Mathematical Modeling and Computation in Finance
With Exercises and Python and MATLAB Computer Codes

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Solutions to exercises from Chapter 1

`https://QuantFinanceBook.com`

Ex 1.1. The cumulative distribution function for the standard normal distribution is given by,

$$F_{\mathcal{N}(0,1)}(x) = \int_{-\infty}^{x} f(z)\mathrm{d}z,$$

where $f(z) = \frac{e^{z^2/2}}{\sqrt{2\pi}}$ is an even function. Changing the integration variable, $y = -z$, gives us,

$$F_{\mathcal{N}(0,1)}(x) = \int_{-x}^{\infty} f(y)\mathrm{d}y.$$

Hence

$$\begin{aligned} F_{\mathcal{N}(0,1)}(x) + F_{\mathcal{N}(0,1)}(-x) &= \left(\int_{-\infty}^{x} f(z)\mathrm{d}z + \int_{x}^{\infty} f(z)\mathrm{d}z\right) \\ &= \int_{-\infty}^{\infty} f(z)\mathrm{d}z = 1. \end{aligned}$$

Ex. 1.3. We have,

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[a+bX] = a + b\mu, \\ \mathbb{V}\mathrm{ar}[Y] &= \mathbb{E}[(bX-b\mu)^2] = \mathbb{V}\mathrm{ar}[bX] = (b\sigma)^2, \\ \mathbb{E}[\mathrm{e}^X] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^x \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}}\mathrm{e}^{\mu+\sigma^2/2}\mathrm{d}x \\ &= \mathrm{e}^{\mu+\sigma^2/2}. \end{aligned}$$

Ex. 1.5. *a.* By linearity, we find,

$$\mathbb{E}[\bar{X}] = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[X_k] = \mu.$$

b.

$$\begin{aligned} \mathbb{V}\mathrm{ar}[\bar{X}] &= \frac{1}{n^2}\mathbb{E}\Big[\sum_i\sum_j X_iX_j\Big] - \mu^2 \\ &= \frac{1}{n^2}\sum_i\left(\mathbb{E}[X_i^2] + \sum_i\sum_{j\neq i}\mathbb{E}[X_iX_j]\right) - \mu^2. \end{aligned}$$

For each variable, $\mathbb{E}[X_i^2] = \mu^2 + \sigma^2$, and since the random variables are independent, it follows that $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j]$. Hence,

$$\begin{aligned}\mathbb{V}\mathrm{ar}[\bar{X}] &= \frac{1}{n^2}\mathbb{E}\Big[\sum_i \sum_j X_i X_j\Big] - \mu^2 \\ &= \frac{1}{n^2}\left(n(\mu^2+\sigma^2) + n(n-1)\mu^2\right) - \mu^2 = \sigma^2/n.\end{aligned}$$

c. We find,

$$\begin{aligned}\sum_{k=1}^{N}(X_k - \bar{X})^2 &= \sum_{k=1}^{N}\left(X_k^2 - 2X_k\bar{X} + \bar{X}^2\right) \\ &= \sum_{k=1}^{N} X_k^2 - 2N\bar{X}^2 + N\bar{X}^2 \\ &= \sum_{k=1}^{N} X_k^2 - N\bar{X}^2.\end{aligned}$$

d. Using the result of part *c*),

$$\mathbb{E}[v_N^2] = \frac{1}{N-1}\left(\sum_{k=1}^{N}\mathbb{E}[X_k^2] - N\mathbb{E}[\bar{X}^2]\right) \quad = \frac{1}{N-1}\left(N(\sigma^2+\mu^2) - N(\sigma^2/N + \mu^2)\right) = \sigma^2.$$

Ex. 1.7. By using Leibniz' rule for $zW(z)$, we obtain,

$$\mathrm{d}(zW(z)) = \mathrm{d}zW(z) + z\mathrm{d}W(z).$$

Integrating both sides gives,

$$\int_0^t W(z)\mathrm{d}z = \int_0^t \mathrm{d}(zW(z)) - \int_0^t z\mathrm{d}W(z).$$

