Mathematical Modeling and Computation in Finance

With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 8

https://QuantFinanceBook.com

Ex. 8.1. The Itô derivative of $Y(t)$ with respect to $X(t)$ is given by,

$$\begin{aligned}
\mathrm{d}Y(t) &= \frac{\partial Y(t)}{\partial X}\mathrm{d}X(t) + \frac{1}{2}\frac{\partial^2 Y(t)}{\partial X^2}(\mathrm{d}X(t)\cdot\mathrm{d}X(t))\\
&= \frac{\mathrm{d}X(t)}{2\sqrt{X(t)}} - \frac{1}{8}\frac{\mathrm{d}X(t)\cdot\mathrm{d}X(t)}{X^{3/2}(t)}\\
&= \left(\mu Y(t) + \frac{3\sigma^2}{8Y(t)}\right)\mathrm{d}t + \frac{\sigma^2}{2}\mathrm{d}W(t).
\end{aligned}$$

Ex. 8.3. In the Schöbel-Zhu model, the volatility process is driven by the following OU dynamics:

$$\mathrm{d}\sigma(t) = \kappa(\bar{\sigma} - \sigma(t))\mathrm{d}t + \gamma\mathrm{d}W_\sigma(t), \quad \sigma(t_0) = \sigma_0.$$

From Section 11.3.1 we know that $\sigma(t)$ is normally distributed $\mathcal{N}(\mu_\sigma(t), \sigma_\sigma^2(t))$ where:

$$\begin{aligned}
\mu_\sigma(t) &:= \mathbb{E}[\sigma(t)] = \sigma_0\mathrm{e}^{-\kappa t} + \bar{\sigma}\left(1 - \mathrm{e}^{-\kappa t}\right),\\
\sigma_\sigma^2(t) &:= \mathbb{V}\mathrm{ar}[\sigma(t)] = \frac{\gamma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa t}\right).
\end{aligned}$$

Now, for $v(t) = \sigma^2(t)$ and by definition of the CDF we have:

$$F_{v(t)}(x) = \mathbb{P}[v(t) \le x] = \mathbb{P}[\sigma^2(t) \le x] = \mathbb{P}[|\sigma(t)| \le \sqrt{x}], \tag{0.1}$$

which further can be written as:

$$F_{v(t)}(x) = \mathbb{P}[|\sigma(t)| \le \sqrt{x}] = \mathbb{P}[-\sqrt{x} \le \sigma(t) \le \sqrt{x}] = F_{\sigma(t)}(\sqrt{x}) - F_{\sigma(t)}(-\sqrt{x}).$$

By differentiating with respect to x, we find:

$$f_{v(t)}(x) := \frac{\mathrm{d}}{\mathrm{d}x}F_{v(t)}(x) = \frac{1}{2\sqrt{x}}\left(f_{\sigma(t)}(\sqrt{x}) + f_{\sigma(t)}(-\sqrt{x})\right), \tag{0.2}$$

which can be further simplified using the properties of normal probability density function.

Ex. 8.5. We start with,

$$\frac{V(t,S,v)}{M(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(t)}V(T,S,v)|\mathcal{F}_t\right].$$

The process is a martingale when the corresponding differential form is free of any $\mathrm{d}t$-terms. With,

$$\mathrm{d}\left(\frac{V}{M}\right) = \left(\frac{\mathrm{d}V}{M} - r\frac{V}{M}\mathrm{d}t\right),$$

the multi-dimensional Itô derivative reads,

$$\mathrm{d}V(S,v;t) = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS\cdot dS) + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}(dv\cdot dv) + \frac{\partial^2 V}{\partial S\partial v}(dS\cdot dv).$$

So,

$$\begin{aligned}\mathrm{d}V(S,v,t) = &\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa(\bar{v} - v(t))\frac{\partial V}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2} + \gamma v S\frac{\partial^2 V}{\partial S\partial v}\right)\mathrm{d}t\\ &+ \sqrt{v}S\frac{\partial V}{\partial S}\mathrm{d}W_x^{\mathbb{Q}} + \gamma\sqrt{v}\frac{\partial V}{\partial v}S\mathrm{d}W_x^{\mathbb{Q}}.\end{aligned}$$

