Mathematical Modeling and Computation in Finance

With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 5

`https://QuantFinanceBook.com`

Ex. 5.1. *a.* We will consider the probability that the first asset $X_{1,mathcalP}$ jumps first. Suppose asset $X_{1,\mathcal{P}}$ jumps the first time at $t+\delta t$. At the same time, it is demanded that asset $X_{2,\mathcal{P}}$ did not jump before $t+\delta t$. This probability is given as,

$$\begin{aligned}&\mathbb{P}[X_{1,\mathcal{P}}(t+\delta t)-X_{1,\mathcal{P}}(t)=1]\mathbb{P}[X_{1,\mathcal{P}}(t)=0]\mathbb{P}[X_{2,\mathcal{P}}(t+\delta t)=0]\\&=\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})t}(\xi_{p,1}\delta t)+\mathcal{O}(\delta t^2).\end{aligned}$$

Hence, the probability density function for such an event reads,

$$f_{X_{\mathcal{P}}}(t)=\xi_{p,1}\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})t}. \tag{0.1}$$

The probability that asset $X_{1,\mathcal{P}}$ jumps first is given by

$$\mathbb{P}[T_{1,X_1}<T_{1,X_2}]=\int\xi_{p,1}\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})t}=\frac{\xi_{p,1}}{\xi_{p,1}+\xi_{p,2}}. \tag{0.2}$$

b. The requested probability is written as follows,

$$\begin{aligned}\mathbb{P}[T_{1,X_1}>t,T_{1,X_2}>t]=\mathbb{P}[X_{1,\mathcal{P}}(t)=0,X_{2,\mathcal{P}}(t)=0]=1-F_{X_{\mathcal{P}}}(t)=\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})t}\\ \implies f_{X_{\mathcal{P}}}(t)=(\xi_{p,1}+\xi_{p,2})\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})t}.\end{aligned}$$

c. We solve this part of the problem in the more general case, by focussing on the probability for the sum of two Poisson process,

$$\begin{aligned}\mathbb{P}[X_{1,\mathcal{P}}+X_{2,\mathcal{P}}(t)=k]&=\sum_{i=0}^{k}\mathbb{P}[X_{1,\mathcal{P}}+X_{2,\mathcal{P}}(t)=k,X_{2,\mathcal{P}}(t)=i]\\&=\sum_{i=0}^{k}\mathbb{P}[X_{1,\mathcal{P}}=k-i,X_{2,\mathcal{P}}(t)=i]\\&=\sum_{i=0}^{k}\mathbb{P}[X_{1,\mathcal{P}}=i]\mathbb{P}[X_{2,\mathcal{P}}=k-i]\\&=\sum_{i=0}^{k}\mathrm{e}^{-\xi_{p,2}t}\frac{(\xi_{p,2}t)^{k-i}}{(k-i)!}\mathrm{e}^{-\xi_{p,1}t}\frac{(\xi_{p,1}t)^i}{i!}\\&=\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})t}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}(\xi_{p,2}t)^{k-i}(\xi_{p,1}t)^i\\&=\mathrm{e}^{-(\xi_{p,1}+\xi_{p,2})}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(\xi_{p,2}t)^{k-i}(\xi_{p,1}t)^i\\&=\frac{((\xi_{p,1}+\xi_{p,2})t)^k}{k!}\cdot\mathrm{e}^{-(\xi_{p,2}+\xi_{p,1})}.\end{aligned}$$

The probability of having at least two jumps is found calculating $1-\mathbb{P}[X_{1,\mathcal{P}}+X_{2,\mathcal{P}}(t) < 2]$.

d The case of no jumps in the two assets in the next two years, where the jumps are modeled by independent Poisson processes, is given below. The intensities for the two years are given by $2\xi_{p,i}, i = 1.2$. So, we have

$$\begin{aligned}\mathbb{P}[X_{1,\mathcal{P}} + X_{2,\mathcal{P}}(t) = 0] &= \mathbb{P}[X_{1,\mathcal{P}} = 0]\mathbb{P}[X_{2,\mathcal{P}}(t) = 0] \\ &= \mathrm{e}^{2\xi_{p,1}+2\xi_{p,2}}.\end{aligned}$$

