Mathematical Modeling and Computation in Finance

With Exercises and Python and MATLAB Computer Codes

C.W. Oosterlee & L.A. Grzelak

Solutions to exercises from Chapter 6

`https://QuantFinanceBook.com`

Ex. 6.1. *a.* The following computer code, under the Python icon, generates results presented in Figure 1.

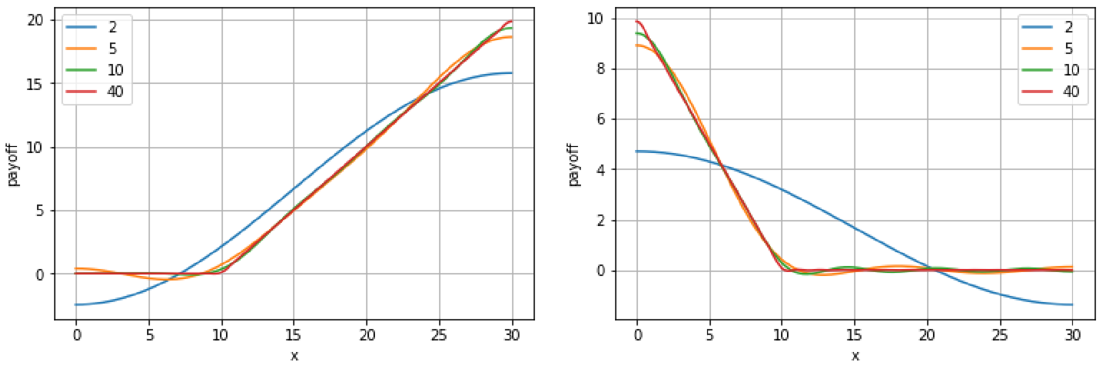

Figure 1: Results for Exercise 6.1 a.

b. The computer code under the next Python icon generates results presented in Figure 2. We observe a faster convergence than in point a. This is due to the smoothness of the approximated function.

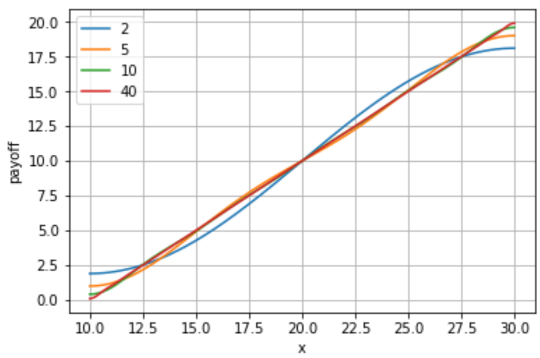

Figure 2: Results for Exercise 6.1 b.

c. The following computer code under the Python icon provides us with the results in Figure 3. Clearly, the sin approximation equals zero at the function's end points.

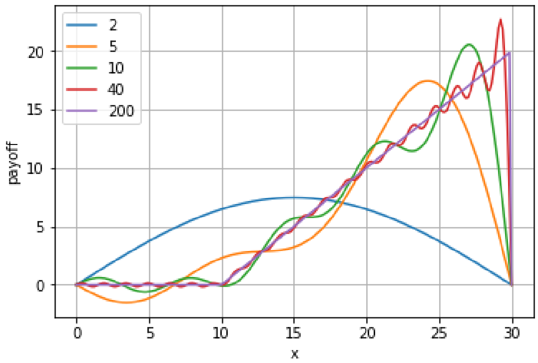

Figure 3: Results for Exercise 6.1 c.

Ex. 6.3. The computer code under the following Python icon provides us the requested results, presented in Figure 4. We need to use 4096 terms in the sum to get accurate results. For the COS price, we use the following setting here, $S_0 = 100$, $r = 0.05$, $\tau = 10$ and $\sigma = 0.15$, $\mu_J = -0.05$, $\sigma_J = 0.3$, $\xi_P = 0.7$.

