Mathematical Modeling and Computation in Finance

With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 3

`https://QuantFinanceBook.com`

Ex. 3.1. The table should be filled as follows,

product	8.5€	9€	9.5	10€	10.5€	11€	11.5€
profit $S(T)$	-%15	-%10	-%5	%0	%5	%10	%15
profit $V_c(T, S(T))$	-%100	-%100	-%100	-%100	-%50	% 0	% 50

Ex. 3.3. The code can be found under the Python icon. We fix $S = S_0$.

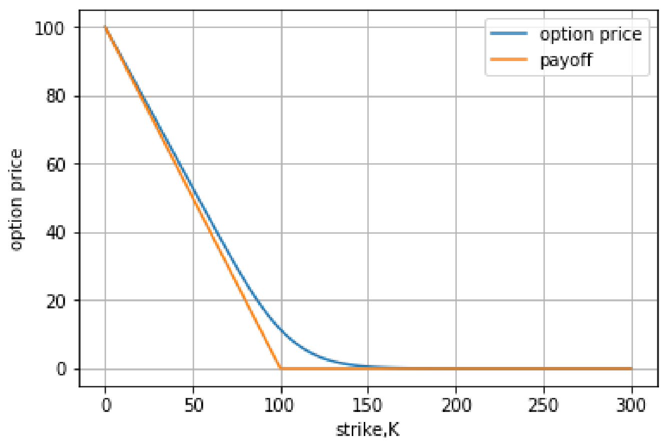

Figure 1: Results for Exercise 3.3

Ex. 3.5. The Δ-hedging strategy cancels out the stochastic movements of the portfolio, simply because,

$$\frac{\partial \Pi}{\partial S} = \frac{\partial V}{\partial S} - \Delta = 0,$$

due to the particular hedging strategy. In this case, $\mathrm{d}\Pi = (\ldots)\mathrm{d}t$, in other words, the infinitesimal change in the portfolio is deterministic. Assume, without loss of generality,

$$\mathrm{d}\Pi - r\Pi\mathrm{d}t = (k(S,t) - r)\Pi\mathrm{d}t.$$

In this case, the system is not arbitrage-free. If $k > r$ at $t = 0$, one could borrow money from the money market with interest rate r, use this to buy the portfolio Π. The portfolio can be sold for an amount $(1 + k\Delta t)\Pi$ at time $t + \Delta t$. Subsequently, the borrowed money is paid back which creates a profit $(k-r)\Pi\Delta t$. In the other case, where $k<r$, one would short-sell the portfolio at $t=0$, simultaneously put the money into a savings account. After a time increment Δt, the portfolio is bought resulting in a final profit of $(r-k)\Delta t\Pi$. So, $\mathrm{d}\Pi = r\Pi\mathrm{d}t$.

Ex. 3.7. The process that gives rise to the equation in the exercise is given by,

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W^Q(t).$$

Furthermore, we use $r=0$. The solution for this differential process is given by

$$X(T) = X(t)\exp\left((\mu-\sigma^2/2)(T-t) + \sigma(W(T)-W(t))\right).$$

The expression $\log(X^2(T))$ then reads,

$$\log X^2(T) = 2\left(\log X(t) + (\mu-\sigma^2/2)(T-t) + \sigma(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t))\right).$$

The Feynman-Kac theorem states that, for this problem,

$$V(X,t) = \mathbb{E}[\log(X^2(T))|\mathcal{F}_t].$$

Brownian motion W_t^Q, under measure $\mathbb{Q}$, is a martingale, and the last term under the expectation operator disappears. So,

$$\mathbb{E}[\log(X^2(T))|\mathcal{F}_t] = 2\left(\log X(t) + (\mu-\sigma^2/2)(T-t)\right).$$

Ex. 3.9. *a.* Without any dividend payment, the value of a call option, with exercise price K and time to maturity T, is given by:

$$V_c(S,t) = SF_{\mathcal{N}(0,1)}(d_1) - Ke^{-r(T-t)}F_{\mathcal{N}(0,1)}(d_2),$$

where

$$d_{1,2} = \frac{\log(S/K) + (r\pm\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

With

$$F_{\mathcal{N}(0,1)}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\mathrm{d}y,$$

the standard cumulative normal distribution function, it follows for the corresponding density that

$$f_{\mathcal{N}(0,1)}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}.$$

Further, it can be found that: $Sf_{\mathcal{N}(0,1)}(d_1) - e^{-r(T-t)}Kf_{\mathcal{N}(0,1)}(d_2) = 0$. From,

$$V_c(S,t) = SF_{\mathcal{N}(0,1)}(d_1) - Ke^{-r(T-t)}F_{\mathcal{N}(0,1)}(d_2),$$

we take derivatives with respect to S and t, Taking the derivative wrt S, gives us,

$$\begin{aligned}\Delta &= F_{\mathcal{N}(0,1)}(d_1) + SF_{\mathcal{N}(0,1)}(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}f_{\mathcal{N}(0,1)}(d_2)\frac{\partial d_2}{\partial S}\\ &= F_{\mathcal{N}(0,1)}(d_1) + \frac{f_{\mathcal{N}(0,1)}(d_1)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\frac{f_{\mathcal{N}(0,1)}(d_2)}{S\sigma\sqrt{T-t}},\end{aligned}$$

which gives us,

$$\Delta = F_{\mathcal{N}(0,1)}(d_1).$$

The derivative with respect to t gives us,

$$\begin{aligned}\frac{\partial V_c}{\partial t} &= \\ & Sf_{\mathcal{N}(0,1)}(d_1)\frac{\partial d_1}{\partial t} - Kre^{-r(T-t)}F_{\mathcal{N}(0,1)}(d_2) - Ke^{-r(T-t)}f_{\mathcal{N}(0,1)}(d_2)\frac{\partial d_2}{\partial t}.\end{aligned}$$

