

Zero-weight scalar fields

Borja Diez

Universidad Arturo Prat

E-mail: borjadiez1014@gmail.com

ABSTRACT: Abstract...

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1 Building the equation of motion

Let us consider a scalar field with zero conformal weight, that is, under an infinitesimal conformal transformation

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \quad \delta_\omega \phi = 0 \quad (1.1)$$

Let us consider the most general second order pseudoscalar constructed from the scalar field ϕ and its derivatives up to second order, together with the metric tensor and its associates curvature

$$\mathcal{E} = \sqrt{-g}E(\phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi, g_{\mu\nu}, R_{\mu\nu}^{\alpha\beta}) = 0 \quad (1.2)$$

The variation of this equation under infinitesimal conformal transformations reads,

$$\delta_\omega \mathcal{E} = \delta_\omega(\sqrt{-g})E + \sqrt{-g}\delta_\omega E \quad (1.3)$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\delta_\omega E \quad (1.4)$$

From $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta_\omega(g_{\mu\nu}g^{\mu\nu}) = 0$ implies that

$$g_{\mu\nu}\delta_\omega g^{\mu\nu} = -g^{\mu\nu}\delta_\omega g_{\mu\nu} \quad (1.5)$$

Furthermore, from (A.3) and (A.15), we have

$$\delta_\omega \mathcal{E} = \frac{\partial E}{\partial \phi} \delta_\omega \phi + \frac{\partial E}{\partial (\nabla_\mu \phi)} \delta_\omega \nabla_\mu \phi + \frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)} \delta_\omega (\nabla_\mu \nabla_\nu \phi) + \frac{\partial E}{\partial g^{\mu\nu}} \delta_\omega g^{\mu\nu} + P_{\alpha\beta}^{\mu\nu} \delta_\omega R_{\mu\nu}^{\alpha\beta} \quad (1.6)$$

where we have defined

$$P_{\alpha\beta}^{\mu\nu} := \frac{\partial E}{\partial R_{\mu\nu}^{\alpha\beta}}, \quad P_\nu^\mu := P_{\alpha\nu}^{\alpha\mu} \quad (1.7)$$

So then

$$\begin{aligned} \delta_\omega \mathcal{E} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)}\delta_\omega (\nabla_\mu \nabla_\nu \phi) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P_{\alpha\beta}^{\mu\nu}\delta_\omega R_{\mu\nu}^{\alpha\beta} \\ &= \sqrt{-g}\frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)}(g_{\nu\beta}\nabla^\rho \phi \nabla_\rho \omega - 2\nabla_{(\mu} \phi \nabla_{\nu)} \omega) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P_{\alpha\beta}^{\mu\nu}\delta_\omega R_{\mu\nu}^{\alpha\beta} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E \\ &= \sqrt{-g}\frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)}(g_{\nu\beta}\nabla^\rho \phi \nabla_\rho \omega - 2\nabla_{(\mu} \phi \nabla_{\nu)} \omega) + \sqrt{-g}P_{\alpha\beta}^{\mu\nu}\delta_\omega R_{\mu\nu}^{\alpha\beta} - 2\omega g^{\mu\nu}\sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) \end{aligned}$$

Note that

$$\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}} \quad (1.8)$$

$$= \sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) \quad (1.9)$$

By defining

$$A^{\mu\nu} := \frac{\partial E}{\partial(\nabla_\mu \nabla_\nu \phi)}, \quad B_{\mu\nu} = -2\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} \quad (1.10)$$

we obtain

$$\delta_\omega \mathcal{E} = \sqrt{-g}A^{\mu\nu}(\gamma_{\mu\nu}\nabla^\rho \phi \nabla_\rho \omega - 2\nabla_\mu \phi \nabla_\nu \omega) + \sqrt{-g}P^{\mu\nu}\delta_\omega R_{\mu\nu}^{\alpha\beta} + \sqrt{-g}\omega B_{\mu\nu}g_{\mu\nu} \quad (1.11)$$

$$= \sqrt{-g}\left(A\nabla^\rho \phi \nabla_\rho \omega - 2A^{\mu\nu}\nabla_\mu \phi \nabla_\nu \omega + \omega B + P_{\alpha\beta}^{\mu\nu}\delta_\omega R_{\mu\nu}^{\alpha\beta}\right) \quad (1.12)$$

