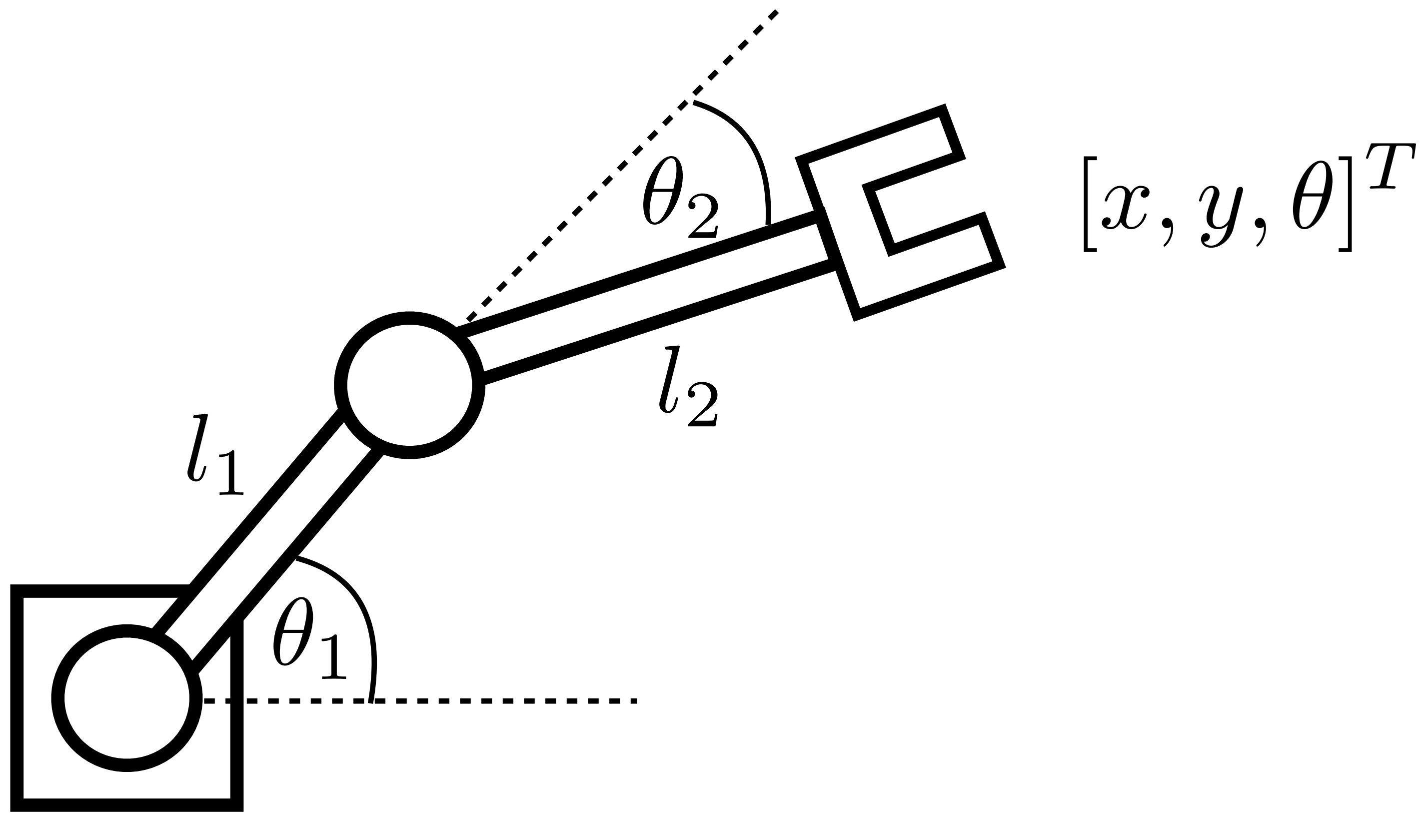
Jacobian Algorithm

One code to rule them all

Trigonometric functions

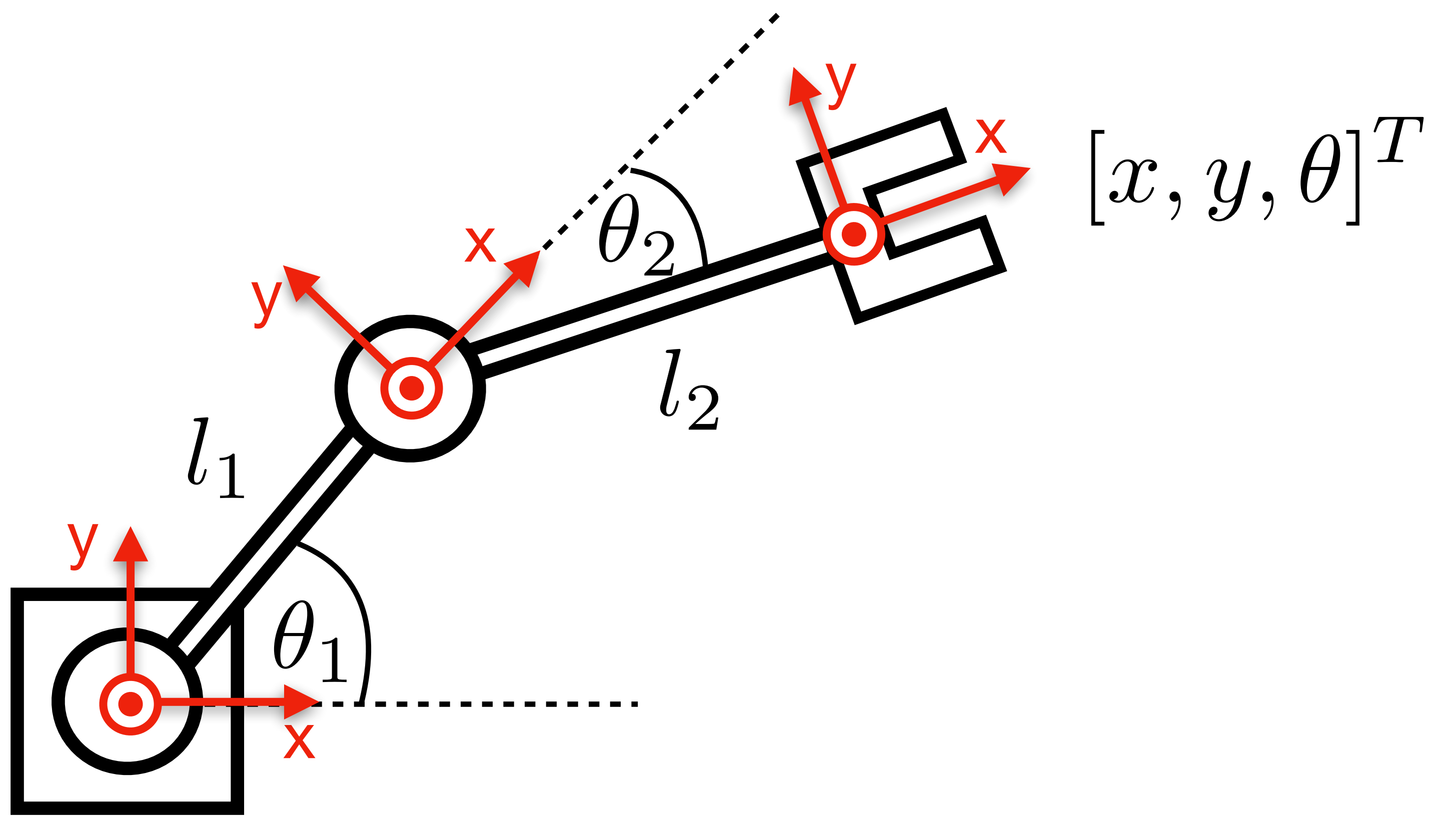


$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$
$$\theta = \theta_1 + \theta_2$$

$$J = \begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{bmatrix}$$

Good for simple (dummy) robots

Homogeneous transformations



1. Denavit-Hartenberg Parameters

	θ	α	r	d
1	θ_1	0	l_1	0
2	θ_2	0	l_2	0

2. Denavit-Hartenberg Matrices

$${}^{n-1}T_n = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n \cos\alpha_n & \sin\theta_n \sin\alpha_n & r_n \cos\theta_n \\ \sin\theta_n & \cos\theta_n \cos\alpha_n & -\cos\theta_n \sin\alpha_n & r_n \sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

3. Transformation from base frame to ee frame

$$T_{ee}^0 = T_1^0 T_2^1 \ldots T_{ee}^{n-1}$$

But how to compute J ????

Good for complex (most real) robots

$$\delta \mathbf{x} = f'(\mathbf{q})\ \delta \mathbf{q}$$

$$\delta \mathbf{x} = J\ \delta \mathbf{q}$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta \phi_x \\ \delta \phi_y \\ \delta \phi_z \end{bmatrix} = \frac{\partial f(\mathbf{q})}{\partial \mathbf{q}} \begin{bmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_3 \\ \vdots \\ \delta q_n \end{bmatrix}$$

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta \phi_x \\ \delta \phi_y \\ \delta \phi_z \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} & \cdots & \frac{\partial f_x}{\partial q_n} \\ \frac{\partial f_y}{\partial q_1} & \frac{\partial f_y}{\partial q_2} & \cdots & \frac{\partial f_y}{\partial q_n} \\ \frac{\partial f_z}{\partial q_1} & \frac{\partial f_z}{\partial q_2} & \cdots & \frac{\partial f_z}{\partial q_n} \\ \frac{\partial f_{\phi_x}}{\partial q_1} & \frac{\partial f_{\phi_x}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_x}}{\partial q_n} \\ \frac{\partial f_{\phi_y}}{\partial q_1} & \frac{\partial f_{\phi_y}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_y}}{\partial q_n} \\ \frac{\partial f_{\phi_z}}{\partial q_1} & \frac{\partial f_{\phi_z}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_z}}{\partial q_n} \end{bmatrix} \begin{bmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_3 \\ \vdots \\ \delta q_n \end{bmatrix}$$

