Electromagnetism VIII: Synthesis

Electromagnetism in matter is covered in chapters 10 and 11 of Purcell, or in sections 6.4 and 7.5 of Wang and Ricardo, volume 2. All other problems combine ideas covered in previous problem sets. For more on dielectrics, see chapters II-10 and II-11 of the Feynman lectures. Electromagnetism in matter is covered in greater detail in chapters 4, 6, and 9 of Griffiths, and chapters I-31 and II-32 through II-37 of the Feynman lectures. There is a total of **81** points.

1 Dielectrics

Idea 1

When a dielectric is placed in an electric field, dipoles inside align with the field, reducing the field value. For the simplest, most symmetrical situations, the field is simply reduced by a factor of the dielectric constant $\kappa = \epsilon/\epsilon_0$. Hence a capacitor filled with dielectric has its capacitance enhanced by κ.

The simple fact above, along with physical intuition, will be enough for most problems. However, it's also sometimes useful to think about what's going on inside a dielectric.

Idea 2

Microscopically, a dielectric carries a polarization $\mathbf{P}$ with units of electric dipole moment density, describing the net effect of its dipoles. This results in a "bound" charge

$$\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$$

within the dielectric, as well a surface bound charge

$$\sigma_{\text{bound}} = \mathbf{P} \cdot \hat{\mathbf{n}}.$$

In a dielectric, the polarization is related to the total electric field by

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \epsilon = \epsilon_0(1 + \chi_e)$$

where χ_e is the electric susceptibility. The tricky thing about using $\mathbf{P}$ is that it depends on the *total* electric field, including the electric field produced by the polarization itself; that is, in general we have to solve for $\mathbf{P}$ in terms of itself.

Example 1

A point charge q is inside a dielectric κ. Find the electric field and charge density.

Solution

By idea 1, the electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon r^2}\hat{\mathbf{r}}.$$

The dielectric simply shields the field by a factor of κ. To find the charge density, note that

$$\mathbf{P} = \frac{q}{4\pi r^2} \frac{\epsilon_0 \chi_e}{\epsilon} \hat{\mathbf{r}}.$$

The divergence of $\mathbf{P}$ is zero everywhere except for the origin, where negative bound charge piles up to cancel some of the charge q. (The compensating positive charge is at infinity, or the outer surface of the dielectric if it is finite.) The total charge at the origin is

$$q = q - q_{\text{bound}} = q\left(1 - \frac{\epsilon_0 \chi_e}{\epsilon}\right) = q\left(1 - \frac{\chi_e}{1+\chi_e}\right) = \frac{q}{\kappa}$$

which is consistent with Gauss's law for $\mathbf{E}$.

Example 2

A dielectric sphere of radius R and dielectric constant κ is placed in a field $\mathbf{E}_0$, and as a result develops a uniform polarization $\mathbf{P}$. Find $\mathbf{P}$ and the field everywhere.

Solution

First let's compute the field due to the sphere. The uniform polarization is equivalent to having two uniformly charged balls of total charge $\pm Q$ displaced by $\mathbf{d}$ so that $Q\mathbf{d} = (4\pi R^3/3)\mathbf{P}$. By the shell theorem, the field inside is uniform,

$$\mathbf{E}_p = -\frac{\mathbf{P}}{3\epsilon_0},$$

and the field outside is exactly a dipole field. Now we compute the magnitude of $\mathbf{P}$. The subtlety is that the atoms in the sphere see not to the applied field $\mathbf{E}_0$, but the total field $\mathbf{E}$,

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}, \quad \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_p.$$

Solving the system, we find

$$\mathbf{E} = \frac{3}{\kappa+2}\mathbf{E}_0, \quad \mathbf{P} = 3\,\frac{\kappa-1}{\kappa+2}\,\epsilon_0 \mathbf{E}_0.$$

The polarizability α of each atom is defined as the dipole moment per applied field,

$$\mathbf{p} = \alpha \mathbf{E}_0$$

so we have shown above that

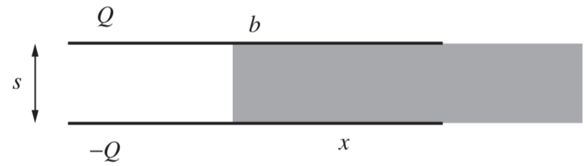
$$\alpha = \frac{3\epsilon_0}{n}\frac{\kappa-1}{\kappa+2}$$

where n is the number density of atoms. This is the Clausius–Mossotti formula; it relates the macroscopically measurable parameter κ to the microscopic parameter α.

