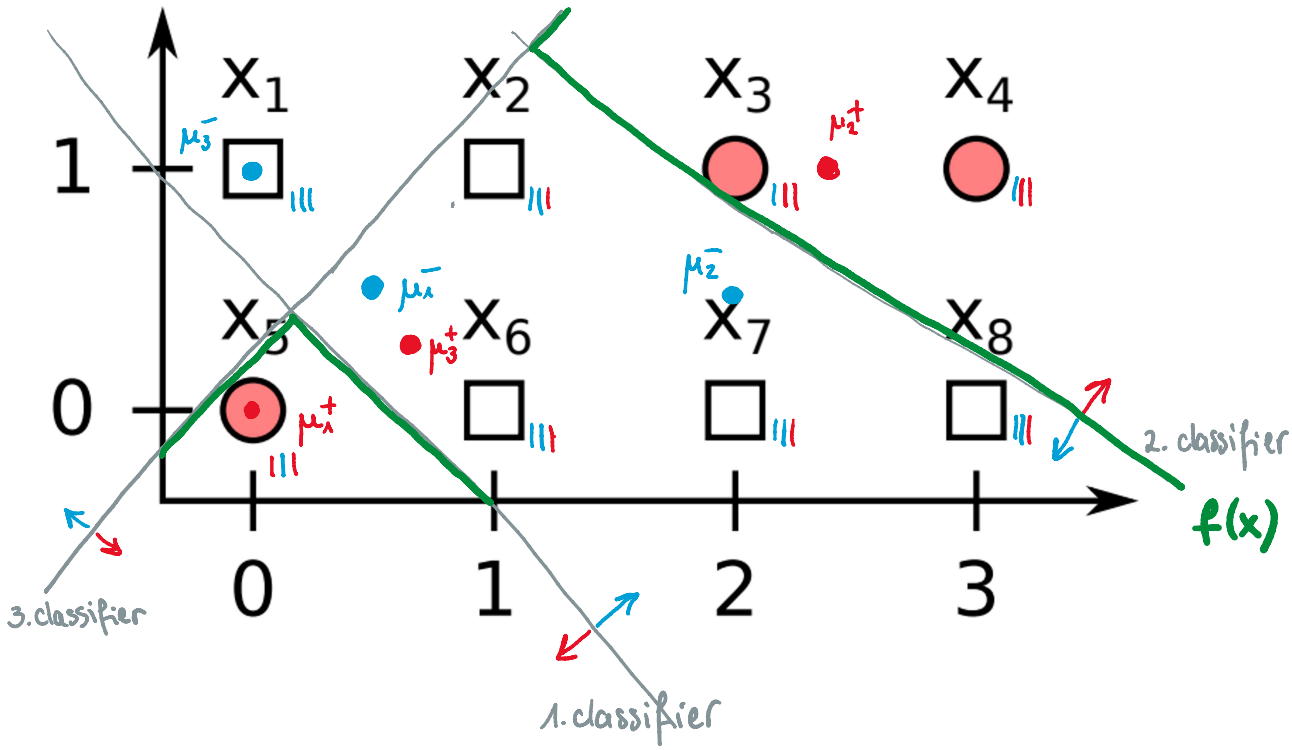
UE 11

Donnerstag, 2. Januar 2025 06:30

Exercise 1 (a) & (b)



1. classifier :

$\mu_1^+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = x_5$

$p_5^{(1)} = 1,$

$p_3^{(1)} = p_4^{(1)} = 0$

$\mu_1^- = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \frac{1}{2}\left((x_1 + x_6) \right)$

$p_1^{(1)} = p_6^{(1)} = \frac{1}{2}$

$p_2^{(1)} = p_7^{(1)} = p_8^{(1)} = 0$

2. classifier :

$\mu_1^+ = \begin{pmatrix} 2.\overline{3} \\ 0.5 \end{pmatrix} = \frac{2}{3} x_3 + \frac{1}{3} x_4$

$p_3^{(2)} = \frac{2}{3} \quad p_4^{(2)} = \frac{1}{3}$

$p_5^{(2)} = 0$

$\mu_2^- = \begin{pmatrix} 2 \\ 0.5 \end{pmatrix} = \frac{1}{2}\left((x_2 + x_8) \right)$

$p_2^{(2)} = p_8^{(2)} = \frac{1}{2}$

$p_1^{(2)} = p_6^{(2)} = p_7^{(2)} = 0$

3. classifier :

$\mu_3^+ = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \frac{1}{3} x_3 + \frac{2}{3} x_5$