$$
\begin{array}{cccc}
q_1 & q_2 & & q_n \\
\left[\begin{array}{cccc}
\frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} & \cdots & \frac{\partial f_x}{\partial q_n} \\
\frac{\partial f_y}{\partial q_1} & \frac{\partial f_y}{\partial q_2} & \cdots & \frac{\partial f_y}{\partial q_n} \\
\frac{\partial f_z}{\partial q_1} & \frac{\partial f_z}{\partial q_2} & \cdots & \frac{\partial f_z}{\partial q_n} \\
\frac{\partial f_{\phi_x}}{\partial q_1} & \frac{\partial f_{\phi_x}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_x}}{\partial q_n} \\
\frac{\partial f_{\phi_y}}{\partial q_1} & \frac{\partial f_{\phi_y}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_y}}{\partial q_n} \\
\frac{\partial f_{\phi_z}}{\partial q_1} & \frac{\partial f_{\phi_z}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_z}}{\partial q_n}
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c}
x \\[1.2em]
y \\[1.2em]
z \\[1.2em]
\phi_x \\[1.2em]
\phi_y \\[1.2em]
\phi_z
\end{array}
\begin{bmatrix}
\frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} & \cdots & \frac{\partial f_x}{\partial q_n} \\
\frac{\partial f_y}{\partial q_1} & \frac{\partial f_y}{\partial q_2} & \cdots & \frac{\partial f_y}{\partial q_n} \\
\frac{\partial f_z}{\partial q_1} & \frac{\partial f_z}{\partial q_2} & \cdots & \frac{\partial f_z}{\partial q_n} \\
\frac{\partial f_{\phi_x}}{\partial q_1} & \frac{\partial f_{\phi_x}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_x}}{\partial q_n} \\
\frac{\partial f_{\phi_y}}{\partial q_1} & \frac{\partial f_{\phi_y}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_y}}{\partial q_n} \\
\frac{\partial f_{\phi_z}}{\partial q_1} & \frac{\partial f_{\phi_z}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_z}}{\partial q_n}
\end{bmatrix}
$$

