \right|.$$

By (37) and Lemma 7,

$$\varlimsup_{n \to \infty} I_n^{1/2n} \leq e^{-w_1 - \varepsilon m_1}, \qquad m_1 = \min_{\gamma_1}(V^{\nu/2} + \psi). \tag{38}$$

On the other hand,

$$\begin{aligned} I_n &= \left| \int_{\gamma_1} (s_n q_n \Psi_n^{\varepsilon})(t) \left(\oint_F (Q_n p_n)(\tau) \frac{\Psi_n(\tau) f(\tau) d\tau}{\tau - t} \right) dt \right| \\ &= \left| \oint_F (Q_n p_n \Psi_n f)(\tau) \left(\int_{\gamma_1} \frac{(s_n q_n \Psi_n^{\varepsilon})(t) dt}{\tau - t} \right) d\tau \right| \\ &= \left| 2\pi i \oint_F (P_n q_n \Psi_n^{1+\varepsilon} f)(\tau)\, d\tau \right|. \end{aligned}$$

Arguing as in the proof of assertion (i) of Theorem 3, we prove that

$$\lim_{n \to \infty} I_n^{1/2n} = e^{-w - \varepsilon m}, \qquad m = \min_F (V^{\nu/2} + \psi).$$

Since $w_1 > w$ and $\varepsilon > 0$ is arbitrary, this contradicts (38). The assertion that (31) does not hold (cf. (36)) thereby leads to a contradiction. Assertion (ii) of the theorem is proved.

9. Let (E, F) be a capacitor, and let φ be an external field given on the plate F (see §1). If λ is the equilibrium charge corresponding to (E, F, φ), then λ_F is the equilibrium measure for (F, ψ), where $\psi = \varphi - V^{\lambda_E}$; it follows from the condition $(E, F, \varphi) \in S$ that $(F, \psi) \in S$. In the conditions of Lemma 1 the sequence ω_{2n+1} was fixed in such a way that $\mu(\omega_{2n+1})/n \to \lambda_E$, and for the sequence $\Psi_n = \Phi_n/\omega_{2n+1}$ we have that

$$\frac{1}{2n} \log \frac{1}{|\Psi_n|} \rightrightarrows \psi = \varphi - V^{\lambda_E}, \qquad z \in \Omega.$$

Lemma 1 is thereby a direct consequence of Theorem 3.

We give a corollary of Theorem 3 relating to more general sequences of multipoint Padé approximants. Let E be an arbitrary compact set in $\hat{\mathbf{C}}$, f a function holomorphic on E, and $\{\alpha_{n,k}\}$, $k = 1,\dots,n$, $n = 1,2,\dots$, a triangular table of points (interpolation nodes) belonging to E. Let R_n, $n = 1,2,\dots$, denote the sequence of (diagonal) Padé approximants of f corresponding to the table $\{\alpha_{n,k}\}$: $R_n = P_n/Q_n$, where P_n and Q_n ($Q_n \not\equiv 0$) are arbitrary polynomials of degree at most n such that $Q_n f - P_n = 0$ at the points of the $(2n+1)$th row of the table $\{\alpha_{n,k}\}$ (counting multiplicity). The next theorem follows from Theorem 3 and relations (11) and (12) in §1.

THEOREM 4. *Assume that the table $\{\alpha_{n,k}\}$ has limit distribution σ:*

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\alpha_{n,k}} \to \sigma \qquad (n \to \infty),$$

F is a compact set in $\hat{\mathbf{C}}\setminus E$ such that $(F, V^{-\sigma}) \in S$, $\mathbf{C}\setminus F$ is connected, and $f \in \mathscr{H}_0(\hat{\mathbf{C}}\setminus F)$. Then the following assertions are true:

(i) *$\mu(Q_n)/n \to \lambda$, where λ is the equilibrium measure for F in the field $\psi = V^{-\sigma}$.*

(ii) *$R_n \xrightarrow{\text{cap}} f$, $z \in D = \hat{\mathbf{C}}\setminus F$, and the degree of convergence is characterized by the relation*

$$|f - R_n|^{1/n} \xrightarrow{\text{cap}} \exp 2(V^{\lambda} - w), \qquad z \in D,\ w = w(F, V^{-\sigma}).$$

In this case the field $\psi = V^{-\sigma}$ is the field of the unit negative charge $-\sigma$, $\sigma \in M(E)$. As already pointed out in subsection 2, in this case λ is the result of balayage of σ onto F; the condition $(F, \psi) \in S$ in this case means that F is a tame compact set, $S(\lambda) = \overline{F}_0$, and

$$\frac{\partial V^{\lambda-\sigma}}{\partial n_+}(\zeta) = \frac{\partial V^{\lambda-\sigma}}{\partial n_-}(\zeta), \qquad \zeta \in F_0.$$

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