[2] **Problem 1** (Purcell 10.10)**.** Assume that the uniform field $\mathbf{E}_0$ that causes the electric field in example 2 is produced by large capacitor plates very far away. The field lines tangent to the sphere hit each of the distant capacitor plates in a circle of radius r. Find r in terms of R and κ.

[2] **Problem 2** (Purcell 10.38). Using a similar method to example 2, consider an infinite cylindrical rod of radius R with a fixed, uniform polarization $\mathbf{P}$, where $\mathbf{P}$ may have any orientation. (Don't worry about where $\mathbf{P}$ comes from; just assume it's "frozen into" the material. That is, the material is "ferroelectric".) Qualitatively describe the electric field everywhere.

[3] **Problem 3** (Purcell 10.2). A rectangular capacitor with side lengths a and b has separation $s \ll a, b$. It is partially filled with a dielectric with dielectric constant κ. The overlap distance is x.



The capacitor is isolated and has constant charge Q.

(a) What is the energy stored in the system?

(b) Using the result of part (a), what is the force on the dielectric? Which direction does it point?

(c) Is your answer to part (b) affected by the presence of fringe fields near the interface?

[3] **Problem 4** (Griffiths 4.28). Two long coaxial cylindrical metal tubes of inner radius a and outer radius b stand vertically in a tank of dielectric oil, with susceptibility χ_e and mass density ρ. The inner one is maintained at potential V, and the outer one is grounded. To what height h does the oil rise in the space between the tubes?

[4] **Problem 5** (Cahn). [A] The region $z < 0$ is filled with a dielectric κ. Find the force on a point charge q a distance d above the origin.

2 Magnetic Materials

Idea 3

When a magnetic material is placed in an magnetic field, dipoles inside align with the field (for a paramagnet) or against the field (for a diamagnet). That is, both dielectrics and diamagnets reduce the applied field within them (the internal fields of electric and magnetic dipoles are opposite).

Idea 4

The configuration of a magnetic material is described by its magnetization $\mathbf{M}$, which has units of dipole moment per unit volume. This results in bound current density

$$\mathbf{J}_{\text{bound}} = \nabla \times \mathbf{M}$$

as well as a surface bound current density

$$\mathbf{K}_{\text{bound}} = \mathbf{M} \times \hat{\mathbf{n}}.$$

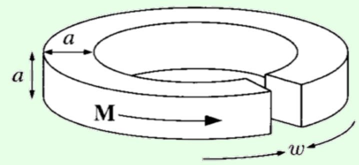
In a magnetic material, the magnetization obeys

$$\mathbf{M} = \frac{1}{\mu_0} \frac{\chi_m}{1 + \chi_m} \mathbf{B}, \quad \mu = \mu_0 (1 + \chi_m)$$

where χ_m is the magnetic susceptibility. (Note that χ_m is not defined the same way as χ_e.) Diamagnets have $\chi_m < 0$ and paramagnets have $\chi_m > 0$. Most common materials are only weakly magnetic, with $\mu \approx \mu_0$. Exceptions include superconductors, which are perfect diamagnets with $\chi_m = -1$ and hence $\mu = 0$, and ferromagnets, which have a frozen-in magnetization even when there's no external field, and hence no meaningful value of μ at all.

Example 3: Griffiths 6.10

An iron rod of length L and square cross section of side a is given a uniform longitudinal magnetization $\mathbf{M}$ and then bent into a circle with a narrow gap of width w.



Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$.

