Mathematical Modeling and Computation in Finance

With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 7

`https://QuantFinanceBook.com`

Ex. 7.1. By the Cholesky decomposition, we can express the dependent random processes in terms of independent processes. The Cholesky decomposition of three random variables is given by $\mathbf{C} = \mathbf{L}\mathbf{L}^T$, with,

$$\mathbf{L} = \begin{bmatrix} \sqrt{C_{11}} & 0 & 0 \\ C_{21}/L_{11} & \sqrt{C_{22} - L_{21}^2} & 0 \\ C_{31}/L_{11} & (C_{32} - L_{31}L_{21})/L_{22} & \sqrt{C_{33} - L_{31}^2 - L_{32}^2} \end{bmatrix}.$$

The correlated random variables are expressed in terms of uncorrelated variables as $\mathbf{L}\tilde{\mathbf{W}} = \mathbf{W}$. Here, the correlation matrix is given by,

$$\mathbf{C} = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}.$$

In this case, $\mathbf{L}$ becomes,

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ \rho & (\rho - \rho^2)/\sqrt{1-\rho^2} & \sqrt{1-2\rho^2} \end{bmatrix}.$$

Hence,

$$\begin{aligned} W_1 &= \tilde{W}_1, \\ W_2 &= \rho\tilde{W}_1 + \sqrt{1-\rho^2}\tilde{W}_2, \\ W_3 &= \rho\tilde{W}_1 + \frac{\rho-\rho^2}{\sqrt{1-\rho^2}}\tilde{W}_2 + \sqrt{1-2\rho^2}\tilde{W}_3. \end{aligned}$$

So, we find,

$$\frac{\mathrm{d}S(t)}{S(t)} = \frac{3}{2}\mathrm{d}t + (1+2\rho)\mathrm{d}\tilde{W}_1 + \frac{1+\rho-2\rho^2}{\sqrt{1-\rho^2}}\mathrm{d}\tilde{W}_2 + \sqrt{1-2\rho^2}\mathrm{d}\tilde{W}_3.$$

Note that $\mathrm{d}(\log(S)) = \frac{\mathrm{d}S}{S} - \frac{1}{2}\frac{\mathrm{d}S^2}{S^2}$. We denote the coefficients of the Brownian motions by α, β, γ, respectively. So,

$$\mathrm{d}X(t) = \left(\frac{3}{2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\right)\mathrm{d}t + \alpha\mathrm{d}\tilde{W}_1 + \beta\mathrm{d}\tilde{W}_2 + \gamma\mathrm{d}\tilde{W}_3,$$

where $X(t) = \log(S(t))$. This gives,

$$X(t) = \log(S_0) + \left(\frac{3}{2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\right) t + \alpha\tilde{W}_1(t) + \beta\tilde{W}_2(t) + \gamma\tilde{W}_3(t).$$

It follows that,

$$X(t) \sim \mathcal{N}(\mu, \sigma^2 t),$$

where,

$$\begin{aligned}\mu &= \log(S_0) + \left(\frac{3}{2} - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\right) t,\\ \sigma^2 &= \alpha^2 + \beta^2 + \gamma^2.\end{aligned}$$

The following computer code, under the Python icon, provides us with the plot presented in Figure 1.

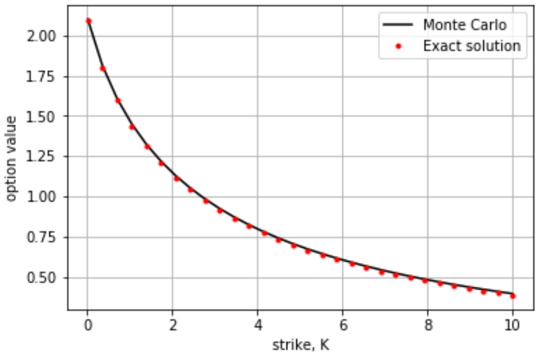

Figure 1: Results for Exercise 7.1.

Ex. 7.3. *a.* We find,

$$\bar{\sigma}(S,t)\bar{\sigma}(S,t) = \sigma(t,.)^2 S^2, \tag{0.1}$$

which indicates that the process does not belong to the class of affine processes.

b. In this case,

$$F_S(x) = \mathbb{E}[S \leq x] = \mathbb{E}[\log(S) \leq \log(x)] = \mathbb{E}[X \leq \log(x)] = F_X(\log(x)).$$

Taking derivatives at both sides, gives us,

$$f_S(x) = \frac{1}{x} f_X(\log(x)).$$

Ex. 7.5. Under the assumption of deterministic interest rates, we have:

$$M(T) = \mathrm{e}^{\int_0^T r(s)\mathrm{d}s},$$

while for the Zero-Coupon Bond $P(0,T)$ we have:

$$P(0,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r(s)\mathrm{d}s}\middle|\mathcal{F}(t_0)\right] = \mathrm{e}^{-\int_0^T r(s)\mathrm{d}s},$$

thus $P(0,T) = \frac{1}{M(T)}$ and, in the more generic setting, $P(t,T) = \frac{M(t)}{M(T)}$.