Ex. 5.3. We start as follows,

$$\mathbb{P}[X_{\mathcal{P}}(t) = i, X_{\mathcal{P}}(s) = j] = \mathbb{P}[X_{\mathcal{P}}(t) = i|X_{\mathcal{P}}(s) = j]\mathbb{P}[X_{\mathcal{P}}(s) = j].$$

The quantity $\mathbb{P}[X_{\mathcal{P}}(t) = i|X_{\mathcal{P}}(s) = j]$ gives the probability of observing i events until time t, given the condition that j events occurred during s. Hence, this probability can be expressed as the probability of observing $i-j$ events in the time period $t-s$.

$$\begin{aligned}\mathbb{P}[X_{\mathcal{P}}(t) = i|X_{\mathcal{P}}(s) = j]\mathbb{P}[X_{\mathcal{P}}(s) = j] &= \mathbb{P}[X_{\mathcal{P}}(t) - X_{\mathcal{P}}(s) = i-j]\mathbb{P}[X_{\mathcal{P}}(s) = j] \\ &= \frac{\mathrm{e}^{-\xi_p(t-s)}(\xi_p(t-s))^{i-j}}{(i-j)!}\frac{\mathrm{e}^{-\xi_p s}(\xi_p s))^j}{j!} \\ &= \frac{\mathrm{e}^{-\xi_p t_j}(\xi_p)^j(t_j-t_i))^{j-i}t_i^i}{j!}.\end{aligned}$$

We further simplify this expression, giving,

$$\mathbb{P}[X_{\mathcal{P}}(t) = i, X_{\mathcal{P}}(s) = j] = \frac{\mathrm{e}^{-\xi_p t}(\xi_p t)^i}{i!} \times \binom{i}{j}\left(1-\frac{s}{t}\right)^{i-j}\left(\frac{s}{t}\right)^j,$$

which is the joint probability of the Poisson process and a binomial distribution. To sum up, we find,

$$\mathbb{P}[X_{\mathcal{P}}(t) = i, X_{\mathcal{P}}(s) = j] = \mathrm{Poiss}(t,i) \times \mathrm{Binom}(i,j;s/t),$$

for $s < t$. Thus,

$$\mathbb{P}[X_{\mathcal{P}}(t_i) = i, X_{\mathcal{P}}(t_j) = j] = \mathbb{P}[X_{\mathcal{P}}(t_j) = j|X_{\mathcal{P}}(t_i) = i]\mathbb{P}[X_{\mathcal{P}}(t_i) = i].$$

The quantity $\mathbb{P}[X_{\mathcal{P}}(t_j) = j|X_{\mathcal{P}}(t_i) = i]$ indicates the probability of observing j events until time t_j given the condition i events occurred until time t_i. Hence, this probability can be expressed as the probability of observing $j-i$ events between $t_j - t_i$,

$$\begin{aligned}\mathbb{P}[X_{\mathcal{P}}(t_j) = j|X_{\mathcal{P}}(t_i) = i]\mathbb{P}[X_{\mathcal{P}}(t_i) = i] &= \mathbb{P}[X_{\mathcal{P}}(t_j) - X_{\mathcal{P}}(t_i) = j-i]\mathbb{P}[X_{\mathcal{P}}(t_i) = i] \\ &= \frac{\mathrm{e}^{-\xi_p(t_j-t_i)}(\xi_p(t_j-t_i))^{j-i}}{(j-i)!}\frac{\mathrm{e}^{-\xi_p t_i}(\xi_p t_i))^i}{i!} \\ &= \frac{\mathrm{e}^{-\xi_p t_j}(\xi_p)^j(t_j-t_i))^{j-i}t_i^i}{j!}.\end{aligned}$$

Ex. 5.5. *a*. This probability is found to be,

$$\mathbb{P}[X_{\mathcal{P}}(T) = 1] = \xi_p T\mathrm{e}^{-\xi_p T}.$$

b. The probability that there is exactly one asset jump in the interval $[0, T/2]$ and no jumps in $[T/2, T]$ is given by,

$$\mathbb{P}[X_{\mathcal{P}}(T/2) = 1]\mathbb{P}[X_{\mathcal{P}}(T) - X_{\mathcal{P}}(T/2) = 0] = \frac{\xi_p T}{2}\mathrm{e}^{-\xi_p T/2} \times \mathrm{e}^{-\xi_p T/2}.$$

