Tema 1- Numeros Reales

A se llama acotado superiormente
$\rightarrow \exists K \in \mathbb{R} : K \geq a, \forall a \in A$

A se llama acotado inferiormente
$\rightarrow \exists K \in \mathbb{R} : K \leq a, \forall a \in A$

A se llama acotado $\rightarrow$ A es acotado superiormente e inferiormente

K y K se llaman cota superior e inferior

S es supremo de A ($S = \sup A$) $\rightarrow$ S es una cota superior de A
$\rightarrow$ $(S-\varepsilon)$ no es una cota superior $\forall \varepsilon > 0$

$S = \sup A$ es la menor de las cotas superiores.

I es ínfimo de A ($I = \inf A$) $\rightarrow$ I és una cota inferior de A
$\rightarrow$ $(I+\varepsilon)$ no es una cota inferior $\forall \varepsilon > 0$

$I = \inf A$ es la mayor de las cotas inferiores.

Si $S \in A \rightarrow$ S se llama máximo de A ($S = \max A$)

Si $I \in A \rightarrow$ I se llama mínimo de A ($I = \min A$)

$$|x| = \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$$

Distancia euclediana $\rightarrow d(x,y) = |x-y|$

Propiedades

1- $|x| \geq 0$, $|x| = 0 \leftrightarrow x = 0$

2- $|x \cdot y| = |x| \cdot |y| \rightarrow |x^n| = |x|^n \quad \forall n \in \mathbb{N}$

3- $|x+y| \leq |x| + |y|$

4- $|x| \leq a \rightarrow -a \leq x \leq a$

4.1- $|x| < a \rightarrow -a < x < a$

4.2- $|x| = a \rightarrow x = \pm a$

4.3- $|x| \geq a \rightarrow x \geq a$ y $x \leq -a$

4.4- $|x| > a \rightarrow x > a$ y $x < -a$

$x \leq y \rightarrow a + x \leq a + y$

$x \leq y \rightarrow a \cdot x \leq a \cdot y$; $x \leq y \rightarrow a \cdot x \geq a \cdot y$

$a \geq 0$ $\qquad$ $a \leq 0$

1-a) $\frac{x-1}{x+1} < 0$

$x-1>0 \leftrightarrow x>1$

$x+1<0 \leftrightarrow x<-1$

$x-1<0 \leftrightarrow x<1$

$x+1>0 \leftrightarrow x>-1$

$\rightarrow -1<x<1$

$x \in (-1,1)$

Fitat superiorment

Suprem $= 1$

~~maxim~~

Fitat inferiorment

Infimo $= -1$

~~infimo~~

$x^2+x \leq 0$

$x \cdot (x+1) \leq 0$ $\quad x \geq -1$

$\frac{1}{x+3} > \frac{1}{4}$

$x > -3 \rightarrow x+3 > 0 \rightarrow \overset{>0}{\frac{1}{x+3}} > \overset{>0}{\frac{1}{4}}$

$4 > x+3 \rightarrow 4-3 > x \rightarrow \boxed{x<1}$

$-3 < x < 1$

$\frac{x-1}{x+1} \leq \frac{x+1}{x-1}$ $\quad x \neq 1$, $x \neq -1$

Si $x<-1 \rightarrow x+1<0$, $x-1<0$

Si $-1<x<1$: $x+1>0$, $x-1<0$

$(x-1)^2 \leq (x+1)^2$

$\frac{x-1}{x+1} \geq \frac{x+1}{x-1}$

$x^2-2x+1 \leq x^2+2x+1$

$0 \leq 4x$

$x \geq 0$

$4x \leq 0$

$x \leq 0$

- Si $x>1 \rightarrow x-1>0$, $x+1>0$ $\quad x \geq 0$

1 Resoleu les desigualtats següents:

a) $\frac{x-1}{x+1} < 0$; b) $\frac{1}{x+3} > \frac{1}{4}$; c) $\frac{x-1}{x+1} \leq \frac{x+1}{x-1}$; d) $x^2+x \leq 0$; e) $1 < x^2 < 4$.

En cada apartat representeu sobre la recta real el conjunt de solucions i digueu si tal conjunt és fitat superiorment (inferiorment). En cas afirmatiu, trobeu-ne el suprem (ínfim).

1 a) $A = (-1,1)$; és un conjunt fitat, $\inf(A) = -1$ i $\sup(A) = 1$.
b) $B = (-3,1)$; és un conjunt fitat, $\sup(B) = 1$ i $\inf(B) = -3$.
c) $C = (-1,0] \cup (1,\infty)$; és un conjunt fitat inferiorment amb $\inf(C) = -1$.
d) $D = [-1,0]$; és un conjunt fitat, $\sup(D) = 0$ i $\inf(D) = -1$.
e) $E = (-2,-1) \cup (1,2)$; fitat superiorment, $\sup(B) = 2$ i fitat inferiorment, $\inf(B) = -2$.

2 Trobeu tots els nombres reals x que satisfan les desigualtats següents:

a) $|2x+7| \geq 3$; b) $|x^2-1| \leq 3$; c) $|x-1| > |x+1|$; d) $|x|+|x+1| < 2$.

En cada apartat representeu sobre la recta real el conjunt de solucions i digueu si tal conjunt és fitat superiorment (inferiorment). En cas afirmatiu, trobeu-ne el suprem (ínfim).

2 a) $A = (-\infty,-5] \cup [-2,+\infty)$; no és un conjunt fitat i no hi ha màxim ni mínim.
b) $B = [-2,2]$; fitat $\sup(B) = 2$ i $\inf(B) = -2$.
c) $C = (-\infty,0)$; fitat superiorment amb $\sup(C) = 0$.
d) $D = (-\frac{3}{2},\frac{1}{2})$; fitat $\sup(D) = 1/2$ i $\inf(D) = -3/2$.

2a) $|2x+7| \geq 3$

$2x+7 \geq 3$

$2x+7 \leq -3$

$2x \geq 3-7$ $\quad$ $2x \leq -3-7$

$x \geq \frac{-4}{2}$ $\quad$ $x \leq \frac{-10}{2}$

$\boxed{x \geq -2}$ $\quad$ $\boxed{x \leq -5}$

$(-\infty, -5] \cup [-2, +\infty)$

2b) $|x^2-1| \leq 3$

$x^2-1 \leq 3 \rightarrow x^2 \leq 4$

$x^2-1 \geq -3$ $\quad$ $x^2 \geq -2$

siempre cierto

$x^2-4 \leq 0$ $\quad$ $x^2 \leq 4$

$-2 < x$ $\quad$ $x \leq \sqrt{4}$

2d) $|x| + |x+1| \leq 2$

$$|x| = \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$$

$$|x+1| = \begin{cases} x+1 & \text{si } x+1 \geq 0 \rightarrow x \geq -1 \\ -x-1 & \text{si } x+1 < 0 \rightarrow x < -1 \end{cases}$$

si $x < -1 \rightarrow -x - x - 1 < 2 \rightarrow -2x < 3$

$-1 \leq x < 0$ $\quad$ $-x + x + 1 < 2 \rightarrow 1 < 2$

$x > 0 \rightarrow x + x + 1 \leq 2 \rightarrow 2x \leq 1$

Tema 2 - Sucesiones de numeros reales

2.1 - Definición, ejemplo y propiedades

Una sucesión de numeros reales és una aplicación

$a: D \in \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$

El conjunto imagen: $\{a_1, a_2, a_3, \dots a_n\}$ Són los terminos de la sucesión.

$\{a_n\}_n \in D$, $(a_n)_n \in D$

Ej: $a_n: \mathbb{N} \rightarrow \mathbb{N}$, $n \rightarrow 2n$ $\quad$ $a_n = 2n \rightarrow \{2, 4, 6 \dots\}$

$a_n: \mathbb{N} \rightarrow \mathbb{N}$, $n \rightarrow (-1)^n$ $\quad$ $a_n = (-1)^n \rightarrow \{-1, 1, -1, 1 \dots\}$

Formas de dar una sucesión

- Todos los terminos de la sucesión $\{2, 4, 6, 8, \dots\}$
- Termino general $a_n = 2n$
- Forma recurrente $a_1 = 2n$, $a_{n+1} = a_n + 2$, $n \geq 1$
- Sucesión de Fibonacci $\rightarrow a_{n+1} = a_n + a_{n-1}$, $n \geq 2$

Exemples de successions conegudes:

- **PROGRESSIÓ ARITMÈTICA**: Cada terme s'obté de l'anterior **sumant** un nº real fix ("diferència")

$$\boxed{a_n = a_{n-1} + d} \quad \forall n \geq 2 \text{ i } a_1 \text{ conegut}$$

Exemple:

Donats: $a_1 = 1$ i $d = 3$

$$\left.\begin{aligned} a_2 &= a_1 + d = 1 + 3 = 4 \\ a_3 &= a_2 + d = 4 + 3 = 7 \\ a_4 &= a_3 + d = 7 + 3 = 10 \\ &= a_1 + (4-1)d = 1 + 3 \cdot 3 = 10 \\ &\ldots\ldots \end{aligned}\right\} \Rightarrow \{1, 4, 7, 10, \ldots\}$$

$+3 \quad +3 \quad +3$

La SUMA dels n primers termes

$$a_1 + a_2 + \ldots + a_n = \sum_{n=1}^{n} a_i = \boxed{S_n = \frac{a_1 + a_n}{2} \cdot n}$$

- **PROGRESSIÓ GEOMÈTRICA**: Cada terme s'obté de l'anterior **multiplicat** un nº real fix ("raó")

$$\boxed{a_n = a_{n-1} \cdot r} \quad \forall n \geq 2 \text{ i } a_1 \text{ conegut}$$

Exemple:

Donats: $a_1 = 1$ i $r = 3$

$$\left.\begin{aligned} a_2 &= a_1 \cdot r = 1 \cdot 3 = 3 \\ a_3 &= a_2 \cdot r = 3 \cdot 3 = 9 \\ a_4 &= a_3 \cdot r = 9 \cdot 3 = 27 \\ &= a_1 \cdot r^{4-1} = 1 \cdot 3^3 = 27 \\ &\ldots\ldots \end{aligned}\right\} \Rightarrow \{1, 3, 9, 27, \ldots\}$$

$\cdot 3 \quad \cdot 3 \quad \cdot 3$

La SUMA dels n primers termes

$$a_1 + a_2 + \ldots + a_n = \sum_{n=1}^{n} a_i = \boxed{S_n = \frac{a_n \cdot r - a_1}{r - 1}}$$

Límite de una Sucesión

Def → Límite Finito ⟷ a_n es Congruente

- $\lim a_n \to l \longleftrightarrow \forall \varepsilon \ \exists n_0 / \forall n > n_0$

$$|x_n - l| < \varepsilon$$

ej: $a_n = \frac{1}{n} \to 0$

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\right\}$

$\varepsilon = 0'001$

$|x_n - 0| < 0'001$

Def → Sucesiones no Convergente ≡ divergente $\begin{cases} +\infty \\ -\infty \\ \nexists \text{ o } \infty \end{cases}$

- $\lim\limits_{n \to +\infty} a_n = +\infty \longleftrightarrow \forall K \ \exists n_0 / \forall n > n_0$

$$x_n > K$$

ej: $a_n = 2n \to +\infty$

$\{2, 4, 6, 8, 10 \dots\}$

$K = 50.000$

$x_n > 50.000$

- $\lim a_n = -\infty \longleftrightarrow \forall K < 0 \ \exists n_0 / \forall n > n_0$

$$x_n < K$$

ej: $a_n = 5 - 2n \to -\infty$

$\{3, 1, -1, -3, -5 \dots\}$

- $\lim a_n = \infty$ o $\nexists \lim$

ej:

$a_n = (-1)^n \cdot 2n \quad \{-2, 4, -6, 8, -10, 12 \dots\}$ → Solución oscilante

$a_n = \cos(n\pi) \quad \{1, -1, 1, -1 \dots\}$ → Solución oscilante

Resumen

a_n → Convergente ($\exists$ límite $\in \mathbb{R}$)

→ Divergente → $\pm\infty$

→ $\nexists \lim (\infty)$ oscilante

Propiedades

- Si $\exists \lim a_n$, és único.
- $a_n \leq b_n \quad \forall n \geq n_0 \to \lim a_n \leq \lim b_n$

  $a_n < b_n \quad \forall n \geq n_0 \to \lim a_n \leq \lim b_n$

Ej: $\left.\begin{array}{l} a_n = 1 \\ b_n = \frac{2}{n} \end{array}\right\}$ $a_n < b_n$

$\lim a_n \leq \lim b_n$

1- a) $\lim \alpha^n = \begin{cases} +\infty & \alpha > 1 \\ 1 & \alpha = 1 \\ 0 & 0 \leq \alpha < 1 \\ 0 & -1 < \alpha < 0 \\ \nexists & \alpha \leq -1 \end{cases}$

b) $\lim n^\alpha = \begin{cases} +\infty & \alpha > 0 \\ 1 & \alpha = 0 \\ -\infty & \alpha < 0 \end{cases}$

Indeterminaciones

$+\infty - \infty$, $\frac{\pm\infty}{\pm\infty}$, $0 \cdot \infty$, $\frac{0}{0}$, 1^∞, 0^0, ∞^0

$\boxed{\frac{\infty}{\infty} \text{ o } \frac{0}{0}}$

Si és $\frac{P(n)}{Q(n)}$: $\begin{cases} \text{grau } P(n) > \text{grau } Q(n) \longrightarrow \pm\infty \\ \text{grau } P(n) < \text{grau } Q(n) \longrightarrow 0 \\ \text{grau } P(n) = \text{grau } Q(n) \longrightarrow \text{Quocient termes de major grau} \end{cases}$

Si no: dividir numerador i denominador pel terme dominant, simplificar.

