

Exercise Sheet 6

Exercise 1: Rotational equivariance (40 P)

In this exercise, we will derive an equivariant function given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that is invariant to the rotation group, i.e.

$$f(\mathbf{x}) = f(g\mathbf{x})$$

for any $g \in \text{SO}(n)$.

Hint: Use the fact that g is orthogonal.

(a) Show that $f(\mathbf{x}) = \|\mathbf{x}\|$ is invariant under $\text{SO}(n)$.

$$\begin{aligned}\|g\mathbf{x}\| &= \sqrt{\mathbf{x}^T g^T g \mathbf{x}} \\ &= \sqrt{\mathbf{x}^T \mathbf{x}} \\ &= \|\mathbf{x}\|\end{aligned}$$

(b) Show that the derivative of any f is equivariant under $\text{SO}(n)$:

$$\nabla f(\mathbf{x}) = \nabla f(g\mathbf{x})$$

for any $g \in \text{SO}(n)$.

Take the derivative on both sides of invariance relation:

$$\begin{aligned}\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}) &= \nabla f(\mathbf{x}) \\ \frac{\partial f \circ g}{\partial \mathbf{x}}(\mathbf{x}) &= g^T \nabla f(g\mathbf{x}) \\ \Rightarrow \nabla f(\mathbf{x}) &= g^T \nabla f(g\mathbf{x}) \\ \Rightarrow g \nabla f(\mathbf{x}) &= g g^T \nabla f(g\mathbf{x}) = \nabla f(g\mathbf{x})\end{aligned}$$

(c) Calculate the gradient $u(\mathbf{x}) = \nabla f(\mathbf{x}) = \nabla \|\mathbf{x}\|$ to obtain an equivariant function $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

$$u(\mathbf{x}) = \nabla f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$$

(d) Analog to (c), derive an equivariant function $v : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$

$$v(\mathbf{x}) = H_f(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x}} \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{I \|\mathbf{x}\|^2 - \mathbf{x} \mathbf{x}^T}{\|\mathbf{x}\|^3}$$

Exercise 2: Programming (60 P)

Download the programming files on ISIS and follow the instructions.