Computational Complexity Theory

Lecture 1: Intro; Turing machines

Department of Computer Science,
Indian Institute of Science

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a. Decision problem

Example: Is vertex t reachable from vertex s in graph G?

Is n a prime number?

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  a. Decision problem

  b. Search problem

Example: Find a satisfying assignment for a Boolean formula.

Find a prime between n and 2n.

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- Computational **problems** come in various flavors:

  a. Decision problem

  b. Search problem

  c. Counting problem

Example: Count the number of cycles in a graph.

Count the number of perfect matchings in a graph.

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- Computational **problems** come in various flavors:

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  c. Counting problem

  d. Optimization problem

Example: Find a minimum size vertex cover in a graph

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- Computational complexity attempts to classify computational problems based on the amount of resources required by algorithms to solve them.
- **Algorithms** are methods for solving problems; they are studied using formal models of computation, like Turing machines.

↓

- a memory with head (like a RAM)
- a finite control (like a processor)

(…more later in this lecture)

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  - Time (bit operations)
  - Space (memory cells)

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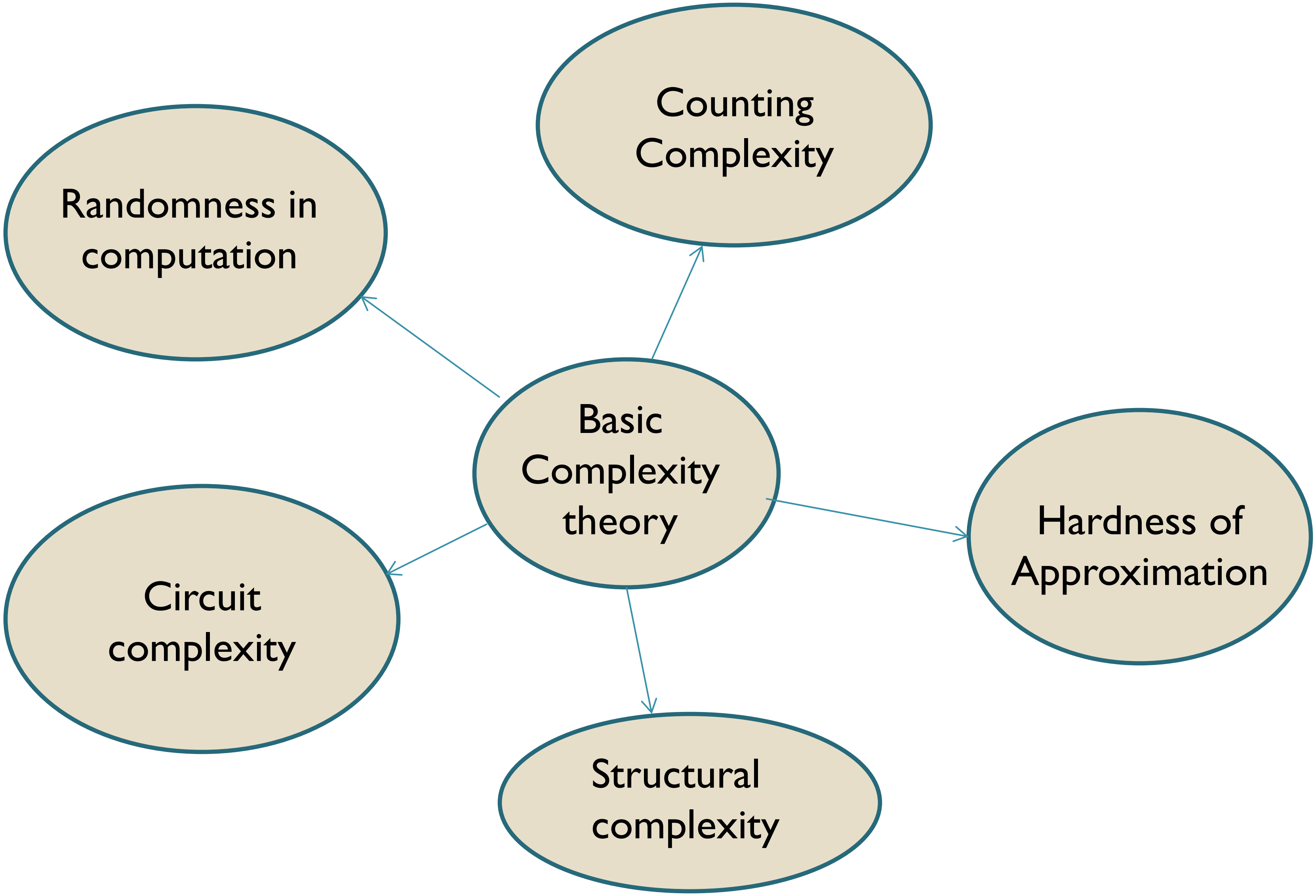
- Computational **resources** (required by models of computation) can be:
  - Time (bit operations)
  - Space (memory cells)
  - Random bits (magic bits: 0 w. p ½ and 1 w.p ½)
  - Communication (bit exchanges)

