Ex 12

①

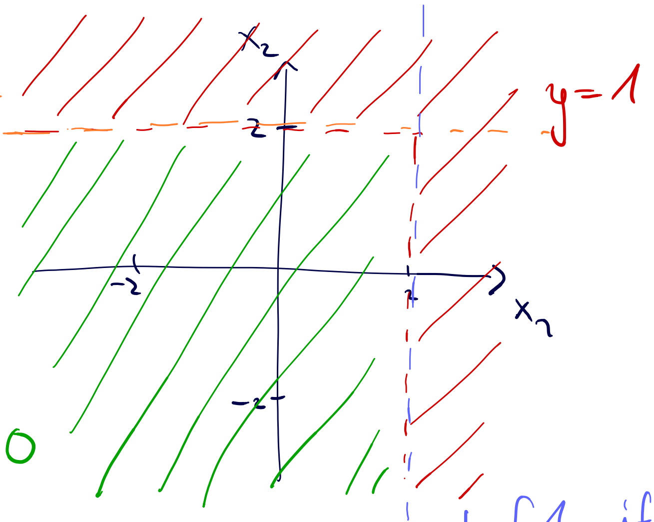
Neurons of type $a_j = \sigma\left(b_j + \sum_i a_i w_{ij}\right)$

with $\sigma(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$

a)

$$y = \begin{cases} 1 & \text{if } \max(x_1, x_2) > 2 \\ 0 & \text{if } \max(x_1, x_2) < 2 \end{cases}$$

$a_2 \overset{!}{=} \begin{cases} 1 & \text{if } x_2 > 2 \\ 0 & \text{else} \end{cases}$



$y = 0$

$a_1 \overset{!}{=} \begin{cases} 1 & \text{if } x_1 > 2 \\ 0 & \text{else} \end{cases}$

$a_2 = \sigma\left(b_2 + \sum_i x_i w_{i2}\right)$

analogous to a_1:

$b_2 = -2$, $w_{12} = 0$, $w_{22} = 1$

$\Rightarrow a_2 = \sigma(x_2 - 2)$

$a_1 = \sigma\left(b_1 + \sum_i x_i w_{i1}\right) \overset{!}{=} \begin{cases} 1 & \text{if } x_1 > 2 \\ 0 & \text{else} \end{cases}$

$\iff b_1 + \sum_i x_i w_{i1} \begin{cases} > 0 & \text{if } x_1 > 2 \\ \leq 0 & \text{else} \end{cases}$

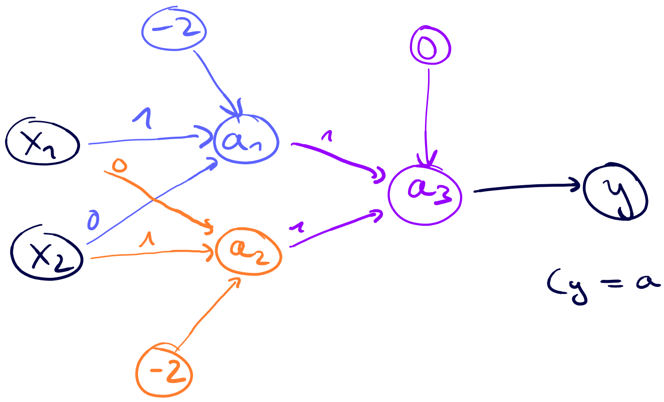
possible solution: $b_1 = -2$, $w_{11} = 1$, $w_{21} = 0$

then $a_1 = \sigma(x_1 - 2)$

Either a_1 or a_2 have to be active. We construct new neuron

$a_3 = \sigma(a_1 + a_2) = \sigma\left(b_3 + \sum_{i=1}^{2} a_i w_{i3}\right)$

with $b_3 = 0$, $w_{13} = 1$, $w_{23} = 1$

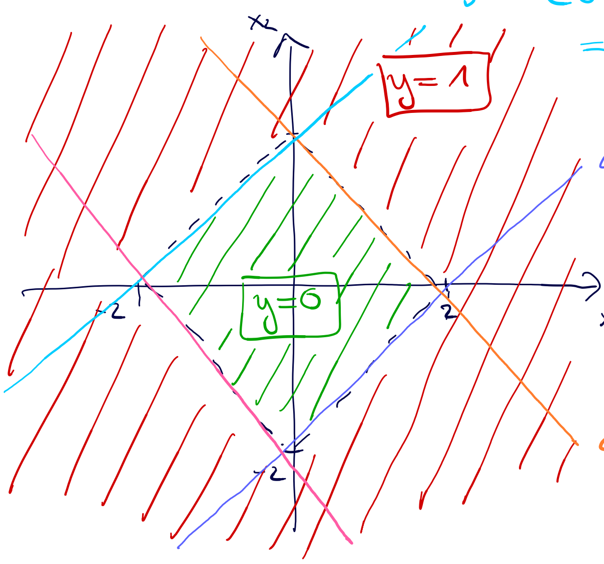


$(y = a_3)$

b)

