Mechanics III: Dynamics

Chapters 3 and 5 of Morin cover dynamics, energy, and momentum. Alternatively, see chapters 2 and 3 of Kleppner and Kolenkow, or chapters 4 and 6 of Wang and Ricardo, volume 1. For fun, see chapters I-9 through I-14 of the Feynman lectures. There is a total of **82** points.

1 Blocks, Pulleys, and Ramps

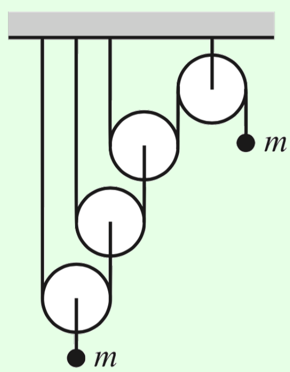
Idea 1

To solve dynamics problems with constraints, it's easiest to first write the constraint in terms of coordinates (e.g. "conservation of string" for pulleys, or stationarity of the CM for an isolated system), then differentiate to get constraints on the velocity and acceleration.

Questions of this type are generally straightforward, as long as you write down the correct equations. The trickiest part is often solving the equations, which can get messy.

Example 1: Morin 3.30

Find the acceleration of the masses in the Atwood's machine shown below.



Neglect friction, and treat all pulleys are massless.

Solution

Let x and x' be the amounts by which the left and right mass have moved down, and number the pulleys 1 through 4 from left to right, and the strings 1 through 3 from left to right. Pulley 4 is stationary, so conservation of string 3 means that pulley 3 moves up by $x'/2$. Next, conservation of string 2 means that pulley 2 moves up by $x'/4$. Finally, conservation of string 1 implies that pulley 1 moves up by $x'/8$, so our final conservation of string constraint is

$$x = -\frac{x'}{8}$$

which upon applying the derivative twice gives

$$a = -\frac{a'}{8}.$$

Now consider the tensions T_i in the strings. We know that

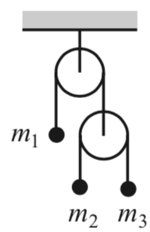
$$a = g - \frac{2T_1}{m}, \quad a' = g - \frac{2T_3}{m}.$$

Since pulley 3 is massless, the forces on it must balance, so $T_2 = 2T_3$. Similarly $T_1 = 2T_2$, so $T_1 = 4T_3$. We hence have a system of three equations in three unknowns (T_1, a, and a'), which can be solved straightforwardly to give

$$a' = \frac{56}{65}\,g, \quad a = -\frac{7}{65}\,g.$$

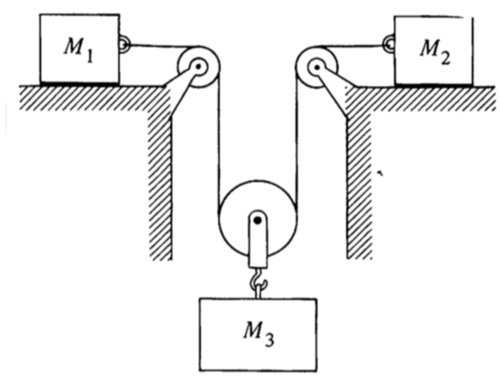
By the way, this arrangement of pulleys is called a Spanish burton. If there are n pulleys chained at the left ($n = 3$ in the above diagram), the mechanical advantage is 2^n, the highest of any possible pulley system. However, in practice such a huge mechanical advantage is rarely useful, since friction would be substantial and the range of motion is small. Instead, people who use pulleys in real life, like sailors, climbers, or auto mechanics, tend to use simpler setups like the block and tackle or chain hoist.

[2] **Problem 1** (Morin 3.2)**.** Consider the double Atwood's machine shown below.



Assuming all pulleys are massless, and neglecting friction, find the acceleration of the mass m_1.

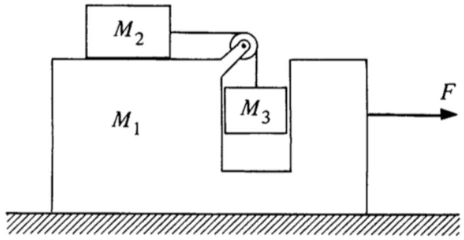
[2] **Problem 2** (KK 2.15)**.** Consider the system of massless pulleys shown below.



The coefficient of friction between the masses and the horizontal surfaces is μ. Show that the tension in the rope is

$$T = \frac{(\mu + 1)g}{2/M_3 + 1/2M_1 + 1/2M_2}.$$

[2] **Problem 3** (KK 2.20)**.** Consider the machine shown below, which we encountered in **M2**.