$\boxed{\infty - \infty}$

Si en el límit hi ha una resta d'arrels quadrades, es multiplica i es divideix per la suma de les arrels i s'aplica $(a+b)\cdot(a-b) = a^2 - b^2$

Es pot utilitzar:

$$a_n - b_n = a_n \cdot b_n \cdot \left(\frac{1}{b_n} - \frac{1}{a_n}\right)$$

$\boxed{1^\infty}$

$$\lim \left(1 + \frac{1}{n}\right)^n = e$$

$$a_n \to 0 \longrightarrow \lim (1 + a_n)^{\frac{1}{a_n}} = e$$

$$\left.\begin{matrix} a_n \to 1 \\ b_n \to +\infty \end{matrix}\right\} \to \lim a_n^{b_n} = \lim (1 + a_n - 1)^{b_n} =$$

$$\lim \left[(1 + a_n - 1)^{\frac{1}{a_n - 1}}\right]^{b_n \cdot (a_n - 1)}$$

$$e^{\lim b_n \cdot (a_n - 1)}$$

$$\left.\begin{matrix} a_n \to 1 \\ b_n \to +\infty \end{matrix}\right\} \to \lim a_n^{b_n} = e^{\lim b_n \cdot (a_n - 1)}$$

$$\lim f(x)^{g(x)} = \lim e^{g(x) \cdot (f(x) - 1)}$$

$\boxed{0 \cdot \infty}$

$\log(10^x) = 10^{\log x}$

$\ln(e^x) = e^{\ln x}$

$\log x^y = y \cdot \log x$

1- $\frac{0}{0} \circ \frac{\infty}{\infty}$

2- 1^∞

$$\left.\begin{array}{l} a_n \to 0 \\ b_n \to \infty \end{array}\right\} \to \left.\begin{array}{l} 1 + a_n \to 1 \\ b_n \to \infty \end{array}\right\} \to \lim (1+a_n)^{b_n}$$

$$e^{\lim b_n \cdot (a_n)}$$

$$\lim b_n \cdot a_n = \ln\left(\lim (1+a_n)^{b_n}\right)$$

$\boxed{0^0 \text{ i } \infty^0}$

Es prenen els logaritmes i es passa a la indeterminació $0 \cdot \infty$

0^0 $\left.\begin{array}{l} a_n \to 0 \\ b_n \to 0 \end{array}\right\} \to \ln a_n^{b_n} = b_n \cdot \ln a_n$

∞^0 $\left.\begin{array}{l} a_n \to +\infty \\ b_n \to 0 \end{array}\right\} \to \ln a_n^{b_n} = b_n \cdot \ln a_n$

Criterios

1- $a_n \neq 0 \quad \forall n \geq n_0 \quad \lim \frac{|a_{n+1}|}{|a_n|} = l \begin{cases} l < 1 \to 0 \\ l > 1 \to +\infty \\ l = 1 \quad ? \end{cases}$

2- $\lim \sqrt[n]{|a_n|} = l \to \begin{cases} l < 1 \to 0 \\ l > 1 \to +\infty \\ l = 1 \quad ? \end{cases}$

3- $a_n \neq 0 \quad \forall n \geq n_0 \quad \lim \frac{|a_{n+1}|}{|a_n|} = l \to \lim \sqrt[n]{|a_n|} = l$

4- Criterio de Sandwitch

$$\underset{\downarrow \atop l}{b_n} \leq a_n \leq \underset{\downarrow \atop l}{c_n} \to \lim a_n = l$$

2a) $\lim \frac{6n^3+4n+1}{2n} \underset{\frac{\infty}{\infty}}{=} \lim \frac{6n^3}{2n} = +\infty$

b) $\lim \frac{n^2-6n-2}{3n^2-9n} \underset{\frac{\infty}{\infty}}{=} \frac{1}{3}$

c) $\lim \left(\sqrt{\frac{n+1}{3n+1}}\right)^{\frac{2n-1}{3n-1}} = \sqrt{\frac{1}{2}}^{\frac{2}{3}} = \boxed{\sqrt[3]{\frac{1}{2}}}$

5a) $\lim \frac{\cos n}{n^2} = 0$

b) $\lim \frac{2^n+3^n}{2^n-3^n} \underset{\frac{+\infty}{-\infty}}{=} \lim \frac{\frac{2^n}{3^n}+\frac{3^n}{3^n}}{\frac{2^n}{3^n}-\frac{3^n}{3^n}} = \lim \frac{\left(\frac{2}{3}\right)^n+1}{\left(\frac{2}{3}\right)^n-1} = \boxed{-1}$

c) $\left(\frac{n+2}{n-3}\right)^{\frac{2n-1}{5}} \underset{1^\infty}{=} \lim e^{\left(\frac{n+2}{n-3}-1\right)\cdot\frac{2n-1}{5}} = \lim e^{\left(\frac{2+3}{n-3}\right)\cdot\frac{2n-1}{5}}$

$= \lim e^{\frac{2n-1}{n-3}}$

$= e^2$

d) $\lim \left(\sqrt{n+1}-\sqrt{n}\right)\cdot\sqrt{\frac{n+1}{2}} = \lim \frac{1}{\sqrt{2}}\sqrt{(n+1)}$

$\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$

$\lim \frac{1}{\sqrt{2}}\sqrt{(n+1)}\cdot\frac{\left(\sqrt{n+1}\right)^2-\sqrt{n}^2}{\sqrt{n+1}+\sqrt{n}}$

$\lim \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{n+1}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{1}{2\sqrt{2}}$

1c) $\lim \sqrt[n]{n} = \lim \frac{a_{n+1}}{a_n} = \frac{n+1}{n} = \boxed{1}$

3- $b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \leq n\cdot\frac{1}{\sqrt{n^2+1}}$

$n\cdot\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+1}}$

$\downarrow$ 1 $\qquad$ $\downarrow$ 1

5a) $\lim \frac{a^n}{n!}$ $\quad |a|>1$

$\lim \frac{\frac{a^{n+1}}{n+1!}}{\frac{a^n}{n!}} = \lim \frac{a}{n+1} = 0$

b) $\lim \frac{n^\alpha}{a^n}$ $\quad |a|>1$

$\lim \frac{\frac{(n+1)^\alpha}{a^{n+1}}}{\frac{n^\alpha}{|a|^n}} = \frac{(n+1)^\alpha\cdot|a|^n}{n^\alpha\cdot a^{n+1}} = \left(\frac{n+1}{n}\right)^\alpha\cdot\frac{1}{a}$

$1\cdot\frac{1}{a}<1$

$= \boxed{0}$

Sucesiones acotadas

Def → $\{a_n\}_n$ esta **acotada superiormente** si $\exists K_1 / a_n \leq K_1 \ \forall n$

$\{a_n\}_n$ esta **acotada inferiormente** si $\exists K_2 / a_n \geq K_2 \ \forall n$

$\{a_n\}_n$ esta **acotada** si $\exists K / |a_n| \leq K \ \forall n$

Def → $\{a_n\}_K$ es **creciente** $a_n \geq a_{n-1} \ \forall n$

$\{a_n\}_n$ es **decreciente** $a_n \leq a_{n-1} \ \forall n$

$\{a_n\}_n$ es **monotona** si es creciente o decreciente

$\{a_n\}_n$ es **estrictamente creciente** si $a_n > a_{n-1}$

$\{a_n\}_n$ es **estrictamente decreciente** si $a_n < a_{n-1}$

ej: $\left\{\frac{1}{n}\right\}_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\right\}$ decreciente

$a_n = \begin{cases} \frac{n+1}{2} & \text{si } n = 2K+1 \\ \frac{n}{2} & \text{si } n = 2K \end{cases} = \{1, 1, 2, 2, 3, 3\}$ creciente

Teorema de Convergencia monotona

$\{a_n\}_n$ **monotona creciente**

$\{a_n\}_n$ **acotada superiormente**

$\Big\}$ $\{a_n\}_n$ es convergente

$\exists \lim$ (finito)

$\{a_n\}_n$ **monotona decreciente**

$\{a_n\}_n$ **acotada inferiormente**

$\Big\}$ $\{a_n\}_n$ es convergente

TEOREMA de la CONVERGÈNCIA MONÒTONA

tota successió $\boxed{\text{FITADA i MONÒTONA}}$ és CONVERGENT

↳ les successions FITADES i CREIXENTS : límit = suprem superiorm.

FITADES i DECREIXENTS : límit = ínfim inferiorm.

6- $a_1 = \frac{-2}{3}$

$3 \cdot a_{n+1} = 2 + a_n^3 \quad n \geq 1$

a) Dem $-2 \leq a_n \leq 1$

b) Dem $\{a_n\}$ creciente

c) Dem $\{a_n\}$ convergente, i lim

a) $a_1 = \frac{-2}{3}$ $\quad -2 \leq \frac{-2}{3} \leq 1$

$a_{n+1} = \frac{2+a_n^3}{3}$

$-2 \leq a_n \leq 1$

$-8 \leq a_n^3 \leq 1$

$-6 \leq 2+a_n^3 \leq 3$

$\frac{-6}{3} \leq \frac{2+a_n^3}{3} \leq \frac{3}{3}$

$-2 \leq a_{n+1} \leq 1$

b) $a_1 = \frac{-2}{3}$ $\quad a_2 \overset{?}{>} a_1$

$a_2 = \frac{2 + \left(\frac{-2}{3}\right)^3}{3} = \frac{46}{81}$

$a_{n-1} \leq a_n$

$(a_{n-1})^3 \leq a_n^3$

$2 + a_{n-1}^3 \leq 2 + a_n^3$

$\frac{2+a_{n-1}^3}{3} \leq \frac{2+a_n^3}{3}$

$a_n \leq a_{n+1}$ ✓

c) $3 \cdot a_{n+1} = 2 + a_n^3$

$3 \cdot L = 2 + L^3 \quad L^3 - 3L + 2 = 0$

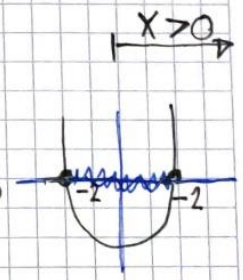
	1	0	-3	2
1		1	1	-2
	1	1	-2	$\boxed{0}$

$L^2 + L - 2 = 0 \quad L = 1, -2$

3- a) $x^3 - 4x < 0$ $\quad (0,2)$

$x \cdot (x^2 - 4) < 0 \rightarrow x > 0$

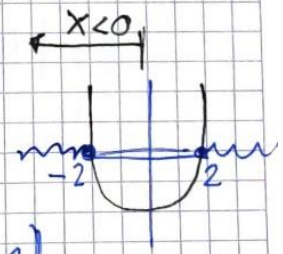
$x^2 - 4 < 0$



$(-\infty, -2) \cup (0, 2)$

$x < 0$

$x^2 - 4 > 0$



$(-\infty, -2)$

b) $x = 2^{-n} \quad x \in \mathbb{R} \quad \exists n \in \mathbb{N}$ $\quad$ Sup = max = $\frac{1}{2}$

