Consider the directed acyclic graph $G = (V, E)$ constructed in class, with vertices s in the upper left corner and t in the lower right corner, whose s-t paths correspond to global alignments between A and B. For a set of edges $F \subset E$, let $c(F)$ denote the total cost of the edges in F. If P is a path in G, let $\Delta(P)$ denote the set of diagonal edges in P (i.e. the *matches* in the alignment).

Let Q denote the s-t path corresponding to the given alignment. Let E_1 denote the horizontal or vertical edges in G (corresponding to indels), E_2 denote the diagonal edges in G that do not belong to $\Delta(Q)$, and $E_3 = \Delta(Q)$. Note that $E = E_1 \cup E_2 \cup E_3$.

Let $\varepsilon = 1/2n$ and $\varepsilon' = 1/4n^2$. We form a graph G' by subtracting ε from the cost of every edge in E_2 and adding ε' to the cost of every edge in E_3. Thus, G' has the same structure as G, but a new cost function c'.

Now we claim that path Q is a minimum-cost s-t path in G' if and only if it is the unique minimum-cost s-t path in G. To prove this, we first observe that

$$c'(Q) = c(Q) + \varepsilon'|\Delta(Q)| \leq c(Q) + \frac{1}{4},$$

and if $P \neq Q$, then

$$c'(P) = c(P) + \varepsilon'|\Delta(P \cap Q)| - \varepsilon|\Delta(P - Q)| \geq c(P) - \frac{1}{2}.$$

Now, if Q was the unique minimum-cost path in G, then $c(Q) \leq c(P)+1$ for every other path P, so $c'(Q) < c'(P)$ by the above inequalities, and hence Q is a minimum-cost s-t path in G'. To prove the converse, we observe from the above inequalities that $c'(Q)-c(Q) > c'(P)-c(P)$ for every other path P; thus, if Q is a minimum-cost path in G', it is the unique minimum-cost path in G.

Thus, the algorithm is to find the minimum cost of an s-t path in G', in $O(mn)$ time and $O(m+n)$ space by the algorithm from class. Q is the unique minimum-cost A-B alignment if and only if this cost matches $c'(Q)$.

[1] ex485.507.165