Tema 1- Numeros reales

A se llama **acotado superiormente**
$\to \exists K \in \mathbb{R}: K \geq a, \forall a \in A$

A se llama **acotado inferiormente**
$\to \exists k \in \mathbb{R}: k \leq a, \forall a \in A$

A se llama **acotado** $\to$ A es acotado superiormente e inferiormente

K y k se llaman **cota superior e inferior**

S es **supremo de A ($S = \text{Sup } A$)**
$\to$ S es una cota superior de A
$\to$ $(S-\varepsilon)$ no es una cota superior $\forall \varepsilon > 0$

$S = \text{Sup } A$ es la menor de las cotas superiores.

I es **ínfimo de A ($I = \inf A$)**
$\to$ I es una cota inferior de A
$\to$ $(I+\varepsilon)$ no es una cota inferior $\forall \varepsilon > 0$

$I = \inf A$ es la mayor de las cotas inferiores.

Si $S \in A \to S$ se llama **máximo de A ($S = \max A$)**

Si $I \in A \to I$ se llama **mínimo de A ($I = \min A$)**

$$|x| = \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$$

Distancia euclediana $\to d(x,y) = |x-y|$

Propiedades

1- $|x| \geq 0$, $|x| = 0 \leftrightarrow x = 0$

2- $|x \cdot y| = |x| \cdot |y| \to |x^n| = |x|^n \quad \forall n \in \mathbb{N}$

3- $|x+y| \leq |x| + |y|$

4- $|x| \leq a \to -a \leq x \leq a$

4.1- $|x| < a \to -a < x < a$

4.2- $|x| = a \to x = \pm a$

4.3- $|x| \geq a \to x \geq a$ y $x \leq -a$

4.4- $|x| > a \to x > a$ y $x < -a$

$x \leq y \to a + x \leq a + y$

$x \leq y \to a \cdot x \leq a \cdot y$ ($a \geq 0$) ; $x \leq y \to a \cdot x \geq a \cdot y$ ($a \leq 0$)

I-a) $\frac{x-1}{x+1} < 0$

$x-1>0 \leftrightarrow x>1$

$x+1<0 \leftrightarrow x<-1$

$x-1<0 \leftrightarrow x<1$

$x+1>0 \leftrightarrow x>-1$ $\rightarrow -1<x<1$

$x \in (-1,1)$

Fitat Superiorment

Suprem $= 1$

~~màxim~~

Fitat inferiorment

ínfimo $= -1$

~~ínfimo~~

$x^2+x \leq 0$

$x \cdot (x+1) \leq 0$ $\quad x \geq -1$

1 Resoleu les desigualtats següents:

a) $\dfrac{x-1}{x+1} < 0$; b) $\dfrac{1}{x+3} > \dfrac{1}{4}$; c) $\dfrac{x-1}{x+1} \leq \dfrac{x+1}{x-1}$; d) $x^2+x \leq 0$; e) $1 < x^2 < 4$.

En cada apartat representeu sobre la recta real el conjunt de solucions i digueu si tal conjunt és fitat superiorment (inferiorment). En cas afirmatiu, trobeu-ne el suprem (ínfim).

1 a) $A = (-1, 1)$; és un conjunt fitat, $\inf(A) = -1$ i $\sup(A) = 1$.
b) $B = (-3, 1)$; és un conjunt fitat, $\sup(B) = 1$ i $\inf(B) = -3$.
c) $C = (-1, 0] \cup (1, \infty)$; és un conjunt fitat inferiorment amb $\inf(C) = -1$.
d) $D = [-1, 0]$; és un conjunt fitat, $\sup(D) = 0$ i $\inf(D) = -1$.
e) $E = (-2, -1) \cup (1, 2)$; fitat superiorment, $\sup(B) = 2$ i fitat inferiorment, $\inf(B) = -2$.

2 Trobeu tots els nombres reals x que satisfan les desigualtats següents:

a) $|2x+7| \geq 3$; b) $|x^2-1| \leq 3$; c) $|x-1| > |x+1|$; d) $|x| + |x+1| < 2$.

En cada apartat representeu sobre la recta real el conjunt de solucions i digueu si tal conjunt és fitat superiorment (inferiorment). En cas afirmatiu, trobeu-ne el suprem (ínfim).

2 a) $A = (-\infty, -5] \cup [-2, +\infty)$; no és un conjunt fitat i no hi ha màxim ni mínim.
b) $B = [-2, 2]$; fitat $\sup(B) = 2$ i $\inf(B) = -2$.
c) $C = (-\infty, 0)$; fitat superiorment amb $\sup(C) = 0$.
d) $D = (-\frac{3}{2}, \frac{1}{2})$; fitat $\sup(D) = 1/2$ i $\inf(D) = -3/2$.

$\frac{1}{x+3} > \frac{1}{4}$

$x > -3 \rightarrow x+3 > 0 \rightarrow \frac{\overset{>0}{1}}{x+3} > \frac{\overset{>0}{1}}{4}$

$4 > x+3 \rightarrow 4-3 > x \rightarrow \boxed{x<1}$

$-3 < x < 1$

$\frac{x-1}{x+1} \leq \frac{x+1}{x-1}$ $\quad x \neq 1$ $\quad x \neq -1$

Si $x<-1 \rightarrow x+1<0$, $x-1<0$

Si $-1<x<1$: $x+1>0$, $x-1<0$

$(x-1)^2 \leq (x+1)^2$

$x^2-2x+1 \leq x^2+2x+1$

$0 \leq 4x$

$x \geq 0$

$\frac{x-1}{x+1} \geq \frac{x+1}{x-1}$

$4x \leq 0$

$x \leq 0$

Si $x>1 \rightarrow x-1>0$, $x+1>0$ $\quad x \geq 0$

2a) $|2x+7| \geq 3$

$2x+7 \geq 3$

$2x+7 \leq -3$

$2x \geq 3-7$ $\quad$ $2x \leq -3-7$

$x \geq \frac{-4}{2}$ $\quad$ $x \leq \frac{-10}{2}$

$\boxed{x \geq -2}$ $\quad$ $\boxed{x \leq -5}$

$(-\infty, -5] \cup [-2, +\infty)$

2b) $|x^2-1| \leq 3$

$x^2-1 \leq 3 \rightarrow x^2 \leq 4$

$x^2-1 \geq -3$ $\quad$ $x^2 \geq -2$

siempre cierto

$x^2-4 \leq 0$ $\quad$ $x^2 \leq 4$

$-2 < x$ $\quad$ $x \leq \sqrt{4}$

2d) $|x| + |x+1| \leq 2$

$|x| = \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$

$|x+1| = \begin{cases} x+1 & \text{si } x+1 \geq 0 \rightarrow x \geq 1 \\ -x-1 & \text{si } x+1 < 0 \rightarrow x < 1 \end{cases}$

si $x < -1 \rightarrow -x - x - 1 < 2 \rightarrow -2x < 3$

$-1 \leq x < 0$ $\quad$ $-x + x + 1 < 2 \rightarrow 1 < 2$

$x > 0 \rightarrow x + x + 1 \leq 2 \rightarrow 2x \leq 1$

Tema 2 - Sucesiones de numeros reales

2.1 - Definición, ejemplo y propiedades

Una sucesión de numeros reales és una aplicación

$a: D \in \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$

El conjunto imagen: $\{a_1, a_2, a_3, \dots a_n\}$ són los terminos de la sucesión.

$\{a_n\}_n \in D$, $(a_n)_n \in D$

Ej: $a_n: \mathbb{N} \rightarrow \mathbb{N}$, $n \rightarrow 2n$ $\quad$ $a_n = 2n \rightarrow \{2, 4, 6 \dots\}$

$a_n: \mathbb{N} \rightarrow \mathbb{N}$, $n \rightarrow (-1)^n$ $\quad$ $a_n = (-1)^n \rightarrow \{-1, 1, -1, 1 \dots\}$

Formas de dar una sucesión

- Todos los terminos de la sucesión $\{2, 4, 6, 8, \dots\}$
- Termino general $a_n = 2n$
- Forma recurrente $a_1 = 2n$, $a_{n+1} = a_n + 2$, $n \geq 1$
- Sucesión de Fibonacci $\rightarrow a_{n+1} = a_n + a_{n-1}$, $n \geq 2$

Exemples de successions conegudes:

- **PROGRESSIÓ ARITMÈTICA**: Cada terme s'obté de l'anterior **sumant** un nº real fix ("**diferència**")

$$\boxed{a_n = a_{n-1} + d} \quad \forall n \geq 2$$

i a_1 conegut

Exemple:

Donats: $a_1 = 1$ i $d = 3$

$$\left.\begin{aligned} a_2 &= a_1 + d = 1+3 = 4 \\ a_3 &= a_2 + d = 4+3 = 7 \\ a_4 &= a_3 + d = 7+3 = 10 \\ &= a_1 + (4-1)d = 1 + 3\cdot 3 = 10 \end{aligned}\right\} \Rightarrow \{1, 4, 7, 10, \dots\}$$

+3 +3 +3

......

La **SUMA** dels **n** primers termes

$$a_1 + a_2 + \dots + a_n = \sum_{n=1}^{n} a_i = \boxed{S_n = \frac{a_1 + a_n}{2} \cdot n}$$

- **PROGRESSIÓ GEOMÈTRICA**: Cada terme s'obté de l'anterior **multiplicat** un nº real fix ("**raó**")

$$\boxed{a_n = a_{n-1} \cdot r} \quad \forall n \geq 2$$

i a_1 conegut

Exemple:

Donats: $a_1 = 1$ i $r = 3$

$$\left.\begin{aligned} a_2 &= a_1 \cdot r = 1\cdot 3 = 3 \\ a_3 &= a_2 \cdot r = 3\cdot 3 = 9 \\ a_4 &= a_3 \cdot r = 9\cdot 3 = 27 \\ &= a_1 \cdot r^{4-1} = 1\cdot 3^3 = 27 \end{aligned}\right\} \Rightarrow \{1, 3, 9, 27, \dots\}$$

·3 ·3 ·3

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La **SUMA** dels **n** primers termes

$$a_1 + a_2 + \dots + a_n = \sum_{n=1}^{n} a_i = \boxed{S_n = \frac{a_n \cdot r - a_1}{r-1}}$$

Limite de una Sucesión

Def → Limite Finito ⟷ a_n es Congruente

- $\lim a_n \to l \longleftrightarrow \forall \varepsilon\ \exists n_0 / \forall n > n_0$

  $|x_n - l| < \varepsilon$

ej: $a_n = \frac{1}{n} \to 0$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\}$

$\varepsilon = 0'001$

$|x_n - 0| < 0'001$

Def → Sucesiones no Convergente ≡ divergente $\begin{cases} +\infty \\ -\infty \\ \nexists \circ \infty \end{cases}$

- $\lim\limits_{n \to +\infty} a_n = +\infty \longleftrightarrow \forall K\ \exists n_0 / \forall n > n_0$

  $x_n > K$

ej: $a_n = 2n \to +\infty$

$\{2, 4, 6, 8, 10 \dots\}$

$K = 50.000$

$x_n > 50.000$

- $\lim a_n = -\infty \longleftrightarrow \forall K < 0\ \exists n_0 / \forall n > n_0$

  $x_n < K$

ej: $a_n = 5 - 2n \to -\infty$

$\{3, 1, -1, -3, -5 \dots\}$

- $\lim a_n = \infty$ o $\nexists \lim$

ej:

$a_n = (-1)^n \cdot 2n \quad \{-2, 4, -6, 8, -10, 12 \dots\} \to$ Solución oscilante

$a_n = \cos(n\pi) \quad \{1, -1, 1, -1 \dots\} \to$ Solución oscilante

Resumen

a_n → Convergente ($\exists$ limite $\in \mathbb{R}$)

a_n → Divergente → $\pm\infty$

Divergente → $\nexists \lim (\infty)$ oscilante

Propiedades

- Si $\exists \lim a_n$, és único.
- $a_n \leq b_n \quad \forall n \geq n_0 \to \lim a_n \leq \lim b_n$

  $a_n < b_n \quad \forall n \geq n_0 \to \lim a_n \leq \lim b_n$

Ej: $\left.\begin{array}{l} a_n = \frac{1}{n} \\ b_n = \frac{2}{n} \end{array}\right\}$ $a_n < b_n$

$\lim a_n \leq \lim b_n$

1- a) $\lim \alpha^n = \begin{cases} +\infty & \alpha > 1 \\ 1 & \alpha = 1 \\ 0 & 0 \leq \alpha < 1 \\ 0 & -1 < \alpha < 0 \\ \nexists & \alpha \leq -1 \end{cases}$

b) $\lim n^\alpha = \begin{cases} +\infty & \alpha > 0 \\ 1 & \alpha = 0 \\ -\infty & \alpha < 0 \end{cases}$

Indeterminaciones

$+\infty - \infty$, $\frac{\pm\infty}{\pm\infty}$, $0 \cdot \infty$, $\frac{0}{0}$, 1^∞, 0^0, ∞^0

$\boxed{\frac{\infty}{\infty} \text{ o } \frac{0}{0}}$

Si és $\frac{P(n)}{Q(n)}$: $\begin{cases} \text{grau } P(n) > \text{grau } Q(n) \rightarrow \pm\infty \\ \text{grau } P(n) < \text{grau } Q(n) \rightarrow 0 \\ \text{grau } P(n) = \text{grau } Q(n) \rightarrow \text{Quocient termes de major grau} \end{cases}$

Si no: dividir numerador i denominador pel terme dominant, simplificar.

$\boxed{\infty - \infty}$

Si en el límit hi ha una resta d'arrels quadrades, es multiplica i es divideix per la suma de les arrels i s'aplica $(a+b)\cdot(a-b) = a^2 - b^2$

Es pot utilitzar:

$$a_n - b_n = a_n \cdot b_n \cdot \left(\frac{1}{b_n} - \frac{1}{a_n}\right)$$

$\boxed{1^\infty}$

$$\lim \left(1 + \frac{1}{n}\right)^n = e$$

$$a_n \rightarrow 0 \rightarrow \lim (1 + a_n)^{\frac{1}{a_n}} = e$$

$$\left.\begin{matrix} a_n \rightarrow 1 \\ b_n \rightarrow +\infty \end{matrix}\right\} \rightarrow \lim a_n^{b_n} = \lim (1 + a_n - 1)^{b_n} =$$

$$\lim \left[(1 + a_n - 1)^{\frac{1}{a_n - 1}}\right]^{b_n \cdot (a_n - 1)}$$

$$e^{\lim b_n \cdot (a_n - 1)}$$

$$\left.\begin{matrix} a_n \rightarrow 1 \\ b_n \rightarrow +\infty \end{matrix}\right\} \rightarrow \lim a_n^{b_n} = e^{\lim b_n \cdot (a_n - 1)}$$

$$\lim f(x)^{g(x)} = \lim e^{g(x) \cdot (f(x) - 1)}$$

$0 \cdot \infty$

$$\log(10^x) = 10^{\log x}$$
$$\ln(e^x) = e^{\ln x}$$
$$\log x^y = y \cdot \log x$$

1- $\frac{0}{0} \circ \frac{\infty}{\infty}$

2- 1^∞

$$\left.\begin{array}{l} a_n \to 0 \\ b_n \to \infty \end{array}\right\} \to \left.\begin{array}{l} 1+a_n \to 1 \\ b_n \to \infty \end{array}\right\} \to \lim (1+a_n)^{b_n}$$

$$e^{\lim b_n \cdot (a_n)}$$

$$\lim b_n \cdot a_n = \ln\left(\lim (1+a_n)^{b_n}\right)$$

0^0 i ∞^0

Es prenen els logaritmes i es passa a la indeterminació $0 \cdot \infty$

0^0 $\quad \left.\begin{array}{l} a_n \to 0 \\ b_n \to 0 \end{array}\right\} \to \ln a_n^{b_n} = b_n \cdot \ln a_n$

∞^0 $\quad \left.\begin{array}{l} a_n \to +\infty \\ b_n \to 0 \end{array}\right\} \to \ln a_n^{b_n} = b_n \cdot \ln a_n$

Criterios

1- $a_n \neq 0 \quad \forall n \geq n_0 \quad \lim \frac{|a_{n+1}|}{|a_n|} = l \begin{cases} l < 1 \to 0 \\ l > 1 \to +\infty \\ l = 1 \quad ? \end{cases}$

2- $\lim \sqrt[n]{|a_n|} = l \to \begin{cases} l < 1 \to 0 \\ l > 1 \to +\infty \\ l = 1 \quad ? \end{cases}$

3- $a_n \neq 0 \quad \forall n \geq n_0 \quad \lim \frac{|a_{n+1}|}{|a_n|} = l \to \lim \sqrt[n]{|a_n|} = l$

4- Criterio de Sandwitch

$$\underset{\downarrow \atop l}{b_n} \leq a_n \leq \underset{\downarrow \atop l}{c_n} \to \lim a_n = l$$

2a) $\lim \dfrac{6n^3+4n+1}{2n} \underset{\frac{\infty}{\infty}}{=} \lim \dfrac{6n^3}{2n} = +\infty$

b) $\lim \dfrac{n^2-6n-2}{3n^2-9n} \underset{\frac{\infty}{\infty}}{=} \dfrac{1}{3}$

c) $\lim \left(\sqrt{\dfrac{n+1}{2n+1}}\right)^{\frac{2n-1}{3n-1}} = \sqrt{\dfrac{1}{2}}^{\frac{2}{3}} = \boxed{\sqrt[3]{\dfrac{1}{2}}}$

