Mechanics Review

For some nice mechanical examples, see this talk and this talk. There is a total of **87** points.

1   Statics and Linear Dynamics

[2] Problem 1. ⟳ Quarterfinal 2000, problem 2.

[3] Problem 2 (Wang)**.** Two massless rigid rods of length ℓ are connected by a joint A, which allows them to freely rotate with respect to each other. The left member is pinned to point O, while the right member is placed on a roller B which can roll frictionlessly on the ground.
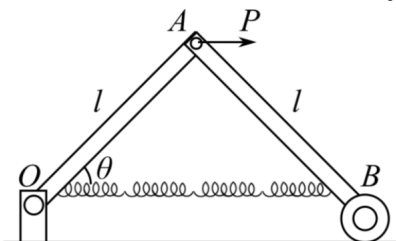
A massless spring of zero relaxed length and spring constant k is stretched between O and B, and a rightward force P is exerted at A. Find the angle θ at equilibrium.

Solution. This problem can be solved directly by introducing variables for the tensions in each rod, then writing down force and torque balance equations for each rod. This can be quite quick if you're clever about choosing your pivot points and systems.

However, the easiest way is to use the Lagrangian-like methods introduced in **M4**. We note that the entire system just has one degree of freedom, the angle θ, and it has a potential energy $V(\theta)$, which includes the potential due to the external force. The static equilibrium point must thus minimize this potential.

If O is the origin, then A is at $(\ell \cos \theta, \ell \sin \theta)$ and B is at $(2\ell \cos \theta, 0)$. So the potential energy is

$$V(\theta) = \frac{1}{2}k(2\ell \cos \theta)^2 - P\ell \cos \theta.$$

Minimizing this expression yields

$$\cos \theta = \frac{P}{4k\ell}.$$

[3] Problem 3. AuPhO 2015, problem 12. A neat real-world problem. See the answer booklet for the diagrams for the final part.

Solution. See the official solutions here.

[3] Problem 4 (FYKOS 34.1)**.** We model a person's head as a sphere of radius R, and a beanie as a circular, massless rubber band of radius r_0 and spring constant k, where $r_0 < R$. The coefficient of friction between the band and head is μ. When is it possible for the person to put the beanie on with one hand?
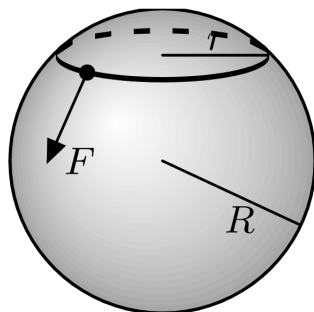
That is, find the conditions for which it is possible to stretch the band so that it lies along the equator of the sphere, by applying forces only at one point at the band, as shown above. Assume for simplicity that the band is always planar.

Solution. This is secretly a statics problem. The problem with trying to put on a beanie this way is that it might slip back up your head, since it wants to contract. If the coefficient of friction is high enough, this slipping won't happen. And as long as slipping doesn't happen, it's possible to pull the beanie down, by just applying an infinitesimal force at some point with a downward component.

Consider the moment where the beanie is a circle with radius r. At each point along the beanie, there is a normal force dN and a friction force df. Balancing the net upward/outward force gives

$$df \, \sin \alpha = dN \, \cos \alpha, \quad \sin \alpha = \frac{r}{R}.$$

Assuming the friction is maximal, we require

$$\mu \geq \cot \alpha.$$

This is harder to satisfy the smaller r is, so the toughest part is when we just start, and $r = r_0$. By solving the relevant right triangle and rearranging, we have

$$\mu \geq \sqrt{(R/r_0)^2 - 1}$$

or equivalently,

$$r_0 \geq \frac{R}{\sqrt{1 + \mu^2}}.$$

The spring constant k doesn't matter, as one could have seen by dimensional analysis.

[**3**] **Problem 5** (MPPP 19). A small pearl moving in deep water experiences a viscous retarding force that is proportional to its speed, by Stokes' law. If a pearl is released from rest under the water, then it reaches a terminal velocity v_1.

(a) Suppose the pearl is instead released horizontally with initial speed v_2. Find the minimum speed of the pearl during the subsequent motion.

(b) If $v_2 < v_1$, for what range of angles can the pearl be released, so that its speed monotonically increases?

Solution. (a) When the speed is at a minimum, $d(v^2)/dt = 0$, so $\mathbf{v} \cdot \mathbf{a} = 0$. The retarding force $-bv$ satisfies $bv_1 = mg$. Newton's laws in each dimension give

$$ma_x = -bv_x \qquad ma_y = -bv_y - mg$$

Solving these equations by separating variables gives

$$v_x = v_2 e^{-bt/m}, \quad v_y = -v_1(1 - e^{-bt/m}).$$

Differentiating, we have

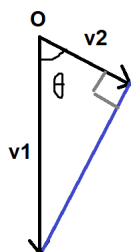$$a_x = -\frac{gv_2}{v_1}e^{-bt/m}, \quad a_y = -ge^{-bt/m}.$$

We want $v_x a_x + v_y a_y = 0$, and letting $\alpha = e^{-bt/m}$, this means,

$$\frac{v_2^2}{v_1}\alpha^2 = v_1(1 - \alpha)\alpha$$

and solving gives

$$\alpha = \frac{v_1^2}{v_1^2 + v_2^2}, \quad v = \sqrt{v_2^2\alpha^2 + v_1^2(1-\alpha)^2} = \frac{v_1 v_2}{\sqrt{v_1^2 + v_2^2}}.$$

(b) Note that in the previous part, a_y/a_x is constant. Let the velocity vector start at O. The end of the velocity vector moves in a straight line since the direction of acceleration is constant (goes from $\mathbf{v}_2$ to $-v_1\hat{\mathbf{y}}$).