Solution

First, suppose there was no gap. Before the iron rod was bent, its uniform magnetization $\mathbf{M}$ corresponded to a bound current density $\mathbf{K}_{\text{bound}} = \mathbf{M}$ everywhere along its surface, directed circumferentially. Since $a \ll L$, this remains approximately true after bending the rod. (A small volume bound current density $\mathbf{J}_{\text{bound}}$ appears, but we neglect this.)

Therefore, the current density is the same as that of a toroidal solenoid with current I and n turns per length, where $M = In$. The field inside is therefore $\mu_0 M$, directed in the $\hat{\boldsymbol{\theta}}$ direction and zero everywhere outside the rod.

Now let's account for the gap. Adding the gap is equivalent to superposing an opposite magnetization at the gap. Since $w \ll a$, we can treat it as an $a \times a$ square current loop, with current $I = mw$. By the Biot–Savart law, the field due to such a loop at the center is

$$B_{\text{loop}} = \frac{2\sqrt{2}\mu_0 M w}{\pi a}.$$

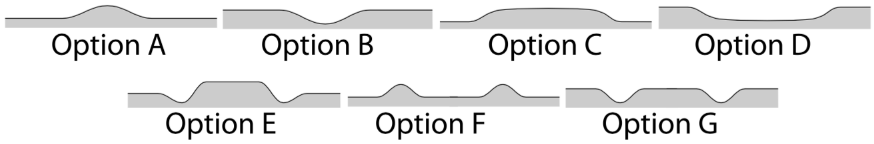
Combining the two gives a total field of

$$B = \mu_0 M \left(1 - \frac{2\sqrt{2}w}{\pi a}\right).$$

Since magnetization can be a bit mathematically nasty, you'll rarely be asked to find explicit fields, as in the above example. What's more important is the conceptual understanding.

[3] **Problem 6** (IPhO 2012)**.** Water is a diamagnetic substance. A powerful cylindrical magnet with field B is placed below the water surface.

(a) Which of the following shows the resulting shape of the water surface?



The magnet is roughly 2/3 as wide as each of these sketches.

(b) Let ρ be the density of the water. If the maximum change in height of the water surface is h, find an approximate expression for $\mu - \mu_0$.

For a closely related problem, see EuPhO 2018, problem 2. Here, the pressure change due to the interaction of water with an extreme magnetic field causes water to boil. As covered in **T3**, this happens roughly when the water pressure hits zero, since the vapor pressure of water at room temperature is small.

[3] **Problem 7.** In **E3** we covered USAPhO 2015, problem B2, which shows that the fields inside magnets differ, depending on whether they are made of "Ampere" or "Gilbert" dipoles. For example, consider a long, thin cylindrical magnet magnetized along its axis. In the Ampere model, the internal field $\mathbf{B}_0$ points along $\mathbf{M}$, while in the Gilbert model the internal field is approximately zero. We can try to distinguish between the models by drilling a hole into the magnet, and measuring the field inside the hole.

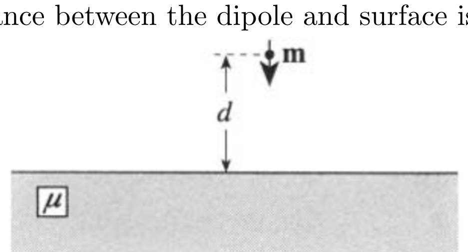
(a) Suppose we drill a long, thin cylindrical hole inside the magnet, parallel to its axis. Show that the field in this hole is the same in both the Ampere and Gilbert models.

(b) Suppose we drill a short, flat cylindrical hole inside the magnet. Show that the field in this hole is the same in both the Ampere and Gilbert models.

(c) Is there *any* way to tell the two models apart, by drilling holes in magnets and measuring the field inside the hole? What if you used different magnet shapes?

[3] **Problem 8.** EFPhO 2004, problem 6. An elegant, tricky problem on permanent magnets.

[4] **Problem 9** (Cahn)**.** [A] A small dipole $\mathbf{m}$ in vacuum points towards the plane surface of a medium with permeability μ. The distance between the dipole and surface is d.