$n = 1, 2 \ldots \rightarrow \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\right\} \rightarrow \left(0, \frac{1}{2}\right]$

Infimo = 0

$n = 0, 1, 2 \ldots \rightarrow \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\right\} \quad (0, 1]$

Supr = max = 1

Infimo = 0

c) $y \in \mathbb{R} \quad \exists x \in \mathbb{R} \quad y = 1 + x^2$

$y \in [1, +\infty)$

Inf = min = 1

7a) $\lim \frac{1}{n^2}+\frac{2}{n^2}+\dots\frac{n}{n^2} = \frac{\infty}{\infty}$

$\lim \frac{1}{n^2}\cdot(1+2+\dots+n)$

$\lim \frac{1}{n^2}\frac{1+n}{2}\cdot n$

$\lim \frac{n+n^2}{2n^2} = \frac{1+n}{2n} = \boxed{\frac{1}{2}}$

8a) $\lim\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\dots\frac{n}{n^2+n}\right)$

$\boxed{\frac{n}{n^2+n}+\frac{n}{n^2+n}+\dots\frac{n}{n^2+n}} \leq \frac{n}{n^2+1}+\frac{n}{n^2+2}+\dots\frac{n}{n^2+n}$

$\lim = 1$

$\leq \frac{n}{n^2+1}+\frac{n}{n^2+1}+\dots\frac{n}{n^2+1}$

$\lim \frac{n}{n^2+n}\cdot n = 1$

$\lim \frac{n}{n^2+1}\cdot n = 1$

7d) $\lim \frac{\sqrt[n]{(n+1)\cdot(n+2)\dots(2n)}}{n}$

$\lim \sqrt{\frac{(n+1)\cdot(n+2)\dots 2n}{n^n}}$

$\lim \frac{|a_n|}{|a_{n-1}|} = l \text{ o } \pm\infty$

$\lim \sqrt[n]{|a_n|} = L$

$\lim \frac{(n+1)\cdot(n+2)\dots(2n)}{(n)^n} = \frac{(2n-1)\cdot(2n)}{n}\cdot\frac{(n-1)^{n-1}}{n^n}$

$\frac{(n-1+1)\cdot(n-1+2)\dots(2(n-1))}{(n-1)^{n-1}}$

$\lim \frac{4n^2-2n}{n}\cdot\lim\frac{(n-1)^{n-1}}{n^n}$

$\lim \frac{4n-2}{n}\cdot\lim\left(\frac{n-1}{n}\right)^{n-1}$

$4\cdot\lim\left(1+\frac{1}{-n}\right)^{-n}\Big)^{\frac{n-1}{-n}}$

$\boxed{4\cdot e^{-1}}$

10- a) $|x|-|x^2| = |x|-x^2$

Si $x\geq 0 \to x-x^2$

Si $x<0 \to -x-x^2$

b) $||x|-1| =$

Si $x\geq 0 \to |x-1| =$ Si $x-1\geq 0 \to x-1$ $[1,+\infty)$; Si $x-1<0 \to -x+1$ $[0,1)$

Si $x<0 \to |-x-1| =$ Si $-x-1\geq 0 \to -x-1$ $(-\infty,-1]$; Si $-x-1<0 \to x+1$ $(-1,0)$

8b) $\lim_{n\to\infty} \frac{5(n+1)^{n+1}}{(3n^2+1)\cdot n^{n+1}} = \frac{\infty}{\infty}$

$\lim \frac{5\cdot(n+1)\cdot n}{3n^2+1}\cdot\frac{(n+1)^n}{n^n}$

$\lim_{n\to\infty} \frac{5n^2+5n}{3n^2+1}\cdot\lim\frac{(n+1)^n}{n^n}$

$\frac{5}{3}\cdot 1^\infty = \frac{5}{3}\cdot\lim\left(1+\frac{1}{n}\right)^n$

$\frac{5}{3}\cdot e$

Tema 3 - Funciones Continuas

Def → $f: \mathbb{R} \to \mathbb{R}$
$x \to y = f(x)$

Def → **Dominio f.** $\text{Dom } f = \{x \in \mathbb{R} / \exists f(x) = y$

Recorregut f. $\text{Rec } f = \{y \in \mathbb{R} / \exists x : f(x) = y$

Grafica de f. $\text{Graf } f = \{(x,y) / y = f(x)$

$f(x) = \frac{1}{x}$ $\quad \text{Dom } f = \mathbb{R} - \{0\}$

$\text{Rec } f = \mathbb{R} - \{0\}$

Def → $\mathbb{R} \to \mathbb{R}$ → si es continua en 'a' si
$x \to y = f(x)$

$\exists f(a)$ → $a \in \text{Dom } f$

$\exists \lim_{x \to a} f(x) = L$ $\quad \left[\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) \right.$

$f(a) = \lim_{x \to a} f(x)$

Def → $\mathbb{R} \to \mathbb{R}$ continua en A si es continua en "a"

Tipos de discontinuidad

1- Discontinuidad evitable → $f(a) \neq \lim f(x)$

$f(x) = \frac{x^2-4}{x-2}$ $\quad$ Dis en $x=2$

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)\cdot(x+2)}{x-2} = \boxed{4}$$

$4 \neq \exists f(2)$

evitamos $g(x) = \begin{cases} f(x) & \text{si } x \neq a \\ \lim_{x \to a} f(x) & \text{si } x = a \end{cases}$

2- Discontinuidad de Salto

2.1- Discontinuidad de Salto finito → $\exists \lim_{x \to a^-} f(x) \neq \exists \lim_{x \to a^+} f(x)$

$f(x) = \begin{cases} x^2+1 & x \leq 1 \\ 3x+2 & x > 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = 2$ ✗ $\lim_{x \to 1^+} f(x) = 5$

2.2- Discontinuidad de Salto infinito →

$\lim_{x \to a^-} f(x)$ o $\lim_{x \to a^+} f(x) = \pm\infty$

$f(x) = \frac{x^2+1}{x-3}$ Disc en $x=3$

$\lim_{x \to 3^-} f = \frac{10}{0^-} = -\infty$

$\lim_{x \to 3^+} f = \frac{10}{0^+} = +\infty$

3- Discontinuidad de 2a especie → $\nexists \lim f(x)$ (Ni $\mathbb{R}$ ni ∞)

$f(x) = \sin \frac{1}{x}$ Disc en $x=0$

Proposición 1

$f: \mathbb{R} \to \mathbb{R}$ elemental, $a \in \text{Dom } f \to f$ continua

Proposicion 2

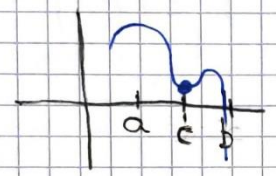
f, g Continua en $a \in \text{dom } f \cap \text{Dom } g \to$ $\begin{cases} f \pm g \text{ Continua} \\ \bullet / f \text{ Continua} \\ fg \text{ continua} \end{cases}$

Teoremas de funciones Continuas

• Teorema de Conservación de Signo

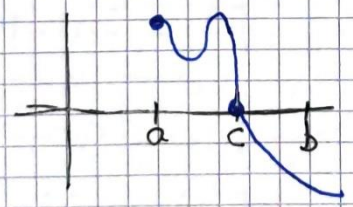
$\left.\begin{array}{l} f \text{ continua en } [a,b] \\ c \in (a,b),\ f(c) \neq 0 \end{array}\right\} \to \exists r > 0 / f(c) \cdot f(x) > 0$

$\forall x \in (c-r, c+r)$



• Teorema de Bolzano

$\left.\begin{array}{l} f \text{ continua en } [a,b] \\ f(a) \cdot f(b) < 0 \end{array}\right\} \to \exists c \in (a,b) / f(c) = 0$



• Teorema de Valor intermedio

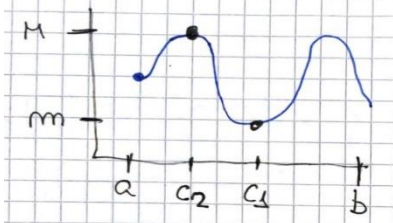
f continua en $[a,b] \to \forall y \in \langle f(a), f(b) \rangle$

$\exists c \in (a,b) / f(c) = y$

• Teorema de Weierstrass

f continua en $[a,b] \longrightarrow \exists M, m \,/\, m \le f(x) \le M \quad \forall x \in (a,b)$

m : minimo
M : maximo



$m = f(c_1)$

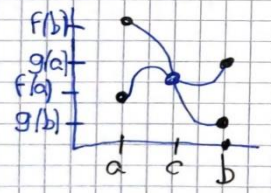
$M = f(c_2)$

$c_1, c_2 \in [a,b]$

2- $a \le b \in \mathbb{R}$

f, g continuas en $[a,b]$

$f(a) < g(a)$, $f(b) > g(b)$ $\Big\}$ $\exists c \in (a,b) \,/\, f(c) = g(c)$



$h(x) = g(x) - f(x)$

$h(a) = g(a) - f(a) > 0$

$h(b) = g(b) - f(b) < 0$

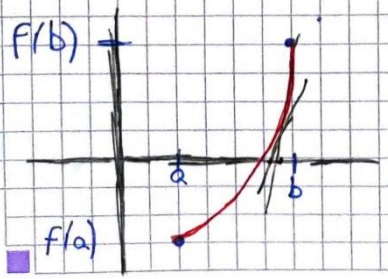
$h(c) = 0$

$g(c) - f(c)$

$\downarrow$

$g(c) = f(c)$

Metodo Newton Raphson (Metodo tangente)



Recta tangente por $(x_n, f(x_n))$

$y = f(x_n) - f'(x_n) \cdot (x - x_n)$

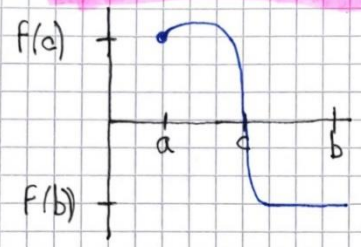
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$|x_{n+1} - x_n| < \varepsilon$

$f(x_{n+1}) < \varepsilon$

Cálculo de raices de una función (ceros)

1- Método bisección



$C_1 = \frac{a+b}{2}$

Si $f(c_1), f(a) < 0$

$$\boxed{\text{Error} = |c - c_n| \le \frac{b-a}{2^n}}$$

$C_2 = \frac{a + c_1}{2}$

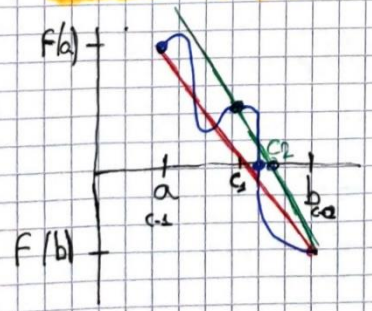
ej $E \le 0'01$

$\frac{b-a}{2^n} \le 0'01$

Si $f(c_1), f(b) < 0$

$C_2 = \frac{b + c_1}{2}$

2- Método Secante



$y = mx + n = 0$

recta $(a, f(a)), (b, f(b))$

$y = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$

$y = f(x_n) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \cdot (x - x_n)$

$$\boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n)}$$

$|x_{n+1} - x_n| < \varepsilon$

$f(x_n) < \varepsilon$

1- $x^3-3x^2+1=0$ tiene Sol. en $[0,2]$

$f(x) = x^3-3x^2+1$

$f(0) = 1 > 0$

$f(2) = -3 < 0$

$\exists c \in (0,2) \,/\, f(c) = 0$

c raíz de la ecuación

3- $f(x) = \frac{x^3}{4} - \sin(\pi x) + 3$

toma el valor 2'5 en $[-2,2]$

$\frac{x^3}{4} - \sin(\pi x) + 3 = 2'5$

$\underbrace{\frac{x^3}{4} - \sin(\pi x) + 0'5}_{g(x)} = 0$

$g(x)$ continua (Pol + Sin)