End effector position $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

End effector orientation $\begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}$

$$
\begin{bmatrix}
\frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} & \cdots & \frac{\partial f_x}{\partial q_n} \\
\frac{\partial f_y}{\partial q_1} & \frac{\partial f_y}{\partial q_2} & \cdots & \frac{\partial f_y}{\partial q_n} \\
\frac{\partial f_z}{\partial q_1} & \frac{\partial f_z}{\partial q_2} & \cdots & \frac{\partial f_z}{\partial q_n} \\
\frac{\partial f_{\phi_x}}{\partial q_1} & \frac{\partial f_{\phi_x}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_x}}{\partial q_n} \\
\frac{\partial f_{\phi_y}}{\partial q_1} & \frac{\partial f_{\phi_y}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_y}}{\partial q_n} \\
\frac{\partial f_{\phi_z}}{\partial q_1} & \frac{\partial f_{\phi_z}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_z}}{\partial q_n}
\end{bmatrix}
$$

$$
\begin{array}{cc}
 & \begin{matrix} q_1 & \quad q_2 & \quad & \quad q_n \end{matrix} \\
\begin{matrix} \text{End effector position} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \\ \text{End effector orientation} \quad \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} \end{matrix} &
\begin{bmatrix}
\frac{\partial f_x}{\partial q_1} & \frac{\partial f_x}{\partial q_2} & \cdots & \frac{\partial f_x}{\partial q_n} \\
\frac{\partial f_y}{\partial q_1} & \frac{\partial f_y}{\partial q_2} & \cdots & \frac{\partial f_y}{\partial q_n} \\
\frac{\partial f_z}{\partial q_1} & \frac{\partial f_z}{\partial q_2} & \cdots & \frac{\partial f_z}{\partial q_n} \\
\frac{\partial f_{\phi_x}}{\partial q_1} & \frac{\partial f_{\phi_x}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_x}}{\partial q_n} \\
\frac{\partial f_{\phi_y}}{\partial q_1} & \frac{\partial f_{\phi_y}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_y}}{\partial q_n} \\
\frac{\partial f_{\phi_z}}{\partial q_1} & \frac{\partial f_{\phi_z}}{\partial q_2} & \cdots & \frac{\partial f_{\phi_z}}{\partial q_n}
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{rc}
 & q_i \\
\begin{array}{r} \\ \text{End effector position} \\ \\ \\ \text{End effector orientation} \\ \\ \end{array} &
\begin{bmatrix}
 & \dfrac{\partial f_x}{\partial q_i} & \\
\cdots & \dfrac{\partial f_y}{\partial q_i} & \cdots \\
 & \dfrac{\partial f_z}{\partial q_i} & \\
 & \dfrac{\partial f_{\phi_x}}{\partial q_i} & \\
\cdots & \dfrac{\partial f_{\phi_y}}{\partial q_i} & \cdots \\
 & \dfrac{\partial f_{\phi_z}}{\partial q_i} &
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{rc}
 & \begin{array}{ccc} & q_i & \end{array} \\
\begin{array}{r} \\ \text{End effector position} \\ \\ \\ \text{End effector orientation} \\ \\ \end{array} &
\left[\begin{array}{ccc}
 & \frac{\partial f_x}{\partial q_i} & \\
\cdots & \frac{\partial f_y}{\partial q_i} & \cdots \\
 & \frac{\partial f_z}{\partial q_i} & \\
 & \frac{\partial f_{\phi_x}}{\partial q_i} & \\
\cdots & \frac{\partial f_{\phi_y}}{\partial q_i} & \cdots \\
 & \frac{\partial f_{\phi_z}}{\partial q_i} &
\end{array}\right]
\end{array}
$$

Can we compute these
automatically
for arbitrary robots?

YES! :)

$$q_i$$

$$\begin{bmatrix} \cdots & \boxed{\begin{matrix} \dfrac{\partial f_x}{\partial q_i} \\ \dfrac{\partial f_y}{\partial q_i} \\ \dfrac{\partial f_z}{\partial q_i} \end{matrix}} & \cdots \\ \cdots & \boxed{\begin{matrix} \dfrac{\partial f_{\phi_x}}{\partial q_i} \\ \dfrac{\partial f_{\phi_y}}{\partial q_i} \\ \dfrac{\partial f_{\phi_z}}{\partial q_i} \end{matrix}} & \cdots \end{bmatrix}$$

FOR PRISMATIC JOINTS

$$R^0_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

FOR REVOLUTE JOINTS

(cross product)

