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Zero-weight scalar fields

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Abstract: Personal compilation of some calculations related to zero-weight conformal scalar fields.

Contents

1 Building the equation of motion

Let us consider a scalar field with zero conformal weight that is, under an infinitesimal conformal transformation

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad \delta_\omega \phi = 0 \tag{1.1}$$

Let us consider the most general second-order pseudoscalar constructed from the scalar field ϕ and its derivatives up to second order, together with the metric tensor and its associates curvature

$$\mathcal{E} = \sqrt{-g}E\left(\phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi, g_{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = 0 \tag{1.2}$$

The variation of this equation under infinitesimal conformal transformations reads,

$$\delta_\omega \mathcal{E} = \delta_\omega(\sqrt{-g})E + \sqrt{-g}\delta_\omega E \tag{1.3}$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\delta_\omega E \tag{1.4}$$

From $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta_\omega(g_{\mu\nu}g^{\mu\nu}) = 0$ implies that

$$g_{\mu\nu}\delta_\omega g^{\mu\nu} = -g^{\mu\nu}\delta_\omega g_{\mu\nu} \tag{1.5}$$

Furthermore, from (A.3) and (A.15), we have

$$\delta_\omega \mathcal{E} = \frac{\partial E}{\partial \phi}\cancelto{0}{\delta_\omega \phi} + \frac{\partial E}{\partial(\nabla_\mu\phi)}\cancelto{0}{\delta_\omega \nabla_\mu \phi} + \frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\delta_\omega(\nabla_\mu\nabla_\nu\phi) + \frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \tag{1.6}$$

where we have defined

$$P^{\mu\nu}_{\alpha\beta} := \frac{\partial E}{\partial R^{\alpha\beta}_{\mu\nu}}, \qquad P^{\mu}_{\nu} := P^{\alpha\mu}_{\alpha\nu} \tag{1.7}$$

So then

$$\begin{aligned}
\delta_\omega \mathcal{E} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu} E + \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\delta_\omega(\nabla_\mu\nabla_\nu\phi) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \\
&= \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_{(\mu}\phi\nabla_{\nu)}\omega\right) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \\
&- \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu} E \\
&= \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_{(\mu}\phi\nabla_{\nu)}\omega\right) + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} - 2\omega g^{\mu\nu}\sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}} && (1.8) \\
&= \sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) && (1.9)
\end{aligned}$$

By defining

$$A^{\mu\nu} := \frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}, \qquad B_{\mu\nu} = -2\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} \tag{1.10}$$

we obtain

$$\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g}A^{\mu\nu}\left(\gamma_{\mu\nu}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_\mu\phi\nabla_\nu\omega\right) + \sqrt{-g}P^{\mu\nu}\delta_\omega R^{\alpha\beta}_{\mu\nu} + \sqrt{-g}\omega B_{\mu\nu}g_{\mu\nu} && (1.11) \\
&= \sqrt{-g}\left(A\nabla^\rho\phi\nabla_\rho\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu}\right) && (1.12)
\end{aligned}$$

Using

$$\delta_\omega R^{\alpha}{}_{\beta\mu\nu} = 2\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta} \tag{1.13}$$

we have

$$\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g}\left(A\nabla^\rho\phi\nabla_\rho\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta}\right) && (1.14) \\
&= \sqrt{-g}\left(A\nabla^\nu\phi\nabla_\nu\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta}\right) && (1.15) \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta}\right] && (1.16)
\end{aligned}$$