$p_3^{(3)} = \frac{1}{3} \quad p_5^{(3)} = \frac{2}{3}$

$p_4^{(3)} = 0$

$\mu_2^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_1$

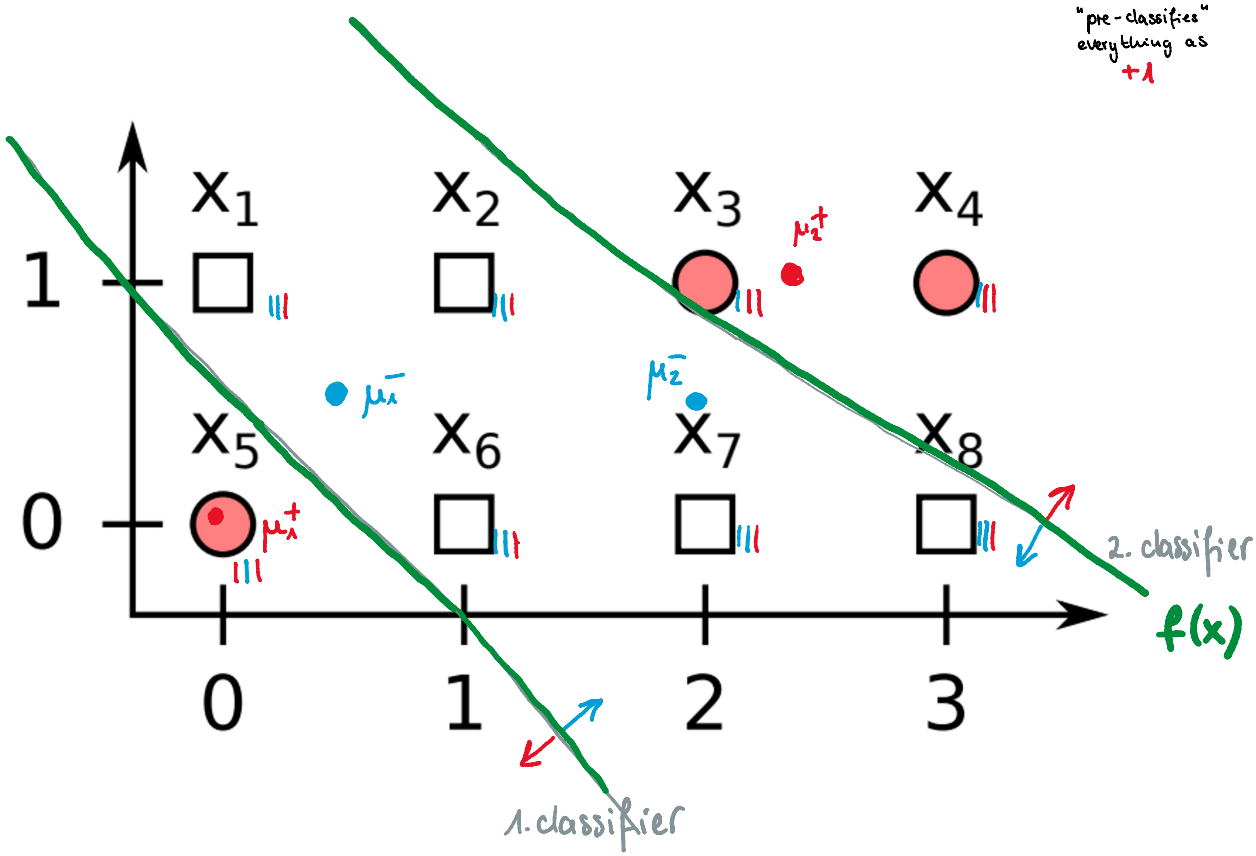
$p_1^{(3)} = 1$

$p_2^{(3)} = p_6^{(3)} = p_7^{(3)} = p_8^{(3)} = 0$

Final classifier combines all classifiers with equal weights $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_0 = 0$.

Alternative solution:

Final classifier only combines classifiers 1 and 2 with equal weights $\alpha_1 = \alpha_2 = \underbrace{\alpha_0 = 1}_{\text{"pre-classifies" everything as } +1}$



Exercise 2

(a)

$$g(\alpha) = N \cdot \left(\prod_{\tau=1}^{t-1} z_\tau\right) \cdot \sum_{i=1}^{N} p_i^{(t)} \cdot \underbrace{\exp\{-y_i \alpha_t h_t(x_i)\}}_{=:\, \exp\{-w_i^{(t)}\}}$$

$$\boxed{* \quad p_i^{(t+1)} = \frac{1}{z_t} p_i^{(t)} \exp\{-w_i^{(t)}\} \iff z_t\, p_i^{(t+1)} = p_i^{(t)} \exp\{-w_i^{(t)}\}}$$