$g(-2) = -2+0'5 = -1'5 < 0$
$g(2) = 2+0'5 = 2'5 > 0$ $\rightarrow g(c) = 0$

b) $|x_{n+1} - x_n| < \varepsilon \; (0'5 \cdot 10^{-3})$

$f(x_n) < \varepsilon$

$c_1 \in (1,2)$
$x_{-1} \quad x_0$

$x_1 = x_0 - \frac{x_0 - x_{-1}}{f(x_0) - f(x_{-1})} \cdot f(x_0) =$

4- $e^{-x^2} = 2x \quad [0,1]$

$e^{-x^2} - 2x = 0$

$f(x)$ continua en $[0,1]$
(Suma exp + polin)

$f(0) = e^0 - 2\cdot 0 = 1$
$f(1) = e^{-1} - 2 < 0$ $\rightarrow f(c) = 0$

$f(0'5) = e^{-0'25} - 2\cdot 0'5 = -0'22 \rightarrow c(0, 0'5)$

$f(0'3) = e^{-0'09} - 2\cdot 0'3 = 0'31 > 0 \rightarrow c \in (0'3, 0'5)$

$c = 0'4$

5- a) Separar las dos Sol reales $x - 3\cdot \ln x = 0$

b) bisección con precisión $0'5\cdot 10^{-3}$ para ambas Sol

c) Secante con 3 decimales correctos para ambas Sol

$f(x) = x - 3\cdot \ln x$

$f(1) = 1 - 3\cdot \ln 1 = 1 > 0$
$f(2) = 2 - 3\cdot \ln 2 = -0'08 < 0$ $\rightarrow c_1 \in (1,2)$

x_n	$f(x_n)$
1	1
2	-0'08
1'9264	-0'0465
1'8496	0'00467
1'85756	-0'00023
1'85718	0'00000012

$f(4) = -0'16 < 0$
$f(5) = 0'17 > 0$ $\rightarrow c_2 \in (4,5)$

b) $0'5\cdot 10^{-3} \rightarrow$ 3 decimales correctos

$\frac{2-1}{2^n} \leq 0'5\cdot 10^{-3}$

$\frac{1}{2^n} \leq 0'5\cdot 10^{-3}$

$\ln 1 - \ln 2^n \leq \ln(0'5\cdot 10^{-3})$

$-n\cdot \ln 2 \leq \ln(0'5\cdot 10^{-3})$

$n \geq \frac{\ln(0'5\cdot 10^{-3})}{-\ln 2}$

$\boxed{n = 11}$

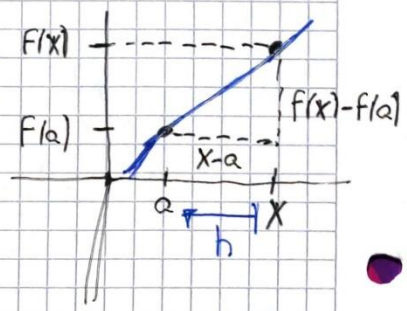
	c_i	$f(c)$
a =	1	1
b =	2	-0'08
c_1 =	1'5	0'28
c_2 =	1'75	

$c_{11} = 1'8569$

Tema 4 – Funciones derivables (una variable)

Def: Derivada de $f(x)$ en el punto a: $f(x)$ continua

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$$



tg α = pendiente de la recta

$f'(a)$: pendiente de la recta tangente a $(a, f(a))$

$$\boxed{y - f(a) = f'(a) \cdot (x-a)}$$ Ecuacion recta tangente

$$\boxed{y - f(a) = \frac{-1}{f'(a)} \cdot (x-a)}$$ Ecuacion recta normal

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

$$\boxed{h = x-a}$$

Proposicion

$f: \mathbb{R} \to \mathbb{R}$ elemental, $a \in \mathrm{Dom}\, f$ $\Big\} \to$ f derivable en a

f, g elementales, $\lambda \in \mathbb{R}$, $a \in \mathrm{Dom}\, f \cap$

λf

$f \pm g$

$f \cdot g$

$\frac{f}{g}$ $g(a) \neq 0$

$\Big\} \to$ Derivables en a

Funcion simple	Derivada
$y = k$	$y' = 0$
$y = x$	$y' = 1$
$y = u(x) + v(x)$	$y' = u'(x) + v'(x)$
$y = k \cdot u(x)$	$y' = k \cdot u'(x)$
$y = u(x) \cdot v(x)$	$y' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$
$y = \frac{u(x)}{v(x)}$	$y' = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v^2(x)}$
$y = x^n$	$y' = n \cdot x^{n-1}$
$y = \ln x$	$y' = \frac{1}{x}$
$y = \log_a x$	$y' = \frac{1}{x} \log_a e$
$y = e^x$	$y' = e^x$
$y = a^x$	$y' = a^x \cdot \ln a$
$y = \mathrm{sen}\, x$	$y' = \cos x$
$y = \cos x$	$y' = -\mathrm{sen}\, x$
$y = \mathrm{tg}\, x$	$y' = \frac{1}{\cos^2 x} = 1 + \mathrm{tg}^2 x$
$y = \mathrm{cotg}\, x$	$y' = \frac{-1}{\mathrm{sen}^2 x} = -[1 + \mathrm{cotg}^2 x]$
$y = \mathrm{arc\,sen}\, x$	$y' = \frac{1}{\sqrt{1-x^2}}$
$y = \mathrm{arc\,cos}\, x$	$y' = \frac{-1}{\sqrt{1-x^2}}$
$y = \mathrm{arc\,tg}\, x$	$y' = \frac{1}{1+x^2}$

Calculo derivadas

$y = \lambda f \to y' = \lambda \cdot f'$

$y = f \pm g \to y' = f' \pm g'$

$y = f \cdot g \to y' = f' \cdot g + f \cdot g'$

$y = \frac{f}{g} \to y' = \frac{f' \cdot g - f \cdot g'}{g^2}$

Regla de la Cadena

$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$

$a \to f(a) \to g(f(a)) = g \circ f$

$$\boxed{(g \circ f)'(a) = g'(f(a)) \cdot f'(a)}$$

$f = \mathrm{Sin}\, x$

$g = e^{x^2}$

$g \circ f = e^{\mathrm{Sin}^2 x}$

$y' = e^{\mathrm{Sin}^2 x} \cdot 2 \cdot \mathrm{Sin}\, x \cdot \mathrm{Cos}\, x$

Derivada funcion exponencial-potencial

$$y = f(x)^{g(x)}$$

$$\ln y = \ln f(x)^{g(x)}$$

$$\ln y = g(x) \cdot \ln f(x)$$

$$y' = f(x)^{g(x)} \cdot \left(g'(x) \cdot \ln f(x) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x) \right)$$

Def → f es **creciente** en $[a,b]$ si

$$x < y \rightarrow f(x) \leq f(y) \quad \forall x, y \in [a,b]$$

f es **estrictamente creciente** en $[a,b]$ si

$$x < y \rightarrow f(x) < f(y)$$

f es **decreciente** en $[a,b]$ si

$$x < y \rightarrow f(x) \geq f(y)$$

f es **estrictamente decreciente** en $[a,b]$ si

$$x < y \rightarrow f(x) > f(y)$$

Def → x_0 es un **mínimo relativo** (local) si

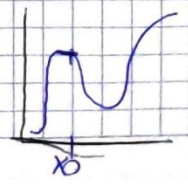
$$\exists I_{x_0} \,/\, f(x_0) \leq f(x) \quad \forall x \in I_{x_0}$$

Intervalo centrado en x_0

$$I_{x_0} = (x_0 - 2, x_0 + 2)$$

x_0 es un **maximo relativo** si

$$\exists I_{x_1} \,/\, f(x_1) \geq f(x) \quad \forall x \in I_{x_1}$$

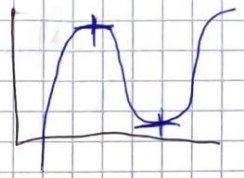


Teorema del extremo interior

(Condicion necesaria extremos relativos)

- f continua en $[a,b]$
- f derivable en (a,b)
- c extremo relativo

$\rightarrow f'(c)=0$



Dem: f derivable en (a,b), c extremo (minimo) $\rightarrow$ f derivable en c:

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

$\rightarrow \exists \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \leq 0$

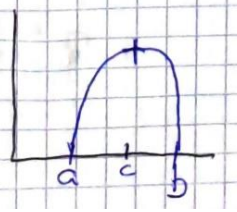
$\rightarrow \exists \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \geq 0$

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$$

Teorema de Rolle

- f continua en $[a,b]$
- f derivable en (a,b)
- $f(a)=f(b)$

$\rightarrow \exists c \in (a,b) / f'(c)=0$

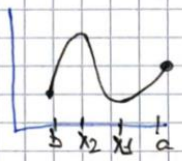


Dem: f continua en $[a,b]$

$\rightarrow \exists x_1, x_2 \in [a,b]$ $\quad f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b]$

$f(x_1) \rightarrow$ m (minimo absoluto)

$f(x_2) \rightarrow$ M (maximo absoluto)



- Si $x_1 \in (a,b) \rightarrow x_1$ minimo relativo $\rightarrow f'(x_1)=0$
- Si $x_2 \in (a,b) \rightarrow x_2$ maximo relativo $\rightarrow f'(x_2)=0$
- Si $x_1, x_2 \notin (a,b) \rightarrow \{x_1, x_2\} = \{a,b\}$

$$f(x_1) \leq f(x) \leq f(x_2)$$

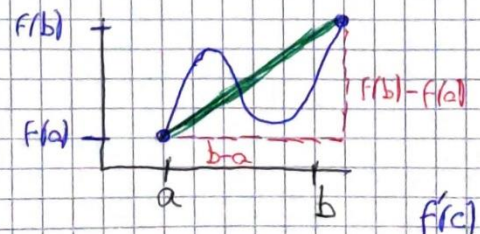
$f(x_1)=f(x_2) \rightarrow f$ constante $\rightarrow f'=0$

Teorema de Lagrange (T. Valor Intermedio)

- f continua en $[a,b]$
- f derivable en (a,b)

$\rightarrow \exists c \in (a,b)$

$$\boxed{\begin{gathered} f'(c) = \frac{f(b)-f(a)}{b-a} \\ f(b)-f(a) = f'(c)\cdot(b-a) \end{gathered}}$$



$f'(c)$ Pendiente recta tangente $(c, f(c))$

$\frac{f(b)-f(a)}{b-a}$, Pendiente recta por $(a,f(a)), (b,f(b))$

Dem:

$$y = f(a) + \frac{f(b)-f(a)}{b-a}\cdot(x-a)$$

$g(x) = f(x) - y \rightarrow$ Continua en $[a,b]$, derivable en (a,b)

$g(b) = f(b) - y(b) = f(b) - f(b) = 0$

$g(a) = f(a) - y(a) = f(a) - f(a) = 0$

$\downarrow$

$\exists c \,/\, g'(c) = 0$

Continuidad de $f(x)$ en x_0

1- $\exists f(x_0)$ $(x_0 \in D(f))$

2- $\exists \lim_{x\to x_0} f = \lim_{x\to x_0^-} f = \lim_{x\to x_0^+} f$

Corolario

- Si $f'(x) > 0$ $\forall x \in (a,b) \rightarrow f$ estrictamente creciente
- Si $f'(x) < 0$ $\forall x \in (a,b) \rightarrow f$ estrictamente decreciente

Regla de l'Hopital

f, g. Funciones derivables en un entorno de a

$[a \in \mathbb{R},\ a = +\infty,\ a = -\infty]$

$$\left.\begin{cases} \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \\ \text{o} \\ \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty \end{cases}\right\} \boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

ej $\lim_{x\to 2}\frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2}\frac{2x-3}{2x} = \boxed{\frac{1}{4}}$