At the limiting angle when the velocity monotonically increases, v_2 will be the minimum velocity, so $\mathbf{v}_2$ is perpendicular to the blue line. That means that the angle θ as measured from the vertical needs to satisfy

$$|\theta| < \arccos\left(\frac{v_2}{v_1}\right).$$

[**3**] **Problem 6** (PPP 42). A uniform rod of mass m and length ℓ is supported horizontally at its ends by two fingers. As the fingers are slowly brought together, the rod alternates between sliding on each finger. The coefficients of friction are $\mu_k < \mu_s$.

(a) Explain why the fingers meet under the center of mass of the rod. (Try it in real life!)

(b) Find the total work done by the fingers during this process.

Solution.   (a) Consider balancing torques about the center of mass of the rod. As a finger moves closer to the center, its lever arm decreases so the normal force increases. Eventually, the maximum possible friction force increases enough so that finger stops sliding, at which point the other finger starts sliding. (For a visual explanation, see this nice video!)

(b) Let x_1 and x_2 denote the distances from the center. Then $F_1 = \frac{mgx_2}{x_1+x_2}$, and similarly for F_2. During the motions, one finger is stationary while the other finger moving from $x_1 = a$ to $x_1 = b$ will do work

$$W = \int_a^b \frac{mg\mu_k x_2}{x_1 + x_2}dx_1 = mg\mu_k x_2 \log\left(\frac{x_2 + a}{x_2 + b}\right)$$

Each finger does work until the ratio of the forces is μ_s/μ_k, and the ratio of the distances is $r = \mu_k/\mu_s$, at which point the finger switches. Then the total work is

$$W = -\frac{1}{2}mg\mu_k\ell\left(\log\left(\frac{1+r}{2}\right) + r\log\left(\frac{r+r^2}{1+r}\right) + r^2\log\left(\frac{r^2+r^3}{r+r^2}\right) + \dots\right)$$

which means

$$\frac{W}{\frac{1}{2}mg\ell} = \log\left(\frac{1+r}{2}\right) + r\log(r) + r^2\log(r) + r^3\log(r) + \dots = \log\left(\frac{1+r}{2}\right) + \frac{r}{1+r}\log(r).$$

Plugging back in for r, we conclude

$$W = \frac{1}{2}mg\mu_k\ell\left(\log\left(\frac{2\mu_s}{\mu_k + \mu_s}\right) + \frac{\mu_k}{\mu_s - \mu_k}\log\left(\frac{\mu_s}{\mu_k}\right)\right).$$

This is a pretty complicated expression, so let's check it with limiting cases. Let $\mu_s = \mu_k(1+\epsilon)$ where $\epsilon \ll 1$. The first logarithm term is $O(\epsilon)$, so

$$W = \frac{1}{2}mg\mu_k\ell\left(\frac{1}{\epsilon}\log(1+\epsilon) + O(\epsilon)\right) = \frac{1}{2}mg\mu_k\ell + O(\epsilon).$$

This makes sense, because in this limit both fingers are sliding almost continuously, moving a distance $\ell/2$ each against a friction force $\mu_k mg/2$.

[3] **Problem 7.** A long rope with linear mass density λ rests on a horizontal table with a small bend.
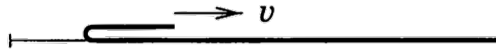
You pull the end of the rope that is near the bend with force F.

(a) Suppose that the bend is very small, so that all of the rope touching the ground is perfectly slack (zero tension). What F is needed to pull the end of the rope with constant velocity v?

(b) Now suppose that the bend is smooth, so that pieces of the rope are gradually accelerated from rest as they pass the bend. What force F is needed to pull the end of the rope with constant velocity v?

(c) In both cases, what force $F(t)$ is needed to pull the rope with constant acceleration a?

Solution.   (a) In this case, energy is not conserved, because the sudden acceleration of a piece of the rope when it hits the bend is inherently inelastic. On the other hand, since the part of the rope touching the ground is slack, there can be no horizontal forces on any part of it,

so the friction force vanishes. As a result, the only horizontal external force is the force you exert. Note that the mass m that is moving is

$$m = \frac{\lambda x}{2}$$

where x is the total distance the bend has moved. Then

$$\frac{dm}{dt} = \frac{\lambda v}{2}$$

and we have

$$F = \frac{dp}{dt} = \frac{dm}{dt} v = \frac{1}{2} \lambda v^2.$$

(b) In this case, momentum is not conserved because of the friction force from the ground, but energy is conserved. We have

$$F = \frac{1}{v} \frac{dE}{dt} = \frac{1}{v} \frac{1}{2} \frac{dm}{dt} v^2 = \frac{1}{4} \lambda v^2.$$

From a momentum standpoint, the "missing" $\lambda v^2/4$ force is provided by friction. To see why this has to be the case, note that the tension is constant within the bend itself. Thus, considering the part of the rope lying on the table as a system, there is a force F to the left, so friction must provide a force F to the right.

