# An Algorithm for the Generalized Symmetric Tridiagonal Eigenvalue Problem

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# Generalized Eigenvalue Problem

**Def**

$T, S \in \mathbb{R}^{n \times n}$. We call $(T, S)$ *pencil*.

We consider *only* $\boxed{\begin{array}{l}\text{symmetric}\\ \text{and}\\ \text{tridiagonal}\end{array}}$ pencil.

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**Def (Problem)**

Find $\lambda$ such that $Tx = \lambda Sx$.
$T, S$ symmetric implies $\lambda \in \mathbb{R}$.

# Algorithm philosophy

We find zeros of the polynomial equation

$$\mathcal{F}_{(T,S)}(\lambda) = \det(T - \lambda S) = 0$$

using an iterative method, living on real line.

# Brainstorming

| | |
|---|---|
| **We want:** | Fast and secure iterative method.<br>Starting points for our method.<br>Scalability. |
| **We have:** | Laguerre's method.<br>Cuppen's divide and conquer method.<br>Symmetric tridiagonal matrices. |
| **We add:** | Unreducible condition.<br>Dynamic programming (Bottom-up).<br>Efficient matrix storing. |

# Rapid tour

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# Unreducible pencil

### Def ( as in [1] )

$(T, S)$ is an *unreducible pencil* if $t_{i,i+1}^2 + s_{i,i+1}^2 \neq 0$ for $i = 1, 2, \ldots, n-1$.

# Matrix storin

$$T = trid(sub, diag, super)$$

But $T$ is symmetric, so $sub = super$. We define and use

```
integer, parameter :: dp = kind(1.d0)
real(dp), dimension(1:n,0:1) :: T, S
```

with $T(:,0) = diag$ and $T(:,1) = super$.

## Oss

We don't use T(n,1) and S(n,1).

# Fast and secure iterative method

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# Fast and secure iterative method

$\mathcal{F}_{T,S}(\lambda)$ is a polynomial with only real zeros; we call them

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

where we count with multiplicity.
If $\lambda_m$ and $\lambda_{m+1}$ are simple zeros (mlt=1), then we consider the quadric

$$g_u(X) = (x - X)^2 \sum_{i=1}^{n} \frac{(u - \lambda_i)^2}{(x - \lambda_i)^2} - (u - X)^2$$

if $\boxed{u \neq x}$ then $g_u(x) < 0$ and $g_u(\lambda_m) > 0$.
So we have two sign change.
Bolzano's Theorem tell us that there are two zeros of $g_u$ between $\lambda_m$ and $\lambda_{m+1}$. We call them $X_-, X_+$.

# Fast and secure iterative method

$$\lambda_m < X_- < x < X_+ < \lambda_{m+1}$$

We have one freedom: the $u$ parameter.
Calling $\hat{X}_- = min_u X_-$ and $\hat{X}_+ = max_u X_+$ we can obtain

$$\lambda_m \approx \hat{X}_- < x < \hat{X}_+ \approx \lambda_{m+1}$$

and with algebraic manipulations:

$$\hat{X}_-, \hat{X}_+ = L_\pm(x) = x + \frac{n}{-\frac{f'}{f} \pm \sqrt{(n-1)[(n-1)(-\frac{f'}{f}) - n\frac{f''}{f}]}}$$

with $\frac{f'}{f} = \frac{(\mathcal{F}_{T,S}(\lambda))'}{\mathcal{F}_{T,S}(\lambda)}$.

# Fast and secure iterative method

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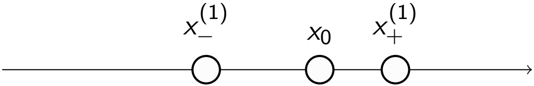

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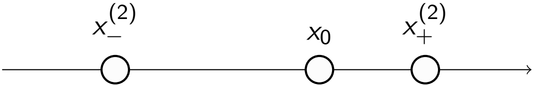

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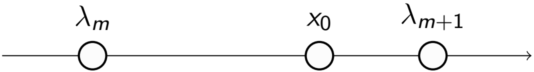

# Fast and secure iterative method

### Def (Laguerre's iteration)

If $mlt(\lambda_m) = mlt(\lambda_{m+1}) = 1$ then

$$x_+^{(k)} = L_+^k(x) = L_+(L_+(\ldots(x_0)))$$
$$x_-^{(k)} = L_-^k(x) = L_-(L_-(\ldots(x_0)))$$

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$$\lambda_m \leftarrow \dots x_-^{(2)} < x_-^{(1)} < x_0 < x_+^{(1)} < x_+^{(2)} \cdots \rightarrow \lambda_{m+1}$$

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So Laguerre's method is secure.