Topics to be covered in this course



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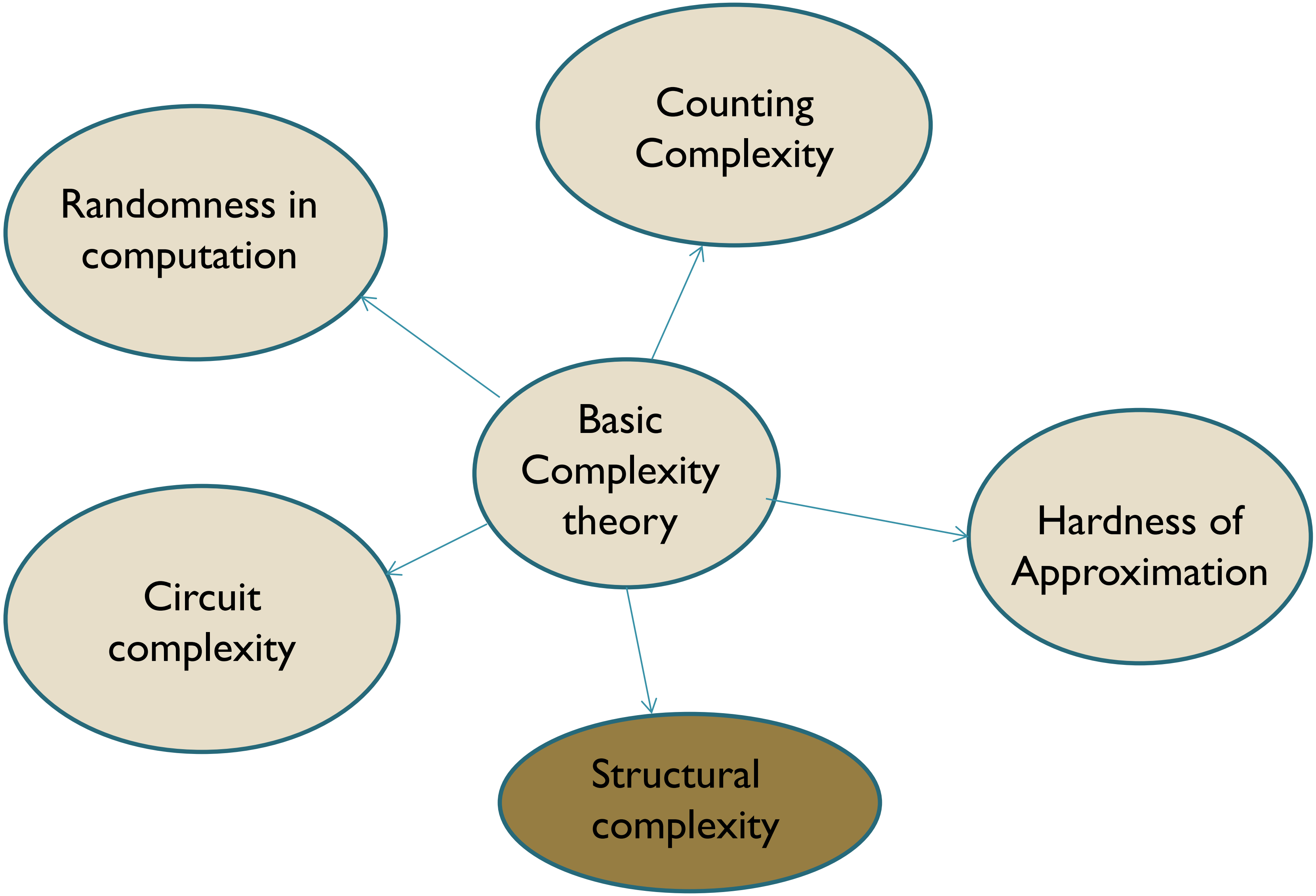