(c) In this case, we can't neglect the dv/dt terms above. For the first part,

$$F = \frac{dm}{dt} v + m \frac{dv}{dt} = \frac{1}{2} \lambda v^2 + \frac{1}{2} \lambda (at)^2$$

where we used $x(t) = at^2/2$. For the second part,

$$F = \frac{1}{v} \frac{dE}{dt} = \frac{1}{2v} \left(\frac{dm}{dt} v^2 + 2mva \right) = \frac{1}{4} \lambda v^2 + \frac{1}{4} \lambda (at)^2.$$

Again, we get precisely half the answer of the other case, for the same reason as in part (b). (Of course, in reality, which answer is closer to the correct one depends on properties of the rope being pulled, and the size of the bend.)

[2] **Problem 8.** ◔ INPhO 2012, problem 1.

Solution. See the official solutions here.

[3] **Problem 9.** ◔ INPhO 2018, problem 4.

Solution. See the official solutions here.

[4] **Problem 10.** ◔ US Theory TST 2022, problem 2. A set of nice exercises.

Solution. See the official solutions here.

[5] **Problem 11.** ◔ APhO 2011, problem 2. This covers "stick-slip", which appears in many real-world contexts and Olympiad problems. You can see it in action on a violin string here. (The calculations in this problem are chosen to be relatively simple, requiring mostly intuition; for some more challenging calculations, see USAPhO 2021, problem A1, which covers a similar setup.)

2 Oscillations

[3] **Problem 12** (EFPhO 2007). Consider a light elastic rod with fixed length ℓ. If one end of the rod is firmly fixed, and a force F is applied to the other end of the rod, perpendicular to the rod at the point of application, then it can be shown that the rod takes a form of a circular arc with radius of curvature $R = k/F$, for a constant k.
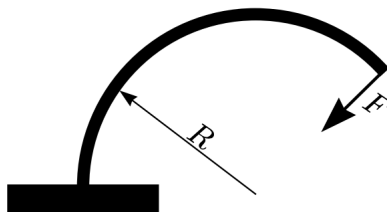
(a) Let the rod be fixed vertically, at its bottom end, and a ball of mass m be attached to its upper end. Find the period of small oscillations, assuming $gm\ell \ll k$.

(b) What is the maximum mass of the ball for the configuration to be stable?

Solution. (a) Suppose the rod bends through a total angle of θ. Its radius of curvature is $R = l/\theta$. The horizontal displacement of the ball is $R(1 - \cos\theta) \approx R\theta^2/2 = l\theta/2$. The horizontal force on the ball is $mg\theta - k/R \approx k/R = k\theta/l$. Hence this is simple harmonic motion with an effective "spring constant" $2k/l^2$, giving period $T = \pi l\sqrt{2m/k}$.

(b) Let the energies be zero at equilibrium. The potential energy in the rod for small angles is $\int F\,dx = \int \frac{k}{r}\,dx$. Since $x = \frac{1}{2}\ell\theta$, $dx = \frac{1}{2}\ell d\theta$ and $\frac{1}{4} = \frac{\theta}{\ell}$,

$$U_R = \int \frac{k\theta}{\ell}(\frac{1}{2}\ell d\theta) = \frac{1}{4}k\theta^2.$$

The gravitational potential energy is

$$U_g = mgr\sin\theta - mg\ell \approx mgr\theta - \frac{1}{6}mgr\theta^3 - mg\ell \approx -\frac{1}{6}mg\ell\theta^2.$$

For a stable equilibrium, the energy should be at a minimum, which is determined by the sign of the coefficient of θ^2 in $U_R + U_g$. The equilibrium is unstable when

$$\frac{1}{4}k - \frac{1}{6}mg\ell < 0$$

which gives a maximum mass of

$$m = \frac{3k}{2g\ell}.$$

[3] **Problem 13.** ⟨⟩ INPhO 2019, problem 7. A nice data analysis problem; bring graph paper.

Solution. See the official solutions here.

3 Rotation

[**2**] **Problem 14.** EFPhO 2015, problem 3.

Solution. See the official solutions here.

[**3**] **Problem 15.** ⊙ USAPhO 2021, problem B1. An elegant rotation problem.

[**3**] **Problem 16** (Morin 8.24)**.** A spherically symmetric ball of radius R initially slides without rotating on a horizontal surface with friction. The initial speed is v_0, and the moment of inertia about the center is $I = \beta m R^2$.

(a) Assuming that the normal force is always applied upward at the bottom of the ball, and that the friction force is always applied horizontally at the bottom of the ball (but assuming nothing about how the friction force varies), find the speed of the ball when it begins to roll without slipping. Also, find the kinetic energy lost while sliding.

(b) Now consider the case where the friction force is standard uniform kinetic friction, $f = -\mu_k N$. Verify that the work done by friction equals the energy loss calculated in part (a).

