# Eficient Quantum Algorithms for Dissipative Nonlinear Differential Eqautions

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# Details of the Differerential Equation

## Term wise analysis

**Guiding question**: Can non-linear differential equations be solved efficiently by Quantum algorithms?

### Non-linear (Quadratic) Differential equation:

$$\frac{du}{dt} = F_2 \cdot u^{\otimes 2} + F_1 \cdot u + F_0(t)$$

- $u = [u_1, ..., u_n]^T \in \mathbb{R}^n$
- $u^{\otimes 2} = [u_1^2, u_1 u_2, ..., u_1 u_n, u_2 u_1, ..., u_n u_{n-1,u_n^2}]^T \in \mathbb{R}^{n^2}$
- $u_j = u_j(t)$ is function of time $t \in [0, T]$
- $F_2 \in \mathbb{R}^{n \times n^2}$ and $F_1 \in \mathbb{R}^{n \times n}$ are time-independent matrices
- $F_0(t) \in \mathbb{R}^n$ is term accounting for inhomogenity

# Defining parameters of the Differential Equation

$$\frac{du}{dt} = F_2 \cdot u^{\otimes 2} + F_1 \cdot u + F_0(t)$$

## Requirements for Quantum advantage

- $F_2, \; F_1 \; F_0$ are s-sparse, i.e., the maximum number of non-zero entries in each row or column $\leq s$
- Parameter $R < 1$ (It is defined below.)

## Parameter R

- Let $\lambda_j$ be eigenvalue of $F_1 \in \mathbb{R}^{n \times n}$ such that $Re(\lambda_n) \leq ... \leq Re(\lambda_1) < 0$
- $R := \frac{1}{|re(\lambda_1)|} \left( \|u_{in}\| \cdot \|F_2\| + \frac{\|F_0\|}{\|u_{in}\|} \right)$

# Meaning of Parameter R in the context

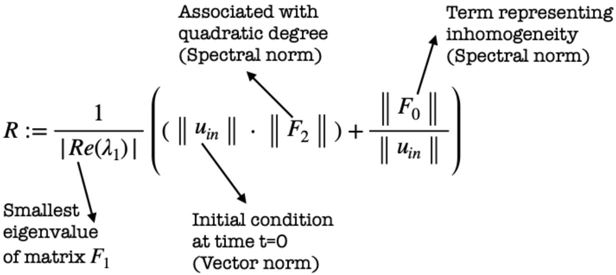


Figure: Parameter R characterize the Differential equation.[1]

## Interpretation of parameter (R)

- ODE is homogenous if $F_0(t) = 0 \implies \mathrm{R} = \frac{||u_{in}||\cdot||F_2||}{|Re(\lambda_1)|}$
- Or, $R = \frac{non-linearity}{linear\ dissipation}$; qualitatively similar to Reynolds number in fluid dynamics

[1] Personal adaptation based on the details in the paper

# When Does Quantum Advantage Exist and How Much?

## If R < 1, then as per Theorem 1 in the paper

- The differential equation has a general case Quantum Algorithm that goes polynomial in simulation time (T).
- Runtime: $\mathcal{O}[T^2 \cdot q \cdot poly(log(T), log(n), log(\frac{1}{\epsilon}))]$

## If R > $\sqrt{2}$, then as per Theorem 2 in the paper

- Then, a general efficient algorithm is not possible. (under the assumption, $BQP \neq PP$)
- Special cases can still have an efficient Quantum algorithm.
- An efficient algorithm for Viscous Burger Equation does exist even though $R \approx 44$

## If 1< R <$\sqrt{2}$

- Status unknown as of October 2023

# Use cases to solve real-world problem

## Derived and discussed in the paper

- SEIR Model of epidemiology
- Certain special cases of Navier-Stokes-type equation.

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\beta \nabla^2 \mathbf{u} + \mathbf{f}$$

- Explicit example: Forced viscous Burgers equation.

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# Quantum Algorithm for the problem

## Algorithmic steps

- Step-I: Convert the dissipative non-linear ODE to an infinite system of linear ODE. Use Quantum Carleman linearization.
- Step-II: Truncate the infinite linear system ODE to a finite system of equations based on allowed approximation error.
- step-III: Discretized the time domain and applied the Forward Euler method to the linear ODE.
- Step-IV: The last step yields a system of linear equations (SLE). Check if SLE fulfils the requirement for an efficient algorithm for Quantum linear system Algorithms (QLSA).
- Step-V: Apply QLSA and use postselection to generate the time-evolved state vector $\mathbf{u(T)}$.[*]

[*] See the caveat on the next page

# Summary of requirements and a few caveats

## Requirements for Quantum advantage

- Dissipation is strong relative to the nonlinear and forcing term: $R<1$
- The solution shall not decay "too rapidly". (Quantified in the paper)

## Important Caveats

- The algorithm produces a state vector that encodes the desired solution without specifying how to extract information from it.
- This is a common caveat to QSLA-based algorithms. In many cases, this is used as a subroutine to other quantum algorithms that require such a state in their intermediate process.

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# Conclusions

- This Quantum algorithm is capable of addressing certain special cases of dissipative nonlinear systems efficiently
- Carleman linearization of nonlinear ODE has been used as the main hammer to deal with such complex system
- This is in line with using Quantum machines to simulate classical phenomena rather than the Qauntum phenomena. The former is known to have a Quantum advantage over any (known) classical computation.

Thank You for Your Attention!