Laws of Valid Reasoning

Traditional "Laws of Thought"

- Concept that rational thought is constrained to follow certain "laws"
- Three traditional "laws" of thought date to Plato, Aristotle, and others
  - **Identity**: Every object is equal to itself
  - **Non-contradiction**: No proposition is both true and not true
  - **Excluded middle**: Every proposition is either true or not true
- Formal expressions of these laws
  - **Identity**: $\forall x, x = x.$ ("For any object x, x is equal to itself")
  - **Non-contradiction**: $\forall x, \neg(x \land \neg x).$ ("For any *proposition*, x, x is either true or not true")
  - **Excluded middle**: $\forall x, x \lor \neg x.$ (For any proposition, P, P is either true or false.)
- These are understood as laws of rational reasoning

The basic idea

- Laws of valid rational thought have to be logically valid
- They have to hold true *independent of any interpretation*
  - E.g., equality: *no matter what x is*, it's equal to itself
  - E.g., non-contradiction: *no matter what proposition x is,* it can't be that it's both true and false
  - E.g., excluded middle: *no matter what proposition x is,* it is either true or it is false
- The logical $\forall$ *(for all) quantifier* is used to assert a *universal generalization*
  - It gives a name, here x, to some *arbitrary but specific* object
  - It then asserts that some proposition (e.g., x = x) is true of *any* such x
  - E.g., translate to English: $\forall$ *(n : integer), even n $\vee$ odd n*
    - We see here the concept of *typed* variables (here n as to be an integer)
    - We also see the concept of a predicate: (even n) is the proposition that *n is even*
- Quantifiers and predicates are central in *predicate (vs propositional) logic*

Inference Rules

- Inference rules are valid rules of deductive reasoning
- Today we recognize a broader range of such rules
- An inference rule is of the form
  - *if you accept propositions, …, to be true,*
  - *then you can deduce that the proposition, …, is true*
- For example,
  - if you accept that x is true, and you accept that y is true,
  - then you can validly deduce that (x ∧ y) is true
- We can write this as follows: *x, y ⊢ x ∧ y*
  - ⊢, the *turnstile* character, is pronounced *entails*
  - *So: x (being true) and y (being true) entails x ∧ y (being true)*
  - If you know or assume x : true, and y : true, you can deduce x ∧ y : true
  - The notation, x : true, represents what we will call a *truth judgment*

Natural deduction

- Natural deduction is the process of reasoning about the truth, or not, of a given proposition by deduction using the rules of inference of a given logic
- For the logics we'll study, these basic rules are organized around connectives
- Suppose * is any logical operator, such as *and, or, not,* or *implies*
- If we want to deduce the truth of a proposition with * as its main connective then we use an inference rule called an *introduction rule* for *
  - Example, $x, y \vdash x \wedge y$ is the introduction rule for $\wedge$. We will call this rule $\wedge$-*introduction*
- If we want to decide the truth of some proposition *from* (or *given the truth of*) a proposition with * as its main connective, then we use an *elimination* rule for *
  - Example/exercise: there are two elimination rules for $\wedge$. Complete their statements here:
    - $x \wedge y \vdash$ ___   ($\wedge$-*elimination-left*)
    - $x \wedge y \vdash$ ___   ($\wedge$-*elimination-right*)

Display (vertical) notation for inference rules

$$\frac{x, \quad y}{x \wedge y} \quad (\wedge \ introduction)$$

$$\frac{x \wedge y}{x} \quad (\wedge \ elimination \ left)$$

$$\frac{x \wedge y}{y} \quad (\wedge \ elimination \ right)$$

Looking ahead: a theorem to prove

To prove: $\wedge$ *is associative*. What this claim means is that, for any propositions X, Y, and Z, the proposition, *(X $\wedge$ Y) $\wedge$ Z, is equivalent to X $\wedge$ (Y $\wedge$ Z)*. *Note the difference in grouping*. By *equivalent to,* we mean that each implies the other. A full proof is built from two smaller proofs, one in each direction. We will prove the forward direction: *(X $\wedge$ Y) $\wedge$ Z → X $\wedge$ (Y $\wedge$ Z)*. Bear in mind that this proposition asserts that <u>if</u> (X $\wedge$ Y) $\wedge$ Z is accepted as true, then *from* this fact one can deduce that *X $\wedge$ (Y $\wedge$ Z)* must also be true. Notably it does not assert that *(X $\wedge$ Y) $\wedge$ Z is true*.

So how can we get from assuming that *(X $\wedge$ Y) $\wedge$ Z is true* to showing that, in the context of this assumption, *X $\wedge$ (Y $\wedge$ Z) must be true?* Let's see.