The probability that there is no asset jump in the interval $[0, T/2]$ and only 1 jump in $[T/2, T]$ is given by,

$$\mathbb{P}[X_{\mathcal{P}}(T/2) = 0]\mathbb{P}[X_{\mathcal{P}}(T) - X_{\mathcal{P}}(T/2) = 1] = \mathrm{e}^{-\xi_p T/2} \times \frac{\xi_p T}{2}\mathrm{e}^{-\xi_p T/2}.$$

The sum of these two possibilities yields the initial probability $\mathbb{P}[X_{\mathcal{P}}(T) = 1]$.

Ex. 5.7. We find,

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{iuJ}] &= \int_{-\infty}^{\infty} \mathrm{e}^{iux} f_J(x)\mathrm{d}x = p_1\alpha_1 \int_0^{\infty} \mathrm{e}^{iux-\alpha_1 x}\mathrm{d}x + p_2\alpha_2 \int_{-\infty}^{0} \mathrm{e}^{iux+\alpha_2 x}\mathrm{d}x \\ &= \frac{p_1\alpha_1}{\alpha_1 - iu} + \frac{p_2\alpha_2}{\alpha_2 + iu}.\end{aligned}$$

Ex. 5.9. Assuming that we are dealing with a Lévy processes we see that by independence between the increments, i.e.: the returns $R(T_i)$ are independent of $R(T_j)$ for $i \neq j$.

With the introduced definitions the following holds:

$$S(T_i) = S(T_0)\mathrm{e}^{R(T_1)+R(T_2)+R(T_3)+\cdots+R(T_i)}. \tag{0.3}$$

This equality can be seen as:

$$\begin{aligned}S(T_i) &= S(T_0)\exp\left(\log\frac{S(T_1)}{S(T_0)} + \log\frac{S(T_2)}{S(T_1)} + \log\frac{S(T_3)}{S(T_2)} \cdots + \log\frac{S(T_i)}{S(T_{i-1})}\right) \\ &= S(T_0)\exp\left[\log\left(\frac{S(T_1)}{S(T_0)}\frac{S(T_2)}{S(T_1)}\frac{S(T_3)}{S(T_2)}\cdots\frac{S(T_{i-1})}{S(T_{i-2})}\frac{S(T_i)}{S(T_{i-1})}\right)\right] \\ &= S(T_0)\exp\left(\log\frac{S(T_i)}{S(T_0)}\right) \equiv S(T_i).\end{aligned}$$

Now, we define the following backward recursive sequence B_i for $i = 1, \ldots, N$:

$$B_i = R(T_{N-i+1}) + \log(1 + \exp(B_{i-1})), \tag{0.4}$$

with the initial condition $B_1 = R(T_N)$. The first few terms of this sequence are the following:

$$\begin{aligned}B_1 &= R(T_N) = \log(\frac{S(T_N)}{S(T_{N-1})}) \\ B_2 &= R(T_{N-1}) + \log\left(1 + \exp(R(T_N))\right) \\ &= \log\frac{S(T_{N-1})}{S(T_{N-2})} + \log\left(1 + \frac{S(T_N)}{S(T_{N-1})}\right) \\ &= \log\left(\frac{S(T_{N-1}) + S(T_N)}{S(T_{N-2})}\right) \\ B_3 &= R(T_{N-2}) + \log\left(1 + \frac{S(T_{N-1}) + S(T_N)}{S(T_{N-2})}\right) \\ &= \log\frac{S(T_{N-2})}{S(T_{N-3})} + \log\left(1 + \frac{S(T_{N-1}) + S(T_N)}{S(T_{N-2})}\right) \\ &= \log\left(\frac{S(T_{N-2}) + S(T_{N-1}) + S(T_N)}{S(T_{N-3})}\right).\end{aligned}$$

We see that the recursion formula for B_i involves two terms which are independent.