Using

$$\delta_\omega R_{\beta\mu\nu}^\alpha = 2\nabla_{[\mu}\delta_\omega \Gamma_{\nu]\beta}^\alpha \quad (1.13)$$

we have

$$\delta_\omega \mathcal{E} = \sqrt{-g}\left(A\nabla^\rho \phi \nabla_\rho \omega - 2A^{\mu\nu}\nabla_\mu \phi \nabla_\nu \omega + \omega B + 2P_\alpha^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega \Gamma_{\nu]\beta}^\alpha\right) \quad (1.14)$$

$$= \sqrt{-g}\left(A\nabla^\nu \phi \nabla_\nu \omega - 2A^{\mu\nu}\nabla_\mu \phi \nabla_\nu \omega + \omega B + 2P_\alpha^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega \Gamma_{\nu]\beta}^\alpha\right) \quad (1.15)$$

$$= \sqrt{-g}\left[(A\nabla^\nu \phi - 2A^{\mu\nu}\nabla_\mu \phi)\nabla_\nu \omega + \omega B + 2P_\alpha^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega \Gamma_{\nu]\beta}^\alpha\right] \quad (1.16)$$

Noting that

$$\delta_\omega \Gamma_{\nu\beta}^\alpha = \frac{1}{2}g^{\alpha\lambda}[\nabla_\nu(\delta_\omega g_{\beta\lambda}) + \nabla_\beta(\delta_\omega g_{\nu\lambda}) - \nabla_\lambda(\delta_\omega g_{\nu\beta})] \quad (1.17)$$

$$= g^{\alpha\lambda}(g_{\beta\lambda}\nabla_\nu \omega + g_{\nu\lambda}\nabla_\beta \omega - g_{\nu\beta}\nabla_\lambda \omega) \quad (1.18)$$

$$= g^{\alpha\lambda}(g_{\beta\lambda}\nabla_\nu \omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega) \quad (1.19)$$

we obtain

$$\begin{aligned} \delta_\omega \mathcal{E} &= \sqrt{-g}\left[(A\nabla^\nu \phi - 2A^{\mu\nu}\nabla_\mu \phi)\nabla_\nu \omega + \omega B + 2P_\alpha^{\beta\mu\nu}\nabla_\mu g^{\alpha\lambda}(g_{\beta\lambda}\nabla_\nu \omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega)\right] \\ &= \sqrt{-g}\left[(A\nabla^\nu \phi - 2A^{\mu\nu}\nabla_\mu \phi)\nabla_\nu \omega + \omega B + 2P^{\lambda\beta\mu\nu}\nabla_\mu (g_{\beta\lambda}\nabla_\nu \omega + 2g_{\nu\lambda}\nabla_\beta \omega)\right] \\ &= \sqrt{-g}\left[(A\nabla^\nu \phi - 2A^{\mu\nu}\nabla_\mu \phi)\nabla_\nu \omega + \omega B + 4P^{\lambda\beta\mu\nu}\nabla_\mu g_{\nu\lambda}\nabla_\beta \omega\right] \\ &= \sqrt{-g}\left[(A\nabla^\nu \phi - 2A^{\mu\nu}\nabla_\mu \phi)\nabla_\nu \omega + \omega B - 4P^{\mu\nu}\nabla_\mu \nabla_\nu \omega\right] \end{aligned}$$

Imposing $\delta_\omega \mathcal{E} = 0$ for all ω , we obtain the following conditions,

$$A\nabla^\nu \phi - 2A^{\mu\nu}\nabla_\mu \phi = 0 \quad (1.20a)$$

$$B = 0 \quad (1.20b)$$

$$P^{\mu\nu} = 0 \quad (1.20c)$$

From (A.21),

$$\frac{\partial E}{\partial R_{\alpha\beta}^{\mu\nu}} = P_{\mu\nu}^{\alpha\beta} = \hat{H}_{\mu\nu}^{\alpha\beta} + \delta_{[\mu}^{[\alpha}\hat{I}_{\nu]}^{\beta]} + J\delta_{[\mu}^\alpha\delta_{\nu]}^\beta \quad (1.21)$$

we notice that since $\hat{H}_{\mu\nu}^{\alpha\beta}$ is the traceless part of $H_{\mu\nu}^{\alpha\beta}$,

$$P_\nu^\beta = P_{\alpha\nu}^{\alpha\beta} = \delta_{[\alpha}^{[\beta}\hat{I}_{\nu]}^{\alpha]} + J\delta_{[\alpha}^\beta\delta_{\nu]}^\alpha \quad (1.22)$$

