M5. Suppose we are given an n-node rooted tree T, such that each node v in T is given a weight $w(v)$. An independent set of T is a subset S of the nodes of T such that no node in S is a child or parent of any other node in S. Design an efficient dynamic programming algorithm to find the maximum-weight independent set of the nodes in T, where the weight of a set nodes is simply the sum of the weights of the nodes in that set. What is the running time of your algorithm? **[12]**

Answer: Suppose MWIS(x) denotes the MWIS of the subtree rooted at node x in T. Let $c_1, \ldots, c_k$ be the children of x and $g_1, \ldots, g_l$ the grandchildren of x. Then either x is in MWIS(x) (in which case, it contains no child of x, and for $1 \leq i \leq l$, MWIS(g_i) $\subset$ MWIS(x)) or does not contain x (in which case, it contains one or more children of x).

This suggests the following algorithm.

Direct every edge of T away from the parent. For each node, form a list of its children and a list of its grandchildren.

Now direct every edge of T towards the parent. Topologically sort T. For the vertices x of T taken in the topological order, do the following. (Maintain two arrays A and B. $A[x]$ is to contain the weight of MWIS(x). Boolean $B[x]$ is to be 0 iff x is in MWIS(x).) If x is a leaf, then set $A[x] = w(x)$ and $B[x] = 0$. Otherwise, set $A[x] = \max\{A[c_1] + \ldots + A[c_k], A[g_1] + \ldots + A[g_l] + w(x)\}$; if $A[x]$ is $A[c_1] + \ldots + A[c_k]$, then set $B[x] = 1$, else set $B[x] = 0$.

To print the MWIS of T, the root of which is r, do the following. If $B[r] = 0$, then print r and recurse with the grandchildren of r; else recurse with the children of r.

As every vertex and parent pointer and grandparent pointer is considered once each, the time complexity is $O(n)$.

(Note that the MWIS of T is not necessarily a superset of the MWIS of the subtree rooted at each child of the root. If the root is picked into MWIS, then that will preclude its children, and so the subtrees at them may have to settle for suboptimal IS-es. For example, consider a three node tree in which the root, its child and grandchild have weights of 100, 10 and 1 respectively.)