Greedy Design Technique

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    - the choice must lead to the feasible solution
      - ✓ expected to be an optimal solution

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- Feasible solutions: solutions that satisfy the constraints.
- Optimal solution: A feasible solution for which the optimization function has the best possible value.
- Greedy is way to construct a feasible solution for such optimization problems, and, sometimes, it leads to an optimal one.

The control abstraction

```
Algorithm Greedy(Type a[], int n)
solution = EMPTY
i=1;
for i = 1 to n
    do
        Type x = select(a);
        if feasible(solution, x)
            solution = solution U x;
    done
return solution
```

The Thirsty Baby problem

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- Based on her experience of taste and desire for nutrition, she also assigns certain satisfying factor, s_i, to the i^{th} liquid.
- If the baby needs to drink t ounces of liquid, how much of each liquid should she drink?

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- Subject constraint: $\sum_{i=1}^{n} x_i = t$ and for all $1 \leq i \leq n$, $o \leq x_i \leq a_i$
- We notice that if $\sum_{i=1}^{n} a_i < t$, then this instance is not solvable.

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  Such that $\sum_{i=1}^{n} s_i x_i$ is maximum optimization function.
- Every set of x_i that satisfies the constraints is a feasible solution, and it is optimal if it further maximizes $\sum_{i=1}^{n} s_i x_i$

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- Any irrevocability?

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- Let $x_i \in \{0, 1\}$. If $x_i = 1$, we will load the i^{th} container, otherwise, we will not load it.
- We wish to assign values to x_i's such that $\sum_{i=1}^{n} w_i \leq c$, and $\sum_{i=1}^{n} x_i$ is maximized.

Knapsack problem

Fractional Knapsack Problem

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- The thief wants to take as valuable a load as possible, but he can carry at most W pounds in his knapsack, for some integer W.
- Which items should he take?

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  - Illustration :
    - $W = 15$
    - $(p_1, p_2, p_3, p_4, p_5, p_6, p_7) = (10, 5, 15, 7, 6, 18, 3)$
    - $(w_1, w_2, w_3, w_4, w_5, w_6, w_7) = (2, 3, 5, 7, 1, 4, 1)$

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    - The solution set is $\left(1, 0, 1, \frac{4}{7}, 0, 1, 0\right)$ and profit $= 47$ units

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- **Greedy Criterion II :**
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      - The solution set is $\left(1, 1, \frac{4}{5}, 0, 1, 1, 1\right)$ and profit $= 54$ units

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      - The solution set is $\left(1,\ \frac{2}{3}, 1, 0, 1, 1, 1\right)$ and profit $= 55.33$ units

Communication and compression

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- This method requires $300{,}000$ bits to code the entire file.

Communication and compression …

- Can we optimize the amount of data to transfer?
  - Can we do better?
    - Use shorter strings for more frequent letters?

Communication and compression …

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    - Use pauses between letters to distinguish
      - Like an extra symbol in encoding

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  - here the 1-bit string 0 represents a, and the 4-bit string 1100 represents f. This code requires $(45 \times 1 \ + 13 \times 3 \ + 12 \times 3 + 16 \times 3 + 9 \times 4 + 5 \times 4) \times 1000 = 224{,}000$ bits.

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- Prefix code
  - The decoding process needs a convenient representation for the prefix code so that we can easily pick off the initial code word. A binary tree whose leaves are the given characters provides one such representation. We interpret the binary code word for a character as the simple path from the root to that character, where 0 means “go to the left child” and 1 means “go to the right child.” Figure shows the trees for the two codes of our example.