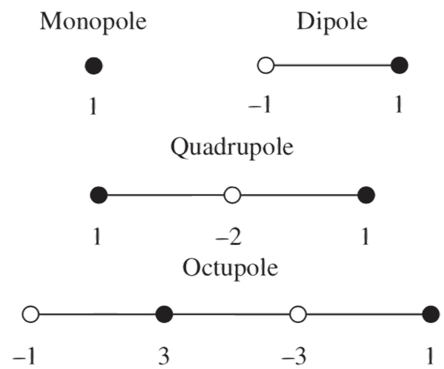
Find the force on the dipole. (If you're not familiar with dipole-dipole forces, see problem 12 first.)

[5] **Problem 10.** Physics Cup 2012, problem 2.

3 Multipoles

In this section, we explore some of the physics of dipoles and higher multipoles.

[3] **Problem 11** (Purcell 10.27). Two monopoles of opposite sign form a dipole, two dipoles of opposite sign for a quadrupole, and so on. Hence we can construct arbitrarily high multipoles using the rows of Pascal's triangle.



The field of a dipole falls as $1/r^3$, a quadrupole as $1/r^4$, and an octupole as $1/r^5$.

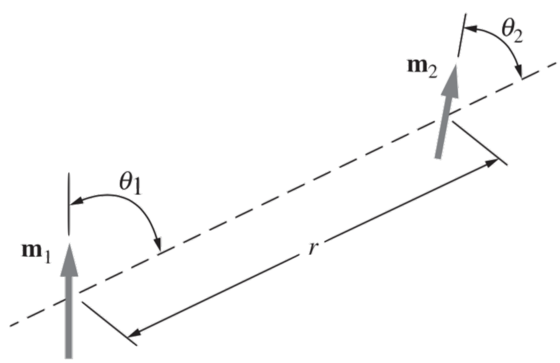
(a) To warm up, verify explicitly that the quadrupole field along the axis of the quadrupole starts at $1/r^4$, i.e. that all lower terms cancel.

(b) Prove that this cancellation occurs for general multipoles along their axis.

(c) [A] The magnitude and orientation of a dipole is specified by a vector, with three components. How many numbers are necessary to specify the magnitude and orientation of a quadrupole? (The linear quadrupoles here are just a special case of a general quadrupole.) Try to generalize to arbitrary multipoles.

To learn how to decompose an arbitrary charge distribution into multipoles, see section 3.4 of Griffiths.

[3] **Problem 12** (Purcell 11.23). Two magnetic dipoles are arranged as shown.



Show that the associated potential energy is

$$U = \frac{\mu_0 m_1 m_2}{4\pi r^3} \left(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2 \right).$$

For what orientations is this potential energy maximized or minimized?

[2] **Problem 13** (Purcell 11.36). Three magnetic compasses are placed at the corners of a horizontal equilateral triangle. As in any ordinary compass, each compass needle is a magnetic dipole constrained to rotate in a horizontal plane. The Earth's magnetic field has been shielded. What

orientation will the compass needles eventually assume? Does your result also hold for regular N-gons?

[3] **Problem 14.** Some questions about forces between dipoles and other multipoles.

(a) Above, you've shown that the force between permanent magnetic dipoles falls off as $1/r^4$. How about two permanent electric dipoles?

(b) How about a permanent dipole and a permanent quadrupole?

(c) How about two permanent quadrupoles?

(d) Now consider an ion and a neutral atom. The electric field of the ion polarizes the atom; the field of that induced dipole then reacts on the ion. Show that the resulting force is attractive and falls as $1/r^5$.

4 Electromagnetic Waves in Matter

In this section, you will work out some of the theory of electromagnetic waves in matter.

Idea 5

In the absence of any free charge or current, Maxwell's equations in matter are identical to Maxwell's equations in vacuum, except that ϵ_0 and μ_0 are related by ϵ and μ, so the waves propagate with speed $1/\sqrt{\epsilon\mu} = c/n$, with $E = (c/n)B$.

