Mathematical Modeling and Computation in Finance

With Exercises and Python and MATLAB Computer Codes

C.W. Oosterlee & L.A. Grzelak

Solutions to exercises from Chapter 11

https://QuantFinanceBook.com

Ex. 11.1. We introduce the process $y(t) = \mathrm{e}^{\lambda t} r(t)$, whose dynamics are given by,

$$\begin{aligned}\mathrm{d}y(t) &= \lambda y(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\mathrm{d}r(t)\\ &= \lambda y(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\left(\lambda(\hat{\theta}(t) - r(t))\mathrm{d}t + \eta \mathrm{d}W(t)\right)\\ &= \lambda\hat{\theta}(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\eta\mathrm{d}W(t),\end{aligned}$$

which implies,

$$r(t) = r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \mathrm{e}^{-\lambda(t-z)}\hat{\theta}(z)\mathrm{d}z + \eta\int_0^t \mathrm{e}^{-\lambda(t-z)}\mathrm{d}W_r(z).$$

This process is a Gaussian normal process. We can see this clearly when discretizing the integral, which turns into sum of normal distributions. Hence, the result of the integration is also a normal process. The mean and variance of the process are given by,

$$\begin{aligned}\mathbb{E}[r(t)|F_0] &= r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \mathrm{e}^{-\lambda(t-z)}\hat{\theta}(z)\mathrm{d}z,\\ \mathbb{V}\mathrm{ar}[r(t)|F_0] &= \eta^2\int_0^t \mathrm{e}^{-2\lambda(t-z)}\mathrm{d}z = \frac{\eta^2}{2\lambda}\left(1 - \mathrm{e}^{-2\lambda t}\right).\end{aligned}$$

Since it is normal, we can use the moment generating function for the normal distribution,

$$\mathbb{E}[\mathrm{e}^{uX}] = \mathrm{e}^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

Plugging in μ and σ^2 of $r(t)$ into the above expression, gives us the desired form.

Ex. 11.3. To prove that the decomposition indeed holds, it is sufficient to show that the solution for $\tilde{r}(t) + \psi(t)$ is equal to $r(t)$. Recall that $\psi(t) = r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \theta(z)\mathrm{e}^{-\lambda(t-z)}\mathrm{d}z$. The derivation of the solution for $\tilde{r}(t)$ is based on the following SDE:

$$\mathrm{d}\tilde{r}(t) = -\lambda\tilde{r}(t)\mathrm{d}t + \eta\mathrm{d}W_r(t).$$

Using the process $y(t) = \mathrm{e}^{\lambda t}\tilde{r}(t)$, whose dynamics are given by,

$$\begin{aligned}\mathrm{d}y(t) &= \lambda y(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\mathrm{d}\tilde{r}(t)\\ &= \lambda y(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\left(-\lambda\tilde{r}(t)\mathrm{d}t + \eta\mathrm{d}W(t)\right)\\ &= \mathrm{e}^{\lambda t}\eta\mathrm{d}W(t),\end{aligned}$$

implies, with $\tilde{r}(t_0) = 0$, that

$$\tilde{r}(t) = \eta\int_0^t \mathrm{e}^{-\lambda(t-z)}\mathrm{d}W_r(z).$$

Therefore,

$$\tilde{r}(t) + \psi(t) \quad = \quad r_0 e^{-\lambda t} + \lambda \int_0^t \theta(z) e^{-\lambda(t-z)} dz + \eta \int_0^t e^{-\lambda(t-z)} dW_r(z), \qquad (0.1)$$

which is equal to the solution for $r(t)$ as derived in Section 11.3.1.

Ex. 11.5. The first two terms in the integral can be expressed in the following way,

$$\int_0^t e^{\lambda u} \frac{1}{\lambda} \frac{\partial}{\partial u} f^r(0,u) + e^{\lambda u} f^r(0,u) du = \int_0^t \frac{1}{\lambda} \frac{\partial}{\partial u} (e^{\lambda u} f^r(0,u)) du = e^{\lambda t} f^r(0,t) - f^r(0,0).$$

The other part of the integral can be expressed as,

$$\frac{\eta^2}{\lambda} e^{-\lambda t} \int_0^t \frac{e^{\lambda u} - e^{-\lambda u}}{2} du = \frac{\eta^2}{\lambda} e^{-\lambda t} \int_0^t \sinh(\lambda u) du = e^{-\lambda t} \frac{\eta^2}{\lambda^2} (\cosh(\lambda t) - 1).$$

Putting all pieces together, gives the desired result.

