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Polinomios de Wheeler en la teoría de Lovelock

Borja Diez B.

Universidad Arturo Pratt, Iquique, Chile

E-mail: borjadiez1014@gmail.com

Abstract: En estas notas iré recopilando los resultados más importantes de mis cálculos de mi tesis de magister. En particular, iré recopilando los polinomios de Wheeler encontrados en diferenter ordenes en la teoría de Lovelock para métricas tipo Eguchi-Hanson con diferentes base manifolds con topología no trivial.

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Capítulo 1

Eguchi-Hanson

1.1. Eguchi-Hanson Einstein gravity

Los polinomios de Wheeler encontrados son los siguientes:
En $4D$ para base manifold $(\mathbb{T}^2)$,$(\mathbb{S}^2)$, $(\mathbb{H}^2)$, $\mathbb{CP}$ y $\mathbb{CH}$ respectivamente son

$$r^4 f(r) + \frac{\Lambda}{r} r^6 + a^4 = 0$$

$$r^4 f(r) + \frac{\Lambda}{6} r^6 - r^4 + a^4 = 0$$

$$r^4 f(r) + \frac{\Lambda}{6} r^6 + r^4 + a^4 = 0$$

$$r^4 f(r) + \frac{\Lambda}{6} r^6 - r^4 + a^4 = 0$$

$$r^4 f(r) + \frac{\Lambda}{6} r^6 + r^4 + a^4 = 0$$

En $6D$ para base manifold $(\mathbb{T}^2)^2$,$(\mathbb{S}^2)^2$, $(\mathbb{H}^2)^2$, $\mathbb{CP}^2$ y $\mathbb{CH}^2$ respectivamente son

$$r^6 f(r) + \frac{\Lambda}{16} r^8 + a^6 = 0$$

$$r^6 f(r) + \frac{\Lambda}{16} r^8 - \frac{2}{3} r^6 + a^6 = 0$$

$$r^6 f(r) + \frac{\Lambda}{16} r^8 + \frac{2}{3} r^6 + a^6 = 0$$

$$r^6 f(r) + \frac{\Lambda}{16} r^8 - \frac{2}{3} r^6 + a^6 = 0$$

$$r^6 f(r) + \frac{\Lambda}{16} r^8 + \frac{2}{3} r^6 + a^6 = 0$$

En $8D$ para base manifold $(\mathbb{T}^2)^3$,$(\mathbb{S}^2)^3$, $(\mathbb{H}^2)^3$, $\mathbb{CP}^3$ y $\mathbb{CH}^3$ respectivamente son

$$r^8 f(r) + \frac{\Lambda}{30} r^{10} + a^8 = 0$$

$$r^8 f(r) + \frac{\Lambda}{30} r^{10} - \frac{1}{2} r^8 + a^8 = 0$$

$$r^8 f(r) + \frac{\Lambda}{30} r^{10} + \frac{1}{2} r^8 + a^8 = 0$$

$$r^8 f(r) + \frac{\Lambda}{30} r^{10} - \frac{1}{2} r^8 + a^8 = 0$$

$$r^8 f(r) + \frac{\Lambda}{30} r^{10} + \frac{1}{2} r^8 + a^8 = 0$$

En $10D$ para base manifold $(\mathbb{T}^2)^4$,$(\mathbb{S}^2)^4$, $(\mathbb{H}^2)^4$, $\mathbb{CP}^4$ y $\mathbb{CH}^4$ respectivamente son

$$r^{10} f(r) + \frac{\Lambda}{48} r^{12} + a^8 = 0$$

$$r^{10} f(r) + \frac{\Lambda}{48} r^{12} - \frac{2}{5} r^{10} + a^8 = 0$$

$$r^{10} f(r) + \frac{\Lambda}{48} r^{12} + \frac{2}{5} r^{10} + a^8 = 0$$

$$r^{10} f(r) + \frac{\Lambda}{48} r^{12} - \frac{2}{5} r^{10} + a^8 = 0$$

$$r^{10} f(r) + \frac{\Lambda}{48} r^{12} + \frac{2}{5} r^{10} + a^8 = 0$$

En $12D$ para base manifold $(\mathbb{S}^2)^5$

$$r^{12} f(r) + \frac{\Lambda}{70} r^{14} - \frac{1}{3} r^{12} + a^{12} = 0$$

Para d dimensiones el polinomio de Wheeler asociado es

$$r^d f(r) + \frac{2\Lambda}{d^2 - 4} r^{d+2} - \frac{4}{d} \gamma r^d + a^d = 0$$

$$r^{2m} f(r) + \frac{\Lambda r^{2(m+1)}}{2(m^2 - 1)} - \frac{2\gamma r^{2m}}{m} + a^{2m} = 0$$

$$\left(f(r) - \frac{4\gamma}{d} \right) r^d + \frac{2\Lambda}{d^2 - 4} r^{d+2} + a^d = 0$$

$$\left(f(r) - \frac{2\gamma}{m} \right) r^{2m} + \frac{\Lambda}{2(m^2 - 1)} r^{2(m+1)} + a^{2m} = 0$$

La función métrica que resuelve

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1.1.1}$$

para dimensión d es

$$f(r) = -\frac{2\Lambda r^2}{d^2 - 4} - \left(\frac{a}{r} \right)^d + \frac{4\gamma}{d} \tag{1.1.2}$$

donde $\gamma = \pm 1, 0$ dependiendo de la curvatura constante del base manifold.

1.2. Eguchi-Hanson en Lovelock cuadrático

Los polinomios de Wheeler encontrados son los siguientes:
En $6D$ para base manifold $(\mathbb{T}^2)^2$,$(\mathbb{S}^2)^2$, $(\mathbb{H}^2)^2$, $\mathbb{CP}^2$ y $\mathbb{CH}^2$ respectivamente son

$$12\alpha f(r)^2 r^4 - r^6 f(r) - \frac{\Lambda r^8}{16} + a^6 = 0$$

$$12\alpha f(r)^2 r^4 + (-r^6 - 16\alpha r^4) f(r) - \frac{\Lambda r^8}{16} + \frac{2}{3} r^6 + 8\alpha r^4 + a^6 = 0$$

$$12\alpha f(r)^2 r^4 + (-r^6 + 16\alpha r^4) f(r) - \frac{\Lambda r^8}{16} - \frac{2}{3} r^6 + 8\alpha r^4 + a^6 = 0$$

$$12\alpha f(r)^2 r^4 + (-r^6 - 16\alpha r^4) f(r) - \frac{\Lambda r^8}{16} + \frac{2}{3} r^6 + \frac{16}{3}\alpha r^4 + a^6 = 0$$

$$12\alpha f(r)^2 r^4 + (-r^6 + 16\alpha r^4) f(r) - \frac{\Lambda r^8}{16} - \frac{2}{3} r^6 + \frac{16}{3}\alpha r^4 + a^6 = 0$$

En $8D$ para base manifold $(\mathbb{T}^2)^3$,$(\mathbb{S}^2)^3$, $(\mathbb{H}^2)^3$, $\mathbb{CP}^3$ y $\mathbb{CH}^3$ respectivamente son

$$32\alpha f(r)^2 r^6 - r^8 f(r) - \frac{\Lambda r^{10}}{30} + a^8 = 0$$

$$32\alpha f(r)^2 r^6 + (-r^8 - 32\alpha r^6) f(r) - \frac{\Lambda r^{10}}{30} + \frac{1}{2} r^8 + \frac{32}{3}\alpha r^6 + a^8 = 0$$

$$32\alpha f(r)^2 r^6 + (-r^8 + 32\alpha r^6) f(r) - \frac{\Lambda r^{10}}{30} - \frac{1}{2} r^8 + \frac{32}{3}\alpha r^6 + a^8 = 0$$

$$32\alpha f(r)^2 r^6 + (-r^8 - 32\alpha r^6) f(r) - \frac{\Lambda r^{10}}{30} + \frac{1}{2} r^8 + 8\alpha r^6 + a^8 = 0$$

$$32\alpha f(r)^2 r^6 + (-r^8 + 32\alpha r^6) f(r) - \frac{\Lambda r^{10}}{30} - \frac{1}{2} r^8 + 8\alpha r^6 + a^8 = 0$$