5a) $\lim \dfrac{\cos n}{n^2} = 0$

b) $\lim \dfrac{2^n+3^n}{2^n-3^n} \underset{\frac{+\infty}{-\infty}}{=} \lim \dfrac{\frac{2n}{3n}+\frac{3^n}{3^n}}{\frac{2n}{3^n}-\frac{3^n}{3^n}} = \lim \dfrac{\left(\frac{2}{3}\right)^n+1}{\left(\frac{2}{3}\right)^n-1} = \boxed{-1}$

c) $\left(\dfrac{n+2}{n-3}\right)^{\frac{2n-1}{5}} \underset{1^\infty}{=} \lim e^{\left(\frac{n+2}{n-3}-1\right)\cdot\frac{2n-1}{5}} = \lim e^{\left(\frac{2+3}{n-3}\right)\cdot\frac{2n-1}{5}}$

$= \lim e^{\frac{2n-1}{n-3}}$

$= e^2$

d) $\lim \left(\sqrt{n+1}-\sqrt{n}\right)\cdot\sqrt{\dfrac{n+1}{2}} = \lim \dfrac{1}{\sqrt{2}}\cdot\sqrt{(n+1)}$

$\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\left(\dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$

$\lim \dfrac{1}{\sqrt{2}}\sqrt{(n+1)}\cdot\dfrac{\left(\sqrt{n+1}\right)^2-\sqrt{n}^2}{\sqrt{n+1}+\sqrt{n}}$

$\lim \dfrac{1}{\sqrt{2}}\cdot\dfrac{\sqrt{n+1}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{2} = \dfrac{1}{2\sqrt{2}}$

1c) $\lim \sqrt[n]{n} = \lim \dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{n} = \boxed{1}$

3- $b_n = \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \dots + \dfrac{1}{\sqrt{n^2+n}} \leq n\cdot\dfrac{1}{\sqrt{n^2+1}}$

$n\cdot\dfrac{1}{\sqrt{n^2+n}} \leq$

$\dfrac{1}{\sqrt{n^2+1}} \leq \dfrac{1}{\sqrt{n^2+1}}$, $\dfrac{1}{\sqrt{n^2+2}} \leq \dfrac{1}{\sqrt{n^2+1}}$, $\dfrac{1}{\sqrt{n^2+1}}$

$n\cdot\dfrac{1}{\sqrt{n^2+n}} \to 1$, $\quad n\cdot\dfrac{1}{\sqrt{n^2+1}} \to 1$

5a) $\lim \dfrac{a^n}{n!} \quad |a|>1$

$\lim \dfrac{\frac{a^{n+1}}{n+1!}}{\frac{a^n}{n!}} = \lim \dfrac{a}{n+1} = 0$

b) $\lim \dfrac{n^\alpha}{a^n} \quad |a|>1$

$\lim \dfrac{\frac{(n+1)^\alpha}{a^{n+1}}}{\frac{n^\alpha}{|a|^n}} = \dfrac{(n+1)^\alpha\cdot|a|^n}{n^\alpha\cdot a^{n+1}} = \left(\dfrac{n+1}{n}\right)^\alpha\cdot\dfrac{1}{a}$

$1\cdot\dfrac{1}{a} < 1$

$= \boxed{0}$

Sucesiones acotadas

Def → $\{a_n\}_n$ esta **acotada superiormente** si $\exists K_1 / a_n \leq K_1 \; \forall n$

$\{a_n\}_n$ esta **acotada inferiormente** si $\exists K_2 / a_n \geq K_2 \; \forall n$

$\{a_n\}_n$ esta **acotada** si $\exists K / |a_n| \leq K \; \forall n$

Def → $\{a_n\}_K$ es **creciente** $a_n \geq a_{n-1} \; \forall n$

$\{a_n\}_n$ es **decreciente** $a_n \leq a_{n-1} \; \forall n$

$\{a_n\}_n$ es **monotona** si es creciente o decreciente

$\{a_n\}_n$ es **extrictamente creciente** si $a_n > a_{n-1}$

$\{a_n\}_n$ es **extrictamente decreciente** si $a_n < a_{n-1}$

ej: $\left\{\frac{1}{n}\right\}_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\right\}$ decreciente

$$a_n = \begin{cases} \frac{n+1}{2} & \text{si } n = 2K+1 \\ \frac{n}{2} & \text{si } n = 2K \end{cases} = \{1,1,2,2,3,3\} \text{ creciente}$$

Teorema de Convergencia monotona

$\{a_n\}_n$ **monotona creciente**
$\{a_n\}_n$ **acotada superiormente**
} $\{a_n\}_n$ es convergente

$\exists$ lim (finito)

$\{a_n\}_n$ **monotona decreciente**
$\{a_n\}_n$ **acotada inferiormente**
} $\{a_n\}_n$ es convergente

TEOREMA de la CONVERGÈNCIA MONÒTONA

tota successió $\boxed{\text{FITADA i MONÒTONA}}$ és CONVERGENT

↳ les successions FITADES i CREIXENTS: límit = suprem
superiorm.

FITADES i DECREIXENTS: límit = ínfim
inferiorm.

6- $a_1 = \frac{-2}{3}$

$3 \cdot a_{n+1} = 2 + a_n^3 \quad n \geq 1$

a) Dem $-2 \leq a_n \leq 1$

b) Dem $\{a_n\}$ creciente

c) Dem $\{a_n\}$ convergente, i lim

a) $a_1 = \frac{-2}{3}$ $\quad -2 \leq \frac{-2}{3} \leq 1$

$a_{n+1} = \frac{2+a_n^3}{3}$

$-2 \leq a_n \leq 1$

$-8 \leq a_n^3 \leq 1$

$-6 \leq 2+a_n^3 \leq 3$

$-\frac{6}{3} \leq \frac{2+a_n^3}{3} \leq \frac{3}{3}$

$-2 \leq a_{n+1} \leq 1$

b) $a_1 = \frac{-2}{3}$ $\quad a_2 \overset{?}{>} a_1$

$a_2 = \frac{2 + \left(-\frac{2}{3}\right)^3}{3} = \frac{46}{81}$

$a_{n-1} \leq a_n$

$(a_{n-1})^3 \leq a_n^3$

$2 + a_{n-1}^3 \leq 2 + a_n^3$

$\frac{2 + a_{n-1}^3}{3} \leq \frac{2 + a_n^3}{3}$

$a_n \leq a_{n+1}$ ✓

c) $3 \cdot a_{n+1} = 2 + a_n^3$

$3 \cdot L = 2 + L^3 \qquad L^3 - 3L + 2 = 0$

	1	0	-3	2
1		1	1	-2
	1	1	-2	$\boxed{0}$

$L^2 + L - 2 = 0 \qquad L = 1, \ -2$

3- a) $x^3 - 4x < 0$ $\quad (0,2)$

$x \cdot (x^2 - 4) < 0 \rightarrow$ $x > 0$, $x^2 - 4 < 0$

$x > 0$, -2, 2

$(-\infty, -2) \cup (0,2)$

$x < 0$, $x^2 - 4 > 0$

$x < 0$, -2, 2

$(-\infty, -2)$

b) $x = 2^{-n} \quad x \in \mathbb{R} \quad \exists n \in \mathbb{N}$ $\quad$ Sup = max = $\frac{1}{2}$

$n = 1, 2 \ldots \rightarrow \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\} \rightarrow (0, \frac{1}{2}]$

Infimo = 0

$n = 0, 1, 2 \ldots \rightarrow \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\}$ $(0, 1]$

Supr = max = 1

Infimo = 0

c) $y \in \mathbb{R} \ \exists x \in \mathbb{R} \qquad y = 1 + x^2$

$y \in [1, +\infty)$

Inf = min = 1

7a) $\lim \frac{1}{n^2}+\frac{2}{n^2}+\dots\frac{n}{n^2} = \frac{\infty}{\infty}$

$\lim \frac{1}{n^2}\cdot(1+2+\dots+n)$

$\lim \frac{1}{n^2}\frac{1+n}{2}\cdot n$

$\lim \frac{n+n^2}{2n^2} = \frac{1+n}{2n} = \boxed{\frac{1}{2}}$

8a) $\lim\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\dots\frac{n}{n^2+n}\right)$

$\boxed{\frac{n}{n^2+n}+\frac{n}{n^2+n}+\dots\frac{n}{n^2+n}} \leq \frac{n}{n^2+1}+\frac{n}{n^2+2}+\dots\frac{n}{n^2+n} \leq \boxed{\frac{n}{n^2+1}+\frac{n}{n^2+1}+\dots\frac{n}{n^2+1}}$

$\lim = 1$

$\lim \frac{n}{n^2+n}\cdot n = 1$

$\lim \frac{n}{n^2+1}\cdot n = 1$

7d) $\lim \sqrt[n]{\frac{(n+1)\cdot(n+2)\dots(2n)}{n}}$

$\lim \sqrt{\frac{(n+1)\cdot(n+2)\dots 2n}{n^n}}$

$\lim \frac{|a_n|}{|a_{n-1}|} = l \text{ o } +\infty$

$\lim \sqrt[n]{|a_n|} = L$

$$\lim \frac{\frac{(n+1)\cdot(n+2)\dots(2n)}{(n)^n}}{\frac{(n-1+1)\cdot(n-1+2)\dots(2(n-1))}{(n-1)^{n-1}}} = \frac{(2n-1)\cdot(2n)}{n}\cdot\frac{(n-1)^{n-1}}{n^n}$$

$\lim \frac{4n^2-2n}{n}\cdot\lim\frac{(n-1)^{n-1}}{n^n}$

$\lim \frac{4n-2}{n}\cdot\lim\left(\frac{n-1}{n}\right)^{n-1}$

$4\cdot\lim\left(\left(1+\frac{1}{-n}\right)^{-n}\right)^{\frac{n-1}{-n}}$

$\boxed{4\cdot e^{-1}}$

10- c) $|x|-|x^2| = |x|-x^2$

Si $x\geq 0 \rightarrow x-x^2$

Si $x<0 \rightarrow -x-x^2$

b) $||x|-1| =$

Si $x\geq 0 \rightarrow |x-1| =$ Si $x-1\geq 0 \rightarrow x-1$ $[1,+\infty)$

Si $x-1<0 \rightarrow -x+1$ $[0,1)$

Si $x<0 \rightarrow |-x-1| =$ Si $-x-1\geq 0 \rightarrow -x-1$ $(-\infty,-1]$

Si $-x-1<0 \rightarrow x+1$ $(-1,0)$

8b) $\lim_{n\to\infty} \frac{5(n+1)^{n+1}}{(3n^2+1)\cdot n^{n+1}} = \frac{\infty}{\infty}$

$\lim \frac{5\cdot(n+1)\cdot n}{3n^2+1}\cdot\frac{(n+1)^n}{n^n}$

$\lim_{n\to\infty} \frac{5n^2+5n}{3n^2+1}\cdot\lim\frac{(n+1)^n}{n^n}$

$\frac{5}{3}\cdot 1^\infty = \frac{5}{3}\cdot\lim\left(1+\frac{1}{n}\right)^n$

$\frac{5}{3}\cdot e$

Tema 3 - Funciones Continuas

Def → $f. \ \mathbb{R} \to \mathbb{R}$

$x \to y = f(x)$

Def → **Dominio f.** $\text{Dom} f = \{x \in \mathbb{R} / \exists f(x) = y$

Recorregut f. $\text{Rec} f = \{y \in \mathbb{R} / \exists x : f(x) = y$

Grafica de f. $\text{Graf} f = \{(x, y) / y = f(x)$

$f(x) = \frac{1}{x}$ $\quad$ $\text{Dom} f = \mathbb{R} - \{0\}$

$\text{Rec} f = \mathbb{R} - \{0\}$

Def → $\mathbb{R} \to \mathbb{R}$ → si es continua en 'a' si

$x \to y = f(x)$

$\exists f(a)$ → $a \in \text{Dom} f$

$\exists \lim_{x \to a} f(x) = L$ $\quad$ $[\ \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

$f(a) = \lim_{x \to a} f(x)$

Def → $\mathbb{R} \to \mathbb{R}$ continua en A si es Continua en "a"

Tipos de discontinuidad

1- **Discontinuidad evitable** → $f(a) \neq \lim f(x)$

$f(x) = \frac{x^2 - 4}{x - 2}$ $\quad$ Dis en $x = 2$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2) \cdot (x+2)}{x - 2} = \boxed{4}$$

$4 \neq \exists f(2)$

evitamos $g(x) = \begin{cases} f(x) & \text{si } x \neq a \\ \lim_{x \to a} f(x) & \text{si } x = a \end{cases}$

2- Discontinuidad de Salto

2.1- **Discontinuidad de Salto Finito** → $\exists \lim_{x \to a^-} f(x) \neq \exists \lim_{x \to a^+} f(x)$

$f(x) = \begin{cases} x^2 + 1 & x \leq 1 \\ 3x + 2 & x > 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = 2$ ✗ $\lim_{x \to 1^+} f(x) = 5$

2.2- Discontinuidad de salto infinito → $\lim\limits_{x\to a^-} f(x)$ o $\lim\limits_{x\to a^+} f(x) = \pm\infty$

$F(x) = \dfrac{x^2+1}{x-3}$ Disc en $x=3$

$$\lim_{x\to 3^-} f = \frac{10}{0^-} = -\infty$$

$$\lim_{x\to 3^+} f = \frac{10}{0^+} = +\infty$$

3- Discontinuidad de 2a especie → $\nexists \lim f(x)$ (ni $\mathbb{R}$ ni ∞)

$f(x) = \sin\dfrac{1}{x}$ Disc en $x=0$

Proposición 1

$f: \mathbb{R} \to \mathbb{R}$ elemental, $a \in \operatorname{Dom} f \to f$ continua

Proposición 2

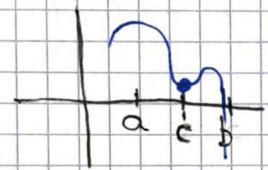
f, g continua en $a \in \operatorname{dom} F \cap \operatorname{Dom} g \to \begin{cases} F \pm g \text{ continua} \\ \cdot/F \text{ continua} \\ Fg \text{ continua} \end{cases}$

Teoremas de funciones continuas

• Teorema de conservación de signo

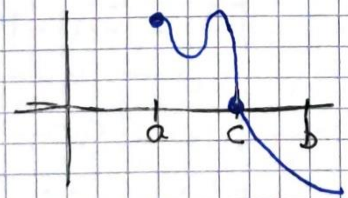
$\left.\begin{array}{l} F \text{ continua en } [a,b] \\ c \in (a,b),\ F(c) \neq 0 \end{array}\right\} \to \exists r > 0 \,/\, F(c)\cdot f(x) > 0$

$\forall x \in (c-r, c+r)$



• Teorema de Bolzano

$\left.\begin{array}{l} f \text{ continua en } [a,b] \\ f(a)\cdot f(b) < 0 \end{array}\right\} \to \exists\, c \in (a,b) \,/\, \boxed{f(c) = 0}$



• Teorema de Valor intermedio

f continua en $[a,b] \to \forall y \in \langle f(a), f(b) \rangle$

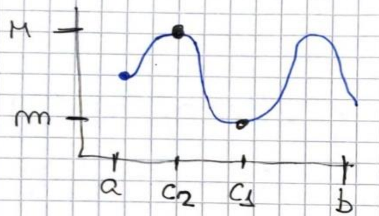
$\exists\, c \in (a,b) \,/\, \boxed{f(c) = y}$

• Teorema de Weierstrass

f continua en $[a,b] \longrightarrow \exists M, m \,/\, m \leq f(x) \leq M$

m : mínimo
M : máximo

$\forall x \in (a,b)$



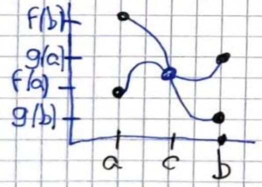
$m = f(c_1)$

$M = f(c_2)$

$c_1, c_2 \in [a,b]$

2- $a \leq b \in \mathbb{R}$

$\left.\begin{array}{l} f, g \text{ continuas en } [a,b] \\ f(a) < g(a),\ f(b) > g(b) \end{array}\right\} \exists c \in (a,b) \,/\, f(c) = g(c)$



$h(x) = g(x) - f(x)$

$h(a) = g(a) - f(a) > 0$

$h(b) = g(b) - f(b) < 0$

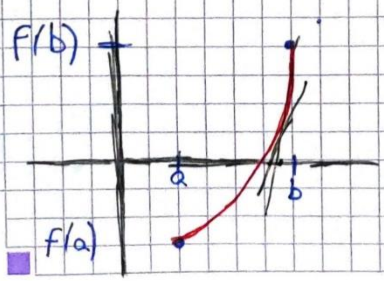
$h(c) = 0$

$g(c) - f(c)$

$\downarrow$

$g(c) = f(c)$

Metodo Newton Raphson (Metodo tangente)



Recta tangente por $(x_n, f(x_n))$

$y = f(x_n) - f'(x_n) \cdot (x - x_n)$

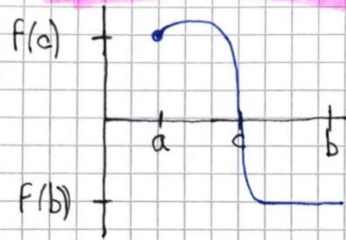
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$|x_{n+1} - x_n| < \varepsilon$

$f(x_{n+1}) < \varepsilon$

Cálculo de raíces de una función (ceros)