# Fast and secure iterative method

We also have un important property:

**Teo**

*If we choose** $\lambda_m < x_0$ *s.t.* $sign(\frac{f^{'}(x_0)}{f(x_0)}) = sign(\lambda_m - x_0)$ *then* $\{x_-^{(k)}\}_{k=1,\ldots}$ *converge monotonically in asymptotically cubic rate to* $\lambda_m$.

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*for $x_0 < \lambda_{m+1}$ s.t. $sign(\frac{f^{'}(x_0)}{f(x_0)}) = sign(\lambda_m - x_0)$ we have $\{x_+^{(k)}\}_{k=1,\ldots}$ conv. mon. cubic to $\lambda_{m+1}$

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So we can exactly define a $\boxed{\text{neighborood "near" } \lambda}$ and in it we have cubic rate convergence (much, much faster then bisection)
The Laguerre's method is fast.

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# Fast and secure iterative method

It's clear that we need a powerfull method to obtain $x_0$ and an algorithm to estimate $mlt(\lambda_m)$.
*Overstimate* $mlt(\lambda_m)$ (as we can read in [2]) causes no trouble, so the most importan aspects of our calculation are:

- Good $x_0$.
- Good evaluation of $L_{\pm}(x)$.

# Searching $x_0$

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## Ideas:

If $n = 1$ we have $t \cdot v = \lambda \cdot s \cdot v$. $s \neq 0$, so $\lambda = \frac{t}{s}$.

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If $n = 1$ we have $t \cdot v = \lambda \cdot s \cdot v$. $s \neq 0$, so $\lambda = \frac{t}{s}$.

If $n = 2$ we have $\begin{bmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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If $n = 1$ we have $t \cdot v = \lambda \cdot s \cdot v$. $s \neq 0$, so $\lambda = \frac{t}{s}$.
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calling*

$$S^{-1}T = \delta^{-1} \cdot \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}$$

we resolve $det[S^{-1}T - \lambda I] = 0$ with

$$\lambda_{1,2} = \frac{1}{2\delta}\left(\alpha + \beta \pm \sqrt{\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta}\right)$$

---

$^*\alpha =, \beta =, \gamma =.$

# Split

Consider $(\hat{T}, \hat{S})$ with

$$\hat{T} = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix}$$
$$\hat{S} = \begin{bmatrix} S_0 & 0 \\ 0 & S_1 \end{bmatrix}$$

and let be

$$\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_n$$

eigenvalues of $(\hat{T}, \hat{S})$, then

# Split

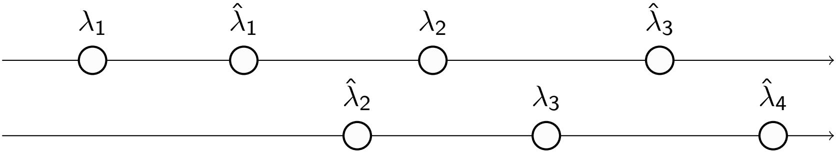
## Teo (A sort of "interlacing")

$$-\infty < \lambda_1 \le \hat{\lambda}_1$$
$$\hat{\lambda}_{i-1} \le \lambda_i \le \hat{\lambda}_{i+1}$$
$$\hat{\lambda}_n \le \lambda_n < \infty$$

*with* $i = 2, 3, \ldots, n-1$.

## Oss

It's possible that



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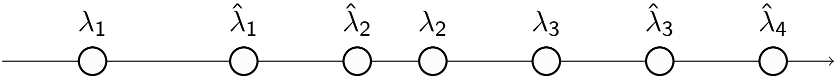

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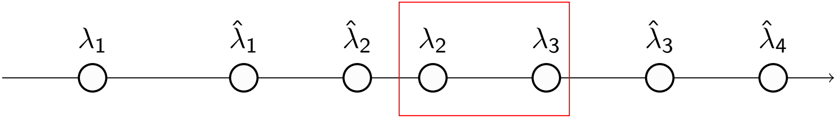

Grazie per l'attenzione.