Show that the acceleration of M_1 when the external force F is zero is

$$a = -\frac{M_2 M_3 g}{M_1 M_2 + M_1 M_3 + 2M_2 M_3 + M_3^2}.$$

[3] **Problem 4.** A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal table.

(a) When the block is released, what is the horizontal acceleration of the plane?

(b) Assume the block starts a distance d above the table. Using results from part (a), what is the horizontal velocity of the block just before it reaches the floor?

(c) Find the speed of the block after it reaches the floor by applying energy and momentum conservation to the entire process.

(d) Your results for parts (b) and (c) should not match. What's going on?

2 Momentum

Idea 2

The momentum of a system is

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{v}_{\text{CM}}.$$

In particular, the total external force on the system is $M\mathbf{a}_{\text{CM}}$, and if there are no external forces, the center of mass moves at constant velocity.

Example 2

A massless rope passes over a frictionless pulley. A monkey hangs on one side, while a bunch of bananas with exactly the same weight hangs from the other side. When the monkey tries to climb up the rope, what happens?

Solution

Remarkably, the answer doesn't depend on how the monkey climbs, whether slowly or quickly, or symmetrically or not! The total vertical force on the monkey is $T - mg$, so the acceleration of the center of mass of the monkey is $T/m - g$. But since the tension is uniform through a massless rope, the acceleration of the bananas is also $T/m - g$. Therefore, the monkey and bananas rise at the same rate, and meet each other at the pulley.

Now here's a question for you: compared to climbing up a rope fixed to the ceiling, climbing up to the pulley takes twice as much work, because the bananas are raised too. But in both cases, isn't the monkey applying the same force through the same distance? Where does the extra work come from? (The answer involves the ideas at the end of this problem set.)

Example 3: KK 3.14 / INPhO 2014.5

Two men, each with mass m, stand on a railway flatcar of mass M initially at rest. They jump off one end of the flatcar with velocity u relative to the car. The car rolls in the opposite direction without friction. Find the final velocities of the flatcar if they jump off at the same time, and if they jump off one at a time. Generalize to the case of $N \gg 1$ men, with a total mass of m_{tot}.

Solution

In the first case, by conservation of momentum, we have

$$Mv + 2m(v - u) = 0$$

where v is the final velocity of the flatcar, so

$$v = \frac{2mu}{M + 2m}.$$

In the second case, by a similar argument, we find that after the first man jumps,

$$v_1 = \frac{mu}{M + 2m}.$$

Now transform to the frame moving with the flatcar. When the second man jumps, he imparts a further velocity $v_2 = mu/(M + m)$ to the flatcar by another similar argument. The final velocity of the flatcar relative to the ground is then

$$v = v_1 + v_2 = mu \left(\frac{1}{M + 2m} + \frac{1}{M + m} \right).$$

It might be a bit disturbing that the final speeds and hence energies of the flatcar are different, even though the men are doing the same thing (i.e. expending the same amount of energy in their legs to jump) in both cases.

The reason for the difference is that in the second case, the second man to jump ends up with less energy, since the velocity he gets from jumping is partially cancelled by the existing velocity v_1. So the extra energy that goes into the flatcar corresponds to less kinetic energy in the men after jumping, which would ultimately have ended up as heat after they slid to a stop. Accounting properly for the kinetic energy of everything in the system solves a lot of paradoxes involving energy, as we'll see below.

In the case of many men, by similar reasoning we have

$$v = \frac{m_{\text{tot}}}{M + m_{\text{tot}}} u$$

in the first case, while in the second case the answer is the sum

$$v = \sum_{i=1}^{N} \frac{m_{\text{tot}} u}{N} \frac{1}{M + (i/N) m_{\text{tot}}}.$$

This can be converted into an integral, by letting $x = i/N$, in which case $\Delta x = 1/N$ and

$$v = \sum_i \Delta x \, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} \approx \int_0^1 dx \, \frac{m_{\text{tot}} u}{M + x m_{\text{tot}}} = \log\left(\frac{M + m_{\text{tot}}}{M}\right) u.$$

Note that this is essentially the rocket equation, which we'll derive in a different way in **M6**.

[2] **Problem 5** (KK 4.11)**.** A flexible chain of mass M and length ℓ is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. Find the reading on the scale when a length of chain x has fallen.

[3] **Problem 6.** Some qualitative questions about momentum.

(a) A box containing a vacuum is placed on a frictionless surface. The box is punctured on its right side. How does it move immediately afterward?