1- Método bisección



$C_1 = \frac{a+b}{2}$

Si $f(c_1), f(a) < 0$

$C_2 = \frac{a + c_1}{2}$

$$\boxed{\text{Error} = |c - c_n| \leq \frac{b-a}{2^n}}$$

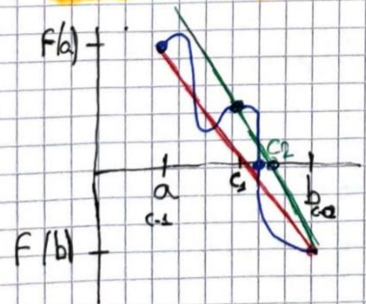
ej $\varepsilon \leq 0'01$

$\frac{b-a}{2^n} \leq 0'01$

Si $f(c_1), f(b) < 0$

$C_2 = \frac{b + c_1}{2}$

2- Método Secante



$y = mx + n = 0$

recta $(a, f(a)), (b, f(b))$

$y = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$

$y = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \cdot (x - x_n)$

$$\boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n)}$$

$|x_{n+1} - x_n| < \varepsilon$

$f(x_n) < \varepsilon$

1- $x^3-3x^2+1=0$ tiene Sol. en $[0,2]$

$f(x) = x^3-3x^2+1$

$f(0) = 1 > 0$

$f(2) = -3 < 0$

$\exists c \in (0,2) \,/\, f(c) = 0$

c raíz de la ecuacion

3- $f(x) = \frac{x^3}{4} - \sin(\pi x) + 3$

toma el valor 2'5 en $[-2,2]$

$\frac{x^3}{4} - \sin(\pi x) + 3 = 2'5$

$\underbrace{\frac{x^3}{4} - \sin(\pi x) + 0'5}_{g(x)} = 0$

g(x) continua (Pol + Sin)

$g(-2) = -2+0'5 = -1'5 < 0$

$g(2) = 2+0'5 = 2'5 > 0$ $\rightarrow g(c)=0$

4- $e^{-x^2} = 2x$ $[0,1]$

$e^{-x^2} - 2x = 0$ f(x) continua en $[0,1]$ (Suma exp + polin)

$f(0) = e^0 - 2\cdot 0 = 1$

$f(1) = e^{-1} - 2 < 0$ $\rightarrow f(c)=0$

$f(0'5) = e^{-0'25} - 2\cdot 0'5 = -0'22 \longrightarrow c(0, 0'5)$

$f(0'3) = e^{-0'09} - 2\cdot 0'3 = 0'31 > 0 \rightarrow c \in (0'3, 0'5)$

$c = 0'4$

5- a) Separar las dos Sol reales $x - 3\cdot \ln x = 0$

b) bisección con precision $0'5\cdot 10^{-3}$ para ambas Sol

c) Secante con 3 decimales correctos para ambas Sol

$f(x) = x - 3\cdot \ln x$

$f(1) = 1 - 3\cdot \ln 1 = 1 > 0$

$f(2) = 2 - 3\cdot \ln 2 = -0'08 < 0$ $\rightarrow c_1 \in (1,2)$

$f(4) = -0'16 < 0$

$f(5) = 0'17 > 0$ $\rightarrow c_2 \in (4,5)$

b) $0'5\cdot 10^{-3} \rightarrow$ 3 decimales correctos

$\frac{2-1}{2^n} \leq 0'5\cdot 10^{-3}$

$\frac{1}{2^n} \leq 0'5\cdot 10^{-3}$

$\ln 1 - \ln 2^n \leq \ln(0'5\cdot 10^{-3})$

$-n\cdot \ln 2 \leq \ln(0'5\cdot 10^{-3})$

$n \geq \frac{\ln(0'5\cdot 10^{-3})}{-\ln 2}$

$\boxed{n = 11}$

	c_i	$f(c)$
a =	1	1
b =	2	-0'08
$c_1 =$	1'5	0'28
$c_2 =$	1'75	

$c_{11} = 1'8569$

b) $|x_{n+1} - x_n| < \varepsilon$ $(0'5\cdot 10^{-3})$

$f(x_n) < \varepsilon$

$c_1 \in (\underset{x_{-1}}{1}, \underset{x_0}{2})$

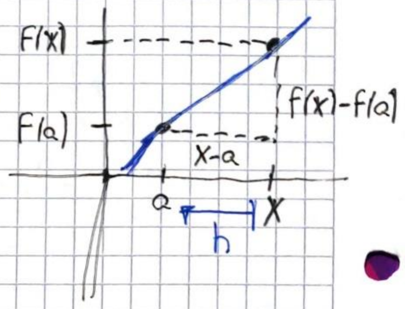
$x_1 = x_0 - \frac{x_0 - x_{-1}}{f(x_0) - f(x_{-1})} \cdot f(x_0) =$

x_n	$f(x_n)$
1	1
2	-0'08
1'9264	-0'0405
1'8496	0'00467
1'85756	-0'00023
1'85718	0'0000012

Tema 4 – Funciones derivables (una variable)

Def: Derivada de $f(x)$ en el punto a: $f(x)$ continua

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$$



$\operatorname{tg} \alpha =$ pendiente de la recta

$f'(a)$: pendiente de la recta tangente a $(a, f(a))$

$$\boxed{y - f(a) = f'(a) \cdot (x-a)}$$ Ecuación recta tangente

$$\boxed{y - f(a) = \frac{-1}{f'(a)} \cdot (x-a)}$$ Ecuación recta normal

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$ $\boxed{h = x-a}$

Proposición

$f: \mathbb{R} \to \mathbb{R}$ elemental, $a \in \operatorname{Dom} \mathbb{R}$ $\Big\} \to f$ derivable en a

f, g elementales, $\lambda \in \mathbb{R}$ $a \in \operatorname{Dom} f \wedge$

λf
$f \pm g$
$f \cdot g$
$\dfrac{f}{g}$ $g(a) \neq 0$

$\Big\} \to$ Derivables en a

Funcion simple	Derivada
$y = k$	$y' = 0$
$y = x$	$y' = 1$
$y = u(x) + v(x)$	$y' = u'(x) + v'(x)$
$y = k \cdot u(x)$	$y' = k \cdot u'(x)$
$y = u(x) \cdot v(x)$	$y' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$
$y = \dfrac{u(x)}{v(x)}$	$y' = \dfrac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v^2(x)}$
$y = x^n$	$y' = n \cdot x^{n-1}$
$y = \ln x$	$y' = \dfrac{1}{x}$
$y = \log_a x$	$y' = \dfrac{1}{x} \log_a e$
$y = e^x$	$y' = e^x$
$y = a^x$	$y' = a^x \cdot \ln a$
$y = \operatorname{sen} x$	$y' = \cos x$
$y = \cos x$	$y' = -\operatorname{sen} x$
$y = \operatorname{tg} x$	$y' = \dfrac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$
$y = \operatorname{cotg} x$	$y' = \dfrac{-1}{\operatorname{sen}^2 x} = -[1 + \operatorname{cotg}^2 x]$
$y = \operatorname{arc\,sen} x$	$y' = \dfrac{1}{\sqrt{1-x^2}}$
$y = \operatorname{arc\,cos} x$	$y' = \dfrac{-1}{\sqrt{1-x^2}}$
$y = \operatorname{arc\,tg} x$	$y' = \dfrac{1}{1+x^2}$

Cálculo derivadas

$y = \lambda f \to y' = \lambda \cdot f'$

$y = f \pm g \to y' = f' \pm g'$

$y = f \cdot g \to y' = f' \cdot g + f \cdot g'$

$y = \dfrac{f}{g} \to y' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$

Regla de la Cadena

$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$

$a \to f(a) \to g(f(a)) = g \circ f$

$$\boxed{(g \circ f)'(a) = g'(f(a)) \cdot f'(a)}$$

$f = \operatorname{Sin} x$
$g = e^{x^2}$
$g \circ f = e^{\operatorname{Sin}^2 x}$
$y' = e^{\operatorname{Sin}^2 x} \cdot 2 \cdot \operatorname{Sin} x \cdot \operatorname{Cos} x$

Derivada funcion exponencial-potencial

$$y = f(x)^{g(x)}$$

$$\ln y = \ln f(x)^{g(x)}$$

$$\ln y = g(x) \cdot \ln f(x)$$

$$y' = f(x)^{g(x)} \cdot \left(g'(x) \cdot \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x) \right)$$

Def → f es **creciente** en $[a,b]$ si

$$x < y \rightarrow f(x) \leq f(y) \quad \forall x, y \in [a,b]$$

f es **extrictamente creciente** en $[a,b]$ si

$$x < y \rightarrow f(x) < f(y)$$

f es **decreciente** en $[a,b]$ si

$$x < y \rightarrow f(x) \geq f(y)$$

f es **extrictamente decreciente** en $[a,b]$ si

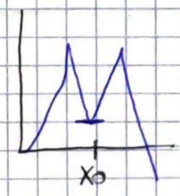
$$x < y \rightarrow f(x) > f(y)$$

Def → x_0 es un **mínimo relativo** (local) si

$$\exists I_{x_0} \;/\; f(x_0) \leq f(x) \quad \forall x \in I_{x_0}$$

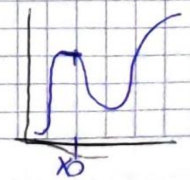
Intervalo centrado en x_0

$$I_{x_0} = (x_0 - 2, x_0 + 2)$$



x_0 es un **maximo relativo** si

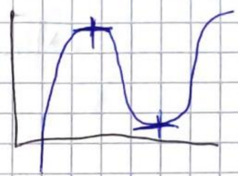
$$\exists I_{x_1} \;/\; f(x_1) \geq f(x) \quad \forall x \in I_{x_1}$$



Teorema del extremo interior — (Condicion necesaria extremos relativos)

- f continua en $[a,b]$
- f derivable en (a,b)
- c extremo relativo

$\rightarrow f'(c)=0$



Dem: f derivable en (a,b), c extremo (minimo) $\rightarrow$ f derivable en c:

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \rightarrow \exists \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\leq 0$$

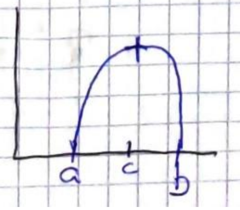
$$\rightarrow \exists \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\geq 0$$

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=0$$

Teorema de Rolle

- f continua en $[a,b]$
- f derivable en (a,b)
- $f(a)=f(b)$

$\rightarrow \exists c \in (a,b) / f'(c)=0$

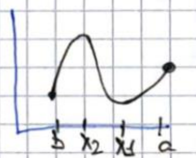


Dem: f continua en $[a,b]$

$\rightarrow \exists x_1, x_2 \in [a,b]$ $f(x_1)\leq f(x)\leq f(x_2)$ $\forall x \in [a,b]$

$f(x_1)$: m (minimo absoluto)

$f(x_2)$: M (maximo absoluto)



- Si $x_1 \in (a,b) \rightarrow x_1$ minimo relativo $\rightarrow f'(x_1)=0$
- Si $x_2 \in (a,b) \rightarrow x_2$ maximo relativo $\rightarrow f'(x_2)=0$
- Si $x_1, x_2 \notin (a,b) \rightarrow \{x_1, x_2\}=\{a,b\}$

$f(x_1)\leq f(x)\leq f(x_2)$

$\downarrow$

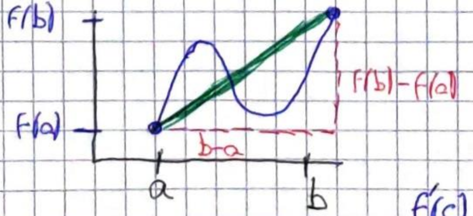
$f(x_1)=f(x_2) \rightarrow f$ constante $\rightarrow f'=0$

Teorema de Lagrange (T. Valor intermedio)

- f continua en $[a,b]$
- f derivable en (a,b)

$\rightarrow \exists c \in (a,b)$

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a} \qquad f(b)-f(a) = f'(c)\cdot(b-a)}$$



$f'(c)$ pendiente recta tangente $(c, f(c))$

$\frac{f(b)-f(a)}{b-a}$, pendiente recta por $(a,f(a)), (b,f(b))$

Dem:

$$y = f(a) + \frac{f(b)-f(a)}{b-a}\cdot(x-a)$$

$g(x) = f(x) - y \rightarrow$ continua en $[a,b]$, derivable en (a,b)

$g(b) = f(b) - y(b) = f(b) - f(b) = 0$

$g(a) = f(a) - y(a) = f(a) - f(a) = 0$

$\downarrow$

$\exists c / g'(c) = 0$

Corolario

- Si $f'(x) > 0 \quad \forall x \in (a,b) \rightarrow f$ extrictamente creciente
- Si $f'(x) < 0 \quad \forall x \in (a,b) \rightarrow f$ extrictamente decreciente

Continuidad de $f(x)$ en x_0

1- $\exists f(x_0) \quad (x_0 \in D(f))$

2- $\exists \lim_{x\to x_0} f = \lim_{x\to x_0^-} f = \lim_{x\to x_0^+} f$

Regla de l'Hopital

f, g. Funciones derivables en un entorno de a

$[a \in \mathbb{R},\ a = +\infty,\ a = -\infty]$

$$\left\{\begin{array}{l} \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \\ \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty \end{array}\right\} \quad \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

ej $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2}\frac{2x-3}{2x} = \boxed{\frac{1}{4}}$

1- Punto de $y = x^2$ en que la recta tangente es paralela al segmento $\overline{AB}$ $[A = (1,1), B = (3,9)]$

Pendiente de $\overline{AB}$ $\quad m = \frac{y_2-y_1}{x_2-x_1} = \frac{9-1}{3-1} = \boxed{4}$

Pendiente de recta tangente $\quad f'(x) = \boxed{2x}$

$4 = 2x \quad \boxed{x=2} \quad (2,4)$

2- Dem $3^{-x} = x$ tiene una única solución. Cuál es la parte entera

$f(x)$ continua (suma exponencial + polinomia)

$f(x) = 3^{-x} - x$

$$\left.\begin{array}{l} f(0) = 1 > 0 \\ f(1) = \frac{1}{3} - 1 < 0 \end{array}\right) \text{T. Bolzano} \rightarrow f(c) = 0$$

Unicidad. Suponemos 2 sol. $f(c_1) = 0 = f(c_2)$

$$\left.\begin{array}{l} f \text{ continua en } [0,1] \\ f \text{ derivable en } (0,1) \\ f(c_1) = f(c_2) \end{array}\right) \text{T. Rolle} \xrightarrow{t \in (c_1, c_2)} f'(t) = 0$$

$$f'(x) = -3^{-x} \cdot \log 3 - 1 = 0$$

$$\underset{\vee\atop 0}{3^{-x}} = \underset{\wedge\atop 0}{\frac{1}{-\log 3}} \qquad \boxed{\text{Contradicción}}$$

b) $f(1,5) = -0'18$

$f(1,3) = 0'01$ $\qquad c \in (1'3, 1'4)$

$f(1'4) = -0'09$

c) $f'(x) = -e^{-x} - \frac{1}{x}$ en $[1, +\infty)$

$$\underset{<0}{-e^{-x}} = \underset{>0}{\frac{1}{x}}$$

d) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

x_n	$f(x_n)$	$f'(x_n)$
1	0'36788	-1'36788
1'26894	0'04295	-1'06919
1'30910	0'00071	-1'03395
1'30979	$2 \cdot 10^{-7}$	-1'03335
1'30980		

4 iteraciones

3 – Dem $f_m(x) = x^3 - 3x + m$ no tiene dos ceros en $[0,1]$ $m \in \mathbb{R}$

$$\left.\begin{array}{l} f \text{ continua} \\ f \text{ derivable} \\ f_m(c_1) = f_m(c_2) = 0 \end{array}\right) f'(t) = 0$$

$f_m'(x) = 3x^2 - 3 = 0$

$x^2 = 1$

$x = \boxed{\pm 1}$ Contradicción no esta en el intervalo

4- $e^{-x} = \ln x$

a) Dem tiene sol en $[1, +\infty)$

b) Dar un intervalo longitud 0'1 que contenga dicha sol

c) Razonar que no tiene 2 sol en $[1, +\infty]$

d) Newton-Raphson con $x_0 = 1$ para obtener la raiz. Detener el calculo si $|x_n - x_{n-1}| < 10^{-4}$

a) $$\left.\begin{array}{l} f(1) = e^{-1} - \ln 1 = e^{-1} > 0 \\ f(2) = e^{-2} - \ln 2 < 0 \end{array}\right\} \text{T. Bolzano } \exists c \in (1,2) / f(c) = 0$$

5- a) $\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to+\infty} \frac{2\sqrt{x}}{x} = 0$

b) $\lim_{x\to0^+} x\cdot\ln x = \lim_{x\to0} \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to0} \frac{-x^2}{x}$

$= \lim_{x\to0} -x = 0$

$\lim_{x\to0} (\sin x)^x = \lim_{x\to0} e^{\ln\cdot[\sin x]^x} = \lim_{x\to0} e^{x\cdot\ln\sin x} = e^0 = 1$

$\lim_{x\to0} x\cdot\ln\cdot\sin x = \lim_{x\to0} \frac{\ln(\sin x)}{\frac{1}{x}} = \frac{\frac{1}{\sin x}\cdot\cos x}{\frac{-1}{x^2}}$

$\lim_{x\to0} \frac{-x^2\cdot\cos x}{\sin x} = \frac{-x^2}{\operatorname{tg} x} = \frac{2x}{\frac{1}{\cos^2 x}} = 0$

10- a) $\operatorname{arctg} x > \frac{x}{1+x^2}$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{\operatorname{arctg} x - \operatorname{arctg} 0}{x-0} = f'(c) = \frac{1}{1+c^2}$;

$\operatorname{arc}\operatorname{tg} x = \frac{x}{1+c^2}$

$c < x$

$c^2 < x^2$

$1+c^2 < 1+x^2$

$\frac{1}{1+c^2} > \frac{1}{1+x^2}$

b) $\ln(1+x) \leq x$

$\frac{\ln(1+x) - \ln(1)}{x-0} = f'(c) = \frac{1}{1+c}$

$\frac{\ln(1+x)}{x} = \frac{1}{1+c} \leftrightarrow \ln(1+x) = \frac{x}{1+c} < x$