En $10D$ para base manifold $(\mathbb{T}^2)^4$,$(\mathbb{S}^2)^4$, $(\mathbb{H}^2)^4$, $\mathbb{CP}^4$ y $\mathbb{CH}^4$ respectivamente son

$$60\alpha f(r)^2 r^8 - f(r) r^{10} - \frac{\Lambda}{48} r^{12} + a^{10} = 0$$

$$60\alpha f(r)^2 r^8 + (-r^{10} - 48\alpha r^8) f(r) - \frac{\Lambda}{48} + \frac{2}{5} r^{10} + 12\alpha r^8 + a^{10} = 0$$

$$60\alpha f(r)^2 r^8 + (-r^{10} + 48\alpha r^8) f(r) - \frac{\Lambda}{48} - \frac{2}{5} r^{10} + 12\alpha r^8 + a^{10} = 0$$

$$60\alpha f(r)^2 r^8 + (-r^{10} - 48\alpha r^8) f(r) - \frac{\Lambda}{48} + \frac{2}{5} r^{10} + \frac{48}{5}\alpha r^8 + a^{10} = 0$$

$$60\alpha f(r)^2 r^8 + (-r^{10} + 48\alpha r^8) f(r) - \frac{\Lambda}{48} - \frac{2}{5} r^{10} + \frac{48}{5}\alpha r^8 + a^{10} = 0$$

En $12D$ para base manifold $(\mathbb{T}^2)^5$,$(\mathbb{S}^2)^5$, $(\mathbb{H}^2)^5$ respectivamente son

$$96\alpha f(r)^2 r^{10} - f(r) r^{12} - \frac{\Lambda}{70} r^{14} + a^{12} = 0$$

$$96\alpha f(r)^2 r^{10} + (-r^{12} - 64\alpha r^{10}) f(r) - \frac{\Lambda}{70} r^{14} + \frac{1}{3} r^{12} + \frac{64}{5}\alpha r^{10} + a^{12} = 0$$

$$96\alpha f(r)^2 r^{10} + (-r^{12} + 64\alpha r^{10}) f(r) - \frac{\Lambda}{70} r^{14} + \frac{1}{3} r^{12} + \frac{64}{5}\alpha r^{10} + a^{12} = 0$$

En $14D$ para base manifold $(\mathbb{S}^2)^6$, $(\mathbb{H}^2)^6$ respectivamente son

$$140\alpha f(r)^2 r^{12} + (-r^{14} - 80\alpha r^{12}) f(r) - \frac{\Lambda}{96} r^{16} + \frac{2}{7} r^{14} + \frac{40}{2}\alpha r^{12} + a^{14} = 0$$

$$140\alpha f(r)^2 r^{12} + (-r^{14} + 80\alpha r^{12}) f(r) - \frac{\Lambda}{96} r^{16} - \frac{2}{7} r^{14} + \frac{40}{2}\alpha r^{12} + a^{14} = 0$$

En d dimensiones el polinomio de Wheeler asociado es

$$\boxed{d(d-4)\alpha f(r)^2 r^{d-2} + f(r)[-r^d - 8(d-4)\gamma r^{d-2}] - \frac{2\Lambda}{d^2-4} r^{d+2} + \frac{4}{d}\gamma r^d + a^d + C\alpha r^{d-2} = 0}$$

o de manera equivalente, considerando $d = 2m$

$$-4m(m-2)\alpha f(r)^2 r^{2(m-1)} + 16 f(r)\gamma(m-2) r^{2(m-1)} + r^{2m} f(r) + \frac{\Lambda r^{2(m+1)}}{2(m^2-1)} - \frac{2\gamma r^{2m}}{m} + a^{2m} - C\alpha r^{2(m-1)} = 0$$

$$\left(f(r) - \frac{2\gamma}{m}\right) r^{2m} + \frac{\Lambda}{2(m^2-1)} r^{2(m+1)} - 4m(m-2)\alpha \left(\left(f(r) - \frac{2\gamma}{m}\right)^2 + \frac{4}{m^2(m-1)}\right) r^{2(m-1)} + a^{2m} = 0$$

$$F(r) r^{2m} - 4m(m-2)\alpha_1 \left(F(r)^2 + \frac{4}{m^2(m-1)}\right) r^{2(m-1)} + \frac{\Lambda}{2(m^2-1)} r^{2(m+1)} + a^{2m} = 0$$

donde $F(r) = f(r) - 2\gamma/m$

1.3. Eguchi-Hanson en Lovelock cúbico

Los polinomios de Wheeler encontrados son los siguientes:
En $8D$ para base manifold $(\mathbb{T}^2)^3$,$(\mathbb{S}^2)^3$, $(\mathbb{H}^2)^3$, $\mathbb{CP}^3$ y $\mathbb{CH}^3$ respectivamente son

$$384\alpha_3 f(r)^3 r^4 - 32\alpha_2 f(r)^2 r^6 + r^8 f(r) + \frac{\Lambda r^{10}}{30} + a^8 = 0$$

$$384\alpha_3 f(r)^3 r^4 + (-32\alpha_2 r^6 - 576\alpha_3 r^4) f(r)^2 + (r^8 + 32\alpha_2 r^6 + 384\alpha_3 r^4) f(r) + \frac{\Lambda r^{10}}{30} - \frac{r^8}{2} - \frac{32\alpha_3 r^6}{3} - 128\alpha_3 r^4 + a^8 = 0$$

$$384\alpha_3 f(r)^3 r^4 + (-32\alpha_2 r^6 + 576\alpha_3 r^4) f(r)^2 + (r^8 - 32\alpha_2 r^6 + 384\alpha_3 r^4) f(r) + \frac{\Lambda r^{10}}{30} + \frac{r^8}{2} - \frac{32\alpha_3 r^6}{3} + 128\alpha_3 r^4 + a^8 = 0$$

$$384\alpha_3 f(r)^3 r^4 + (-32\alpha_2 r^6 - 576\alpha_3 r^4) f(r)^2 + (r^8 + 32\alpha_2 r^6 + 288\alpha r^4) f(r) + \frac{\Lambda r^{10}}{30} - \frac{r^8}{2} - 8\alpha_2 r^6 - 48\alpha_3 r^4 + a^8 = 0$$

$$384\alpha_3 f(r)^3 r^4 + (-32\alpha_2 r^6 + 576\alpha_3 r^4) f(r)^2 + (r^8 - 32\alpha_2 r^6 + 288\alpha r^4) f(r) + \frac{\Lambda r^{10}}{30} + \frac{r^8}{2} - 8\alpha_2 r^6 + 48\alpha_3 r^4 + a^8 = 0$$

