

# Tensores

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# Capítulo 1

## Tensor Algebra

### 1.1. Introduction

<sup>1</sup> To work effectively in Newtonian theory, one really needs the language of vectors. This language, first of all, is more succinct, since it summarizes a set of three equations in one. Moreover, the formalism of vectors helps to solve certain problems more readily, and, most important of all, the language reveals structure and thereby offers insight. In exactly the same way, in relativity theory, one needs the language of tensors. Again, the language helps to summarize sets of equations succinctly and to solve problems more readily, and it reveals structure in the equations. This part is devoted to learning the formalism of tensors which is a pre-condition for the rest.

The approach we adopt is to concentrate on the technique of tensors without taking into account the deeper geometrical significance behind the theory. We shall be concerned more with what you do with tensors rather than what tensors actually are. There are two distinct approaches to the teaching of tensors: the abstract or index-free (coordinate-free) approach and the conventional approach based on indices. There has been a move in recent years in some quarters to introduce tensors from the stars using the more modern abstract approach (although some have subsequently changed their mind and reverted to the conventional approach). The main advantage of this approach is that it offers deeper geometrical insight. However, it has two disadvantages. First of all, it requires much more of a mathematical background, which in turn takes time to develop. The other disadvantage is that, for all its elegance, when one wants to do real calculation with tensors, as one frequently needs to, then recourse has to be made to indices. We shall adopt the more conventional index approach, because it will prove faster and more practical. However, we advise those who wish to take their study of the subject further to look at the index-free approach at the first opportunity.

### 1.2. Manifolds and coordinates

\*\*\* Quizás se podría complementar algo más \*\*\*

We shall start by working with tensors defined in  $n$  dimensions since, and it is part of the power of the formalism, there is little extra effort involved. A tensor is an object defined on a geometric entity called a (differential) **manifold**. We shall not define a manifold precisely because

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<sup>1</sup>Notes taken from Introducing Einstein's relativity by Ray D'Inverno

it would involve us too much of a digression. But, in simple terms, a manifold is something which 'locally' looks like a bit of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$

We shall simply take an  $n$ -dimensional manifold  $M$  to be a set of points such that each point possesses a set of  $n$  **coordinates**  $x^1, x^2, \dots, x^n$ , where each coordinate ranges over a subset of the reals, which may, in particular, range from  $-\infty$  to  $+\infty$ . To start off with, we can think of these coordinates as corresponding to distances or angles in Euclidean space.

### 1.3. Curves and surfaces

We shall frequently define these curves and surfaces parametrically

$$x^a = x^a(u), \quad a = 1, \dots, n \quad (1.1)$$

$$f(x^1, x^2, \dots, x^n) = 0 \quad (1.2)$$

Points in an  $m$ -dimensional subspace ( $m < n$ ) must satisfy  $n - m$  constraints

$$\begin{aligned} f^1(x^1, \dots, x^n) &= 0 \\ &\vdots \\ f^{n-m}(x^1, \dots, x^n) &= 0 \end{aligned} \quad (1.3)$$

### 1.4. Transformation of coordinates

We need to find out how quantities behave when we go from one coordinate system to another one. We therefore consider the change of coordinates  $x^a \rightarrow x'^a$  given by the  $n$  equations

$$x'^a = f^a(x^1, \dots, x^n), \quad a = 1, \dots, n \quad (1.4)$$

we can write (1.4) more succinctly as  $x'^a = f^a(x)$ , or more simply

$$\boxed{x'^a = x'^a(x)} \quad (1.5)$$

We next contemplate differentiating (1.5) with respect to each coordinate  $x^b$

$$\left[ \frac{\partial x'^a}{\partial x^b} \right]$$

the determinant  $J'$  of this matrix is called the **Jacobian** of the transformation

$$J' = \left| \frac{\partial x'^a}{\partial x^b} \right| \quad (1.6)$$

Assume that this is non-zero. Then we can solve (1.5) for the old coordinates  $x^a$  and obtain the **inverse** transformation

$$\begin{aligned} x^a &= x^a(x') \\ J &= \left| \frac{\partial x^a}{\partial x'^b} \right| \quad (\text{Jacobian of the inverse transformation}) \\ J &= \frac{1}{J'} \end{aligned}$$