Now, we consider the dynamics for $S^F(t,T) = \frac{S(t)}{P(t,T)}$:

$$\mathrm{d}S^F(t,T) \quad := \quad \mathrm{d}\left(\frac{S(t)}{P(t,T)}\right) = \frac{\mathrm{d}S(t)}{P(t,T)} - \frac{S(t)}{P^2(t,T)}\mathrm{d}P(t,T). \tag{0.2}$$

Since $\mathrm{d}P(t,T) = -r(t)P(t,T)\mathrm{d}t$, we can write:

$$\begin{aligned}\mathrm{d}S^F(t,T) &= \frac{\mathrm{d}S(t)}{P(t,T)} - \frac{S(t)}{P^2(t,T)}\mathrm{d}P(t,T)\\ &= \frac{\mathrm{d}S(t)}{P(t,T)} - \frac{S(t)}{P^2(t,T)}\mathrm{d}P(t,T) = \frac{\mathrm{d}S(t)}{P(t,T)} - \frac{S(t)P'(t,T)}{P^2(t,T)}\mathrm{d}t,\end{aligned}$$

where $P'(t,T) = -r(t)P(t,T)$. The process for $S(t)$ is defined as:

$$\mathrm{d}S(t) = \sigma(\vartheta S(t) + (1-\vartheta)S_0)\mathrm{d}W^{\mathbb{P}}(t) \tag{0.3}$$

thus after substitution we have:

$$\mathrm{d}S^F(t,T) = \frac{1}{P(t,T)}\sigma(\vartheta S(t) + (1-\vartheta)S_0)\mathrm{d}W^{\mathbb{P}}(t) - \frac{S(t)P'(t,T)}{P^2(t,T)}\mathrm{d}t,$$

We know that $S^F(t,T)$ is a martingale under the $T-$forward measure, so it needs to be a process without drift under $\mathbb{Q}^T$. This implies the following measure change:

$$\frac{1}{P(t,T)}\sigma(\vartheta S(t) + (1-\vartheta)S_0)\mathrm{d}W^{\mathbb{P}}(t) - \frac{S(t)P'(t,T)}{P^2(t,T)}\mathrm{d}t = \frac{1}{P(t,T)}\sigma(\vartheta S(t) + (1-\vartheta)S_0)\mathrm{d}W^T(t),$$

which after simplifications becomes:

$$\mathrm{d}W^{\mathbb{P}}(t) - \frac{S(t)P'(t,T)}{P(t,T)\sigma(\vartheta S(t) + (1-\vartheta)S_0)}\mathrm{d}t = \mathrm{d}W^T(t).$$

By applying a measure change $\frac{S(t)}{P(t,T)}$ should be a martingale and the dynamics $\mathrm{d}S(t)$ under the $T-$forward measure are given by,

$$\begin{aligned}\mathrm{d}S(t) &= \sigma(\vartheta S(t) + (1-\vartheta)S_0)\left(\mathrm{d}W^T(t) + \frac{S(t)P'(t,T)}{P(t,T)\sigma(\vartheta S(t) + (1-\vartheta)S_0)}\mathrm{d}t\right)\\ &= \frac{S(t)P'(t,T)}{P(t,T)}\mathrm{d}t + \sigma(\vartheta S(t) + (1-\vartheta)S_0)\mathrm{d}W^T(t).\end{aligned}$$

Ex. 7.7. Under the log transformation for the stock, $X(t) = \log S(t)$, we find,

$$\begin{aligned}\mathrm{d}X(t) &= -\frac{1}{2}\sigma^2(t)\mathrm{d}t + \sigma(t)\mathrm{d}\widetilde{W}_1(t),\\ \mathrm{d}\sigma(t) &= \gamma\sigma(t)\left[\rho\mathrm{d}\widetilde{W}_1(t) + \sqrt{1-\rho^2}\mathrm{d}\widetilde{W}_2(t)\right].\end{aligned}$$

By integrating the SDEs, we get,

$$\begin{aligned}X(T) &= X_0 - \frac{1}{2}\int_0^T \sigma^2(t)\mathrm{d}t + \int_0^T \sigma(t)\mathrm{d}\widetilde{W}_1(t),\\ \sigma(T) &= \sigma(t_0) + \gamma\rho\int_0^T \sigma(t)\mathrm{d}\widetilde{W}_1(t) + \gamma\sqrt{1-\rho^2}\int_0^T \sigma(t)\mathrm{d}\widetilde{W}_2(t).\end{aligned}$$

The integrated variance w.r.t. the Brownian motions is now given by,

$$\int_0^T \sigma(t)\mathrm{d}\widetilde{W}_1(t) = \frac{1}{\gamma\rho}\left[\sigma(T) - \sigma(t_0) - \gamma\sqrt{1-\rho^2}\int_0^T \sigma(t)\mathrm{d}\widetilde{W}_2(t)\right].$$

After substitutions, we obtain,

$$X(T) \quad = \quad X_0 - \frac{1}{2}\int_0^T \sigma^2(t)\mathrm{d}t + \frac{1}{\gamma\rho}\left[\sigma(T) - \sigma(t_0) - \gamma\sqrt{1-\rho^2}\int_0^T \sigma(t)\mathrm{d}\widetilde{W}_2(t)\right],$$

The problem at this point is that we have three unknowns at the right-hand side of the equation above, $\sigma(T)$, $\int_0^T \sigma^2(t)\mathrm{d}t$ and $\int_0^T \sigma(t)\mathrm{d}\widetilde{W}_2(t)$.