$$R^0_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d^0_n - d^0_{i-1})$$

(n = no. joints

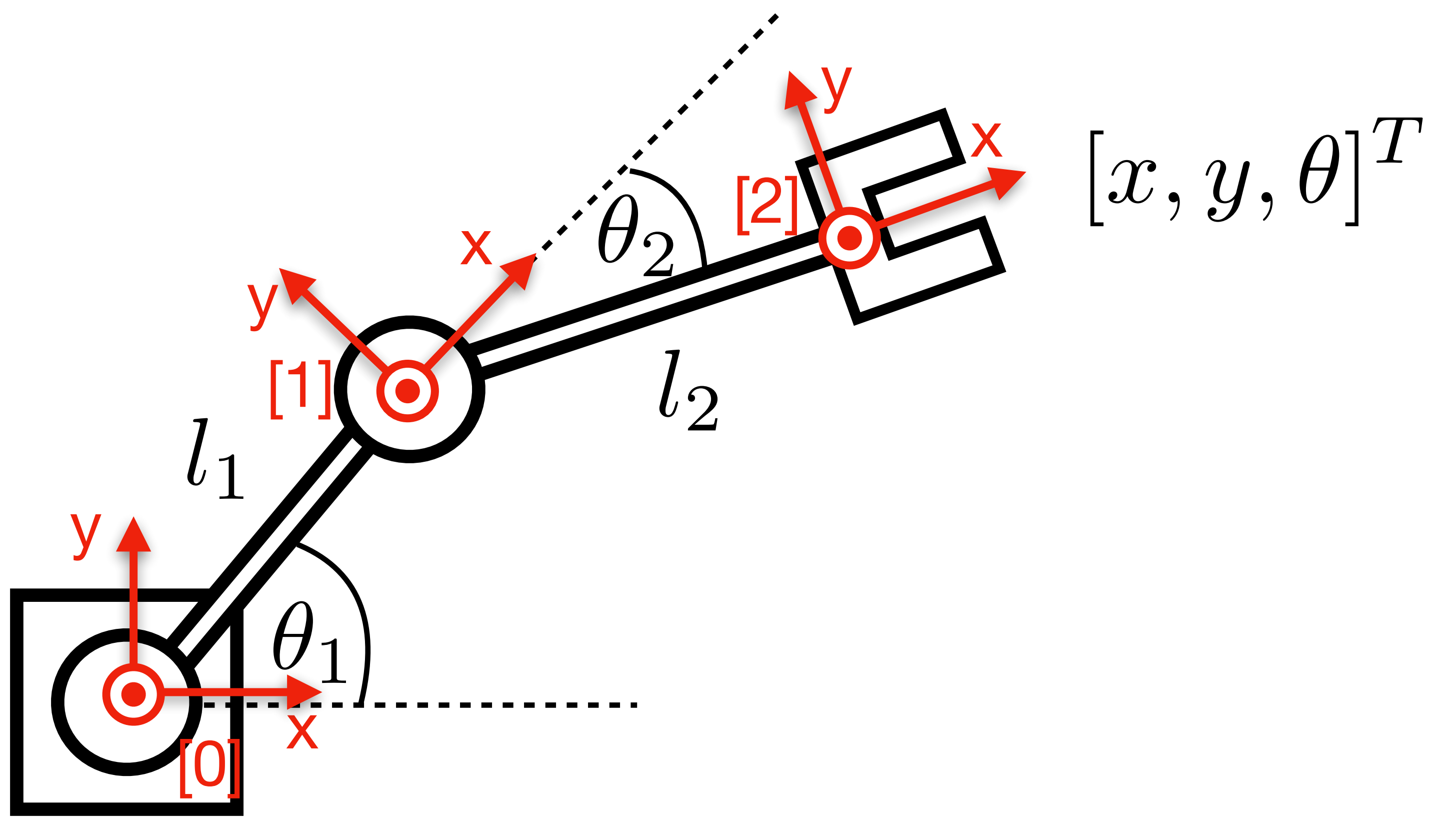
Remember:

$$T^0_i = \begin{bmatrix} \boxed{\quad R^0_i \quad} & \boxed{d^0_i} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$T^0_i = T^0_0 T^0_1 T^1_2 \dots T^{i-1}_i$$

$$T^0_0 = I$$

$$R^0_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



cross product

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Denavit-Hartenberg Parameters

	θ	α	r	d
1	θ_1	0	l_1	0
2	θ_2	0	l_2	0

Denavit-Hartenberg Matrices

$$^{n-1}T_n = \left[\begin{array}{ccc|c} \cos\theta_n & -\sin\theta_n \cos\alpha_n & \sin\theta_n \sin\alpha_n & r_n \cos\theta_n \\ \sin\theta_n & \cos\theta_n \cos\alpha_n & -\cos\theta_n \sin\alpha_n & r_n \sin\theta_n \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T_0^0 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad T_1^0 = \left[\begin{array}{ccc|c} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$T_2^0 = T_1^0 T_2^1$$

$$T_2^0 = \left[\begin{array}{ccc|c} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{122} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ J = \end{array} \begin{array}{c} q_i \\ \left[\begin{array}{c} \boxed{\begin{array}{c} \frac{\partial f_x}{\partial q_i} \\ \frac{\partial f_y}{\partial q_i} \\ \frac{\partial f_z}{\partial q_i} \end{array}} \\ \boxed{\begin{array}{c} \frac{\partial f_{\phi_x}}{\partial q_i} \\ \frac{\partial f_{\phi_y}}{\partial q_i} \\ \frac{\partial f_{\phi_z}}{\partial q_i} \end{array}} \end{array} \right] \end{array}$$

(cross product)

$$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n^0 - d_{i-1}^0)$$

(n = no. joints)

$$T_i^0 = \left[\begin{array}{ccc|c} & R_i^0 & & d_i^0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