Ex. 11.7. The value of the ZCB valued under the T-forward measure is given by,

$$V^{ZCB}(t_0, T) = P(t_0, T) e^{A(\tau)} E^T [\max(K - e^{r(T)B(\tau)})].$$

In Exercise 11.1, the mean and variance for $r(t)$ have been derived. The distribution of the normal process, $z(\tau, T) := r(T)B(\tau)$, is given by,

$$z(\tau, T) \sim N(B(\tau)\mu_r(T), B^2(\tau)\sigma_r^2(T)). \qquad (0.2)$$

The value of a put option on a zero coupon bond is now given by,

$$\begin{aligned}
V^{ZCB}(t_0, T) &= P(t_0, T) e^{A(\tau)} \mathbb{E}^T [\max\left(K - e^{Z(T,\tau)}, 0\right)] \\
&= P(t_0, T) e^{A(\tau)} \int_{-\infty}^{\infty} \max\left(K - e^{B(\tau)\mu_r(T) + B(\tau)\sigma_r(T)x}\right) f_{N(0,1)}(x) dx \\
&= P(t_0, T) e^{A(\tau)} \int_a^{\infty} \left(K - e^{B(\tau)\mu_r(T) + B(\tau)\sigma_r(T)x}\right) f_{N(0,1)}(x) dx,
\end{aligned}$$

where $a = \frac{\log(K) - B(\tau)\mu_r(T)}{B(\tau)\sigma_r(T)}$. The first integral is the CDF of the standard normal distribution evaluated at a. To evaluate the second term, we rewrite the expression, as follows,

$$\begin{aligned}
-\frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{B(\tau)\mu_r(T) + B(\tau)\sigma_r(T)x} e^{-x^2/2} dx &= -\frac{e^{B(\tau)\mu_r(T) + \frac{1}{2}B^2(\tau)\sigma_r^2(T)}}{\sqrt{2\pi}} \int_a^{\infty} e^{-\frac{1}{2}(x - B(\tau)\sigma_r(T))^2} dx \\
&= -\frac{e^{B(\tau)\mu_r(T)}}{\sqrt{2\pi}} \int_{a - B(\tau)\sigma_r(T)}^{\infty} e^{-\frac{1}{2}y^2} dy \\
&= -e^{B(\tau)\mu_r(T) + \frac{1}{2}B^2(\tau)\sigma_r^2(T)} \left(1 - F_{\mathcal{N}(0,1)}(a - B(\tau)\sigma_r(T))\right).
\end{aligned}$$

Adding the first term, we obtain,

$$V^{ZCB}(t_0, T) = P(t_0, T) e^{A(\tau)} \left(K F_{\mathcal{N}(0,1)}(a) - e^{B(\tau)\mu_r(T) + \frac{1}{2}B^2(\tau)\sigma_r^2(T)} - F_{\mathcal{N}(0,1)}(B(\tau)\sigma_r(T) - a)\right).$$

Ex. 11.9. *a*. To determine the weights we solve the following system of equations:

$$\begin{aligned}
\omega_1 + \omega_2 &= 1 \\
\omega_1 P(t_0, 1y) + \omega_2 P(t_0, 20y) &= P(t_0, 10y),
\end{aligned}$$

whose solution is given by:

$$\omega_1 = \frac{P(t_0, 10y) - P(t_0, 20y)}{P(t_0, 1y) - P(t_0, 20y)}.$$

b. The first step in determining the optimal weights ω_1 and ω_2 is discretizing the integral:

$$\mathbb{V}\text{ar}\left[\int_0^{10y} \omega_1 P(t,1y) + \omega_2 P(t,20y)\mathrm{d}t\right] = \mathbb{V}\text{ar}\left[\omega_1 \int_0^{10y} P(t,1y)\mathrm{d}t + \omega_2 \int_0^{10y} P(t,20y)\mathrm{d}t\right].$$

