Mathematical Modeling and Computation in Finance
With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 10

`https://QuantFinanceBook.com`

Ex. 10.1. We find,

$$\begin{aligned}
V^{\text{fwd}} &= \mathrm{e}^{-rT_2}\int_{-\infty}^{\infty}\max(K^* - \mathrm{e}^{(r-0.5\sigma^2)(T_2-T_1)+\sigma\sqrt{T_2-T_1}x})f(x)\mathrm{d}x.\\
&= \mathrm{e}^{-rT_2}\int_{-\infty}^{a}(K^* - \mathrm{e}^{(r-0.5\sigma^2)(T_2-T_1)+\sigma\sqrt{T_2-T_1}x})f(x)\mathrm{d}x.\\
&= \mathrm{e}^{-rT_2}K^*F_{\mathcal{N}(0,1)}(a) - \mathrm{e}^{-rT_2}\int_{-\infty}^{a}\mathrm{e}^{(r-0.5\sigma^2)(T_2-T_1)+\sigma\sqrt{T_2-T_1}x}f(x)\mathrm{d}x,
\end{aligned}$$

where,

$$a = \frac{1}{\sigma\sqrt{T_2-T_1}}(\log K^* - (r-0.5\sigma^2)(T_2-T_1)),$$

and,

$$\mathrm{e}^{-rT_2}\int_{-\infty}^{a}\mathrm{e}^{(r-0.5\sigma^2)(T_2-T_1)+\sigma\sqrt{T_2-T_1}x}f(x)\mathrm{d}x = \mathrm{e}^{-T_1}F_{\mathcal{N}(0,1)}(a-\sigma\sqrt{T_2-T_1}).$$

Hence, the overall expression becomes,

$$V^{\text{fwd}} = \mathrm{e}^{-rT_2}K^*F_{\mathcal{N}(0,1)}(a) - \mathrm{e}^{-T_1}F_{\mathcal{N}(0,1)}(a-\sigma\sqrt{T_2-T_1}).$$

Replacing $d_1 = \sigma\sqrt{T_2-T_1} - a$, we find,

$$V^{\text{fwd}} = \mathrm{e}^{-rT_2}K^*F_{\mathcal{N}(0,1)}(-d_2) - \mathrm{e}^{-T_1}F_{\mathcal{N}(0,1)}(-d_1),$$

which is the value of the forward start put option.

Ex. 10.3. *a.* To derive the partial differential equation for the characteristic function, we use the Feynman-Kac Theorem. Let's first provide the theorem in its most general form. Consider the following PDE, subject to the terminal condition $V(T,x) = H(x)$,

$$\frac{\partial V}{\partial t} + \mu(t,x)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial V^2}{\partial x^2} - r(t,x)V + g(t,x) = 0.$$

The solution to this equation is given by,

$$V(t,x) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-\int_t^u r(z,X)\mathrm{d}z}g(u,X)\mathrm{d}u + \mathrm{e}^{-\int_t^T r(X,z)\mathrm{d}z}H(X(T))|X=x\right],$$

where

$$\mathrm{d}X(t) = \mu(t,X)\mathrm{d}t + \sigma(t,X)\mathrm{d}W(t).$$

We will apply this general theorem to the particular case at hand. In this case,

$$\begin{aligned}
\mathrm{d}X(t) &= -\frac{1}{2}\sigma^2(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \quad r(t,X) = 0,\\
g(X,t) &= 0, \quad H(X(T)) = \mathrm{e}^{iuX(T)}, \quad \phi(t,x) = V(t,x).
\end{aligned}$$

In this case, the partial differential equation reads,

$$\frac{\partial \phi(x,t)}{\partial t} - \frac{1}{2}\sigma^2(x,t)\frac{\partial \phi(x,t)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial \phi^2(x,t)}{\partial x^2} = 0.$$

b. By inserting (10.41) into (10.40) with $\bar{A} := \bar{A}(u,\tau)$ and $\bar{B} := \bar{B}(u,\tau)$, we get the following equation

$$\phi^{T_2}\left(\frac{\partial \bar{A}}{\partial t} + x\frac{\partial \bar{B}}{\partial t}\right) - \frac{1}{2}\bar{\sigma}_F^2(t)\bar{B}\phi^{T_2} + \frac{1}{2}\bar{\sigma}_F^2(t)\bar{B}^2\phi^{T_2} = 0.$$

Dividing both sides by ϕ^{T_2}, gives us,

$$\frac{\partial \bar{A}}{\partial t} + x\frac{\partial \bar{B}}{\partial t} - \frac{1}{2}\bar{\sigma}_F^2(t)\bar{B} + \frac{1}{2}\bar{\sigma}_F^2(t)\bar{B}^2 = 0.$$