Communication and compression …

- Prefix code

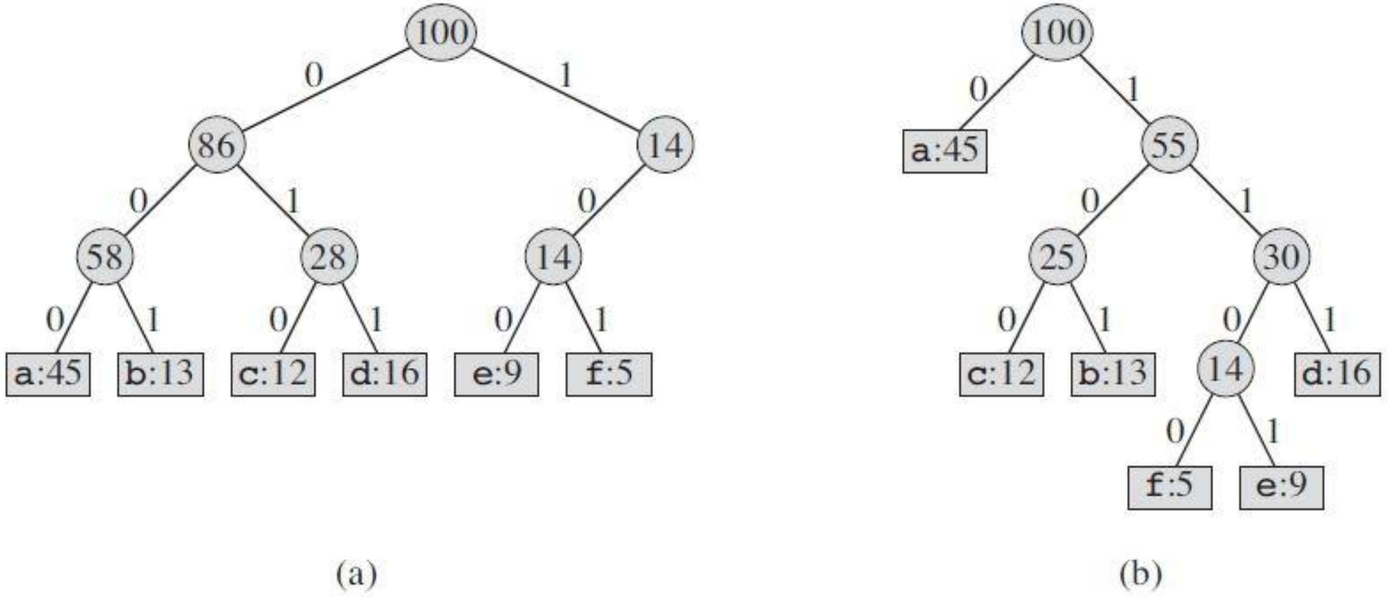

Figure Trees corresponding to the coding schemes in Figure . Each leaf is labeled with a character and its frequency of occurrence. Each internal node is labeled with the sum of the frequencies of the leaves in its subtree. **(a)** The tree corresponding to the fixed-length code a = 000, . . . , f = 101. **(b)** The tree corresponding to the optimal prefix code a = 0, b = 101, . . . , f = 1100.

Communication and compression …

- Prefix code
  - Given a tree T corresponding to a prefix code, we can easily compute the number of bits required to encode a file. For each character c in the alphabet C, let the attribute $c.freq$ denote the frequency of c in the file and let $d_T(c)$ denote the depth of c's leaf in the tree. Note that $d_T(c)$ is also the length of the code word for character c. The number of bits required to encode a file is thus

  $B(T) = \sum_{c \epsilon C} c.freq \, . d_T(c)$, which we define as the cost of tree T.

  - But, How to construct optimal prefix code?

Communication and compression …

- Huffman code

Communication and compression …

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  - Huffman invented a greedy algorithm that constructs an optimal prefix code called a ***Huffman code***.

Communication and compression …

- Huffman code
  - In the pseudo code that follows, we assume that C is a set of n characters and that each character $c \epsilon C$ is an object with an attribute $c.freq$ giving its frequency. The algorithm builds the tree T corresponding to the optimal code in a bottom-up manner. It begins with a set of $|C|$ leaves and performs a sequence of $|C| - 1$ “merging” operations to create the final tree. The algorithm uses a min-priority queue Q, keyed on the $freq$ attribute, to identify the two least-frequent objects to merge together. When we merge two objects, the result is a new object whose frequency is the sum of the frequencies of the two objects that were merged.