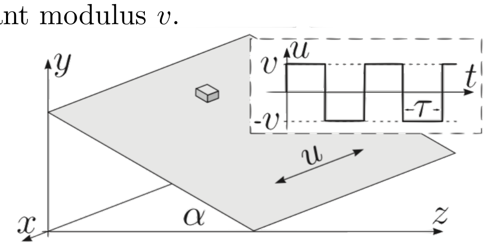
(b) You are riding forward on a sled across frictionless ice. Snow falls vertically (in the frame of the ice) on the sled. Which of the following makes the sled go the fastest or the slowest?

1. You sweep the snow off the sled, directly to the left and right in your frame.
2. You sweep the snow off the sled, directly to the left and right in the ice frame.
3. You do nothing.

(c) An hourglass is made by dividing a cylinder into two identical halves, separated by a small orifice. Initially, the top half is full of sand and the bottom half is empty. The hourglass is placed on a scale, and then the orifice is opened. The total weight of the hourglass and sand is W. How does the scale reading compare to W shortly after the sand starts falling, shortly after it finishing falling, and in between? (For concreteness, assume the surfaces of the sand in the top and bottom halves are always horizontal, and that the sand passes through the orifice at a constant rate.)

[3] **Problem 7.** USAPhO 2018, problem A1.

[3] **Problem 8** (Kalda)**.** A block is on a ramp with angle α and coefficient of friction $\mu > \tan\alpha$. The ramp is rapidly driven back and forth so that its velocity vector $\mathbf{u}$ is parallel to both the slope and the horizontal and has constant modulus v.



The direction of $\mathbf{u}$ reverses abruptly after each time interval τ, where $g\tau \ll v$. Find the average velocity $\mathbf{w}$ of the block. (Hint: as mentioned in **M1**, it's best to work in the frame of the ramp, because it causes the friction, even though this introduces fictitious forces.)

[4] **Problem 9** (Morin 5.21). A sheet of mass M moves with speed V through a region of space that contains particles of mass m and speed v. There are n of these particles per unit volume. The sheet moves in the direction of its normal. Assume $m \ll M$, and assume that the particles do not interact with each other.

(a) If $v \ll V$, what is the drag force per unit area on the sheet?

(b) If $v \gg V$, what is the drag force per unit area on the sheet? Assume for simplicity that the component of every particle's velocity in the direction of the sheet's motion is exactly $\pm v/2$.

(c) Now suppose a cylinder of mass M, radius R, and length L moves through the same region of space with speed V, and assume $v = 0$ and $m \ll M$. The cylinder moves in a direction perpendicular to its axis. What is the drag force on the cylinder?

Parts (a) and (b) are a toy model for the two regimes of drag, mentioned in **M1**. However, it shouldn't be taken too seriously, because as we'll see in **M7**, the typical velocity that separates the two types of behavior doesn't have to be of order v. Instead, it depends on how strongly the particles interact with each other.

3 Energy

Idea 3

The work done on a point particle is

$$W = \int \mathbf{F} \cdot d\mathbf{x}$$

and is equal to the change in kinetic energy, as you showed in **P1**.

Remark: Dot Products

The dot product of two vectors is defined in components as

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z$$

and is equal to $|\mathbf{v}|\,|\mathbf{w}|\cos\theta$ where θ is the angle between them. For example, if $\mathbf{A}$ and $\mathbf{B}$ are the sides of a triangle, the other side is $\mathbf{C} = \mathbf{A} - \mathbf{B}$, and

$$C^2 = |\mathbf{A} - \mathbf{B}|^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = A^2 + B^2 - 2AB\cos\theta$$

which proves the law of cosines. (Or, if you accept the law of cosines, you could regard this as a proof that the dot product depends on $\cos\theta$ as claimed.)

Like the ordinary product, the dot product obeys the product rule. For example,

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{w}) = \dot{\mathbf{v}} \cdot \mathbf{w} + \mathbf{v} \cdot \dot{\mathbf{w}}.$$

Using this, it's easy to generalize the derivation of the work-kinetic energy theorem in **P1** to three dimensions; we have

$$\frac{1}{2}d(v^2) = \frac{1}{2}d(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot d\mathbf{v} = \frac{d\mathbf{x}}{dt} \cdot d\mathbf{v} = \frac{d\mathbf{v}}{dt} \cdot d\mathbf{x} = \mathbf{a} \cdot d\mathbf{x}$$

and this is equivalent to the desired theorem. As you can see, it's all basically the same, since the product and chain rule manipulations work the same way for vectors and scalars.