(c) In reality, the conclusions above can be modified by "rolling resistance". Any real material will slightly deform when the ball rolls on it. We can crudely account for this by thinking of the normal force as applied not at the bottom of the ball, but at a point slightly forward from the bottom. The horizontal component of this normal force is defined to be $f_r = -\mu_r N_y$ where N_y is the vertical normal force, and $\mu_r \ll 1$. In addition, kinetic friction is still present, as in part (b). Under these assumptions, find the velocity of the ball once it stops slipping. Is more or less energy lost than in part (b)?

Solution. (a) The point here is that we can relate the linear and angular impulses without having to know how the force behaves in time. If there's a net impulse J on the ball, the net change in angular momentum about the center of the ball is $\int RF\,dt = RJ$. When the ball is rolling without slipping, $v = \omega R$. Thus

$$J = mv_0 - mv_f = L/R = \beta m R\omega$$

which gives

$$v_f = \frac{v_0}{1+\beta}.$$

The kinetic energy lost is

$$\Delta K = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_f^2 - \frac{1}{2}\beta m R^2\omega^2 = \frac{1}{2}m(v_0^2 - (1+\beta)v_f^2) = \frac{1}{2}\frac{\beta}{1+\beta}mv_0^2.$$

(b) Here, $f = -\mu_k mg$ and acts for a time of $t = J/f$. Since the acceleration is constant, the ball travels a distance of $\frac{1}{2}(v_0 + v_f)t$ while sliding. However, it will turn a distance of $R\theta = \frac{1}{2}\omega Rt = \frac{1}{2}v_f t$ in the other direction, so the relative distance traveled between the surface of the ball and the ground is $L = \frac{1}{2}v_0 t$.

$$\Delta K = fL = \frac{1}{2}v_0 J = \frac{1}{2}m(v_0 - v_f)v_0 = \frac{1}{2}\frac{\beta}{1+\beta}mv_0^2$$

as desired.

(c) Now the angular and linear accelerations are

$$\alpha = \frac{\mu_k g}{\beta R}, \quad a = -(\mu_r + \mu_k)g$$

where the rolling resistance doesn't affect the angular acceleration, because the overall normal force always exerts no torque about the center of mass of the ball. (Note that this conclusion would have been changed if we accounted for the deformation of the ball itself, which would give a second, additional type of rolling resistance. Here we are assuming that the ball is much harder than the surface it rolls on, though there are plenty of situations where the reverse is true, such as when a bike tire rolls on concrete.)

Thus, by similar reasoning to that of part (a),

$$v_f = \frac{v_0}{1 + \beta(1 + \mu_r/\mu_k)}.$$

This is smaller than the result of part (a), so more energy is lost. The reason is that the rolling resistance dissipates additional energy. Notice that even once slipping stops, rolling resistance will continue to dissipate energy, causing the ball to eventually come to a stop.

Remark

In the early 1800s, some said it was impossible for a train engine to pull anything heavier than the engine itself. As the argument went, the force that moves the train forward is friction between the engine car's wheels and the track. If the engine car has mass M, the maximum friction force is μM. If the rest of the train has mass M', however, then it requires a force $\mu M'$ to get it started moving, so the train can't start if $M' > M$.

Problem 16 explains why this reasoning is wrong. The maximum forward frictional force on the engine car wheels is determined by the coefficient of static friction μ_s, while the force needed to get the rest of the train moving is determined by the coefficient of rolling friction μ_r. So we only need $\mu_s M > \mu_r M'$, which is possible since μ_r can be very small. For steel train wheels on steel rail, we might have $\mu_s \sim 0.5$ but $\mu_r \lesssim 10^{-3}$.

On the other hand, early trains could have trouble going up inclines. This led to several innovative concepts, such as trains propelled by atmospheric pressure or pushed by mechanical legs. All the mechanical systems we're familiar with today, whose design might seem obvious at first glance, actually gradually evolved through many intermediate forms. For instance, most people think they know how a bicycle works, but actually don't, because it's really quite tricky. Accordingly, it took over a century for the modern bicycle design to evolve.

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[4] **Problem 17** (IPhO 1998). A hexagonal pencil with mass M and side length R is pushed and rolls down a ramp of inclination θ. For some values of θ, the pencil will roll down the plane with some terminal speed, never losing contact with the ramp. In order to avoid a complicated moment of inertia calculation, we will assume the cross section looks like a wheel with six equally spaced massless spokes and no rim, with all the mass on the axle.

(a) The pencil does not speed up indefinitely, but rather reaches a steady state. Explain why, and compute the speed the pencil's axis has immediately after each collision, in the steady state.

(b) Find the minimum θ so that rolling spontaneously starts, without needing a push.

(c) Find the minimum θ so that, once the pencil has been pushed to start rolling, it never stops.

(d) Find the maximum θ so that a rolling pencil always remains in contact with the plane.

Solution.   (a) Each time the pencil rolls through an angle $\pi/3$, a new vertex of the pencil hits the plane. In this moment, that vertex suddenly becomes the new pivot point about which the pencil is rotated, which means energy is dissipated in an inelastic collision. This is the reason that the pencil reaches a steady state, instead of accelerating indefinitely. You can see this very nicely depicted in this video.