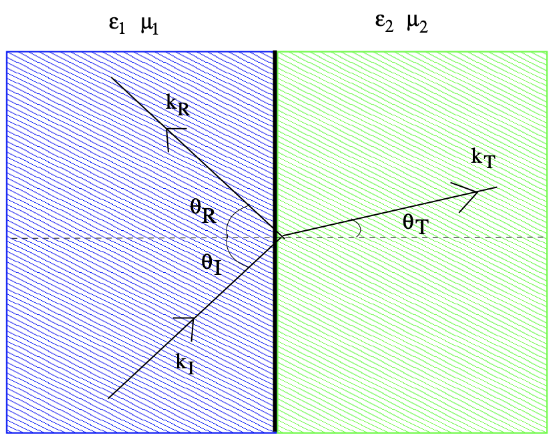
[5] **Problem 15.** Suppose the regions $x < 0$ and $x > 0$ are filled with material with permittivities ϵ_1 and ϵ_2, both with permeability μ_0. (This is typical; if you don't count permanent magnets, most objects have permeability about μ_0.) We send in an incident wave from the left with electric field

$$\mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega_i t)}.$$

The wave will be both transmitted and reflected at the interface, so the total electric field is

$$\mathbf{E} = \begin{cases} \mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega_i t)} + \mathbf{E}_r e^{i(\mathbf{k}_r \cdot \mathbf{x} - \omega_r t)} & x < 0, \\ \mathbf{E}_t e^{i(\mathbf{k}_t \cdot \mathbf{x} - \omega_t t)} & x > 0. \end{cases}$$

The angles with the normal are θ_i, θ_r, and θ_t as shown.



(a) Argue that by continuity of the field at the boundary,

$$\omega_i = \omega_r = \omega_t.$$

(b) Suppose the y-axis is oriented so that $\mathbf{k}_i \cdot \hat{\mathbf{y}} = 0$. Argue that

$$\mathbf{k}_r \cdot \hat{\mathbf{y}} = \mathbf{k}_t \cdot \hat{\mathbf{y}} = 0, \quad \mathbf{k}_i \cdot \hat{\mathbf{z}} = \mathbf{k}_r \cdot \hat{\mathbf{z}} = \mathbf{k}_t \cdot \hat{\mathbf{z}}.$$

From these conditions, derive the laws of reflection and refraction,

$$\theta_i = \theta_r, \quad n_1 \sin\theta_i = n_2 \sin\theta_t.$$

Note that neither this part nor the previous part require Maxwell's equations; they hold for *all* kinds of waves as long as we define $n_i \propto 1/v_i$.

(c) Argue that at the boundary, $\mathbf{E}_{\parallel}$ and $B_{\perp}$ must be continuous in general. In this case, because both sides have the same permittivity μ_0, there is no bound current, so $\mathbf{B}_{\parallel}$ is also continuous.

(d) Now suppose the electric fields are polarized along the $\mathbf{y}$ axis, so $\mathbf{E}_i$, $\mathbf{E}_r$, and $\mathbf{E}_t$ are all parallel to the y-axis. Then continuity of $\mathbf{E}_{\parallel}$ gives

$$E_i + E_r = E_t.$$

Using continuity of $\mathbf{B}_{\parallel}$, show that

$$\frac{E_r}{E_i} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}, \quad \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}.$$

These are the Fresnel equations for normal polarized light. (Hint: this is a bit messy, so you can warm up with the case $\theta_i = 0$.)

(e) If $n_1 > n_2$, then total internal reflection occurs when

$$\sin\theta_i > \frac{n_2}{n_1}$$

and the wave is totally reflected. Nonetheless, E_t is nonzero in this regime. To make sense of this, show that $\mathbf{k}_t \cdot \mathbf{x}$ is imaginary in this regime, indicating that the "transmitted" wave does not propagate in the region $x > 0$, but rather exponentially decays.

[5] **Problem 16.** In most common materials, $\mu \approx \mu_0$ while ϵ depends on frequency. We'll investigate the origin of this frequency dependence below.