$c \in (0,x)$ $c>0$

$1+c > 1$

$\frac{1}{1+c} < 1$

c) $\arccos x > \frac{\pi}{2} - \frac{x}{\sqrt{1-x^2}}$

$\frac{\arccos x - \arccos 0}{x-0} = f'(c) = \frac{1}{\sqrt{1-c^2}}$

$\frac{\arccos x - \frac{\pi}{2}}{x} = \frac{-1}{\sqrt{1-c^2}}$; $\arccos x = \frac{-x}{\sqrt{1-c^2}} + \frac{\pi}{2}$

$\sqrt{1-c^2} > \sqrt{1-x^2}$

$\frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-x^2}}$

6- a) Demostreu que $\ln x = x^2-4x$ té una solució en $[1,\infty)$

$f(x) = \ln x - x^2 + 4x$

f continuas en $(0,+\infty)$

$f(1) = \ln 1 - 1^2 + 4\cdot1 = 3 > 0$

$f(2) = \ln 2 - 2^2 + 4\cdot2 > 0$

$f(3) = \ln 3 - 3^2 + 4\cdot3 > 0$

$f(4) = \ln 4 - 4^2 + 4\cdot4 > 0$

$f(5) = \ln 5 - 5^2 + 4\cdot5 < 0$

$\exists c \in (4,5)$

$f(c) = 0$

$\ln c = c^2 - 4c$

7a) $f(x) = \frac{1}{1-e^{\frac{1}{x}}}$ en $[-1,1]$

f (0) no es continua

7b) $f(x) + \frac{1}{2} = 0$ un interval de longitud igual o menor que $\frac{1}{3}$

$g(x) = \frac{1}{1-e^{\frac{1}{x}}} + \frac{1}{2}$

$g(x) = \frac{2-e^{\frac{1}{x}}}{2-2\cdot e^{\frac{1}{x}}}$

$g(-1) = 2'082 > 0$

$\lim_{x\to0^-} \frac{1}{1-e^{\frac{1}{0}}} + \frac{1}{2} = \frac{1}{1-e^{-\infty}} + \frac{1}{2} = \frac{1}{1-\frac{1}{e^\infty}} + \frac{1}{2}$

$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2} > 0$

$g(1) = -0'082 < 0$

$\lim_{x\to0^+} \frac{1}{1-e^\infty} + \frac{1}{2} = \frac{1}{2} > 0$

Tema 5 - Polinomio de Taylor

$f(x)$, $x_0 \in \text{Dom } f$

Queremos aproximar la funcion por polinomio en un entorno de x_0

ej: $f(x) = e^x \qquad x_0 = 0$

$P_0(x) = 1$

$P_1(x) = f(0) + f'(0) \cdot (x-0)$

P_2 - misma curvatura

Polinomio Taylor de grado n, de la función $f(x)$ en un entorno de x_0

$$\boxed{P_{n,f(x),x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k}$$

$f^{(k)} \to$ derivada

ej: $f(x) = \ln(x+1) \quad x_0 = 0$

$f'(x) = \frac{1}{x+1} = 1$

$f''(x) = -1 \cdot (x+1)^{-2} = -1$

$f'''(x) = 2 \cdot (x+1)^{-3} = 2$

$f''''(x) = -6 \cdot (x+1)^{-4} = -6$

$P_{4,f,0}(x) = 0 + 1x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4$

$$\boxed{P^{(k)}_{n,f,x_0}(x_0) = f^{(k)}(x_0)}$$

Def $\to$ Termino Complementario, resto o residuo

del polinomio de taylor de orden n de la funcion f en el punto x_0

$$\boxed{R_{n,f,x_0}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}}$$

c está entre $(0,x)$

$$\boxed{f(x) - P_{n,f,x_0}(x) = R_{n,f,x_0}(x)}$$

Aproximación y acotación del error

$f: \mathbb{R} \to \mathbb{R}$

$f, f', f'' \ldots f^n$ continuas en $[a,b]$

$\exists f^{(n+1)}$ continuas $[a,b]$, $x_0 \in [a,b]$

$\exists M_{n+1} = \max_{t \in (x_0,x)} f^{(n+1)}(t)$

$$F(x) - P_n(x) = R_n(x) = \frac{F^{n+1}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}$$

$$\leq \frac{M}{(n+1)!} \cdot (x-x_0)^{n+1}$$

1- $f(x) = \sqrt[3]{1728+x}$ de grado 2

Para calcular $\sqrt[3]{1731}$. Acotar el error $\sqrt[3]{1731} \simeq P_2(3)$

$x_0 = 0$

$$P_2(x) = F(x_0) + F'(x_0) \cdot (x - x_0) + \frac{F''(x_0)}{2} \cdot (x-x_0)^2$$

$$P_2(x) = \sqrt[3]{1728} + F'(x_0) \cdot x + \frac{F''(x_0)}{2} \cdot x^2$$

$$F'(x_0) = \frac{1}{3}(1728+x)^{-\frac{2}{3}} = \frac{1}{432}$$

$$F''(x_0) = -\frac{1}{3} \cdot \frac{2}{3} (1728+x)^{-\frac{5}{3}} = \frac{-1}{1119744}$$

$$F'''(x) = \frac{10}{27}(1728+x)^{-\frac{7}{3}}$$

$$P_2(x) = 12 + \frac{x}{432} - \frac{x^2}{2239488}$$

$$\boxed{P_2(3) = 12'00694042}$$

$c \in \langle 0,3 \rangle$ $\boxed{\frac{5}{3 \cdot 12^7}}$

$$R_2(x) = \frac{F'''(c)}{3!} x^3 = \frac{10 \cdot 3^2}{27 \cdot 6 (\sqrt[3]{1728+c})^7} < \frac{5}{3(\sqrt[3]{1728+0})^7} =$$

3- Cota superior del error de la fórmula

$e \simeq 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ usando Taylor de e^x

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$P_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!}$$

$$\text{Error} = R_4(1) = \frac{F^5(c)}{5!} 1^5 = \frac{e^c}{5!} \leq \boxed{\frac{e}{5!}}$$

$c \in (0,1)$

2- $F(x) = \ln(1-x)$

a) Hallar los 5 primeros términos no nulos del polinomio de Taylor ($x_0 = 0$) y el resto de Lagrange

b) Hallar el grado del polinomio para obtener $P_n(0'75)$ con error menor que 10^{-3}

$$P_n(x) = F(x_0) + F'(x_0) \cdot (x - x_0) + \frac{F''(x_0)}{2} \cdot (x-x_0)^2 + \dots$$

$$+ \frac{F^{(n)}(x_0)}{n!} (x-x_0)^n$$

$$F'(x) = \frac{-1}{+1-x}$$

$$\boxed{P_5(x) = -x - \frac{1}{2}x^2 - \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \frac{24x^5}{5!}}$$

$$F''(x) = -(1-x)^{-2}$$

$$F'''(x) = -2 \cdot (1-x)^{-3}$$

$$\boxed{R_5(x) = \frac{-120}{6! \cdot (1-c)^6} \cdot x^6}$$

$$F''''(x) = -6 \cdot (1-x)^{-4}$$

$$F'''''(x) = -24 \cdot (1-x)^{-5}$$

b) $R_n(x) = \frac{+n!}{(1-x)^{n+1} \cdot (n+1)!} \cdot x^{n+1} < 10^{-3}$

$P_n(0'75) \simeq \ln(0'75) = F(0'25)$

$$R_n(0'25) = \frac{0'25^{n+1}}{(n+1) \cdot (1-c)^{n+1}}$$

Aplicaciones polinomio Taylor

1- Monotonia y extremos

f (n+1) veces derivable

$f'(a) = f''(a) = f'''(a) = f^{n-1}(a) = 0$

$f^{n}(a) \neq 0$

$f(x) = P_n(x) + R_n(x)$

$f(x) = f(a) + \dfrac{f^{n}(a)}{n!}(x-a)^n + R_n(x)$

n par: $f^{n}(a) > 0 \rightarrow f(x) - f(a) > 0 \rightarrow f(x) > f(a)$ ↓ a es un minimo

$f^{n}(a) < 0 \rightarrow f(x) - f(a) < 0$ ↓ $f(x) < f(a) \rightarrow$ a es un máximo

n impar: $f^{n}(a) > 0 \rightarrow x - a > 0 \rightarrow f(x) - f(a) > 0$ ↓ f es creciente ↑ $x - a < 0 \rightarrow f(x) - f(a) < 0$

~~$f^{n}(a) >$~~

$f^{n}(a) < 0 \rightarrow x - a > 0 \rightarrow f(x) - f(a) < 0$ ↓ f es decreciente ↑ $x - a < 0 \rightarrow f(x) - f(a) > 0$

ej $f(x) = 2x^3 - 9x^2 + 12x + 5$

$f' = 0$

$f' = 6x^2 - 18x + 12 = 0$

$f'' = 12x - 18$

$f''(1) = -6 < 0 \rightarrow x = 1 \rightarrow$ Maximo

$f''(2) = 6 > 0$ x: 2 minimo

$f(x) = x^9 - 25$

$f' = 9x^8 = 0$

$x = 0$

$f'' = 56x^6$

$f''(0) = 0$

$f''' = 56 \cdot 6 \cdot x^5$

$f^4 = 56 \cdot 6 \cdot 5 \cdot x^4$

$f^5 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot x^3$

$f^6 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot 3\, x^2$

$f^7 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2\, x$

$f^8 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$

x = 0: minimo

2- Curvatura y puntos inflexion

f convexa  ⟷ 

$f(x) \leq$ recta secante

recta tangente $\leq f(x)$

f concava: No convexa

f n+1 veces derivable

$f''(a) = f'''(a) = f^{n-1}(a) = 0$

$f^{n}(a) \neq 0$

$f(x) - \underbrace{f(a) + f'(a)\cdot(x-a)}_{\text{recta tangente}} = \dfrac{f^{n}(a)}{n!}(x-a)^n + R_n$

n par: $f^{n}(a) > 0 \rightarrow$ f es convexa

$f^{n}(a) < 0 \rightarrow$ f es concava

n impar: $f^{n}(a) > 0 \rightarrow x > a \rightarrow$ convexa; $x < a \rightarrow$ concava → a punto de inflexion

$f^{n}(a) < 0 \rightarrow x > a \rightarrow$ concava; $x < a \rightarrow$ convexa → a punto de inflexión

6- $f: [0,1] \to [0,1]$

f continua derivable

$f'(x) \neq 1$

Demostrar que existeix x_0: $f(x_0) = x_0$

$g(x) = f(x) - x$

$g(x)$ continua

$g(x)$ derivable

$g(0) = f(0) - 0 = f(0) \geq 0$

$g(1) = f(1) - 1 < 0$

$\exists x_0 \in (0,1)$

$g(x_0) = 0$

$f(x_0) - x_0 = 0$

$\boxed{f(x_0) = x_0}$

x_0 únic - T. Rolle: $g(x)$ continua en $[0,1]$

$g(x)$ derivable en $[0,1]$

$g(a) = g(b)$

$\exists c \in (a,b)$ $g'(c) = 0$

$g(x_0) = g(x_1) = 0$

$g'(x) = f'(x) - 1$

$f'(c) - 1 = 0$

$f'(c) \neq 1$

7- $e^x = \frac{1}{2}x + 2$

a) Demonstrar 2 solucions en $[-5, 2]$ → Positiva → Negativa

$f(x) = e^x - \frac{1}{2}x - 2$

$f(-5) = e^{-5} - \frac{1}{2}(-5) - 2 > 0$

$f(0) = e^0 - \frac{1}{2}0 - 2 < 0$

$\to \exists x_1 : f(x_1) = 0$, $x_1 \in (-5, 0)$

$f(2) = e^2 - \frac{1}{2} \cdot 2 - 2 > 0$

$\to \exists x_2 : f(x_2) = 0$, $x_2 \in (0, 2)$

b) Demostreu que l'equació té només 2 solucions

$f'(x) = e^x - \frac{1}{2}$

$f'(x_0) = 0$

$x_0 = \ln\left(\frac{1}{2}\right) = \boxed{-0'7}$

c) n? nº d'iteraccions mètode Bisecció per calcular la solució positiva amb un error menor que 10^{-8}

$\frac{b-a}{2^n} < n$

$\frac{2-0}{2^n} < 10^{-8}$

$\log 2^{1-n} < \log 10^{-8}$

$(1-n) \cdot \log 2 < -8 \log 10$

$\log 2 - n \cdot \log 2 < -8$

$\log 2 + 8 < n \cdot \log 2$

$\frac{\log 2 + 8}{\log 2} < n$

$27'57 < n$

$\boxed{n \geq 28}$

Tema 6 - Calculo integral

def $\rightarrow$ $f: \mathbb{R} \rightarrow \mathbb{R}$

$F: \mathbb{R} \rightarrow \mathbb{R}$

$a < b \in \mathbb{R}$; $[a,b] < \text{Dom } F \cap \text{Dom } F'$

$F(x)$ es una primitiva de $f(x)$ si $F'(x) = f(x)$

ej $\left.\begin{array}{l} f(x) = 3x^2 \\ F(x) = x^3 \end{array}\right\} \rightarrow F' = f$

$\left.\begin{array}{l} f(x) = 3x^2 \\ F(x) = x^3+5 \end{array}\right\} \rightarrow F' = f$

Proposición

F, g primitivas de f $\rightarrow$ $F = G + C$ $\quad C \in \mathbb{R}$

Inmediatas	*Cuasi inmediatas (con funciones)*
$\int x^n dx = \frac{x^{n+1}}{n+1} + k$ se suma 1 al expo y se divide por lo mismo	$\int f^n(x) \cdot f'(x) dx = \frac{f^{n+1}(x)}{n+1} + k$
$\int \frac{1}{x} dx = \ln\|x\| + k$	$\int \frac{f'(x)}{f(x)} dx = \ln\|f(x)\| + k$
$\int e^x dx = e^x + k$ se queda igual	$\int e^{f(x)} f'(x) dx = e^{f(x)} + k$
$\int a^x dx = \frac{a^x}{\ln a} + k$	$\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\ln a} + k$
$\int \text{sen}x dx = -\cos x + k$	$\int \text{sen}(f(x)) f'(x) dx = -\cos(f(x)) + k$
$\int \cos x dx = \text{sen}x + k$	$\int \cos(f(x)) f'(x) dx = \text{sen}(f(x)) + k$
$\int \frac{1}{\cos^2 x} dx = \int (1 + tg^2 x) dx = tgx + k$	$\int \frac{f'(x)}{\cos^2(f(x))} dx = \int (1 + tg^2(f(x))) f'(x) dx = tg(f(x)) + k$
$\int \frac{1}{\text{sen}^2 x} dx = \int (1 + \cot g^2 x) dx = -ctgx + k$	$\int \frac{f'(x)}{\text{sen}^2(f(x))} dx =$ $\int (1 + ctg^2(f(x))) f'(x) dx = -ctg(f(x)) + k$
$\int \frac{1}{\sqrt{1 - x^2}} dx = \text{arcsen}x + k$	$\int \frac{f'(x)}{\sqrt{1 - f^2(x)}} dx = \text{arcsen}(f(x)) + k$
$\int \frac{1}{1 + x^2} dx = \text{arctg}x + k$	$\int \frac{f'(x)}{1 + f^2(x)} dx = \text{arctg}(f(x)) + k$

Propiedades y métodos de calcular

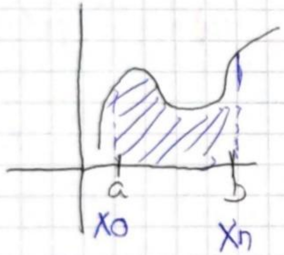
- En la suma/resta: La integral de la suma es la suma de las integrales: $\int (u + v)\, dx = \int u\, dx + \int v\, dx$

Integral indefinida de f

Def → $\int f(x)\,dx = \{F(x) \,/\, F'(x) = f(x)\} = F(x)+C$

Integral definida

$f \geq 0$ en $[a,b]$



$$x_i = \frac{b-a}{n}$$

Suma inferior

$$L_n = \sum_{x\in[x_0,x_n]} \min f(x)\cdot(x_i - x_{i-1}) \leq \text{Area}$$

Suma Superior

$$U_n = \sum_{x\in[x_0,x_n]} \max f(x)\cdot(x_i - x_{i-1}) \geq \text{Area} = \int_a^b f(x)\,dx$$

primitiva $F(x) = x^3+2x$

primitiva $F(x)+1 = x^3+2x+1$

primitiva $F(x)+2 = x^3+2x+2$

primitiva $F(x)-1 = x^3+2x-1$

⋮

integrar ←

derivar →

$F'(x) = f(x) = 3x^2+2$

$(F(x)+1)' = f(x) = 3x^2+2$

$(F(x)+2)' = f(x) = 3x^2+2$

$(F(x)-1)' = f(x) = 3x^2+2$

"INTEGRAL INDEFINIDA"

es ↓ representa

$$\boxed{F(x)+C = \int f(x)\,dx = \int F'(x)\,dx}$$

$$\underbrace{x^3+2x+C}_{\text{integral indef.}} = \int \underbrace{(3x^2+2)}_{\text{derivada}}\,dx$$

* propietats: 1. $\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx$

2. $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

→ amb funcions compostes

tenim en compte la **REGLA** de la **CADENA**

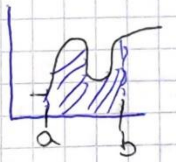
$$F[g(x)]+C = \int (F[g(x)])'\,dx = \int F'(g(x))\cdot g'(x)\,dx$$

Exemple: $\int \underbrace{\cos x^2}_{\sin' x^2} \cdot \underbrace{2x}_{(x^2)'}\,dx = \sin(x^2)+C$