En $10D$ para base manifold $(\mathbb{T}^2)^4$,$(\mathbb{S}^2)^4$, $(\mathbb{H}^2)^4$ respectivamente son

$$1920\alpha_3 f(r)^3 r^6 - 60\alpha_2 f(r)^2 r^8 + r^{10} f(r) + \frac{\Lambda r^{12}}{48} + a^{10} = 0$$

$$1920\alpha_3 f(r)^3 r^6 + (-60\alpha_2 r^8 - 2304\alpha_3 r^6) f(r)^2 + (r^{10} + 48\alpha_2 r^8 + 1152\alpha_3 r^6) f(r) + \frac{\Lambda r^{12}}{48} - \frac{2r^{10}}{5} - 12\alpha_2 r^8 - 256\alpha_3 r^6 + a^{10} = 0$$

$$1920\alpha_3 f(r)^3 r^6 + (-60\alpha_2 r^8 + 2304\alpha_3 r^6) f(r)^2 + (r^{10} - 48\alpha_2 r^8 + 1152\alpha_3 r^6) f(r) + \frac{\Lambda r^{12}}{48} + \frac{2r^{10}}{5} - 12\alpha_2 r^8 + 256\alpha_3 r^6 + a^{10} = 0$$

En $12D$ para base manifold $(\mathbb{S}^2)^5$ es

$$5760\alpha_3 f(r)^3 r^8 + (-96\alpha_2 r^{10} - 5760\alpha_3 r^8) f(r)^2 + (r^{12} + 64\alpha_2 r^{10} + 2304\alpha_3 r^8) f(r) + \frac{\Lambda r^{14}}{70} - \frac{r^{12}}{3} - \frac{64\alpha_2 r^{10}}{5} - 384\alpha_3 r^8 + a^{12} = 0$$

El polinomio de Wheeler asociado para d-dimensiones donde $d = 2m$ va como

$$16m(m-1)(m-2)(m-3)\alpha_3 f(r)^3 r^{2(m-2)} - 4m(m-2)\alpha_2 f(r)^2 r^{2(m-1)} + f(r)\left(r^{2m} + 16\gamma(m-2)\alpha_2 r^{2(m-1)}\right) + \frac{\Lambda r^{2(m+1)}}{2(m^2-1)} - \frac{2\gamma r^{2m}}{m} + a^{2m} = 0$$

1.4. Generalizaciones

El polinomio de Wheeler hasta Lovelock cúbico para $d = 2m$ dimensiones, pareciera ser

$$a^{2m} + \frac{\Lambda r^{2(m+1)}}{2(m^2-1)} + \left(f(r) - \frac{2\gamma}{m}\right) r^{2m} - 4m(m-2)\alpha_2 \left(\left(f(r) - \frac{2\gamma}{m}\right)^2 + \frac{4\gamma^2}{m^2(m-1)}\right) r^{2(m-1)}$$

$$+ 16m(m-1)(m-2)(m-3)\alpha_3$$

$$\times \left(\left(f(r) - \frac{2\gamma}{m}\right)^3 - \frac{12 f(r)\gamma^2}{m^2} + \frac{8\gamma^3}{m^3} + \frac{12 f(r)\gamma}{m(m-1)} - \frac{8\gamma^3}{m(m-1)(m-2)}\right)^{2(m-2)} = 0$$

o de forma más compacta, podemos considerar la siguiente serie

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(m-k)!}{m!} f(r)^{n-k} (2\gamma)^k \tag{1.4.1}$$

las cuales se convierte en

$$\sum_{l=0}^{n} \frac{\alpha_l (-4)^{l-1} r^{2(m-l+1)} (m!)^2}{(m-l-1)!(m-1+1)!} \frac{1}{m(m-1)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(m-k)!}{m!} f(r)^{n-k} (2\gamma)^k = a^{2m}$$

El polinomio de Wheeler para orden n en la serie de Lovelock para $d = 2m$ dimensiones, pareciera ser

$$a^{2m} + \sum_{l=0}^{n} \frac{\alpha_l (-4)^{l-1} m!(m-2)!}{(m-l-1)!(m-l+1)!} F_l(r) r^{2(m-l+1)} = 0$$

con

$$F_l(r) = \left(f(r) - \frac{2\gamma}{m}\right)^l + \sigma \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(r)^{l-k} \left(\frac{2\gamma}{m}\right)^k \left(\frac{m^k (m-k)!}{m!} - 1\right)$$

donde $\gamma = 0, \pm 1$ para base manifolds productos de $(\mathbb{T}^2)$,$(\mathbb{S}^2)$ y $(\mathbb{H}^2)$ respectivamente.

Para el caso cargado, considerando un ansatz de Maxwell alineado en la dirección del fibrado $U(1)$, se tiene

$$A = A_\mu \mathrm{d}x^\mu = \left(\frac{p}{r^{2(m-1)}} + qr^2\right)(\mathrm{d}t + \mathcal{B})$$

$$a^{2m} + \sum_{l=0}^{n} \frac{p(-4)^{l-1} m!(m-2)!}{(m-l-1)!(m-l+1)!} F_l(r) r^{2(m-l+1)} + \frac{4(m-2)}{m-1} \left(\frac{p^2}{r^{2(m-1)}} + \frac{r^{2(m+1)}}{m+1} q^2\right) + 16pqr^2 = 0$$

Capítulo 2

Taub-NUT

2.1. Taub-NUT Einstein gravity

Los polinomios de Wheeler encontrados son los siguientes:
Para base manifold $(\mathbb{T}^2)$,$(\mathbb{T}^2)^2$,$(\mathbb{T}^2)^3$,$(\mathbb{T}^2)^4$,$(\mathbb{T}^2)^5$ son

$$\left(n^2 - r^2\right) f(r) + \Lambda \left(n^4 + 2n^2r^2 - \frac{1}{3}r^4\right) + a^4 r = 0$$

$$2\left(-r+n\right)^2 \left(n+r\right)^2 f(r) + \Lambda \left(n^6 + 3n^4r^2 - n^2r^4 + \frac{1}{5}r^6\right) + a^6 r = 0$$

$$3\left(-r+n\right)^3 \left(n+r\right)^3 f(r) + \Lambda \left(n^8 + 4n^6r^2 - 2n^4r^4 + \frac{4}{5}n^2r^6 - \frac{1}{7}r^8\right) + a^8 r = 0$$

$$4\left(-r+n\right)^4 \left(n+r\right)^4 f(r) + \Lambda \left(n^{10} + 5n^8r^2 - \frac{10}{3}n^6r^4 + 2n^4r^6 - \frac{5}{7}n^2r^8 + \frac{1}{9}r^{10}\right) + a^{10} r = 0$$

$$5\left(n-r\right)^5 \left(n+r\right)^5 f(r) + \Lambda \left(n^{12} + 6n^{10}r^2 - 5n^8r^4 + 4n^6r^6 - \frac{15}{7}n^4r^8 + \frac{2}{3}n^2r^{10} - \frac{1}{11}r^{12}\right) + a^{10} r = 0$$

Una propuesta tentativa es

$$a^{2m} r + (m-1)\left(n^2 - r^2\right)^{m-1} f(r) + \Lambda(\text{algo})$$

donde ese algo va como potencias de $(n^2 + r^2)$.