Including the term $-r\frac{V}{M}dt$ and imposing the martingale condition, i.e., all dt-terms should vanish, gives us the option pricing PDE for the Heston model,

$$\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa(\bar{v} - v(t))\frac{\partial V}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2} + \gamma v S\frac{\partial^2 V}{\partial S\partial v} - rV\right) = 0.$$

Ex. 8.7. In the computation of Δ and Γ, we use the COS method and the expressions presented in Section 6.2.2, where it was shown that Δ and Γ can be easily computed by adjusting the coefficients in the COS method.

The computer code under the Python icon provides us with the plot presented in Figure 1.

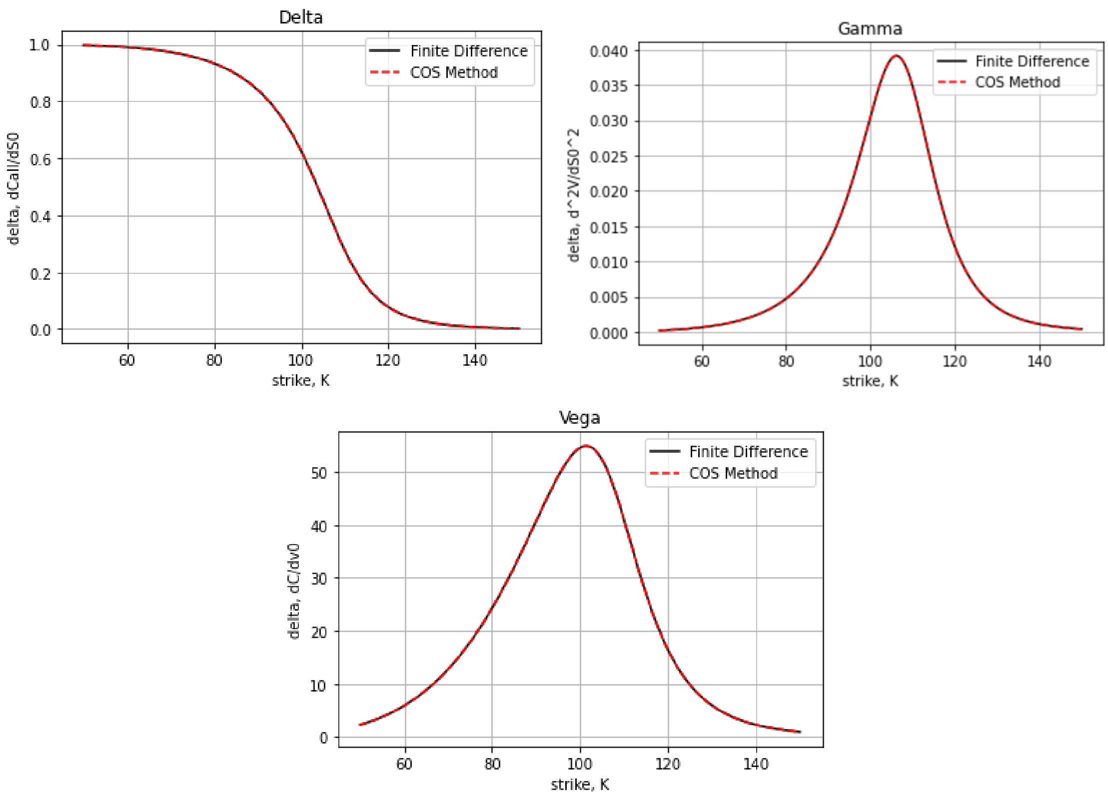

Figure 1: Results for Exercise 8.7 (Delta, Gamma and Vega computed with finite differences and with the COS method.