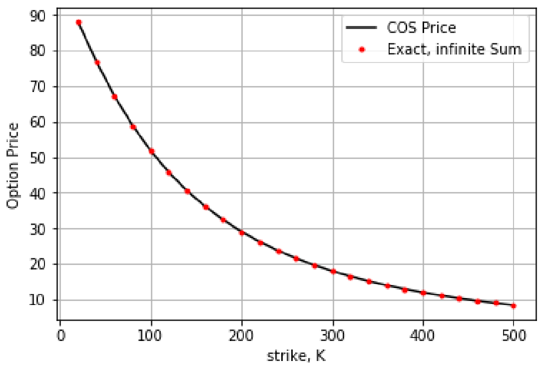

Figure 4: Results for Exercise 6.3

Ex. 6.5. We start with the following expression,

$$\mathbb{E}[\mathrm{e}^{iuY}] = \int_{-\infty}^{\infty} k_1 \mathrm{e}^{iuy} \mathrm{e}^{k_1(k_2-y)-\mathrm{e}^{k_1(k_2-y)}} \,\mathrm{d}y. \tag{0.1}$$

Making a change of variables, $x = y + k_2$, gives us,

$$\mathbb{E}[\mathrm{e}^{iuY}] = \mathrm{e}^{ik_2u} \int_{-\infty}^{\infty} k_1 \mathrm{e}^{iux} \mathrm{e}^{-k_1x-\mathrm{e}^{-k_1x}} \,\mathrm{d}x. \tag{0.2}$$

Making another change of variables, $t = \mathrm{e}^{-k_1 x}$, gives us the requested result,

$$\mathbb{E}[\mathrm{e}^{iuY}] = -\mathrm{e}^{ik_2 u} \int_{\infty}^{0} t^{\frac{-iu}{k_1}} \mathrm{e}^{-t} \mathrm{d}t \tag{0.3}$$

$$= \mathrm{e}^{ik_2 u} \int_{0}^{\infty} t^{\frac{-iu}{k_1}} \mathrm{e}^{-t} \mathrm{d}t \tag{0.4}$$

$$= \Gamma\left(1 - \frac{u}{ik_1}\right) \mathrm{e}^{ik_2 u}. \tag{0.5}$$

Ex. 6.7. The exercise is trivial when it comes to just taking the derivatives. Here, we will calculate the cumulants by employing Python's symbolic computation package Sympy. See the code under the Python icon.

The results are presented below.

```
-1.0*t*(A*C*Y*(G*M**Y - G**Y*M) - G*M*(A*C*(G**Y + M**Y
- (G + 1)**Y - (M - 1)**Y) + r - 0.5*s**2))/(G*M)
1.0*t*(A*C*G**2*M**Y*Y**2 - A*C*G**2*M**Y*Y
+ A*C*G**Y*M**2*Y**2 - A*C*G**Y*M**2*Y
+ G**2*M**2*s**2)/(G**2*M**2)
-1.0*A*C*Y*t*(1.0*G**3*M**Y*Y**2 - 3.0*G**3*M**Y*Y
 + 2.0*G**3*M**Y - 1.0*G**Y*M**3*Y**2 + 3.0*G**Y*M**3*Y - 2.0*G**Y*M**3)/(G**3*M**3)
1.0*A*C*Y*t*(1.0*G**4*M**Y*Y**3 - 6.0*G**4*M**Y*Y**2
+ 11.0*G**4*M**Y*Y - 6.0*G**4*M**Y
+ 1.0*G**Y*M**4*Y**3 - 6.0*G**Y*M**4*Y**2 + 11.0*G**Y*M**4*Y
- 6.0*G**Y*M**4) /(G**4*M**4) -1.0*M**Y*Y/M
```

We can confirm this result analytically. The CGMY-BM model has the following ChF,

$$\begin{aligned}\phi_{X_{CGMY}}(u) &= \exp\left[iu\left(\log S(0) + r + \bar{\omega} - \frac{1}{2}\sigma^2_{CGMY}\right)t - \frac{1}{2}\sigma^2_{CGMY}u^2 t\right] \times \varphi_{CGMY}(u,t) \\ &=: \psi(u) \times \varphi_{CGMY}(u,t).\end{aligned}$$

Under the log-transform, it reads,

$$\log \phi_{X_{CGMY}}(u) = \log \psi(u) + \log \varphi(u,t), \tag{0.6}$$

thus:

$$\frac{\partial}{\partial u} \log \phi_{X_{CGMY}}(u) = \frac{\partial}{\partial u} \log \psi(u) + \frac{\partial}{\partial u} \log \varphi(u,t). \tag{0.7}$$

The first derivative equals:

$$\frac{\partial \log \psi}{\partial u} = i\left(\log S(0) + r + \bar{\omega} - \frac{1}{2}\sigma^2_{CGMY}\right)t - \sigma^2_{CGMY} ut,$$

and at $u = 0$ we have:

$$\left.\frac{\partial \log \psi}{\partial u}\right|_{u=0} = i\left(\log S(0) + r + \bar{\omega} - \frac{1}{2}\sigma^2_{CGMY}\right)t.$$