Example 4: IPhO 1996 1(b)

A skier starts from rest at point A and slowly slides down a hill with coefficient of friction μ, without turning or braking, and stops at point B. At this point, his horizontal displacement is s. What is the height difference h between points A and B?

Solution

Since the skier begins and ends at rest, the change in height is the total energy lost to friction,

$$mgh = \int f_{\text{fric}}\, ds$$

where the integral over ds goes over the skier's path. Since the skier is always moving slowly, the normal force is approximately $mg\cos\theta$. (More generally, there would be another contribution to provide the centripetal acceleration.) But then

$$\int f_{\text{fric}}\, ds = \int \mu mg \cos\theta\, ds = \int \mu mg\, dx = \mu mgs$$

which gives an answer of $h = \mu s$. (If the skier's path turned around, then this would still hold as long as s denotes the total horizontal distance traveled.)

[3] **Problem 10** (MPPP 16)**.** On a windless day, a cyclist going "flat out" can ride uphill at a speed of $v_1 = 12\,\text{km/h}$ and downhill at $v_2 = 36\,\text{km/h}$ on the same inclined road. We wish to find the cyclist's top speed on a flat road if their maximal effort is independent of the speed at which the bike is traveling. Note that in this regime, the air drag force is quadratic in the speed.

(a) Solve the problem assuming that "maximal effort" refers to the force exerted on the pedals by the rider, and that the rider never changes gears.

(b) Solve the problem assuming that "maximal effort" refers to the rider's power.

[3] **Problem 11.** USAPhO 2016, problem B1.

[2] **Problem 12.** Alice steps on the gas pedal on her car. Bob, who is standing on the sidewalk, sees Alice's car accelerate from rest to 10 mph. Charlie, who is passing by in another car, sees Alice's car accelerate from 10 mph to 20 mph. Hence Charlie sees the kinetic energy of Alice's car increase by three times as much. How is this compatible with energy conservation, given that the same amount of gas was burned in both frames?

[3] **Problem 13** (KK 4.8)**.** A block of mass M is attached to a spring of spring constant k. It is pulled a distance L from its equilibrium position and released from rest. The block has a small coefficient of friction μ with the ground. Find the number of cycles the mass oscillates before coming to rest.

[3] **Problem 14** (Morin 5.4)**.** A massless string of length 2ℓ connects two hockey pucks that lie on frictionless ice. A constant horizontal force F is applied to the midpoint of the string, perpendicular

to it. The pucks eventually collide and stick together. How much kinetic energy is lost in the collision?

Idea 4

If a problem can be solved using either momentum conservation or energy conservation alone, it usually means one of the two isn't actually conserved. In particular, many processes are inherently inelastic and inevitably dissipate energy. For more about inherently inelastic processes, see section 5.8 of Morin.

[2] **Problem 15** (KK 4.20)**.** Sand falls slowly at a constant rate dm/dt onto a horizontal belt driven at constant speed v.

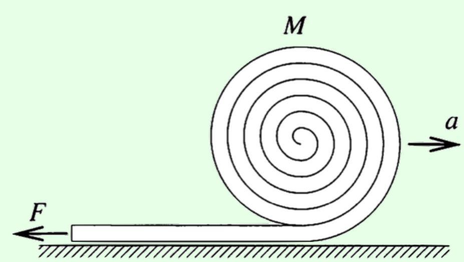
(a) Find the power P needed to drive the belt.

(b) Show that the rate of increase of the kinetic energy of the sand is only $P/2$.

(c) We can explain this discrepancy exactly. Argue that in the reference frame of the belt, the rate of heat dissipation is $P/2$. Since temperature is the same in all frames, the rate of heat dissipation is $P/2$ in the original frame as well, accounting for the missing energy.

Example 5: PPP 108

A fire hose of mass M and length L is coiled into a roll of radius R. The hose is sent rolling along level ground, with its center of mass given initial speed $v_0 \gg \sqrt{gR}$. The free end of the hose is held fixed.



The hose unrolls and becomes straight. How long does this process take to complete?

Solution

First, we need to find what is conserved. The horizontal momentum is not conserved, because there is an external horizontal force needed to keep the end of the hose in place. On the other hand, the energy *is* conserved, even though this process looks inelastic. The hose "sticks" to the floor as it unrolls, but this process dissipates no energy because the circular part of the hose rolls without slipping, so the bottom of this part always has zero velocity.