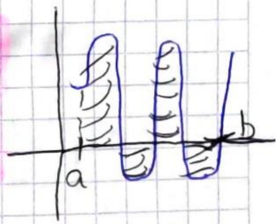
Def $\rightarrow \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$

$$\int_a^b f(x)\,dx = \lim L_n = \lim U_n$$

Def $\rightarrow \int_a^b f(x)\,dx = \text{Area}$

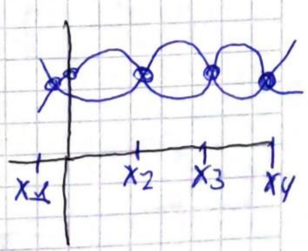


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$$



Area entre dos curvas (f, g)

$$f = g \begin{cases} x_1 \\ x_2 \\ x_3 \\ x_4 \end{cases}$$



E | A_2

$$A = \int_{x_1}^{x_2} f - g + \int_{x_2}^{x_3} g - f + \ldots$$

Teorema Fundamental del Calculo : TFC

$f: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

$F: \mathbb{R} \to \mathbb{R}$

$x \to \int_a^x f(t)\,dt$

- $F(x)$ continua en $[a,b]$ derivable en (a,b)
- $F'(x) = f(x)$

$$F(x) = \left(\int_a^x f(t)\,dt\right)' = f(x)$$

Regla de Barrow

$f: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

$F: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

F derivable en (a,b) y $F'(x) = f(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Teorema fonamental del Càlcul

Si f contínua en $[a,b]$ i $F(x)=\int_a^x f(t)dt$ $\Rightarrow$ F contínua en $[a,b]$ i és una primitiva de f: $F'=f$

Regla de Barrow

Si f contínua en $[a,b]$ i $F(x)$ contínua en $[a,b]$ i derivable en (a,b) tal que $F'=f$ $\Rightarrow$ $\int_a^b f(x)dx = F(b) - F(a)$!

aplicació

Calcular integrals definides

Exemple

PAS 1. Calcular $\int f(x)dx = F$ indefinida

PAS 2. substituir $F(x=b)$, $F(x=a)$

PAS 3. restar $F(b) - F(a)$

$$\int_1^4 f(x)dx = \int_1^4 x^2dx = \frac{x^3}{3}\Big|_1^4 =$$ PAS 1

$$= \frac{4^3}{3} - \frac{1^3}{3} =$$ PAS 2

$$= \frac{63}{3} = \underline{\underline{21}}$$ PAS 3

ej: $$\int_1^3 x^2dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \boxed{\frac{26}{3}}$$

Corolario

$f: \mathbb{R}\to\mathbb{R}$ continua en $[a,b]$

$$\left(\int_x^a f(x)dx\right)' = -f(x)$$

$$\left(\int_a^x f(x)dx\right)' = f(x)$$

Corolario

$f: \mathbb{R}\to\mathbb{R}$ continua en $[a,b]$

$g: \mathbb{R}\to\mathbb{R}$ derivable en (a,b)

$$\left(\int_a^{g(x)} f(t)dt\right)' = f(g(x))\cdot g'(x)$$

Corolario

$f: \mathbb{R}\to\mathbb{R}$ continua en $[a,b]$

$g_1, g_2: \mathbb{R}\to\mathbb{R}$ derivables en (a,b)

$$\left(\int_{g_2(x)}^{g_1(x)} f(t)dt\right)' = f(g_1(x))\cdot g_1'(x) - f(g_2(x))\cdot g_2'(x)$$

2- b) $\lim\limits_{x \to 0} \dfrac{x \cdot \int_0^x e^{t^2} dt}{\int_0^x e^{t^2} \cdot \sin t \, dt} = \dfrac{1 \cdot \int_0^x e^{t^2} dt + x \cdot e^{x^2}}{e^{x^2} \cdot \sin x}$

$$= \frac{e^{x^2} + 1 \cdot e^{x^2} + x \cdot e^{x^2} \cdot 2x}{e^{x^2} \cdot 2x \cdot \sin x + e^{x^2} \cdot \cos x}$$

$$= 2$$

3- $F : (0, +\infty) - \{1\} \to \mathbb{R}$

$F(x) = \int_x^{x^2} \frac{dt}{\ln t}$

$F'(x) = \frac{1}{\ln x^2} \cdot 2x - \frac{1}{\ln x} \cdot 1 = \frac{2x}{2 \ln x} - \frac{1}{\ln x} = \frac{x-1}{\ln x}$

$(0,1) \to > 0$

$(1, +\infty) \to > 0$

F par si $F(-x) = F(x)$

F impar si $F(-x) = -F(x)$

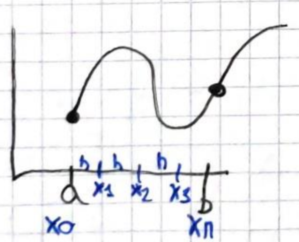
Integración numerica (Aprox integrales)

$F : \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

$F \geq 0$

Queremos calcular $\int_a^b f(x)$ con la precision deseada

Metodo de los Trapecios (T_n)



$y = f(x)$

Partición de $[a,b]$

$\{ x_0 = a$

$h = \frac{b-a}{n}$; $x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$x_i = x_0 + i \cdot h$

$$\boxed{A = \frac{f(x_{i-1}) + f(x_i)}{2} \cdot h}$$

$$\boxed{\int_a^b f(x) \simeq \sum_{i=1}^{n} \frac{h}{2} \cdot (f(x_i) + f(x_{i-1}))}$$

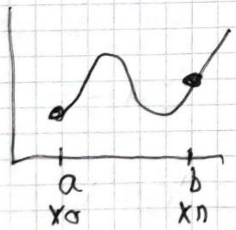
$$\text{Error} = \int_a^b f(x) - T_n = \frac{(b-a)^3}{12n^2} \cdot f''(c)$$

Cota error $\to M_2 \geq \max f''(x) \to$

$$\boxed{\text{Error} \leq \frac{(b-a)^3}{12n^2} \cdot M_2}$$

$$\boxed{T_n = h \cdot \left(\frac{f(x_0) + f(x_n)}{2} + f(x_1) + f(x_2) \ldots \right)}$$

Metodo de Simpson (S_n)



Particion de $[a,b]$

$x_0 = a$

$x_i = x_0 + i \cdot h$ n par

$x_n = b$

$h = \dfrac{b-a}{n}$

$$A = \int_{x_{n-1}}^{x_{n+1}} P_2(x) = \frac{h}{3} \cdot \left[f(x_{i-1}) + 4 \cdot (f(x_i)) + f(x_{i+1}) \right]$$

$$\boxed{S_n = \frac{h}{3} \cdot \left[f(x_0) + 4 \cdot \underbrace{[f(x_1) + f(x_3) \dots]}_{\text{impars}} + 2 \cdot \underbrace{[f(x_2) + f(x_4)]}_{\text{pars}} + f(x_n) \right]}$$

$$\text{Error} = \int_a^b f - S_n = \frac{(b-a)^5}{180 \cdot n^4} \cdot f^4(c)$$

f 4 veces derivable en $[a,b]$

$M_4 \geq \max f^4(x)$

$$\boxed{\text{Error} \leq \frac{(b-a)^5}{180 \cdot n^4} \cdot M_4}$$

TEMA : INTEGRALES

Ej 7: Integrales inmediatas

D) $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \dfrac{x^{3/2+1}}{3/2+1} = \dfrac{2}{5} x^{5/2} + K = \dfrac{2}{5}\sqrt{x^5} = \dfrac{2}{5} x^2 \sqrt{x} + K$

$\int [u(x)]^r dx = \dfrac{[u(x)]^{r+1}}{r+1} + K$

$\int \dfrac{1}{u(x)} \cdot u'(x)\,dx = \ln|u(x)| + K$

Separar en 3 integrals

a) $\int \left(\dfrac{1-x}{x}\right)^2 dx = \int \dfrac{\boxed{1-2x+x^2}}{x^2} dx = \int \dfrac{dx}{x^2} - \int \dfrac{2x}{x^2} dx + \int \dfrac{x^2}{x^2} dx = \int x^{-2} dx - 2\int \boxed{\dfrac{dx}{x}} + \int dx =$

$= \dfrac{x^{-2+1}}{-2+1} - 2\ln|x| + x + K = -\dfrac{1}{x} - 2\ln|x| + x + K$

b) $\int \dfrac{x^3}{\boxed{x^4+1}} dx = \dfrac{1}{4}\int \dfrac{1}{x^4+1} 4x^3\,dx = \dfrac{1}{4}\ln|x^4+1| + K$

NO SEPARAR

$(2x^2)' = 4x$

f) $\int x \cdot 5^{2x^2} dx = \int 5^{\boxed{2x^2}} x\,dx = \dfrac{1}{4}\int 5^{2x^2} 4x\,dx = \dfrac{1}{4} \cdot \dfrac{1}{\ln 5} 5^{2x^2} + K = \dfrac{1}{4\ln 5} 5^{2x^2} + K$

$\int a^{u(x)} \cdot u'(x)\,dx = \dfrac{1}{\ln a} a^{u(x)}$

c) $\int \sqrt{\dfrac{\arcsen x}{1-x^2}}\,dx = \int \sqrt{\arcsen x} \cdot \dfrac{1}{\sqrt{1-x^2}} dx = \int (\arcsen x)^{1/2} \cdot \dfrac{1}{\sqrt{1-x^2}} dx =$

$= \dfrac{(\arcsen x)^{1/2+1}}{1/2+1} + K = \dfrac{2}{3}(\arcsen x)^{3/2} + K$ $(\arcsen x)' = \dfrac{1}{\sqrt{1-x^2}}$

e) $\int \dfrac{dx}{x\ln x} = \int \dfrac{1}{\ln x} \cdot \dfrac{1}{x} dx = \int (\ln x)^{-1} \cdot \dfrac{1}{x} dx = \ln|\ln x| + K$

$u'(x) = (\ln x)' = \dfrac{1}{x}$

g) $\int \dfrac{1}{1+16x^2} dx = \int \dfrac{1}{1+(4x)^2} = \dfrac{1}{4}\int \dfrac{1}{1+(4x)^2} 4\,dx = \dfrac{1}{4}\operatorname{arctg}(4x) + K$

$\int \dfrac{1}{1+[u(x)]^2} \cdot u'(x)\,dx = \operatorname{arctg}[u(x)] + K$

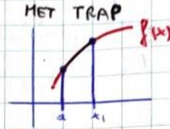
INTEGRALES

* INTEGRACIÓ NUMÈRICA → Ejs 12, 13, 14

* TFC: f contínua, $F(x)=\int_a^x f(t)\,dt$ $\Rightarrow$ F contínua en $[a,b]$, F derivable (a,b), $F'(x)=f(x)$

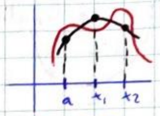
serveix per:
- → Punt crític (x_0) de F: $F'(x_0)=0$
- → $\lim_{x} \frac{F=\int}{g(x)} = \text{indet} \overset{LH}{=} \frac{F'}{g'}$

MET TRAP



$f(x)$ → Polinomi 1r grau (RECTA)

MET SIMPSON



$f(x)$ → Polinomi 2n grau (PARABOLA)

INTEGRACIÓ NUMÈRICA

(TIPUS EXERCICIS):

TIPUS 1 → Donen número intervals
→ CALCULAR → Integral
→ Error

* Exercici 12

TIPUS 2 → Donen fita ERROR
→ CALCULAR → Nº intervals
→ Integral

* Exercici 13

Ex 12 a:

$\int_0^1 e^{x^2}\,dx = ?$ Donen $n=4$

$\boxed{h=\frac{b-a}{n}} = \frac{1-0}{4} = \frac{1}{4}$

$\boxed{x_i = a+ih} = 0 + i\cdot 1/4 = i/4$

$f(x)=e^{x^2}$

$f'(x)=2xe^{x^2}$

$f''(x)=2e^{x^2}+2x\cdot 2xe^{x^2}=2e^{x^2}+4x^2e^{x^2}$ → ERROR TRAPEZIS

$f'''(x)=2\cdot 2xe^{x^2}+4\cdot 2x\cdot e^{x^2}+4x^2\cdot 2x\cdot e^{x^2}=12xe^{x^2}+8x^3e^{x^2}$

$f^{iv}(x)=12e^{x^2}+12\cdot 2x\cdot e^{x^2}+8\cdot 3x^2\cdot e^{x^2}+8x^3\cdot 2x\cdot e^{x^2}=12e^{x^2}+48x^2e^{x^2}+16x^4e^{x^2}$ → ERROR SIMPSON

E.TRAP $< \frac{(b-a)^3}{12n^2}\cdot \max_{[a,b]}|f''(x)|$

E.SIMPSON $< \frac{(b-a)^5}{180n^4}\cdot \max_{[a,b]}|f^{iv}(x)|$

$f'''(x_0)=0 \rightsquigarrow 4xe^{x^2}\cdot(3+2x^2)=0$ → $\boxed{x_0=0}$; $3+2x^2=0 \to x_0=\pm\sqrt{-3/2}\notin\mathbb{R}$

h) $\int \tan^2 x\,dx = \int\left[\frac{\sin^2 x}{\cos^2 x}+1-1\right]dx = \int\left(\frac{\sin^2x+\cos^2x}{\cos^2x}-1\right)dx \overset{\sin^2x+\cos^2x=1}{=} \underbrace{\int\left(\frac{1}{\cos^2x}-1\right)dx}_{\text{Separar 2 integrals}} =$

$= \int\frac{dx}{\cos^2x}-\int dx = \operatorname{tg}x - x + K$

$\int \frac{1}{\cos^2[u(x)]}\cdot u'(x)\,dx = \operatorname{tg}x+K$

Ej 8: Integrar per PARTS:

TEORIA

$\int[u(x)\cdot v(x)]' = \int u'(x)\cdot v(x) + \int u(x)\cdot v'(x)$

$u(x)\cdot v(x) = \int u'(x)\cdot v(x) + \boxed{\int u(x)\cdot v'(x)} \to \int u(x)\cdot v'(x) = u(x)\,v(x) - \int v(x)\cdot u'(x)$

d) $\int x\sin(2x)\,dx = x\cdot\frac{-1}{2}\cos(2x) - \int -\frac{1}{2}\cos(2x)\,dx = \frac{-x}{2}\cos(2x)+\frac{1}{2\cdot 2}\int\cos(2x)\cdot 2\,dx =$

$u=x \to du=dx$

$dv=\sin(2x)dx \to v=\int dv=\int\sin(2x)dx = -\frac{1}{2}\cos(2x)$

$= -\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+K$

b) $\int\frac{\ln x}{\sqrt{x}}dx = \int \ln x\frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - \int 2\sqrt{x}\frac{dx}{x} = 2\sqrt{x}\ln x - 2\int x^{-1/2}dx = 2\sqrt{x}\ln x - 2\frac{x^{1/2}}{1/2}+K$

$= 2\sqrt{x}\ln x - 4\sqrt{x}+K =$

$= 2\sqrt{x}(\ln x - 2)+K$

$u=\ln x \to du=\frac{dx}{x}$

$dv=\frac{dx}{\sqrt{x}} \to v=\int dv=\int\frac{dx}{\sqrt{x}}=\int x^{-1/2}dx=\frac{x^{1/2}}{1/2}=2\sqrt{x}$

a) $\int e^{2x}\cdot\sin x\,dx \overset{①}{=} -e^{2x}\cos x - \int -\cos x\,2e^{2x}dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \overset{③}{=}$

① $u=e^{2x} \to du=2e^{2x}dx$; $dv=\sin x\,dx \to v=\int dv=\int\sin x\,dx=-\cos x$

④ $u=\cos x$; $dv=e^{2x}dx$

② $u=\sin x$; $dv=e^{2x}dx$

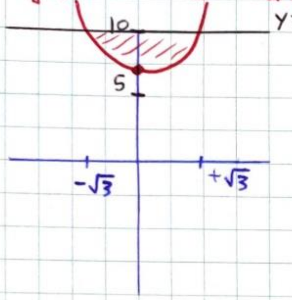
③ $u=e^{2x}\to du=2e^{2x}dx$; $dv=\cos x \to v=\int dv=\int\cos x = -\sin x$

$\overset{③}{=} -e^{2x}\cos x + 2\left[e^{2x}\sin x - 2\underbrace{\int e^{2x}\cdot\sin x\,dx}_{A}\right]$

$A = -e^{2x}\cos x + 2e^{2x}\operatorname{sen} x - 4A \rightarrow A + 4A = -e^{2x}\cos x + 2e^{2x}$

$$A = \frac{1}{5}e^{2x}\left[2\operatorname{sen} x - \cos x\right] + K$$

Ej 9: Area limitada per la paràbola $y = x^2+7$ y la recta $y = 10$



PAS 1 → Abscisses dels PUNTS DE TALL

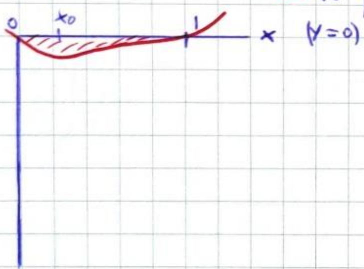
$$x^2+7 = 10 \rightarrow x^2 = 3 \rightarrow x = \pm\sqrt{3}$$

PAS 2 → AREA

$$AREA = \left|\int_{-\sqrt{3}}^{\sqrt{3}} \left[10-(x+7)\right]dx\right| = -\int_{-\sqrt{3}}^{\sqrt{3}} x^2 dx + 3\int_{-\sqrt{3}}^{\sqrt{3}} dx = -\frac{x^3}{3}\Bigg|_{-\sqrt{3}}^{\sqrt{3}} + 3x\Bigg|_{-\sqrt{3}}^{\sqrt{3}} =$$

$$= -\left[\frac{(\sqrt{3})^3}{3} - \frac{(-\sqrt{3})^3}{3}\right] + \left[3\left[\sqrt{3}-(-\sqrt{3})\right]\right] = -\frac{2\sqrt{3}}{3} + 3\cdot 2\sqrt{3} = -\frac{2}{3}3\sqrt{3} + 6\sqrt{3} = 4\sqrt{3} \text{ unitats area}$$