The first integral on the right-hand side can be expressed as,

$$\int_0^t \mathrm{d}(zW(z)) = tW(t) = t\int_0^t \mathrm{d}W(z).$$

Hence

$$\int_0^t W(z)\mathrm{d}z = \int_0^t (t-z)\mathrm{d}W(z).$$

Ex. 1.9. *a.* The SDE must not contain a drift term. The differential process of the integral equation is found to be,

$$\begin{aligned}\mathrm{d}X(t) &= \mathrm{d}(g(t)W(t)) - g'(t)W(t)\mathrm{d}t \\ &= g'(t)W(t)\mathrm{d}t - g(t)\mathrm{d}W(t) - g'(t)W(t)\mathrm{d}t \\ &= -g(t)\mathrm{d}W(t),\end{aligned}$$

where the differentiation, denoted by d, is in the Itô sense. Hence, the process is a martingale.

b. Express the term $\mathrm{e}^{2t}W(t)$ in its integral form. as follows,

$$\begin{aligned}\mathrm{d}(\mathrm{e}^{2t}W(t)) &= 2\mathrm{e}^{2t}W(t)\mathrm{d}t + \mathrm{e}^{2t}\mathrm{d}W(t) \\ \mathrm{e}^{2T}W(T) &= \int_{t=T}^{t=0}(2\mathrm{e}^{2t}W(t)\mathrm{d}t + \mathrm{e}^{2t}\mathrm{d}W(t)),\end{aligned}$$

where $W(0)=0$ is used in the second equation. Taking expectations at both sides and using $\mathbb{E}[\int_{t=T}^{t=0} \mathrm{e}^{2t}\mathrm{d}W(t)]=0$, because infinitesimal increments of Brownian motion are governed by a normal distribution with zero mean, gives,

$$\mathbb{E}[\mathrm{e}^{2T}W(T)] = \mathbb{E}\big[\int_{t=T}^{t=0} 2\mathrm{e}^{2t}W(t)\mathrm{d}t\big].$$

Ex. 1.11. Integrating the differential process $\mathrm{d}X(t)$, we get

$$X(T) = \mu T + \sigma W(T) + x_0.$$

Substituting this into the integral, gives us,

$$\int_0^T X(t)\mathrm{d}t = \mu T^2/2 + x_0 T + \int_0^T \sigma W(t)\mathrm{d}t.$$

We now use the identity derived in Exercise 1.7,

$$\int_0^T X(t)\mathrm{d}t = \mu T^2/2 + x_0 T + \sigma\int_0^T (T-t)\mathrm{d}W(t).$$

The integral $\int_0^T (T-t)\mathrm{d}W(t)$ can be discretized, giving,

$$\int_0^T (T-t)\mathrm{d}W(t) = \lim_{N\to\infty}\sum_{i=0}^{N}(T-t)\Delta W_i,$$

where $\Delta W_i = W_{i+1}-W_i$, and each of these increments follows a normal distribution. The sum of normal random variables is also a normal random variable. Its mean and variance are given by,

$$\mathbb{E}\big[\int_0^T X(t)\mathrm{d}t\big] = \mu T^2/2 + x_0 T.$$

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\big[\int_0^T X(t)\mathrm{d}t\big] &= \mathbb{E}\left[\left(\mu T^2/2 + x_0 T + \sigma\int_0^T (T-t)\mathrm{d}W(t)\right)^2\right] - (\mu T^2/2 + x_0 T)^2 \\
&= (\mu T^2/2 + x_0 T)^2 + \sigma(\mu T^2/2 + x_0 T)\mathbb{E}\Big[\int_0^T (T-t)\mathrm{d}W(t)\Big] \\
&+ \mathbb{E}\left[\left(\sigma\int_0^T (T-t)\mathrm{d}W(t)\right)^2\right].
\end{aligned}$$

The second term in the second equation vanishes. Finally, with the Itô isometry on the final integral, we find,

$$\mathbb{E}\left[\left(\sigma\int_0^T (T-t)\mathrm{d}W(t)\right)^2\right] = \sigma^2\int_0^T \mathbb{E}[(T-t)^2]\mathrm{d}t = \frac{1}{3}\sigma^2 T^3.$$

Therefore,

$$\int_0^T X(t)\mathrm{d}t \sim \mathcal{N}\left(x_0 T + \frac{1}{2}\mu T^2, \frac{1}{3}\sigma^2 T^3\right).$$