Notes on

The asymptotic structure of spacetime: a Hamiltonian perspective

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Contents

1 Brief introduction: BMS, matching conditions and spatial infinity

- Boundary conditions originally given $D = 4$ din not exhibit the BMS group bot only Poincaré at i^0 [1].
- BMS diffeos preserve boundary conditions at $\mathcal{I}$ (exact symmetries of General Relativity). The should appear independently of the description (incluiding slcings adapted to i^0).
- Invariance of the gravitational S-matrix under BMS is based on the assumption of *antipodal matching conditions* of th fields and charges between $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ (clearly involves i^0).
- Connecting i^0 with $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ is a non-trivial and subtle question. Evolution of reasonable Cauchy data make null infinity not so smooth. Metric and Weyl tensor develop logarithmic singularities

BMS symetry emerges at i^0 in $D = 4$ through the reconsideration of the parity conditions [2].

- The central ingredients are finiteness and invariance of the off-shell action: boundary conditions that make the kinetic term finite (well-defines symplectic structure).
- Symmetries are canonical: we can assoaciate to any symmetry a charge-generator.
- Matching conditions imposeed by Strominger are a consequence of the boundary conditions imposed at i^0 for having a well-defines action printiple.

The symmary of this lecture will be the following:

- Review of the asymptotic analysis on spacelike hypersurfaces that are asymptotically flat through the Hamiltonian approach. Based on: [1], [2], [3].
- Logarithmic supertranslations and supertranslation-invariant Lorentz charges. Based on: [4], [5].

2 BMS symmetry at spatial infinity

2.1 Einstein gravity in Hamiltonian form

In the ADM decomposition one decompose the spacetime in space-like hypersurfaces at constant time. This hypersurfaces are separated by a $N\mathrm{d}t$ distance where N is knwon as the *lapse* function and measure the separetaion between two slides. In the other hand, if we have a point in a initial hypersurface, we can measure if this point move along the surface a distance $N^i\mathrm{d}t$ which is captured by the *shift* function N^i. Now we can compute how the line element change form one surface to the other one,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t). \tag{2.1}$$

From here we can read how is the ADM decomposition of the metric,

$$g_{\mu\nu} = \begin{pmatrix} N^iN^i - N^2 & N^j \\ & \\ N^i & g_{ij} \end{pmatrix} \tag{2.2}$$

and its inverse

$$g^{\mu\nu} = \begin{pmatrix} -\dfrac{1}{N^2} & \dfrac{N^i}{N^2} \\ & \\ \dfrac{N^j}{N^2} & g^{ij} - \dfrac{N^iN^j}{N^2} \end{pmatrix} \tag{2.3}$$

The Einstein-Hilbert action in Hamiltonian form, using the ADM decomposition is given by

$$S[g_{ij}, \pi^{ij}, N^{\perp}, N^i] = \int \mathrm{d}t \left[\int \mathrm{d}^3x \left(\pi^{ij} \dot{g}_{ij} - N^{\perp}\mathcal{H}_{\perp} - N^i \mathcal{H}_i \right) + B_{\infty} \right] \tag{2.4}$$

where g_{ij} corresponds to the spatial component of the metric and B_{∞} is a boundary term at infinity which is going to be fixed by the fall-off of the fields (g_{ij}, π^{ij}) and the boundary conditions that we impose under the following criteria are

- incluiding as many solutions as possible,
- making the action finite and,
- yielding finite/integrable canonical generators

and the fall-off of the constraints are

$$\mathcal{H}_{\perp} = \frac{1}{\sqrt{g}} \left(\pi^{ij}\pi_{ij} - \frac{\pi^2}{2} \right) - \sqrt{g} R, \qquad \mathcal{H}_i = -2\nabla^j \pi_{ij} \tag{2.5}$$

where $\sqrt{g}$ is the square root of the determinant of the spatial part of the metric which is related with de square root of the determinant of the metric manifold as $\sqrt{-g} = N\sqrt{g}$.

2.2 Diffeomorphisms in Hamiltonian description

Symmetries? Diffeomorphisms that leave the action invariant up to surface integrals at the time boundarie: $(g_{ij}, \pi^{ij}) \to (g'_{ij}, \pi'^{ij})$. In the Hamiltonian description this can be found by takin thecanonical Poisson bracket

$$\delta_{\xi, \xi^i} \Phi = \left\{ \Phi, \int \mathrm{d}^3x (\xi^{\perp}\mathcal{H}_{\perp} + \xi^i \mathcal{H}_i) \right\} \tag{2.6}$$

with

$$\{g_{ij}(x), \pi^{kl}(x')\} = \delta^{(k}_{(i}\delta^{l)}_{j)} \delta^{(3)}(x - x') \tag{2.7}$$

Then, we will obtain the following transformation for the fields

$$\delta_{\xi,\xi^i} g_{ij} = \frac{2\xi}{\sqrt{g}} \left(\pi_{ij} - \frac{1}{2} g_{ij}\pi \right) + \mathcal{L}_{\xi} g_{ij} \tag{2.8}$$

$$\delta_{\xi,\xi^i} \pi^{ij} = -\xi\sqrt{g} \left(R^{ij} - \frac{1}{2} g^{ij} R \right) + \frac{\xi}{2\sqrt{g}} g^{ij} \left(\pi^{mn}\pi_{mn} - \frac{\pi^2}{2} \right) \tag{2.9}$$

$$- \frac{2\xi}{\sqrt{g}} \left(\pi^{im}\pi^{j}_{m} - \frac{1}{2}\pi^{ij}\pi \right) + \sqrt{g} \left(\xi^{|ij} - g^{ij}\xi^{|m}{}_{|m} \right) + \mathcal{L}_{\xi}\pi^{ij} \tag{2.10}$$

2.3 Asymptotic analysis: Regge-Teitelboim boundary conditions

Let us consider the satandar Regge-Teitelmboim (RT) boundary conditions:

$$g_{ij} = \delta_{ij} + \frac{\bar{h}_{ij}}{r} + \mathcal{O}(r^{-2}), \qquad \pi^{i} = \frac{\bar{\pi}^{ij}}{r^2} + \mathcal{O}(r^{-3}) \tag{2.11}$$

where

$$\bar{h}_{ij}(-n^i) = \bar{h}_{ij}(n^i) = \text{even} \qquad \bar{\pi}_{ij}(-n^i) = -\bar{\pi}_{ij}(n^i) = \text{odd} \tag{2.12}$$

References

[1] T. Regge and C. Teitelboim, *Role of Surface Integrals in the Hamiltonian Formulation of General Relativity*, *Annals Phys.* **88** (1974) 286.

[2] M. Henneaux and C. Troessaert, *BMS Group at Spatial Infinity: the Hamiltonian (ADM) approach*, *JHEP* **03** (2018) 147 [1801.03718].

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[4] O. Fuentealba, M. Henneaux and C. Troessaert, *Logarithmic supertranslations and supertranslation-invariant Lorentz charges*, *JHEP* **02** (2023) 248 [2211.10941].

[5] O. Fuentealba, M. Henneaux and C. Troessaert, *Asymptotic Symmetry Algebra of Einstein Gravity and Lorentz Generators*, *Phys. Rev. Lett.* **131** (2023) 111402 [2305.05436].