Since the variation with respect to the Weyl tensor does not contribute so the first trace (1.21), the contribution of the Riemann tensor to E has to be through its traceless part,

$$\mathcal{E} = \sqrt{-g}E(\phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi, g_{\mu\nu}, C_{\mu\nu}^{\alpha\beta}) = 0. \quad (1.23)$$

Example 1.1. *Let us consider the following action principle*

$$S[\phi, g] = \int d^D x \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right)^{D/2} \quad (1.24)$$

$$= \int d^D x \sqrt{-g} \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{D/2} \quad (1.25)$$

$$= \int d^D x \sqrt{-g} X^{D/2} \quad (1.26)$$

where we have defined $X := -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$.

Now, we must to find E . Varying with respect to ϕ ,

$$\delta_\phi S = - \int d^D x \sqrt{-g} \frac{D}{2} \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \nabla_\mu \delta \phi \quad (1.27)$$

$$= \int d^D x \sqrt{-g} \frac{D}{2} \nabla_\mu \left[\left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \right] \delta \phi + \text{b.t} \quad (1.28)$$

therefore,

$$E = \nabla_\mu \left[\left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \right] \quad (1.29)$$

$$= \frac{D-2}{2} \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-4}{2}} (-1) \nabla^\alpha \phi \nabla_\mu \nabla_\alpha \phi \nabla^\mu \phi + \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \square \phi \quad (1.30)$$

which can be rewritten as

$$E = X^{\frac{D-2}{2}} \square \phi - \frac{D-2}{2} X^{\frac{D-4}{2}} \nabla_\mu \nabla_\nu \phi \nabla^\mu \phi \nabla^\nu \phi \quad (1.31)$$

Let's see how (1.20a) looks,

$$A^{\mu\nu} = \frac{\partial}{\partial (\nabla_\mu \phi \nabla_\nu \phi)} \left[X^{\frac{D-2}{2}} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{D-2}{2} X^{\frac{D-4}{2}} \nabla_\mu \nabla_\nu \phi \nabla^\mu \phi \nabla^\nu \phi \right] \quad (1.32)$$

$$= X^{\frac{D-2}{2}} g^{\mu\nu} - \frac{D-2}{2} X^{\frac{D-4}{2}} \nabla^\mu \phi \nabla^\nu \phi \quad (1.33)$$

and its trace yields

$$A = DX^{\frac{D-2}{2}} - \frac{D-2}{2} X^{\frac{D-4}{2}} (\nabla \phi)^2 \quad (1.34)$$

$$= DX^{\frac{D-2}{2}} + (D-2) X^{\frac{D-4}{2}} \left(-\frac{1}{2} (\nabla \phi)^2 \right) \quad (1.35)$$

$$= DX^{\frac{D-2}{2}} + (D-2) X^{\frac{D-2}{2}} \quad (1.36)$$

$$= 2(D-1) X^{\frac{D-2}{2}} \quad (1.37)$$

Pluggin into (1.20a),

$$\begin{aligned}
A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi &= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}g^{\mu\nu}\nabla_\mu\phi + (D-2)X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla_\nu\phi\nabla_\mu\phi \\
&= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}\nabla^\nu\phi + (D-2)X^{\frac{D-4}{2}}(\nabla\phi)^2\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-4}{2}}X\right]\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-2}{2}}2\right]\nabla^\nu\phi \\
&= 0 \quad \checkmark
\end{aligned}$$

2 Building the auxiliar metric

We know the conditions that the most general second order equation of motion for the zero conformal weight scalar field must satisfy. They are given by (1.20). Now, the question is: how do we construct an auxiliary metric $\tilde{g}_{\mu\nu}$ such that $\delta_\omega\tilde{g}_{\mu\nu} = 0$?

Remember that in the exponential frame for the scalar field, the conformal transformations look like

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = e^{2\omega} g_{\mu\nu}, \quad \phi \rightarrow \bar{\phi} = \phi \quad (2.1)$$

Since the inverse metric transforms as

$$g^{\mu\nu} \rightarrow \bar{g}^{\mu\nu} = e^{-2\omega} g^{\mu\nu} \quad (2.2)$$

the kinetic term for the scalar field, defined as

$$X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \quad (2.3)$$

transforms as

$$\bar{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = -\frac{1}{2}e^{-2\omega}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = e^{-2\omega}X \quad (2.4)$$

Thus, the auxiliary metric defined as

$$\tilde{g}_{\mu\nu} = Xg_{\mu\nu} = -\frac{1}{2}(\nabla\phi)^2g_{\mu\nu} \implies \delta_\omega\tilde{g}_{\mu\nu} = 0 \quad (2.5)$$

is conformally invariant.