In the experiment, we defined a maximum absolute error and we varied ΔS. For different "shock sizes" we checked the impact of the size on the convergence of Delta. The results presented below show that very good results are obtained for $\Delta S < 0.01$.

```
Delta Computation: Max ABS Error for 10.000 is equal to 0.18188
Delta Computation: Max ABS Error for 8.890 is equal to 0.16353
Delta Computation: Max ABS Error for 7.780 is equal to 0.14464
Delta Computation: Max ABS Error for 6.670 is equal to 0.12524
Delta Computation: Max ABS Error for 5.560 is equal to 0.10535
Delta Computation: Max ABS Error for 4.450 is equal to 0.08501
Delta Computation: Max ABS Error for 3.340 is equal to 0.06429
Delta Computation: Max ABS Error for 2.230 is equal to 0.04321
Delta Computation: Max ABS Error for 1.120 is equal to 0.02182
Delta Computation: Max ABS Error for 0.010 is equal to 0.00020
```

The same strategy is applied for the computation of Gamma. The results clearly show that gamma is less sensitive to the size of ΔS. Already for $\Delta S = 1.12$ the results are excellent. From a practical perspective, we prefer to keep ΔS of the same size as for the computation of Delta. Therefore, the recommended size would be < 0.01.

```
Gamma Computation: Max ABS Error for 10.000 is equal to 0.00467
Gamma Computation: Max ABS Error for 8.890 is equal to 0.00386
Gamma Computation: Max ABS Error for 7.780 is equal to 0.00309
Gamma Computation: Max ABS Error for 6.670 is equal to 0.00236
Gamma Computation: Max ABS Error for 5.560 is equal to 0.00170
Gamma Computation: Max ABS Error for 4.450 is equal to 0.00112
Gamma Computation: Max ABS Error for 3.340 is equal to 0.00065
Gamma Computation: Max ABS Error for 2.230 is equal to 0.00030
Gamma Computation: Max ABS Error for 1.120 is equal to 0.00008
Gamma Computation: Max ABS Error for 0.010 is equal to 0.00002
```

In the final experiment, we varied the shock size Δv_0 and checked the impact on Vega. Note, however, that the concept of vega is not properly defined under the Heston model. One might assume that Vega represents the sensitivity of an option value to the initial variance v_0 this concept is however not unique. The reason is that the variance process in the Heston model is connected to different parameters (not only to v_0).

Nevertheless, in this experiment we varied Δv_0 and the impact on the sensitivity to initial variance (Vega) is presented below. Note that in order to obtain satisfactory results, one should use a shock size < 0.001.

```
Vega Computation: Max ABS Error for 0.010 is equal to 1.62306
Vega Computation: Max ABS Error for 0.009 is equal to 1.45266
Vega Computation: Max ABS Error for 0.008 is equal to 1.28035
Vega Computation: Max ABS Error for 0.007 is equal to 1.10608
Vega Computation: Max ABS Error for 0.006 is equal to 0.92982
Vega Computation: Max ABS Error for 0.004 is equal to 0.75153
Vega Computation: Max ABS Error for 0.003 is equal to 0.57116
Vega Computation: Max ABS Error for 0.002 is equal to 0.38866
Vega Computation: Max ABS Error for 0.001 is equal to 0.20399
Vega Computation: Max ABS Error for 0.0001 is equal to 0.01710
```

Ex. 8.9. We start with the following CIR process,

$$\mathrm{d}v(t) = \kappa\left(v(t_0) - v(t)\right)\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W(t), \tag{0.3}$$

with the solution given by:

$$v(t) \quad = \quad v(t_0) + \gamma\int_0^t \mathrm{e}^{-\kappa(t-s)}\sqrt{v(s)}\mathrm{d}W(s).$$

