Mathematical Modeling and Computation in Finance
With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 2

`https://QuantFinanceBook.com`

Ex 2.1. *a.* Applying Itô's lemma on $g(t) = S^2(t)$, gives us,

$$\begin{aligned} dg(t) &= 2S(t)\mathrm{d}S(t) + \mathrm{d}S(t)\cdot\mathrm{d}S(t) \\ &= g(t)(2\mu+\sigma^2)\mathrm{d}t + 2g(t)\sigma\mathrm{d}W(t). \end{aligned}$$

The dynamics of the process are given by,

$$g(t) = e^{(2\mu-\sigma^2)t+2\sigma W(t)}.$$

b. We find,

$$dg(t) = \ln 2 \cdot 2^{W(t)}\mathrm{d}W(t) + \frac{(\ln 2)^2}{2}2^{W(t)}\mathrm{d}t.$$

Due to the appearance of a drift term, the process is not a martingale.

Ex. 2.3. Let $X(t), Y(t)$ be stochastic variables. Then, we have,

$$\begin{aligned} (X(t_{k+1} - X(t_k))(Y(t_{k+1}) - Y(t_k)) &= X(t_{k+1})Y(t_{k+1}) - X(t_k)Y(t_k) \\ &- X(t_k)(Y(t_{k+1}) - Y(t_k)) \\ &- Y(t_k)(X(t_{k+1}) - X(t_k)). \end{aligned}$$

Therefore, we have, with m time steps,

$$\begin{aligned} \sum_{k=1}^{m}(X(t_{k+1}) - X(t_k))(Y(t_{k+1}) - Y(t_k)) &= X(t_m)Y(t_m) - X(t_1)Y(t_1) \\ &- \sum_{k=1}^{m} X(t_k)(Y(t_{k+1}) - Y(t_k)) \\ &- \sum_{k=1}^{m} Y(t_k)(X(t_{k+1}) - X(t_k)). \end{aligned}$$

With the time step, $\Delta t \to 0$, the discrete sums become Itô integrals, leading to the requested form.

Ex. 2.5. We have

$$\mathbb{E}[S(t)] = \int_0^\infty s f(s) ds,$$

where $f(s)$ is the lognormal density function. Hence,

$$\mathbb{E}[S(t)] = \frac{1}{\sigma\sqrt{2\pi t}}\int_0^\infty \exp\left(\frac{-(\log(s/S_0) - (\mu-\sigma^2/2)t)^2}{2\sigma^2 t}\right) ds.$$

With a change of variables, $s = S_0 e^x$, it follows that,

$$\begin{aligned}
\mathbb{E}[S(t)] &= \frac{S_0}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^x \exp\left(\frac{-(x-(\mu-\sigma^2/2)t)^2}{2\sigma^2 t}\right) dx \\
&= \frac{S_0 e^{(\mu-1/2\sigma^2)t}}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^x \exp\left(\frac{-x^2}{2\sigma^2 t}\right) dx \\
&= \frac{S_0 e^{\mu t}}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2 t}\right) dx \\
&= \frac{S_0 e^{\mu t}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx \\
&= S_0 e^{\mu t}.
\end{aligned}$$

This result is the same when using the moment generating function. Furthermore,

$$\begin{aligned}
\mathbb{V}\text{ar}[S(t)] &= \frac{S_0^2}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{2x} \exp\left(\frac{-(x-(\mu-\sigma^2/2)t)^2}{2\sigma^2 t}\right) dx - (S_0 e^{\mu t})^2 \\
&= \frac{S_0^2 e^{2(\mu-\sigma^2/2)t}}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{2x} \exp\left(\frac{-x^2}{2\sigma^2 t}\right) dx - (S_0 e^{\mu t})^2 \\
&= \frac{S_0^2 e^{2(\mu+\sigma^2/2)t}}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2 t}\right) dx - (S_0 e^{\mu t})^2 \\
&= S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right).
\end{aligned}$$

Once again, it is possible to check this result using the moment generating function.

Ex. 2.7. By definition,

$$F_S(x) = \mathbb{P}[S \leq x] = \mathbb{P}[\log S \leq \log x] = F_X(\log x).$$

Hence,

$$F_S(x) = F_X(\log x).$$

The probability density function is defined as $f_S(x) := \frac{d}{dx}F_S(x)$. Hence, by taking derivatives at both sides, we find,

$$f_S(x) = \frac{1}{x} f_X(\log x),$$

which completes the proof.

Ex. 2.9. One can approach this exercise in two ways. First of all, one can find the expressions for $X(t), Y(t)$ and combine them to determine $Z(t)$. The other approach is based on the dynamics of $Z(t)$ in its differential form by means of Itô's lemma. We solve the problem by the first approach and check the result using the second.

a.

$$\begin{aligned}
X(t) &= e^{\sigma W(t) - \frac{\sigma^2}{2}t + 0.04t}, \\
\mathrm{d}X(t) &= 0.04X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t).
\end{aligned}$$

$$\begin{aligned}
Y(t) &= e^{0.15W(t) - \frac{0.15^2}{2}t + \beta t}, \\
\mathrm{d}Y(t) &= \beta Y(t)\mathrm{d}t + 0.15Y(t)\mathrm{d}W(t).
\end{aligned}$$

Using the expressions for $X(t)$ and $Y(t)$, we get,

$$Z(t) = 2e^{(\sigma-0.15)W(t) + (0.04 + \frac{0.15^2}{2} - \beta - \frac{\sigma^2}{2})t} - \lambda t.$$

b. A martingale process does not contain a drift term. We have,

$$\mathrm{d}Z(t) = (Z + \lambda t)(0.15^2 + 0.04 - \beta - 0.15\sigma)\mathrm{d}t - \lambda \mathrm{d}t + (Z + \lambda t)(\sigma - 0.15)\mathrm{d}W(t).$$

With β and σ constant, and $\lambda \in \mathbb{R}^+$, necessary conditions for a vanishing drift term are $\lambda = 0$ and

$$0.15^2 + 0.04 - \beta - 0.15\sigma = 0 \implies \beta = 0.04 - 0.15\sigma + 0.15^2.$$

To check this result we employ the Itô's derivative rules for multivariable functions, i.e.,

$$\begin{aligned}\mathrm{d}Z(t) &= 2\left(\frac{\mathrm{d}X(t)}{Y(t)} - \frac{X(t)\mathrm{d}Y(t)}{Y^2(t)} - \frac{\mathrm{d}X(t)\mathrm{d}Y(t)}{Y^2(t)} + \frac{X(t)\mathrm{d}Y^2(t)}{Y^3(t)}\right) - \lambda \mathrm{d}t \\ &= (Z(t) + \lambda t)\big((0.04 - \beta - 0.15\sigma + 0.15^2)\mathrm{d}t + (\sigma - 0.15)\mathrm{d}W(t)\big) - \lambda \mathrm{d}t,\end{aligned}$$

which yields the same constraints. Hence, $\lambda = 0$ and $\beta = 0.04 - 0.15\sigma + 0.15^2$.