1 Impedance Control

This document provides a rigorous mathematical derivation of a control-law for implementing an impedance controller.

1.1 Equation of Motion

The following equation of motion (EOM) describes how the actual, physical robot behaves in the real world:

$$M(q)\ddot{q} + C(\dot{q}, q) + H(\dot{q}, q) + G(q) + \tau_{\text{ext}} = \tau_{\text{cmd}} \tag{1}$$

More specifically, this behavior is determined by the robot's dynamics, including inertia $M(q)$, coriolis and centrifugal forces $C(\dot{q}, q)$, its inherent stiffness and friction $H(\dot{q}, q)$, and gravity $G(q)$. These dynamics are defined in configuration space and depend on the current joint angles q, joint velocities $\dot{q}$ and joint accelerations $\ddot{q}$. Furthermore, the system's behavior is affected by torques τ_{ext} that stem from contact forces between the robot and the environment, and by control torques τ_{cmd} which result from actuation.

1.2 Control-Law Partitioning

Since the control torque τ_{cmd} results from execution of control commands, we can define τ_{cmd} to be whatever we want. We define τ_{cmd} in a way so that it cancels out the dynamics of the physical system, and at the same time realizes an impedance to the environment. We can achieve this through control-law partitioning:

$$\tau_{\text{cmd}} = \alpha\, \ddot{q}^{\text{cmd}} + \beta \tag{2}$$

Here, $\ddot{q}^{\text{cmd}}$ denotes the *servo portion* of the control law which encodes the desired behavior of a unit-mass-system, and α and β denote the *model portion* which cancels out the dynamics of the actual, physical system. We define the terms α and β as follows:

$$\begin{aligned}
&\alpha = \hat{M}(q^{\text{fb}}) \\
&\beta = \hat{C}(\dot{q}^{\text{fb}}, q^{\text{fb}}) + \hat{H}(\dot{q}^{\text{fb}}, q^{\text{fb}}) + \hat{G}(q^{\text{fb}}) + \tau_{\text{ext}}^{\text{fb}}
\end{aligned}$$

The control torque τ_{cmd} is specified by a computer program. This term does not depend on the physical system in the real world, but instead, it depends on a model of this system which requires feedback information. Here, $\hat{M}$, $\hat{C}$, $\hat{H}$, and $\hat{G}$ refer to the model parameters relating to inertia, coriolis and centrifugal forces, inherent stiffness and friction, and gravity, respectively. Feedback on the robot's current joint position q^{fb} and velocity $\dot{q}^{\text{fb}}$ is provided by the robot's joint encoders, and feedback on contact torques $\tau_{\text{ext}}^{\text{fb}}$ is provided either by a force-torque sensor at the end effector (and mapped to the joint space via the Jacobian matrix) or by torque sensors at the joints.

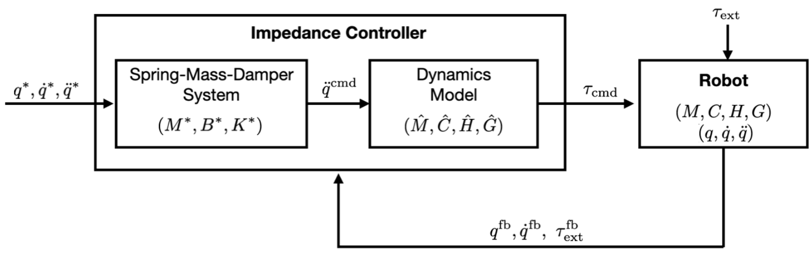

Abbildung 1: Flow diagram of impedance control: Based on model parameters, the impedance controller turns sensory feedback about the current state of robot and about contact forces/torques into control torques that let the robot follow a desired virtual trajectory while realizing a desired impedance, as specified by a spring-mass-damper system.

1.3 Closed-Loop Dynamics

By substituting the control torque τ_{cmd} in the equation of motion (Eq. 1) by the model-based control torque we specified above through control-law partitioning (Eq. 2), we obtain the closed-loop dynamics of the system:

$$\begin{aligned} & M(q)\ddot{q} + C(\dot{q}, q) + H(\dot{q}, q) + G(q) + \tau_{\text{ext}} \\ = & \hat{M}(q^{\text{fb}})\ddot{q}^{\text{cmd}} + \hat{C}(\dot{q}^{\text{fb}}, q^{\text{fb}}) + \hat{H}(\dot{q}^{\text{fb}}, q^{\text{fb}}) + \hat{G}(q^{\text{fb}}) + \tau_{\text{ext}}^{\text{fb}} \end{aligned}$$

We now make the assumption that our dynamics model accurately predicts the robot's behavior and that the sensory feedback on the current state of the robot and on the contact forces/torques is accurate. We therefore assume the following:

$$\begin{aligned} (\hat{M}, \hat{C}, \hat{H}, \hat{G}) &= (M, C, H, G) \\ (q^{\text{fb}}, \dot{q}^{\text{fb}}) &= (q, \dot{q}) \\ \tau_{\text{ext}}^{\text{fb}} &= \tau_{\text{ext}} \end{aligned}$$

Based on these assumptions, we can drastically simplify the closed loop dynamics and obtain a unit-mass-system which has no dynamic properties:

$$\ddot{q} = \ddot{q}^{\text{cmd}}$$

Thus, the model portion of the control-law canceled out the dynamics of the physical robot and the contact forces between the end effector and the environment. The behavior of the actual, physical system in the real world is now completely defined by this unit-mass-system.