Natural Deduction Proof

Proof: We need to show that _if_ (X ∧ Y) ∧ Z is true, then so is X ∧ (Y ∧ Z). We'll start by _assuming_ that (X ∧ Y) ∧ Z is true. Now let's see what we can deduce with this assumption in our context.

1. (X ∧ Y) ∧ Z ⊢ (X ∧ Y)   and-elim left gives us X ∧ Y : true
2. (X ∧ Y) ∧ Z ⊢ Z   and-elim right gives us Z : true
3. (X ∧ Y)   ⊢ X   and-elim left gives us X : true
4. (X ∧ Y)   ⊢ Y   and-elim right gives us Y : true
5. Y, Z ⊢ (Y ∧ Z)   and-intro applied to Y, Z
6. X, (Y ∧ Z) ⊢ X ∧ (Y ∧ Z)   and-intro. _QED ("yay!" in mathspeak)_

English language proof

The great thing about an English proof (*proof summary* would be more accurate) is we can skip details that we can reasonably assume readers will understand. Here's an example.

Prove: For any propositions, X, Y, and Z, *(X ∧ Y) ∧ Z → X ∧ (Y ∧ Z)*

Proof: To start ,we'll <u>assume</u> (X ∧ Y) ∧ Z. From this we can deduce X, Y, and Z (by applying and-elimination several times). From Y, and Z, we can deduce (Y ∧ Z). (by and-introduction). Next, from X and (Y ∧ Z), we can deduce X ∧ (Y ∧ Z). And so what we have shown is that <u>if</u> (X ∧ Y) ∧ Z is true, *then* so must be X ∧ (Y ∧ Z), which is what we mean by *(X ∧ Y) ∧ Z → X ∧ (Y ∧ Z).* QED.

A bit more reasoning

- On the last pages, we proved a universal generalization: for *any propositions, X, Y,* and *Z,* it is the case that *(X ∧ Y) ∧ Z → X ∧ (Y ∧ Z).*
- We can write this more concisely as: ∀ (X Y Z), *(X ∧ Y) ∧ Z → X ∧ (Y ∧ Z)*
- This is a shorthand for ∀ X, ∀ Y, ∀ Z, *(X ∧ Y) ∧ Z → X ∧ (Y ∧ Z)*
- Clearly the converse is also true: ∀ (X Y Z), *X ∧ (Y ∧ Z) → (X ∧ Y) ∧ Z.*
- We can now apply (new information!) the bi-implication introduction rule to conclude ∀ (X Y Z), *(X ∧ Y) ∧ Z ↔ X ∧ (Y ∧ Z).* NB: the arrow goes both ways.
- What we have proved is that *no matter what propositions X, Y, and Z are, the two expressions are equivalent: you can always move parentheses to group elements connected by ∧ without changing the meaning of the expression.*

The power of theorems!

- We now have that ∀ (X Y Z), $(X \land Y) \land Z \leftrightarrow X \land (Y \land Z)$ is a _theorem_
- A theorem is a mathematical statement *that has been proved to be valid*
- The theorem we've proved here shows that _∧ is associative_
  - Theorems are given names to we can refer to them later. Let's call this _∧-associative_
  - Note: Any operator, *, is said to be associative iff $(X * Y) * Z = X * (Y * Z)$
  - Note: we're conflating equivalence of propositions (↔) and equality (=) here – a detail
- Key point: We can now apply a theorem as if it were a new inference rule
  - Suppose we know (SkyBlue ∧ *WaterGreen)* ∧ *EarthBrown*
  - Suppose we want to prove SkyBlue ∧ (*WaterGreen* ∧ *EarthBrown*)
  - All we have to say is that this is true by applying the theorem that states that ∧ *is associative*
- Every time we prove a theorem, it adds a new tool to our reasoning toolkit

Why Natural Deduction Proofs When We Can Use Z3?

- Suppose that X, Y, and Z, are propositional (Boolean) variables
- Now proving $(X \land Y) \land Z \leftrightarrow X \land (Y \land Z)$ means showing that it is *valid*
- We know that we can use Z3 as a SAT solver to check such problems for
  - Satisfiability – there is *at least one* interpretation that makes the formula true (a model)
  - Unsatisfiability – that *no* interpretation makes the formula true
- You can easily check for *validity* (all interpretations are models) by checking to see if the *negation* of a formula is _unsatisfiable_. Fill in the blank.
- The logics of SAT and SMT solvers are limited in what they can express
- There are no general-purpose "solvers" for finding models or proving the validity of arbitrary propositions (otherwise we wouldn't need mathematicians)
- It's in this context of more expressive logics that proof by natural deduction becomes essential