Ej 11: Area limitada
- → Semieix positiu abscisses $\boxed{y=0}$
- → Corba $y = (x^2-x)e^{-x}$
- → 4t quadrant



PAS 1

$$(x^2-x)e^{-x} = 0 \Leftrightarrow x^2-x = 0$$
$$x(x-1) = 0 \rightarrow x=0,\ x=1$$

Estudiem $y = (x^2-x)e^{-x}$ $\forall x \in (0,1) \rightarrow y(x) < 0$

$$y' = (2x-1)e^{-x} - e^{x}(x^2-x)$$

$$y'(x_0) = 0 \leftrightarrow (-x_0^2+3x_0-1)e^{-x} = 0 \rightarrow$$

$\rightarrow x_0 \simeq 2{,}62 \notin [0,1]$

$\rightarrow x_0 \simeq 0{,}38 \in [0,1]$

PAS 2 → $A = \int_0^1 \left[0-(x^2-x)e^{-x}\right]dx = -\int_0^1 x^2 e^{-x}dx + \int_0^1 xe^{-x}dx$ ⇝ ① INTEGRAL INDEF ② BARROW → INTEGRAL DEF

Error TRAP → $\frac{(1-0)^3}{12\cdot 4^2}\cdot 6e = \frac{e}{32} = 0{,}084946$

$$\int_0^1 e^{x^2}dx \overset{TRAP}{\simeq} h\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right] = h\left[\frac{f(a)+f(b)}{2} + f(x_1)+f(x_2)+f(x_3)\right] =$$

$$= \frac{1}{4}\left[\frac{e^0+e^1}{2} + e^{(1/4)^2} + e^{(2/4)^2} + e^{(3/4)^2}\right] \simeq 1{,}49067$$

ERROR SIMPSON → $f^{IV}(x)$ creixent en $[0,1]$ → $\frac{(1-0)^5}{180\cdot 4^4}\cdot 76e = 0{,}004483$

$$\int_0^1 e^{x^2}dx \overset{SIMPSON}{\simeq} \frac{h}{3}\left[f(a)+f(b) + \underset{\text{Impar}}{4f(x_1)} + \underset{\text{Par}}{2f(x_2)} + \underset{\text{Impar}}{4f(x_3)}\right] = \frac{1/4}{3}\left[e^0+e^1+4\cdot e^{(1/4)^2} + 2\cdot e^{(2/4)^2} + 4\cdot e^{(3/4)^2}\right] \simeq 1{,}46371$$

↳ Cifras correctas

Ex 13 b:

Simpson (Trapezis) $\int_0^1 \cos(x^2)\,dx$ Donen error $< 0{,}5\cdot 10^{-2} \rightarrow n = ?$

⊛ $f(x) = \cos(x^2)$

$f'(x) = -2x \sin(x^2)$

$f''(x) = -2\sin(x^2) - 2x\cdot 2x\cos(x^2) = -2\sin(x^2) - 4x^2\cos(x^2)$

$f'''(x) = -2\cdot 2x\cos(x^2) - 4\cdot 2x\cos(x^2) - 4x^2\cdot 2x(-\sin(x^2)) = -12x\cos(x^2) + 8x^3\sin(x^2)$

$f^{IV}(x) = -12\cos(x^2) - 12x\cdot 2x(-\sin(x^2)) + 8\cdot 3x^2\sin(x^2) + 8x^3\cdot 2x\cos(x^2) =$

$= -12\cos(x^2) + 48x^2\sin(x^2) + 16x^4\cos(x^2)$

$$|f^{IV}(x)| = |\underbrace{16x^4\cos(x^2)}_{\geq 0 \text{ en } [a,b]} + \underbrace{48x^2\sin(x^2)}_{\geq 0} \underbrace{-12\cos(x^2)}_{<0}| < 16x^4\underbrace{\cos(x^2)}_{1} + 48x^2\underbrace{\sin(x^2)}_{1} < 16+48 = 64$$

$$\text{Error Simpson} = \frac{(b-a)^5}{180n^4}\max_{[0,1]}|f^{IV}(x)| = \frac{1}{180n^4}64 < 0{,}5\cdot 10^{-2} \rightarrow n \geq \sqrt[4]{\frac{32}{45}\cdot 10^2} = 2{,}90 \rightarrow 3$$

n = parell SIMPSON $\rightarrow \boxed{n=4}$

$$\int_0^1 \cos(x^2)\,dx \overset{\text{SIMPSON}}{\simeq} \frac{h}{3}\left[f(a)+f(b)+4f(x_1)+2f(x_2)+4f(x_3)\right] =$$

$$= \frac{1/4}{3}\left[f(0)+f(1)+4f(1/4)+2f(2/4)+4f(3/4)\right] = \textbf{RADIANS} =$$

$$= \frac{1}{12}\left[\cos(0)+\cos(1)+4\cos\left[(1/4)^2\right]+2\cos\left[(2/4)^2\right]+4\cos\left[(3/4)^2\right]\right] \simeq 0{,}90450 \rightarrow S(n=4) \simeq 0{,}905$$

TFC i f contínua $[a,b]$

$F(x) = \int_0^x f \Downarrow$ $\boxed{\begin{array}{l} F \text{ contínua } [a,b] \\ F \text{ derivable } (a,b) \end{array}}$ $\rightarrow F'(x) = f(x) \quad \forall x \in (a,b)$

Versió: $F(x) = \int_{u(x)}^{v(x)} f(t)\,dt \rightarrow F'(x) = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)$

Ej 15:

$$\left.\begin{array}{l} F = \int_1^{x^2+2} \boxed{\frac{e^t}{t}}\,dt \\ f = \frac{e^t}{t} \text{ contínua } t \in \mathbb{R}-\{0\} \rightarrow \text{contínua } \forall x \in [1, x^2+2] \end{array}\right\} \xRightarrow[f \text{ contínua}]{\text{TFC}} F \begin{cases} \text{Contínua} \\ \text{Derivable} \end{cases}$$

$$F'(x) = \frac{e^{x^2+2}}{x^2+2}\cdot 2x - \frac{e^1}{1}\cdot 0$$

a) Comproveu que $\boxed{x=0}$ és un PUNT CRÍTIC de F

$F'(x=0) = 0 \rightarrow \frac{e^{0+2}}{0+2}\cdot 2\cdot 0 \overset{?}{=} 0 \rightarrow \frac{e^2}{2}\cdot 0 = 0$ Com volíem demostrar

Examen 16/11/2010

$$\lim_{x\to 0^+} \frac{\int_0^{x^2}\sin(\sqrt{t})\,dt}{x^3} = \frac{0}{0} \overset{LH}{=} \lim_{x\to 0^+}\frac{F'}{3x^2} = \lim_{x\to 0^+}\frac{2x\sin x}{3x^2} = \lim_{x\to 0^+}\frac{2}{3}\boxed{\frac{\sin x}{x}} = \frac{2}{3}\cdot 1 = \frac{2}{3}$$

$$\left.\begin{array}{l} F = \int_0^{x^2} f(t) = \sin(\sqrt{t})\,dt \\ f(t) = \sin(\sqrt{t}) \text{ contínua } \mathbb{R} \end{array}\right\} \xRightarrow[f \text{ contínua}]{\text{TFC}} F \begin{cases} \text{contínua} \\ \text{derivable} \end{cases}$$

$$F' = \sin(\sqrt{x^2})\cdot 2x - \cancel{\sin(\sqrt{0})}^{\,0}\cdot 0 = 2x\sin|x|$$

Tema 7 - Funciones de Varias Variables

Conicas

Ecuación de 2^n grado en $\mathbb{R}^2$

ej $x^2+2xy+y^2-3x+2y=0$

- Circumferencia de centro (a,b) y radio r:

$$(x-x_0)^2+(y-y_0)^2=r^2$$

- elipse ejes a,b centro (y_0,x_0)

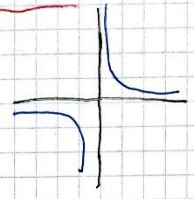
$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

- hiperbole centro (x_0,y_0) ejes a,b

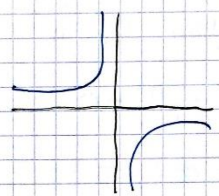
$$y=\frac{b}{a}x \qquad \frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1$$

hipérbole equilatera

$xy=K$



$K>0$

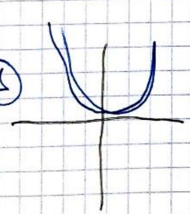


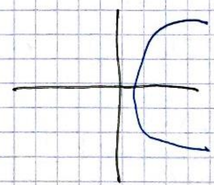
$K<0$

Parabolas: $y=ax^2+bx+c$ (1)

$x=ay^2+by+c$ (2)

(1)





(2)

ej $x^2+y^2=4$

Circumferencia $C=(0,0)$ $r=2$

$2x^2+2y^2+3x+5y-5=0$

$$\left(x+\frac{3}{4}\right)^2+\left(y+\frac{5}{4}\right)^2=\frac{74}{16}$$

$2x^2+3x \quad +2y^2+5y=5$

$$x^2+\frac{3}{2}x+\frac{9}{16}+y^2+\frac{5}{2}y=\frac{5}{2}+\frac{9}{16}+\frac{25}{16}$$

$x^2+4y^2=100$

$$\frac{x^2}{100}+\frac{y^2}{25}=1 \qquad C=(0,0)$$

ejes $(10,5)$

$y^2-4y-6x+5=0$

$$x=\frac{y^2-4y+5}{6}$$

$f:\mathbb{R}\to\mathbb{R}$

$x\to y=f(x)$

$f:\mathbb{R}^2\to\mathbb{R}$

$(x,y)\to z=f(x,y)$

Topologias en el espació n-dimensional

$x=(x_1,x_2,\dots x_n)\in\mathbb{R}^n$

def $\to$ Norma de $x=(x_1,\dots x_n)\in\mathbb{R}^n$

$$\|x\|=\sqrt{x_1^2+x_2^2+\dots x_n^2}$$

Def → Frontera de A : $Fr(a) = \partial A$ conjunto de todos los puntos frontera.

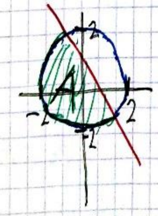
Interior de A : $\mathring{A}$, Conjunto de todos los puntos interiores

Adherencia de A : $Adh(A) = \bar{A}$ Menor conjunto cerrado que contiene a A

A abierto si $A = \mathring{A}$

A cerrado si $Fr(A) \subset A \leftrightarrow \bar{A} = A$ $[\bar{A} = A \cup Fr(A)]$

ej: $A = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 \leq 4,\ x+y<1\}$



$A^o = x^2+y^2 < 4,\ x+y<1$

$F_r(a) = x^2+y^2=4,\ x+y \leq 1$

$\cup\ x+y = 1,\ x^2+y^2 < 4$

A acotado

A no compacto

$\bar{A} = A \cup Fr(a)$

$A \neq A^o \rightarrow$ no Abierto

$A \neq \bar{A} \rightarrow$ no Cerrado

$B_r(a) = \{x \in \mathbb{R}^n / d(x,a) < r\}$

Bola de centro a y radio r

Def → Complementario de A: $A^c = \{x \in \mathbb{R},\ x \notin A\}$

$\mathbb{R}^n - A$

Def → $A \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$

- a es interior a A si $\exists r\ B_r(a) \subset A$
- a es exterior a A si $\exists r\ B_r(a) \subset A^c$ $[B_r(a) \cap A = \emptyset]$
- a es punto Flotante si $\forall r \begin{cases} B_r(a) \cap A \neq \emptyset \\ B_r(a) \cap A^c \neq \emptyset \end{cases}$

$B_r(a)$ Contiene puntos de A, A^c

Def → $A \subset \mathbb{R}^n$

A es abierto ⟷ a interior $\forall a \in A$

A es cerrado ⟷ A contiene todos los puntos frontera

A es acotado ⟷ $\exists r > 0 / B_r(a) \supset A$

A es Compacto ⟷ A es cerrado y acotado

Propiedades → 1) $||x|| \geq 0$

$||x|| = 0 \leftrightarrow x=0$

2) $||\lambda x|| = |\lambda| \cdot ||x||$

3) $||x+y|| \leq ||x|| + ||y||$

Def → $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$

$y = (y_1, y_2, \dots y_n) \in \mathbb{R}^n$

distancia x,y : $d(x,y) = ||x-y|| = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$

$||x|| = d(x,0)$

Propiedades → 1) $d(x,y) \geq 0$

$d(x,y) = 0 \rightarrow x=y$

2) $d(x,y) = d(y,x)$

3) $d(x,y) \leq d(x,z) + d(z,y)$

Def → $a = (a_1, a_2 \dots a_n) \in \mathbb{R}^n$

Bola abierta de Centro a y radio $r > 0$

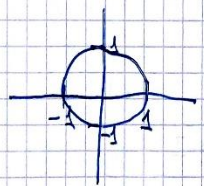
$B_r(a) = \{x \in \mathbb{R}^n / d(a,x) < r\}$

$Ext(A) = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 > 4 \lor x+y > 1$

1- $A = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 < 1\}$

$B = \{(x,y) \in \mathbb{R}^2 / y \leq x^2, y \neq 0, x \in [-2,2]\}$

a) $A = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 < 1$



$F_r(A) = \{x,y \in \mathbb{R}^2 / x^2+y^2 = 1$

$A^\circ = \{x,y \in \mathbb{R}^2 / x^2+y^2 < 1$

$\bar{A} = A \cup F_r(A) = x^2+y^2 \leq 1$

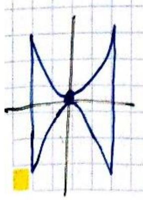
$A = \mathring{A} =$ A es abierto

$A \neq \bar{A} =$ A no es cerrado

A no es compacto

B) $|y| \leq x^2 \to -x^2 \leq y \leq x^2$

$F_r(B) = \{(x,y) \in \mathbb{R}^2 / |y| = x^2, x \in [-2,2]\}$



$\cup \{(x,y) \in \mathbb{R}^2 / x = (2,-2), |y| = x^2\}$

$\cup \{(x,y) \in \mathbb{R}^2 / y = 0, x \in [-2,2]\}$

$\bar{B} = |y| \leq x^2, x \in [-2,2]$ $\quad \mathring{B} = |y| < x^2, y \neq 0, x \in (-2,2)$

$B \neq \mathring{B}$ no abierto

$\bar{B} \neq B \to B$ no cerrado

B no compacto

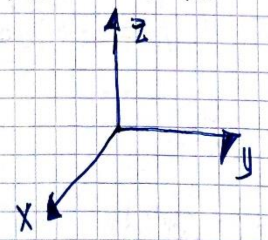
B acotado

Def → $f: \mathbb{R}^n \to \mathbb{R}$

$x \to y = f(x) = f(x_1, x_2 \dots x_n)$

ej: $f: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to f(x,y) = z$



A : dom f

f(A) : Rec f o Im f

(Recorrido) (Imagen)

$(x, f(x))$ gráfica de f

Def → Curva de nivell : Curva donde f = constante

(conecta todos los puntos donde f tenga el mismo valor constante

ej: $f(x,y) = x^2+4y^2$ $\quad z = -1, 0, -2, 0, 1, 2$

(5 curvas de nivel)

$z=0 \to x^2+4y^2 = 0 \to x=0, y=0$

$z=1 \to x^2+4y^2 = 1 \to$ elipse $C=(0,0)$ $a=1$, $b=\frac{1}{2}$

$z=2 \to x^2+4y^2 = 2 \to \frac{x^2}{2} + \frac{y^2}{\frac{1}{2}} = 1 \to$ elipse $C=(0,0)$

$a=\sqrt{2}$

$b=\sqrt{\frac{1}{2}}$

$z=-1 \quad -1 = x^2+4y^2 \quad \nexists$

$z=-2 \quad -2 = x^2+4y^2 \quad \nexists$

Def → $A \subset \mathbb{R}^n$

$a \in A$

$f: A \to \mathbb{R}$ ($a \in$ Dom?)