Let the pencil's center of mass have speed v_i just before an impact, and v_f just after the impact. The impact involves a singular impact force at the new pivot point, which means angular momentum is conserved about that point. Thus,

$$M v_i R \cos 60° = M v_f R$$

from which we conclude

$$v_f = \frac{v_i}{2}.$$

In the steady state, the kinetic energy gained from rolling from one vertex to another balances the energy lost in the collision, so conserving energy immediately after a collision and immediately before a next one gives

$$\frac{1}{2} M v_f^2 + M g R \sin \theta = \frac{1}{2} M (2 v_f)^2$$

from which we conclude

$$\boxed{v_f = \sqrt{\frac{2 g R \sin \theta}{3}}.}$$

By the way, the original formulation of the question gave the pencil a moment of inertia $C M R^2$ about its center of mass. The solution with general C is not much harder. Now the angular momentum conservation condition is

$$M v_i R \cos 60° + C M R^2 \omega_i = (C + 1) M R^2 \omega_f$$

where $\omega_i = v_i / R$ and $\omega_f = v_f / R$. Thus,

$$v_f = \frac{2C + 1}{C + 1} \frac{v_i}{2}.$$

Likewise, the energy balance equation for the steady state becomes

$$\frac{1}{2} M (C + 1) v_f^2 + M g R \sin \theta = \frac{1}{2} M (C + 1) (2 v_f)^2$$

from which we conclude

$$v_f = \sqrt{\frac{2 g R \sin \theta (C + 1/2)^2}{(C + 1)\left((C + 1)^2 - (C + 1/2)^2\right)}}.$$

(b) This is a basic statics problem. The rolling must start if the center of mass of the hexagon is not above its support, which implies $\theta = 30°$.

(c) Between two vertex transitions, the maximum potential energy of the pencil is when the center of mass is directly above the vertex at a height R. It will start out at a height of $h_0 = R\sin(\theta + 60°)$ above the vertex, and fall down to a height $h_f = R\sin(60° - \theta)$ above the vertex.

In order for it to roll indefinitely, potential energy from height R to h_f followed by the inelastic collision must leave enough kinetic energy for the pencil to go from height h_0 to height R. Earlier we found that $\omega_f = \omega_0(C + 1/2)/(C + 1)$, so the kinetic energy will be reduced by a factor of $\alpha = (\omega_f/\omega_0)^2$. Thus the energy equation for indefinite rolling is

$$\alpha Mg(R - h_f) = Mg(R - h_0).$$

This implies

$$\frac{1 - \sin(\theta + 60°)}{1 - \sin(60° - \theta)} = \alpha.$$

In our case, $\alpha = 1/4$. The solution of the above equation can be found using either binary search or by the "plug in" method, i.e. repeatedly calculating

$$\arcsin\left(1 - \frac{1 - \sin(60° - \text{Ans})}{4}\right) - 60°.$$

Both methods give an answer of $\theta = 10.21°$.

(d) The pencil leaves the ramp when gravity isn't strong enough to provide the needed centripetal acceleration for the rotation about a vertex. Right before the next vertex transition, the pencil is moving the fastest, and the radial component of gravity is the smallest, so the pencil most readily leaves the ramp at that point. Using part (a)'s notation (ω_0 is the angular velocity right before the next transition), the leaving condition is $g\cos(30° + \theta) = g\sin(60° - \theta) = \omega_0^2 R$, where $30° + \theta$ is the angle between the vertical and line connecting the center of mass to the vertex. Using our expression for $\omega_0 = \frac{v_f}{R}(C + 1)/(C + 1/2)$ found in part (a),

$$\sin(60° - \theta) = \frac{2\sin\theta(C + 1)}{((C + 1)^2 - (C + 1/2)^2)}$$

With $C = 0$, we have

$$\sin(60° - \theta) = \frac{8\sin\theta}{3}.$$

We can binary search for the answer or repeatedly plug in

$$\arcsin\left(\frac{3\sin(60° - \text{Ans})}{8}\right)$$

to find that the maximum angle for it to stay on the ramp is $\theta = 15.3°$. So the range of angles where the rolling will never stop, but also keep the pencil on the ramp, is quite narrow!

This famous question has appeared on the IPhO, BAUPC, and Morin's mechanics book, and papers have even experimentally confirmed its results. If you would like to know even more about it, see the extended analysis here.

[**4**] **Problem 18.** ⊙ USAPhO 2017, problem B1. A tough rotation problem.

The next two questions are about three-dimensional rotation, covered in **M8**.

[**3**] **Problem 19** (BAUPC). A frictionless fixed cone stands on its tip.
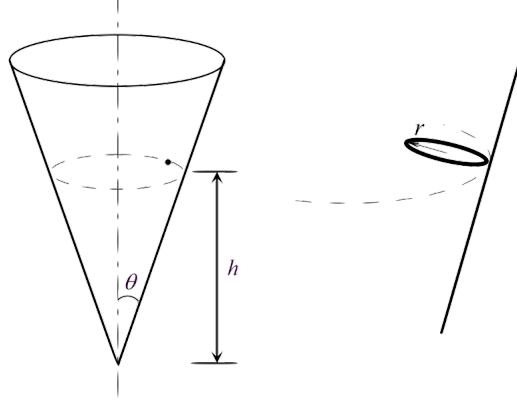
(a) A particle slides on the inside surface of the cone at height h above the tip, as shown at left above. Find the frequency of the circular motion.