Noting that

$$\begin{aligned}
\delta_\omega\Gamma^{\alpha}{}_{\nu\beta} &= \frac{1}{2}g^{\alpha\lambda}\left[\nabla_\nu(\delta_\omega g_{\beta\lambda}) + \nabla_\beta(\delta_\omega g_{\nu\lambda}) - \nabla_\lambda(\delta_\omega g_{\nu\beta})\right] && (1.17) \\
&= g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + g_{\nu\lambda}\nabla_\beta\omega - g_{\nu\beta}\nabla_\lambda\omega\right) && (1.18) \\
&= g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega\right) && (1.19)
\end{aligned}$$

we obtain

$$
\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_\mu g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega\right)\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P^{\lambda\beta\mu\nu}\nabla_\mu\left(\cancel{g_{\beta\lambda}}\nabla_\nu\omega + 2g_{\nu\lambda}\nabla_\beta\omega\right)\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 4P^{\lambda\beta\mu\nu}\nabla_\mu g_{\nu\lambda}\nabla_\beta\omega\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B - 4P^{\mu\nu}\nabla_\mu\nabla_\nu\omega\right]
\end{aligned}
$$

Imposing $\delta_\omega\mathcal{E} = 0$ for all ω, we obtain the following conditions,

$$A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi = 0 \tag{1.20a}$$

$$B = 0 \tag{1.20b}$$

$$P^{\mu\nu} = 0 \tag{1.20c}$$

From (A.21),

$$\frac{\partial E}{\partial R^{\mu\nu}_{\alpha\beta}} = P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu}\hat{I}^{\beta]}_{\nu]} + J\delta^\alpha_{[\mu}\delta^\beta_{\nu]} \tag{1.21}$$

we notice that since $\hat{H}^{\alpha\beta}_{\mu\nu}$ is the traceless part of $H^{\alpha\beta}_{\mu\nu}$,

$$P^\beta_\nu = P^{\alpha\beta}_{\alpha\nu} = \delta^{[\alpha}_{[\alpha}\hat{I}^{\beta]}_{\nu]} + J\delta^\alpha_{[\alpha}\delta^\beta_{\nu]} \tag{1.22}$$

Since the variation with respect to the Weyl tensor does not contribute to the first trace (1.21), the contribution of the Riemann tensor to E has to be through its traceless part,

$$\mathcal{E} = \sqrt{-g}E\left(\phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi, g_{\mu\nu}, C^{\alpha\beta}_{\mu\nu}\right) = 0\,. \tag{1.23}$$

Example 1.1. *Let us consider the following action principle*

$$S[\phi, g] = \int \mathrm{d}^Dx\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right)^{D/2} \tag{1.24}$$

$$= \int \mathrm{d}^Dx\sqrt{-g}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{D/2} \tag{1.25}$$

$$= \int \mathrm{d}^Dx\sqrt{-g}X^{D/2} \tag{1.26}$$

where we have defined $X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$.

Now, we must to find E*. Varying with respect to* ϕ*,*

$$\delta_\phi S = -\int \mathrm{d}^Dx\sqrt{-g}\frac{D}{2}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\nabla_\mu\delta\phi \tag{1.27}$$

$$= \int \mathrm{d}^Dx\sqrt{-g}\frac{D}{2}\nabla_\mu\left[\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\right]\delta\phi + \mathrm{b.t} \tag{1.28}$$

therefore,

$$E = \nabla_\mu \left[\left(-\frac{1}{2}(\nabla\phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \right] \tag{1.29}$$

$$= \frac{D-2}{2} \left(-\frac{1}{2}(\nabla\phi)^2 \right)^{\frac{D-4}{2}} (-1)\nabla^\alpha\phi\nabla_\mu\nabla_\alpha\phi\nabla^\mu\phi + \left(-\frac{1}{2}(\nabla\phi)^2 \right)^{\frac{D-2}{2}} \Box\phi \tag{1.30}$$

which can be rewritten as

$$E = X^{\frac{D-2}{2}}\Box\phi - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi \tag{1.31}$$