$$\begin{aligned}
&= N \cdot \sum_{i=1}^{N} \left(\prod_{\tau=1}^{t-1} z_\tau\right) p_i^{(t)} \cdot \exp\{-w_i^{(t)}\} \\
&= N \cdot \sum_{i=1}^{N} \left(\prod_{\tau=1}^{t-2} z_\tau\right) z_{t-1}\, p_i^{(t)} \cdot \exp\{-w_i^{(t)}\} \\
&\overset{*}{=} N \cdot \sum_{i=1}^{N} \left(\prod_{\tau=1}^{t-2} z_\tau\right) p_i^{(t-1)} \cdot \exp\{-w_i^{(t-1)}\} \cdot \exp\{-w_i^{(t)}\} \\
&= N \cdot \sum_{i=1}^{N} \left(\prod_{\tau=1}^{t-3} z_\tau\right) z_{t-2}\, p_i^{(t-1)} \exp\{-w_i^{(t-1)}\} \cdot \exp\{-w_i^{(t)}\} \\
&\overset{*}{=} N \cdot \sum_{i=1}^{N} \left(\prod_{\tau=1}^{t-3} z_\tau\right) p_i^{(t-2)} \exp\{-w_i^{(t-2)}\} \exp\{-w_i^{(t-1)}\} \cdot \exp\{-w_i^{(t)}\} \\
&= \dots \\
&= N \cdot \sum_{i=1}^{N} p_i^{(1)} \cdot \prod_{\tau=1}^{t} \exp\{-w_i^{(\tau)}\} \\
&\overset{p_i^{(1)} = 1/N}{=} N \cdot \frac{1}{N} \sum_{i=1}^{N} \prod_{\tau=1}^{t} \exp\{-w_i^{(\tau)}\} \\
&= \sum_{i=1}^{N} \prod_{\tau=1}^{t} \exp\{-w_i^{(\tau)}\} \\
&= \sum_{i=1}^{N} \exp\left\{\sum_{\tau=1}^{t} -w_i^{(\tau)}\right\} \\
&= \sum_{i=1}^{N} \exp\left\{\sum_{\tau=1}^{t} -y_i \alpha_\tau h_\tau(x_i)\right\} \\
&= \sum_{i=1}^{N} \exp\left\{-y_i \sum_{\tau=1}^{t} \alpha_\tau h_\tau(x_i)\right\} \\
&= \sum_{i=1}^{N} \exp\{-y_i \cdot f_{\alpha,t}(x_i)\} \\
&= G(\alpha)
\end{aligned}$$

(b)

$$\mathcal{G}(\alpha) = N \cdot \left(\prod_{\tau=1}^{t-1} z_\tau\right) \sum_{i=1}^{N} p_i^{(t)} \exp\{-y_i \alpha_t h_t(x_i)\}$$

$$\frac{\partial \mathcal{G}(\alpha)}{\partial \alpha_t} = N \cdot \left(\prod_{\tau=1}^{t-1} z_\tau\right) \cdot \sum_{i=1}^{N} p_i^{(t)} \cdot \exp\{-y_i \alpha_t h_t(x_i)\} \cdot (-y_i h_t(x_i)) \overset{!}{=} 0$$

$$\Leftrightarrow \sum_{i=1}^{N} p_i^{(t)} \exp\{-y_i \alpha_t h_t(x_i)\} \cdot (-y_i h_t(x_i)) = 0$$

$h_t(x_i), y_i \in \{+1, -1\}$ so that $(-y_i \cdot h_t(x_i)) = \begin{cases} +1 & , \text{if } y_i \neq h_t(x_i) \\ -1 & , \text{if } y_i = h_t(x_i) \end{cases}$

$$\Leftrightarrow \sum_{i=1}^{N} p_i^{(t)} \cdot \left(\mathbb{1}_{(y_i = h_t(x_i))} \cdot \left(-\exp\{-\alpha_t\}\right) + \mathbb{1}_{(y_i \neq h_t(x_i))} \cdot \left(\exp\{\alpha_t\}\right)\right) = 0$$

$$\Leftrightarrow -\exp\{-\alpha_t\} \sum_{i:\, y_i = h_t(x_i)} p_i^{(t)} + \exp\{\alpha_t\} \sum_{i:\, y_i \neq h_t(x_i)} p_i^{(t)} = 0$$

$$\Leftrightarrow \exp\{\alpha_t\} \sum_{i:\, y_i \neq h_t(x_i)} p_i^{(t)} = \exp\{-\alpha_t\} \sum_{i:\, y_i = h_t(x_i)} p_i^{(t)}$$

$$\Leftrightarrow \exp\{\alpha_t\} \sum_{i:\, y_i \neq h_t(x_i)} p_i^{(t)} = \exp\{-\alpha_t\} \left(1 - \sum_{i:\, y_i \neq h_t(x_i)} p_i^{(t)}\right)$$

$$\Leftrightarrow \exp\{\alpha_t\} \, \mathbb{E}_{p^{(t)}}\left[\mathbb{1}_{(y_i \neq h_t(x_i))}\right] = \exp\{-\alpha_t\} \left(1 - \mathbb{E}_{p^{(t)}}\left[\mathbb{1}_{(y_i \neq h_t(x_i))}\right]\right)$$

$$\Leftrightarrow \exp\{\alpha_t\} \, \varepsilon_t = \exp\{-\alpha_t\} (1 - \varepsilon_t)$$

$$\Leftrightarrow \frac{\exp\{\alpha_t\}}{\exp\{-\alpha_t\}} = \frac{(1-\varepsilon_t)}{\varepsilon_t}$$

$$\Leftrightarrow \log\{\exp\{2\alpha_t\}\} = \log\left\{\frac{(1-\varepsilon_t)}{\varepsilon_t}\right\}$$

$$\Leftrightarrow \alpha_t = \frac{1}{2} \log\left\{\frac{(1-\varepsilon_t)}{\varepsilon_t}\right\}$$