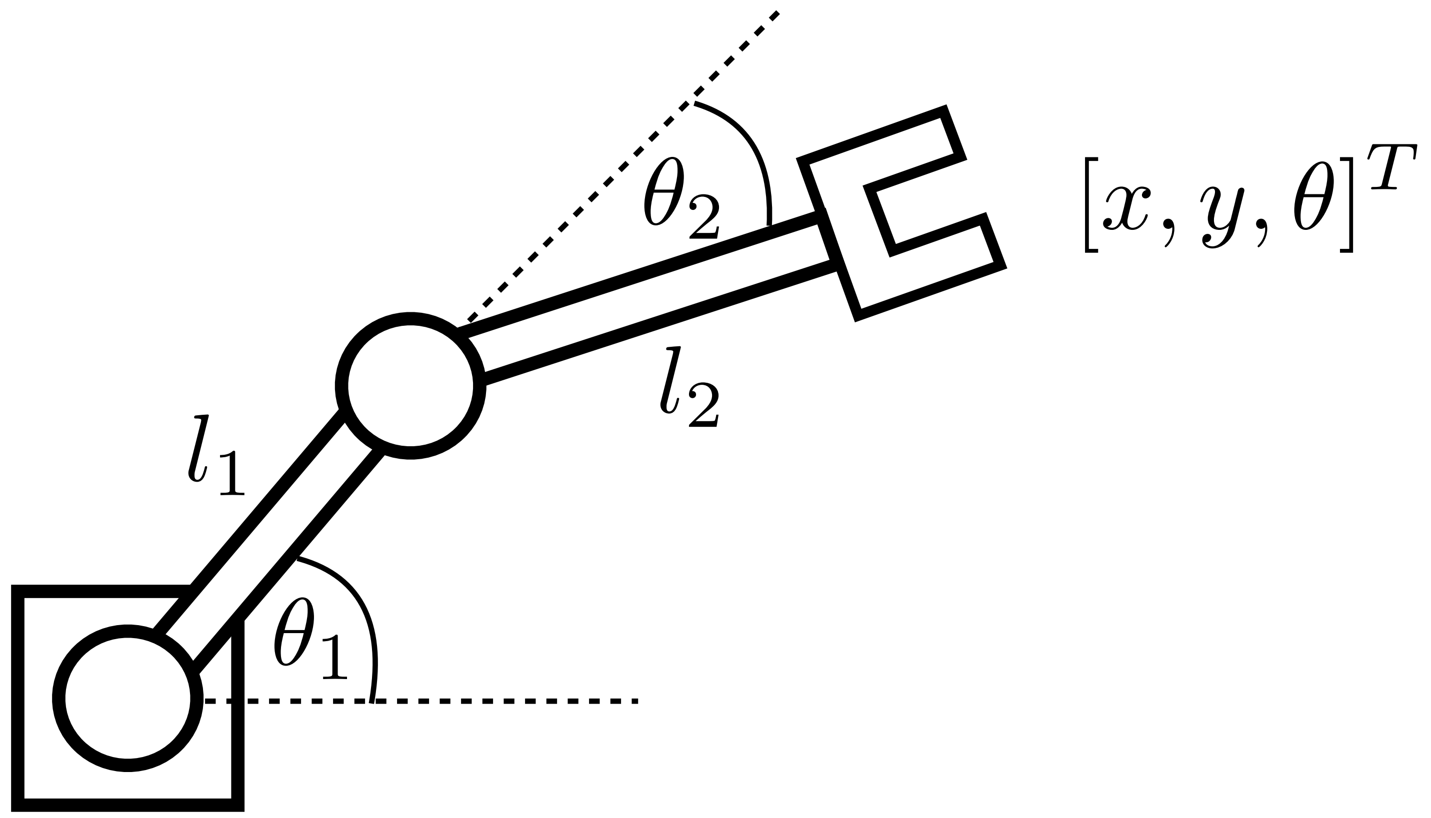
$$T_i^0 = T_0^0 T_1^0 T_2^1 \ldots T_i^{i-1}$$

$$T_0^0 = I$$

$$R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

FOR REVOLUTE JOINTS

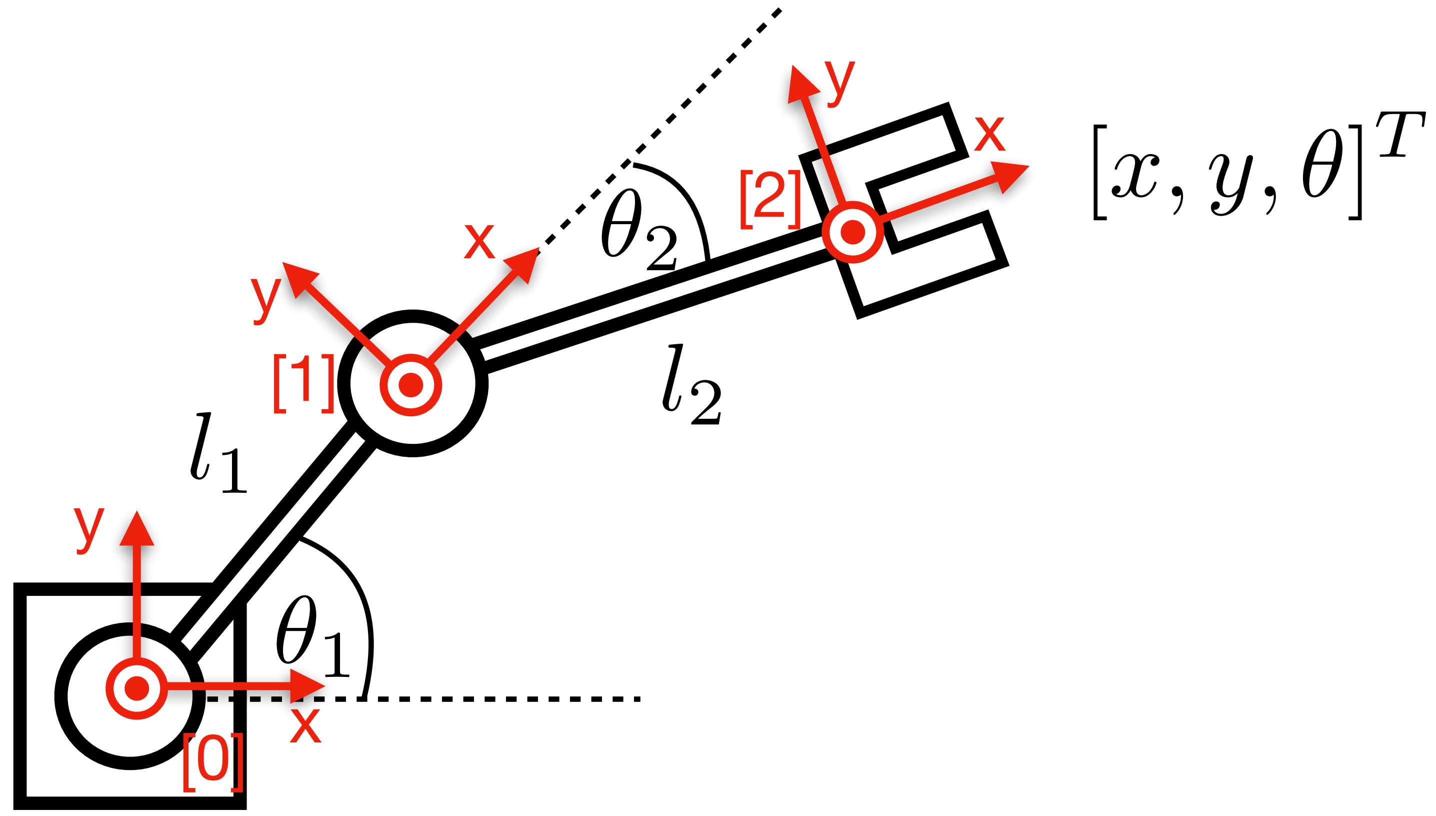
Trigonometric functions



$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$
$$\theta = \theta_1 + \theta_2$$

$$J = \begin{array}{c} \\ \end{array}\begin{bmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{bmatrix} \begin{array}{c} x \\ y \\ \phi_z \end{array}$$