Let's see how (1.20a) *looks,*

$$A^{\mu\nu} = \frac{\partial}{\partial(\nabla_\mu\phi\nabla_\nu\phi)} \left[X^{\frac{D-2}{2}} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi \right] \tag{1.32}$$

$$= X^{\frac{D-2}{2}} g^{\mu\nu} - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi \tag{1.33}$$

and its trace yields

$$A = DX^{\frac{D-2}{2}} - \frac{D-2}{2}X^{\frac{D-4}{2}}(\nabla\phi)^2 \tag{1.34}$$

$$= DX^{\frac{D-2}{2}} + (D-2)X^{\frac{D-4}{2}}\left(-\frac{1}{2}(\nabla\phi)^2 \right) \tag{1.35}$$

$$= DX^{\frac{D-2}{2}} + (D-2)X^{\frac{D-2}{2}} \tag{1.36}$$

$$= 2(D-1)X^{\frac{D-2}{2}} \tag{1.37}$$

Pluggin' into (1.20a),

$$\begin{aligned} A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi &= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}g^{\mu\nu}\nabla_\mu\phi + (D-2)X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\phi \\ &= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}\nabla^\nu\phi + (D-2)X^{\frac{D-4}{2}}(\nabla\phi)^2\nabla^\nu\phi \\ &= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-4}{2}}X \right]\nabla^\nu\phi \\ &= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-2}{2}}2 \right]\nabla^\nu\phi \\ &= 0 \qquad \checkmark \end{aligned}$$

2 Building the auxiliary metric

We know the conditions that the most general second-order equation of motion for the zero conformal weight scalar field must satisfy. They are given by (1.20). Now, the question is: how do we construct an auxiliary metric $\tilde{g}_{\mu\nu}$ such that $\delta_\omega\tilde{g}_{\mu\nu} = 0$?

Remember that in the exponential frame for the scalar field, the conformal transformations look like

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \mathrm{e}^{2\omega}g_{\mu\nu}, \qquad \phi \to \bar{\phi} = \phi \tag{2.1}$$

Since the inverse metric transforms as

$$g^{\mu\nu} \to \bar{g}^{\mu\nu} = \mathrm{e}^{-2\omega} g^{\mu\nu} \tag{2.2}$$

the kinetic term for the scalar field, defined as

$$X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{2.3}$$

transforms as

$$\bar{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = -\frac{1}{2}\mathrm{e}^{-2\omega}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \mathrm{e}^{-2\omega}X \tag{2.4}$$

Thus, the auxiliary metric is defined as

$$\tilde{g}_{\mu\nu} = X g_{\mu\nu} = -\frac{1}{2}(\nabla\phi)^2 g_{\mu\nu} \quad \Longrightarrow \quad \delta_\omega \tilde{g}_{\mu\nu} = 0 \tag{2.5}$$

is conformally invariant.

Let us consider a pseudoscalar built from the zero weight conformal scalar field and its derivatives up to second-order and the conformally invariant geometry,

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}) \tag{2.6}$$

where we used the fact that for zero-weight, the $P^{\mu\nu} = 0$ condition implies that the only dependence on the curvature is through the Weyl tensor, which is conformally invariant.

We notice that $\delta_\omega\mathcal{E} = 0$. Indeed

$$\begin{aligned}\delta_\omega\mathcal{E} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}E\cancel{\delta_\omega\tilde{g}_{\mu\nu}} + \sqrt{-\tilde{g}}\left(\frac{\partial E}{\partial\phi}\cancel{\delta_\omega\phi} + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\omega(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right.\\ \left. + \frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\cancel{\delta_\omega\tilde{g}^{\mu\nu}} + \tilde{P}^{\mu\nu}_{\alpha\beta}\cancel{\delta_\omega\tilde{C}^{\alpha\beta}_{\mu\nu}}\right)\end{aligned}$$

but

$$\delta_\omega\tilde{\nabla}_\mu\phi = \delta_\omega\partial_\mu\phi = \partial_\mu\delta_\omega\phi = 0 \tag{2.7}$$

and

$$\begin{aligned}\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) &= \delta_\omega(\tilde{\nabla}_\mu\partial_\nu\phi) && (2.8)\\ &= \delta_\omega(\partial_\mu\partial_\nu\phi - \tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi) && (2.9)\\ &= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi - \tilde{\Gamma}^\lambda_{\mu\nu}\partial_\lambda\delta_\omega\phi && (2.10)\\ &= -\delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi && (2.11)\\ &= 0 && (2.12)\end{aligned}$$

since $\delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu} = 0$. Therefore,

$$\delta_\omega\mathcal{E} = 0\,. \tag{2.13}$$

3 Fréchet derivative

Now, let's try to find an action principle from which it emerges (**??**). To do that, we will use the formalism used in [1]. Hence, first we must to introduce the concept of Fréchet derivative.

