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Torsional Stealth

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Abstract: Abstract...

Contents

1 Conexión de Lorentz

La parte sin torsión de la conexión se obtiene como sigue. Dado que $\mathrm{d}e^a$ es una 2-forma, deben existir coeficientes $\Omega^a{}_{bc}(x)$ tales que podamos escribir

$$\mathrm{d}e^a = -\frac{1}{2}\Omega^a{}_{bc}e^b \wedge e^c \tag{1.1}$$

en donde claramente $\Omega^a{}_{bc}$ es antisimétrico en los índices contraídos con los vielbeins

$$\Omega^a{}_{bc} = -\Omega^a{}_{cb} \tag{1.2}$$

La condición de torsión nula

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \tag{1.3}$$

puede reescribirse como

$$\frac{1}{2}\Omega^a{}_{bc}e^b \wedge e^c + \omega^a{}_b \wedge e^b = 0 \tag{1.4}$$

$$\frac{1}{2}\Omega^a{}_{bc}e^b \wedge e^c + \omega^a{}_{bc}e^c \wedge e^b = 0 \tag{1.5}$$

$$\frac{1}{2}\Omega^a{}_{bc}e^b \wedge e^c - \omega^a{}_{bc}e^b \wedge e^c = 0 \tag{1.6}$$

es decir,

$$\omega^a{}_{bc}e^b \wedge e^c = \frac{1}{2}\Omega^a{}_{bc}e^b \wedge e^c \tag{1.7}$$

Usando (1.2), y expandiendo el producto cuña, se tiene

$$\frac{1}{2}\omega^a{}_{bc}(e^b \otimes e^c - e^c \otimes e^b) = \frac{1}{4}\Omega^a{}_{bc}(e^b \otimes e^c - e^c \otimes e^b) \tag{1.8}$$

$$(\omega^a{}_{bc} - \omega^a{}_{cb})e^b \otimes e^c = \Omega^a{}_{bc}e^b \wedge e^c \tag{1.9}$$

luego

$$\omega_{abc} - \omega_{acb} = \Omega_{abc} \tag{1.10}$$

Considerando permutaciones cíclicas de esta última ecuación,

$$\omega_{abc} - \omega_{acb} = \Omega_{abc} \tag{1.11}$$

$$\omega_{bca} - \omega_{bac} = \Omega_{bca} \tag{1.12}$$

$$\omega_{cab} - \omega_{cba} = \Omega_{cab} \tag{1.13}$$

Sumando estas 3 ecuaciones en la forma $(++-)$, obtenemos

$$\omega_{abc} - \omega_{acb} + \omega_{bca} - \omega_{bac} - \omega_{cab} + \omega_{cba} = \Omega_{abc} + \Omega_{bca} - \Omega_{cab} \tag{1.14}$$

Usando el hecho de que escritas de esta manera, ω_{abc} son antisimétricas en dos primeros 2 índices, i.e, $\omega_{abc} = -\omega_{bac}$, se tiene

$$2\omega_{abc} = \Omega_{abc} + \Omega_{bca} - \Omega_{cab} \tag{1.15}$$

Despejando,

$$\omega_{abc} = \frac{1}{2}(\Omega_{abc} + \Omega_{bca} - \Omega_{cab}) \tag{1.16}$$

Finalmente, se tiene

$$\boxed{\omega_{ab} = \frac{1}{2}(\mathcal{C}_{bac} + \mathcal{C}_{acb} - \mathcal{C}_{cba})e^c} \tag{1.17}$$

1.1 Curvatura de Lorentz

Notemos que al separar la conexión como $\omega^{ab} = \mathring{\omega}^{ab} + \kappa^{ab}$ induce tambien una separación en la curvatura. En efecto

$$\begin{aligned}
R^{ab} &= \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} && (1.18)\\
&= \mathrm{d}\big(\mathring{\omega}^{ab} + \kappa^{ab}\big) + (\mathring{\omega}^a{}_c + \kappa^a{}_c) \wedge (\mathring{\omega}^{cb} + \kappa^{cb}) && (1.19)\\
&= \mathrm{d}\mathring{\omega}^{ab} + \mathrm{d}\kappa^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathring{\omega}^a{}_c \wedge \kappa^{cb} + \kappa^a{}_c \wedge \mathring{\omega}^{cb} + \kappa^a{}_c \wedge \kappa^{cb} && (1.20)\\
&= \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathrm{d}\kappa^{ab} + \mathring{\omega}^a{}_c \wedge \kappa^{cb} + \mathring{\omega}^b{}_c \wedge \kappa^{ac} + \kappa^a{}_c \wedge \kappa^{cb} && (1.21)\\
&= \mathring{R}^{ab} + \mathring{\mathrm{D}}\kappa^{ab} + \kappa^a{}_c \wedge \kappa^{cb} && (1.22)
\end{aligned}$$

Así

$$\boxed{R^{ab} = \mathring{R}^{ab} + \mathring{\mathrm{D}}\kappa^{ab} + \kappa^a{}_c \wedge \kappa^{cb}} \tag{1.23}$$

2 Transformación conforme

Consideremos una transformación conforme de la forma

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \mathrm{e}^{2\sigma(x)} g_{\mu\nu} \tag{2.1}$$

donde $\sigma(x)$ es una función arbitraria de las coordenadas del espacio-tiempo. De aquí es fácil ver que bajo esta transformación, el vielbein transforma como

$$e^a \to \bar{e}^a = \mathrm{e}^{\sigma(x)} e^a \tag{2.2}$$

De (1.1) podemos escribir

$$\mathrm{d}e^a = \frac{1}{2}\mathcal{C}^a{}_{bc} e^b \wedge e^c \tag{2.3}$$

donde los coeficientes $\mathcal{C}_{abc}$ son los parámetros de estructura y se relacionan con la parte sin torsión de la conexión de Lorentz según (1.17), i.e.,

$$\omega_{ab} = \frac{1}{2}(\mathcal{C}_{bac} + \mathcal{C}_{acb} - \mathcal{C}_{cba})e^c \tag{2.4}$$

Se define el operador $\mathrm{I}_a : \Omega^p \to \Omega^{p-1}$ según [1]

$$\mathrm{I}_a = - * (e_a \wedge * \tag{2.5}$$

Calculemos $\mathrm{d}\bar{e}^a$,

$$\begin{aligned}
\mathrm{d}\bar{e}^a &= \mathrm{d}(\mathrm{e}^\sigma e^a) &(2.6)\\
&= \mathrm{e}^\sigma(\mathrm{d}\sigma e^a + \mathrm{d}e^a) &(2.7)
\end{aligned}$$

Notemos que podemos escribir la 1-forma $\mathrm{d}\sigma$ como

$$\mathrm{d}\sigma = \xi_a e^a, \qquad \text{donde } \xi_a = \mathrm{I}_a \mathrm{d}\sigma \tag{2.8}$$