1- Punto de $y = x^2$ en que la recta tangente es paralela al segmento $\overline{AB}$ $[A=(1,1),\ B=(3,9)]$

Pendiente de $\overline{AB}$ $\quad m = \frac{y_2-y_1}{x_2-x_1} = \frac{9-1}{3-1} = \boxed{4}$

Pendiente de recta tangente $\quad f'(x) = \boxed{2x}$

$4 = 2x \quad \boxed{x=2} \quad (2,4)$

2- Dem $3^{-x} = x$ tiene una única solución. Cuál es la Parte entera

$f(x)$ continua (Suma exponencial + polinomia)

$f(x) = 3^{-x} - x$

$f(0) = 1 > 0$

$f(1) = \frac{1}{3} - 1 < 0$

T. Bolzano $\rightarrow f(c) = 0$

Unicidad. Suponemos 2 sol. $f(c_1) = 0 = f(c_2)$

f continua en $[0,1]$

f derivable en $(0,1)$

$f(c_1) = f(c_2)$

T. Rolle $\rightarrow f'(t) = 0$ $\quad t \in (c_1, c_2)$

$f'(x) = -3^{-x} \cdot \log 3 - 1 = 0$

$3^{-x} = \frac{1}{-\log 3}$ $\quad$ Contradicción

$\vee \quad\quad \wedge$

$0 \quad\quad 0$

b) $f(1{,}5) = -0'18$

$f(1{,}3) = 0'01$ $\quad c \in (1'3, 1'4)$

$f(1'4) = -0'09$

c) $f'(x) = -e^{-x} - \frac{1}{x}$ en $[1, +\infty)$

$\underset{<0}{-e^{-x}} = \underset{>0}{\frac{1}{x}}$

d) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

x_n	$f(x_n)$	$f'(x_n)$
1	0'36788	-1'36788
1'26894	0'04295	-1'06919
1'30910	0'00071	-1'03395
1'30979	$2 \cdot 10^{-7}$	-1'03335
1'30980		

4 iteraciones

3- Dem $f_m(x) = x^3 - 3x + m$ no tiene dos ceros en $[0,1]$ $m \in \mathbb{R}$

f continua

f derivable

$f_m(c_1) = f_m(c_2) = 0$

$f'(t) = 0$

$f'_m(x) = 3x^2 - 3 = 0$

$x^2 = 1$

$x = \boxed{\pm 1}$ Contradicción no esta en el intervalo

4- $e^{-x} = \ln x$

a) Dem tiene sol en $[1, +\infty)$

b) Dar un intervalo longitud 0'1 que contenga dicha sol

c) Razonar que no tiene 2 sol en $[1, +\infty]$

d) Newton-Raphson con $x_0 = 1$ para obtener la raiz. Detener el calculo si $|x_n - x_{n-1}| < 10^{-4}$

a) $f(1) = e^{-1} - \ln 1 = e^{-1} > 0$

$f(2) = e^{-2} - \ln 2 < 0$

T. Bolzano $\exists c \in (1,2) / f(c) = 0$

5- a) $\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = 0$

b) $\lim_{x\to 0^+} x\cdot\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0} \frac{-x^2}{x}$

$= \lim_{x\to 0} -x = 0$

$\lim_{x\to 0} (\sin x)^x = \lim_{x\to 0} e^{\ln\cdot[\sin x]^x} = \lim_{x\to 0} e^{x\cdot\ln\sin x} = e^0 = 1$

$\lim_{x\to 0} x\cdot\ln\sin x = \lim_{x\to 0} \frac{\ln(\sin x)}{\frac{1}{x}} = \frac{\frac{1}{\sin x}\cdot\cos x}{\frac{-1}{x^2}}$

$\lim_{x\to 0} \frac{-x^2\cdot\cos x}{\sin x} = \frac{-x^2}{\operatorname{tg} x} = \frac{2x}{\frac{1}{\cos^2 x}} = 0$

10- a) $\operatorname{arctg} x > \frac{x}{1+x^2}$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\frac{\operatorname{arctg} x - \operatorname{arctg} 0}{x-0} = f'(c) = \frac{1}{1+c^2}$;

6- a) Demostreu que $\ln x = x^2-4x$ té una Solució en $[1,\infty)$

$f(x) = \ln x - x^2 + 4x$

f continuas en $(0,+\infty)$

$f(1) = \ln 1 - 1^2 + 4\cdot 1 = 3 > 0$

$f(2) = \ln 2 - 2^2 + 4\cdot 2 > 0$

$f(3) = \ln 3 - 3^2 + 4\cdot 3 > 0$

$f(4) = \ln 4 - 4^2 + 4\cdot 4 > 0$

$f(5) = \ln 5 - 5^2 + 4\cdot 5 < 0$

$\exists\, c \in (4,5)$

$f(c) = 0$

$\ln c = c^2 - 4c$

7a) $f(x) = \frac{1}{1-e^{\frac{1}{x}}}$ en $[-1,1]$

$f(0)$ no es continua

7b) $f(x) + \frac{1}{2} = 0$ un interval de longitud igual o menor que $\frac{1}{3}$

$g(x) = \frac{1}{1-e^{\frac{1}{x}}} + \frac{1}{2}$

$g(x) = \frac{2-e^{\frac{1}{x}}}{2-2\cdot e^{\frac{1}{x}}}$

$g(-1) = 2'082 > 0$

$\lim_{x\to 0^-} \frac{1}{1-e^{1/0}} + \frac{1}{2} = \frac{1}{1-e^{-\infty}} + \frac{1}{2} = \frac{1}{1-\frac{1}{e^{\infty}}} + \frac{1}{2}$

$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2} > 0$

$g(1) = -0'082 < 0$

$\lim_{x\to 0^+} \frac{1}{1-e^{\infty}} + \frac{1}{2} = \frac{1}{2} > 0$

$\operatorname{arc\,tg} x = \frac{x}{1+c^2}$

$c < x$

$c^2 < x^2$

$1+c^2 < 1+x^2$

$\frac{1}{1+c^2} > \frac{1}{1+x^2}$

b) $\ln(1+x) \le x$

$\frac{\ln(1+x) - \ln(1)}{x-0} = f'(c) = \frac{1}{1+c}$

$\frac{\ln(1+x)}{x} = \frac{1}{1+c} \leftrightarrow \ln(1+x) = \frac{x}{1+c} < x$

$c \in (0,x)$ $c > 0$

$1+c > 1$

$\frac{1}{1+c} < 1$

c) $\arccos x > \frac{\pi}{2} - \frac{x}{\sqrt{1-x^2}}$

$\frac{\arccos x - \arccos 0}{x-0} = f'(c) = \frac{1}{\sqrt{1-c^2}}$

$\sqrt{1-c^2} > \sqrt{1-x^2}$

$\frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-x^2}}$

$\frac{\arccos x - \frac{\pi}{2}}{x} = \frac{-1}{\sqrt{1-c^2}}$; $\arccos x = \frac{-x}{\sqrt{1-c^2}} + \frac{\pi}{2}$

Tema 5 - Polinomio de Taylor

$f(x)$, $x_0 \in \text{Dom } f$

Queremos aproximar la funcion por polinomio en un entorno de x_0

ej: $f(x) = e^x \qquad x_0 = 0$

$P_0(x) = 1$

$P_1(x) = f(0) + f'(0) \cdot (x-0)$

P_2 - misma curvatura

Polinomio Taylor de grado n, de la función $f(x)$ en un entorno de x_0

$$\boxed{P_{n,f(x),x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k}$$

$f^{(k)}$ ← derivada

ej: $f(x) = \ln(x+1) \quad x_0 = 0$

$f'(x) = \frac{1}{x+1} = 1$

$f''(x) = -1 \cdot (x+1)^{-2} = -1$

$f'''(x) = 2 \cdot (x+1)^{-3} = 2$

$f''''(x) = -6 \cdot (x+1)^{-4} = -6$

$P_{4,f,0}(x) = 0 + 1x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4$

$$\boxed{P^{(k)}_{n,f,x_0}(x_0) = f^{(k)}(x_0)}$$

Def → Termino Complementario, Resto o Residuo

del polinomio de taylor de orden n de la funcion f en el punto x_0

$$\boxed{R_{n,f,x_0}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}}$$

c está entre $(0,x)$

$$\boxed{f(x) - P_{n,f,x_0}(x) = R_{n,f,x_0}(x)}$$

Aproximación y acotación del error

$f : \mathbb{R} \to \mathbb{R}$

$f, f', f'' \ldots f^n$ continuas en $[a,b]$

$\exists f^{(n+1)}$ continuas $[a,b]$, $x_0 \in [a,b]$

$\exists M_{n+1} = \max_{t \in (x_0,x)} f^{(n+1)}(t)$

$$F(x) - P_n(x) = R_n(x) = \frac{F^{n+1}(c)}{(n+1)!} \cdot (x-x_0)^{n+1}$$

$$\leq \frac{M}{(n+1)!} \cdot (x-x_0)^{n+1}$$

1- $f(x) = \sqrt[3]{1728+x}$ de grado 2

Para calcular $\sqrt[3]{1731}$. Acotar el error $\sqrt[3]{1731} \simeq P_2(3)$

$x_0 = 0$

$$P_2(x) = F(x_0) + F'(x_0) \cdot (x-x_0) + \frac{F''(x_0)}{2} \cdot (x-x_0)^2$$

$$P_2(x) = \sqrt[3]{1728} + F'(x_0) \cdot x + \frac{F''(x_0)}{2} \cdot x^2$$

$$F'(x_0) = \frac{1}{3}(1728+x)^{\frac{-2}{3}} = \frac{1}{432}$$

$$F''(x_0) = -\frac{1}{3} \cdot \frac{2}{3}(1728+x)^{\frac{-5}{3}} = \frac{-1}{11197414}$$

$$F'''(x) = \frac{10}{27}(1728+x)^{\frac{-7}{3}}$$

$$P_2(x) = 12 + \frac{x}{432} - \frac{x^2}{22394488}$$

$$\boxed{P_2(3) = 12'00694642}$$

$c \in \langle 0,3 \rangle$ $\boxed{\frac{5}{3 \cdot 12^7}}$

$$R_2(x) = \frac{F'''(c)}{3!} x^3 = \frac{10 \cdot 3^2}{27 \cdot 6(\sqrt[3]{1728+c})^7} \leq \frac{5}{3(1728)^7}$$

3- Cota Superior del error de la formula

$e \simeq 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ usando taylor de e^x

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

$$P_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!}$$

$$\text{Error} = R_4(1) = \frac{F^5(c)}{5!} 1^5 = \frac{e^c}{5!} \leq \boxed{\frac{e}{5!}}$$

$c \in (0,1)$

2- $F(x) = \ln(1-x)$

a) Hallar los 5 primeros terminos no nulos del polinomio de Taylor ($x_0=0$) y el resto de Lagrange

b) Hallar el grado del polinomio para obtener $P_n(0'75)$ con error menor que 10^{-3}

$$P_n(x) = F(x_0) + F'(x_0) \cdot (x-x_0) + \frac{F''(x_0)}{2} \cdot (x-x_0)^2 + \dots$$

$$\frac{+F^{(n)}(x_0)}{n!} (x-x_0)^n$$

$$F'(x) = \frac{-1}{+1-x}$$

$$F''(x) = -(1-x)^{-2}$$

$$F'''(x) = -2 \cdot (1-x)^{-3}$$

$$F''''(x) = -6 \cdot (1-x)^{-4}$$

$$F'''''(x) = -24 \cdot (1-x)^{-5}$$

$$\boxed{P_5(x) = -x - \frac{1}{2}x^2 - \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \frac{24x^5}{5!}}$$

$$\boxed{R_5(x) = \frac{-120}{6! \cdot (1-c)^6} \cdot x^6}$$

b) $$R_n(x) = \frac{+n!}{(1-x)^{n+1} \cdot (n+1)!} \cdot x^{n+1} < 10^{-3}$$

$P_n(0'75) \simeq \ln(0'75) = F(0'25)$ $R_n(0'25) = \frac{0'25^{n+1}}{(n+1) \cdot (1-0'25)^{n+1}}$