Counting Complexity

Randomness in computation

Basic Complexity theory

Hardness of Approximation

Circuit complexity

Structural complexity

Structural Complexity

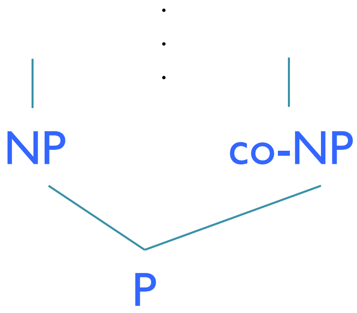
- Classes P, NP, co-NP… NP-completeness.
  - How hard is it to check satisfiability of a Boolean formula?
  - What if the formula has exactly one or no satisfying assignment?

Structural Complexity

- Classes P, NP, co-NP… NP-completeness.
- Space bounded computation.
    - How much space is required to check s-t connectivity?

Structural Complexity

- Classes P, NP, co-NP… NP-completeness.
- Space bounded computation.
- Polynomial Hierarchy (PH).



- How hard is it to check if the largest independent set in G has size k ?

- How hard is it to check if there is a circuit of size k that computes the same Boolean function as a given Boolean circuit C ?

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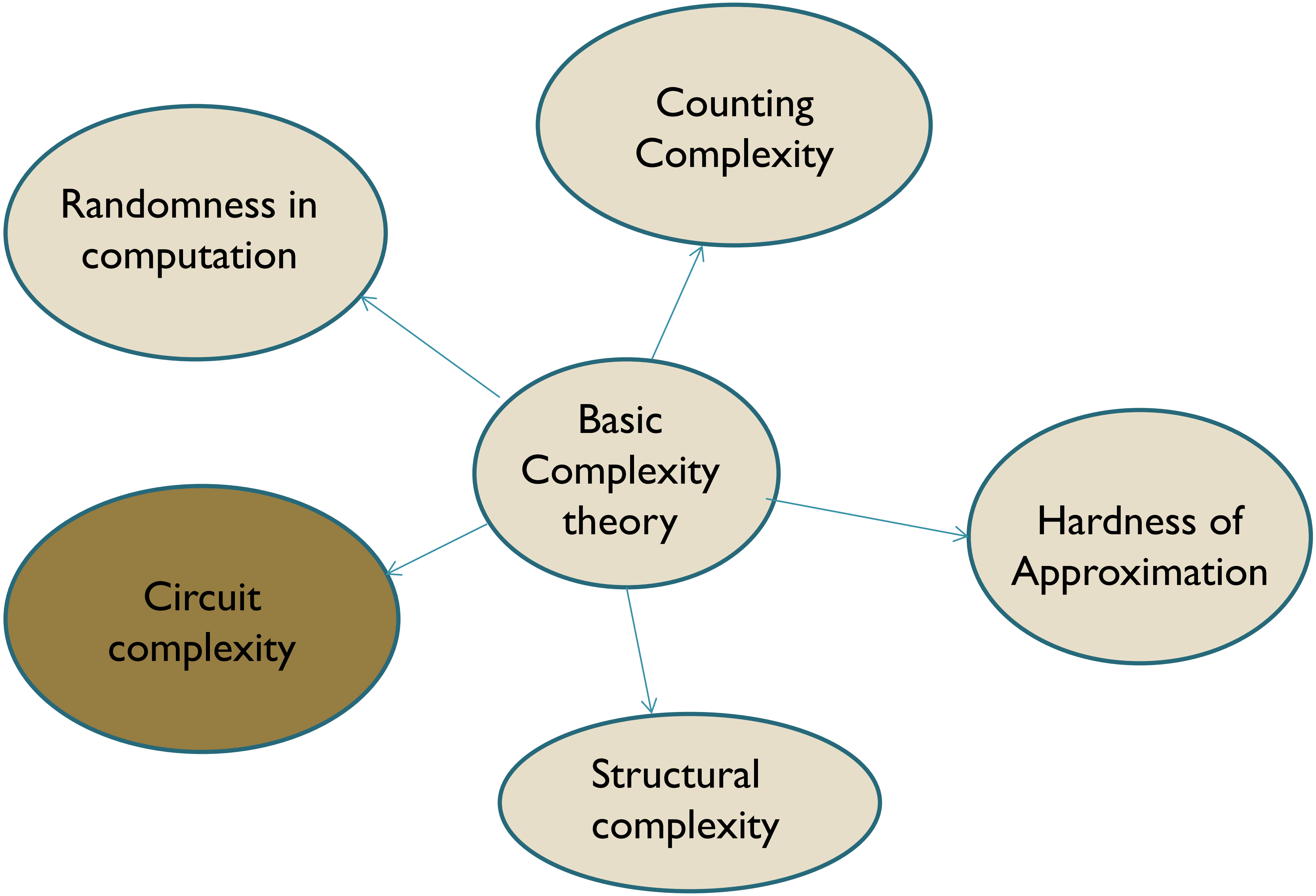