Now, the expectation in question reads:

$$\begin{aligned}\mathbb{E}[v(t)v(s)] &= \mathbb{E}\left[\left(v(t_0)+\gamma\int_0^t \mathrm{e}^{-\kappa(t-z)}\sqrt{v(z)}\mathrm{d}W(z)\right)\left(v(t_0)+\gamma\int_0^s \mathrm{e}^{-\kappa(s-z)}\sqrt{v(z)}\mathrm{d}W(z)\right)\right]\\ &= v^2(t_0)+\gamma^2\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{-\kappa(t-z)}\sqrt{v(z)}\mathrm{d}W(z)\right)\left(\int_0^s \mathrm{e}^{-\kappa(s-z)}\sqrt{v(z)}\mathrm{d}W(z)\right)\right].\end{aligned}$$

For $s<t$, we have:

$$\begin{aligned}\mathbb{E}[v(t)v(s)] &= v^2(t_0)+\gamma^2\mathrm{e}^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^t \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right)\left(\int_0^s \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right)\right]\\ &= v^2(t_0)+\gamma^2\mathrm{e}^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^s \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)+\int_s^t \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right)\left(\int_0^s \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right)\right]\\ &= v^2(t_0)+\gamma^2\mathrm{e}^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^s \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right)^2+\int_s^t \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\int_0^s \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right],\end{aligned}$$

whichw, due to the independent increments of Brownian motion, becomes:

$$\mathbb{E}[v(t)v(s)] = v^2(t_0)+\gamma^2\mathrm{e}^{-\kappa(t+s)}\mathbb{E}\left[\left(\int_0^s \mathrm{e}^{\kappa z}\sqrt{v(z)}\mathrm{d}W(z)\right)^2\right],$$

and by Itô's isometry, we find:

$$\begin{aligned}\mathbb{E}[v(t)v(s)] &= v^2(t_0)+\gamma^2\mathrm{e}^{-\kappa(t+s)}\int_0^s \mathrm{e}^{2\kappa z}\mathbb{E}[v(z)]\mathrm{d}z\\ &= v^2(t_0)+\gamma^2 v(t_0)\mathrm{e}^{-\kappa(t+s)}\int_0^s \mathrm{e}^{2\kappa z}\mathrm{d}z\\ &= v^2(t_0)+\gamma^2 v(t_0)\mathrm{e}^{-\kappa(t+s)}\frac{\mathrm{e}^{2\kappa s}-1}{2\kappa} = v^2(t_0)+\gamma^2 v(t_0)\mathrm{e}^{-\kappa t}\frac{\mathrm{e}^{\kappa s}-\mathrm{e}^{-\kappa s}}{2\kappa}.\end{aligned}$$

Ex. 8.11. When the parameters μ_J, σ_J, ξ_p increase in value, the corresponding probability density functions spread out and get fatter tails. The effect of the individual parameters on the distribution is presented below. The following computer code, under the Python icon, generates the plot in Figure 2. We use the following set of parameters, $v_0 = 0.02$, $S_0 = 100.0$, $r = 0.05$, $T = 5.0$, $\kappa = 0.5$, $\gamma = 0.3$ $\bar{v} = 0.3$, $\rho = -0.7$, $\xi_P = 0.2$, $\mu_J = 0.1$, $\sigma_J = 0.1$. The impact of the jump parameters on the tails of the PDF is presented in Figure 3.