Aplicaciones polinomio Taylor

1- Monotonía y extremos

f (n+1) veces derivable

$f'(a) = f''(a) = f'''(a) = f^{n-1}(a) = 0$

$f^{n}(a) \neq 0$

$f(x) = P_n(x) + R_a(x)$

$f(x) = f(a) + \frac{f^{n}(a)}{n!}(x-a)^n + R_n(x)$

n: par: $f^{n}(a) > 0 \rightarrow f(x)-f(a) > 0 \rightarrow f(x) > f(a)$ → a es un mínimo

$f^{n}(a) < 0 \rightarrow f(x)-f(a) < 0$

$f(x) < f(a) \rightarrow$ a es un máximo

n impar: $f^{n}(a) \gtrless 0 \rightarrow x-a > 0 \rightarrow f(x)-f(a) > 0$ → f es creciente

$\rightarrow x-a < 0 \rightarrow f(x)-f(a) < 0$

$f^{n}(a) < 0 \rightarrow x-a > 0 \rightarrow f(x)-f(a) < 0$ → f es decreciente

$x-a < 0 \rightarrow f(x)-f(a) > 0$

ej: $f(x) = 2x^3 - 9x^2 + 12x + 5$

$f' = 0$

$f' = 6x^2 - 18x + 12 = 0$

$f'' = 12x - 18$

$f''(2) = 6 > 0$

x: 2 mínimo

$f''(1) = -6 < 0 \rightarrow x = 1 \rightarrow$ Máximo

$f(x) = x^9 - 25$

$f' = 9x^8 = 0$

$x = 0$

$f'' = 56x^6$

$f''(0) = 0$

$f''' = 56 \cdot 6 \cdot x^5$

$f^4 = 56 \cdot 6 \cdot 5 \cdot x^4$

$f^5 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot x^3$

$f^6 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot 3\, x^2$

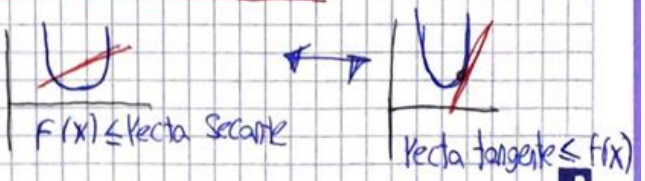
$f^7 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2\, x$

$f^8 = 56 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$

x = 0: mínimo

2- Curvatura y puntos inflexión

f convexa



f cóncava: No convexa

f n+1 veces derivable

$f''(a) = f'''(a) = f^{n-1}(a) = 0$

$f^{n}(a) \neq 0$

$f(x) - \underbrace{f(a) + f'(a)\cdot(x-a)}_{\text{recta tangente}} = \frac{f^{n}(a)}{n!}(x-a)^n + R_n$

n par: $f^{n}(a) > 0 \rightarrow$ f es convexa

$f^{n}(a) < 0 \rightarrow$ f es cóncava

n impar: $f^{n}(a) > 0 \rightarrow x > a \rightarrow$ convexa

$\rightarrow x < a \rightarrow$ cóncava

→ a punto de inflexión

$f^{n}(a) < 0 \rightarrow x > a \rightarrow$ cóncava

$\rightarrow x < a \rightarrow$ convexa

→ a punto de inflexión

6- $f: [0,1] \to [0,1]$

f contínua derivable

$f'(x) \neq 1$

Demostrar que existeix x_0: $f(x_0) = x_0$

$g(x) = f(x) - x$

g(x) contínua

g(x) derivable

$g(0) = f(0) - 0 = f(0) \geq 0$

$g(1) = f(1) - 1 < 0$

$\exists x_0 \in (0,1)$ $g(x_0) = 0$

$f(x_0) - x_0 = 0$

$\boxed{f(x_0) = x_0}$

x_0 únic – T. Rolle: g(x) contínua en $[0,1]$, g(x) derivable en $[0,1]$, $g(a) = g(b)$ $\Big\}$ $\exists c \in (a,b)$ $g'(c) = 0$

$g(x_0) = g(x_1) = 0$

$g'(x) = f'(x) - 1$

$f'(c) - 1 = 0$

$f'(c) \neq 1$

7- $e^x = \frac{1}{2}x + 2$

a) Demostrar 2 solucions en $[-5, 2]$ → Positiva, → Negativa

$f(x) = e^x - \frac{1}{2}x - 2$

$f(-5) = e^{-5} - \frac{1}{2}\cdot(-5) - 2 > 0$

$f(0) = e^0 - \frac{1}{2}0 - 2 < 0$

$\exists x_1 : f(x_1) = 0$, $x_1 \in (-5, 0)$

$f(2) = e^2 - \frac{1}{2}\cdot 2 - 2 > 0$

$\exists x_2 : f(x_2) = 0$, $x_2 \in (0, 2)$

b) Demostreu que l'equació té només 2 solucions

$f'(x) = e^x - \frac{1}{2}$

$f'(x_0) = 0$

$x_0 = \ln\left(\frac{1}{2}\right) = \boxed{-0'7}$

c) n? nº d'iteracions mètode Bisecció per calcular la solució positiva amb un error menor que 10^{-8}

$\frac{b-a}{2^n} < n$

$\frac{2-0}{2^n} < 10^{-8}$

$\log 2^{1-n} < \log 10^{-8}$

$(1-n)\cdot\log 2 < -8\log 10$

$\log 2 - n\cdot\log 2 < -8$

$\log 2 + 8 < n\cdot\log 2$

$\frac{\log 2 + 8}{\log 2} < n$

$27'57 < n$

$\boxed{n \geq 28}$

Tema 6 - Calculo integral

def $\rightarrow$ $F: \mathbb{R} \rightarrow \mathbb{R}$

$F: \mathbb{R} \rightarrow \mathbb{R}$

$a < b \in \mathbb{R}$; $[a,b] \subset \text{Dom } F \cap \text{Dom } F'$

$F(x)$ es una primitiva de $f(x)$ si $F'(x) = f(x)$

ej $\left.\begin{array}{l} f(x) = 3x^2 \\ F(x) = x^3 \end{array}\right\} \rightarrow F' = f$

$\left.\begin{array}{l} f(x) = 3x^2 \\ F(x) = x^3 + 5 \end{array}\right\} \rightarrow F' = f$

Proposición

F, g primitivas de f $\rightarrow$ $F = G + C$ $C \in \mathbb{R}$

Inmediatas	*Cuasi inmediatas (con funciones)*
$\int x^n dx = \frac{x^{n+1}}{n+1} + k$ se suma 1 al expo y se divide por lo mismo	$\int f^n(x) \cdot f'(x) dx = \frac{f^{n+1}(x)}{n+1} + k$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + k$	$\int \frac{f'(x)}{f(x)} dx = \ln\lvert f(x)\rvert + k$
$\int e^x dx = e^x + k$ se queda igual	$\int e^{f(x)} f'(x) dx = e^{f(x)} + k$
$\int a^x dx = \frac{a^x}{\ln a} + k$	$\int a^{f(x)} f'(x) dx = \frac{a^{f(x)}}{\ln a} + k$
$\int \text{sen}x dx = -\cos x + k$	$\int \text{sen}(f(x)) f'(x) dx = -\cos(f(x)) + k$
$\int \cos x dx = \text{sen}x + k$	$\int \cos(f(x)) f'(x) dx = \text{sen}(f(x)) + k$
$\int \frac{1}{\cos^2 x} dx = \int (1 + tg^2 x) dx = tgx + k$	$\int \frac{f'(x)}{\cos^2(f(x))} dx = \int (1 + tg^2(f(x)) f'(x) dx = tg(f(x)) + k$
$\int \frac{1}{\text{sen}^2 x} dx = \int (1 + \text{cotg}^2 x) dx = -\text{ctg}x + k$	$\int \frac{f'(x)}{\text{sen}^2(f(x))} dx =$ $\int (1 + \text{ctg}^2(f(x)) f'(x) dx = -\text{ctg}(f(x)) + k$
$\int \frac{1}{\sqrt{1 - x^2}} dx = \text{arcsen}x + k$	$\int \frac{f'(x)}{\sqrt{1 - f^2(x)}} dx = \text{arcsen}(f(x)) + k$
$\int \frac{1}{1 + x^2} dx = \text{arctg}x + k$	$\int \frac{f'(x)}{1 + f^2(x)} dx = \text{arctg}(f(x)) + k$

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Propiedades y métodos de calcular

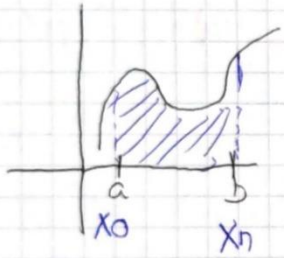
- <u>En la suma/resta</u>: La integral de la suma es la suma de las integrales: $\int (u + v)\, dx = \int u\, dx + \int v\, dx$

Integral indefinida de f

Def $\rightarrow \int f(x)\,dx = \{F(x) \,/\, F'(x) = f(x)\} = F(x) + c$

Integral definida

$f \geq 0$ en $[a,b]$



$$x_i = \frac{b-a}{n}$$

Suma inferior

$$L_n = \sum \min_{x \in [x_0, x_n]} f(x) \cdot (x_i - x_{i-1}) \leq \text{Area}$$

Suma Superior

$$U_n = \sum \max_{x \in [x_0, x_n]} f(x) \cdot (x_i - x_{i-1}) \geq \text{Area} = \int_a^b f(x)\,dx$$

primitiva	$F(x) = x^3 + 2x$	integrar ←	$F'(x) = f(x) = 3x^2 + 2$
primitiva	$F(x) + 1 = x^3 + 2x + 1$	derivar →	$(F(x)+1)' = f(x) = 3x^2 + 2$
primitiva	$F(x) + 2 = x^3 + 2x + 2$		$(F(x)+2)' = f(x) = 3x^2 + 2$
primitiva	$F(x) - 1 = x^3 + 2x - 1$		$(F(x)-1)' = f(x) = 3x^2 + 2$
⋮	⋮		

"INTEGRAL INDEFINIDA"

es ↓ representa

$$\boxed{F(x) + c = \int f(x)\,dx = \int F'(x)\,dx}$$

$$x^3 + 2x + c = \int (3x^2 + 2)\,dx$$

integral indef. — derivada

* propietats:

1. $\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$
2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

→ amb funcions compostes

tenim en compte la **REGLA** de la **CADENA**

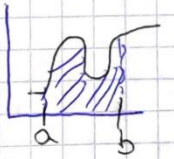
$$F[g(x)] + c = \int (F[g(x)])'\,dx = \int F'(g(x)) \cdot g'(x)\,dx$$

Exemple: $\int \underbrace{\cos x^2}_{\sin' x^2} \cdot \underbrace{2x}_{(x^2)'}\,dx = \sin(x^2) + c$

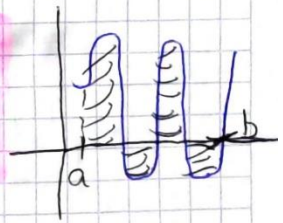
Def $\rightarrow$ $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$

$$\int_a^b f(x)\,dx = \lim L_n = \lim U_n$$

Def $\rightarrow$ $\int_a^b f(x)\,dx = \text{Area}$

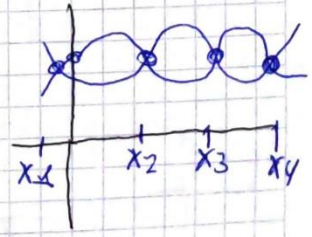


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$$



Area entre dos curvas (f, g)

$$f = g \begin{cases} x_1 \\ x_2 \\ x_3 \\ x_4 \end{cases}$$



El A_2

$$A = \int_{x_1}^{x_2} f - g + \int_{x_2}^{x_3} g - f + \dots$$

Teorema Fundamental del Calculo : TFC

$f: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

$F: \mathbb{R} \to \mathbb{R}$

$x \to \int_a^x f(t)\,dt$

- $F(x)$ continua en $[a,b]$ derivable en (a,b)
- $F'(x) = f(x)$

$$F(x) = \left(\int_a^x f(t)\,dt\right)' = f(x)$$

Regla de Barrow

$f: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

$F: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

F derivable en (a,b) y $F'(x) = f(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Teorema fonamental del Càlcul