Since $\ddot{q}^{\text{cmd}}$ is a component of the control torque $\tau_{\text{cmd}} = \alpha\ \ddot{q}^{\text{cmd}} + \beta$ which is specified by a computer program, we can choose $\ddot{q}^{\text{cmd}}$ to be whatever we want. Thus, we can choose the behavior of the actual, physical robot in the real world to be whatever we want, irrespective of its inherent dynamic properties.

1.4 Impedance Model

We want the robot to impose an impedance to the environment. Therefore, contact forces/torques between the robot's end effector and the environment should follow a spring-mass-damper system with desired inertia M^*, friction B^*, and stiffness K^*:

$$\tau_{\text{ext}}^{\text{fb}} = M^*(\ddot{q}^* - \ddot{q}^{\text{cmd}}) + B^*(\dot{q}^* - \dot{q}^{\text{fb}}) + K^*(x^* - q^{\text{fb}}) \tag{3}$$

Here, q^*, $\dot{q}^*$, and $\ddot{q}^*$ refer to the desired joint positions, velocities, and accelerations, respectively. These values are specified by a virtual trajectory. Furthermore, the impedance model requires sensory feedback on the current joint positions q^{fb} and velocities $\dot{q}^{\text{fb}}$, and on the contact force/torques $\tau_{\text{ext}}^{\text{fb}}$. We can reformulate this equation to obtain the *servo-portion* of the control-law partitioning:

$$\ddot{q}^{\text{cmd}} = \ddot{q}^* - (M^*)^{-1}[B^*(\dot{q}^{\text{fb}} - \dot{q}^*) + K^*(q^{\text{fb}} - q^*) + \tau_{\text{ext}}^{\text{fb}}] \tag{4}$$

We now have defined all components of the control-law based on control-law partitioning $\tau_{\text{cmd}} = \alpha\ \ddot{q}^{\text{cmd}} + \beta$, resulting in the following control torque:

$$\begin{aligned}\tau_{\text{cmd}} = \hat{M}(q^{\text{fb}})\{\ddot{q}^* - (M^*)^{-1}[B^*(\dot{q}^{\text{fb}} - \dot{q}^*) + K^*(q^{\text{fb}} - q^*) + \tau_{\text{ext}}^{\text{fb}}]\}\\ + \hat{C}(\dot{q}^{\text{fb}}, q^{\text{fb}}) + \hat{H}(\dot{q}^{\text{fb}}, q^{\text{fb}}) + \hat{G}(q^{\text{fb}}) + \tau_{\text{ext}}^{\text{fb}}\end{aligned}$$

When the actual, physical robot now interacts with the environment in the real world, the resulting contact forces/torques are no longer governed by the robot's inherent dynamics. Instead, they are determined by the model parameters M^*, B^*, and K^* of the spring-mass-damper system which realizes a desired impedance to the environment.

1.5 Trivial Example

Let's assume an actual, physical robot that for some reason has no dynamics. Thus, its behavior can be described by this very simple equation of motion:

$$\ddot{q} + \tau_{\text{ext}} = \tau_{\text{cmd}}$$

Now let's specify a control-law $\tau_{\text{cmd}} = \alpha\ \ddot{q}^{\text{cmd}} + \beta$ based on control-law partitioning, where the model portion α and β cancel out the (non-existent) dynamics, and the contact forces/torques. Therefore:

$$\begin{aligned}\alpha &= 1\\ \beta &= \tau_{\text{ext}}^{\text{fb}}\end{aligned}$$

We now want this robot (although it actually has no dynamics) to behave as if its end effector moves inside honey (let's say this corresponds to $B^* = 100$), has a mass of $M^* = 1000$ kg, and a stiffness of $K^* = 100$ (these are actually matrices and not just numbers, since the pose of the end effector has six dimensions). Thus, we define the servo portion $\ddot{q}^{\text{cmd}}$ of the control-law according to a spring-mass damper system (Eq. 4) which exhibits these desired dynamical properties:

$$\ddot{q}^{\text{cmd}} = \ddot{q}^* - (M^*)^{-1}[B^*(\dot{q}^{\text{fb}} - \dot{q}^*) + K^*(q^{\text{fb}} - q^*) + \tau_{\text{ext}}^{\text{fb}}]$$

Since now we have defined all components of the control-law partitioning, we obtain the following control-law:

$$\tau_{\text{cmd}} = \ddot{q}^* - (M^*)^{-1}[B^*(\dot{q}^{\text{fb}} - \dot{q}^*) + K^*(q^{\text{fb}} - q^*) + \tau_{\text{ext}}^{\text{fb}}] + \tau_{\text{ext}}^{\text{fb}}$$

We can verify that this control-law indeed realizes the desired impedance. To do to this, we plug this control torque back into the robot's original equation of motion and obtain the system's closed-loop dynamics:

$$\ddot{q} + \tau_{\text{ext}} = \ddot{q}^* - (M^*)^{-1}[B^*(\dot{q}^{\text{fb}} - \dot{q}^*) + K^*(q^{\text{fb}} - q^*) + \tau_{\text{ext}}^{\text{fb}}] + \tau_{\text{ext}}^{\text{fb}}$$

By assuming that the feedback on contact forces/torques $\tau_{\text{ext}}^{\text{fb}}$ is identical to the actual contact forces/torques τ_{ext}, we can reformulate the equation above and obtain the desired spring-mass-damper system:

$$M^*(\ddot{q}^* - \ddot{q}) + B^*(\dot{q}^* - \dot{q}^{\text{fb}}) + K^*(q^* - q^{\text{fb}}) = \tau_{\text{ext}}^{\text{fb}}$$