Let us consider a pseudoscalar built from the zero wight conformal scalar field and its derivatives up to second order, and the conformally invariant geometry,

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{R}_{\mu\nu}^{\alpha\beta}) \quad (2.6)$$

We notice that $\delta_\omega\mathcal{E} = 0$. Indeed

$$\begin{aligned}
\delta_\omega\mathcal{E} &= -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}E\delta_\omega\tilde{g}_{\mu\nu} + \sqrt{-\tilde{g}}\left(\frac{\partial E}{\partial\phi}\delta_\omega\phi + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\omega(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) \right. \\
&\quad \left. + \frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\omega\tilde{g}^{\mu\nu} + \tilde{R}_{\alpha\beta}^{\mu\nu}\delta_\omega\tilde{R}_{\mu\nu}^{\alpha\beta}\right)
\end{aligned}$$

but

$$\delta_\omega\tilde{\nabla}_\mu\phi = \delta_\omega\partial_\mu\phi = \partial_\mu\delta_\omega\phi = 0 \quad (2.7)$$

and

$$\delta_\omega(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi) = \delta_\omega(\tilde{\nabla}_\mu \partial_\nu \phi) \quad (2.8)$$

$$= \delta_\omega(\partial_\mu \partial_\nu \phi - \tilde{\Gamma}^\lambda_{\mu\nu} \partial_\lambda \phi) \quad (2.9)$$

$$= \partial_\mu \partial_\nu \delta_\omega \phi - \delta_\omega \tilde{\Gamma}^\lambda_{\mu\nu} \partial_\lambda \phi - \tilde{\Gamma}^\lambda_{\mu\nu} \partial_\lambda \delta_\omega \phi \quad (2.10)$$

$$= -\delta_\omega \tilde{\Gamma}^\lambda_{\mu\nu} \partial_\lambda \phi \quad (2.11)$$

$$= 0 \quad (2.12)$$

since $\delta_\omega \tilde{\Gamma}^\lambda_{\mu\nu} = 0$. Therefore,

$$\delta_\omega \mathcal{E} = 0. \quad (2.13)$$

Since from conformal invariance for zero weight scalar field, the $P^{\mu\nu} = 0$ condition implies that the explicit dependence of the Riemann tensor in the equation of motion is through the Weyl tensor, then, for a equation of motion built from the conformally invariant geometry we have

$$E = E(\phi, \tilde{\nabla}_\mu \phi, \tilde{\nabla}_\mu \nabla_\nu \phi, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}) = 0 \quad (2.14)$$

[Is this last true?]

A Some useful calculations

A.1 Variation of the Christoffel symbols

The Christoffel symbols in terms of the metric are given by

$$\Gamma_{\mu\beta}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}g_{\beta\rho} + \partial_{\beta}g_{\mu\rho} - \partial_{\rho}g_{\mu\beta})$$

Varying both sides, we have

$$\begin{aligned}\delta\Gamma_{\mu\beta}^{\lambda} &= \frac{1}{2}\delta g^{\lambda\rho}(\partial_{\mu}g_{\beta\rho} + \partial_{\beta}g_{\mu\rho} - \partial_{\rho}g_{\mu\beta}) + \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}\delta g_{\beta\rho} + \partial_{\beta}\delta g_{\mu\rho} - \partial_{\rho}\delta g_{\mu\beta}) \\ &= -\frac{1}{2}g^{\lambda\sigma}g^{\rho\tau}(\delta g_{\sigma\tau})(\partial_{\mu}g_{\beta\rho} + \partial_{\beta}g_{\mu\rho} - \partial_{\rho}g_{\mu\beta}) + \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}\delta g_{\beta\rho} + \partial_{\beta}\delta g_{\mu\rho} - \partial_{\rho}\delta g_{\mu\beta}) \\ &= -g^{\lambda\sigma}(\delta g_{\sigma\tau})\Gamma_{\mu\beta}^{\tau} + \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}\delta g_{\beta\rho} + \partial_{\beta}\delta g_{\mu\rho} - \partial_{\rho}\delta g_{\mu\beta})\end{aligned}$$