Once we figure out energy is conserved, the problem is straightforward. The assumption $v_0 \gg \sqrt{gR}$ means we can neglect the change in gravitational potential energy as the hose unrolls. After the hose travels a distance x,

$$\frac{1}{2}\left(1+\frac{1}{2}\right)Mv_0^2 = \frac{1}{2}\left(1+\frac{1}{2}\right)mv^2$$

where the 1/2 terms are from rotational kinetic energy. Since $m(x) = M(1-x/L)$, we have

$$v(x) = \frac{v_0}{\sqrt{1-x/L}}$$

which gives a total time

$$T = \int_0^L \frac{dx}{v(x)} = \frac{L}{v_0}\int_0^1 \sqrt{1-u}\,du = \frac{2L}{3v_0}.$$

Evidently, the hose accelerates as it unrolls.

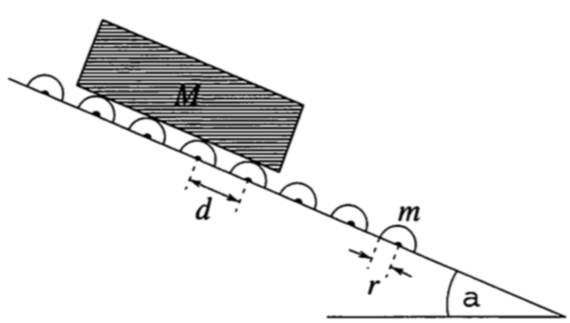
[4] **Problem 16.** Consider the following related problems; in all parts, neglect friction.

(a) A uniform rope of length ℓ lies stretched out flat on a table, with a tiny portion $\ell_0 \ll \ell$ hanging through a hole. The rope is released from rest, and all points on the rope begin to move with the same speed. Since this motion is smooth, energy is conserved. Find the speed of the rope when the end goes through the hole.

(b) $\star$ For practice, repeat part (a) by solving for $x(t)$ explicitly. (Hint: this is best done using the generalized coordinate techniques of **M4**.)

(c) Now suppose a flexible uniform chain of length ℓ is placed loosely coiled close to the hole. Again, a tiny portion $\ell_0 \ll \ell$ hangs through the hole, and the chain is released from rest. In this case, the unraveling of the chain is an inherently inelastic process, because each link of the chain sits still until it is suddenly jerked into motion. Find the speed of the chain when the last link goes through the hole. (Hint: you should get a nonlinear differential equation, which can be solved by guessing $x(t) = At^n$.)

[3] **Problem 17** (PPP 95)**.** A long slipway, inclined at an angle α to the horizontal, is fitted with many identical rollers, consecutive ones being a distance d apart. The rollers have horizontal axles and consist of rubber-covered solid steel cylinders each of mass m and radius r. A plank of mass M, and length much greater than d, is released at the top of the slipway.



Find the terminal speed $v_{\max}$ of the plank. Ignore air drag and friction at the pivots of the rollers.

4 Elastic Collisions

Idea 5

Any temporary interaction between two objects that conserves energy and momentum is a perfectly elastic collision. In one dimension, such collisions are "trivial": their outcome is fully determined by energy and momentum conservation, because there are two final velocities and two conservation laws. In two dimensions, there are four final velocity components and three conservation laws (energy and 2D momentum), so we need one more number to describe what happens, such as the angle of deflection. In a two-dimensional collision, the outcome depends on the details, such as how the objects approach each other, and the force between them. The same holds in three dimensions.

Example 6

Two masses are constrained to a line. The mass m_1 moves with velocity v_1, and the mass m_2 moves with velocity v_2. The masses collide perfectly elastically. Find their speeds afterward.

Solution

The usual method is to directly invoke conservation of energy and momentum, which leads to a quadratic equation. A slicker method is to work in the center of mass frame instead. (This is useful for collision problems in general, and it'll become even more useful for the relativistic collisions covered in **R2**.)

The center of mass of the system has speed

$$v_{\text{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

Moreover, by momentum conservation, the center of mass never accelerates. Now we boost into the frame moving with the center of mass. Since the total momentum is by definition zero in the center of mass frame, the momenta of the particles cancel out. The only way for this to remain true after the collision is if we multiply their velocities by the same number. Energy is only conserved if this number is ± 1, with the latter representing no collision at all.