En efecto, si escribimos $\mathrm{d}\sigma = \alpha_f e^f$, se tiene

$$\begin{aligned}
\mathrm{I}_a\mathrm{d}\sigma &= - * (e_a \wedge * \mathrm{d}\sigma) &(2.9)\\
&= - * \left(e_a \wedge \alpha_f \frac{1}{3!} \epsilon^{fbcd} e^b \wedge e^c \wedge e^d \right) &(2.10)\\
&= - * \left(\alpha_f \frac{1}{3!} \epsilon^{fbcd} e^a \wedge e^b \wedge e^c \wedge e^d \right) &(2.11)\\
&= -\alpha_f \frac{1}{3!} \epsilon^{fbcd} \epsilon_{abcd} &(2.12)\\
&= \alpha_f \frac{1}{3!} 3! &(2.13)\\
&= \alpha_f &(2.14)
\end{aligned}$$

luego, $\mathrm{d}\sigma = \mathrm{I}_a \mathrm{d}\sigma e^a = \xi_a e^a$. Reemplazando en (2.7),

$$\begin{aligned}
\mathrm{d}\bar{e}^a &= \mathrm{e}^\sigma(\mathrm{d}\sigma e^a + \mathrm{d}e^a) &(2.15)\\
&= \mathrm{e}^\sigma \left(\xi_m e^m \wedge e^a - \frac{1}{2} \mathcal{C}^a{}_{bc} e^b \wedge e^c \right) &(2.16)\\
&= \mathrm{e}^{-\sigma} \left(\xi_m \bar{e}^m \wedge \bar{e}^a - \frac{1}{2} \mathcal{C}^a{}_{bc} \bar{e}^b \wedge \bar{e}^c \right) &(2.17)\\
&= \mathrm{e}^{-\sigma} \left(\xi_m \delta^m_{[b} \delta^a_{c]} - \frac{1}{2} \mathcal{C}^a{}_{bc} \right) \bar{e}^b \wedge \bar{e}^c &(2.18)\\
&= \mathrm{e}^{-\sigma} \left(\frac{1}{2} \xi_m \delta^{ma}_{bc} - \frac{1}{2} \mathcal{C}^a{}_{bc} \right) \bar{e}^b \wedge \bar{e}^c &(2.19)\\
&= \mathrm{e}^{-\sigma} \left(-\frac{1}{2} \xi_m \delta^{am}_{bc} - \frac{1}{2} \mathcal{C}^a{}_{bc} \right) \bar{e}^b \wedge \bar{e}^c &(2.20)\\
&= -\frac{1}{2} \bar{\mathcal{C}}^a{}_{bc} \bar{e}^b \wedge \bar{e}^c &(2.21)
\end{aligned}$$

Así,

$$\bar{\mathcal{C}}^a{}_{bc} = \mathrm{e}^{-\sigma} \left(\mathcal{C}^a{}_{bc} + \xi_m \delta^{am}_{bc} \right) \tag{2.22}$$

Notemos que

$$\begin{aligned}
\bar{\mathcal{C}}_{abc} &= \mathrm{e}^{-\sigma} \left(\mathcal{C}_{abc} + \xi_m \eta_{ab} \delta^m_c - \xi_m \delta^m_b \eta_{ac} \right) &(2.23)\\
&= \mathrm{e}^{-\sigma}(\mathcal{C}_{abc} + \xi_c \eta_{ab} - \xi_b \eta_{ac}) &(2.24)
\end{aligned}$$

De manera que

$$\begin{aligned}
\bar{\mathcal{C}}_{bac} &= \mathrm{e}^{-\sigma}(\mathcal{C}_{bac} + \xi_c \eta_{ba} - \xi_a \eta_{bc}) &(2.25a)\\
\bar{\mathcal{C}}_{acb} &= \mathrm{e}^{-\sigma}(\mathcal{C}_{acb} + \xi_a \eta_{cb} - \xi_b \eta_{ca}) &(2.25b)\\
\bar{\mathcal{C}}_{cba} &= \mathrm{e}^{-\sigma}(\mathcal{C}_{cba} + \xi_a \eta_{cb} - \xi_b \eta_{ca}) &(2.25c)
\end{aligned}$$

De (1.17) es fácil ver que

$$\bar{\omega}_{ab} = \frac{1}{2}(\bar{\mathcal{C}}_{bac} + \bar{\mathcal{C}}_{acb} - \bar{\mathcal{C}}_{cba})\bar{e}^c \tag{2.26}$$

Reemplazando (2.25) en (2.26) se tiene

$$\bar{\omega}_{ab} = \frac{1}{2}\mathrm{e}^{-\sigma}(\mathcal{C}_{bac} + \xi_c\eta_{ba} - \xi_a\eta_{bc} + \mathcal{C}_{acb} + \xi_a\eta_{cb} - \xi_b\eta_{ca} - \mathcal{C}_{cba} - \xi_a\eta_{cb} + \xi_b\eta_{ca})\bar{e}^c \tag{2.27}$$

$$= \frac{1}{2}\mathrm{e}^c(\mathcal{C}_{bac} + \mathcal{C}_{acb} - \mathcal{C}_{cba}) + \frac{1}{2}e^c(\xi_c\eta_{ba} - \xi_a\eta_{bc} + \xi_a\eta_{cb} - \xi_b\eta_{ca} - \xi_a\eta_{cb} + \xi_b\eta_{ca}) \tag{2.28}$$

$$= \omega_{ab} + \frac{1}{2}(\cancel{\xi_c\eta_{ba}e^c} - \xi_a e_b + \xi_b e_a - \cancel{\xi_c\eta_{ab}e^c} - \xi_a e_b + \xi_b e_a) \tag{2.29}$$

$$= \omega_{ab} + e_a\xi_b - e_b\xi_a \tag{2.30}$$

Así,

$$\bar{\omega}_{ab} = \omega_{ab} + \theta_{ab} \tag{2.31}$$

donde hemos definido $\theta_{ab} = e_a\xi_b - e_b\xi_a$.

Una importante pregunta que concierne a las transformaciones conforme en el formalismo de primer orden está relacionada con el tipo de transformación de la parte sin torsión de la conexión.

Primero veamos el el caso **canónico** en el cual la estructura torsional es invariante bajo las transformaciones conforme, es decir,

$$\kappa_{ab} \to \bar{\kappa}_{ab} = \kappa_{ab} \tag{2.32}$$

Esto implica que la conexión de spin transforma según (2.31),

$$\bar{\omega}_{ab} = \omega_{ab} + \theta_{ab} \tag{2.33}$$

Para ver cómo transforma la torsión en este caso, usamos el hecho que

$$T^a = \kappa^a{}_b \wedge e^b \tag{2.34}$$

luego,

$$\bar{T}^a = \bar{\kappa}^a{}_b \wedge \bar{e}^b \tag{2.35}$$

$$= \kappa^a{}_b \wedge (\mathrm{e}^\sigma e^b) \tag{2.36}$$

$$= \mathrm{e}^\sigma T^a \tag{2.37}$$

es decir,

$$\boxed{\bar{T}^a = \mathrm{e}^\sigma T^a} \tag{2.38}$$

Veamos como transforma la curvatura R^{ab}. De (1.23),

$$\bar{R}^{ab} = \bar{\mathring{R}}^{ab} + \bar{\mathring{\mathrm{D}}}\bar{\kappa}^{ab} + \bar{\kappa}^a{}_c \wedge \bar{\kappa}^{cb} \tag{2.39}$$

$$= \mathrm{d}\bar{\mathring{\omega}}^{ab} + \bar{\mathring{\omega}}^a{}_c \wedge \bar{\mathring{\omega}}^{cb} + \mathrm{d}\bar{\kappa}^{ab} + \bar{\mathring{\omega}}^a{}_c \wedge \bar{\kappa}^{cb} + \bar{\mathring{\omega}}^b{}_c \wedge \bar{\kappa}^{ac} + \bar{\kappa}^a{}_c \wedge \bar{\kappa}^{cb} \tag{2.40}$$

$$= \mathrm{d}\big(\mathring{\omega}^{ab} + \theta^{ab}\big) + (\mathring{\omega}^a{}_c + \theta^a{}_c) \wedge (\mathring{\omega}^{cb} + \theta^{cb}) + \mathrm{d}\kappa^{ab} + (\mathring{\omega}^a{}_c + \theta^a{}_c) \wedge \kappa^{cb} \tag{2.41}$$

$$+ (\mathring{\omega}^b{}_c + \theta^b{}_c) \wedge \kappa^{ac} + \kappa^a{}_c \wedge \kappa^{cb} \tag{2.42}$$

$$= \mathrm{d}\mathring{\omega}^{ab} + \mathrm{d}\theta^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathring{\omega}^a{}_c \wedge \theta^{cb} + \theta^a{}_c \wedge \mathring{\omega}^{ab} + \theta^a{}_c \wedge \theta^{cb} + \mathrm{d}\kappa^{ab} \tag{2.43}$$