For the second term, we find,

$$\begin{aligned}\frac{\partial \log \varphi_{CGMY}(u,t)}{\partial u} &= tC\Gamma(-Y)\frac{\partial}{\partial u}\left[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right] \\ &= itC\Gamma(-Y)\left[Y(G+iu)^{Y-1} - Y(M-iu)^{Y-1}\right],\end{aligned}$$

which at $u = 0$ reads,

$$\frac{\partial \log \varphi_{CGMY}(u,t)}{\partial u}\Big|_{u=0} \quad = \quad itC\Gamma(-Y)\left[Y(G)^{Y-1} - Y(M)^{Y-1}\right].$$

The cumulants are defined as

$$\zeta_n = \frac{1}{i^n}\frac{\partial^n \log \phi(u)}{\partial u^n}\Big|_{u=0}.$$

Thus,

$$\begin{aligned}
\zeta_1 &= \frac{1}{i}\frac{\partial \log \phi(u)}{\partial u}\Big|_{u=0} \\
&= \left(\log S(0) + r + \bar{\omega} - \frac{1}{2}\sigma^2_{CGMY}\right)t + tCY\Gamma(-Y)\left[G^{Y-1} - M^{Y-1}\right].
\end{aligned}$$

For the second cumulant, we differentiate the first derivative,

$$\begin{aligned}
\frac{\partial^2}{\partial u^2}\log \phi_{X_{CGMY}}(u) &= \frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\log \psi(u)\right) + \frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\log \varphi(u,t)\right) \\
&= \frac{\partial}{\partial u}\left(i\left(\log S(0) + r + \bar{\omega} - \frac{1}{2}\sigma^2_{CGMY}\right)t - \sigma^2_{CGMY}ut\right) \\
&\quad + \frac{\partial}{\partial u}\left(itC\Gamma(-Y)\left[Y(G+iu)^{Y-1} - Y(M-iu)^{Y-1}\right]\right) \\
&= -\sigma^2_{CGMY}t + itCY\Gamma(-Y)\frac{\partial}{\partial u}\left[(G+iu)^{Y-1} - (M-iu)^{Y-1}\right] \\
&= -\sigma^2_{CGMY}t - (Y-1)tCY\Gamma(-Y)\left[(G+iu)^{Y-2} + (M-iu)^{Y-2}\right].
\end{aligned}$$

For $u = 0$, this gives,

$$\frac{\partial^2}{\partial u^2}\log \phi_{X_{CGMY}}(u)\Big|_{u=0} \quad = \quad -\sigma^2_{CGMY}t - (Y-1)tCY\Gamma(-Y)\left[G^{Y-2} + M^{Y-2}\right].$$

So, the second cumulant is equal to,

$$\begin{aligned}
\zeta_2 &= -\frac{\partial^2 \log \phi(u)}{\partial u^2}\Big|_{u=0} \\
&= \sigma^2_{CGMY}t + (Y-1)tCY\Gamma(-Y)\left[G^{Y-2} + M^{Y-2}\right].
\end{aligned}$$

From the gamma function we know, $\Gamma(2-Y) = Y(Y-1)\Gamma(Y)$, which simplifies the second cumulant,

$$\zeta_2 \quad = \quad \sigma^2_{CGMY}t + tC\Gamma(2-Y)\left[G^{Y-2} + M^{Y-2}\right].$$

For the third cumulant, we have,

$$\begin{aligned}
\frac{\partial^3}{\partial u^3}\log \phi_{X_{CGMY}}(u) &= \frac{\partial}{\partial u}\left(-\sigma^2_{CGMY}t - (Y-1)tCY\Gamma(-Y)\left[(G+iu)^{Y-2} + (M-iu)^{Y-2}\right]\right) \\
&= -(Y-1)tCY\Gamma(-Y)\frac{\partial}{\partial u}\left[(G+iu)^{Y-2} + (M-iu)^{Y-2}\right] \\
&= -i(Y-2)(Y-1)tCY\Gamma(-Y)\left[(G+iu)^{Y-3} - (M-iu)^{Y-3}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta_3 &= i\frac{\partial^3 \log \phi(u)}{\partial u^3}\Big|_{u=0} \\
&= (Y-2)(Y-1)tCY\Gamma(-Y)\left[G^{Y-3} - M^{Y-3}\right].
\end{aligned}$$