Homogeneous transformations



$$J = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{array}{c} x \\ y \\ z \\ \phi_x \\ \phi_y \\ \phi_z \end{array}$$

Jacobian Algorithm

Needs to be done manually — 1. Create DH parameter table

Can be automated:

2. Derive DH matrices from DH parameter table
3. Derive transformations T_i^0 from DH matrices (for $i \in [0, n]$)
4. Derive $J_T(q_i)$ and $J_R(q_i)$ from transformations, with:

if joint i is prismatic:

$$J_T(q_i) = R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_R(q_i) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if joint i is revolute:

$$J_T(q_i) = R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d_n^0 - d_{i-1}^0)$$

$$J_R(q_i) = R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Why this works:

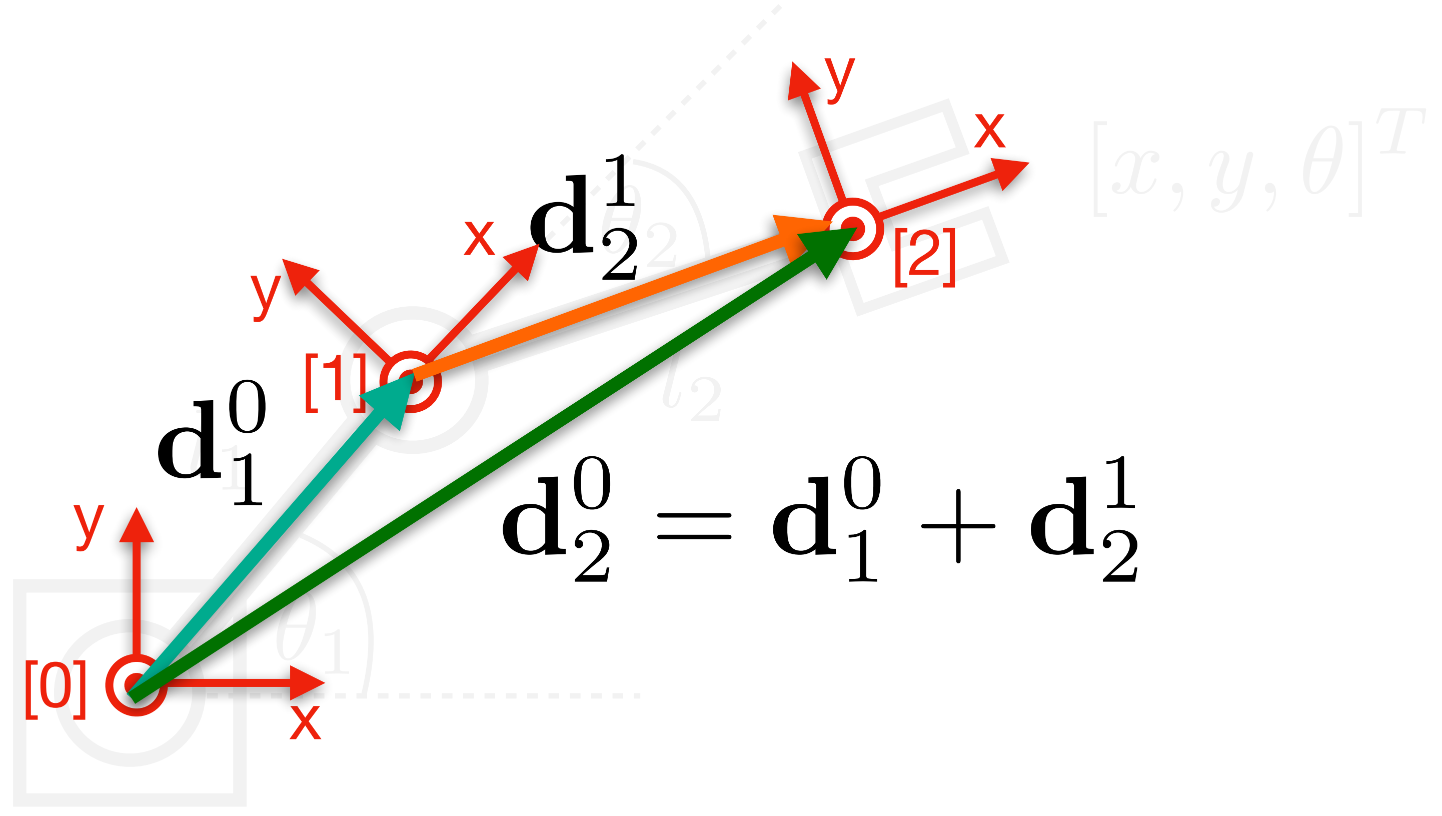
$$\mathbf{J}(q_2)$$
$$i = 2$$

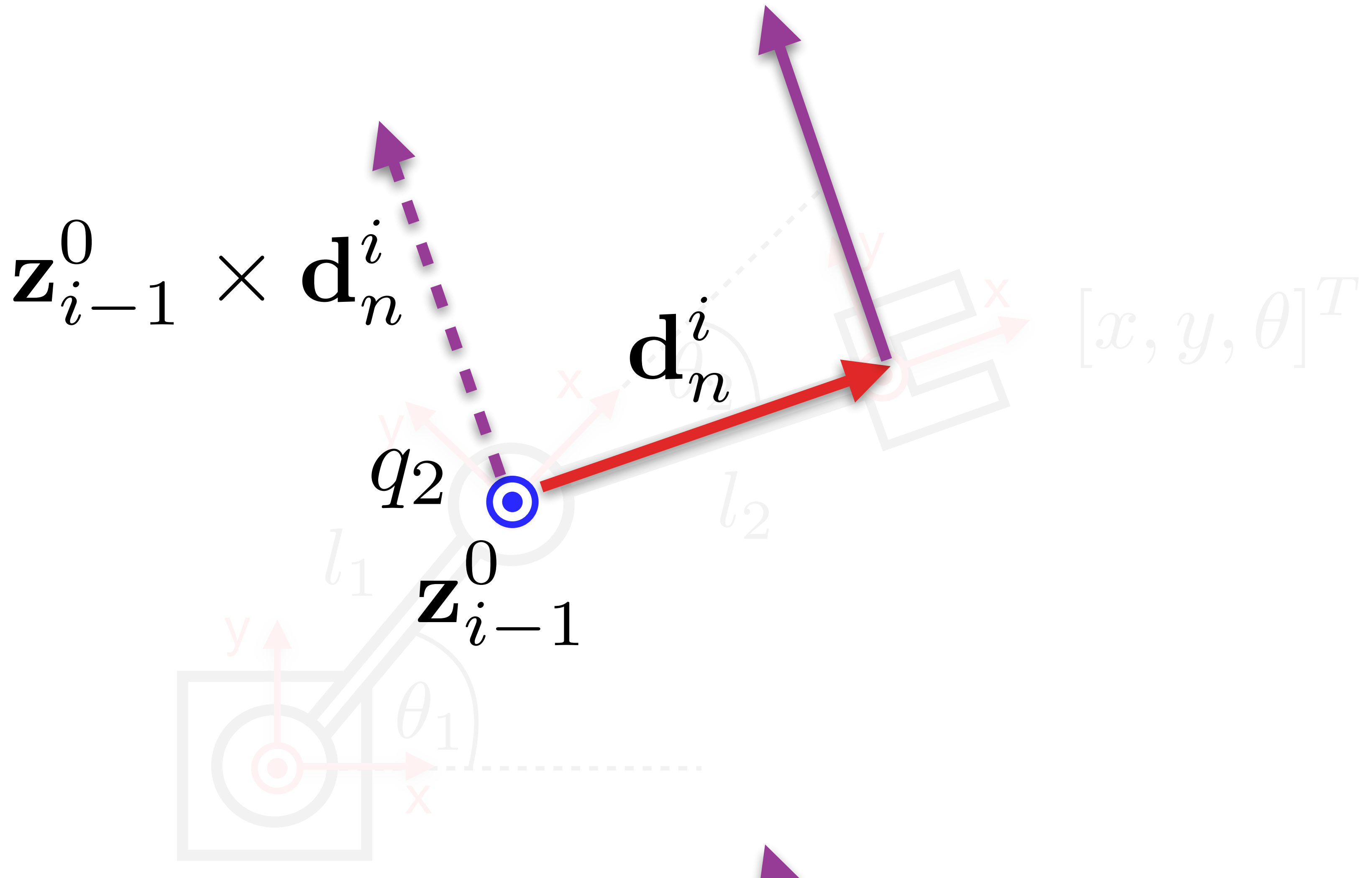
$$\overbrace{R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix}}^{\mathbf{z}^0_{i-1}} \times \overbrace{(d^0_n - d^0_{i-1})}^{\mathbf{d}^i_n}$$