In 3 dimensions, the equation of a surface is given by  $z = f(x, y)$ , then its total differential is defined to be

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Then, in an analogous manner, starting from (1.5) we define the total differential

$$\begin{aligned} dx'^a &= \frac{\partial x'^a}{\partial x^1} dx^1 + \cdots + \frac{\partial x'^a}{\partial x^n} dx^n \\ dx'^a &= \sum_{b=1}^n \frac{\partial x'^a}{\partial x^b} dx^b \end{aligned} \quad (1.7)$$

introducing the **Einstein summation convention**

$$\boxed{dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b} \quad (1.8)$$

It defines the Kronecker delta as

$$\delta_b^a = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases} \quad (1.9)$$

It therefore follow directly from the definition of partial differentiaton that

$$\frac{\partial x'^a}{\partial x'^b} = \frac{\partial x^a}{\partial x^b} = \delta_b^a \quad (1.10)$$

## 1.5. Contravariant tensors

We shall start with a prototype and then give general definition.

Consider two neighboring points in the manifold  $P$  and  $Q$  with coordinates  $x^a$  and  $x^a + dx^a$  respectively. The two points define an **infinitesimal displacement** or **infinitesimal vector**  $\overrightarrow{PQ}$ . The components of this vector in the  $x^a$ -coordinate system are  $dx^a$ . The components in another coordinate system, say the  $x'^a$ -coordinate system, are  $dx'^a$  which are conected to  $dx^a$  by (1.8)

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b \quad (1.11)$$

The transformation matrix appearing in this equation is to be regarded as being evaluated at the point  $P$ , i.e, strictly speaking we should write

$$dx'^a = \left[ \frac{\partial x'^a}{\partial x^b} \right]_P dx^b \quad (1.12)$$

A **contravariant vector** or **contravariant tensor of rank (order) 1** is a set of quantities, written  $X^a$  in the  $x^a$ -coordinates system, associated with a point  $P$ , which trasform under a change of coordinates according to

$$\boxed{X'^a = \frac{\partial x'^a}{\partial x^b} X^b} \quad (1.13)$$

where the transform matrix is evaluated at  $P$ . The infinitesimal vector  $dx^a$  is a special case of (1.13) where the components  $X^a$  are infinitesimal.

A **contravariant tensor of rank 2** is a set of  $n^2$  quantities associated with a point  $P$ , denoted by  $X^{ab}$  in the  $x^a$ -coordinate system, which transform according to

$$X'^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} X^{cd} \quad (1.14)$$

An important case is a tensor of zero rank, called a **scalar** or **scalar invariant**  $\phi$ , which transform according to

$$\boxed{\phi' = \phi} \quad (1.15)$$

at  $P$ .

## 1.6. Covariant and mixed tensors

Let

$$\phi = \phi(x^a) \quad (1.16)$$

be a real-valued function on the manifold (at every point  $P$  in the manifold,  $\phi(P)$  produces a real number). Also assume that  $\phi$  is continuous and differentiable.

Remembering from (??),  $x^a$  can be thought of as a function of  $x'^b$ , (1.16) can be written equivalently as

$$\phi = \phi(x^a(x'))$$

Remembering Differentiating this with respect to  $x'^b$ , we obtain

$$\frac{\partial \phi}{\partial x'^b} = \frac{\partial \phi}{\partial x^a} \frac{\partial x^a}{\partial x'^b}$$

Then changing the order of the terms, the dummy index, and the free index (from  $b$  to  $a$ ) gives

$$\frac{\partial \phi}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} \frac{\partial \phi}{\partial x^b} \quad (1.17)$$

This is the prototype equation we are looking for. Notice that it involves the inverse transformation matrix  $\partial x^b / \partial x'^a$ . Thus, a **covariant vector** or **covariant tensor of rank (order) 1** is a set of quantities, which transform according to

$$\boxed{X'_a = \frac{\partial x^b}{\partial x'^a} X_b} \quad (1.18)$$