

We first do this under the assumption that all edge costs are distinct. In this case, we can solve (a) as follows. Let $e = (v, w)$ be the new edge being added. We represent T using an adjacency list, and we find the v - w path P in T in time linear in the number of nodes and edges of T , which is $O(|V|)$. If every node on this path in T has cost less than c , then the Cycle Property implies that the new edge $e = (v, w)$ is not in the minimum spanning tree, since it is the most expensive edge on the cycle C formed from P and e , so the minimum spanning tree has not changed. On the other hand, if some edge on this path has cost greater than c , then the Cycle Property implies that the most expensive such edge f cannot be in the minimum spanning tree, and so T is no longer the minimum spanning tree.

For (b), we replace the heaviest edge on the v - w path P in T with the edge $e = (v, w)$, obtaining a new spanning tree T' . We claim that T' is a minimum spanning tree. To prove this, we consider any edge e' not in T' , and show that we can apply the Cycle Property to conclude that e' is not in any minimum spanning tree. So let $e' = (v', w')$. Adding e' to T' gives us a cycle C' consisting of the v' - w' path P' in T' , plus e' . If we can show e' is the most expensive edge on C' , we are done.

To do this, we consider one further cycle: the cycle K formed by adding e' to T . By the Cycle Property, e' is the most expensive edge on K . So now there are three cycles to think about: C , C' , and K . Edge f is the most expensive edge on C , and edge e' is the most expensive edge on K . Now, if the new edge e does not belong to C' , then $C' = K$, and so e' is the most expensive edge on C' . Otherwise, the cycle K includes f (since C' needed to use e instead), and C' uses a portion of C (including e) and a portion of K . In this case, e' is more expensive than f (since f lies on K), and hence it is more expensive than everything on C (since f is the most expensive edge on C). It is also more expensive than everything else on K , and so it is the most expensive edge on C' , as desired.

Now, if the edge costs are not all distinct, we apply the approach in the chapter: we first perturb all edge costs by extremely small amounts so they become distinct. Moreover, we do this so we add a very small quantity ϵ to the new edge e , and we perturb the costs of all other edges f by even much smaller, distinct, quantities δ_f . For a tree T , let $c(T)$ denote its real (original) cost, and let $c'(T)$ denote its perturbed cost.

Now we use the above solution with distinct edge costs. Our perturbation has the following two properties.

- (i) First, for trees T_1 and T_2 , if $c'(T_2) < c'(T_1)$, then $c(T_2) \leq c(T_1)$.
- (ii) Second, if $c(T_1) = c(T_2)$, and T_2 contains e but T_1 doesn't, then $c(T_2) > c(T_1)$.

It follows from these two properties that our conclusion in (a) is correct: since $c'(T') < c'(T)$, and T' contains e but T doesn't, property (i) implies $c(T') \leq c(T)$, and then property (ii) implies $c(T') < c(T)$. Now, in (b), we compute a minimum spanning tree with respect to the perturbed costs which, by property (i), is also one of (possibly several) minimum spanning trees with respect to the real costs.

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