$$\mathring{\omega}^a{}_c \wedge \kappa^{cb} + \theta^a{}_c \wedge \kappa^{cb} + \mathring{\omega}^b{}_c \wedge \kappa^{ac} + \theta^b{}_c \wedge \kappa^{ac} + \kappa^a_c \wedge \kappa^{cb} \tag{2.44}$$

$$= \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathrm{d}\kappa^{ab} + \mathring{\omega}^a{}_c \wedge \kappa^{cb} + \mathring{\omega}^b{}_c \wedge \kappa^{ac} + \kappa^a{}_c \wedge \kappa^{cb} \tag{2.45}$$

$$+ \mathrm{d}\theta^{ab} + \mathring{\omega}^a{}_c \wedge \theta^{cb} + \kappa^a{}_c \wedge \theta^{cb} + \mathring{\omega}^b{}_c \wedge \theta^{ac} + \kappa^b{}_c \wedge \theta^{ac} + \theta^a{}_c \wedge \theta^{cb} \tag{2.46}$$

$$= R^{ab} + \omega^a{}_c \wedge \theta^{cb} + \omega^b{}_c \wedge \theta^{ac} \tag{2.47}$$

$$= R^{ab} + \mathrm{D}\theta^{ab} + \theta^a{}_c \wedge \theta^{cb} \tag{2.48}$$

es decir,

$$\boxed{\bar{R}^{ab} = R^{ab} + \mathrm{D}\theta^{ab} + \theta^a{}_c \wedge \theta^{cb}} \tag{2.49}$$

Por otro lado tenemos el caso **exótico**, donde la conexión de spin queda invariante bajo las transformaciones conformes,

$$\omega_{ab} \to \bar{\omega}_{ab} = \omega_{ab} \tag{2.50}$$

Veamos como transforma la contorsión κ_{ab},

$$\bar{\omega}_{ab} = \omega_{ab} \tag{2.51}$$
$$\mathring{\bar{\omega}}_{ab} + \bar{\kappa}_{ab} = \mathring{\omega}_{ab} + \kappa_{ab} \tag{2.52}$$
$$\mathring{\omega}_{ab} + \theta_{ab} + \bar{\kappa}_{ab} = \mathring{\omega}_{ab} + \kappa_{ab} \tag{2.53}$$
$$\theta_{ab} + \bar{\kappa}_{ab} = \kappa_{ab} \tag{2.54}$$

luego,

$$\boxed{\bar{\kappa}_{ab} = \kappa_{ab} - \theta_{ab}} \tag{2.55}$$

Veamos como transforma la curvatura en este caso

$$\begin{aligned}
\bar{R}^{ab} &= \mathring{\bar{R}}^{ab} + \mathring{\bar{\mathrm{D}}}\bar{\kappa}^{ab} + \bar{\kappa}^a{}_c \wedge \bar{\kappa}^{cb} \\
&= \mathrm{d}\mathring{\bar{\omega}}^{ab} + \mathring{\bar{\omega}}^a{}_c \wedge \mathring{\bar{\omega}}^{cb} + \mathrm{d}\bar{\kappa}^{ab} + \mathring{\bar{\omega}}^a{}_c \wedge \bar{\kappa}^{cb} + \mathring{\bar{\omega}}^b{}_c \wedge \bar{\kappa}^{ac} + \bar{\kappa}^a{}_c \wedge \bar{\kappa}^{cb} \\
&= \mathrm{d}\mathring{\bar{\omega}}^{ab} + \cancel{\mathrm{d}\theta^{ab}} + (\mathring{\omega}^a{}_c + \theta^a{}_c) \wedge (\mathring{\omega}^{cb} + \theta^{cb}) + \mathrm{d}\kappa^a - \cancel{\mathrm{d}\theta^{ab}} \\
&\quad + (\mathring{\omega}^a{}_c + \theta^a{}_c) \wedge (\kappa^{cb} - \theta^{cb}) + (\mathring{\omega}^b{}_c + \theta^b{}_c) \wedge (\kappa^{ac} - \theta^{ac}) + (\kappa^a{}_c - \theta^a{}_c) \wedge (\kappa^{cb} - \theta^{cb}) \\
&= \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathring{\omega}^a{}_c \wedge \theta^{cb} + \theta^a{}_c \wedge \mathring{\omega}^{cb} + \theta^a{}_c \wedge \theta^{cb} + \mathrm{d}\kappa^{ab} + \mathring{\omega}^a{}_c \wedge \kappa^{cb} - \mathring{\omega}^a{}_c \wedge \theta^{cb} \\
&\quad + \theta^a{}_c \wedge \kappa^{cb} - \theta^a{}_c \wedge \theta^{cb} + \mathring{\omega}^b{}_c \wedge \kappa^{ac} - \mathring{\omega}^b{}_c \wedge \theta^{ac} + \theta^b{}_c \wedge \kappa^{ac} - \theta^b{}_c \wedge \theta^{ac} + \kappa^a{}_c \wedge \kappa^{cb} \\
&\quad - \kappa^a{}_c \wedge \theta^{cb} - \theta^a{}_c \wedge \kappa^{cb} + \theta^a{}_c \wedge \theta^{cb} \\
&= \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathrm{d}\kappa^{ab} + \mathring{\omega}^a{}_c \wedge \kappa^{cb} + \mathring{\omega}^b{}_c \wedge \kappa^{ac} + \kappa^a{}_c \wedge \kappa^{cb} \\
&= R^{ab}
\end{aligned}$$

donde es fácil ver que los demás términos se cancelan entre sí. Finalmente

$$\boxed{\bar{R}^{ab} = R^{ab}} \tag{2.56}$$

Veamos ahora como transforma la torsión,

$$\begin{aligned}
\bar{T}^a &= \bar{\kappa}^a_b \wedge \bar{e}^b \\
&= (\kappa^a{}_b - \theta^a{}_b) \wedge \mathrm{e}^\sigma e^b \\
&= \mathrm{e}^\sigma (\kappa^a{}_b - \theta^a{}_b) \wedge e^b \\
&= \mathrm{e}^\sigma (\kappa^a{}_b \wedge e^b - \theta^a{}_b \wedge e^b) \\
&= \mathrm{e}^\sigma (T^a - \theta^a{}_b \wedge e^b)
\end{aligned} \tag{2.57}$$

Notemos que

$$\begin{aligned}
\theta^a{}_b \wedge e^b &= (e^a \xi_b - e_b \xi^a) \wedge e^b \\
&= e^a \xi_b \wedge e^b \\
&= e^a \wedge \mathrm{d}\sigma
\end{aligned} \tag{2.58}$$

Luego,

$$\boxed{\bar{T}^a = e^\sigma (T^a - e^a \wedge \mathrm{d}\sigma)} \tag{2.59}$$

Sin embargo, podemos considerar un rango más amplio de transformaciones interpolando entre ambos casos. Definimos el parámetro real $\lambda \in \mathbb{R}$, de manera que

$$\bar{e}^a = \mathrm{e}^{\sigma} e^a \tag{2.60a}$$
$$\bar{\omega}^{ab} = \omega^{ab} + \lambda\theta^{ab} \tag{2.60b}$$
$$\bar{\kappa}^{ab} = \kappa^{ab} + (\lambda - 1)\theta^{ab} \tag{2.60c}$$

así, el límite $\lambda = 1$ corresponde al caso canónico y $\lambda = 0$ al exótico. Veamos como queda la torsión y la curvatura bajo las transformaciones (2.60). Para la torsión tenemos

$$\bar{T}^a = \bar{\kappa}^a{}_b \wedge \bar{e}^b \tag{2.61}$$
$$= \mathrm{e}^{\sigma}\left[\kappa^a{}_b \wedge e^b + (\lambda - 1)\theta^a{}_b \wedge e^b\right] \tag{2.62}$$
$$= \mathrm{e}^{\sigma}\left[T^a + (\lambda - 1)e^a \wedge \mathrm{d}\sigma\right] \tag{2.63}$$

donde para pasar a la última línea se usó (2.57). Expandiendo a primer orden la exponencial, tenemos

$$\bar{T}^a = T^a + (\lambda - 1)e^a \wedge \mathrm{d}\sigma + \sigma T^a + \sigma(\lambda - 1)e^a \wedge \mathrm{d}\sigma \tag{2.64}$$

luego, la transformación infinitesimal de la torsión es

$$\boxed{\delta T^a = \sigma T^a + (\lambda - 1)e^a \wedge \mathrm{d}\sigma} \tag{2.65}$$

ya que el último término es de segundo orden en σ.

