Exercise 1: Markov Model Forward Problem (20 P)

A Markov Model can be seen as a joint distribution over states at each time step $q_1, \dots, q_T$ where $q_t \in \{S_1, \dots, S_N\}$, and where the probability distribution has the factored structure:

$$(1) \qquad P(q_1, \dots, q_T) = P(q_1) \cdot \prod_{t=2}^{T} P(q_t | q_{t-1})$$

$$q_1 \to q_2 \to q_3 \to \cdots \to q_{t-1} \to q_t \to q_{t+1}$$

Factors are the probability of the initial state and conditional distributions at every time step.

(a) *Show* that the following relation holds:

$$P(q_{t+1} = S_j) = \sum_{i=1}^{N} P(q_t = S_i) P(q_{t+1} = S_j | q_t = S_i)$$

for $t \in \{1, \dots, T-1\}$ and $j \in \{1, \dots, N\}$.

$$P(q_{t+1} = S_j) = \sum_{q_1}\sum_{q_2}\cdots\sum_{q_{t-1}}\sum_{q_t} p(q_1, \dots, q_t, q_{t+1} = S_j)$$

$$\overset{(1)}{=} \sum_{q_1}\sum_{q_2}\cdots\sum_{q_{t-1}}\sum_{q_t} p(q_1) \cdot p(q_2|q_1) \cdot p(q_3|q_2) \cdots p(q_{t+1} = S_j | q_t)$$

$$= \sum_{q_t} p(q_{t+1} = S_j | q_t) \cdot \underbrace{\sum_{q_1} \cdots \sum_{q_{t-1}} p(q_1, \dots q_t)}_{p(q_t)}$$

Exercise 2: Hidden Markov Model Forward Problem (20 P)

A Hidden Markov Model (HMM) can be seen as a joint distribution over hidden states $q_1, \dots, q_T$ at each time step and corresponding observation $O_1, \dots, O_T$. Like for the Markov Model, we have $q_t \in \{S_1, \dots, S_N\}$. The probability distribution of the HMM has the factored structure:

$$(2) \quad P(q_1, \dots, q_T, O_1, \dots, O_T) = P(q_1) \cdot \prod_{t=2}^{T} P(q_t | q_{t-1}) \cdot \prod_{t=1}^{T} P(O_t | q_t)$$

$$\begin{array}{ccccccccccccc} q_1 & \to & q_2 & \to & q_3 & \to & \cdots & \to & q_{t-1} & \to & q_t & \to & q_{t+1} \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ O_1 & & O_2 & & O_3 & & & & O_{t-1} & & O_t & & O_{t+1} \end{array}$$

Factors are the probability of the initial state and conditional distributions at every time step.

(a) *Show* that the following relation holds:

$$P(O_1, \dots, O_t, O_{t+1}, q_{t+1} = S_j) = \sum_{i=1}^{N} P(O_1, \dots, O_t, q_t = S_i) P(q_{t+1} = S_j | q_t = S_i) P(O_{t+1} | q_{t+1} = S_j)$$

for $t \in \{1, \dots, T-1\}$ and $j \in \{1, \dots, N\}$. (2) + marginalization

$$p(O_1, \dots O_{t+1}, q_{t+1} = S_j) = \sum_{q_1}\sum_{q_2}\cdots\sum_{q_t} p(q_1) p(O_1|q_1) \cdot p(q_2|q_1) \cdot p(O_2|q_2) \cdots p(q_{t+1} = S_j | q_t) \cdot p(O_{t+1} | q_{t+1} = S_j)$$

$$= \sum_{q_t} p(q_{t+1} = S_j | q_t) p(O_{t+1} | q_{t+1} = S_j) \cdot \sum_{q_1}\sum_{q_2}\cdots\sum_{q_{t-1}} p(q_1) p(O_1|q_1) \cdots p(q_t|q_{t-1}) p(O_t|q_t)$$

$$= \sum_{q_1} p(q_{t+1} = S_j | q_t) p(O_{t+1} | q_{t+1} = S_j) \cdot p(O_1, \dots O_t, q_t)$$