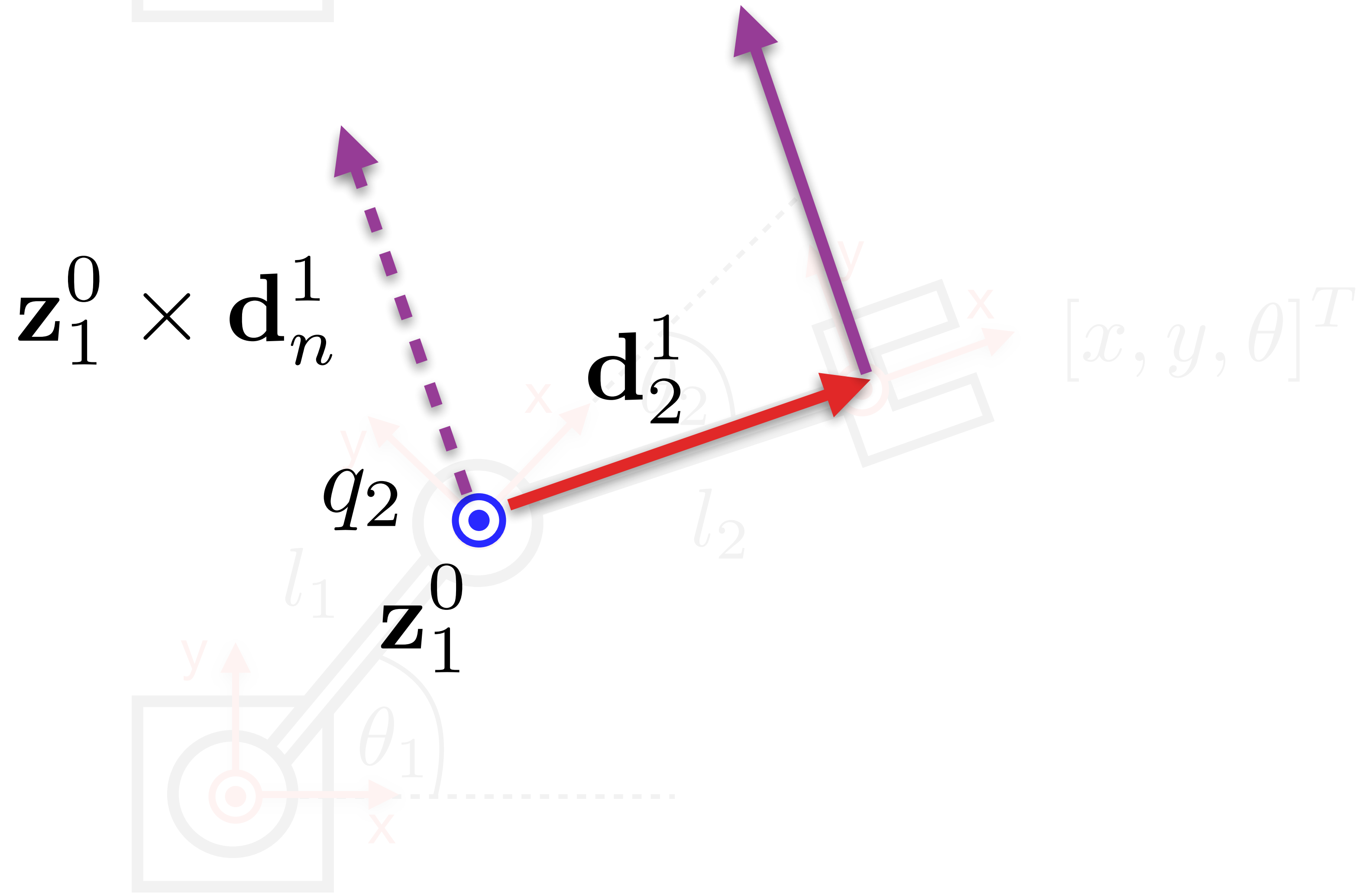
Extract from homogeneous transformation

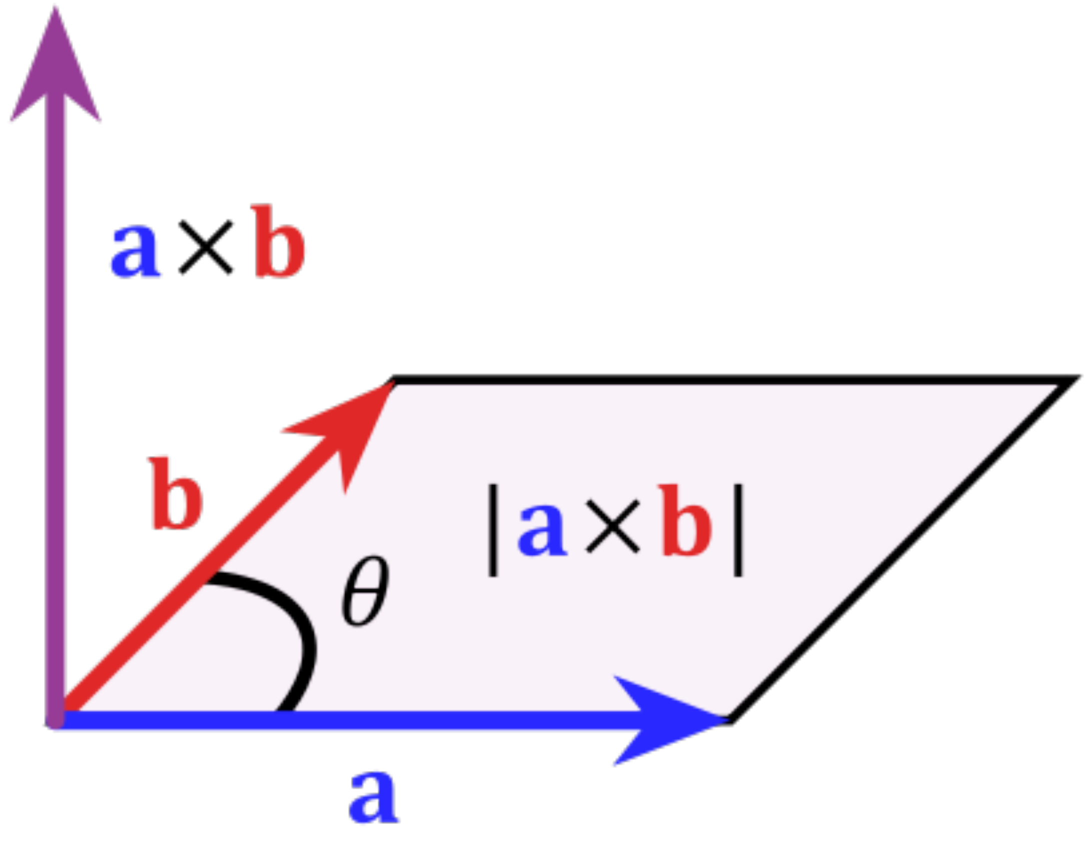
$$T^0_i = \begin{bmatrix} & R^0_i & & d^0_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^0_i = \begin{bmatrix} \mathbf{x}^0_i & \mathbf{y}^0_i & \mathbf{z}^0_i & d^0_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$









cross product

Happy Coding!

$$
\begin{array}{c}
\quad q_i \quad \\
\left[\begin{array}{ccc}
\cdots & \boxed{\begin{array}{c} \dfrac{\partial f_x}{\partial q_i} \\ \dfrac{\partial f_y}{\partial q_i} \\ \dfrac{\partial f_z}{\partial q_i} \end{array}} & \cdots \\
\cdots & \boxed{\begin{array}{c} \dfrac{\partial f_{\phi_x}}{\partial q_i} \\ \dfrac{\partial f_{\phi_y}}{\partial q_i} \\ \dfrac{\partial f_{\phi_z}}{\partial q_i} \end{array}} & \cdots
\end{array}\right]
\end{array}
$$

FOR PRISMATIC JOINTS

$$R^0_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

FOR REVOLUTE JOINTS

(cross product)

$$R^0_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times (d^0_n - d^0_{i-1})$$

(n = no. joints)

Remember:

$$T^0_i = \begin{bmatrix} \boxed{\quad R^0_i \quad} & \boxed{d^0_i} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$T^0_i = T^0_0 T^0_1 T^1_2 \dots T^{i-1}_i$$

$$T^0_0 = I$$

$$R^0_{i-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$