Based on this results we conclude that (the proof for this recursion still needs to be incorporated)

$$\begin{aligned}A &= \frac{1}{N}S(T_0)\mathrm{e}^{B_N} \\ &= \frac{1}{N}S(T_0)\mathrm{e}^{R(T_1)+\log(1+B_{N-1})}.\end{aligned}$$

It is easy to check the equation above. If, for example, take $N = 3$ then we have:

$$\begin{aligned}A &= \frac{1}{N}S(T_0)\mathrm{e}^{B_3} \\ &= \frac{1}{N}S(T_0)\exp\left[\log\left(\frac{S(T_1) + S(T_2) + S(T_3)}{S(T_0)}\right)\right] \\ &= \frac{1}{N}S(T_0)\mathrm{e}^{-\log(S(T_0))}\mathrm{e}^{\log(S(T_1)+S(T_2)+S(T_3))} \\ &= \frac{1}{N}\left(S(T_1) + S(T_2) + S(T_3)\right).\end{aligned}$$

Ex. 5.11. *a*. For set 1 we have the following convergence for $K = 40$, $T = 0.1$:

```
for n=1 value = 1.879279278955442
for n=2 value = 2.175090188405151
for n=3 value = 2.199844883233802
for n=4 value = 2.2012449990014047
for n=5 value = 2.201304960271733
for n=6 value = 2.2013070292619217
```

For set 2 we have the following convergence for $K = 40$, $T = 0.1$:

```
for n=1 value = 2.645529494206406
for n=2 value = 3.542803064574649
for n=3 value = 3.730578978057041
for n=4 value = 3.757992258032897
for n=5 value = 3.761060035284701
for n=6 value = 3.7613383781095355
```

For set 3 we have the following convergence for $K = 40$, $T = 0.1$:

```
for n=1 value = 1.2429527538910146
for n=2 value = 1.24495185788295
for n=3 value = 1.244953021913773
for n=4 value = 1.2449530223806056
```

Results for the convergence for all the strikes are presented in Figure 1. The following