Veamos ahora cómo queda la curvatura bajo (2.60),

$$\begin{aligned}
\bar{R}^{ab} &= \bar{\mathring{R}}^{ab} + \bar{\mathring{\mathrm{D}}}\bar{\kappa}^{ab} + \bar{\kappa}^a{}_c \wedge \bar{\kappa}^{cb} \\
&= \mathrm{d}\bar{\mathring{\omega}}^{ab} + \bar{\mathring{\omega}}^a{}_c \wedge \bar{\mathring{\omega}}^{cb} + \mathrm{d}\bar{\kappa}^{ab} + \bar{\mathring{\omega}}^a{}_c \wedge \bar{\kappa}^{cb} + \bar{\mathring{\omega}}^b{}_c \wedge \bar{\kappa}^{ac} + \bar{\kappa}^a{}_c \wedge \bar{\kappa}^{cb} \\
&= \mathrm{d}\mathring{\omega}^{ab} + \mathrm{d}\theta^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb} + \mathring{\omega}^a{}_c \wedge \theta^{cb} + \theta^a{}_c \wedge \mathring{\omega}^{cb} + \theta^a{}_c \wedge \theta^{cb} + \mathrm{d}\kappa^{ab} + (\lambda - 1)\mathrm{d}\theta^{ab} + \mathring{\omega}^a{}_c \wedge \kappa^{cb} \\
&\quad + (\lambda - 1)\mathring{\omega}^a{}_c \wedge \theta^{cb} + \theta^a{}_c \wedge \kappa^{cb} + (\lambda - 1)\theta^a{}_c \wedge \theta^{cb} + \mathring{\omega}^b{}_c \wedge \kappa^{ac} + (\lambda - 1)\mathring{\omega}^b{}_c \wedge \theta^{ac} + \theta^c{}_c \wedge \kappa^{ac} \\
&\quad + (\lambda - 1)\theta^b{}_c \wedge \theta^{ac} + \kappa^a{}_c \wedge \kappa^{cb} + (\lambda - 1)\kappa^a{}_c \wedge \theta^{cb} + (\lambda - 1)\theta^a{}_c \wedge \kappa^{cb} + (\lambda - 1)^2\theta^a{}_c \wedge \theta^{cb} \\
&= R^{ab} + \mathrm{D}\theta^{ab} + (\lambda - 1)\mathrm{D}\theta^{ab} + (\lambda^2 - 2\lambda + 1)\theta^a{}_c \wedge \theta^{cb} + (2\lambda - 1)\theta^a{}_c \wedge \theta^{cb} \\
&= R^{ab} + \lambda\mathrm{D}\theta^{ab} + \lambda^2\theta^a{}_c \wedge \theta^{cb}
\end{aligned}$$

donde se usó (2.31). Así

$$\boxed{\delta R^{ab} = \lambda\mathrm{D}\theta^{ab} + +\lambda^2\theta^a{}_c \wedge \theta^{cb}} \tag{2.66}$$

ya que el último término es de segundo orden en σ.

Además, el la transformación del campo escalar tal que sea invariante conforme en d-dimensones es [2]

$$\phi \to \bar{\phi} = \exp\left[\sigma\left(\frac{2-d}{2}\right)\right]\phi \tag{2.67}$$

En resúmen, tenemos

$$\delta e^a = \sigma e^a \tag{2.68a}$$
$$\delta\phi = \left(\frac{2-d}{2}\right)\sigma\phi \tag{2.68b}$$
$$\delta\omega^a = \lambda\theta^{ab} \tag{2.68c}$$
$$\delta R^{ab} = \lambda D\theta^{ab} \tag{2.68d}$$
$$\delta T^a = \sigma T^a + (\lambda - 1)e^a \wedge \mathrm{d}\sigma \tag{2.68e}$$

2.1 Campos auxiliares

Definimos los siguientes campos auxiliares

$$\tilde{e}^a = \phi^{\frac{2}{d-2}} e^a \tag{2.69}$$

$$\tilde{\omega}^{ab} = \omega^{ab} + \lambda \Sigma^{ab} \tag{2.70}$$

$$\tilde{\kappa}^{ab} = \kappa^{ab} + (\lambda - 1)\Sigma^{ab} \tag{2.71}$$

donde

$$\Sigma^{ab} = \frac{2}{(d-2)} \frac{1}{\phi} \chi^{ab} \tag{2.72}$$

y

$$\chi^{ab} \equiv e^a z^b - z^a e^b, \qquad z^a = \mathrm{I}^a \mathrm{d}\phi \tag{2.73}$$

Veamos cómo es su variacion infinitesimal bajo transformaciones conformes infinitesimales usando (2.68). Para $\tilde{e}^a$, tenemos

$$\delta \tilde{e}^a = \frac{2}{d-2} \phi^{\frac{4-d}{d-2}} \left(\frac{2-d}{2} \right) \sigma \phi e^a + \phi^{\frac{2}{d-2}} \sigma e^a \tag{2.74}$$

$$= -\phi^{\frac{2}{d-2}} \sigma e^a + \phi^{\frac{2}{d-2}} \sigma e^a \tag{2.75}$$

$$= 0 \tag{2.76}$$

Luego, $\tilde{e}^a$ es invariante conforme.

Veamos ahora cómo varia $\tilde{\omega}^{ab}$,

$$\delta \tilde{\omega}^{ab} = \delta \omega^{ab} + \lambda \delta \Sigma^{ab} \tag{2.77}$$

$$= \lambda \theta^{ab} + \lambda \delta \Sigma^{ab} \tag{2.78}$$

Calculemos $\delta \Sigma^{ab}$,

$$\delta \Sigma^{ab} = -\frac{2}{(d-2)} \frac{1}{\phi^2} \delta \phi \chi^{ab} + \frac{2}{(d-2)} \frac{1}{\phi} \delta \chi^{ab} \tag{2.79}$$

$$= -\frac{2}{(d-2)} \frac{1}{\phi^2} \left(\frac{2-d}{d} \right) \sigma \phi \chi^{ab} + \frac{2}{(d-2)} \frac{1}{\phi} \delta \chi^{ab} \tag{2.80}$$

$$= \frac{\sigma}{\phi} \chi^{ab} + \frac{2}{(d-2)} \frac{1}{\phi} \delta \chi^{ab} \tag{2.81}$$

Calculemos $\delta \chi^{ab}$,

$$\delta \chi^{ab} = \delta e^a z^b + e^a \delta z^b - \delta z^a e^b - z^a \delta e^b \tag{2.82}$$

$$= \sigma (e^a z^b - z^a e^b) + e^a \delta z^b - \delta z^a e^b \tag{2.83}$$

$$= \sigma \chi^{ab} + e^a \delta z^b - \delta z^a e^b \tag{2.84}$$

Calculemos δz^a. Para ello, notemos que (Por demostrar!)