Basic Complexity theory

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Structural complexity

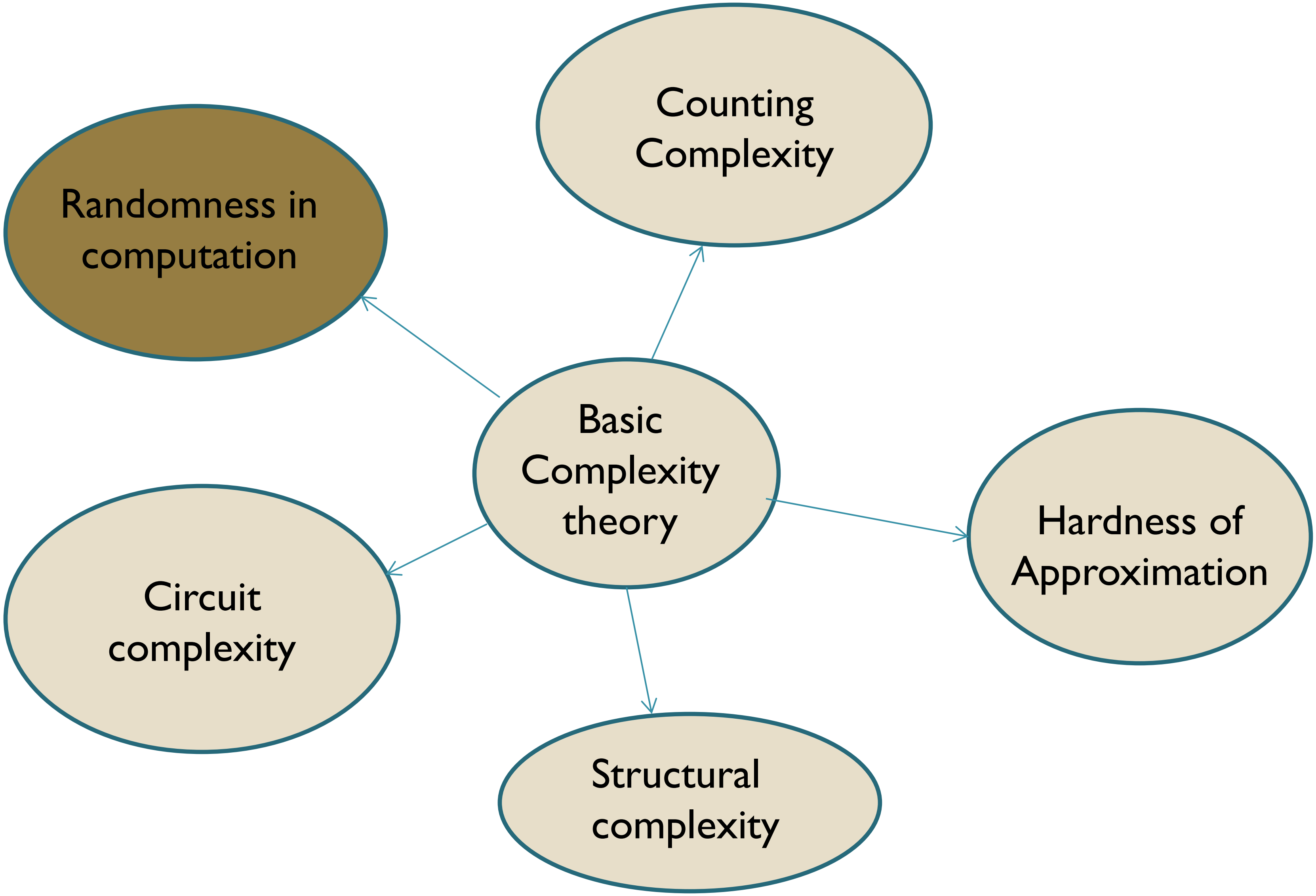
Circuit Complexity

- The internal workings of an algorithm can be viewed as a Boolean circuit -- a nice combinatorial model of computation that is closely related to Turing Machines.
- The size, depth & width of a circuit correspond to the sequential, parallel & space complexity, respectively, of the algorithm that it represents.