This discretization can be written as:

$$\mathbb{V}\text{ar}\left[\omega_1 X + (1-\omega_1)Y\right] = \mathbb{V}\text{ar}\left[Z\right].$$

To numerically determine the optimal ω_1, define the following minimization problem:

$$\min_{\omega_1} \left(\mathbb{V}\text{ar}\left[\omega_1 X + (1-\omega_1)Y\right] - \mathbb{V}\text{ar}\left[Z\right]\right)^2,$$

where $X = \int_0^{10y} P(t,1y)\mathrm{d}t$, $Y = \int_0^{10y} P(t,20y)\mathrm{d}t$ $Z = \int_0^{10y} P(t,10y)\mathrm{d}t$, which can be solved by means of the Newton-Raphson algorithm.
In Figure 1, the value of the target function depending on ω_1 is presented.

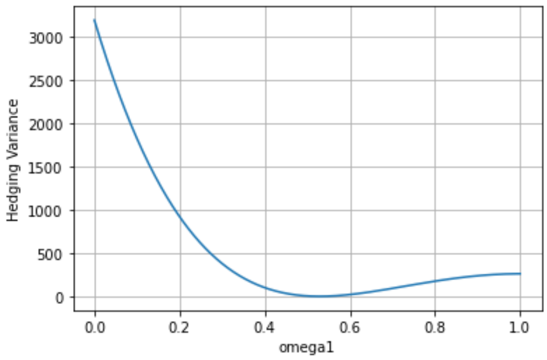

Figure 1: Results for Exercise 11.9.

The approach presented in this method is better suitable than the approach presented in item *a*), as the volatility is incorporated. As shown below, the optimal ω_1 is approximately 0.52.

```
optimal omega =       fun: 2.601210214589423e-13
 hess_inv: array([[0.00010651]])
      jac: array([5.85996932e-08])
  message: 'Optimization terminated successfully.'
     nfev: 16
      nit: 5
     njev: 8
   status: 0
  success: True
        x: array([0.52881557])
```

c. A change of measure won't affect the results obtained under item *b*). The price of a financial derivative is affected by the model parameters.

Ex. 11.11. *a*. We have,

$$f^r(0,t) = -\frac{\partial}{\partial t}\log P_{mkt}(t).$$

For the case at hand,

$$f^r(0,t) = ae^{at}.$$

Furthermore, θ becomes,

$$\theta(t) = a\mathrm{e}^{at}\left(1+\frac{a}{\lambda}\right)+\frac{\eta^2}{2\lambda^2}\left(1-\mathrm{e}^{-2\lambda t}\right).$$

b. The bond price is given by,

$$P(0,t) = \exp\left(-\int_0^t \psi(z)\mathrm{d}z + A(0,t)\right),$$

where $\psi(t)$ is given by,

$$\psi(t) = r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \theta(z)\mathrm{e}^{-\lambda(t-z)}\mathrm{d}z.$$

Based on the expression for θ, we find that,

$$\psi(t) = \mathrm{e}^{-\lambda t}\left[r_0 + \mathrm{e}^{\lambda t}f^r(0,t) - f^r(0,0) + \frac{\eta^2}{\lambda^2}(\cosh(\lambda t)-1)\right].$$

We substitute these expressions in the mathematical model and compare the result with market observations. In Figure 2 the results from the experiments are presented. As can be seen from the figure below, the market observations fit the theoretical values well.

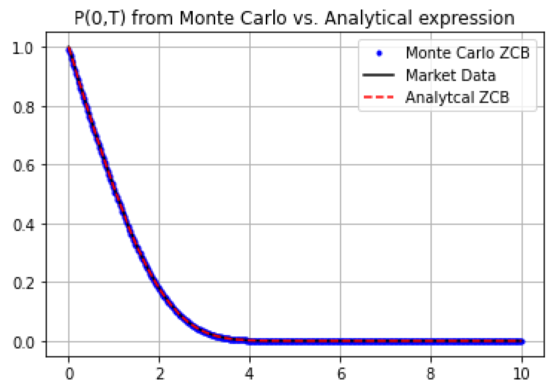

Figure 2: Results for Exercise 11.11.