This implies the following system of equations to be solved:

$$x\frac{\partial \bar{B}}{\partial t} = 0, \quad \text{and} \quad \frac{\partial \bar{A}}{\partial t} - \frac{1}{2}\bar{\sigma}_F^2(t)\bar{B} + \frac{1}{2}\bar{\sigma}_F^2(t)\bar{B}^2 = 0.$$

With $\tau = T - t$, dividing the first equation by x, gives,

$$\frac{\partial \bar{B}}{\partial \tau} = 0, \quad \text{and} \quad \frac{\partial \bar{A}}{\partial \tau} = -\frac{1}{2}\bar{\sigma}_F^2(T-\tau)\bar{B} + \frac{1}{2}\bar{\sigma}_F^2(T-\tau)\bar{B}^2,$$

with initial conditions $\bar{B}(0,u) = iu$ and $\bar{A}(0,u) = 0$ at $\tau = 0$. The solution of the system is given by,

$$\bar{B}(u,\tau) = iu, \quad \text{and} \quad \bar{A}(u,\tau) = -\frac{1}{2}u(i+u)\int_0^\tau \bar{\sigma}_F^2(T-\tau)\mathrm{d}\tau.$$

Ex. 10.5. *a.* In the first part of the exercise, we perform a Monte Carlo simulation and compute the expectation by conditioning on a small interval around y_1. In this part, there is dependence of the error on the number of paths.

Result for Set 1:

```
Analytical expression for y1=1.75 yields E[Y2|Y1=1.75] = 0.0319274171186213
Monte Carlo, for y1=1.75 yields E[Y2|Y1=1.75] = 0.03304459258769258
```

Result for Set 2:

```
Analytical expression for y1=1.75 yields E[Y2|Y1=1.75] = 0.008276871935581484
Monte Carlo, for y1=1.75 yields E[Y2|Y1=1.75] = 0.011217891765021708
```

When the same experiment is repeated for the other set, with $\sigma_1 = \sigma_2 = 0.9$, errors increase.

b. In the second part, a nonparametric method with bins is implemented. The method provides a much better accuracy, even when the number of paths is considerably lower than the number of paths used in the basic Monte Carlo simulation. Moreover, there appears to be an optimal ratio of the number of paths and the number of bins. Numerical results are presented in Figure 1.

The codes for points *a.* and *b.* are available below the following Python icon:

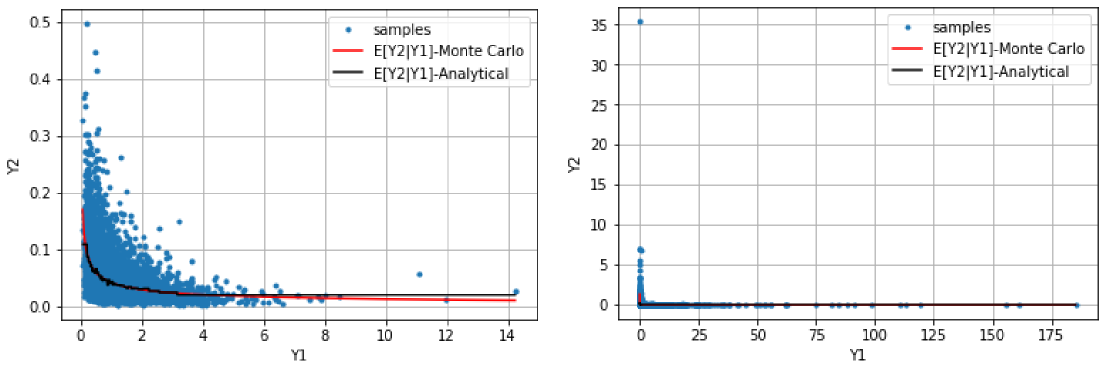

Figure 1: Results for Exercise 10.5.

Ex. 10.7. For the Bates model, only the $\bar{A}(u,\tau)$ function in the Heston model needs to be modified. The value of the forward start option changes in the same manner, i.e., only the function $\bar{A}(u,\tau)$ is modified.