Circuit Complexity

- The internal workings of an algorithm can be viewed as a Boolean circuit -- a nice combinatorial model of computation that is closely related to Turing Machines.
- The size, depth & width of a circuit correspond to the sequential, parallel & space complexity, respectively, of the algorithm that it represents.
- Proving $\mathrm{P} \neq \mathrm{NP}$ reduces to showing circuit lower bounds.
    - We will see lower bounds for restricted classes of circuits.

Topics to be covered in this course



Randomness in Computation

- Probabilistic complexity classes (BPP, RP, co-RP).
  - Does randomization help in improving efficiency?
  - Quicksort has O(n log n) expected time but O(n^2) worst case time.
  - Can SAT be solved in polynomial time using randomness?

    Theorem (Schoening, 1999): 3SAT can be solved in *randomized* $O((4/3)^n)$ time.

  - Access to random bits can help improve computational efficiency… but, to what extent?

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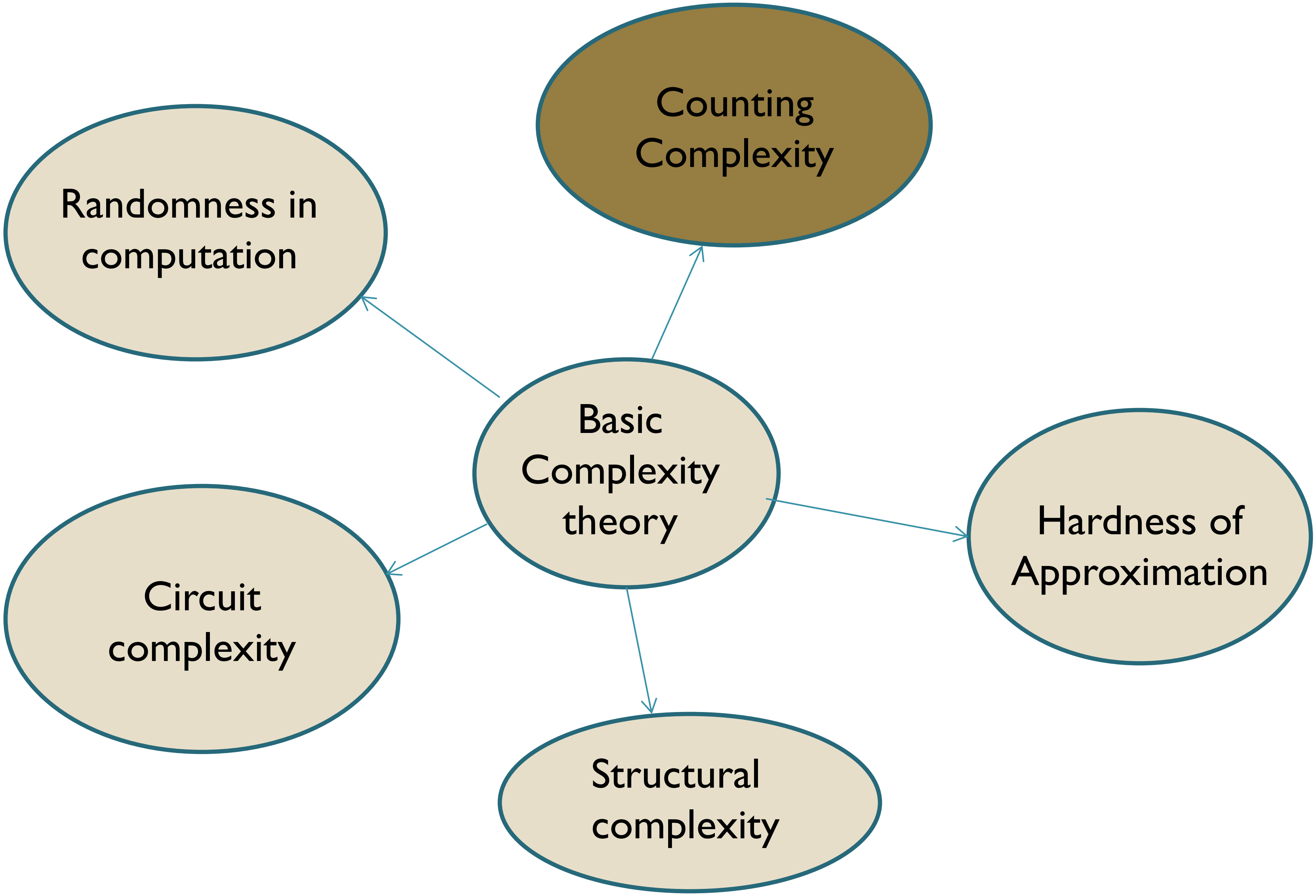