Changing the dumb indice σ by ρ ,

$$\begin{aligned}\delta\Gamma_{\mu\beta}^{\lambda} &= -g^{\lambda\rho}(\delta g_{\rho\tau})\Gamma_{\mu\beta}^{\tau} + \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}\delta g_{\beta\rho} + \partial_{\beta}\delta g_{\mu\rho} - \partial_{\rho}\delta g_{\mu\beta}) \\ &= \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}\delta g_{\beta\rho} + \partial_{\beta}\delta g_{\mu\rho} - \partial_{\rho}\delta g_{\mu\beta} - 2\delta g_{\rho\tau}\Gamma_{\mu\beta}^{\tau}) \\ &= \frac{1}{2}g^{\lambda\rho}(\partial_{\mu}\delta g_{\beta\rho} - \Gamma_{\mu\beta}^{\tau}\delta g_{\rho\tau} - \Gamma_{\rho\mu}^{\tau}\delta g_{\tau\beta} + \partial_{\beta}\delta g_{\mu\rho} - \Gamma_{\mu\beta}^{\tau}\delta g_{\rho\tau} - \Gamma_{\rho\beta}^{\tau}\delta g_{\tau\mu} \\ &\quad - \partial_{\rho}\delta g_{\mu\beta} + \Gamma_{\mu\rho}^{\tau}\delta g_{\tau\beta} + \Gamma_{\beta\rho}^{\tau}\delta g_{\mu\tau}) \\ &= \frac{1}{2}g^{\lambda\rho}(\nabla_{\mu}\delta g_{\beta\rho} + \nabla_{\beta}\delta g_{\mu\rho} - \nabla_{\rho}\delta g_{\mu\beta})\end{aligned}\tag{A.1}$$

A.2 Variation of the Riemann tensor

The Riemann tensor is given by

$$R_{\lambda\mu\nu}^{\rho} = \partial_{\mu}\Gamma_{\lambda\nu}^{\rho} - \partial_{\nu}\Gamma_{\lambda\mu}^{\rho} + \Gamma_{\tau\mu}^{\rho}\Gamma_{\lambda\nu}^{\tau} - \Gamma_{\tau\nu}^{\rho}\Gamma_{\lambda\mu}^{\tau}$$

Varying both sides,

$$\begin{aligned}\delta R_{\lambda\mu\nu}^{\rho} &= \partial_{\mu}\delta\Gamma_{\lambda\nu}^{\rho} - \partial_{\nu}\delta\Gamma_{\lambda\mu}^{\rho} + \delta\Gamma_{\tau\mu}^{\rho}\Gamma_{\lambda\nu}^{\tau} + \Gamma_{\tau\mu}^{\rho}\delta\Gamma_{\lambda\nu}^{\tau} - \delta\Gamma_{\tau\nu}^{\rho}\Gamma_{\lambda\mu}^{\tau} - \Gamma_{\tau\nu}^{\rho}\delta\Gamma_{\lambda\mu}^{\tau} \\ &= \partial_{\mu}\delta\Gamma_{\nu\lambda}^{\rho} + \Gamma_{\tau\mu}^{\rho}\delta\Gamma_{\nu\lambda}^{\tau} - \Gamma_{\mu\lambda}^{\tau}\delta\Gamma_{\tau\nu}^{\rho} - \partial_{\nu}\delta\Gamma_{\mu\lambda}^{\rho} + \Gamma_{\nu\lambda}^{\tau}\delta\Gamma_{\tau\mu}^{\rho} - \Gamma_{\tau\nu}^{\rho}\delta\Gamma_{\mu\lambda}^{\tau}\end{aligned}$$

Adding a convenient zero of the form $\Gamma_{\mu\nu}^{\tau}\delta\Gamma_{\tau\lambda}^{\rho} - \Gamma_{\mu\nu}^{\tau}\delta\Gamma_{\tau\lambda}^{\rho}$, and using the fact that $\delta\Gamma_{\mu\nu}^{\lambda}$ is a tensor, we have

$$\delta R_{\lambda\mu\nu}^{\rho} = \nabla_{\mu}\delta\Gamma_{\nu\lambda}^{\rho} - \nabla_{\nu}\delta\Gamma_{\mu\lambda}^{\rho} = 2\nabla_{[\mu}\delta\Gamma_{\nu]\lambda}^{\rho}\tag{A.2}$$