Ex. 7.9. The value of the exchange option is the expected value of the discounted payoff function under the risk neutral measure,

$$V^{ex}(t, S_1(t), S_2(t)) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\max(S_1(T) - S_2(T), 0)|\mathcal{F}(t)\right].$$

Changing the numeraire to become the second asset in the problem, gives us,

$$V^{ex}(t, S_1(t), S_2(t)) = \mathbb{E}^{S_2}\left[\frac{\lambda_{S_2}^{\mathbb{Q}}(T)S_2(T)}{M(T)}\max(S_1(T)/S_2(T) - 1, 0)|\mathcal{F}(t)\right].$$

The Radon-Nikodym derivative $\lambda_{S_2}^{\mathbb{Q}}$ is given by,

$$\lambda_{S_2}^{\mathbb{Q}} = \frac{M(T)S_2(t_0)}{M(t_0)S_2(T)},$$

and the value of the option becomes,

$$V^{ex}(t, S_1(t), S_2(t)) = \frac{S_2(t_0)}{M(t_0)}\mathbb{E}^{S_2}\left[\max(S_1(T)/S_2(T) - 1, 0)|\mathcal{F}(t)\right].$$

We make sure that all processes are martingales under measure S_2, i.e.,

$$\begin{aligned} \mathrm{d}\left(\frac{M}{S_2}\right) &= \frac{M}{S_2}(\sigma_2^2\mathrm{d}t - \sigma_2\mathrm{d}W_2^{\mathbb{Q}}) \\ \implies \mathrm{d}W_2^{S_2} &= -\sigma_2\mathrm{d}t + \mathrm{d}W_2^{\mathbb{Q}}. \end{aligned}$$

Under this measure, the dynamics for process S_2 become,

$$\mathrm{d}S_2 = (r + \sigma_2^2)S_2\mathrm{d}t + \sigma_2 S_2\mathrm{d}W_2^{S_2}.$$

The other process that needs to be a martingale is S_1/S_2. In other words,

$$\begin{aligned} \mathrm{d}\left(\frac{S_1}{S_2}\right) &= \frac{S_1}{S_2}\left((\sigma_2^2 - \rho\sigma_1\sigma_2)\mathrm{d}t + \sigma_1\mathrm{d}W_1^Q - \sigma_2\mathrm{d}W_2^Q\right) \\ &= \frac{S_1}{S_2}\left(-\rho\sigma_1\sigma_2\mathrm{d}t + \sigma_1\mathrm{d}W_1^Q - \sigma_2\mathrm{d}W_2^{S_2}\right) \\ \implies \mathrm{d}W_1^{S_2} &= -\rho\sigma_2\mathrm{d}t + \mathrm{d}W_1^Q. \end{aligned}$$

We find,

$$\begin{aligned} \mathrm{d}\left(\frac{S_1}{S_2}\right) &= \frac{S_1}{S_2}\left(\sigma_1\mathrm{d}W_1^{S_2} - \sigma_2\mathrm{d}W_2^{S_2}\right) \\ \mathrm{d}S_1 &= (r + \rho\sigma_1\sigma_2)S_1\mathrm{d}t + \sigma_1 S_1\mathrm{d}W_1^{S_2}. \end{aligned}$$

The dynamics $\mathrm{d}\left(\frac{S_1}{S_2}\right)$ can be expressed in terms of a single Brownian motion with variance. Note that,

$$\mathbb{V}\mathrm{ar}[X + Y] = \mathbb{V}\mathrm{ar}[X] + \mathbb{V}\mathrm{ar}[Y] + 2\mathrm{Cov}[X, Y],$$

giving,

$$\sigma_{(W_1^{S_2}+W_2^{S_2})} = \sqrt{\sigma_1^2 + 2\sigma_1\sigma_2\rho + \sigma_2^2}.$$

The expected value can be calculated by means of the Black-Scholes equation, with $r = 0$ and $\sigma = \sqrt{\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2}$. So,

$$V^{ex}(t_0, S_1(t_0), S_2(t_0)) = S_{1,0}F_{\mathcal{N}(0,1)}(d_1) - S_{2,0}F_{\mathcal{N}(0,1)}(d_2),$$

where d_1, d_2 have the same form as in the Black-Scholes solution, with the parameters $r \to 0$, $\sigma \to \sqrt{\sigma_1^2 - 2\sigma_1\sigma_2\rho + \sigma_2^2}$ and $S_0 \to \frac{S_{1,0}}{S_{2,0}}$.