El polinomio de Wheeler para la generalización de Taub-NUT en Einstein gravity es

$$(m-1)(n^2 - r^2)^{m-1} f(r) + \Lambda W_m + \gamma(m-1)W_{m-1} + a^{2m} r = 0$$

donde

$$W_m \equiv \sum_{i=0}^{m} (-1)^{m-1} \binom{m}{i} \left(-\frac{n^2}{r^2}\right)^{m-i} \frac{2m-1}{2i-1} r^{2m}$$

La función métrica de Taub-NUT en Einsteing gravity es

$$f(r)=\frac{1}{(n^2-r^2)^{m-1}}\left[-\frac{\Lambda}{m-1}W_m-\gamma W_{m-1}+\frac{a^{2m}}{m-1}r\right]$$

donde

$$W_m \equiv = (-1)^{m-1}\binom{m}{i}\left(-\frac{n^2}{r^2}\right)^{m-i}\frac{r^{2m}}{2i-1}$$

2.2. Taub-NUT Einstein-Gauss-Bonnet gravity

Los polinomios de Wheeler encontrados son los siguientes:
Para base manifold $(\mathbb{T}^2)^2$,$(\mathbb{T}^2)^3$,$(\mathbb{T}^2)^4$,$(\mathbb{T}^2)^5$ son

$$\left(n^6+3n^4r^2-n^2r^4+\frac{1}{5}r^6\right)\Lambda-12\alpha\left(n^2+r^2\right)f(r)^2+2\left(n^2-r^2\right)^2f(r)+a^6r=0$$

$$\left(n^8+4r^2n^6-2r^4n^4+\frac{4}{5}r^6n^2-\frac{1}{7}r^8\right)\Lambda+\alpha\left(-36n^4-24n^2r^2+60r^4\right)f(r)^2+3\left(n^2-r^2\right)^3f(r)+a^8r=0$$

$$\left(n^{10}+5n^8r^2-\frac{10}{3}n^6r^4+2n^4r^6-\frac{5}{7}n^2r^8+\frac{1}{9}r^{10}\right)\Lambda-72\left(n^2-r^2\right)^2\left(n^2+\frac{7r^2}{3}\right)\alpha f(r)^2$$
$$+4\left(n^2-r^2\right)^4f(r)+a^{10}r=0$$

$$\left(n^{12}+6n^{10}r^2-5n^8r^4+4n^6r^6-\frac{15}{7}n^4r^8+\frac{2}{3}n^2r^{10}-\frac{1}{11}r^{12}\right)\Lambda$$
$$-120\alpha\left(n^2+3r^2\right)\left(n^2-r^2\right)^3f(r)^2+5\left(n^2-r^2\right)^5f(r)+a^{12}r=0$$

Debemos encontrar una relación entre los siguientes factores

$$\begin{aligned}&12(n^2+r^2)\\&12(n^2-r^2)(3n^2+5r^2)\\&72(n^2-r^2)^2\\&120(n^2+3r^2)(n^2-r^2)^3\end{aligned}$$

Podemos considerar $\Lambda=0$ y $a=0$ y factorizar los términos restantes y vemos que quedala siguiente estructura

$$\begin{aligned}&-2f(r)\left(6f(r)\,\alpha\,n^2+6f(r)\,\alpha\,r^2-n^4+2n^2r^2-r^4\right)\\&-3f(r)\,(n-r)\,(n+r)\left(12f(r)\,\alpha\,n^2+20f(r)\,\alpha\,r^2-n^4+2n^2r^2-r^4\right)\\&-4f(r)\,(n-r)^2\,(n+r)^2\left(18f(r)\,\alpha\,n^2+42f(r)\,\alpha\,r^2-n^4+2n^2r^2-r^4\right)\\&-5f(r)\,(n-r)^3\,(n+r)^3\left(24f(r)\,\alpha\,n^2+72f(r)\,\alpha\,r^2-n^4+2n^2r^2-r^4\right)\end{aligned}$$

Es claro ver la relación entre los primeros factores. El desafío ahora es encontrar la relación entre los paréntesis.

$$-(m-1)f(r)(n^2-r^2)^{m-3}(algo-(n^2-r^2)^2)$$

The Weeler polynomial for plane geometries ($\gamma = 0$) in Einstein-Gauss-Bonnet gravity is

$$a^{2m}r+\Lambda W_m+(m-1)(n^2-r^2)^{m-1}f(r)-\frac{12(m-1)(m-2)}{2}(n^2-r^2)^{m-3}\left(n^2+\frac{(2m-3)}{3}r^2\right)f(r)^2=0$$

The Wheeler polynomial for $\sigma = 0$ in Einstein-Gauss-Bonnet gravity is

$$\begin{aligned}&a^{2m}r+\Lambda W_m+\alpha_1(m-1)\left[(n^2-r^2)^{m-1}f(r)+\gamma W_{m-1}\right]\\&-6\alpha_2(m-1)(m-2)\left[(n^2-r^2)^{m-3}\left[n^2+\frac{(2m-3)}{3}r^2\right]f(r)^2+\gamma\left(\frac{2}{3}(n^2-r^2)^{m-2}f(r)+\gamma\frac{(m-1)}{3m}W_{m-2}\right)\right]\end{aligned}$$

The Wheeler polynomial for $\sigma = 1$ in Einstein-Gauss-Bonnet gravity is

$$\begin{aligned}&a^{2m}r+\Lambda W_m+\alpha_1(m-1)\left[(n^2-r^2)^{m-1}f(r)+\gamma W_{m-1}\right]\\&-6\alpha_2(m-1)(m-2)\left[(n^2-r^2)^{m-3}\left[n^2+\frac{(2m-3)}{3}r^2\right]f(r)^2+\gamma\left(\frac{2}{3}(n^2-r^2)^{m-2}f(r)+\gamma\frac{1}{3}W_{m-2}\right)\right]=0\end{aligned}$$

For the $\mathbb{CP}$'s case, the contribution of the third order is

$$\begin{aligned}&12(m-1)(m-2)(m-3)\alpha_3\left((n^2-r^2)^{m-5}[5n^4+(4m-10)n^2r^2+\left(\frac{4(m-2)^2}{3}-\frac{1}{3}\right)r^4]f(r)^3\right.\\&\left.+\gamma(3n^2+(2m-5)r^2)(n^2-r^2)^{m-4}f(r)^2+\frac{m-1}{m}(n^2-r^2)^{m-3}f(r)+\frac{(m-1)(m-2)}{3m^2}W_{m-3}\right)\end{aligned}$$

For the $\mathbb{S}$'s case, the contribution of the third order is

$$\begin{aligned}&12(m-1)(m-2)(m-3)\alpha_3\left((n^2-r^2)^{m-5}[5n^4+(4m-10)n^2r^2+\left(\frac{4(m-2)^2}{3}-\frac{1}{3}\right)r^4]f(r)^3\right.\\&\left.+\gamma(3n^2+(2m-5)r^2)(n^2-r^2)^{m-4}f(r)^2+(n^2-r^2)^{m-3}f(r)+\frac{1}{3(2m-7)}W_{m-3}\right)\end{aligned}$$

2.3. Wheeler polynomial

$$a^{2m}r+\sum_{p=0}^{n}\frac{(-1)^{p-1}}{2}\frac{(2p)!}{p!}\frac{(m-1)!}{(m-1-p)!}F_p(r)\alpha_p=0 \tag{2.3.1}$$

where

$$F_p(r)\equiv\sum_{i=0}^{p}\frac{2p-1-(p-1)i}{2p-1}(n^2-r^2)^{m-2p+1+i}V_{p-i}f(r)^{p-i}\gamma^i \tag{2.3.2}$$

$$V_p\equiv\sum_{i=0}^{p}\frac{\binom{p-1}{i}\left(\frac{n^2}{r^2}\right)^{p-i-1}r^{2(p-1)}(2m-1-2p+2p)!(m-p-1)!}{\binom{i+2(p-1)}{2(p-1)}(2m-1-2p)!(m-1-p+i)!2^i} \tag{2.3.3}$$

$$V_0\equiv W_m=\sum_{i=0}^{m}(-1)^{m-1}\binom{m}{i}\left(-\frac{n^2}{r^2}\right)^{m-i}\frac{1}{2i-1}r^{2m} \tag{2.3.4}$$