$$\delta_e z^a = -\mathrm{I}^n (\delta e^a) z_n \tag{2.85}$$

luego,

$$\delta z^a = \delta_a z^a + \delta_\phi z^a \tag{2.86}$$
$$= -\mathrm{I}^n(\delta e^a) z_n + \mathrm{I}^a\,\mathrm{d}(\delta\phi) \tag{2.87}$$
$$= -\mathrm{I}^n(\sigma e^a) z_n + \left(\frac{2-d}{2}\right)\mathrm{I}^a\,\mathrm{d}(\sigma\phi) \tag{2.88}$$
$$= -\sigma\mathrm{I}^n(e^a) z_n + \left(\frac{2-d}{2}\right)(\phi\mathrm{d}\sigma + \sigma\mathrm{d}\phi) \tag{2.89}$$
$$= -\sigma\eta^{na} z_n + \left(\frac{2-d}{2}\right)(\phi\xi^a + \sigma z^a) \tag{2.90}$$
$$= -\frac{d}{2}\sigma z^a + \left(\frac{2-d}{2}\right)\phi\xi^a \tag{2.91}$$

Reemplazando en (2.84)

$$\delta\chi^{ab} = \sigma\chi^{ab} + e^a\left[-\frac{d}{2}\sigma z^b + \left(\frac{2-d}{2}\right)\phi\xi^b\right] - \left[-\frac{d}{2}\sigma z^a + \left(\frac{2-d}{2}\right)\phi\xi^a\right]e^b \tag{2.92}$$
$$= \sigma\chi^{ab} - \sigma\frac{d}{2}\chi^{ab} + \left(\frac{2-d}{2}\right)\phi\theta^{ab} \tag{2.93}$$
$$= \sigma\left(\frac{2-d}{2}\right)\chi^{ab} + \left(\frac{2-d}{2}\right)\phi\theta^{ab} \tag{2.94}$$

Reemplazando en (2.95),

$$\begin{aligned}
\delta\Sigma^{ab} &= \frac{\sigma}{\phi}\chi^{ab} + \frac{2}{(d-2)}\frac{1}{\phi}\delta\chi^{ab} \\
&= \frac{\sigma}{\phi}\chi^{ab} + \left(\frac{2}{d-2}\right)\frac{1}{\phi}\left[\sigma\left(\frac{2-d}{2}\right)\chi^{ab} + \left(\frac{2-d}{2}\right)\phi\theta^{ab}\right] \\
&= \frac{\sigma}{\phi}\chi^{ab} - \frac{\sigma}{\phi}\chi^{ab} - \theta^{ab} \\
&= -\theta^{ab}
\end{aligned} \tag{2.95}$$

Así, de (2.96),

$$\delta\tilde{\omega}^{ab} = \lambda\theta^{ab} + \lambda\delta\Sigma^{ab} \tag{2.96}$$
$$= \lambda\theta^{ab} - \lambda\theta^{ab} \tag{2.97}$$
$$= 0 \tag{2.98}$$

es decir, $\tilde{\omega}^{ab}$ es invariante conforme.

Finalmente calaculemos la variación de $\tilde{\kappa}^{ab}$,

$$\delta\bar{\kappa}^{ab} = \delta\kappa^{ab} + (\lambda-1)\delta\Sigma^{ab} \tag{2.99}$$
$$= \delta\kappa^{ab} - (\lambda-1)\theta^{ab} \tag{2.100}$$
$$= (\lambda-1)\theta^{ab} - (\lambda-1)\theta^{ab} \tag{2.101}$$
$$= 0 \tag{2.102}$$

donde usamos (2.95).

Veamos ahora cómo queda la torsión y la curvartura definidas a partir de estos campos auxiliares. Para la torsión tenemos

$$\begin{aligned}
\tilde{T}^a &= \tilde{\kappa}^a{}_b \wedge \tilde{e}^b && (2.103)\\
&= \left[\kappa^a{}_b + (\lambda-1)\Sigma^a{}_b\right] \wedge \left[\phi^{\frac{2}{d-2}} e^b\right] && (2.104)\\
&= \phi^{\frac{2}{d-2}} \left[\kappa^a{}_b \wedge e^b + (\lambda-1)\Sigma^a{}_b \wedge e^b\right] && (2.105)\\
&= \phi^{\frac{2}{d-2}} \left[T^a + (\lambda-1)\Sigma^a{}_b \wedge e^b\right] && (2.106)
\end{aligned}$$

pero

$$\begin{aligned}
\Sigma^a{}_b \wedge e^b &= \frac{2}{(d-2)}\frac{1}{\phi}\chi^a{}_b \wedge e^b && (2.107)\\
&= \frac{2}{(d-2)}\frac{1}{\phi}\left(e^a z_b - z^a e_b\right) \wedge e^b && (2.108)\\
&= \frac{2}{(d-2)}\frac{1}{\phi} e^a \wedge \mathrm{d}\phi && (2.109)
\end{aligned}$$

luego

$$\begin{aligned}
\tilde{T}^a &= \phi^{\frac{2}{d-2}} \left[T^a + (\lambda-1)\Sigma^a{}_b \wedge e^b\right] && (2.110)\\
&= \phi^{\frac{2}{d-2}} \left[T^a + (\lambda-1)\frac{2}{(d-2)}\frac{1}{\phi} e^a \wedge \mathrm{d}\phi\right] && (2.111)\\
&= \phi^{\frac{2}{d-2}} T^a + \frac{2}{(d-2)}\phi^{\frac{4-d}{d-2}}(\lambda-1) e^a \wedge \mathrm{d}\phi && (2.112)
\end{aligned}$$

Calculemos la curvatura construida a partir de los campos auxiliares

$$\begin{aligned}
\tilde{R}^{ab} &= \mathrm{d}\tilde{\omega}^{ab} + \tilde{\omega}^a{}_c \wedge \tilde{\omega}^{cb} && (2.113)\\
&= \mathrm{d}\big(\omega^{ab} + \lambda\Sigma^{ab}\big) + \left(\omega^a{}_c + \lambda\Sigma^a{}_c\right) \wedge \big(\omega^{cb} + \lambda\Sigma^{cb}\big) && (2.114)\\
&= \mathrm{d}\omega^{ab} + \lambda\mathrm{d}\Sigma^{ab} + \omega^a{}_c \wedge \omega^{cb} + \lambda\omega^a{}_c \wedge \Sigma^{cb} + \lambda\Sigma^a{}_c \wedge \omega^{cb} + \lambda^2\Sigma^a{}_c \wedge \Sigma^{cb} && (2.115)\\
&= R^{ab} + \lambda(\mathrm{d}\Sigma^{ab} + \omega^a{}_c \wedge \Sigma^{cb} + \omega^b{}_c \wedge \Sigma^{ac}) + \lambda^2\Sigma^a{}_c \wedge \Sigma^{cb} && (2.116)\\
&= R^{ab} + \lambda\mathrm{D}\Sigma^{ab} + \lambda^2\Sigma^a{}_c \wedge \Sigma^{cb} && (2.117)
\end{aligned}$$

Debido a que estas cantidades están construidas a partir de objetos que son conformalmente invariantes, ellas también lo son.

En resumen, tenemos los siguientes campos auxiliares conformalmente invariantes

$$\begin{aligned}
\tilde{e}^a &= \phi^{\frac{2}{d-2}} e^a && (2.118)\\
\tilde{\omega}^{ab} &= \omega^{ab} + \lambda\Sigma^{ab} && (2.119)\\
\tilde{\kappa}^{ab} &= \kappa^{ab} + (\lambda-1)\Sigma^{ab} && (2.120)\\
\tilde{T}^a &= \phi^{\frac{2}{d-2}} T^a + \frac{2}{(d-2)}\phi^{\frac{4-d}{d-2}}(\lambda-1) e^a \wedge \mathrm{d}\phi && (2.121)\\
\tilde{R}^{ab} &= R^{ab} + \lambda\mathrm{D}\Sigma^{ab} + \lambda^2\Sigma^a{}_c \wedge \Sigma^{cb} && (2.122)
\end{aligned}$$

donde

$$\Sigma^{ab} = \frac{2}{(d-2)}\frac{1}{\phi}\left(e^a z^b - z^a e^b\right), \qquad z^a = \mathrm{I}^a \mathrm{d}\phi \qquad (2.123)$$