A.3 Variation of derivatives of ϕ w.r.t ω

Let's compute the infinitesimal variations of the covariant derivatives of the scalar field. First, let us calculate $\delta_{\omega}\nabla_{\mu}\phi$:

$$\delta_{\omega}\nabla_{\mu}\phi = \nabla_{\mu}\delta_{\omega}\phi = 0\tag{A.3}$$

Now, let's compute $\delta_{\omega}(\nabla_{\mu}\nabla_{\nu}\phi)$:

$$\delta_{\omega}(\nabla_{\mu}\nabla_{\nu}\phi) = \delta_{\omega}\nabla_{\mu}(\partial_{\nu}\phi)\tag{A.4}$$

$$= \delta_{\omega}(\partial_{\mu}\partial_{\nu}\phi - \Gamma_{\nu\mu}^{\lambda}\partial_{\lambda}\phi)\tag{A.5}$$

$$= \partial_{\mu}\partial_{\nu}\delta_{\omega}\phi - \delta_{\omega}\Gamma_{\nu\mu}^{\lambda}\partial_{\lambda}\phi - \Gamma_{\nu\mu}^{\lambda}\delta_{\omega}\partial_{\lambda}\phi\tag{A.6}$$

$$= \partial_{\mu}\partial_{\nu}\delta_{\omega}\phi - \delta_{\omega}\Gamma_{\nu\mu}^{\lambda}\partial_{\lambda}\phi - \Gamma_{\nu\mu}^{\lambda}\partial_{\lambda}\delta_{\omega}\phi\tag{A.7}$$

$$= -\partial_{\lambda}\phi\delta_{\omega}\Gamma_{\nu\mu}^{\lambda}\tag{A.8}$$

Using that the variation of the Christoffel connection is

$$\delta\Gamma_{\mu\beta}^{\lambda} = \frac{1}{2}g^{\lambda\rho}(\nabla_{\mu}\delta g_{\beta\rho} + \nabla_{\beta}\delta g_{\mu\rho} - \nabla_{\rho}\delta g_{\mu\beta}) \quad (\text{A.9})$$

we have

$$\delta_{\omega}(\nabla_{\mu}\nabla_{\nu}\phi) = -\partial_{\lambda}\phi\frac{1}{2}g^{\lambda\rho}(\nabla_{\nu}\delta_{\omega}g_{\mu\rho} + \nabla_{\mu}\delta_{\omega}g_{\nu\rho} - \nabla_{\rho}\delta_{\omega}g_{\nu\beta}) \quad (\text{A.10})$$

but $\delta_{\omega}g_{\nu\mu} = 2\omega g_{\nu\mu}$, so then

$$\delta_{\omega}(\nabla_{\mu}\nabla_{\nu}\phi) = -\partial_{\lambda}\phi g^{\lambda\rho}[\nabla_{\nu}(\omega g_{\mu\rho}) + \nabla_{\mu}(\omega g_{\nu\rho}) - \nabla_{\rho}(\omega g_{\nu\beta})] \quad (\text{A.11})$$

Using the metric compatibility condition $\nabla_{\mu}g_{\alpha\beta} = 0$ and $\nabla_{\alpha}\phi = \partial_{\alpha}\phi$, we obtain

$$\delta_{\omega}(\nabla_{\mu}\nabla_{\nu}\phi) = -\partial_{\lambda}\phi g^{\lambda\rho}(g_{\mu\rho}\nabla_{\nu}\omega + g_{\nu\rho}\nabla_{\mu}\omega - g_{\nu\beta}\nabla_{\rho}\omega) \quad (\text{A.12})$$

$$= -\partial^{\rho}\phi(g_{\mu\rho}\nabla_{\nu}\omega + g_{\nu\rho}\nabla_{\mu}\omega - g_{\nu\beta}\nabla_{\rho}\omega) \quad (\text{A.13})$$

$$= -\nabla_{\mu}\phi\nabla_{\nu}\omega - \nabla_{\nu}\phi\nabla_{\mu}\omega + g_{\nu\beta}\nabla^{\rho}\phi\nabla_{\rho}\omega \quad (\text{A.14})$$

$$= -2\nabla_{(\mu}\phi\nabla_{\nu)}\omega + g_{\nu\beta}\nabla^{\rho}\phi\nabla_{\rho}\omega \quad (\text{A.15})$$