For the fourth cumulant, we find,

$$
\begin{aligned}
\frac{\partial^4}{\partial u^4}\log\phi_{X_{CGMY}}(u) &= \frac{\partial}{\partial u}\left(-i(Y-2)(Y-1)tCY\Gamma(-Y)\left[(G+iu)^{Y-3}-(M-iu)^{Y-3}\right]\right) \\
&= (Y-3)(Y-2)(Y-1)tCY\Gamma(-Y)\left[(G+iu)^{Y-4}+(M-iu)^{Y-4}\right].
\end{aligned}
$$

So, the fourth cumulant is equal to,

$$
\begin{aligned}
\zeta_4 &= \frac{\partial^4\log\phi(u)}{\partial u^4}\Big|_{u=0} \\
&= (Y-3)(Y-2)(Y-1)tCY\Gamma(-Y)\left[G^{Y-4}+M^{Y-4}\right],
\end{aligned}
$$

which, with $(Y-3)(Y-2)(Y-1)Y\Gamma(-Y)=\Gamma(4-Y)$, simplifies to,

$$
\zeta_4 = tC\Gamma(4-Y)\left[G^{Y-4}+M^{Y-4}\right].
$$

Ex. 6.9. The following computer code, under the Python icon, provides the results presented in Figure 5, Figure 6 and Figure 7.

python

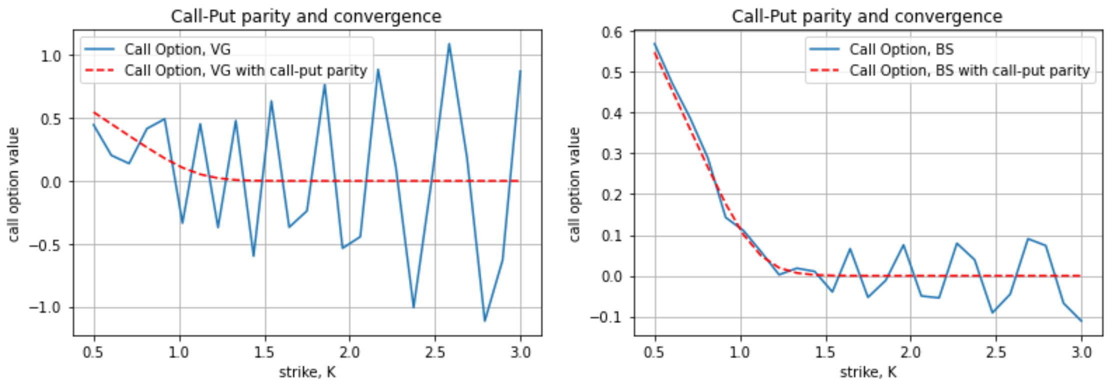

Figure 5: Results for Exercise 6.9- COS method evaluated for pricing of Call Option with $N = 500$. Left: VG, Right:BS.

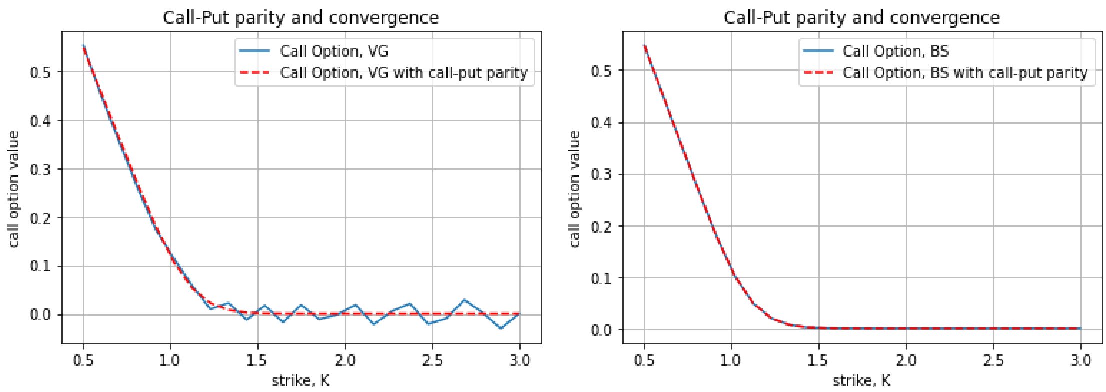

Figure 6: Results for Exercise 6.9- COS method evaluated for pricing of Call Option with $N = 600$. Left: VG, Right:BS.

In the experiment the number of expansion terms varied: $N = 500$, $N = 600$ and $N = 700$. It is clear from the results in Figure 5, Figure 6 and Figure 7 that the put option converges much better than the call. Hence, using the COS method on put options, together with the put-call parity relation, yields much better results for calls.