Counting Complexity

- Counting complexity classes (class #P).
  - How hard is it to count the number of perfect matchings in a graph?
  - How hard is it to count the number of cycles in a graph?
  - Can we compute the number of simple paths between s and t in G efficiently?
  - Is counting much harder than the corresponding decision problem?

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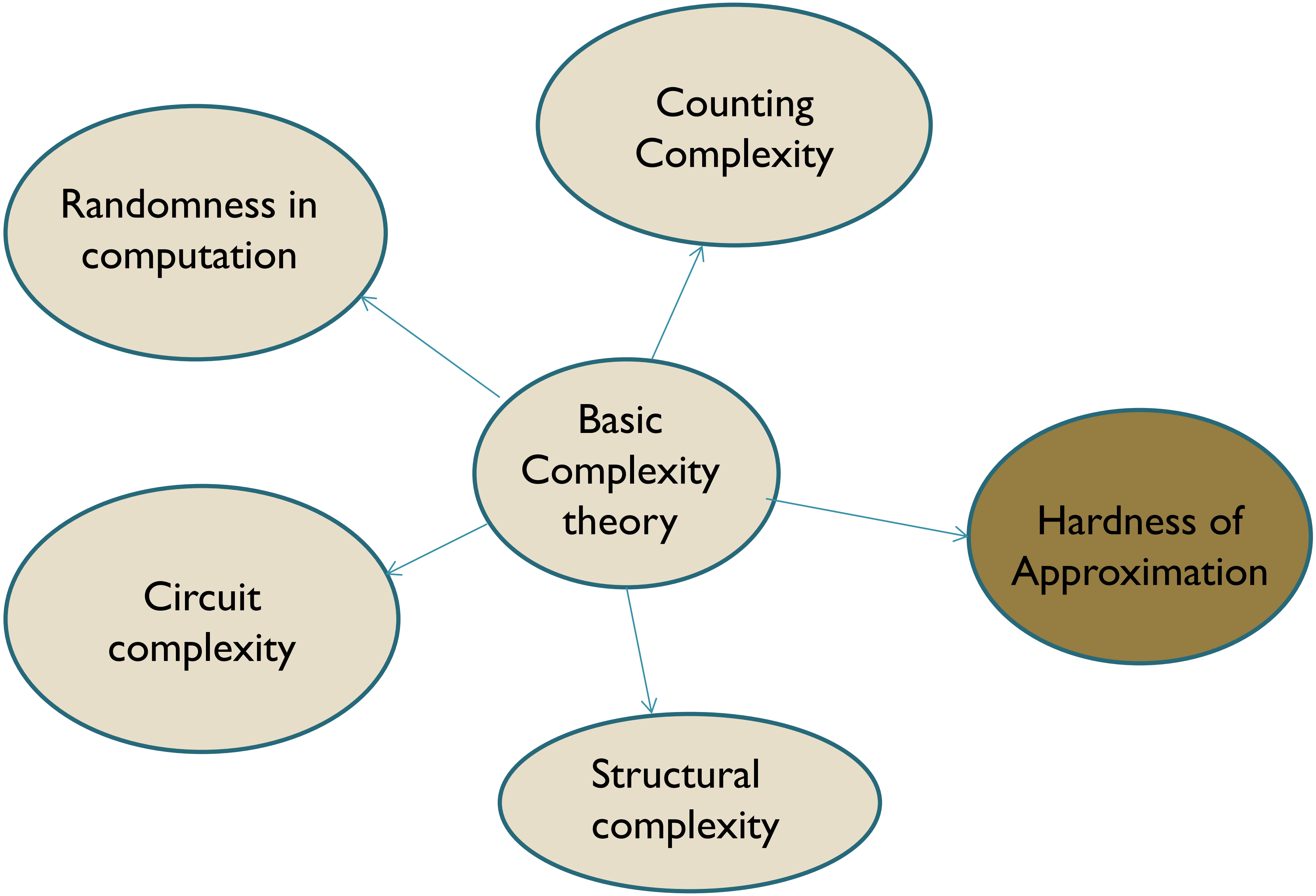

Basic Complexity theory

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Hardness of Approximation

Hardness of Approximation

- Probabilistically Checkable Proofs (PCPs).