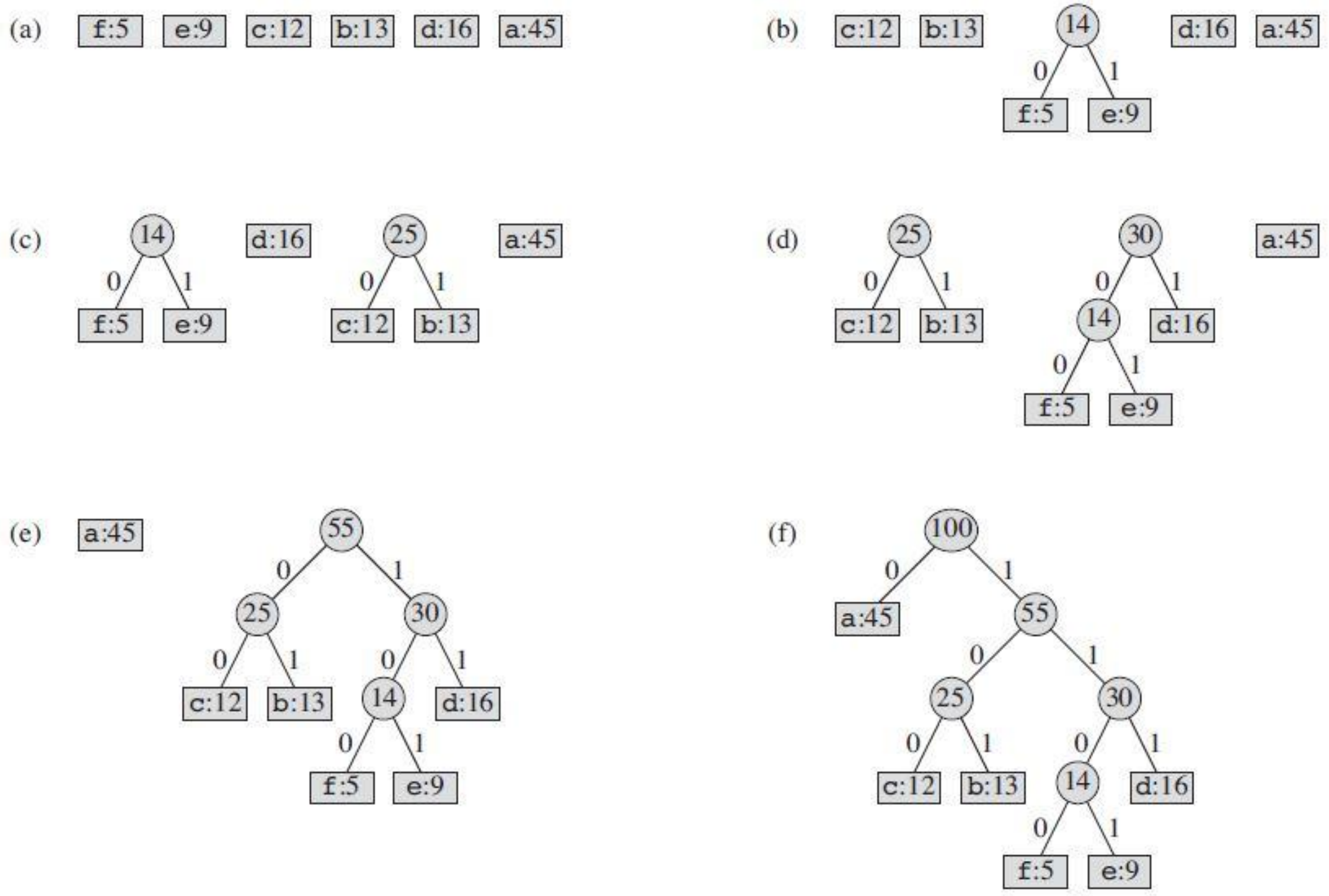
Communication and compression …

- Huffman code

```
HUFFMAN(C)
n = |C|
Q = C
for i = 1 to n − 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)      // return the root of the tree
```

Communication and compression …

- Huffman code



Communication and compression …

- Huffman's Algorithm Analysis
  - we assume that Q is implemented as a binary min-heap.
  - For a set C of n characters, we can initialize Q in line 2 in $O(n)$ time using the BUILD-MIN-HEAP procedure.
  - The for loop in lines $3-8$ executes exactly $n-1$ times, and since each heap operation requires time $O(\lg n)$, the loop contributes $O(n \lg n)$ to the running time.
  - Thus, the total running time of HUFFMAN on a set of n characters is $O(n \lg n)$.