(b) Now suppose the cone has friction, and a small ring of negligible radius rolls on the surface without slipping at the same height. Also assume that the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone, as shown at right above. Find the frequency of the circular motion.

(c) How general were our assumptions in part (b)? Specifically, would the described motion had been possible if the plane of the ring were at a different angle? Is a slightly smaller or larger angle to the horizontal possible? Would it be possible if the ring were exactly horizontal?

Solution. (a) The centripetal force $m\omega^2 h \tan\theta$ is horizontal, which equals to $N\cos\theta$. The particle must be vertically balanced, so $N\sin\theta = mg$, giving

$$\omega^2 h \tan\theta = g\cot\theta$$

and an answer of

$$\omega = \cot\theta\sqrt{\frac{g}{h}}.$$

(b) Let the ring have moment of inertia βmr^2 (where $\beta = 1$) and moves in a circle of radius $R = h\tan\theta \gg r$. The no slip condition is $\omega r = \Omega R$. When taken about the point of contact, the ring has a horizontal angular momentum $L_h = (1+\beta)mr^2\omega\sin\theta$. Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, then about the point of contact (so the confusing normal and friction forces can be ignored) the torque is $\tau = mgr\cos\theta$ and $|d\mathbf{L}|/dt = \Omega L_h$. Thus

$$mgr\cos\theta = \Omega(1+\beta)mr^2\omega\Omega\sin\theta$$

$$\frac{g}{r} = (1+\beta)\frac{\Omega^2 R}{r}\tan\theta$$

$$\Omega = \cot\theta\sqrt{\frac{g}{2h}}$$

(c) There are two constraints in this problem: force balance and torque balance. As we saw in part (b), considering the torque of gravity about the contact point alone fixes the angular frequency Ω of the circular motion. That in turn gives the force balance equations (vertical force is zero, horizontal force is centripetal), and since the coefficient of friction is high enough to prevent slipping, there's always some combination of normal and frictional forces that makes the problem work out. Since none of this depends very sensitively on the angle, we could change the angle and the problem would still work.

There's only one exception: you can't have a horizontal ring. In that case, the angular momentum of the ring does not change at all (because it's always spinning in a horizontal plane), so the torque balance equation can't be satisfied. As a real-life example, when motorcyclists ride along the equator of the globe of death, they always tilt a bit above the horizontal.

[3] **Problem 20.** Richard Feynman used to tell the following story, here reproduced verbatim.

I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.

I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!

I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it... the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate.

Feynman was right about quantum electrodynamics, but was he right about the plate?

Solution. For concreteness, take the angular momentum of the plate to point upward. From the problem statement, the axis of symmetry of the plate is a small angle θ away from this direction.
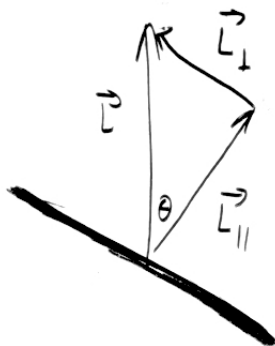
As in **M8**, we decompose the angular momentum into parallel and perpendicular components, and

$$L_\parallel = L, \quad L_\perp = \theta L$$

by the small angle approximation, and hence

$$\omega_\parallel = \frac{L_\parallel}{I_\parallel} = \frac{L}{MR^2/2}, \quad \omega_\perp = \frac{L_\perp}{I_\perp} = \frac{\theta L}{MR^2/4}$$

where the last step is by the perpendicular axis theorem. Now we need to think more about the physical motion of the plate. The component $\omega_\parallel$ of angular velocity parallel to the axis of rotation is the part that makes the medallion go around,

$$\omega_{\text{med}} = \omega_\parallel.$$

The component $\omega_\perp$ makes the orientation of the plate itself rotate. Specifically, the entire setup drawn above rotates about the axis of $\mathbf{L}$ with angular velocity ω_{wob}. Imagine the path taken by the unit normal $\hat{\mathbf{n}}$ to the plate. The tip of this vector goes in a circle of circumference $2\pi\theta$, but the speed of the tip of the vector is $\omega_\perp$. Therefore, the angular velocity of the vector along the circle is

$$\omega_{\text{wob}} = \frac{\omega_\perp}{\theta}.$$

The answer to the question is

$$\frac{\omega_{\text{med}}}{\omega_{\text{wob}}} = \frac{1}{2}.$$

So it's the opposite of what Feynman says – actually the wobbling goes twice as fast. Sometimes, when you tell a story too many times, you forget details like this.

4   Gravity

[3] **Problem 21** (Morin 5.65). Let the Earth's radius be R, its average density be ρ, and its angular frequency of rotation be ω. Consider a long rope with uniform mass density extending radially from just above the surface of the Earth out to a radius ηR. Show that if the rope is to remain above the same point on the equator at all times, then we must have

$$\eta^2 + \eta = \frac{8\pi G \rho}{3\omega^2}.$$

What is the numerical value of η, and where does the tension in the rope achieve its maximum value? Such a rope would function as a space elevator, allowing objects to be lifted to space much more cheaply. It was conceived by the science fiction writer R. A. Heinlein, who called it a "skyhook".