Si f contínua en $[a,b]$
i $F(x) = \int_a^x f(t)\,dt$ $\Rightarrow$ F contínua en $[a,b]$
i és una primitiva de f: $F' = f$

Regla de Barrow

Si f contínua en $[a,b]$
i $F(x)$ contínua en $[a,b]$
i derivable en (a,b) tal que $F' = f$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a) \quad !$$

aplicació

Calcular integrals definides

Exemple

PAS 1. Calcular $\int f(x)\,dx = F$ indefinida

PAS 2. substituir $F(x=b)$
$F(x=a)$

PAS 3. restar $F(b) - F(a)$

ej: $\int_1^4 f(x)\,dx = \int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 =$ **PAS 1**

$= \frac{4^3}{3} - \frac{1^3}{3} =$ **PAS 2**

$= \frac{63}{3} = 21$ **PAS 3**

ej: $\int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \boxed{\frac{26}{3}}$

Corolari

$f: \mathbb{R} \to \mathbb{R}$ contínua en $[a,b]$

$$\left(\int_x^a f(x)\,dx\right)' = -f(x)$$

$$\left(\int_a^x f(x)\,dx\right)' = f(x)$$

Corolari

$f: \mathbb{R} \to \mathbb{R}$ contínua en $[a,b]$

$g_1, g_2: \mathbb{R} \to \mathbb{R}$ derivables en (a,b)

$$\left(\int_{g_2(x)}^{g_1(x)} f(t)\,dt\right)' = f(g_1(x)) \cdot g_1'(x) - f(g_2(x)) \cdot g_2'(x)$$

Corolari

$f: \mathbb{R} \to \mathbb{R}$ contínua en $[a,b]$

$g: \mathbb{R} \to \mathbb{R}$ derivable en (a,b)

$$\left(\int_a^{g(x)} f(t)\,dt\right)' = f(g(x)) \cdot g'(x)$$

2- b) $\lim_{x\to 0} \dfrac{x \cdot \int_0^x e^{t^2}\,dt}{\int_0^x e^{t^2}\cdot \sin t\,dt} = \dfrac{1\cdot \int_0^x e^{t^2}\,dt + x\cdot e^{x^2}}{e^{x^2}\cdot \sin x}$

$= \dfrac{e^{x^2} + 1\cdot e^{x^2} + x\cdot e^{x^2}\cdot 2x}{e^{x^2}\cdot 2x\cdot \sin x + e^{x^2}\cdot \cos x}$

$= 2$

3- $f:(0,+\infty) \to \{1\} \to \mathbb{R}$

$f(x) = \int_x^{x^2} \dfrac{dt}{\ln t}$

$f'(x) = \dfrac{1}{\ln x^2}\cdot 2x - \dfrac{1}{\ln x}\cdot 1 = \dfrac{2x}{2\ln x} - \dfrac{1}{\ln x} = \dfrac{x-1}{\ln x}$

$(0,1) \to >0$

$(1,+\infty) \to >0$

$$\text{Error} = \int_a^b f(x) - T_n = \frac{(b-a)^3}{12n^2}\cdot f^{n}(c)$$

Cota error $\to M_2 \geq \max f''(x) \to$

$$\boxed{\text{Error} \leq \frac{(b-a)^3}{12n^2}\cdot M_2}$$

$$\boxed{T_n = h\cdot\left(\frac{f(x_0)+f(x_n)}{2} + f(x_1) + f(x_2)\dots\right)}$$

f par si $f(-x) = f(x)$

f impar si $f(-x) = -f(x)$

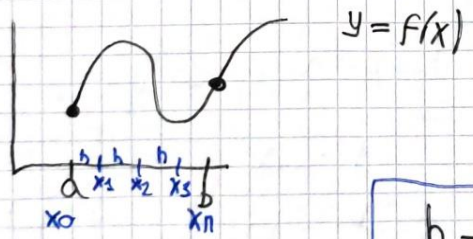
Integración numerica (Aprox integrales)

$f: \mathbb{R} \to \mathbb{R}$ continua en $[a,b]$

$f \geq 0$

Queremos calcular $\int_a^b f(x)$ con la precision deseada

Metodo de los Trapecios (T_n)



Partición de $[a,b]$

$\{ x_0 = a$

$$\boxed{A = \frac{f(x_{i-1})+f(x_i)}{2}\cdot h}$$

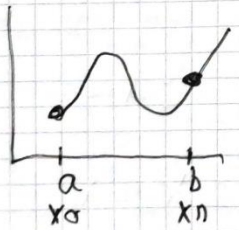
$$\boxed{h = \frac{b-a}{n}\ ;\ x_i = x_0 + i\cdot h}$$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$$\boxed{\int_a^b f(x) \simeq \sum_{i=1}^{n} \frac{h}{2}\cdot\left(f(x_i) + f(x_{i-1})\right)}$$

Metodo de Simpson (S_n)



Particion de $[a,b]$

$x_0 = a$

$x_i = x_0 + i \cdot h$ $\quad$ n par

$x_n = b$

$h = \frac{b-a}{n}$

$$A = \int_{x_{n-1}}^{x_{n+1}} P_2(x) = \frac{h}{3} \cdot \left[f(x_{i-1}) + 4 \cdot (f(x_i)) + f(x_{i+1}) \right]$$

$$\boxed{S_n = \frac{h}{3} \cdot \left[f(x_0) + 4 \cdot \left[\underbrace{f(x_1) + f(x_3) \dots}_{\text{impares}} \right] + 2 \cdot \left[\underbrace{f(x_2) + f(x_4)}_{\text{pares}} \right] + f(x_n) \right]}$$

$$\text{Error} = \int_a^b f - S_n = \frac{(b-a)^5}{180 \cdot n^4} \cdot f^4(c)$$

f 4 veces derivable en $[a,b]$

$M_4 \geq \max f^4(x)$

$$\boxed{\text{Error} \leq \frac{(b-a)^5}{180 \cdot n^4} \cdot M_4}$$

TEMA : INTEGRALES

Ej 7: Integrales inmediatas

D) $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{x^{3/2+1}}{3/2+1} = \frac{2}{5} x^{5/2} + K = \frac{2}{5}\sqrt{x^5} = \frac{2}{5} x^2 \sqrt{x} + K$

$\int [u(x)]^r dx = \frac{[u(x)]^{r+1}}{r+1} + K$

$\int \frac{1}{u(x)} \cdot u'(x)\,dx = \ln|u(x)| + K$

a) $\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{\boxed{1-2x+x^2}}{x^2}\,dx = \int \frac{dx}{x^2} - \int \frac{2}{x^2}\,dx + \int \frac{\cancel{x}}{\cancel{x}}\,dx = \int x^{-2}\,dx - 2\int \boxed{\frac{dx}{x}} + \int dx =$

Separar en 3 integrales

$= \frac{x^{-2+1}}{-2+1} - 2\ln|x| + x + K = -\frac{1}{x} - 2\ln|x| + x + K$

b) $\int \frac{x^3}{\boxed{x^4+1}}\,dx = \frac{1}{4}\int \frac{1}{x^4+1}\,4x^3\,dx = \frac{1}{4}\ln|x^4+1| + K$

NO SEPARAR

f) $\int x \cdot 5^{2x^2}\,dx = \int 5^{\boxed{2x^2}} x\,dx = \frac{1}{4}\int 5^{2x^2}\,4x\,dx = \frac{1}{4} \cdot \frac{1}{\ln 5} 5^{2x^2} + K = \frac{1}{4\ln 5} 5^{2x^2} + K$

$(2x^2)' = 4x$

$\int a^{u(x)} \cdot u'(x)\,dx = \frac{1}{\ln a} a^{u(x)}$

c) $\int \sqrt{\frac{\arcsin x}{1-x^2}}\,dx = \int \sqrt{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}\,dx = \int (\arcsin)^{1/2} \cdot \frac{1}{\sqrt{1-x^2}}\,dx =$

$= \frac{(\arcsin x)^{1/2+1}}{1/2+1} + K = \frac{2}{3}(\arcsin x)^{3/2} + K$ $\quad$ $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

e) $\int \frac{dx}{x \ln x} = \int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx = \int (\ln x)^{-1} \cdot \frac{1}{x}\,dx = \ln|\ln x| + K$

$u'(x) = (\ln x)' = \frac{1}{x}$

g) $\int \frac{1}{1+16x^2}\,dx = \int \frac{1}{1+(4x)^2} = \frac{1}{4}\int \frac{1}{1+(4x)^2}\,4\,dx = \frac{1}{4}\,\text{arctg}(4x) + K$

$\int \frac{1}{1+[u(x)]^2} \cdot u'(x)\,dx = \text{arctg}[u(x)] + K$

INTEGRALES

* INTEGRACIÓN NUMÈRICA → Ejs 12, 13, 14

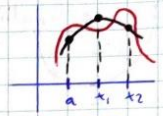
* TFC $\left.\begin{array}{l} 1\ f \text{ continua} \\ F(x)=\int_a^x f(t)\,dt \end{array}\right\} \Rightarrow$ F continua en $[a,b]$, F derivable (a,b), $F'(x)=f(x)$

Serveix per
→ Punt crític (x_0) de F $\quad F'(x_0)=0$
→ $\lim_x \frac{F=\int}{g(x)} = \text{indet} \overset{LH}{=} \frac{F'}{g'}$

MET TRAP

$f(x)$ → Polinomi 1r grau (RECTA)

MET SIMPSON



$f(x)$ → Polinomi 2n grau (PARABOLA)

INTEGRACIÓ NUMÈRICA

(TIPUS EXERCICIS):

TIPUS 1 → Donen número intervals
→ CALCULAR → Integral
→ Error

* Exercici 12

TIPUS 2 → Donen fita ERROR
→ CALCULAR → Nº intervals
→ Integral

* Exercici 13

Ex 12 a:

$\int_0^1 e^{x^2}\,dx = ?$ $\quad$ Donen $n=4$

$\boxed{h=\frac{b-a}{n}} = \frac{1-0}{4} = \frac{1}{4}$

$\boxed{x_i = a + ib} = 0 + i\cdot 1/4 = i/4$

(*) $f(x)=e^{x^2}$

$f'(x)=2xe^{x^2}$

$f''(x)=2e^{x^2}+2x\cdot 2xe^{x^2}=2e^{x^2}+4x^2e^{x^2}$ → ERROR TRAPEZIS

$f'''(x)=2\cdot 2xe^{x^2}+4\cdot 2x\cdot e^{x^2}+4x^2\cdot 2x\cdot e^{x^2}=12xe^{x^2}+8x^3e^{x^2}$

$f^{IV}(x)=12e^{x^2}+12\cdot 2x\cdot e^{x^2}+8\cdot 3x^2\cdot e^{x^2}+8x^3\cdot 2x\cdot e^{x^2}=12e^{x^2}+48x^2e^{x^2}+16x^4e^{x^2}$ → ERROR SIMPSON

E.TRAP $< \frac{(b-a)^3}{12n^2}\cdot \max_{[a,b]}|f''(x)|$ $\qquad$ E.SIMPSON $< \frac{(b-a)^5}{180n^4}\cdot \max_{[a,b]}|f^{IV}(x)|$

$f'''(x_0)=0 \rightsquigarrow 4xe^{x^2}\cdot(3+2x^2)=0$ → $\boxed{x_0=0}$; $3+2x^2=0 \to x_0=\pm\sqrt{-3/2} \notin \mathbb{R}$

h) $\int \tan^2 x\,dx = \int\left[\frac{\sin^2 x}{\cos^2 x}+1-1\right]dx = \int\left(\frac{\sin^2 x+\cos^2 x}{\cos^2 x}-1\right)dx \overset{\sin^2 x+\cos^2 x=1}{=} \underbrace{\int\left(\frac{1}{\cos^2 x}-1\right)dx}_{\text{Separar 2 integrals}} =$