2.2 Invariantes en $2+1$

En $(2+1)$-dimensiones estas cantidades se reducen a

$$\tilde{e}^a = \phi^2 e^a \tag{2.124}$$
$$\tilde{\omega}^{ab} = \omega^{ab} + \lambda\Sigma^{ab} \tag{2.125}$$
$$\tilde{\kappa}^{ab} = \kappa^{ab} + (\lambda-1)\Sigma^{ab} \tag{2.126}$$
$$\tilde{T}^a = \phi^2 T^a + 2\phi(\lambda-1)e^a \wedge \mathrm{d}\phi \tag{2.127}$$
$$\tilde{R}^{ab} = R^{ab} + \lambda \mathrm{D}\Sigma^{ab} + \lambda^2 \Sigma^a{}_c \wedge \Sigma^{cb} \tag{2.128}$$

donde

$$\Sigma^{ab} = \frac{2}{\phi}\left(e^a z^b - z^a e^b\right) = \frac{4}{\phi} e^{[a} z^{b]}, \qquad z^a = \mathrm{I}^a \mathrm{d}\phi \tag{2.129}$$

Calculemos los invariantes que nos podemos construir a partir de estas cantidades en $(2+1)$. Primero

$$\epsilon_{abc}\tilde{R}^{ab} \wedge \tilde{e}^c = \epsilon_{abc}\left(R^{ab} + \lambda \mathrm{D}\Sigma^{ab} + \lambda^2\Sigma^a{}_d \wedge \Sigma^{db}\right) \wedge \left(\phi^2 e^c\right) \tag{2.130}$$
$$= \phi^2 \epsilon_{abc}\left(R^{ab} \wedge e^c + \lambda \mathrm{D}\Sigma^{ab} \wedge e^c + \lambda^2 \Sigma^a{}_d \wedge \Sigma^{db} \wedge e^c\right) \tag{2.131}$$

Trabajemos un poco más los últimos dos términos[1].

Primero, integremos por partes $\epsilon_{abc}\phi^2 \mathrm{D}\Sigma^{ab} e^c$. Notemos que

$$\epsilon_{abc}\mathrm{D}(\phi^2\Sigma^{ab}e^c) = 2\epsilon_{abc}\phi \mathrm{d}\phi \Sigma^{ab} e^c + \epsilon_{abc}\phi^2 \mathrm{D}\Sigma^{ab}e^c - \epsilon_{abc}\phi^2\Sigma^{ab}T^c \tag{2.132}$$

Luego,

$$\epsilon_{abc}\phi^2 \mathrm{D}\Sigma^{ab} e^c = \epsilon_{abc}\phi^2\Sigma^{ab}T^c - 2\phi\epsilon_{abc}\mathrm{d}\phi\Sigma^{ab}e^c + \mathrm{b.t} \tag{2.133}$$
$$= 4\phi\epsilon_{abc} z^a T^b e^c - 8\phi\epsilon_{abc}\mathrm{d}\phi e^a z^b e^c \tag{2.134}$$

Además,

$$\epsilon_{abc}\phi^2\Sigma^a{}_d \Sigma^{db} e^c = -\frac{4}{3}\frac{1}{\phi^2} z^2 e^a e^b e^c \tag{2.135}$$

donde hemos usado que

$$2\det(e)\partial_\mu\phi\partial^\mu\delta \mathrm{d}^3x = \frac{1}{3}\epsilon_{abc} z^2 e^a e^b e^c \tag{2.136}$$

Finalmente,

$$\epsilon_{abc}\tilde{R}^{ab}\tilde{e}^c = \epsilon_{abc}\phi^2 R^{ab}e^c + 4\lambda\phi\epsilon_{abc}z^a T^b e^c - 8\lambda\epsilon_{abc}\phi \mathrm{d}\phi e^a e^b e^c - \frac{4\lambda^2}{3}\epsilon_{abc}z^2 e^a e^b e^c \tag{2.137}$$

Usando (2.136),

$$\epsilon_{abc}\phi \mathrm{d}\phi e^a z^b e^c = -2\det(e)\partial_\mu\phi\partial^\mu \mathrm{d}^3x = -\frac{1}{3}\epsilon_{abc}z^2 e^a e^b e^c \tag{2.138}$$

Luego,

$$\epsilon_{abc}\tilde{R}^{ab}\tilde{e}^c = \epsilon_{abc}\phi^2 R^{ab}e^c + 4\lambda\phi\epsilon_{abc}z^a T^b e^c + \frac{8\lambda}{3}\epsilon_{abc}z^2 e^a e^b e^c - \frac{4\lambda^2}{3}\epsilon_{abc}z^2 e^a e^b e^c \tag{2.139}$$

$$\boxed{\epsilon_{abc}\tilde{R}^{ab}\tilde{e}^c = \epsilon_{abc}\phi^2 R^{ab}e^c + \frac{8\lambda}{3}\left(1-\frac{\lambda}{2}\right)\epsilon_{abc}z^2 e^a e^b e^c + 4\lambda\phi\epsilon_{abc}z^a T^b e^c} \tag{2.140}$$

[1] De aquí en adelante para simplificar notación, consideraremos el *wedge product* implícito.

Este término es el análogo en $(2+1)$ que el que aparecee en la acción de [3].

Otro invariante que nos podemos construir es

$$\tilde{T}^a \wedge \tilde{e}_a = \big(\phi^2 T^a + 2\phi(\lambda-1)e^a \wedge \mathrm{d}\phi\big) \wedge \big(\phi^2 e_a\big) \tag{2.141}$$

$$= \phi^4 T^a \wedge e_a \tag{2.142}$$

Otro más

$$\epsilon_{abc}\tilde{e}^a \wedge \tilde{e}^b \wedge \tilde{e}^c = \phi^6 \epsilon_{abc} e^a \wedge e^b \wedge e^c \tag{2.143}$$

El otro invariante que podemos construir en $(2+1)$ es el Chern-Simons a partir de los campos auxiliares . Calculemos

$$\tilde{\omega}^{ab}\mathrm{d}\tilde{\omega}_{ab} + \frac{2}{3}\tilde{\omega}^a{}_b\tilde{\omega}^b{}_c\tilde{\omega}^c{}_a \tag{2.144}$$

Veamos primero cómo queda $\tilde{\omega}^{ab}\mathrm{d}\tilde{\omega}_{ab}$. Para ello, recordemos que

$$\Sigma^{ab} = \frac{4}{\phi}e^{[a}z^{b]}, \qquad \mathrm{d}\Sigma^{ab} = -\frac{4}{\phi^2}\mathrm{d}\phi e^{[a}z^{b]} + \frac{4}{\phi}\mathrm{d}e^{[a}z^{b]} - \frac{4}{\phi}e^{[a}\mathrm{d}z^{b]} \tag{2.145}$$

Así,

$$\begin{aligned}\tilde{\omega}_{ab}\mathrm{d}\tilde{\omega}^{ab} &= (\omega_{ab} + \lambda\Sigma_{ab}) \wedge \mathrm{d}\big(\omega^{ab} + \lambda\Sigma^{ab}\big) \\ &= \omega_{ab}\mathrm{d}\omega^{ab} + \lambda\omega_{ab}\left(-\frac{4}{\phi^2}\mathrm{d}\phi e^a z^b + \frac{4}{\phi}\mathrm{d}e^a z^b - \frac{4}{\phi}e^a\mathrm{d}z^b\right) + \frac{4\lambda}{\phi}e^a z^b \mathrm{d}\omega_{ab} + \frac{16\lambda^2}{\phi^2}z^2 e_a \mathrm{d}e^a\end{aligned}$$