$$y = \begin{cases} 1 & \text{if } \|x\| > 2 \\ 0 & \text{if } \|x\| < 2 \end{cases}$$

$$a_3 = \begin{cases} 1 & \text{if } (-x_1) + x_2 > 2 \\ 0 & \text{else} \end{cases}$$
$$= \sigma(-x_1 + x_2 - 2)$$



$$a_1 = \begin{cases} 1 & \text{if } x_1 + (-x_2) > 2 \\ 0 & \text{else} \end{cases}$$
$$= \sigma(x_1 - x_2 - 2)$$

$$a_2 = \begin{cases} 1 & \text{if } x_1 + x_2 > 2 \\ 0 & \text{else} \end{cases}$$
$$= \sigma(x_1 + x_2 - 2)$$

$$a_4 = \begin{cases} 1 & \text{if } (-x_1) + (-x_2) > 2 \\ 0 & \text{else} \end{cases}$$
$$= \sigma(-x_1 - x_2 - 2)$$

$a_1 = \sigma(x_1 - x_2 - 2)$

$\Rightarrow w_{11} = 1,\ w_{21} = -1,\ b_1 = -2$

$a_3 = \sigma(-x_1 + x_2 - 2)$

$\Rightarrow w_{13} = -1,\ w_{23} = 1,\ b_3 = -2$

$a_2 = \sigma(x_1 + x_2 - 2)$

$\Rightarrow w_{12} = 1,\ w_{22} = 1,\ b_2 = -2$

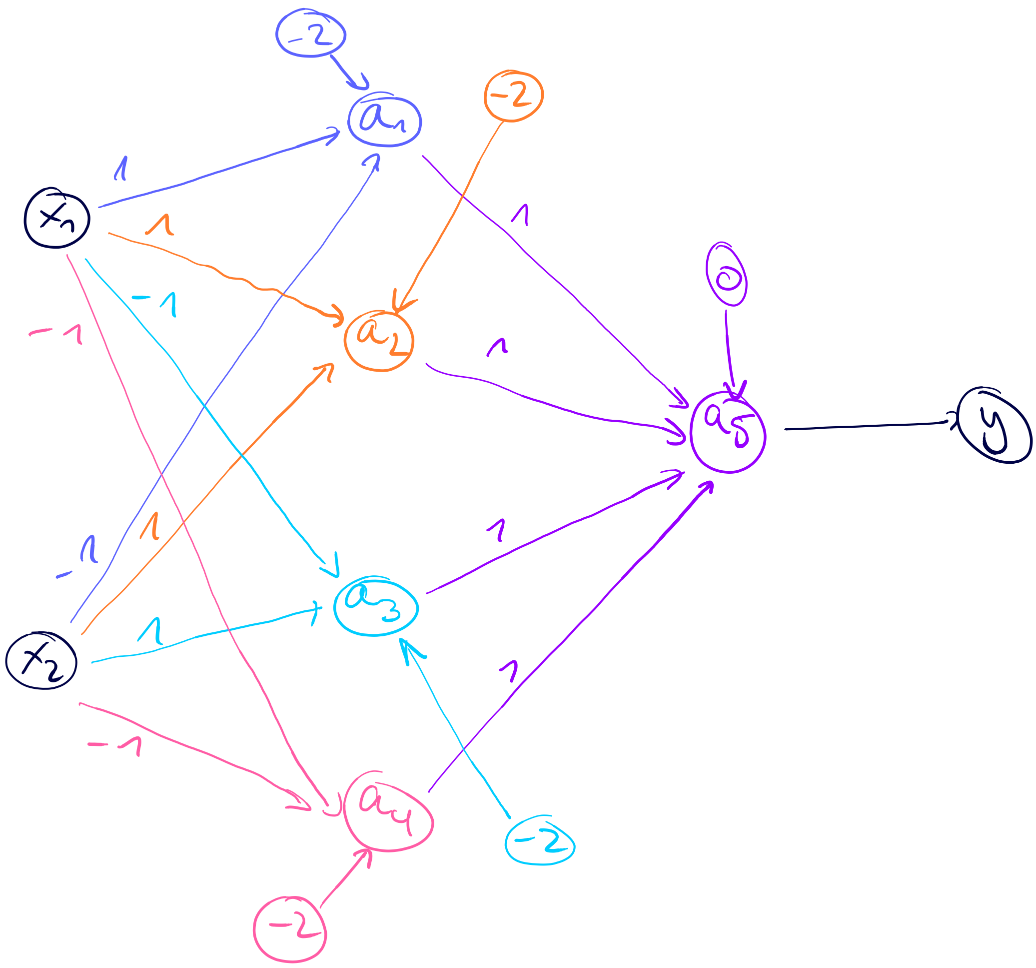
$a_4 = \sigma(-x_1 - x_2 - 2)$

$\Rightarrow w_{14} = -1,\ w_{24} = -1,\ b_4 = -2$

Again only one of $\{a_1, a_2, a_3, a_4\}$ has to be active for the condition to hold. Thus we construct $y = a_5$ similarly as above:

$$a_5 = \sigma(a_1 + a_2 + a_3 + a_4)$$

$$\Rightarrow w_{15} = w_{25} = w_{35} = w_{45} = 1 \quad , \quad b_5 = 0$$



②

a) $$\mathcal{E}(w) = \alpha \|w\|^2 + \frac{1}{N}\sum_{i=1}^{N} (w^T x_i - t)^2$$

$$H(w) = \frac{\partial^2 \mathcal{E}(w)}{\partial w^2}$$

$$\frac{\partial \mathcal{E}(w)}{\partial w} = \frac{\partial\left[\alpha\|w\|^2 + \frac{1}{N}\sum_{i=1}^{N}(w^T x_i - t)^2\right]}{\partial w}$$

$$\left| \frac{\partial f(x)+g(x)}{\partial x} = \frac{\partial f(x)}{\partial x} + \frac{\partial g(x)}{\partial x} \right.$$

$$= \frac{\partial \alpha\|w\|^2}{\partial w} + \frac{\partial \frac{1}{N}\sum_{i=1}^{N}(w^T x_i - t)^2}{\partial w}$$

| take constant factor out of derivative

$$= \alpha \frac{\partial \sum_{j=1}^{d} w_j^2}{\partial w} + \frac{1}{N} \cdot \frac{\partial \sum_{i=1}^{N}(w^T x_i - t)^2}{\partial w}$$

$$\left| \frac{\partial \sum_{j=1}^{d} w_j^2}{\partial w_k} = 2w_k \Rightarrow \frac{\partial \sum_{j=1}^{d} w_j^2}{\partial w} = 2w \right.$$

$$= 2\alpha w + \frac{1}{N}\sum_{i=1}^{N}\frac{\partial (w^T x_i - t)^2}{\partial w}$$

| chain rule

$$= 2\alpha w + \frac{1}{N}\sum_{i=1}^{N} 2(w^T x_i - t)\cdot \underbrace{\frac{\partial w^T x_i - t}{\partial w}}_{= x_i}$$

$$= 2\alpha w + \frac{1}{N}\sum_{i=1}^{N} 2(w^T x_i - t)\, x_i$$

$$\frac{\partial^2 \mathcal{E}(w)}{\partial w^2} = \frac{\partial\left[2\alpha w + \frac{1}{N}\sum_{i=1}^{N} 2(w^T x_i - t)\, x_i\right]}{\partial w}$$

$$= \frac{\partial 2\alpha w}{\partial w} + \frac{\partial \frac{1}{N}\sum_{i=1}^{N} 2(w^T x_i - t)\, x_i}{\partial w}$$

$$= 2\alpha \frac{\partial w}{\partial w} + 2\cdot\frac{1}{N}\sum_{i=1}^{N}\frac{\partial (w^T x_i - t) x_i}{\partial w}$$

$$= 2\alpha I + 2\cdot\frac{1}{N}\sum_{i=1}^{N}\frac{\partial (w^T x_i)\cdot x_i - t\cdot x_i}{\partial w}$$

$$= 2\alpha I + 2\frac{1}{N}\sum_{i=1}^{N}\frac{\partial\, x_i x_i^T w - t x_i}{\partial w}$$

$$= 2\alpha I + 2\underbrace{\frac{1}{N}\sum_{i=1}^{N} x_i x_i^T}_{=\Sigma}$$

$$= 2(\Sigma + \alpha I)$$

b) Let $\hat{\lambda}_1, \dots, \hat{\lambda}_d$ be the eigenvalues of $2(\Sigma + \alpha I)$ sorted in decreasing order. Then we have

$$c = \frac{\hat{\lambda}_1}{\hat{\lambda}_d}$$

We will now show that $\hat{\lambda}_i = k \cdot (\lambda_i + \alpha)$ for some constant k.