Solution. The gravitational field from Earth will be

$$g = \frac{4}{3}G\pi\rho R^3/r^2$$

which works with the tension to provide the centripetal acceleration $\omega^2 r$. For a small piece of rope of length dr and mass $dm = \mu\, dr$, the force balance gives

$$\omega^2 r\, dm = g\, dm - dT$$

$$dT = \frac{4}{3}\frac{G\pi\rho\mu R^3}{r^2}dr - \mu\omega^2 r\, dr$$

Integrating from $r = R$ to $r = \eta R$ gives

$$T(\eta R) - T(R) = \frac{4}{3}G\pi\rho\mu R^2 \left(1 - \frac{1}{\eta}\right) - \frac{1}{2}\mu\omega^2 R^2(\eta^2 - 1).$$

At both ends of the rope, the tension must be zero since they're not connected to anything, so

$$\frac{8\pi G\rho}{3\omega^2}\frac{\eta-1}{\eta} = (\eta-1)(\eta+1)$$

which gives

$$\eta^2 + \eta = \frac{8\pi G\rho}{3\omega^2} = 579$$

and solving the quadratic numerically gives

$$\eta = 23.6.$$

The maximum value of the tension occurs when $dT/dr = 0$, which is when

$$r^3 = \frac{4\pi G\rho R^3}{3\omega^2}, \quad r = R\left(\frac{4\pi G\rho}{3\omega^2}\right)^{1/3} = 6.62R.$$

This radius has a physical meaning: since the gravitational and centrifugal forces on a piece of mass balance here, it's the radius where a satellite can stay in geostationary orbit.

[2] **Problem 22** (Morin 10.7). A puck slides with a small speed v on frictionless ice. The surface is "level" in the sense that it is orthogonal to $\mathbf{g}_{\text{eff}}$ at all points, where $\mathbf{g}_{\text{eff}}$ includes the centrifugal acceleration. Show that the puck moves in a circle, as seen in the Earth's rotating frame. Find its radius and the frequency and direction of the motion, in terms of the Earth's angular velocity ω_0 and the latitude ϕ of the puck.

Solution. Since the surface is level with gravity and the centrifugal acceleration, the normal force will cancel out the effects from those, so the only remaining force is the Coriolis force $-2m\mathbf{\Omega} \times \mathbf{v}$. The component of Earth's angular velocity normal to the ground at latitude ϕ is $\Omega \sin\phi$, so

$$2\omega_0 v \sin\phi = v^2/r, \quad r = \frac{v}{2\omega_0 \sin\phi}, \quad \omega = 2\omega_0 \sin\phi.$$

The puck will travel clockwise in the Northern hemisphere and counterclockwise in the Southern hemisphere. (You might wonder why this is opposite the direction hurricanes turn. The difference is that in a hurricane, the center has low pressure, and the Coriolis force provides a outward force which opposes the inward pressure force, so that the system doesn't immediately collapse. By contrast, here the Coriolis force must be inward since it is the only source of centripetal force.)

[3] **Problem 23.** In this problem, you will explore yet another slick derivation of Kepler's first law. As usual, suppose the orbit lies in the xy plane. This derivation resolves around writing $\mathbf{r}(t) = r(t)\hat{\mathbf{r}}(\theta)$ where $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$, and solving a differential equation for $\mathbf{v}(\theta)$.

(a) Show that $d\mathbf{v}/d\theta$ is proportional to $\hat{\mathbf{r}}$, times constants and conserved quantities.

(b) Integrate this result to find $\mathbf{v}(\theta)$. What nice geometric property does it have?

(c) Plug this result into $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ to show that $r(\theta)$ is a conic section.

Solution. (a) Using the usual variables, we have

$$\frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \quad L = mr^2\frac{d\theta}{dt}.$$

The desired quantity is

$$\frac{d\mathbf{v}}{d\theta} = \frac{d\mathbf{v}}{dt}\frac{dt}{d\theta} = -\frac{GMm}{L}\hat{\mathbf{r}}.$$

(b) Performing a straightforward integration, we find

$$\mathbf{v}(\theta) = -\frac{GMm}{L}(\sin\theta + c_1, -\cos\theta + c_2)$$

where c_1 and c_2 are constants of integration. The nice result is that $\mathbf{v}(\theta)$ is a circle! That's the hidden reason why this derivation will be so quick.

(c) Carrying out the cross product, the magnitude of the angular momentum is

$$L = -\frac{GMm^2r}{L}((\cos\theta)(-\cos\theta + c_2) - (\sin\theta)(\sin\theta + c_1)).$$

We can easily solve this to get

$$\frac{1}{r} = \frac{GMm^2}{L^2}(1 + c_1\sin\theta - c_2\cos\theta).$$

This can be put into the standard form for a conic section by just shifting θ to get rid of the $\sin\theta$ term.

[3] **Problem 24.** ⏲ USAPhO 2021, problem B3. A simple setup with many nice lessons.