(a) Model an electron in an atom as a mass m with charge q attached to a spring, with natural frequency ω_0 and a damping force $-m\gamma\mathbf{v}$, in an electric field $\mathbf{E}_0 e^{-i\omega t}$. Write down the equation of motion for the electron.

(b) The atomic polarizability is $\mathbf{p} = \alpha\mathbf{E}$. Show that

$$\alpha = \frac{q^2/m}{-\omega^2 + \omega_0^2 - i\gamma\omega}.$$

(c) For a gas with small number density n, the Clausius–Mossotti formula reduces to

$$\epsilon = \epsilon_0 + n\alpha.$$

Therefore, the permittivity is generally a complex number. The wavevector and frequency are related by $k^2 = \mu\epsilon\omega^2$. Explain why the fact that ϵ is complex indicates that waves can be absorbed.

(d) What frequency maximizes the absorption rate of the electromagnetic waves? Roughly how many wavelengths does a wave propagate at this frequency before being absorbed?

(e) What frequency maximizes the speed of the electromagnetic waves, and what is that speed?

(f) Transparent objects such as glass can be modeled as having a very high resonant frequency, much higher than that of visible light. Does blue light or red light refract more when passing from air to glass?

The intuitive reason that these electrons can affect the propagation speed of light is because they emit secondary electromagnetic waves that are out of phase with the original wave; this "pushes" the phase of the composite wave forward or backward, affecting the phase velocity. A complete explanation can be found in chapter I.31 of the Feynman lectures.

[5] **Problem 17.** IPhO 2002, problem 1. A neat application of electromagnetic waves in matter.

[5] **Problem 18.** APhO 2007, problem 2. A problem on an exotic negative index of refraction.

Remark

Above, we considered the response of a medium composed of atoms, obeying $p = \alpha E$. However, this relation is just an approximation, like Hooke's law. For larger electric fields, higher order terms are necessary,

$$p = \alpha E + \alpha' E^2 + \ldots$$

which lead to strange effects, studied in the field of nonlinear optics. For example, suppose we send in light of frequency ω. Then

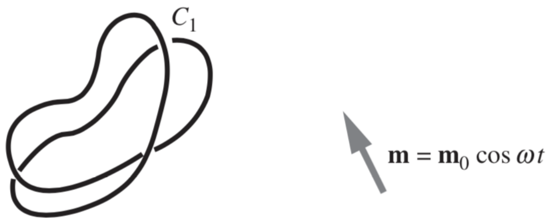
$$E^2 \propto \cos^2(\omega t) = \frac{1 + \cos(2\omega t)}{2}.$$

That means that a nonlinear medium can respond to light at frequency ω by oscillating, and hence emitting light, at frequency 2ω. This phenomenon is called frequency doubling, or second-harmonic generation, and converts red light to blue. Similarly, for a cubic nonlinearity, you can use trigonometric identities to show that frequency tripling can occur.

5 Electromagnetic Systems

In this section we'll consider problems that use everything we've covered, with a focus on technological applications and systems with multiple moving parts.

[2] **Problem 19** (Purcell 11.19)**.** A magnetic dipole $\mathbf{m}$ oscillates so that $\mathbf{m}(t) = \mathbf{m}_0 \cos \omega t$. Some of its flux links the nearby circuit C_1, inducing an electromotive force $\mathcal{E}_1 \sin \omega t$.



If a current I_1 flowed in C_1, then the resulting field at the location of the dipole would be $\mathbf{B}_1$. Show that $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$. (Hint: recall the results involving mutual inductance in **E5**.)

[3] **Problem 20.** EFPhO 2007, problem 3. A problem on focusing particles with electric fields.

[4] **Problem 21.** IPhO 2004, problem 3. A practical problem which also reviews damped/driven oscillations.

[4] **Problem 22.** EFPhO 2014, problem 1. A challenging problem about a complex nonlinear circuit.

[5] **Problem 23.** Physics Cup 2020, problem 1.