Therefore, during an elastic collision, the velocities in the center of mass frame simply reverse. The initial velocities in that frame are

$$v_{1,\text{CM}} = v_1 - v_{\text{CM}}, \quad v_{2,\text{CM}} = v_2 - v_{\text{CM}}.$$

The final velocities in that frame are

$$v'_{1,\text{CM}} = -v_1 + v_{\text{CM}}, \quad v'_{2,\text{CM}} = -v_2 + v_{\text{CM}}.$$

Finally, going back to the original frame gives the final velocities

$$v'_1 = -v_1 + 2v_{\text{CM}}, \quad v'_2 = -v_2 + 2v_{\text{CM}}.$$

There are many special cases we can check. For example, if $m_1 = m_2$, then the two masses simply swap their velocities, as if they just passed through each other. As another check, consider the case where the second mass is initially at rest, $v_2 = 0$. Then

$$v_1' = v_1 \frac{m_1 - m_2}{m_1 + m_2}, \quad v_2' = v_1 \frac{2m_1}{m_1 + m_2}.$$

When $m_1 = m_2$, the first mass gives all its velocity to the second. When m_2 is large, the first mass just rebounds off with velocity $-v_1$. When m_1 is large, the first mass keeps on going and the second mass picks up velocity $2v_1$. Finally, when $m_1 = m_2/3$, then the final speeds are $v_1' = -v_1/2$ and $v_2' = v_1/2$, a nice result which is worth committing to memory.

Idea 6

The kinetic energy of a set of masses m_i with total mass M can be decomposed as

$$\sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M v_{\text{CM}}^2 + \sum_i \frac{1}{2} m_i (v_i - v_{\text{CM}})^2$$

where the first term is the "center of mass" contribution, and the second term is the amount of kinetic energy in the center of mass frame. (This statement also holds true for multidimensional collisions, if the squares are replaced with vector magnitudes.) The first term can be rewritten as $P^2/2M$ where P is the total momentum of the masses. Therefore, the kinetic energy of a system of masses with fixed total momentum is minimized when the second term is zero, i.e. when all the masses are traveling with the same velocity. This implies, for instance, that a totally inelastic collision dissipates the highest possible amount of kinetic energy.

Example 7

Three balls of mass M are initially at rest. Then an explosion occurs, giving the system a fixed total kinetic energy. What is the maximum possible fraction of this energy that one ball can carry by itself?

Solution

Suppose we want to maximize the energy of the first ball, and let p_0 be the magnitude of its final momentum. Since the total momentum is zero, the other two balls also have a total momentum of magnitude p_0. As shown in the above idea, the energy of those two balls is minimized if they travel at the same speed. Therefore, the optimal scenario is to have the first ball come out with speed v_0 and have both of the other two come out the other direction with equal speed $v_0/2$. Then the first ball has $2/3$ of the total energy.

This is the simplest possible "optimal collision" problem; we'll see more in **R2**. Many can be solved with the basic idea that some of the outgoing masses should have the same velocity.

[2] **Problem 18** (Morin 5.23)**.** A tennis ball with mass m_2 sits on top of a basketball with a mass $m_1 \gg m_2$. The bottom of the basketball is a height h above the ground. When the balls are dropped, how high does the tennis ball bounce?

[3] **Problem 19** (PPP 46)**.** A Newton's cradle consists of three suspended steel balls of masses m_1, m_2, and m_3 arranged in that order with their centers in a horizontal line. The ball of mass m_1 is drawn aside in their common plane until its center has been raised by h and is then released. If all collisions are elastic, how much m_2 be chosen so that the ball of mass m_3 rises to the greatest possible height, and what is this height? (Neglect all but the first two collisions.)

[3] **Problem 20.** Here's a variety problem involving some "clean" mathematical results. All three parts can be solved without lengthy calculation.

(a) Consider n identical balls confined to a line. Assuming all collisions are perfectly elastic, what is the maximum number of collisions that could happen? Assume no triple collisions happen.

(b) A billiard ball hits an identical billiard ball initially at rest in a perfectly elastic collision. Show that the balls exit at a right angle to each other.

(c) A mass M collides elastically with a stationary mass m. If $M > m$, show that the maximum possible angle of deflection of M is $\sin^{-1}(m/M)$.

[3] **Problem 21** (PPP 72)**.** Beads of equal mass m are strung at equal distances d along a long, horizontal, infinite wire. The beads are initially at rest but can move without friction. The first bead is continuously accelerated towards the right by a constant force F. After some time, a "shock wave" of moving beads will propagate towards the right.

(a) Find the speed of the shock wave, assuming all collisions are completely inelastic.

(b) Do the same, assuming all collisions are completely elastic. What is the average speed of the accelerated bead in this case?

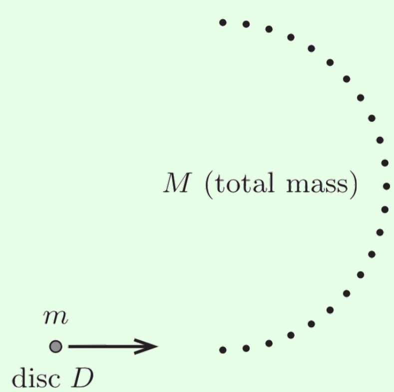
If you're having trouble visualizing this, try plotting all the masses' positions $x(t)$ over time.