Calculemos ahora $\frac{2}{3}\tilde{\omega}^{ab}\tilde{\omega}_{bc}\tilde{\omega}^{cd}\eta_{ca}$. Primero

$$\tilde{\omega}^{ab}\tilde{\omega}_{bc} = \omega^{ab}\omega_{bc} + \frac{4\lambda}{\phi}\omega^{ab}e_{[b}z_{c]} + \frac{4\lambda}{\phi}e^{[a}z^{b]}\omega_{bc} + \frac{4\lambda^2}{\phi^2}(e^a\mathrm{d}\phi z_c - z^2 e^a e_c + z^a \mathrm{d}\phi e_c) \tag{2.146}$$

Ahora $\tilde{\omega}^{ab}\tilde{\omega}_{bc}\tilde{\omega}^{cd}$,

$$\tilde{\omega}^{ab}\tilde{\omega}_{bc}\tilde{\omega}^{cd} = \omega^{ab}\omega_{bc}\omega^{cd} + \frac{4\lambda}{\phi}\omega^{ab}\omega_{bc}e^{[c}z^{d]} + \frac{4\lambda}{\phi}\omega^{ab}e_{[b}z_{c]}\omega^{cd} + \frac{16\lambda^2}{\phi^2}\omega^{ab}e_{[b}z_{c]}e^{[c}z^{d]} \tag{2.147}$$

$$+ \frac{4\lambda}{\phi}e^{[a}z^{b]}\omega_{bc}\omega^{cd} + \frac{16\lambda^2}{\phi^2}e^{[a}z^{b]}\omega_{bc}e^{[c}z^{d]} - \frac{4\lambda^2}{\phi^2}z^2e^ae_c\omega^{cd} - \frac{16\lambda^3}{\phi^3}z^2eae_ce^{[c}z^{d]} \tag{2.148}$$

$$+ \frac{4\lambda^2}{\phi^2}z^a\mathrm{d}\phi e_c\omega^{cd} + \frac{16\lambda^3}{\phi^3}z^a\mathrm{d}\phi e_c e^{[c}z^{d]} \tag{2.149}$$

Bajando el último índice con una η_{da},

$$\begin{aligned}\tilde{\omega}^{ab}\tilde{\omega}_{bc}\tilde{\omega}^c{}_a &= \omega^{ab}\omega_{bc}\omega^c{}_a + \frac{2\lambda}{\phi}\omega^{ab}\omega_{bc}(e^cz_a - z^ce_a) + \frac{2\lambda}{\phi}\omega^{ab}(e_bz_c - z_be_c)\omega^c{}_a + \frac{2\lambda}{\phi}(e^az^b - z^ae^b)\omega_{bc}\omega^c{}_a \\ &+ \frac{4\lambda^2}{\phi^2}\omega^{ab}(2z_ae_b\mathrm{d}\phi + z^2e_ae_b) + \frac{4\lambda^2}{\phi^2}\omega_{bc}(2z^be^c\mathrm{d}\phi + z^2e^be^c) + \frac{4\lambda^2}{\phi^2}\omega^c{}_a(2z_ce^a\mathrm{d}\phi + z^2e_ce^a) \\ &= \omega^{ab}\omega_{bc}\omega^c{}_a + \frac{6\lambda}{\phi}\omega^{ab}\omega_{bc}(e^cz_a - z^ce_a) + \frac{12\lambda^2}{\phi^2}\omega^{ab}(2z_ae_b\mathrm{d}\phi + z^2e_ae_b)\end{aligned}$$

Luego,

$$\begin{aligned}\tilde{\omega}^{ab}\mathrm{d}\tilde{\omega}_{ab} + \frac{2}{3}\tilde{\omega}^a{}_b\tilde{\omega}^b{}_c\tilde{\omega}^c{}_a &= \omega_{ab}\mathrm{d}\omega^{ab} + \lambda\omega_{ab}\left(-\frac{4}{\phi^2}\mathrm{d}\phi e^az^b + \frac{4}{\phi}\mathrm{d}e^az^b - \frac{4}{\phi}e^a\mathrm{d}z^b\right) + \frac{4\lambda}{\phi}e^az^b\mathrm{d}\omega_{ab} + \frac{16\lambda^2}{\phi^2}z^2e_a\mathrm{d}e^a \\ &+ \frac{2}{3}\omega^{ab}\omega_{bc}\omega^c{}_a + \frac{4\lambda}{\phi}\omega^{ab}\omega_{bc}(e^cz_a - z^ce_a) + \frac{8\lambda^2}{\phi^2}\omega^{ab}(2z_ae_b\mathrm{d}\phi + z^2e_ae_b)\end{aligned}$$

3 Ecuaciones de movimiento

Calculemos las ecuaciones de movimiento asociadas los invariantes usando que

$$\delta R^{ab} = \mathrm{D}\delta\omega^{ab} \tag{3.1}$$

y

$$\delta T^{a} = \mathrm{D}\delta e^{a} + \delta\omega^{a}{}_{b} \tag{3.2}$$

3.1 Primer caso

Consideremos el siguiente Lagrangeano

$$L[e,\omega,\phi] = L[e,\omega]_{EC\Lambda} - L[e,\omega,\phi]_{M} \tag{3.3}$$

donde

$$\boxed{L_M = \frac{1}{16}\epsilon_{abc}\left(\phi^2 R^{ab}e^c + \frac{4\lambda}{3}(2-\lambda)z^2 e^a e^b e^c + 4\lambda\phi z^a T^b e^c + V(\phi)e^a e^b e^c\right)} \tag{3.4}$$

Variaciones con respecto a e:
El primer término queda

$$\delta_e\left(\frac{1}{16}\epsilon_{abc}\phi^2 R^{ab}e^c\right) = \frac{1}{16}\epsilon_{abc}\phi^2 R^{ab}\delta e^c \tag{3.5}$$

El segundo

$$\delta_e\left(\frac{\lambda}{12}(2-\lambda)z^2 e^a e^b e^c\right) = \frac{\lambda(\lambda-2)}{2}\epsilon_{abc}z_d z^a e^b e^c \delta e^d + \frac{\lambda}{4}(2-\lambda)\epsilon_{abc}z^2 e^a e^b e^c \tag{3.6}$$

donde hemos usado que

$$\epsilon_{abc}z_d I^n(\delta e^d)z_n e^a e^b e^c = 3\lambda(\lambda-2)\epsilon_{abc}z_d z^a e^b e^c \delta e^d \tag{3.7}$$

El tercero

$$\delta_e\left(\frac{\lambda}{4}\epsilon_{abc}\phi z^a T^b e^c\right) = \frac{\lambda}{4}\epsilon_{abc}\phi z^a T^b \delta e^c - \frac{\lambda}{4}\epsilon_{abc}\phi I^n T^a e^b \delta e^c + \frac{\lambda}{4}\epsilon_{abc}z^a \mathrm{d}\phi e^b \delta e^c \tag{3.8}$$

$$- \frac{\lambda}{4}\epsilon_{abc}\phi \mathrm{D}z^a e^d \delta e^c \tag{3.9}$$

Finalmente,

$$\tau_c = \frac{1}{4}\epsilon_{abc}\left(\frac{\phi^2}{4}R^{ab} + \lambda\phi z^a T^b\right) + \frac{\lambda}{4}(2-\lambda)\epsilon_{abc}z^2 e^a e^b + \frac{3}{16}\epsilon_{abc}V(\phi)e^a e^b \tag{3.10}$$

$$- \frac{\lambda}{4}\epsilon_{abc}\left[\phi \mathrm{D}z^a e^b + \phi z_n I^n T^a e^b - z^a \mathrm{d}\phi e^b\right] + \frac{\lambda(\lambda-2)}{2}\epsilon_{abc}z_d z^a e^b e^c \delta e^d \tag{3.11}$$

Variaciones con respecto a $\delta\omega^{ab}()$:
El primer término queda:

$$\delta_\omega\left(\frac{1}{16}\epsilon_{abc}\phi^2 R^{ab}e^c\right) = \frac{1}{8}\epsilon_{abc}\phi \mathrm{d}\phi e^c - \frac{1}{16}\epsilon_{abc}\phi^2 T^c \tag{3.12}$$