Let v_i be an eigenvector of Σ with corresponding eigenvalue λ_i, i.e. $\Sigma v_i = \lambda_i v_i$. Then, v_i is also an eigenvector of $2(\Sigma + \alpha I)$:

$$2(\Sigma + \alpha I) v_i = 2\underbrace{\Sigma v_i}_{=\lambda_i v_i} + 2\alpha \underbrace{I v_i}_{= v_i}$$

$$= 2\lambda_i v_i + 2\alpha v_i$$

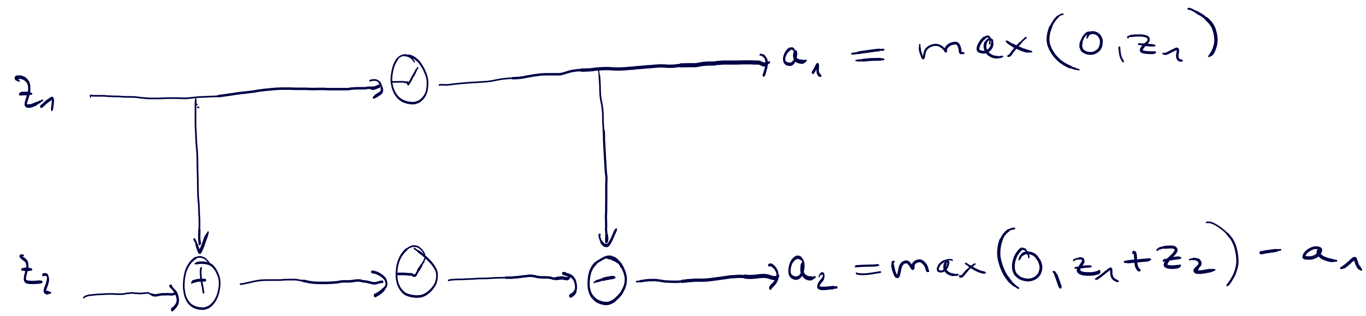
$$= \underbrace{2(\lambda_i + \alpha)}_{= \hat{\lambda}_i} v_i$$

with eigenvalue $\hat{\lambda}_i = 2(\lambda_i + \alpha)$

Thus, we have

$$c = \frac{\hat{\lambda}_1}{\hat{\lambda}_d} = \frac{2(\lambda_1 + \alpha)}{2(\lambda_d + \alpha)} = \frac{\lambda_1 + \alpha}{\lambda_d + \alpha}$$

③



a) Compute $\frac{\partial \mathcal{E}}{\partial z_1}$ and $\frac{\partial \mathcal{E}}{\partial z_2}$ assuming we already know $\frac{\partial \mathcal{E}}{\partial a_1}$ and $\frac{\partial \mathcal{E}}{\partial a_2}$.

$$\frac{\partial \mathcal{E}}{\partial z_1} = \frac{\partial \mathcal{E}}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1} + \frac{\partial \mathcal{E}}{\partial a_2} \cdot \frac{\partial a_2}{\partial z_1}$$

$$\frac{\partial \mathcal{E}}{\partial z_2} = \frac{\partial \mathcal{E}}{\partial a_2} \cdot \frac{\partial a_2}{\partial z_2} + \underbrace{\frac{\partial \mathcal{E}}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_2}}_{=0}$$

$$\frac{\partial a_1}{\partial z_1} = \frac{\partial \max(0, z_1)}{\partial z_1} = \mathbb{1}_{\{z_1 > 0\}}$$

$$\frac{\partial a_2}{\partial z_1} = \frac{\partial \max(0, z_1 + z_2) - a_1}{\partial z_1}$$

$$= \frac{\partial \max(0, z_1 + z_2)}{\partial (z_1 + z_2)} \cdot \underbrace{\frac{\partial (z_1 + z_2)}{\partial z_1}}_{= \frac{\partial z_1}{\partial z_1} + \frac{\partial z_2}{\partial z_1} = 1 + 0} - \underbrace{\frac{\partial a_1}{\partial z_1}}_{= \mathbb{1}_{\{z_1 > 0\}}}$$

$$= \mathbb{1}_{\{z_1 + z_2 > 0\}} \cdot 1 - \mathbb{1}_{\{z_1 > 0\}}$$

$$\frac{\partial a_2}{\partial z_2} = \frac{\partial \max(0, z_1+z_2) - a_1}{\partial z_2} = \frac{\partial \max(0, z_1+z_2)}{\partial (z_1+z_2)} \cdot \frac{\partial (z_1+z_2)}{\partial z_2} - \underbrace{\frac{\partial a_1}{\partial z_2}}_{=0}$$

$$= \mathbb{1}_{\{z_1+z_2>0\}} \cdot \left(\underbrace{\frac{\partial z_1}{\partial z_2}}_{=0} + \underbrace{\frac{\partial z_2}{\partial z_2}}_{=1} \right)$$

$$= \mathbb{1}_{\{z_1+z_2>0\}}$$