Hardness of approximation results.

Theorem (Hastad, 1997): If there's a poly-time algorithm to compute an assignment that satisfies at least 7/8 + ε fraction of the clauses of an input 3SAT, for any constant ε > 0, then P = NP.

Course Info

- **Course no.:** E0 224 **Credits:** 3:1
- **Instructor:** Chandan Saha
- **Lecture time:** M, W 2-3:30 pm. **Venue:** CSA 112

- **Course homepage:**

https://www.csa.iisc.ac.in/~chandan/courses/complexity22/home.html

Course Info

- **Prerequisites**: Basic familiarity with algorithms; Mathematical maturity.
- **Primary reference:** Computational Complexity – A Modern Approach by Sanjeev Arora and Boaz Barak.
- **Lectures:** Slides will be posted on the course homepage.
- **Number of lectures:** ~27.

Course Info

- **Grading policy**: Three assignments - 45%
  One presentation - 25%
  Final exam - 30%

Assignments

- **First assignment:** Will posted on Aug 31; due date will be Sep 14.
- **Second assignment:** Will posted on Sep 30; due date will be Oct 14.
- **Third assignment:** Will posted on Oct 31; due date will be Nov 14.
- **Mode:** Assignments will be posted on the course homepage. You need to e-mail me your assignment as a pdf file (use Latex).

Presentations

- A group of 2 students would present a paper/result.
- **Duration of a presentation:** 1-1.5 hr.
- **Mode:** In class, use slides.

- I will start giving topics to present from mid-Sep. All topics will be handed out by mid-Oct.
- You will get ~4 weeks to prepare a presentation.
- We will finish all the presentations by Nov 23 (Wed).

Final exam

- Would be a 3 hr long written test.

- **When?** First week of Dec.

Let’s begin…

Turing Machines

- An algorithm is a set of instructions or rules.
- To understand the performance of an algorithm we need a model of computation. Turing machine is one such *natural* model (introduced by Alan Turing in 1936).

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- Turing machines ↔ A mathematical way to describe algorithms.

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(e.g. of a physical realization of a TM is a simple adder)

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- δ is a function from $Q \times \Gamma^k$ to $Q \times \Gamma^k \times \{L,S,R\}^k$

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known as transition function; it captures the dynamics of M

Turing Machines: Computation

- Start configuration.
  - All tapes other than the input tape contain blank symbols.
  - The input tape contains the input string.
  - All the head positions are at the start of the tapes.
  - The machine is in the start state q_{start} .

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- Computation.
  - A **step of computation** is performed by applying δ.
- Halting.
  - Once the machine enters q_{halt} it stops computation.

Turing Machines: Running time

- Let f: $\{0,1\}^* \rightarrow \{0,1\}^*$ and T: N → N and M be a Turing machine.

- Definition. We say M ***computes*** f if on every x in $\{0,1\}^*$, M halts with f(x) on its output tape beginning from the start configuration with x on its input tape.

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- Definition. M computes f *in T(|x|)* ***time***, if for every x in {0,1}*, M halts within T(|x|) steps of computation and outputs f(x).

Turing Machines

- In this course, we would be dealing with
  - Turing machines that halt on every input.
  - Computational problems that can be solved by Turing machines.

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  - ➢ Turing machines that halt on every input.
  - ➢ Computational problems that can be solved by Turing machines.

- Can every computational problem be solved using Turing machines?

Turing Machines: Uncomputability

- There are problems for which there exists *no* TM that halts on every input instances of the problem and outputs the correct answer.

  - Input: A system of polynomial equations in many variables with integer coefficients.
  - Output: Check if the system has integer solutions .
  - Question: Is there an algorithm to solve this problem?

Turing Machines: Uncomputability

- There are problems for which there exists *no* TM that halts on every input instances of the problem and outputs the correct answer.

  - A typical input instance:

$$\left.\begin{array}{l} x^2y + 5y^3 = 3 \\[2em] x^2 + z^5 - 3y^2 = 0 \\[2em] y^2 - 4z^6 = 0 \end{array}\right\} \text{Integer solutions for x, y, z?}$$

Turing Machines: Uncomputability

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(Hilbert's tenth problem, 1900)

Turing Machines: Uncomputability

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  - Input: A system of polynomial equations in many variables with integer coefficients.
  - Output: Check if the system has integer solutions .
  - Question: Is there an algorithm to solve this problem?