The pricing of forward start options, under the assumption of constant interest rates, boils down to finding an expression for the following characteristic function (Section 10.1.3):

$$\phi_x(u) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-iuX(T_1)}\mathrm{e}^{r(T_2-T_1)}\psi_X(u,T_1,T_2)\Big|\mathcal{F}(t_0)\right]. \tag{0.1}$$

Under the Bates model, $\psi_X(u,T_1,T_2)$ reads:

$$\psi_X(u,T_1,T_2) = \mathrm{e}^{\bar{A}_{\text{Bates}}(u,\tau)+\bar{B}(u,\tau)X(T_1)+\bar{C}(u,\tau)v(T_1)}. \tag{0.2}$$

Thus, for $\bar{B}(u,\tau) = iu$, we find,

$$\begin{aligned}\phi_x(u) \quad &= \quad \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-iuX(T_1)}\mathrm{e}^{r(T_2-T_1)}\mathrm{e}^{\bar{A}_{\text{Bates}}(u,\tau)+iuX(T_1)+\bar{C}(u,\tau)v(T_1)}\Big|\mathcal{F}(t_0)\right]\\ &= \quad \mathrm{e}^{r(T_2-T_1)}\mathrm{e}^{\bar{A}_{\text{Bates}}(u,\tau)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\bar{C}(u,\tau)v(T_1)}\Big|\mathcal{F}(t_0)\right],\end{aligned} \tag{0.3}$$

with

$$\bar{A}_{\text{Bates}}(u,\tau) = \bar{A}(u,\tau) - \xi_p iu\tau\left(\mathrm{e}^{\mu_J+1/2\sigma_J^2}-1\right) + \xi_p\tau\left(\mathrm{e}^{iu\mu_J-1/2\sigma_J^2u^2}-1\right).$$

Note that the expression in (0.3) is derived analytically and can be found in Section 10.1.3.

The impact of the model parameters is presented in Figure 2.

Below, the codes to generate numerical results can be found. In the experiments, the first and the second maturity have been verified.

Ex. 10.9. The computer code under the Python icon below provides us with the plot in Figure 3. Note that in this exercise we perform the hedging experiment with the Black-Scholes model while the market data is generated with the Heston model. In such a configuration, it is important to "calibrate" the Black-Scholes volatility parameter σ. In the experiment, we take $\sigma = \bar{v}$, however, in practice, this parameter would be calibrated.

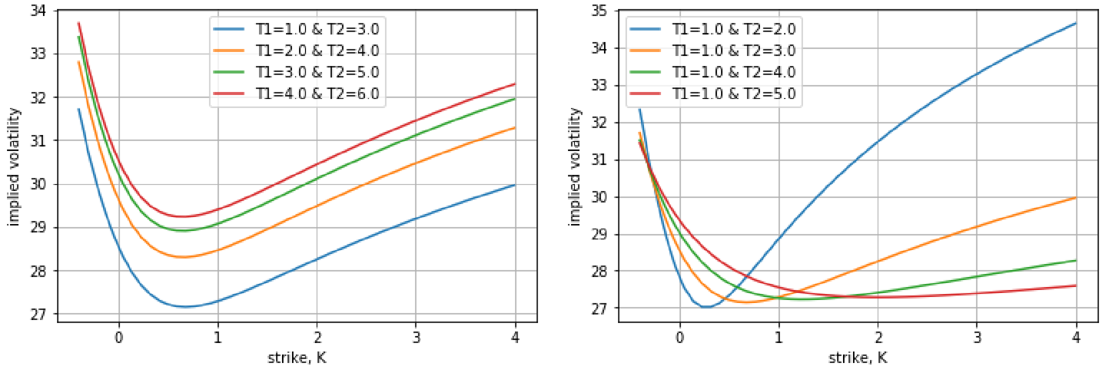

Figure 2: Results for Exercise 10.7.

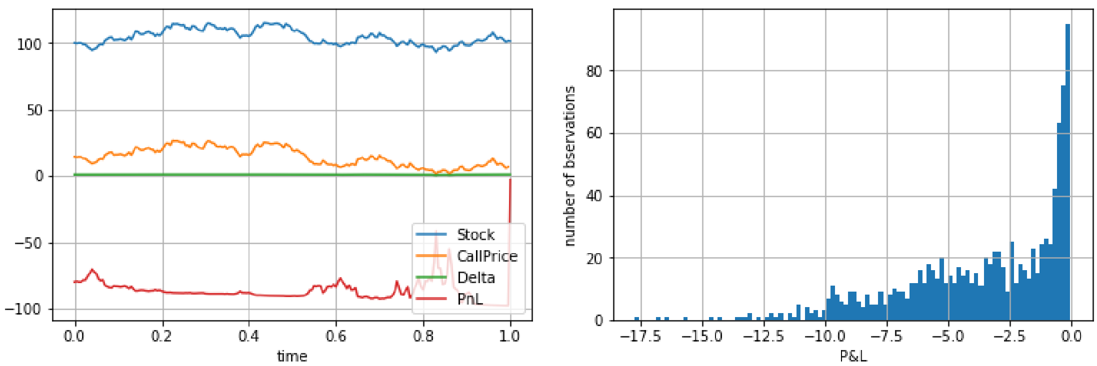

Figure 3: Results for Exercise 10.9.