El segundo término queda:

$$\delta_\omega\left(\frac{\lambda}{4}\epsilon_{abc}\phi z^a T^b e^c\right) = \frac{\lambda}{4}\epsilon_{fac}\phi z^f e_b e^c - \frac{\lambda}{4}\epsilon_{fbc}\phi z^f e_a e^c \tag{3.13}$$

Variaciones con respecto a $\delta\phi()$:
El primer término queda:

$$\delta\phi\left(\frac{1}{16}\phi^2 R^{ab}e^c\right) = \frac{1}{6}\epsilon_{abc}\phi R^{ab}e^c \tag{3.14}$$

El cuarto término queda:

$$\delta\phi\left(\frac{1}{16}V(\phi)e^a e^b e^c\right) = \frac{1}{16}\frac{\mathrm{d}V}{\mathrm{d}\phi}e^a e^b e^c \tag{3.15}$$

3.2 Segundo caso

Consideremos el siguiente Lagrangeano

$$L = \alpha_0\omega_a\left(\mathrm{d}\omega^a + \frac{2}{3}\epsilon^{abc}\omega_b\omega_c\right) + \alpha_1 T^a e_a - \alpha_2\phi^4 T^a e_a \tag{3.16}$$

Variando con respecto al vielbein, tenemos

$$\delta_e L = (\alpha_1 - \alpha_2\phi^4)(\mathrm{D}\delta e^a e_a + T^a\delta e_a) \tag{3.17}$$

Integrando por partes,

$$\mathrm{d}\left[(\alpha_1 - \alpha_2\phi^4)\delta e^a e_a\right] = -4\alpha_2\phi^3\mathrm{d}\phi\delta e^a e_a + (\alpha_1 - \alpha_2\phi^4)\mathrm{D}\delta e^a e_a - (\alpha_1 - \alpha_2\phi^4)\delta e^a e_a \tag{3.18}$$

Luego

$$\begin{aligned}\delta_e L &= 4\alpha_2\phi^3\mathrm{d}\phi\delta e^a e_a + (\alpha_1 - \alpha_2\phi^4)\delta e^a T_a + (\alpha_1 - \alpha_2\phi^4)T^a\delta e_a & (3.19)\\ &= -4\alpha_2\phi^3\mathrm{d}\phi e^a\delta e_a + 2(\alpha_1 - \alpha_2\phi^4)T^a\delta e_a & (3.20)\end{aligned}$$

$$\boxed{\frac{\delta L}{\delta e_a} = 2(\alpha_1 - \alpha_2\phi^4)T^a - 4\alpha_2\phi^3\mathrm{d}\phi e^a = 0} \tag{3.21}$$

De donde se puede despejar algebraícamente la torsión en términos de ϕ,

$$\boxed{T^a = \frac{2\alpha_2\phi^3\mathrm{d}\phi e^a}{(\alpha_1 - \alpha_2\phi^4)}} \tag{3.22}$$

Variando con respecto a ϕ se tiene

$$\boxed{\frac{\delta L}{\delta\phi} = -4\alpha_2\phi^3 T^a e_a = 0} \tag{3.23}$$

esta ecuación se satisface automáticamente de (3.22).

4 Bichos en el dual del grupo

Una propiedad interesante que aparece en (2+1) es que podemos simplificar un poco lo cálculos trabajando con la conexión de Lorentz (y los demás objetos) en el dual del grupo.

La convención a seguir será

$$\epsilon_{012} = -\epsilon^{012} = 1 \tag{4.1}$$
$$\epsilon_{abc}\epsilon^{def} = -\delta^{def}_{abc} \tag{4.2}$$
$$\omega_{ab} = \epsilon_{abc}\omega^c \tag{4.3}$$

Multiplicando (4.3) por ϵ^{abf}, tenemos

$$\epsilon^{abf}\omega_{ab} = \epsilon^{abf}\epsilon_{abc}\omega^c \tag{4.4}$$
$$\epsilon^{abf}\omega_{ab} = -\delta^{abf}_{abc}\omega^c \tag{4.5}$$
$$\epsilon^{abf}\omega_{ab} = -2!\delta^f_c\omega^c \tag{4.6}$$
$$\epsilon^{abf}\omega_{ab} = -2\omega^f \tag{4.7}$$

Así,

$$\omega^f = -\frac{1}{2}\epsilon^{abf}\omega_{ab} \tag{4.8}$$

o de manera equivalente

$$\boxed{\omega_a = \frac{1}{2}\epsilon_{abc}\omega^{bc}} \tag{4.9}$$

Veamos como queda la torsión escrita en términos de estas conexiones duales,

$$\begin{aligned} T^a &= \mathrm{d}e^a + \omega^a{}_b \wedge e^b && (4.10)\\ &= \mathrm{d}e^a + \omega^{ab} \wedge e_b && (4.11)\\ &= \mathrm{d}e^a - \epsilon^{abc}\omega_c \wedge e_b && (4.12)\\ &= \mathrm{d}e^a - \epsilon^{acb}\omega_b \wedge e_c && (4.13)\\ &= \mathrm{d}e^a + \epsilon^{abc}\omega_b \wedge e_c && (4.14) \end{aligned}$$

Es decir,

$$\boxed{T^a = \mathrm{d}e^a + \epsilon^{abc}\omega_b \wedge e_c} \tag{4.15}$$

Veamos ahora cómo queda la curvatura de Lorentz en el dual de grupo $R_a = \frac{1}{2}\epsilon_{abc}R^{bc}$. Notemos que

$$\begin{aligned} R^{bc} &= \mathrm{d}\omega^{bc} + \omega^b{}_d \wedge \omega^{dc} && (4.16)\\ &= \mathrm{d}\big(-\epsilon^{bcf}\omega_f\big) + \eta^{bg}\omega_{gd} \wedge \omega^{bc} && (4.17)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f - \eta^{bg}\epsilon_{gdm}\omega^m \wedge (-\epsilon^{dcl}\omega_l) && (4.18)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f - \eta^{bg}\epsilon_{gdm}\epsilon^{dcl}\omega^m \wedge \omega_l && (4.19)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f + \eta^{bg}\epsilon_{dgm}\epsilon^{dcl}\omega^m \wedge \omega_l && (4.20)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f + \eta^{bg}\delta^{cl}_{gm}\omega^m \wedge \omega_l && (4.21)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f + \cancel{\eta^{bg}\delta^c_g\delta^l_m\omega^m \wedge \omega_l} - \eta^{bg}\delta^l_g\delta^c_m\omega^m \wedge \omega_l && (4.22)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f - \omega^c \wedge \omega^b && (4.23)\\ &= -\epsilon^{bcf}\mathrm{d}\omega_f + \omega^b \wedge \omega^c && (4.24) \end{aligned}$$

Luego,

$$
\begin{aligned}
\epsilon_{abc}R^{bc} &= \epsilon_{abc}\left(-\epsilon^{bcf}\mathrm{d}\omega_f + \omega^b \wedge \omega^c\right) && (4.25)\\
&= -\epsilon_{abc}\epsilon^{bcf}\mathrm{d}\omega_f + \epsilon_{abc}\omega^b \wedge \omega^c && (4.26)\\
&= -\epsilon_{abc}\epsilon^{fbc}\mathrm{d}\omega_f + \epsilon_{abc}\omega^b \wedge \omega^c && (4.27)\\
&= \delta^{bfc}_{abc}\mathrm{d}\omega_f + \epsilon_{abc}\omega^b \wedge \omega^c && (4.28)\\
&= 2\mathrm{d}\omega_a + \epsilon_{abc}\omega^b \wedge \omega^c && (4.29)
\end{aligned}
$$

Así,

$$
\begin{aligned}
R_a &= \frac{1}{2}\epsilon_{abc}R^{bc} && (4.30)\\
&= \mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c && (4.31)
\end{aligned}
$$

$$
\boxed{R^a = \mathrm{d}\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c} \tag{4.32}
$$

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