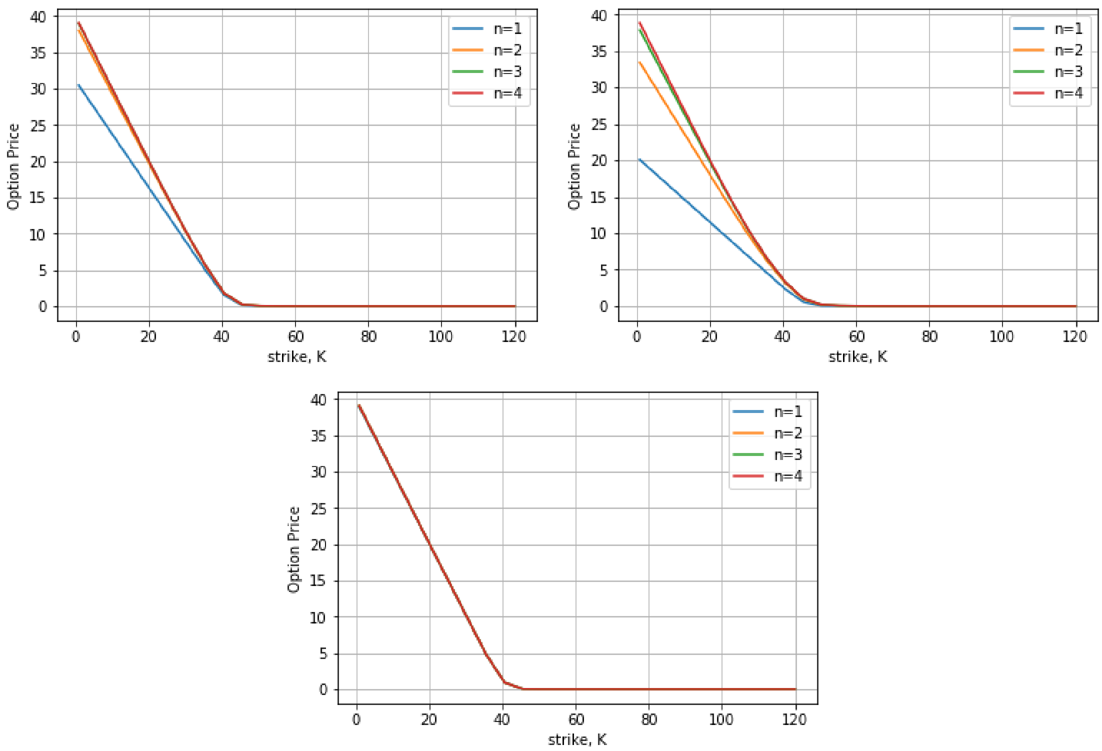

Figure 1: Results for Exercise 5.11, top left: set 1, top right: set 2, bottom: set 3.

computer code provides us with the answers and plots.

python

b. Black-Scholes price is given by Black-Scholes price = 1.1302297359820521 and it is generated by the code in point *a*.

c. For set 1 we have the following convergence for $K = 50$, $T = 0.1$:

```
for n=1 value = 0.00265247403143192
for n=2 value = 0.041938969590529354
for n=3 value = 0.046894598634580606
for n=4 value = 0.047229209104888904
for n=5 value = 0.04724508185782476
for n=6 value = 0.04724566787447372
Black-Scholes price = 0.0002144631682042715
```

For set 2 we have the following convergence for $K = 50$, $T = 0.1$:

```
for n=1 value = 0.0553722108843448
for n=2 value = 0.21563539503317633
for n=3 value = 0.25996143404492433
for n=4 value = 0.2673180539638608
for n=5 value = 0.2682066563813788
for n=6 value = 0.26829154013870804
Black-Scholes price = 0.0002144631682042715
```

For set 3 we have the following convergence for $K = 50$, $T = 0.1$:

```
for n=1 value = 0.00029835201260477334
for n=2 value = 0.0009022576205817332
for n=3 value = 0.0009026785623522327
for n=4 value = 0.0009026787435970574
for n=5 value = 0.0009026787436550093
for n=6 value = 0.0009026787436550241
Black-Scholes price = 0.0002144631682042715
```

The results for $T = 1$ and $T = 10$ can be generated with the accompanying python code.

Ex. 5.13. *a*. Solution of the SDE in (5.75) simply reads:

$$S(t) = S(t_0)\mathrm{e}^{\int_0^t \left(r - \frac{1}{2}J^2\right)\mathrm{d}t + JW(t)} \tag{0.5}$$

Now to price a European call option we use conditional expectation property therefore we have:

$$\begin{aligned}
V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-rT}\max(S(T) - K, 0)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-rT}\max(S(t_0)\mathrm{e}^{\int_0^t \left(r - \frac{1}{2}J^2\right)\mathrm{d}t + JW(t)} - K, 0)\right] \\
&= \mathbb{E}\left\{\mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max(S(t_0)\mathrm{e}^{\int_0^t \left(r - \frac{1}{2}J^2\right)\mathrm{d}t + JW(t)} - K, 0)\Big| J\right]\right\} \\
&= \sum_{i=1}^{2}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-rT}\max(S(t_0)\mathrm{e}^{\int_0^t \left(r - \frac{1}{2}J^2\right)\mathrm{d}t + JW(t)} - K, 0)\Big| J = \sigma_i\right]\mathbb{Q}\left[J = \sigma_i\right]
\end{aligned}$$

We immediately notice the expectation on the RHS to be simple the Black-Scholes price of the European call option.

b. The following computer code provides us with the plot presented in Figure 2.

python™

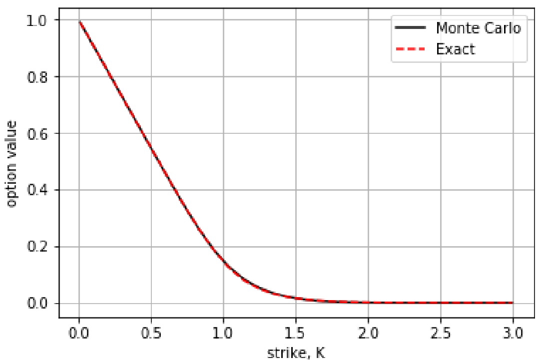

Figure 2: Results for Exercise 5.13 b