2.4. Wheeler polynomial v2

$$\alpha_0 W_m + \alpha_1 \sum_{k=0}^{1} \frac{1}{m^{1-k}} \frac{m!}{(m-1+k)!} (n^2 - r^2)^{m-1-k+1} f(r)^k \tag{2.4.1}$$

$$+ \alpha_2 \left[\sum_{k=0}^{1} \frac{1}{m^{2-k}} \frac{m!}{(m-2+k)!} (n^2 - r^2)^{m-2-k+1} f(r)^k + H_{2,2}(n^2 - r^2)^{m-2-2+1} f(r)^2 \right] \tag{2.4.2}$$

$$+ \alpha_3 \left[\sum_{k=0}^{1} \frac{1}{m^{3-k}} \frac{m!}{(m-3+k)!} (n^2 - r^2)^{m-3-k+1} f(r)^k + \sum_{k=2}^{3} H_{3,k}(n^2 - r^2)^{m-3-k+1} f(r)^k \right] \tag{2.4.3}$$

$$\mu r + \alpha_0 W_m + \sum_{p=1}^{N} \alpha_p \left(\sum_{k=0}^{1} \frac{1}{m^{p-k}} \frac{m!}{(m-p+k)!} (n^2 - r^2)^{m-p-k+1} f(r)^k + \sum_{k=2}^{p} H_{p,k}(n^2 - r^2)^{m-p-k+1} f(r)^k \right)$$

$$H_{p,k} \equiv \sum_{j=1}^{k} \frac{(2m-1-2j-2(p-k))!!}{(2m-1-2k-2(p-k))!!} r^{2(k-j)} n^{2(j-1)} \tag{2.4.4}$$

$$W_m = \sum_{i=0}^{m} (-1)^{m-1} \binom{m}{i} \left(-\frac{n^2}{r^2} \right)^{m-i} \frac{1}{2i-1} r^{2m} \tag{2.4.5}$$

2.5. Wheeler polynomial v3

Para $\sigma = 0$:

$$\mu r + \sum_{p=0}^{N} \frac{(-1)^{p-1}}{2} \frac{(2p)!}{p!} \frac{(m-1)!}{(m-1-p)!} \alpha_p \left(\sum_{k=0}^{1} G_{p,k}(n^2 - r^2)^{m-p-k+1} f(r)^k + (1-\sigma) \sum_{k=2}^{p} H_{p,k}(n^2 - r^2)^{m-p-k+1} f \right. \tag{2.5.1}$$

donde

$$H_{p,k} \equiv \sum_{j=1}^{k} \frac{[2(m-j-p+k)-1]!!}{[2(m-p)-1]!!} r^{2(k-j)} n^{2(j-1)} \tag{2.5.2}$$

$$G_{p,k} \equiv \frac{1}{m^{p-k}} \frac{m!}{(m-p+k)!} W_{(p,m-p)}^{1-k} \tag{2.5.3}$$

$$W_{p,m} \equiv \sum_{j=0}^{m} (-1)^{j-1} \binom{m}{j} n^{2(m-j)} r^{2j} \frac{1}{2j-1} (n^2 - r^2)^{p-m-1} \tag{2.5.4}$$

2.6. Wheeler polynomial v4

Para $\sigma = 0$:

$$\mu r + \sum_{p=0}^{N} \frac{(-1)^{p-1}}{2} \frac{(2p)!}{p!} \frac{(m-1)!}{(m-1-p)!} \alpha_p \left((1-\sigma) \sum_{k=0}^{1} \gamma^{p-k} G_{p,k} + \sum_{k=2}^{p} \gamma^{p-k} H_{p,k} \right) = 0 \tag{2.6.1}$$

donde

$$H_{p,k} \equiv (n^2 - r^2)^{m-p-k+1} f(r)^k \sum_{j=1}^{k} \frac{[2(m-j-p+k)-1]!!}{[2(m-p)-1]!!} r^{2(k-j)} n^{2(j-1)} \tag{2.6.2}$$

$$G_{p,k} \equiv \frac{1}{m^{p-k}} \frac{m!}{(m-p+k)!} W^{1-k}_{(p,m-p)} (n^2 - r^2)^{m-p-k+1} f(r)^k \tag{2.6.3}$$

$$W_{p,m} \equiv \sum_{j=0}^{m} (-1)^{j-1} \binom{m}{j} n^{2(m-j)} r^{2j} \frac{1}{2j-1} (n^2 - r^2)^{-(m+1)} \tag{2.6.4}$$

2.7. Wheeler polynomial v5

Para $\sigma = 0$:

$$\mu r + \sum_{p=0}^{N} \frac{(-1)^{p-1}}{2} \frac{(2p)!}{p!} \frac{(m-1)!}{(m-1-p)!} \alpha_p \left(\sum_{k=0}^{1} \gamma^k H_{p,p-k} + \sum_{k=2}^{p-1} \gamma^k G_{p,k} \right) = 0 \tag{2.7.1}$$

donde

$$H_{p,k} \equiv (n^2 - r^2)^{m-p-k+1} f(r)^k \sum_{j=1}^{k} \frac{[2(m-j-p+k)-1]!!}{[2(m-p)-1]!!} r^{2(k-j)} n^{2(j-1)} \tag{2.7.2}$$

$$G_{p,k} \equiv \frac{m!}{m^k (m-k)!} (n^2 - r^2)^{m-p-k+1} f(r)^k \tag{2.7.3}$$

2.8. Wheeler polynomial v7

$$\mu r + \sum_{p=0}^{N} \frac{(-1)^{p-1}}{2} \frac{(2p)!}{p!} \frac{(m-1)!}{(m-1-p)!} \alpha_p \left(\sum_{k=0}^{p} \gamma^k H_{p,p-k} F_{p,k} + \sum_{k=2}^{p} \{(1-\sigma) G_{p,k} + \sigma S_{p,k} - H_{p,p-k}\} \gamma^k F_{p,k} \right) = 0 \tag{2.8.1}$$

o de manera equivalente

$$\mu r + \sum_{p=0}^{N} \frac{(-1)^{p-1}}{2} \frac{(2p)!}{p!} \frac{(m-1)!}{(m-1-p)!} \alpha_p \left(\sum_{k=0}^{1} \gamma^k H_{p,p-k} F_{p,k} + \sum_{k=2}^{p} \{(1-\sigma) G_{p,k} + \sigma S_{p,k}\} \gamma^k F_{p,k} \right) = 0 \tag{2.8.2}$$

donde

$$H_{p,k} := \sum_{j=1}^{k} \frac{[2(m-j-p+k)-1]!!}{[2(m-p)-1]!!} r^{2(k-j)} n^{2(j-1)} \tag{2.8.3}$$

$$G_{p,k} := \frac{m!}{m^k (m-k)!} \tag{2.8.4}$$

$$S_{p,k} := \frac{[2(m-k-p)+3]!!}{[2(m-p)-1]!!} \tag{2.8.5}$$

$$F_{p,k} := (1 - \delta_{p,k}) \left[(n^2 - r^2)^{m-2p+k+1} f(r)^{p-k} \right] + \delta_{p,k} W_{m-p} \tag{2.8.6}$$

$$W_m := \sum_{i=0}^{m} (-1)^{m-1} \binom{m}{i} \left(-\frac{n^2}{r^2} \right)^{m-i} \frac{1}{2i-1} r^{2m} \tag{2.8.7}$$