A.4 Variation of E w.r.t Riemann tensor

In order to see what (1.20c) implies, let us split the dependence of the Riemann tensor in terms of its traceless part $C_{\mu\nu}^{\alpha\beta}$, the traceless part of the Ricci tensor S_{β}^{α} , and the scalar curvature R . So we have

$$E(g^{\mu\nu}, R_{\mu\nu}^{\alpha\beta}) = E(g^{\mu\nu}, C_{\mu\nu}^{\alpha\beta}, S_{\beta}^{\alpha}, R)$$

The variation w.r.t the Riemann tensor yields

$$\begin{aligned} \delta_{\text{Riem}}E &= P_{\mu\nu}^{\alpha\beta}\delta R_{\alpha\beta}^{\mu\nu} \\ &= H_{\mu\nu}^{\alpha\beta}\delta_{\text{Riem}}C_{\alpha\beta}^{\mu\nu} + I_{\beta}^{\alpha}\delta_{\text{Riem}}S_{\alpha}^{\beta} + J\delta_{\text{Riem}}R \end{aligned} \quad (\text{A.16})$$

where

$$H_{\mu\nu}^{\alpha\beta} \equiv \frac{\partial E}{\partial C_{\alpha\beta}^{\mu\nu}}, \quad I_{\beta}^{\alpha} \equiv \frac{\partial E}{\partial S_{\alpha}^{\beta}}, \quad \text{y} \quad J \equiv \frac{\partial E}{\partial R}$$

Since $P_{\alpha\beta}^{\mu\nu}$ has the same algebraic symmetries as the Riemann tensor, its traceless part is given by

$$\hat{P}_{\alpha\beta}^{\mu\nu} = P_{\alpha\beta}^{\mu\nu} - \frac{4}{D-2}\delta_{[\alpha}^{\mu}P_{\beta]}^{\nu]} + \frac{2}{(D-2)(D-1)}P\delta_{[\alpha}^{\mu}\delta_{\beta]}^{\nu]} \quad (\text{A.17})$$

Let us note that

$$\begin{aligned} J\delta_{\text{Riem}}R &= J\delta_{\text{Riem}}(R_{\mu\nu}^{\alpha\beta}\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu}) \\ &= J\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu}R_{\mu\nu}^{\alpha\beta} \end{aligned} \quad (\text{A.18})$$

Writing S_{β}^{α} in terms of the Riemann,

$$\begin{aligned} S_{\nu}^{\beta} &= R_{\nu}^{\beta} - \frac{1}{D}R\delta_{\nu}^{\beta} \\ &= R_{\mu\nu}^{\alpha\beta}\delta_{\alpha}^{\mu} - \frac{1}{D}\delta_{\nu}^{\beta}R\delta_{\mu\lambda}^{\alpha\gamma}\delta_{\alpha}^{\mu}\delta_{\gamma}^{\lambda} \end{aligned}$$

then,

$$\delta_{\text{Riem}}\tilde{S}_{\nu}^{\beta} = \delta_{\alpha}^{\mu}\delta\tilde{R}_{\mu\nu}^{\alpha\beta} - \frac{1}{D}\delta_{\nu}^{\beta}\delta\tilde{R}_{\mu\lambda}^{\alpha\gamma}\delta_{\alpha}^{\mu}\delta_{\gamma}^{\lambda}$$

Hence,

$$\begin{aligned}
I_\beta^\nu \delta_{\text{Riem}} S_\nu^\beta &= I_\beta^\nu \delta_\alpha^\mu \delta R_{\mu\nu}^{\alpha\beta} - \frac{1}{D} I_\beta^\nu \delta_\nu^\beta \delta_\alpha^\mu \delta_\gamma^\lambda \delta R_{\mu\lambda}^{\alpha\gamma} \\
&= I_\beta^\nu \delta_\alpha^\mu \delta \tilde{R}_{\mu\nu}^{\alpha\beta} - \frac{1}{D} I \delta_\alpha^\mu \delta R_{\mu\lambda}^{\alpha\gamma} \\
&= \delta_\alpha^\mu \delta R_{\mu\nu}^{\alpha\beta} \left(I_\beta^\nu - \frac{1}{D} I \delta_\beta^\nu \right) \\
&= \delta_\alpha^\mu \hat{I} \delta R_{\mu\nu}^{\alpha\beta}
\end{aligned} \tag{A.19}$$