[3] **Problem 22.** USAPhO 2019, problem A1.

[3] **Problem 23.** USAPhO 2009, problem B1.

Example 8: MPPP 42

There are N identical tiny discs lying on a table, equally spaced along a semicircle, with total mass M. Another disc D of mass m is very precisely aimed to bounce off all of the discs in turn, then exit opposite the direction it came.



In the limit $N \to \infty$, what is the minimal value of M/m for this to be possible? Given this value, what is the ratio of the final and initial speeds of the disc?

Solution

The reason there is a lower bound on M is that, by problem 20(c), there is a maximal angle that each tiny disc can deflect the disc D. For large N, the deflection is π/N for each disc, so

$$\frac{\pi}{N} = \sin^{-1}\frac{M/N}{m} \approx \frac{M}{Nm}$$

which implies that $M/m \geq \pi$.

To see how much energy is lost in each collision, work in the center of mass frame and consider the first collision. In this frame, the disc D is initially approximately still, and the tiny disc comes in horizontally with speed v. To maximize the deflection angle in the table's frame, the tiny disc should rebound vertically, as this provides the maximal vertical impulse to the disc D.

Thus, going back to the table's frame, where the disc D has speed v, the tiny disc scatters with speed $\sqrt{v^2+v^2} = \sqrt{2}v$. By conservation of energy,

$$\Delta\left(\frac{1}{2}mv^2\right) = -\frac{1}{2}\frac{M}{N}(\sqrt{2}v)^2.$$

This simplifies to

$$\frac{\Delta v}{v} = -\frac{\pi}{N}$$

which means that after N collisions, we have the cute result

$$\frac{v_f}{v_i} = \left(1-\frac{\pi}{N}\right)^N \approx e^{-\pi}$$

where in the last step we used a result from **P1**.

Example 9: EFPhO 2003.1

A spherical volleyball of radius r and mass m is inflated with excess pressure ΔP. If it is dropped from the ceiling and hits the ground, estimate how long the subsequent elastic collision takes.

Solution

Answering this question requires making a simplified physical model of how the collision occurs. Let's say that when the volleyball hits the ground, it will keep going straight down, deforming the part that touches the ground into a flat circular face. Specifically, when the ball has moved a distance y into the ground, the flat face has area

$$A = \pi\left(\sqrt{r^2-(r-y)^2}\right)^2 = \pi y(2r-y) \approx 2\pi ry$$

where we assumed that $y \ll r$ at all times, which is reasonable as long as the ball's initial speed is not enormous. As a result, the pressure of the volleyball exerts a force

$$F = 2\pi r \, \Delta P \, y$$

on the ground. This assumes the pressure inside the volleyball remains uniform, and that the rest of the volleyball stays approximately spherical, which is again reasonable as long as the initial speed is not huge.

Assuming the initial velocity is not too small, gravity is negligible during the collision, so during the collision the force on the volleyball is effectively that of an ideal spring. The collision lasts for half a period, giving

$$\tau = \pi \sqrt{\frac{m}{k_{\text{eff}}}} = \sqrt{\frac{\pi m}{2r \, \Delta P}}.$$

If we plug in realistic numbers, the result is of order 10 ms, which is plausible.

5 Continuous Systems

Example 10

As shown in **M2**, a hanging chain takes the form of a catenary. Suppose you pull the chain down in the middle. How does the center of mass of the chain move? Does the answer depend on how hard you pull?

Solution

No matter how hard you pull, or in what direction, the height of the center of mass always goes up! This is because this quantity measures the total gravitational potential energy of the chain. If you pull a chain in equilibrium, in any direction whatsoever, you will do work on it. So this raises its potential energy, and hence the center of mass.

Another way of saying this is that the equilibrium position, without the extra pull you supply, is already in the lowest energy state, and hence already has the lowest possible center of mass. Changing this shape in any way raises the center of mass.

[2] **Problem 24.** A uniform half-disc of radius R is nailed to a wall at the center of its circle and allowed to come to equilibrium. The half-disc is then rotated by an angle $d\theta$. By calculating the energy needed to do this in two different ways, find the distance from the pivot point to the center of mass.

