

We claim that a minimum spanning tree, computed with each edge cost equal to the negative of its bandwidth, has this property; and so it is enough to compute a minimum spanning tree.

We first prove this with the assumption that all edge costs are distinct, and then show that it remains true even when this assumption is lifted. (Note that the algorithm remains the same either way; it is just the analysis that has to be extended.) So suppose the claim is not true. Then there is some pair of nodes u, v for which the path P in the minimum spanning tree does not have a bottleneck rate as high as some other u - v path P' . Let $e = (x, y)$ be an edge of minimum bandwidth on the path P ; note that $e \notin P'$. Moreover, e has the smallest bandwidth of any edge in $P \cup P'$. Now, using the edges in $P \cup P'$ other than e , it is possible to travel from x to y (for example, by going from x back to u via P , then to v via P' , then to y via P). Thus, there is a simple path from x to y using these edges, and so there is a cycle C on which e has the minimum bandwidth.

This means that in our minimum spanning tree instance, e has the maximum cost on the cycle C ; but e belongs to the minimum spanning tree, contradicting the cycle property.

Now, if the edge costs are not all distinct, we apply the approach in the chapter: we perturb all edge bandwidths by extremely small amounts so they become distinct, and then find a minimum spanning tree. We therefore refer, for each edge e , to a *real bandwidth* b_e and a *perturbed bandwidth* b'_e . In particular, we choose perturbations small enough so that if $b_e > b_f$ for edges e and f , then also $b'_e > b'_f$. Our tree has the best bottleneck rate for all pairs, under the perturbed bandwidths. But suppose that the u - v path P in this tree did not have the best bottleneck rate if we consider the original, real bandwidths; say there is a better path P' . Then there is an edge e on P for which $b_e > b_f$ for all edges f on P' . But the perturbations were so small that they did not cause any edges with distinct bandwidths to change the relative order of their bandwidth values, so it would follow that $b'_e > b'_f$ for all edges f on P' , contradicting our conclusion that P had the best bottleneck rate with respect to the perturbed bandwidths.

¹ex152.208.224