python™

As a verification, we integrate the densities and obtain the following numerical results,

```
Integral over the density of f_X is equal to =0.9998975505286268;
Integral over the density of f_S is equal to =0.999964226528772;
Expected stock, from density = 128.40659747114492;
Expected stock, exact = 128.40254166877415.
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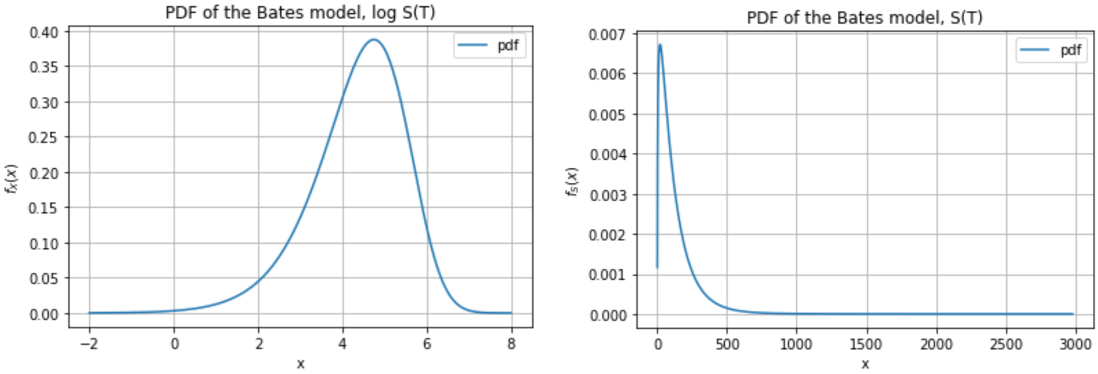

Figure 2: Exercise 8.11- plot of the densities for the Bates model, $\log S(T)$ and $S(T)$.

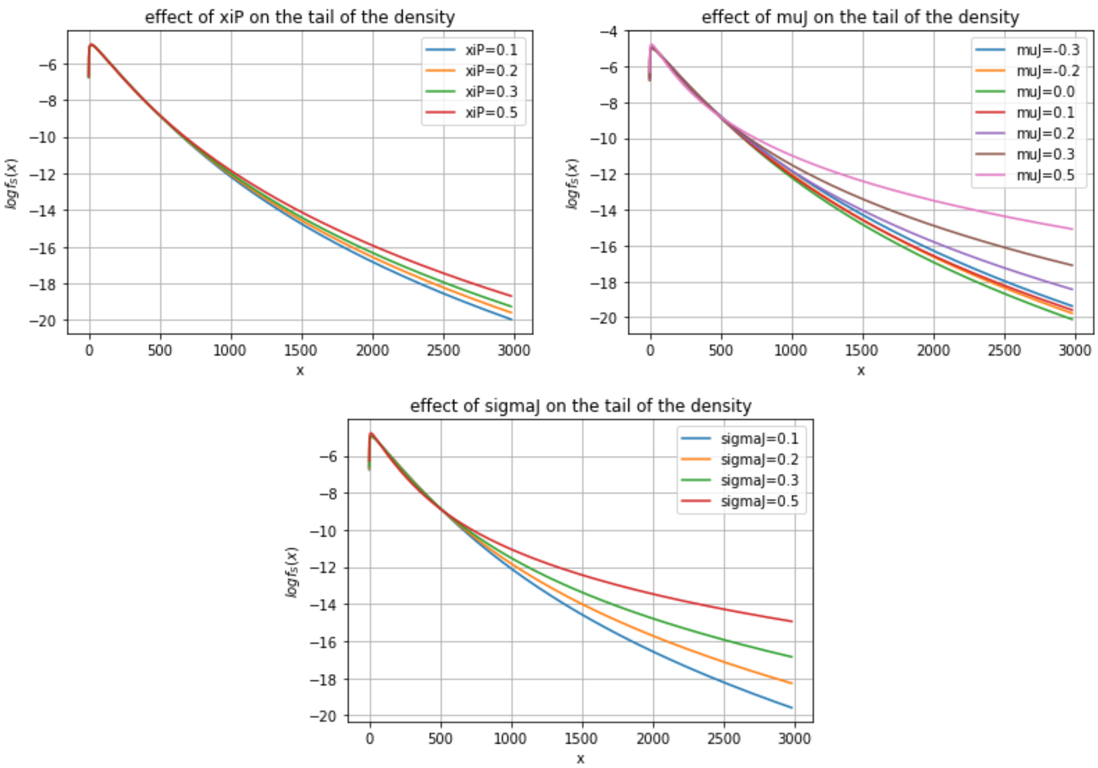

Figure 3: Exercise 8.11-impact of the parameters on the tails of the distribution.