$\lim f(x) = l \to \forall \varepsilon > 0 \ \exists r > 0 \ \forall x \in B_r(a) \setminus \{a\}$

$|f(x) - l| < \varepsilon$

Def → $f: A \subset \mathbb{R}^N \to \mathbb{R}$

$a \longrightarrow f(a)$

f continua en $a \to \lim\limits_{x \to a} f(x) = f(a)$

$\varphi(a) = \{$ funciones continuas en "a" $\}$

Prop → $f,g : A^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}$, f,g continuas en $a \in A$

- $f \pm g \in \varphi(a)$
- $\lambda f \in \varphi(a) \quad \lambda \in \mathbb{R}$
- $\dfrac{f}{g} \in \varphi(a)$ si $g(a) \neq 0$
- $\left.\begin{array}{l} f \in \varphi(a) \\ f(a) = b \\ h \in \varphi(b) \end{array}\right\} (h \circ f) \in \varphi(a)$
- $\left.\begin{array}{l} \exists \lim f(x) = b \\ h \in \varphi(b) \end{array}\right\} \to$

ej Calcular Dom f (Dibujar y escribir)

$z = f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1} \qquad f: \mathbb{R}^2 \to \mathbb{R}$

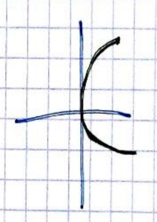
$x+y+1 > 0 \to x+y+1 = 0 \; ; \; y = -x-1$

$x-1 \neq 0 \to x-1 \neq 0 \; ; \; x \neq 1$

$z = x \cdot \ln(y^2 - x)$

$y^2 - x > 0 \to y^2 - x = 0$

$x = y^2$



2-b) $g(x,y) = \sqrt{y \cdot \sin x}$ (Hallar y representar dom)

$y \cdot \sin x \geq 0 \begin{cases} y = 0 \to x \in \mathbb{R} \\ y > 0 \to \sin x \geq 0 \; ; \; [2n\pi, (2n+1)\pi] \\ y < 0 \to \sin x \leq 0 \; ; \; [(2n-1)\pi, 2n\pi] \end{cases}$

$\text{Dom } g = \{(x,y), y=0\} \cup \{(x,y), y>0, x \in [2n\pi, (2n+1)\pi]\}$

$\cup \{y<0, x \in [(2n-1)\pi, 2n\pi]\}$

3- Dibujar curvas nivel para $z = -2, -1, 0, 1, 2$

a) $z(x,y) = x^2 - y^2$

$z = -2 \qquad -2 = x^2 - y^2$

$\dfrac{x^2}{-2} + \dfrac{y^2}{2} = 1$

Hiperbole

$z = -1 \qquad x^2 - y^2 = -1$ Hiperbole vertical

$-x^2 + y^2 = 1$

$z = 0 \qquad x^2 - y^2 = 0$

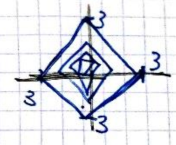
$y^2 = \pm x \to$ 2 rectas

$z = 1 \qquad x^2 - y^2 = 1$ hiperbola horizontal

$z = 2 \qquad x^2 - y^2 = 2$

$\dfrac{x^2}{2} - \dfrac{y^2}{2} = 1$ hiperbole horizontal

3- b) $z(x,y) = 1 - |x| - |y|$



$z = -2 \qquad 1 - |x| - |y| = -2$

a) $x \geq 0, y \geq 0 \qquad -x+3 = y$

b) $x \geq 0, y < 0 \qquad y = x-3$

c) $x < 0, y \geq 0 \qquad y = x+3$

d) $x < 0, y < 0 \qquad y = -x-3$

Tema 8: Derivadas parciales y direccionales

Def → $f: A \subset \mathbb{R}^n \to \mathbb{R}$ $a \in A$ abierto

$\vec{v} \in \mathbb{R}^n$ unitario $\|\vec{v}\| = 1$

$\|\vec{v}\| = \sqrt{v_1^2 + \dots v_n^2}$

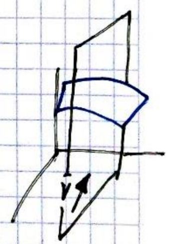
Derivada direccional de f en el punto "a" y direccion $\vec{v}$:

$$D_{\vec{v}} f(a) = \lim_{\lambda \to 0} \frac{f(a+\lambda v) - f(a)}{\lambda}$$

Int geometrica

$z = f(x,y)$ superficie

r recta pasa por a, direccion $\vec{v}$



$\pi: \begin{cases} r \subset \pi \\ \pi \parallel z \end{cases} \quad (\pi \perp xy)$

$\pi \cap (z = f(x,y)) =$ curva

$D_v f(a) =$ pendiente recta tangente de la curva

Derivadas parciales

- base canonica $\{e_1, e_2, \dots e_n\}$

$e_1 = (1,0\dots0)$

$e_2 = (0,1,0\dots0)$

$e_n = (0\dots,0,1)$

en $\mathbb{R}^2$ $e_1(1,0) = \vec{i}$

$e_2(0,1) = \vec{j}$

Cojo $v = e_i$

$D_{e_i} f(a)$: Derivada parcial de f respecto la iesima variable

$$\boxed{D_{e_i} f(a) = D_i f(a) = \frac{\partial f}{\partial x_i}(a) = \partial_{x_i} f(a) = f_{x_i}}$$

en $\mathbb{R}^2$ $x_1 \to x$

$x_2 \to y$

$D_1 f = \frac{\partial f}{\partial x} = f_x$

$D_2 f = \frac{\partial f}{\partial y} = f_y$

$$D_i f(a) = \lim_{\lambda \to 0} \frac{f(a+\lambda e_i) - f(a)}{\lambda}$$

$$D_i f(a) = \lim_{\lambda \to 0} \frac{f[(a_1,a_2\dots a_i,a_n) + \lambda[0,0\dots1\dots0]] - f(a_1,\dots a_n)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{f(a_1,a_2\dots a_{i-1}, a_i+\lambda, a_{i+1}\dots a_n) - f(a_1,\dots a_n)}{\lambda}$$

$D_i f$ supone constantes las $n-1$ componentes $\neq i$ y supone x_i unica variable

ej $f(x,y) = x^2 \cdot y + x \cdot \sin y$

$D_1 f(x,y) = \frac{\partial f}{\partial x}(x,y) = 2xy + \sin y$

$D_2 f(x,y) = \frac{\partial f}{\partial y} = x^2 + x \cdot \cos y$

ej: $f(x,y) = y \cdot \sin(x^2 y)$

$$F_x(x,y) = y \cdot \cos(x^2 \cdot y) \cdot 2xy = 2xy^2 \cdot \cos(x^2 \cdot y)$$

$$F_y(x,y) = 1 \cdot \sin(x^2 y) + y \cdot \cos(x^2 y) \cdot x^2$$

Interpretación geométrica

$\frac{\partial f}{\partial x}$: Pendiente de f en la dirección "x"

$\frac{\partial f}{\partial y}$: Pendiente de f en la dirección "y"

1- $f(x,y) = (\sin x)^{\sin y}$ Derivadas parciales

$$F_x = \sin y \cdot (\sin x)^{(\sin y - 1)}$$

$$F_y = (\sin x)^{\sin y} \cdot \ln(\sin x) \cdot \cos y$$

2- $f(x,y) = x^2 + y^2$

$D_v f(P)$ con $\vec{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$, $P = (2,3)$

a) $D_v f(P) = \lim_{\lambda \to 0} \frac{f(P+\lambda v) - f(P)}{\lambda} = \lim_{\lambda \to 0} \frac{f\left((2,3) + \lambda\left(\frac{3}{5}, \frac{4}{5}\right)\right) - f(2,3)}{\lambda}$

$$\lim_{\lambda \to 0} \frac{\left(2 + 3 \cdot \frac{\lambda}{5}\right)^2 + \left(3 + \frac{4\lambda}{5}\right)^2 - (2^2 + 3^2)}{\lambda}$$

$$\lim_{\lambda \to 0} \frac{\frac{12\lambda}{5} + \frac{9\lambda^2}{25} + \frac{24\lambda}{5} + \frac{16\lambda^2}{25}}{\lambda}$$

$$\lim_{\lambda \to 0} \frac{\frac{36\lambda}{5} + \lambda^2}{\lambda} = \lim_{\lambda \to 0} \frac{\lambda \cdot \left(\frac{36}{5} + \lambda\right)}{\lambda} = \boxed{\frac{36}{5}}$$

ej: Hallar las pendientes en las direcciones x, y

$$f(x,y) = \frac{-x^2}{2} - y^2 + \frac{25}{8} \quad \text{en } \left(\frac{1}{2}, 1, 2\right)$$

$$f\left(\frac{1}{2}, 1\right) = \frac{-\left(\frac{1}{2}\right)^2}{2} - 1^2 + \frac{25}{8} = 2$$

$$F_x\left(\frac{1}{2}, 1\right) = -x = -\frac{1}{2}$$

$$F_y\left(\frac{1}{2}, 1\right) = -2y = -2$$

Vector gradiente

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad a \in A$ (abierto)

f tiene todas las derivadas parciales en "a"

$$\text{grad } f(a) = \nabla f(a) = \left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a)\right)$$

ej: $f \to \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y,z) = x^2 \cdot y \cdot \cos(xz) + x \cdot e^{yz} - 7z^3 \quad Df(1,2,0)$$

$$F_x = y \cdot (2x \cdot \cos(xz) - x^2 \cdot \sin(xz) \cdot z) + e^{yz}$$

$$F_y = x^2 \cdot \cos(xz) + x \cdot e^{yz} \cdot z$$

$$F_z = -x^2 \cdot y \cdot \sin(xz) \cdot x + x \cdot e^{yz} \cdot y - 21z^2$$

$$\boxed{\nabla f(1,2,0) = (5,1,2)}$$

Generalizando

$f: \mathbb{R}^n \to \mathbb{R}^m$

ej $f: \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \to (2x+y, x^2 y, x-y)$

$(1,2) \to (4,2,-1)$

Matriz Jacobiana

$$DJ(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \Big/ (a_1, a_2 .. a_n)$$

Plano tangente y recta normal a una superficie en un punto

Def → $F: A \subset \mathbb{R}^N \to \mathbb{R}$ A abierto $a \in A$

F es C^0 en a si F continua en a

F es C^1 en a si ∃ todas derivadas parciales y son continuas

F es C^n en a si ∃ todas las derivadas parciales de orden n y son continuas

Superficie → $z = F(x,y)$ → forma explícita

→ $F(x,y,z) = 0$ → forma implícita

Forma explícita

Plano tangente en $(a, b, f(a,b))$

$$\boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)}$$

Recta normal en $(a,b,f(a,b))$

$$\frac{x-a}{\frac{\partial f}{\partial x}(a,b)} = \frac{y-b}{\frac{\partial f}{\partial y}(a,b)} = \frac{z-f(a,b)}{-1}$$

Forma implícita

Plano tangente en (a,b,c)

$$\boxed{\frac{\partial F}{\partial x}(a,b,c)\cdot(x-a) + \frac{\partial F}{\partial y}(a,b,c)\cdot(y-b) + \frac{\partial F}{\partial z}(a,b,c)\cdot(z-c) = 0}$$

Recta normal en (a,b,c)

$$\boxed{\frac{x-a}{\frac{\partial F}{\partial x}(a,b,c)} = \frac{y-b}{\frac{\partial F}{\partial y}(a,b,c)} = \frac{z-c}{\frac{\partial F}{\partial z}(a,b,c)}}$$

5- a) $z = x^2 + y^2$ en $(1,2,5)$

$5 \in Sup \to 5 = F(1,2)$

$\nabla F(1,2) = (2x, 2y) = (2,4)$

$z = 5 + (2,4)\cdot(x-1, y-2)$

$z = 5 + 2\cdot(x-1) + 4\cdot(y-2)$

$$\boxed{2x + 4y - z = 5}$$

$$\boxed{\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-5}{-1}}$$

b) $z = \text{arctg}\,\frac{y}{x}$ en $\left(1, 1, \frac{\pi}{4}\right)$

$F(1,1) \overset{\checkmark}{=} \frac{\pi}{4}$

$\nabla F(x,y) = \left(\frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot -y\cdot x^{-2}, \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}\right)$

$\nabla F(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

$z = \frac{\pi}{4} + \left(-\frac{1}{2}, \frac{1}{2}\right)\cdot(x-1, y-1)$

$$\boxed{\frac{1}{2}x - \frac{1}{2}y + z = \frac{\pi}{4}}$$

$$\frac{x-1}{-\frac{1}{2}} = \frac{y-1}{\frac{1}{2}} = \frac{z-\frac{\pi}{4}}{-1}$$

$$\boxed{-2x + 2 = 2y - 2 = -z + \frac{\pi}{4}}$$

Corolarios

1- $\nabla f(a)=0 \rightarrow D_v f(a)=0$

2- $D_v f(a,b) = \nabla f(a)\cdot v = \|\nabla f(a)\|\cdot\|v\|\cdot\cos\alpha$

$= \|\nabla f(a)\|\cdot\cos\alpha$

$\rightarrow \alpha=\frac{\pi}{2} \rightarrow D_v f(a)=0$ $(v \perp \nabla f(a))$

$\rightarrow \alpha=0 \rightarrow D_v f(a)$ es máxima (el valor máximo de $D_v f(a)$ está en la dirección y sentido de $\nabla f(a)$) y vale $\|\nabla f(a)\|$

$\rightarrow \alpha=\pi \rightarrow D_v f(a)$ es mínima (el valor de $D_v f(a)$ está en dirección y sentido de $-\nabla f(a)$ y vale $-\|\nabla f(a)\|$

3- $\nabla f(a)$ es ortogonal a las curvas de nivel en $(a, f(a))$

ej $f(x,y)=x^2+y^2$

$D_v f(2,3)$ $\vec{v}=\left(\frac{3}{5},\frac{4}{5}\right)$ $\|v\|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=1$

$D_v f(a,b)=\nabla f(a,b)\cdot\vec{v}$

$\nabla f(2,3)=(2x,2y)=(4,6)$

$D_v f(2,3)=\frac{36}{5}=(4,6)\cdot\left(\frac{3}{5},\frac{4}{5}\right)$

Derivada de $z=x^2-y^2$ en $(1,1)$ con ángulo $\frac{\pi}{3}$ con dirección positiva

$\vec{v}=(\|v\|\cdot\cos\alpha, \|v\|\cdot\sin\alpha)$
$=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

$D_v f(1,1)=(2,-2)\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$=\boxed{1-\sqrt{3}}$

$\nabla f(1,1)=(2x,-2y)=(2,-2)$

$\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$ en $(-2,1,-3)$

$f(-2,1)\overset{\checkmark}{=}-3$

$\nabla f(-2,1)=\left(\frac{2x}{4},2y,\frac{2z}{9}\right)=\left(-1,2,\frac{-2}{3}\right)$

$\left(-1,2,-\frac{2}{3}\right)\cdot(x+2,y-1,z+3)=0$

$$\boxed{-x-2+2y-2-\frac{2}{3}z-2=0}$$

$$\boxed{\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-\frac{2}{3}}}$$

Propiedades del gradiente

Proposición $\rightarrow$ $f:\mathbb{R}^n\rightarrow\mathbb{R}$ $a\in Int(Dom f)$

$f\in C^1(a)$

$v\in\mathbb{R}^n$ unitario $(\|v\|=1)$

$$\boxed{D_v f(a)=\nabla f(a)\cdot v}$$

4- a, b, c $F(x,y,z) = axy^2 + byz + cz^2x^3$
en $(1,2,-1)$ tenga valor maximo de 64 en una direccion paralela al eje OZ

$D_v F(1,2,-1) \rightarrow$ valor maximo 64

$\nabla F = (\ , \ , \ ) \rightarrow \| \ . \ \| = 64$
$\rightarrow$ Paralelo eje Z $(0,0,1)$

$\nabla F = (ay^2 + 3cz^2x^2, 2axy + bz, by + 2cx^3z)$ $\quad \nabla F(0,0,\lambda)$

$\nabla F(1,2,-1) = (4a + 3c, 4a - b, 2b - 2c)$ $\quad \sqrt{0^2 + 0^2 + \lambda^2} = 64$

$4a + 3c = 0 \rightarrow c = -\frac{4}{3}a$
$4a - b = 0 \rightarrow b = 4a$
$2b - 2c = \pm 64 \rightarrow 8a + \frac{8}{3}a = \pm 64$

$\lambda = \pm 64$

$\nabla F = (0, 0, \pm 64)$

$$\boxed{\begin{aligned} a &= \pm 6 \\ b &= \pm 8 \\ c &= \pm 24 \end{aligned}}$$

6- $F(x,y) = 4x + 2y - x^2 + xy - y^2$
Calcula puntos de la superficie $z = f(x,y)$ tales que $\underbrace{\text{su plano tangente sea paralelo al plano xy}}_{z = K}$

$\frac{\partial f}{\partial x}(a,b) = (4 - 2x + y) = 0$
$\frac{\partial f}{\partial y}(a,b) = (2 + x - 2y) = 0$
$\rightarrow \boxed{x = \frac{10}{3}, \quad y = \frac{8}{3}}$

$$\boxed{\text{Pto}\left(\frac{10}{3}, \frac{8}{3}, \frac{28}{3}\right)}$$

$F\left(\frac{10}{3}, \frac{8}{3}\right) = \frac{28}{3}$

Tema 9 Formula de Taylor y los extremos relativos en diversas variables

Formula de Taylor

Def $\rightarrow F: \mathbb{R}^n \rightarrow \mathbb{R}$ $n \geq 2$ $i \in \{1, \dots n\}$

La función i-ésima derivada parcial de F
$D_i \quad \mathbb{R}^n \rightarrow \mathbb{R}$ funcion
$(a_1, a_n) \longrightarrow D_i F(a_1, a_n)$

Esta funcion puede admitir derivadas parciales

$D_{ij} \ F(a_1, \dots a_n) = D_j [D_i F(a_1, \dots a_n)] \ i \leq i, j \leq n$

$$\begin{cases} \dfrac{\partial^2 F(a_1, a_n)}{\partial x_j x_i} = \dfrac{\partial}{\partial x_j} \left[\dfrac{\partial F}{\partial x_i}(a_1, a_n)\right] & i \neq j \\[2ex] \dfrac{\partial^2 F}{\partial x_i^2}(a_1 \dots a_n) = \dfrac{\partial}{\partial x_i}\left[\dfrac{\partial F}{\partial x_i}(a_1 \dots a_n)\right] & i = j \end{cases}$$

ej: $F: \mathbb{R}^3 \rightarrow \mathbb{R}$
$(x,y,z) \rightarrow F(x,y,z) = xe^{yz} - 7z^3$

$\frac{\partial f}{\partial x} = e^{yz}$ $\quad \frac{\partial f}{\partial y} = x \cdot e^{yz} \cdot z$ $\quad \frac{\partial f}{\partial z} = x \cdot e^{yz} \cdot y - 21z^2$

$$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = z \cdot e^{yz}$$

$$\frac{\partial^2 f}{\partial y \partial x} = z \cdot e^{yz} \qquad \frac{\partial^2 f}{\partial y^2} = xz^2 \cdot e^{yz}$$

$$\frac{\partial^2 f}{\partial z \partial x} = y \cdot e^{yz} \qquad \frac{\partial^2 f}{\partial z \partial y} = x \cdot (e^{yz} + z \cdot y \cdot e^{yz}$$

$$\frac{\partial^2 f}{\partial x \partial z} = y \cdot e^{yz}$$

$$\frac{\partial^2 f}{\partial y \partial z} = x \cdot (e^{yz} + e^{yz} \cdot y \cdot z)$$

$$\frac{\partial^2 f}{\partial^2 z} = xy^2 \cdot e^{yz} - 4yz$$

Def → $F: \mathbb{R}^n \to \mathbb{R}$ $n \geq 1$

$A \subset Int(Dom\, f)$

$f \in C^k$ si f tiene derivadas parciales de orden k y son continuas

$f \in C^\infty$ si f tiene todas derivadas parciales de cualquir orden y son continuas