2.9. Wheeler polynomial F's

$$\mu r + \sum_{p=0}^{N} \frac{(-1)^{p-1}}{2} \frac{(2p)!}{p!} \frac{(m-1)!}{(m-1-p)!} \alpha_p F_p(r) = 0 \tag{2.9.1}$$

donde

$$
\begin{aligned}
F_0(r) &= W_m \\
F_1(r) &= (n^2 - r^2)^{m-1} f(r) + \gamma W_{m-1} \\
F_2(r) &= (n^2 - r^2)^{m-3} \left[n^2 + \frac{(2m-3)}{3} r^2 \right] f(r)^2 + \gamma \frac{2}{3} (n^2 - r^2)^{m-2} f(r) + \left((1-\sigma) \frac{(m-1)}{3m} + \frac{\sigma}{3} \right) W_{m-2} \\
F_3(3) &= (n^2 - r^2)^{m-5} \left[\frac{(2m-3)(2m-5)}{3} r^4 + 2(2m-5) n^2 r^2 + 5n^4 \right] f(r)^3 \\
&\quad + \gamma (n^2 - r^2)^{m-4} [3n^2 + (2m-5) r^2] f(r)^2 + \left(\frac{(1-\sigma)(m-1)}{m} + \sigma \right) (n^2 - r^2)^{m-3} f(r) \\
&\quad + \gamma \left(\frac{(1-\sigma)(m-1)(m-2)}{3m^2} + \frac{\sigma}{3(2m-7)} \right) W_{m-3} \\
W_m &:= \sum_{i=0}^{m} (-1)^{m-1} \binom{m}{i} \left(-\frac{n^2}{r^2} \right)^{m-i} \frac{1}{2i-1} r^{2m}
\end{aligned}
$$

2.10. Polynomial constraint

A property of the $\mathbb{CP}^k$ spaces is that they are Lovelock constant, that is, each tensor of the equations of motion is proportional to the metric. Here are some examples:

For $\mathbb{CP}^2$ in Einstein gravity we have

$$\begin{aligned} -\frac{\alpha_0}{2}\delta^\mu_\nu &= -\frac{\alpha_0}{2}\delta^\mu_\nu \\ \alpha_1\left(R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\right) &= -\alpha_1\delta^\mu_\nu \end{aligned}$$

For $\mathbb{CP}^3$ in Einstein-Gauss-Bonnet gravity,

$$\begin{aligned} -\frac{\alpha_0}{2}\delta^\mu_\nu &= -\frac{\alpha_0}{2}\delta^\mu_\nu \\ \alpha_1\left(R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\right) &= -2\alpha_1\delta^\mu_\nu \\ \alpha_2 H^\mu_\nu &= -3\alpha_2\delta^\mu_\nu \end{aligned}$$

For $\mathbb{CP}^4$ in cubic Lovelock,

$$\begin{aligned} -\frac{\alpha_0}{2}\delta^\mu_\nu &= -\frac{\alpha_0}{2}\delta^\mu_\nu \\ \alpha_1\left(R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\right) &= -3\alpha_1\delta^\mu_\nu \\ \alpha_2 H^\mu_\nu &= -\frac{48}{5}\alpha_2\delta^\mu_\nu \\ \alpha_3 M^\mu_\nu &= -\frac{288}{25}\alpha_3\delta^\mu_\nu \end{aligned}$$

where

$$H^\mu_\nu = -\frac{1}{8}\delta^{\mu\mu_1\ldots\mu_4}_{\nu\nu_1\ldots\nu_4} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} R^{\nu_3\nu_3}{}_{\mu_3\mu_4}$$

and

$$M^\mu_\nu = -\frac{1}{16}\delta^{\mu\mu_1\ldots\mu_6}_{\nu\nu_1\ldots\nu_6} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} R^{\nu_3\nu_3}{}_{\mu_3\mu_4} R^{\nu_5\nu_6}{}_{\mu_5\mu_6}$$

In this way, the $\mathbb{CP}^k$ spaces are solution of the Lovelock equations in $D = 2k$ dimensions up to a polynomial constraint on the Lovelock couplings α_p. For intance, the following are the constraints to order $1, 2$ and 3 in the Lovelock series for $\mathbb{CP}^2, \mathbb{CP}^3$, $\mathbb{CP}^4, \mathbb{CP}^5$ and $\mathbb{CP}^6$ respectively

$$\begin{aligned} \frac{\alpha_0}{2} + \alpha_1 &= 0 \\ \frac{\alpha_0}{2} + 2\alpha_1 + 3\alpha_2 &= 0 \\ \frac{\alpha_0}{2} + 3\alpha_1 + \frac{48}{5}\alpha_2 + \frac{288}{25}\alpha_3 &= 0 \\ \frac{\alpha_0}{2} + 4\alpha_1 + 20\alpha_2 + \frac{160}{3}\alpha_3 &= 0 \\ \frac{\alpha_0}{2} + 5\alpha_1 + \frac{240}{7}\alpha_2 + \frac{7200}{49}\alpha_3 &= 0 \end{aligned}$$

In general, for Einstein gravity we have that $\mathbb{CP}^k$ spaces satisfy the the equations of motion up to the following polynomial constraint

$$\frac{\alpha_0}{2} + (k-1)\alpha_1 = 0$$

The polynomial constraint for cubic Lovelock gravity is

$$\frac{\alpha_0}{2} + (k-1)\alpha_1 + \frac{2k(k-1)(k-2)}{k+1}\alpha_2 + \frac{4k(k-1)^2(k-2)(k-3)}{(k+1)^2}\alpha_3 = 0$$

or equivantly

$$\frac{\alpha_0}{2k}\binom{k}{1} + \frac{2\alpha_1}{k}\binom{k}{2} + \frac{12\alpha_2}{k+1}\binom{k}{3} + \frac{96\alpha_3(k-1)}{(k+1)^2}\binom{k}{4} = 0$$

In a more compact way, the polynomial constraint so that the $\mathbb{CP}^k(\gamma = 1)$ and $\mathbb{CH}^k(\gamma = -1)$ satisfy the Lovelock equations is

$$\frac{1}{k}\sum_{p=0}^{n}\binom{k}{k-p}^2 \frac{l!^2(k-p)}{(k-p+1)}\left(\frac{2}{k+1}\right)^{p-1}\alpha_p = 0$$

or equivantely

$$\frac{k!^2}{k}\sum_{p=0}^{n}\frac{\alpha_p}{(k-p-1)!(k-p+1)!}\left(\frac{2\gamma}{k+1}\right)^{p-1} = 0$$

The Lovelock constant property reads as

$$-\frac{1}{2^{p+1}}\delta^{\mu\mu_1\ldots\mu_{2p}}_{\nu\nu_1\ldots\nu_{2p}} R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_{2p-1}\nu_{2p}}_{\phantom{\nu_{2p-1}\nu_{2p}}\mu_{2p-1}\mu_{2p}} = \frac{k!^2}{k}\frac{1}{(k-p-1)!(k-p+1)!}\left(\frac{2}{k+1}\right)^{p-1}\delta^{\mu}_{\nu}$$

In a more compact way, the polynomial constraint so that the $\mathbb{CP}^k(\gamma = 1)$ and $\mathbb{CH}^k(\gamma = -1)$ satisfy the Lovelock equations is

$$\frac{k!^2}{k}\sum_{p=0}^{n}\frac{\alpha_p}{(k-p-1)!(k-p+1)!}\left(\frac{2\gamma}{k+1}\right)^{p-1} = 0$$

The Einstein-Hilbert equations of motion are

$$-\sum_{p=0}^{1}\frac{\alpha_p}{2^{p+1}}\delta^{\mu\mu_1\mu_2}_{\nu\nu_1\nu_2}R^{\nu_1\nu_2}_{\mu_1\mu_2}=-\frac{\alpha_0}{2}\delta^{\mu}_{\nu}+\alpha_1\left(R^{\mu}_{\nu}-\frac{1}{2}R\delta^{\mu}_{\nu}\right)$$

but they can also be written as

$$-\frac{1}{4}\delta^{\mu\mu_1\mu_2}_{\nu\nu_1\nu_2}\left(R^{\nu_1\nu_2}_{\mu_1\mu_2}+c_1^{(1)}\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\right)=R^{\mu}_{\nu}-\frac{1}{2}R\delta^{\mu}_{\nu}-10c_1^{(1)}\delta^{\mu}_{\nu}$$