$= \int\frac{dx}{\cos^2 x} - \int dx = \operatorname{tg} x - x + K$

$\int \frac{1}{\cos[u(x)]}\cdot u'(x)\,dx = \operatorname{tg} x + K$

Ej 8: Integrar per PARTS:

TEORIA

$\int[u(x)\cdot v(x)]' = \int u'(x)\cdot v(x) + \int u(x)\cdot v'(x)$

$u(x)\cdot v(x) = \int u'(x)\cdot v(x) + \boxed{\int u(x)\cdot v'(x)} \to \int u(x)\cdot v'(x) = u(x)\,v(x) - \int v(x)\cdot u'(x)$

d) $\int x\sin(2x)\,dx = x\cdot -\frac{1}{2}\cos(2x) - \int -\frac{1}{2}\cos(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{2\cdot 2}\int\cos(2x)\cdot 2\,dx =$

$u=x \longrightarrow du=dx$

$dv=\sin(2x)dx \to v=\int dv=\int\sin(2x)dx = -\frac{1}{2}\cos(2x)$

$= -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + K$

b) $\int\frac{\ln x}{\sqrt{x}}\,dx = \int \ln x\,\frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - \int 2\sqrt{x}\,\frac{dx}{x} = 2\sqrt{x}\ln x - 2\int x^{-1/2}dx = 2\sqrt{x}\ln x - 2\frac{x^{1/2}}{1/2} + K$

$u=\ln x \to du=\frac{dx}{x}$

$dv=\frac{dx}{\sqrt{x}} \to v=\int dv=\int\frac{dx}{\sqrt{x}}=\int x^{-1/2}dx=\frac{x^{1/2}}{1/2}=2\sqrt{x}$

$= 2\sqrt{x}\ln x - 4\sqrt{x} + K =$

$= 2\sqrt{x}(\ln x - 2) + K$

a) $\int e^{2x}\cdot\sin x\,dx \overset{①}{=} -e^{2x}\cos x - \int -\cos x\,2e^{2x}dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \overset{③}{=}$

① $u=e^{2x} \longrightarrow du=2e^{2x}dx$
$dv=\sin x\,dx \to v=\int dv=\int\sin x\,dx=-\cos x$

④ $u=\cos x$
$dv=e^{2x}dx$

② $u=\sin x$
$dv=e^{2x}dx$

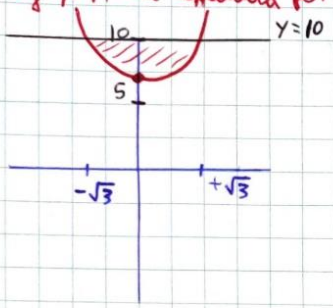
③ $u=e^{2x} \longrightarrow du=2e^{2x}dx$
$dv=\cos x \to v=\int dv=\int\cos x=-\sin x$

$\overset{③}{=} -e^{2x}\cos x + 2\left[e^{2x}\sin x - 2\underbrace{\int e^{2x}\cdot\sin x\,dx}_{A}\right]$

$A = -e^{2x}\cos x + 2e^{2x}\sin x - 4A \rightarrow A + 4A = -e^{2x}\cos x + 2e^{2x}$

$A = \frac{1}{5}e^{2x}\left[2\sin x - \cos x\right] + K$

Ej 9: Area limitada per la paràbola $y = x^2+7$ y la recta $y = 10$



PAS 1 → Abscisses dels PUNTS DE TALL

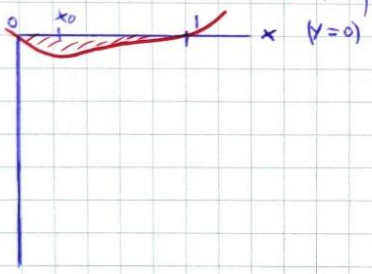
$x^2+7 = 10 \rightarrow x^2 = 3 \rightarrow x = \pm\sqrt{3}$

PAS 2 → AREA

$$\text{AREA} = \left|\int_{-\sqrt{3}}^{\sqrt{3}} \left[10 - (x+7)\right] dx\right| = -\int_{-\sqrt{3}}^{\sqrt{3}} x^2 dx + 3\int_{-\sqrt{3}}^{\sqrt{3}} dx = -\frac{x^3}{3}\Big|_{-\sqrt{3}}^{\sqrt{3}} + 3x\Big|_{-\sqrt{3}}^{\sqrt{3}} =$$

$$= -\left[\frac{(\sqrt{3})^3}{3} - \frac{(-\sqrt{3})^3}{3}\right] + \left[3\left[\sqrt{3} - (-\sqrt{3})\right]\right] = -\frac{2\sqrt{3}}{3} + 3\cdot 2\sqrt{3} = -\frac{2}{3}3\sqrt{3} + 6\sqrt{3} = 4\sqrt{3}$$ unitats area

Ej 11: Area limitada
- → Semieix positiu abscisses $\boxed{y=0}$
- → Corba $y = (x^2-x)e^{-x}$
- → 4t quadrant



PAS 1

$(x^2-x)e^{-x} = 0 \Leftrightarrow x^2 - x = 0$

$x(x-1) = 0 \rightarrow x = 0$, $x = 1$

Estudiem $y = (x^2-x)e^{-x}$ $\forall x \in (0,1) \rightarrow y(x) < 0$

$y' = (2x-1)e^{-x} - e^{-x}(x^2-x)$

$y'(x_0) = 0 \Leftrightarrow (-x_0^2 + 3x_0 - 1)e^{-x} = 0 \rightarrow$

$\rightarrow x_0 \simeq 2{,}62 \notin [0,1]$

$\rightarrow x_0 \simeq 0{,}38 \in [0,1]$

PAS 2 → $A = \int_0^1 \left[0 - (x^2-x)e^{-x}\right] dx = -\int_0^1 x^2 e^{-x} dx + \int_0^1 x e^{-x} dx$ ⇝ ① INTEGRAL INDEF ② BARROW → INTEGRAL DEF

Error TRAP → $\frac{(1-0)^3}{12\cdot 4^2}\cdot 6e = \frac{e}{32} = 0{,}084946$

$$\int_0^1 e^{x^2}dx \overset{\text{TRAP}}{\approx} h\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right] = h\left[\frac{f(a)+f(b)}{2} + f(x_1) + f(x_2) + f(x_3)\right] =$$

$$= \frac{1}{4}\left[\frac{e^0+e^1}{2} + e^{(1/4)^2} + e^{(2/4)^2} + e^{(3/4)^2}\right] \simeq 1{,}49067$$

ERROR SIMPSON → $f^{IV}(x)$ creixent en $[0,1]$ → $\frac{(1-0)^5}{180\cdot 4^4}\cdot 76e = 0{,}004483$

$$\int_0^1 e^{x^2}dx \overset{\text{SIMPSON}}{\approx} \frac{h}{3}\left[f(a)+f(b) + \overset{\text{impar}}{4f(x_1)} + \overset{\text{par}}{2f(x_2)} + \overset{\text{impar}}{4f(x_3)}\right] = \frac{1/4}{3}\left[e^0+e^1+4\cdot e^{(1/4)^2} + 2\cdot e^{(2/4)^2} + 4\cdot e^{(3/4)^2}\right] \simeq 1{,}46371$$

↳ 6 cifras correctes

Ex 13 b:

Simpson (Trapezis) $\int_0^1 \cos(x^2)\,dx$ Demen error $< 0{,}5\cdot 10^{-2} \to n = ?$

(*) $f(x) = \cos(x^2)$

$f'(x) = -2x \sin(x^2)$

$f''(x) = -2\sin(x^2) - 2x\cdot 2x\cos(x^2) = -2\sin(x^2) - 4x^2\cos(x^2)$

$f'''(x) = -2\cdot 2x\cos(x^2) - 4\cdot 2x\cos(x^2) - 4x^2\cdot 2x(-\sin(x^2)) = -12x\cos(x^2) + 8x^3\sin(x^2)$

$f^{IV}(x) = -12\cos(x^2) - 12x\cdot 2x(-\sin(x^2)) + 8\cdot 3x^2\sin(x^2) + 8x^3\cdot 2x\cos(x^2) =$

$= -12\cos(x^2) + 48x^2\sin(x^2) + 16x^4\cos(x^2)$

$$|f^{IV}(x)| = |\underbrace{16x^4\cos(x^2)}_{\geq 0 \text{ en } [a,b]} + \underbrace{48x^2\sin(x^2)}_{\geq 0} \underbrace{-12\cos(x^2)}_{<0}| < 16x^4\underbrace{\cos(x^2)}_{1} + 48x^2\underbrace{\sin(x^2)}_{1} < 16+48 = 64$$

$$\text{Error Simpson} = \frac{(b-a)^5}{180\,n^4}\max_{[0,1]}|f^{IV}(x)| = \frac{1}{180n^4}\,64 < 0{,}5\cdot 10^{-2} \to n \geq \sqrt[4]{\frac{32}{45}\cdot 10^2} = 2{,}90 \to 3$$

$\boxed{n=4}$ ← n = parell SIMPSON

$$\int_0^1 \cos(x^2)\,dx \overset{\text{SIMPSON}}{\simeq} \frac{h}{3}\left[f(a)+f(b)+4f(x_1)+2f(x_2)+4f(x_3)\right] =$$

$$= \frac{1/4}{3}\left[f(0)+f(1)+4f(1/4)+2f(2/4)+4f(3/4)\right] = \text{RADIANS} =$$

$$= \frac{1}{12}\left[\cos(0)+\cos(1)+4\cos\left[(1/4)^2\right]+2\cos\left[(2/4)^2\right]+4\cos\left[(3/4)^2\right] \simeq 0{,}90450 \to S(n=4) \simeq 0{,}905$$

TFC i f continua $[a,b]$

$F(x) = \int_0^x f \Downarrow$

$\boxed{\begin{array}{l} F \text{ continua } [a,b] \\ F \text{ derivable } (a,b) \end{array}}$

$\downarrow$

$F'(x) = f(x) \quad \forall x \in (a,b)$

Versió

$$F(x) = \int_{u(x)}^{v(x)} f(t)\,dt \to F'(x) = f(v(x))\cdot v'(x) - f(u(x))\cdot u'(x)$$

Ej 15:

$$F = \int_1^{x^2+2} \boxed{\frac{e^t}{t}}\,dt$$

$f = \frac{e^t}{t}$ continua $t \in \mathbb{R}-\{0\} \to$ continua $\forall x \in [1, x^2+2]$

$\xrightarrow[f \text{ continua}]{\text{TFC}}$ F < Continua, Derivable

$$F'(x) = \frac{e^{x^2+2}}{x^2+2}\cdot 2x - \frac{e^1}{1}\cdot 0$$

a) Comproveu que $\boxed{x=0}$ és un PUNT CRÍTIC de F

$$F'(x=0) = 0 \to \frac{e^{0+2}}{0+2}\,2\cdot 0 \overset{?}{=} 0 \to \frac{e^2}{2}\cdot 0 = 0$$ Com volíem demostrar

Examen 16/11/2010

$$\lim_{x\to 0^+} \frac{\int_0^{x^2}\sin(\sqrt{t})\,dt}{x^3} = \frac{0}{0} \overset{LH}{=} \lim_{x\to 0^+}\frac{F'}{3x^2} = \lim_{x\to 0^+}\frac{2x\sin x}{3x^2} = \lim_{x\to 0^+}\frac{2}{3}\boxed{\frac{\sin x}{x}} = \frac{2}{3}\cdot 1 = \frac{2}{3}$$

$F = \int_0^{x^2} f(t) = \sin(\sqrt{t})\,dt$

$f(t) = \sin(\sqrt{t})$ continua $\mathbb{R}$

$\xrightarrow[f \text{ continua}]{\text{TFC}}$ F < continua, derivable

$$F' = \sin(\sqrt{x^2})\cdot 2x - \cancel{\sin(\sqrt{0})}^{\,0}\cdot 0 = 2x\sin|x|$$