

Problema 1

Calculemos primero la derivada parcial de r con respecto a x e y :

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}[(x^2 + y^2)^{1/2}] = \frac{2x}{2}(x^2 + y^2)^{-1/2} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}[(x^2 + y^2)^{1/2}] = \frac{2y}{2}(x^2 + y^2)^{-1/2} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

Luego

$$\begin{aligned} \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 &= \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 \\ &= \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} \\ &= \frac{x^2 + y^2}{x^2 + y^2} \\ &= 1 \end{aligned}$$

Así

$$\boxed{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1} \quad (1)$$

Calculemos ahora las segundas derivadas parciales de r con respecto a x e y

$$\begin{aligned} \frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x} \left[\frac{x}{\sqrt{x^2 + y^2}} \right] \\ &= \frac{\sqrt{x^2 + y^2} - x \frac{x}{\sqrt{x^2 + y^2}}}{x^2 + y^2} \\ &= \frac{\sqrt{x^2 + y^2} - \frac{x^2}{\sqrt{x^2 + y^2}}}{x^2 + y^2} \\ &= \frac{\frac{x^2 + y^2 - x^2}{\sqrt{x^2 + y^2}}}{x^2 + y^2} \\ &= \frac{y^2}{(x^2 + y^2)\sqrt{x^2 + y^2}} \end{aligned}$$

De manera similar tenemos

$$\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y} \left[\frac{y}{\sqrt{x^2 + y^2}} \right] = \frac{x^2}{(x^2 + y^2)\sqrt{x^2 + y^2}}$$

Luego,

$$\begin{aligned}
 \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} &= \frac{y^2}{(x^2 + y^2)\sqrt{x^2 + y^2}} + \frac{x^2}{(x^2 + y^2)\sqrt{x^2 + y^2}} \\
 &= \frac{x^2 + y^2}{(x^2 + y^2)\sqrt{x^2 + y^2}} \\
 &= \frac{1}{\sqrt{x^2 + y^2}} \\
 &= \frac{1}{r}
 \end{aligned}$$

Es decir,

$$\boxed{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}} \quad (2)$$

Calculemos ahora las segundas derivadas parciales de u con respecto a x e y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

Luego

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left[f'(r) \frac{\partial r}{\partial x} \right] = \frac{\partial f'(r)}{\partial x} \frac{\partial r}{\partial x} + f'(r) \frac{\partial^2 r}{\partial x^2} \\
 &= \frac{\partial r}{\partial x} \frac{\partial f'(r)}{\partial r} \frac{\partial r}{\partial x} + f'(r) \frac{\partial^2 r}{\partial x^2} \\
 &= f''(r) \left(\frac{\partial r}{\partial x} \right)^2 + f'(r) \frac{\partial^2 r}{\partial x^2}
 \end{aligned} \quad (3)$$

De manera similar tenemos

$$\frac{\partial^2 u}{\partial y^2} = f''(r) \left(\frac{\partial r}{\partial y} \right)^2 + f'(r) \frac{\partial^2 r}{\partial y^2} \quad (4)$$

Sumando (??) y (??) obtenemos

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f''(r) \left(\frac{\partial r}{\partial x} \right)^2 + f'(r) \frac{\partial^2 r}{\partial x^2} + f''(r) \left(\frac{\partial r}{\partial y} \right)^2 + f'(r) \frac{\partial^2 r}{\partial y^2} \\
 &= f''(r) \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right] + f'(r) \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} \right]
 \end{aligned}$$

Utilizando (??) y (??) tenemos

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)}$$