5 Fluids

[2] **Problem 25** (BAUPC)**.** Two cylindrical containers, A and B, have the same shape and contain equal volumes of water. In addition to the water, B contains an immersed balloon, attached to the bottom with a string. Consider the following reasoning.

> The total upward force exerted by the bottom of container A is equal to the weight of the water in A, likewise for B. Since the areas of the bottoms are the same, and the weights of the water are the same, the pressures at the bottoms are the same.

Is this reasoning correct or incorrect? Explain your answer.

Solution. The reasoning is incorrect. As usual, the pressure depends only on the depth of the water, which is greater for container B. The net forces still balance: the greater upward force of the bottom on the water in container B is compensated for by the downward force of the balloon on the water.

[4] **Problem 26.** ⏲ IPhO 1997, problem 3. A simple, neat question about how a plane works.

Remark

As you saw in the above problem, a plane works by having a wing angled so that air passing next to it gets pushed downward; momentum conservation then implies there is an upward force on the wing. Concretely, this upward force occurs because there is a higher pressure on the bottom of the wing, which by Bernoulli's principle means there must be a higher air velocity on the top of the wing.

This is all standard knowledge among engineers. However, in outdated or poor physics textbooks, an incorrect explanation is sometimes given. The story is that because the top

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side of the wing is curved, the air at the top of the wing must be moving faster, so that it can "catch up" to the air at the bottom. Then by Bernoulli's principle there is a higher pressure on the bottom of the wing, causing lift.

The second step of this argument is right, but the first step doesn't make sense: there is no reason air should want to "catch up" with the air it used to be next to. For example, airplanes can fly upside down as long as the wing is angled the right way. In this case, the air at the top of the wing still moves faster, but now it's the bottom side that's curved.

There's much more to say about flight, which is a whole field of study. For some nice further discussion, see section 4.6 of The Art of Insight, or this paper and this paper by the same author. And for a perspective from a cantankerous old aerodynamicist, see this talk.

[5] **Problem 27.** Physics Cup 2023, problem 1.

Solution. See the official solutions here.

[3] **Problem 28.** [A] The Navier–Stokes equation, governing the velocity $\mathbf{v}(\mathbf{x}, t)$ of an incompressible fluid of density ρ and dynamic viscosity μ, are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2 \mathbf{v}.$$

Let's consider a fixed fluid, with a given ρ and μ, and see how potential solutions to this equation behave under symmetry transformations.

(a) Given a velocity $\mathbf{v}(\mathbf{x}, t)$, we can define a time-reversed velocity profile $\mathbf{v}'(\mathbf{x}, t) = -\mathbf{v}(\mathbf{x}, -t)$. If we replace $\mathbf{v}$ with $\mathbf{v}'$ in the above equation, then the $\partial \mathbf{v}/\partial t$ term stays the same, because both the sign of the velocity and the sign of the time derivative get flipped. Show that among the other three terms, two of them stay the same and one flips sign, and interpret your result.

We can schematically describe the above transformation as "$t \to -t$ and $\mathbf{v} \to -\mathbf{v}$". Next, we'll consider a scaling, which simultaneously takes $\mathbf{x} \to \lambda \mathbf{x}$ and $t \to \lambda^2 t$.

(b) Find how this scaling affects $\mathbf{v}$, P, time derivatives, and space derivatives.

(c) Show that under this scaling, all the terms in the Navier–Stokes equation scale as λ^{-n}, for an integer n you should find.

In other words, you will have shown that the Navier–Stokes equation has scale symmetry: if something can happen in a fluid, then it can also happen at a scale 2 times larger and 4 times slower. Remarkably, this also explains why there aren't many additional terms in the equation! In principle there must be terms with more time or space derivatives, since the microscopic dynamics of a fluid are very complex. However, all such terms would be suppressed by more than λ^{-n} as we scale a solution up, which means that they wouldn't have any significant effect on macroscopic scales. The Navier–Stokes equation contains precisely the terms that remain relevant.

Solution.   (a) The other term on the left-hand side gets two sign flips, because it has two powers of $\mathbf{v}$. On the right-hand side, the fluid properties ρ and μ stay the same, as does the pressure

P, so the ∇P term is unchanged, while the $\nabla^2 \mathbf{v}$ term flips sign. The interpretation is that viscosity is the only effect in the equations that cares about the direction of time, because it arises from dissipation and thus has a direction set by the second law of thermodynamics. In the absence of viscosity, the dynamics of a fluid are time reversal symmetric.

(b) Since $\mathbf{v}$ is a distance per time, it must scale as $\mathbf{v} \to \lambda^{-1} \mathbf{v}$. The spatial derivative transforms in the opposite way as the space: if a configuration gets stretched out, then its change per length is smaller, so its derivative gets smaller. Thus we conclude $\nabla \to \lambda^{-1} \nabla$, and by similar reasoning $\partial/\partial t \to \lambda^{-2} \partial/\partial t$.

Pressure is the tricky one. You can find it by thinking about how pressure emerges in kinetic theory, or by dimensional analysis: P/ρ is a speed squared, and ρ is fixed, so $P \to \lambda^{-2} P$.

(c) By plugging in our results to part (b), it's straightforward to show $n = 3$.