Finally, let us write the Weyl tensor in terms of the Riemann,

$$\begin{aligned}
\tilde{C}_{\mu\nu}^{\alpha\beta} &= \tilde{R}_{\mu\nu}^{\alpha\beta} - \frac{4}{D-2} \delta_{[\mu}^{\alpha} \tilde{R}_{\nu]}^{\beta]} + \frac{2}{(D-1)(D-2)} \tilde{R} \delta_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} \\
&= \tilde{R}_{\mu\nu}^{\alpha\beta} - \frac{4}{D-2} \delta_\gamma^\lambda \delta_{[\mu}^{\alpha} \tilde{R}_{\nu]\lambda}^{\beta]\gamma} + \frac{2}{(D-1)(D-2)} \delta_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} \delta_\rho^\tau \delta_\sigma^\lambda \tilde{R}_{\tau\lambda}^{\rho\sigma}
\end{aligned}$$

Varying with respect to $R_{\mu\nu}^{\alpha\beta}$,

$$\delta_{\text{Riem}} C_{\mu\nu}^{\alpha\beta} = \delta R_{\mu\nu}^{\alpha\beta} - \frac{4}{D-2} \delta_\gamma^\lambda \delta_{[\mu}^{\alpha} \delta R_{\nu]\lambda}^{\beta]\gamma} + \frac{2}{(D-1)(D-2)} \delta_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} \delta_\rho^\tau \delta_\sigma^\lambda \delta R_{\tau\lambda}^{\rho\sigma}$$

Then,

$$\begin{aligned}
H_{\alpha\beta}^{\mu\nu} \delta_{\text{Riem}} C_{\mu\nu}^{\alpha\beta} &= H_{\alpha\beta}^{\mu\nu} \left[\delta R_{\mu\nu}^{\alpha\beta} - \frac{4}{D-2} \delta_\gamma^\lambda \delta_{[\mu}^{\alpha} \delta R_{\nu]\lambda}^{\beta]\gamma} + \frac{2}{(D-1)(D-2)} \delta_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} \delta_\rho^\tau \delta_\sigma^\lambda \delta R_{\tau\lambda}^{\rho\sigma} \right] \\
&= H_{\alpha\beta}^{\mu\nu} \delta R_{\mu\nu}^{\alpha\beta} - \frac{4}{D-2} H_{\gamma\beta}^{\lambda\nu} \delta_\alpha^\mu \delta_\lambda^\gamma \delta R_{\nu\mu}^{\beta\alpha} + \frac{2}{(D-1)(D-2)} H_{\rho\sigma}^{\tau\lambda} \delta_\tau^\rho \delta_\lambda^\sigma \delta_\alpha^\mu \delta_\beta^\nu \delta R_{\mu\nu}^{\alpha\beta} \\
&= \delta R_{\mu\nu}^{\alpha\beta} \left[H_{\alpha\beta}^{\mu\nu} - \frac{4}{D-2} H_\beta^\nu \delta_\alpha^\mu + \frac{2}{(D-1)(D-2)} H \right] \\
&= \hat{H}_{\alpha\beta}^{\mu\nu} \delta R_{\mu\nu}^{\alpha\beta}
\end{aligned} \tag{A.20}$$

where the indices have been renamed in a convenient way and has been used the fact that $H_{\mu\nu}^{\alpha\beta}$ has the same algebraic symmetries as the Riemann tensor.

In this way, plugging (A.18), (A.19) and (A.20) into (A.16), we obtain

$$P_{\mu\nu}^{\alpha\beta} \delta R_{\alpha\beta}^{\mu\nu} = \hat{H}_{\mu\nu}^{\alpha\beta} \delta R_{\alpha\beta}^{\mu\nu} + \delta_\mu^\alpha \hat{I}_\nu^\beta \delta R_{\alpha\beta}^{\mu\nu} + J \delta_\mu^\alpha \delta_\nu^\beta \delta R_{\alpha\beta}^{\mu\nu}$$

Hence,

$$P_{\mu\nu}^{\alpha\beta} = \hat{H}_{\mu\nu}^{\alpha\beta} + \delta_{[\mu}^{\alpha} \hat{I}_{\nu]}^{\beta]} + J \delta_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} \tag{A.21}$$

References