Teorema de Schwarz

$f: \mathbb{R}^n \to \mathbb{R}$ $n \geq 2$ $A \subseteq Int(Dom\, f)$

$$f \in C^2(A) \to \frac{\partial^2 f}{\partial x_i \partial x_j}(a_1, a_n) = \frac{\partial^2 f}{\partial x_j x_i}(a_1, a_n)$$

$$D_{ij} = D_{ji}$$

Matriz Hessiana

$$Hf(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

$$= \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_n}\right)$$

$$P_2 f(x_0,y_0)(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y-y_0) + \frac{1}{2!} \cdot \left[\frac{\partial^2 f}{\partial x^2}(x_0,y_0) \cdot (x-x_0)^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \cdot (x-x_0) \cdot (y-y_0) + \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \cdot (y-y_0)^2\right]$$

$$R_2(x,y) = \frac{1}{3!} \cdot \left[\frac{\partial^3 f}{\partial x^3}(c_1,c_2) \cdot (x-x_0) + \cdots \frac{\partial^3 f}{\partial y^3}(c_1,c_2) \cdot (y-y_0)\right.$$

$$R_2(x,y) = \frac{1}{3!} \cdot \sum_{i=0}^{3} \frac{\partial^3 f\,(c,d)}{\partial x^{3-i} \partial y^i} (x-a)^{3-i} \cdot (y-b)^i$$

$f(x,y) = \ln(1+2x+3y)$ $\quad f(0,0) = 0$

$f(x) = \frac{2}{1+2x+3y}$ $\quad f_x(0,0) = 2$

$f_y = \frac{3}{1+2x+3y}$ $\quad f_y(0,0) = 3$

$f_{xx} = -4(1+2x+3y)^{-2}$

$f_{xy} = -6(1+2x+3y)^{-2}$

$f_{yy} = -9(1+2x+3y)^{-2}$

$f_{xx}(0,0) = -4$

$f_{xy}(0,0) = -6$

$f_{yy}(0,0) = -9$

$$P_2(x,y) = 0 + 2x + 3y + \frac{1}{2}\left(-4x^2 + 2\cdot(-6)\cdot x\cdot y + (-9)\cdot y^2\right)$$

$$= \boxed{2x + 3y - 2x^2 - 6xy - \frac{9}{2}y^2}$$

$$f(0{,}1, 0{,}1) \approx P_2(0{,}1, 0{,}1) = \boxed{0{,}375}$$

$f = \sqrt[3]{xy} = (xy)^{\frac{1}{3}}$ $\quad f(1,1) = 1$

$f_x = \frac{1}{3}(xy)^{-2/3}\, y$ $\quad f_x(1,1) = \frac{1}{3}$

$f_y = \frac{1}{3}(xy)^{-2/3}\, x$ $\quad f_y(1,1) = \frac{1}{3}$

$$\boxed{z = 1 + \frac{1}{3}(x-1) + \frac{1}{3}(y-1)}$$

$$z(0{,}99, 1{,}01) = \boxed{1}$$

$$f_{xx} = \frac{1}{3}y\cdot\left(-\frac{2}{3}\right)\cdot(xy)^{-5/3}\cdot y = \frac{-2}{9}y^2\cdot(xy)^{-5/3}$$

$$f_{xy} = \frac{1}{3}x^{-2/3}\cdot\frac{1}{3}y^{-2/3} = \frac{1}{9}\cdot x^{-2/3}\cdot y^{-2/3}$$

$$f_{yy} = \frac{1}{3}x\cdot\left(\frac{-2}{3}\right)\cdot(xy)^{-5/3}\cdot x = \frac{-2}{9}x^2\cdot(xy)^{-5/3}$$

$$R_1 = \frac{1}{2}\cdot\left[\frac{-2}{9}d^2(cd)^{-5/3}(0{,}01)^2 + 2\cdot\frac{1}{9}c^{-2/3}d^{-2/3}\cdot(0{,}01)^2\right.$$

$$= \boxed{0{,}5\cdot 10^{-4}}$$

9.2 Puntos críticos. Cálculo extremos:

Def $\rightarrow$ $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $a = (a_1, a_2, \dots a_n) \in \text{Int}(\text{Dom} f)$

f tiene un máximo relativo en a si

$\exists r > 0 \quad \forall x = (x_1 \dots x_n) \in B_r(a)$

$f(a) \geq f(x)$

f tiene un mínimo relativo en "a" si

$\exists r > 0 \quad \forall x = (x_1, x_2, x_n) \in B_r(a)$

$f(a) \leq f(x)$

$f \in C'$ en un entorno de "a"

"a es un punto crítico de f si

$\nabla f(a) = 0$;

Proposición: (Condición necesaria de extremo relativo)

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ $\quad a = (a_1 \dots a_n) \in \text{Int}(\text{Dom} f)$

$f \in C'$

f tiene un extremo en "a" $\rightarrow$ a es un punto crítico

$\nabla f(a) = 0$

Def $\rightarrow$ $f: \mathbb{R}^n \rightarrow \mathbb{R}$ $\quad a \in \text{Int}(\text{Dom} f)$

f tiene un punto de silla (Punt de Sella)

a no es extremo pero es punto crítico.

$f'(x) = 0$ $\rightarrow a_1$, $\rightarrow a_2$

$f''(a_1) > 0$ mínim

$f''(a_2) < 0$ màxim

Proposición: (Condición suficiente de extremo relativo)

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$ $\quad (a,b) \in \text{Int}(\text{Dom} f)$ $\quad \nabla f(a,b) = 0$

$f \in C^2$

Matriz Hessiana
$$Hf(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial^2 y}(a,b) \end{pmatrix}$$

$\text{Det } Hf(a,b) = \Delta_2$

$\Delta_2 < 0 \rightarrow$ f tiene un punto de silla en (a,b)

$\Delta_2 > 0 \rightarrow \begin{cases} \dfrac{\partial^2 f}{\partial x^2}(a,b) > 0 \rightarrow \text{mínimo} \\ \dfrac{\partial^2 f}{\partial x^2}(a,b) < 0 \rightarrow \text{máximo} \end{cases}$

$\Delta_2 = 0 \rightarrow$ Estudio local

3-a) $f(x,y) = x^3 + y^3 - 9xy + 27$

$$\begin{aligned} f_x &= 3x^2 - 9y = 0 \\ f_y &= 3y^2 - 9x = 0 \end{aligned}$$

$3x^2 - 3y^2 + 9x \cancel{\mp} 9y = 0$

$3 \cdot (x^2 - y^2) - 9 \cdot (y - x) = 0$

$3 \cdot (x+y)(x-y) + 9 \cdot (x-y) = 0$

$3 \cdot (x-y) \cdot ((x+y)+3) = 0 \rightarrow x - y = 0$
$x = y$

$3x^2 - 9x = 0 \rightarrow \boxed{\begin{matrix} x = 0 \\ x = 3 \end{matrix}}$ $(0,0)$ $(3,3)$

$x \cdot (3x - 9) = 0$

$x + y + 3 = 0 \quad y = -x - 3$

$3x^2 - 9 \cdot (-x-3) = 0$

$\sqrt{\ } \; \nexists$

$f_{xx} = 6x$

$f_{xy} = -9$

$f_{yy} = 6y$

$Hf = \begin{pmatrix} 6x & -9 \\ -9 & 6y \end{pmatrix}$

$Hf(0,0) = \begin{vmatrix} 0 & -9 \\ -9 & 0 \end{vmatrix} = -81$

Punto de Silla

$Hf(3,3) = \begin{vmatrix} 18 & -9 \\ -9 & 18 \end{vmatrix} > 0$

$A_1 > 0 \rightarrow$ mínimo en 3,3

b) $f(x,y) = (x^2 - 2x + 4y^2 - 8y)^2$

$f_x = 2 \cdot (x^2 - 2x + 4y^2 - 8y) \cdot (2x - 2)$

$f_y = 2 \cdot (x^2 - 2x + 4y^2 - 8y) \cdot (8y - 8)$

$(x^2 - 2x + 4y^2 - 8y) = 0$

$\left.\begin{matrix} 2x - 2 = 0 \\ 8y - 8 = 0 \end{matrix}\right\} \rightarrow \begin{matrix} x = 1 \\ y = 1 \end{matrix}$ $(1,1)$

$f_{xx} = 2 \cdot (2x-2)^2 + 2(x^2 - 2x + 4y^2 - 8y) \cdot 2$

$f_{xy} = 2 \cdot (2x-2) \cdot (8y-8)$

$f_{yy} = 2 \cdot (8y-8)^2 + 2 \cdot (x^2 - 2x + 4y^2 - 8y) \cdot 8$

$Hf(1,1) = \begin{pmatrix} -20 & 0 \\ 0 & -80 \end{pmatrix} = 1600 > 0$

$A_1 =$ negativo

máximo

$Hf \quad x^2 - 2x + 4y^2 - 8y = 0 = \begin{pmatrix} 2 \cdot (2x-2)^2 & 2 \cdot (2x-2) \cdot (8y-8) \\ 2 \cdot (2x-2) \cdot (8y-8) & 2 \cdot (8y-8)^2 \end{pmatrix}$

$f(\text{entorno}) \geq 0$

mínimo en $x^2 + 2x + 4y^2 - 8y = 0$ $\quad A_2 = 0$

4- $F(x,y) = ax^3 + 3bxy^2 - 15a^2x - 12y + 5$
tenga mínimo en $(2,1)$

$\nabla f(2,1) = 0$

$F_x = 3ax^2 + 3by^2 - 15a^2$

$F_y = 6bxy - 12$

$F_x(2,1) = 12a + 3b - 15a^2 = 0$

$F_y(2,1) = 12b - 12 = 0 \rightarrow \boxed{b = 1}$

$a = 1$
$a = -\frac{1}{5}$

$A_2 = \det Hf(2,1) = \begin{vmatrix} 12a & 6b \\ 6b & 12b \end{vmatrix} = 144ab - 36b$

$F_{xx} = 6ax$

$F_{xy} = 6by$

$F_{yy} = 6bx$

$\boxed{\begin{matrix} a=1 \\ b=1 \end{matrix}} \rightarrow A_2 = 144 - 36 > 0$

$A_1 = 12 \cdot 1 > 0$

$a = -\frac{1}{5} \quad b = 1$

$A_2 = -\frac{144}{5} - 36 < 0 \rightarrow$ Punto de Silla

Tema 10 Extremos Condicionados

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$
$(x,y) \rightarrow f(x,y) = 4 - x^2 - y^2$

Condición $\rightarrow y = 2 - x \rightarrow y = 2 - 1 = 1$

$F(x, 2-x) = 4 - x^2 - 4 + 4x - x^2 = -2x^2 + 4x$

$F'(x, 2-x) = -4x + 4 = 0 \quad x = 1$

$F''(x, 2-x) = -4 < 0 \rightarrow \boxed{\text{máximo}}$

$L(x, y, \lambda) = f(x,y) + \lambda \cdot g(x,y)$

$\exists \lambda_0 \in \mathbb{R} \quad \nabla L(a, b, \lambda_0) = (0,0,0)$

$$\boxed{\nabla f(a,b) = -\lambda_0 \nabla g(a,b)}$$

Ej: $F(x,y) = 2x + 2y$
$x \cdot y = S$

$L(x, y, \lambda) = 2x + 2y + \lambda \cdot (x \cdot y - S)$

$2 + \lambda y \rightarrow y = \frac{-2}{\lambda}$

$2 + \lambda x \rightarrow x = \frac{-2}{\lambda}$

$xy - S \rightarrow \frac{+4}{\lambda^2} - S = 0$

$\lambda^2 S - 4 = 0$

$\lambda^2 = \sqrt{\frac{4}{S}}$

$\lambda = \frac{2}{\sqrt{S}}$

$= -\frac{2}{\sqrt{S}} \rightarrow \begin{matrix} x = \sqrt{S} \\ y = \sqrt{S} \end{matrix}$

$\begin{vmatrix} 0 & y & x \\ y & 0 & \lambda \\ x & \lambda & 0 \end{vmatrix} = -4\sqrt{S} < 0$

$\boxed{\text{mínimo}}$

$$f(x,y) = x^2 + y^2$$

$$y + x^2 = 1$$

$$y = 1 - x^2$$

$$f(x,y) = x^2 + 1 - 2x^2 + x^4$$

$$= x^4 - x^2 + 1$$

$$f'(x) = 4x^3 - 2x = 0$$

$$2x \cdot (2x^2 - 1) = 0 \rightarrow x = 0$$

$$2x^2 - 1 = 0$$

$$x = \pm\sqrt{\frac{1}{2}} \rightarrow x = \frac{\sqrt{2}}{2}$$

$$x = -\frac{\sqrt{2}}{2}$$

$$f''(x) = 12x^2 - 2$$

$f''(0) = -2 < 0$ maximo

$f''\left(\pm\frac{\sqrt{2}}{2}\right) = 6 - 2 > 0$ minimo

$$\boxed{\begin{matrix} x = 0 & & x = \pm\frac{\sqrt{2}}{2} \\ y = 1 & & y = \frac{1}{2} \end{matrix}}$$

Teorema de Weierstrass

$F: \mathbb{R}^n \rightarrow \mathbb{R}$ funcion

$\vec{x} \rightarrow f(\vec{x})$

$D \subseteq \text{Dom } f$, D compacto

f continua en D

$$\boxed{f(\vec{x}_m) \leq f(\vec{x}) \leq f(\vec{x}_M)}$$

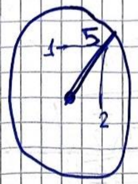
3- $f(x,y) = x^2 + y^2 - 12x - 8y + 50$

$$\text{Dom} = x^2 + y^2 - 4x - 2y \leq 20$$

$$x^2 - 4x + y^2 - 2y \leq 20$$

$$(x-2)^2 - 4 + (y-1)^2 - 1 \leq 20$$

$$(x-2)^2 + (y-1)^2 \leq 25$$



Compacto, ya que es cerrado y acotado

$$\frac{\partial f}{\partial x} = 2x - 12 = 0 \quad x = 6$$

$$\frac{\partial f}{\partial y} = 2y - 8 \quad y = 4$$

$$(6-2)^2 + (4-1)^2 < 25$$

$$25 < 25$$

$$(6,4) \notin$$

$$L(x,y,\lambda) = x^2 + y^2 - 12x - 8y + 50 + \lambda \cdot ((x-2)^2 + (y-1)^2 - 25)$$

$$\frac{\partial f}{\partial x} = 2x - 12 + 2\lambda \cdot (x-2) = 0 \quad x = \frac{12 + 4\lambda}{2 + 2\lambda}$$

$$\frac{\partial L}{\partial y} = 2y - 8 + 2\lambda \cdot (y-1) \quad y = \frac{8 + 2\lambda}{2 + 2\lambda}$$

$$\frac{\partial L}{\partial \lambda} = (x-2)^2 + (y-1)^2 - 25$$

$$\frac{36 + 24\lambda + 4\lambda^2}{(\lambda+1)^2} - 4 \cdot \frac{(6 + 2\lambda)}{\lambda + 1} + 4 + \frac{16 + 8\lambda + \lambda^2}{(1+\lambda)}$$

$f(-1,-1) = \boxed{-1} \rightarrow$ menor

$f(0,0) = 0$

$f(-3,0) = \boxed{6} \rightarrow$ mayor

$f(0,-3) = \boxed{6}$

$f\left(-\frac{1}{2},0\right) = -\frac{1}{4}$

$f\left(0,-\frac{1}{2}\right) = -\frac{1}{4}$

$f=\left(\frac{-3}{2},\frac{-3}{2}\right) = \frac{-3}{4}$

vertices $(0,0),(-3,0),(0,-3)$

$y=0$ con $-3 \leq x \leq 0$

$f(x,0) = x^2+x \rightarrow f'(x,0) = 2x+1 = 0$

$\boxed{y=0}$ $\boxed{x=\frac{-1}{2}}$

$x=0$ con $-3 \leq y \leq 0$

$f(0,y) = y^2+y \rightarrow \boxed{x=0}$ $\boxed{y=\frac{-1}{2}}$

$y=-3-x$ con $-3 \leq x \leq 0$

$f(x,-3-x) = x^2+(-3-x)^2+3x+x^2+x-3-x$

$= 3x^2+9x+6$

$= x^2+3x+3 = 0$

$2x+3=0$

$\boxed{x=\frac{-3}{2}}$ $\boxed{y=\frac{-3}{2}}$

$= -25\lambda^2 - 50\lambda = 0$

$-25\lambda\cdot(\lambda+2) = 0$

$\rightarrow \lambda = 0 \rightarrow x=6,\ y=4$

$\rightarrow \lambda = 2 \rightarrow x = \cancel{10}\ -2,\ y=-2$

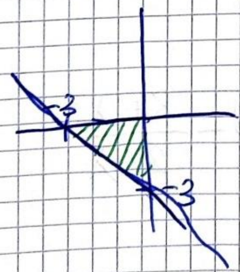
$\boxed{(6,4),(-2,-2)}$

$f(6,4) = \boxed{-2}$

$f(-2,-2) = \boxed{98}$

4- $f(x,y) = x^2+y^2-xy+x+y$

Dom $x \leq 0,\ y \leq 0,\ x+y \geq -3$



$\frac{\partial f}{\partial x} = 2x - y + 1 = 0$

$\frac{\partial f}{\partial y} = 2y - x + 1 = 0$

$y = 2x+1$

$\boxed{x=-1 \quad y=-1}$