2.11. Otro approach

Consideremos $n=0$, para Einstein grav tenemos

$$(m-1)\left(f(r)\,r^{2m-2}-\frac{r^{2m-2}}{2m-3}\right) \tag{2.11.1}$$

2.12. Lovelock constant

Capítulo 3

First order formalism

Let us consider the metric ansatz constructed as the $U(1)$ fibration of Einstein-Kähler manifolds

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\mathrm{d}r^2}{f(r)} + f(r)h(r)(\mathrm{d}\tau + \mathcal{B})^2 + N(r)\,\mathrm{d}\Sigma^2\ , \tag{3.0.1}$$

where $\mathcal{B} = \mathcal{B}_i \mathrm{d}\bar{x}^i$ denotes the Kähler potential 1-form that defines the symplectic structure $\Omega = \mathrm{d}\mathcal{B}$ associated to the $(2m-2)$-dimensional Einstein-Kähler trnasverse with line element

$$\mathrm{d}\Sigma^2 = g_{ij}(\bar{x})\mathrm{d}\bar{x}^i \mathrm{d}\bar{x}^j \tag{3.0.2}$$

Here, barred coordinates with Latin indices represent those that belong to the transverse section. Thus, coordinates with greek indices can be $x^\mu = \{r, t, \bar{x}^i\}$.

It is straightforward to see that the non-zero metric components are given by

$$g_{rr} = \frac{1}{f}, \quad g_{tt} = fh, \quad g_{ti} = fh\mathcal{B}_i, \quad g_{ij} = fh\mathcal{B}_i\mathcal{B}_j + N\bar{g}_{ij} \tag{3.0.3}$$

In order to find the components of the inverse metric, we use the fact that $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$, resulting in the following system of equations

$$\begin{cases} 1 &= g^{rr}g_{rr} \\ 1 &= fh(g^{tt} + g^{ti}\mathcal{B}_i) \\ \delta^i_j &= fh\mathcal{B}_j(g^{ti} + g^{ik}\mathcal{B}_k) + Ng^{ik}\bar{g}_{kj} \\ 0 &= fh\mathcal{B}_i(g^{tt} + g^{tk}\mathcal{B}_k) + Ng^{tk}\bar{g}_{ki} \end{cases} \tag{3.0.4}$$

whose solution is given by

$$g^{rr} = 1 \tag{3.0.5}$$

$$g^{tt} = \frac{\mathcal{B}_i\mathcal{B}_j}{N}\bar{g}^{ij} + \frac{1}{fh} \tag{3.0.6}$$

$$g^{ti} = -\frac{1}{N}\mathcal{B}_j\bar{g}^{ij} \tag{3.0.7}$$

$$g^{ij} = \frac{1}{N}\bar{g}^{ij} \tag{3.0.8}$$

We define the orthonormal noncoordinate frame basis of the metric () as

$$e^0 = \frac{\mathrm{d}r}{\sqrt{f(r)}}\,, \qquad e^1 = \sqrt{f(r)h(r)}(\mathrm{d}\tau + \mathcal{B})\,, \qquad e^A = \sqrt{N(r)}\,\bar{e}^A\,, \tag{3.0.9}$$

$$\omega^{01} = -\frac{(fh)'}{2h\sqrt{f}}e^1\,, \quad \omega^{0i} = -\frac{\sqrt{f}N'}{2N}e^i\,, \quad \omega^{1i} = \frac{\sqrt{fh}}{2N}\Omega^i{}_j e^j\,, \quad \omega^{ij} = \bar{\omega}^{ij} - \frac{\sqrt{fh}}{2N}\Omega^{ij}e^1\,. \tag{3.0.10}$$

$$E_0 = \sqrt{f}\partial_r \tag{3.0.11}$$

$$E_1 = \frac{1}{\sqrt{fh}}\partial_t \tag{3.0.12}$$

$$E_A = -\frac{1}{\sqrt{N}}B_j\bar{E}^j_A\partial_t + \frac{1}{\sqrt{N}}\bar{E}^j_A\partial_j \tag{3.0.13}$$

$$R^{rt}_{rt} = -\frac{1}{2\sqrt{h}}\left[\frac{(fh)'}{\sqrt{h}}\right]' \tag{3.0.14}$$

$$R^{ij}_{rt} = -\frac{1}{2N}\left[\frac{fh}{N}\right]'\Omega^{ij} \tag{3.0.15}$$

$$R^{rt}_{ij} = -\frac{N}{2h}\left[\frac{fh}{N}\right]'\Omega_{ij} \tag{3.0.16}$$

$$R^{ri}_{rj} = -\frac{1}{2}\sqrt{\frac{f}{N}}\left[\frac{\sqrt{f}N'}{\sqrt{N}}\right]'\delta^i_j \tag{3.0.17}$$

$$R^{\tau i}_{\tau j} = \frac{fh}{4N^2}\Omega_k{}^i\Omega^k{}_j - \frac{N'(fh)'}{4Nh}\delta^i_j \tag{3.0.18}$$

$$R^{ri}_{tj} = -\frac{f}{4}\left[\frac{fh}{N}\right]'\Omega^i{}_j \tag{3.0.19}$$

$$R^{\tau i}_{rj} = \frac{1}{4fh}\left[\frac{fh}{N}\right]'\Omega^i{}_j \tag{3.0.20}$$

$$R^{ij}_{kl} = \frac{1}{N}\bar{R}^{ij}_{kl} - \frac{fN'^2}{2N^2}\delta^i_{[k}\delta^j_{l]} - \frac{fh}{4N^2}\left(\Omega^{ij}\Omega_{kl} + 4\Omega^{[i}{}_{[k}\Omega^{j]}{}_{l]}\right) \tag{3.0.21}$$

$$R^{ij}_{\tau k} \overset{?}{=} 0 \tag{3.0.22}$$

$$R^{ri}_{r\tau} = 0 \tag{3.0.23}$$

3.1. Connection components

$$\Gamma^r{}_{\tau\tau} = -\frac{1}{2}f(fh)' \checkmark \tag{3.1.1}$$

$$\Gamma^r{}_{rr} = -\frac{f'}{2f} \checkmark \tag{3.1.2}$$

$$\Gamma^r{}_{ij} = -\frac{f(fh)'}{2}\mathcal{B}_i\mathcal{B}_j - \frac{fN'}{2}\bar{g}_{ij} \tag{3.1.3}$$

$$\Gamma^r_{\tau i} = -\frac{f}{2}B_i(fh)' \tag{3.1.4}$$

$$\Gamma^\tau{}_{\tau r} = \frac{1}{2}\frac{(fh)'}{fh} \tag{3.1.5}$$

$$\Gamma^\tau{}_{\tau i} = -\frac{fh}{2N}B^k\Omega_{ik} \tag{3.1.6}$$

$$\Gamma^\tau{}_{ri} = \frac{B_i}{2}\left[\frac{(fh)'}{fh} - \frac{N'}{N}\right] = \frac{B_i}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\ln\left(\frac{fh}{N}\right)\right] \tag{3.1.7}$$

$$\Gamma^\tau_{ij} = \partial_{(i}B_{j)} - B_k\bar{\Gamma}^k{}_{ij} + \frac{fh}{N}\Omega_{k(i}\mathcal{B}_{j)}\mathcal{B}^k \tag{3.1.8}$$

$$\Gamma^i{}_{\tau j} = -\frac{fh}{2N}\Omega^i{}_j \tag{3.1.9}$$

$$\Gamma^i{}_{rj} = \frac{1}{2}\frac{N'}{N}\delta^i_j \tag{3.1.10}$$

$$\Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk} + \frac{fh}{N}\mathcal{B}_{(j}\Omega_{k)}^{\ \ i} \tag{3.1.11}$$

where $\Omega_{ij} = 2\partial_{[i}B_{j]}$.