- Theorem. There doesn't exist any algorithm (realizable by a TM) to solve this problem. (Davis, Putnam, Robinson, Matiyasevich 1970)

Why Turing Machines?

- TMs are natural and intuitive.
- Church-Turing thesis: *"Every physically realizable computation device – whether it's based on silicon, DNA, neurons or some other alien technology – can be simulated by a Turing machine".*

--- [quoted from Arora-Barak's book]

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- Several other computational models can be simulated by TMs.

Why Turing Machines?

- TMs are natural and intuitive.

- Strong Church-Turing thesis: *“Every physically realizable computation device – whether it’s based on silicon, DNA, neurons or some other alien technology – can be simulated efficiently by a Turing machine”.*

Possible exception: Quantum machines!

Basic facts about TMs

Turing Machines

- Time constructible functions. A function $T: N \rightarrow N$ is *time constructible* if $T(n) \geq n$ and there's a TM that computes the function that maps x to $\underbrace{T(|x|)}_{\text{in binary}}$ in $O(T(|x|))$ time.

- Examples: $T(n) = n^2$, or 2^n, or $n \log n$

Turing Machines: Robustness

- Let f: {0,1}* → {0,1}* and T: N → N be a time constructible function.

- Binary alphabets suffice.
  - If a TM M computes f in T(n) time using Γ as the alphabet set, then there's another TM M' that computes f in time 4.log |Γ| .T(n) using {0, 1, blank} as the alphabet set.

Turing Machines: Robustness

- Let $f: \{0,1\}^* \rightarrow \{0,1\}^*$ and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a time constructible function.

- Binary alphabets suffice.
  - If a TM M computes f in $T(n)$ time using Γ as the alphabet set, then there's another TM M' that computes f in time $4.\log |\Gamma| .T(n)$ using $\{0, 1, \text{blank}\}$ as the alphabet set.

- A single tape suffices.
  - If a TM M computes f in $T(n)$ time using k tapes then there's another TM M' that computes f in time $5k \,.\, T(n)^2$ using a single tape that is used for input, work and output.

Turing Machines: As strings

- Every TM can be represented by a finite string over {0,1}.

…simply encode the description of the TM.

Turing Machines: As strings

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- Every string over {0,1} represents some TM.

  …invalid strings map to a fixed, trivial TM.

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- Every TM has infinitely many string representations.

  … allow padding with arbitrary number of 0's

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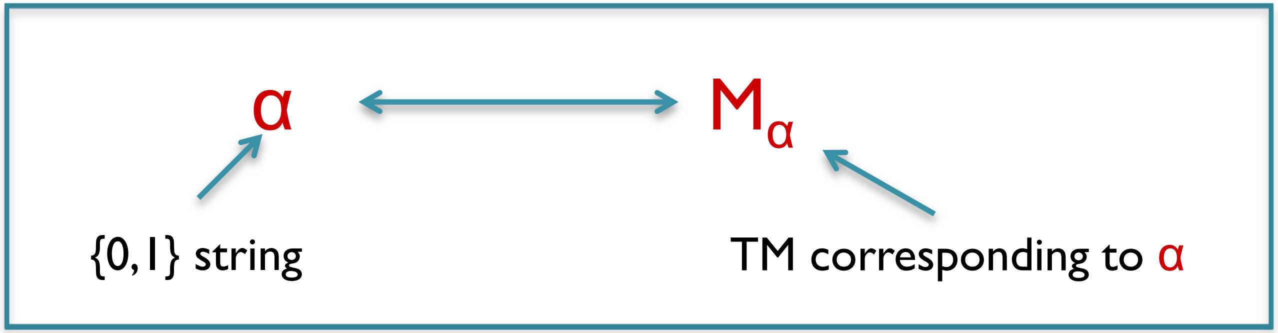

α $\longleftrightarrow$ M_α

{0,1} string

TM corresponding to α

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- Every TM has infinitely many string representations.

- A TM (i.e., its string representation) can be given as an input to another TM !!

Universal Turing Machines

- Theorem. There exists a TM U that on every input x, α in $\{0,1\}^*$ outputs $M_\alpha(x)$.

- Further, if M_α halts within T steps then U halts within $C. T. \log T$ steps, where C is a constant that depends only on M_α 's alphabet size, number of states and number of tapes.

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- Physical realization of UTMs are modern day electronic computers.