It follows that,

$$\frac{\partial d_1}{\partial t} = \frac{\partial d_2}{\partial t} - \frac{1}{2}\frac{\sigma}{\sqrt{T-t}},$$

nd with $Sf_{\mathcal{N}(0,1)}(d_1) - e^{-r(T-t)}Kf_{\mathcal{N}(0,1)}(d_2) = 0$, we find,

$$\frac{\partial V_c}{\partial t} = -Kre^{-r(T-t)}F_{\mathcal{N}(0,1)}(d_2) - \frac{Sf_{\mathcal{N}(0,1)}(d_1)\sigma}{2\sqrt{T-t}}$$

Further, we find for Γ,

$$\Gamma = \frac{\partial \Delta}{\partial S} = f_{\mathcal{N}(0,1)}(d_1)\frac{\partial d_1}{\partial S} = \frac{f_{\mathcal{N}(0,1)}(d_1)}{S\sigma\sqrt{T-t}}.$$

Substituting all derived expressions in the Black-Scholes equation gives us,

$$\begin{aligned}\frac{\partial V_c}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_c}{\partial S^2} + r\frac{\partial V_c}{\partial S}S - rV_c &= \frac{-S\sigma}{2\sqrt{T-t}}f_{\mathcal{N}(0,1)}(d_1) - rKe^{-r(T-t)}F_{\mathcal{N}(0,1)}(d_2)\\ &\quad + \frac{1}{2}\sigma^2 S^2\frac{f_{\mathcal{N}(0,1)}(d_1)}{S\sigma\sqrt{T-t}} + rSF_{\mathcal{N}(0,1)}(d_1)\\ &\quad - r\left(SF_{\mathcal{N}(0,1)}(d_1) - Ke^{-r(T-t)}F_{\mathcal{N}(0,1)}(d_2)\right)\\ &= 0.\end{aligned}$$

At the same time, it is easy to deduce that

$$\lim_{t\to T} V_c(t,S) = \max(S-K,0),$$

and that the solution also satisfies the boundary conditions for $S=0$ and $S\to\infty$.

b. We substitute the analytic expressions for each term in the BS equation,

$$\frac{\partial V}{\partial t} = rV + Ke^{-r(t-t)}F'_{\mathcal{N}(0,1)}(d_2)\frac{\partial d_2}{\partial t}.$$

Together with the term $-rV$, we find

$$\begin{aligned}\frac{\partial V}{\partial t} - rV &= Ke^{-r(t-t)}f_{\mathcal{N}(0,1)}(d_2)\left(-\frac{r-\sigma^2/2}{\sigma\sqrt{T-t}} + \frac{d_2}{2(T-t)}\right)\\ rS\frac{\partial V}{\partial S} &= rS\left(Ke^{-r(t-t)}\frac{\partial d_2}{\partial S}F'_{\mathcal{N}(0,1)}(d_2)\right) = Ke^{-r(t-t)}f_{\mathcal{N}(0,1)}(d_2)\frac{r}{\sigma\sqrt{T-t}}\\ \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} &= -\frac{1}{2}Ke^{-r(T-t)}f_{\mathcal{N}(0,1)}(d_2)\left(\frac{\sigma}{\sqrt{T-t}} + \frac{d_2}{T-t}\right),\end{aligned}$$

where $f_{\mathcal{N}(0,1)}(x)$ stands for the standard normal probability density function. All terms cancel and thus the BS equation is satisfied.

Ex. 3.11. *a.* See the code.

As seen in the figure, there are three ranges in the pay-off at the expiry date. At the left side of the diagram, the put option pay-off is positive, in the middle part the pay-off equals zero, and at the high asset value side one profits from the call option.

b. It can be valued via the BS equation with modified boundary conditions.

$$\begin{aligned}V_s(S,T) &= \max(S-K_1,0) + \max(K_2-S,0).\\ V_S(0,t) &= K_2e^{-r(T-t)}.\\ V_S(S,t) &= S - \mathrm{K}e^{-r(T-t)},\ S(t)\to\infty.\end{aligned}$$

c. If the volatility of the asset prices is high and investment profits are uncertain (because it is unknown in which direction the asset will move), an investor can buy a strangle.

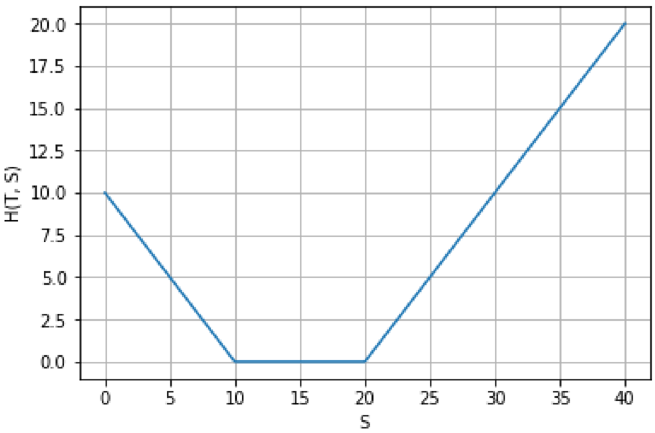

Figure 2: Results for Exercise 3.11a