Problema 1

Calculemos primero la derivada parcial de r con respecto a x e y:

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}[(x^2+y^2)^{1/2}] = \frac{2x}{2}(x^2+y^2)^{-1/2} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}[(x^2+y^2)^{1/2}] = \frac{2y}{2}(x^2+y^2)^{-1/2} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

Luego

$$\begin{aligned}\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 &= \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 \\ &= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \\ &= \frac{x^2+y^2}{x^2+y^2} \\ &= 1\end{aligned}$$

Así

$$\boxed{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1} \tag{1}$$

Calculemos ahora las segundas derivadar parciales de r con respcto a x e y

$$\begin{aligned}\frac{\partial^2 r}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2}}\right] \\ &= \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} \\ &= \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} \\ &= \frac{\frac{x^2+y^2-x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} \\ &= \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}\end{aligned}$$

De manera similar tenemos

$$\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{y}{\sqrt{x^2+y^2}}\right] = \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Luego,

$$\begin{aligned}
\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} &= \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{x^2}{(x^2+y^2\sqrt{x^2+y^2})} \\
&= \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}} \\
&= \frac{1}{\sqrt{x^2+y^2}} \\
&= \frac{1}{r}
\end{aligned}$$

Es decir,

$$\boxed{\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}} \tag{2}$$

Calculemos ahora las segundas derivadas parciales de u con respecto a x e y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

Luego

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right] &= \frac{\partial f'(r)}{\partial x}\frac{\partial r}{\partial x} + f'(r)\frac{\partial^2 r}{\partial x^2} \\
&= \frac{\partial r}{\partial x}\frac{\partial f'(r)}{\partial r}\frac{\partial r}{\partial x} + f'(r)\frac{\partial^2 r}{\partial x^2} \\
&= f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2} \qquad (3)
\end{aligned}$$

De manera similar tenemos

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2} \tag{4}$$

Sumando (3) y (4) obtenemos

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2} + f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2} \\
&= f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]
\end{aligned}$$

Utilizando (1) y (2) tenemos

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)}$$