[4] **Problem 25** (Morin 5.31)**.** Assume that a cloud consists of tiny water droplets suspended (uniformly distributed, and at rest) in air, and consider a raindrop falling through them. Assume the raindrop is initially of negligible size, remains spherical at all times, and collides perfectly inelastically with the droplets. It turns out that the raindrop accelerates uniformly; assuming this, find the acceleration.

[3] **Problem 26** (Kvant)**.** Half of a flexible pearl necklace lies on a horizontal frictionless table, while the other half hangs down vertically at the edge. If the necklace is released from rest, it will slide off the table. At some point, the hanging part of the necklace will begin to whip back and forth. What fraction of the necklace is on the table when this begins? (Hint: we are considering a pearl necklace with no empty string between adjacent pearls; as a result, all the pearls accelerate smoothly. To solve the problem, think about the vertical forces. There is an important related problem in **M2**.)

[4] **Problem 27** (BAUPC 2002)**.** A small ball is attached to a massless string of length L, the other end of which is attached to a very thin pole. The ball is thrown so that it initially travels in a horizontal circle, with the string making an angle θ_0 with the vertical. As time goes on, the string wraps itself around the pole. Assume that (1) the pole is thin enough so that the length of string in the air decreases very slowly, and (2) the pole has enough friction so that the string does not slide on the pole, once it touches it. Show that the ratio of the ball's final speed (right before it hits the pole) to initial speed is $\sin\theta_0$.

When dealing with an extended system whose parts all move in different ways, conservation of energy is occasionally useless. However, the somewhat obscure idea of "center of mass energy" may become useful instead. For more about this concept, see section 13.5 of Halliday and Resnick.

Idea 7: Center of Mass Energy

The work done on a part of a system is

$$dW = F\,dx$$

where F is the force on that specific part of the system, and dx is its displacement. Then $dW = dE$ where E is the total energy of the system.

Similarly, the "center of mass work" done on a system is

$$dW_{\text{cm}} = F\,dx_{\text{cm}}$$

where F is the total force on the system and dx_{cm} is the displacement of the center of mass. Then $dW_{\text{cm}} = dE_{\text{cm}}$ where the "center of mass energy" is defined as $E_{\text{cm}} = Mv_{\text{cm}}^2/2$.

It should be noted that, like regular energy and work, center of mass energy and work depend on the reference frame you're using.

Example 11

Consider a cyclist who pedals their bike to accelerate. The wheels roll without slipping on the ground. The cyclist moves a distance d, with the bike experiencing a constant friction force f from the ground. Analyze the situation using both energy and center of mass energy.

Solution

Since the wheels roll without slipping, their contact point with the ground is always zero, so the friction force does exactly zero work. Thus the net energy of the cyclist/bike system is conserved. The additional kinetic energy of the cyclist/bike comes from the chemical energy of the cyclist, which ultimately came from what they ate. So conservation of energy is correct, but it doesn't tell us anything useful at all.

Now consider center of mass energy. Considering the cyclist/bike system, the center of mass work is fd, which is the change in $Mv_{\text{cm}}^2/2$. This allows us to compute the change in velocity of the cyclist/bike.

Example 12

Consider the same setup as in the previous example, but now the cyclist brakes hard. The wheels slip on the ground, and experience a friction force $-f$ while the cyclist moves a distance d. Analyze the situation using both energy and center of mass energy.

Solution

The center of mass work equation tells us about the overall deceleration of the cyclist/bike, just as in the previous example.

On the other hand, the work done by the friction force is indeterminate! It can be any quantity between zero and $-fd$. When it is 0, the total energy of the cyclist/bike system is again conserved, which means all the kinetic energy lost is dissipated as heat inside the bike itself. When it is $-fd$, all the kinetic energy lost is dissipated as heat in the *ground*, and hence energy is removed from the cyclist/bike system. In general, the work will be an intermediate value, meaning that both the ground and the bike heat up, but we can't calculate what it is without a microscopic model of how the friction works. It depends on, e.g. how easily the ground and bike tire surface deform.

[1] **Problem 28.** Alice and Bob stand facing each other with their arms bent and hands touching on an ice skating rink. Bob has his back against a wall.

(a) Suppose Bob extends his arms, pushing Alice through a distance d with a force F. Analyze what happens to Alice in terms of both work and center of mass work.

(b) Suppose Alice extends her arms, pushing herself through a distance d with a force F. Repeat the analysis; what is different and what is the same?

(c) Suppose a spherical balloon is compressed uniformly from all sides. Is there work done on the balloon? How about center of mass work?

[4] **Problem 29.** USAPhO 2013, problem B1. This problem is quite tricky! Once you're done, carefully read the official solution, which describes how center of mass work is applied.