3.2. Curvature components

$$R^r{}_{trt} = \partial_r\left[-\frac{1}{2}f(fh)'\right] + \frac{f'(fh)'}{4} + \frac{(fh)'^2}{4h} \tag{3.2.1}$$

3.3. Invariant of the transverse section

We consider

$$\bar{\mathcal{L}}^{(1)} = \bar{R} \tag{3.3.1}$$

For the case of 2-sphere products, we have

$$(\mathbb{S}^2)^1 : \mathcal{L}^{(1)} = 2 \tag{3.3.2}$$
$$(\mathbb{S}^2)^2 : \mathcal{L}^{(1)} = 4 \tag{3.3.3}$$
$$(\mathbb{S}^2)^3 : \mathcal{L}^{(1)} = 6 \tag{3.3.4}$$
$$(\mathbb{S}^2)^4 : \mathcal{L}^{(1)} = 8 \tag{3.3.5}$$
$$\vdots \tag{3.3.6}$$
$$(\mathbb{S}^2)^k : \mathcal{L}^{(1)} = 2k \tag{3.3.7}$$

For the case of $\mathbb{CP}^k$, we have

$$\mathbb{CP}^1 : \mathcal{L}^{(1)} = 2 \tag{3.3.8}$$
$$\mathbb{CP}^2 : \mathcal{L}^{(1)} = 4 \tag{3.3.9}$$
$$\mathbb{CP}^3 : \mathcal{L}^{(1)} = 6 \tag{3.3.10}$$
$$\vdots \tag{3.3.11}$$
$$\mathbb{CP}^k : \mathcal{L}^{(1)} = 2k \tag{3.3.12}$$

For the case of $(\mathbb{H}^2)^k$ we obtain $\bar{R} = -2k$.

Now, we consider

$$\bar{\mathcal{L}}^{(2)} = \bar{R}^{\mu\nu}_{\lambda\rho}\bar{R}^{\lambda\rho}_{\mu\nu} - 4\bar{R}^{\mu}_{\nu}\bar{R}^{\nu}_{\mu} + \bar{R}^2$$

and we obtain For the case of 2-sphere products, we have

$$(\mathbb{S}^2)^1 : \mathcal{L}^{(2)} = 0 \tag{3.3.13}$$
$$(\mathbb{S}^2)^2 : \mathcal{L}^{(2)} = 8 \tag{3.3.14}$$
$$(\mathbb{S}^2)^3 : \mathcal{L}^{(2)} = 24 \tag{3.3.15}$$
$$(\mathbb{S}^2)^4 : \mathcal{L}^{(2)} = 48 \tag{3.3.16}$$
$$(\mathbb{S}^2)^5 : \mathcal{L}^{(2)} = 80 \tag{3.3.17}$$
$$\vdots \tag{3.3.18}$$
$$(\mathbb{S}^2)^k \text{ y } (\mathbb{H}^2)^k : \mathcal{L}^{(2)} = 4k(k-1) \tag{3.3.19}$$

For the case of $\mathbb{CP}^k$, we have

$$\mathbb{CP}^1 : \mathcal{L}^{(2)} = 0 \tag{3.3.20}$$
$$\mathbb{CP}^2 : \mathcal{L}^{(2)} = \frac{16}{3} \tag{3.3.21}$$
$$\mathbb{CP}^3 : \mathcal{L}^{(2)} = 18 = \frac{72}{4} \tag{3.3.22}$$
$$\mathbb{CP}^4 : \mathcal{L}^{(2)} = \frac{192}{5} \tag{3.3.23}$$
$$\mathbb{CP}^5 : \mathcal{L}^{(2)} = \frac{200}{3} = \frac{400}{6} \tag{3.3.24}$$
$$\mathbb{CP}^6 : \mathcal{L}^{(2)} = \frac{200}{3} = \frac{720}{7} \tag{3.3.25}$$
$$\vdots \tag{3.3.26}$$
$$\mathbb{CP}^k \text{ y } \mathbb{CH}^k : \mathcal{L}^{(2)} = \frac{4k^2(k-1)}{k+1} \tag{3.3.27}$$

Now, we consider

$$\bar{\mathcal{L}}^{(3)}$$

For the case of 2-sphere products, we have

$$(\mathbb{S}^2)^1 : \mathcal{L}^{(1)} = 0 \tag{3.3.28}$$
$$(\mathbb{S}^2)^2 : \mathcal{L}^{(1)} = 0 \tag{3.3.29}$$
$$(\mathbb{S}^2)^3 : \mathcal{L}^{(1)} = 48 \tag{3.3.30}$$
$$(\mathbb{S}^2)^4 : \mathcal{L}^{(1)} = 192 \tag{3.3.31}$$
$$(\mathbb{S}^2)^5 : \mathcal{L}^{(1)} = 480 \tag{3.3.32}$$
$$(\mathbb{S}^2)^6 : \mathcal{L}^{(1)} = 960 \tag{3.3.33}$$
$$\vdots \tag{3.3.34}$$
$$(\mathbb{S}^2)^k : \mathcal{L}^{(3)} = 8k(k-1)(k-2) = 8(k-1)[(k-1)^2 - 1] \tag{3.3.35}$$

For the case of 2-hyperboloids products, we have

$$(\mathbb{H}^2)^1 : \mathcal{L}^{(1)} = 0 \tag{3.3.36}$$
$$(\mathbb{H}^2)^2 : \mathcal{L}^{(1)} = 0 \tag{3.3.37}$$
$$(\mathbb{H}^2)^3 : \mathcal{L}^{(1)} = -48 \tag{3.3.38}$$
$$(\mathbb{H}^2)^4 : \mathcal{L}^{(1)} = -192 \tag{3.3.39}$$
$$\vdots \tag{3.3.40}$$
$$(\mathbb{H}^2)^k : \mathcal{L}^{(3)} = -8k(k-1)(k-2) = -8(k-1)[(k-1)^2 - 1] \tag{3.3.41}$$

For the case of $\mathbb{CP}^k$ products, we have

$$\mathbb{CP}^1 : \mathcal{L}^{(3)} = 0 \tag{3.3.42}$$

$$\mathbb{CP}^2 : \mathcal{L}^{(3)} = 0 \tag{3.3.43}$$

$$\mathbb{CP}^3 : \mathcal{L}^{(3)} = 18 = \frac{288}{16} \tag{3.3.44}$$

$$\mathbb{CP}^4 : \mathcal{L}^{(3)} = \frac{2304}{25} \tag{3.3.45}$$

$$\mathbb{CP}^5 : \mathcal{L}^{(3)} = \frac{800}{3} = \frac{9600}{36} \tag{3.3.46}$$

$$\mathbb{CP}^6 : \mathcal{L}^{(3)} = \frac{28800}{49} \tag{3.3.47}$$

$$\vdots \tag{3.3.48}$$

$$\mathbb{CP}^k : \mathcal{L}^{(3)} = \frac{8k^2(k-1)^2(k-2)}{(k+1)^2} \tag{3.3.49}$$

For the case of $\mathbb{CH}^k$ products, we have

$$\mathbb{CH}^k : \mathcal{L}^{(3)} = -\frac{8k^2(k-1)^2(k-2)}{(k+1)^2} \tag{3.3.50}$$