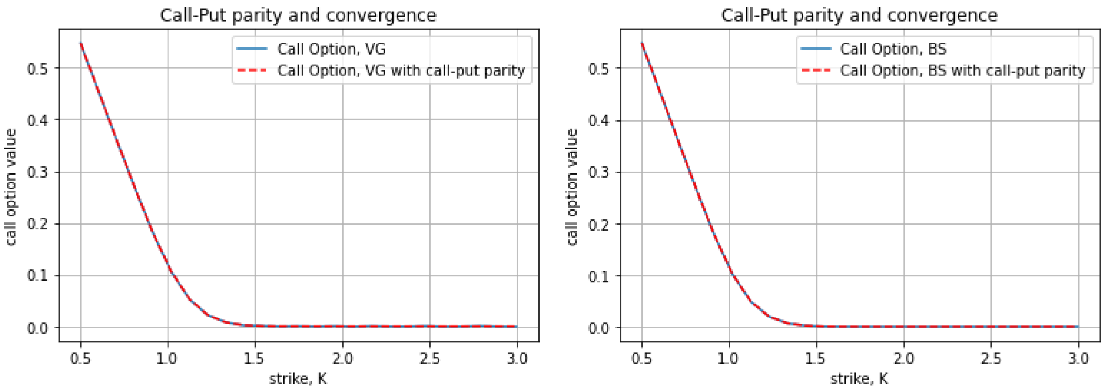

Figure 7: Results for Exercise 6.9- COS method evaluated for pricing of Call Option with $N = 700$. Left: VG, Right:BS.

Ex. 6.11. The following computer code, under the Python icon, provides the results presented in Figure 8.

python

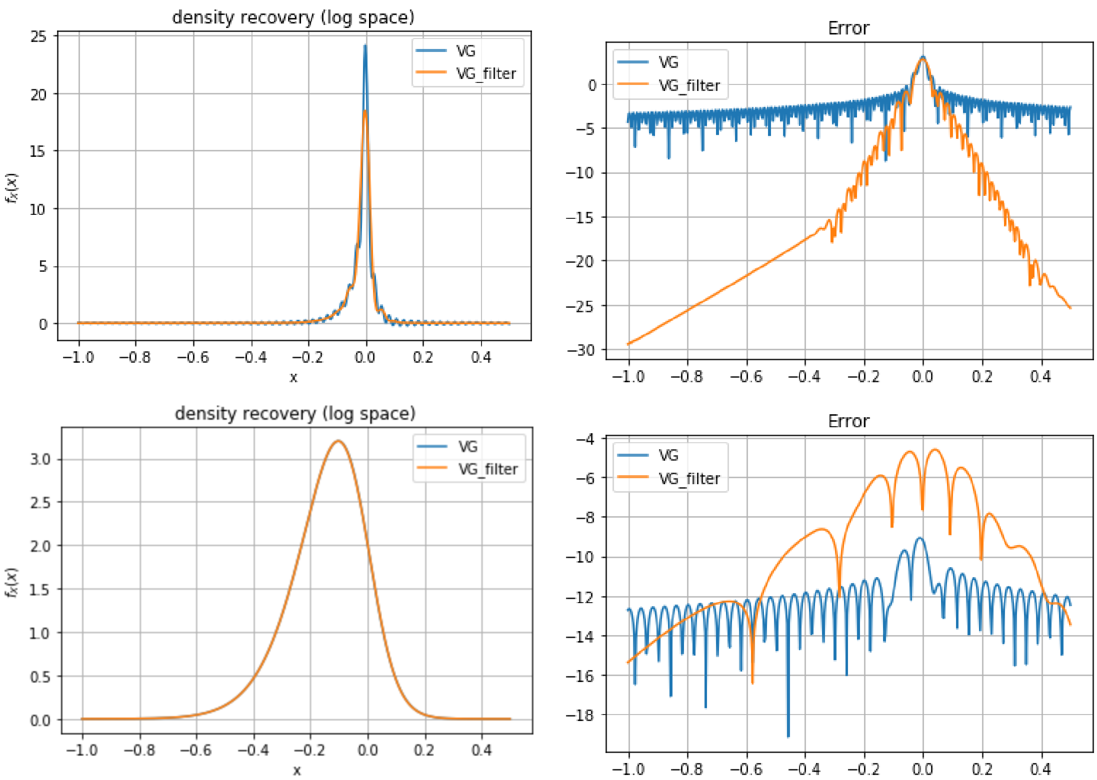

Figure 8: Results for Exercise 6.11: Top row shows results for $T = 0.1$ bottom two figures are for $T = 1.0$.