Let $P[u] = P\left(x, u^{(n)}\right)$ be a differential function, i.e. that depends in the point x, the function u together with its derivatives $u^{(n)}$. Consider now its variation under a one-parameter family of functions. After interchanging the variation with derivatives (and without integrating by parts) we end with a differential operator acting on an arbitrary variation δu called the *Fréchet derivative* of P,

$$\delta P = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, P[u + \varepsilon \delta u]\bigg|_{\varepsilon=0} := D_P(\delta u) \tag{3.1}$$

Here, we consider the second order conformally invariant psudoscalar defined by

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu \phi, \tilde{\nabla}_\mu \nabla_\nu \phi, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}) \tag{3.2}$$

which is the natural quantity that could be derived from a convariant action. The role of he dependent function u is played by the zero weight conformally invariant scalar field ϕ, and hence the Fréchet derivative of $\mathcal{E}$ can be calculated from

$$D_{\mathcal{E}}(\delta\phi) = \delta_\phi \mathcal{E}\,. \tag{3.3}$$

From (3.2) and using that $\delta\phi = 0$, we have

$$\begin{aligned}
\delta_\phi \mathcal{E} = &-\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi \tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left(\frac{\partial E}{\partial \phi}\delta\phi \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\phi(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right. \\
&\left. + \frac{\partial E}{\partial \tilde{g}^{\mu\nu}}\delta_\phi \tilde{g}^{\mu\nu} + \tilde{P}^{\mu\nu}_{\alpha\beta}\delta_\phi \tilde{C}^{\alpha\beta}_{\mu\nu}\right) \\
= &-\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi \tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial \tilde{g}^{\mu\nu}}\delta_\phi \tilde{g}^{\mu\nu} + E_\phi \delta\phi + E^\mu_\phi \delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi \delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) + \tilde{P}^{\mu\nu}_{\alpha\beta}\cancel{\delta_\phi \tilde{C}^{\alpha\beta}_{\mu\nu}}\right] \\
= &-\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi \tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial \tilde{g}^{\mu\nu}}\delta_\phi \tilde{g}^{\mu\nu} + E_\phi \delta\phi + E^\mu_\phi \delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi \delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right]
\end{aligned}$$

where we have defined

$$E_\phi := \frac{\partial E}{\partial \phi}, \qquad E^\mu_\phi := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}, \qquad E^{\mu\nu}_\phi := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} \tag{3.4}$$

in order to reduce some notation and we used the fact that the Weyl tensor is conformally invariant, $\tilde{C}^{\alpha\beta}_{\mu\nu} = C^{\alpha\beta}_{\mu\nu}$, so that $\delta_\phi C^{\alpha\beta}_{\mu\nu} = 0$. Then,

$$\delta_\phi \mathcal{E} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi \tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial \tilde{g}^{\mu\nu}}\delta_\phi \tilde{g}^{\mu\nu} + E_\phi \delta\phi + E^\mu_\phi \delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi \delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right] \tag{3.5}$$

Using (1.9), we can write

$$\delta_\phi \mathcal{E} = \sqrt{-\tilde{g}}\left[\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}E)}{\partial \tilde{g}^{\mu\nu}}\delta_\phi \tilde{g}^{\mu\nu} + E_\phi \delta\phi + E_\phi^\mu \delta_\phi(\tilde{\nabla}_\mu \phi) + E_\phi^{\mu\nu}\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right] \tag{3.6}$$

$$= \sqrt{-\tilde{g}}\left[-\frac{1}{2}E_{\mu\nu} + E_\phi \delta\phi + E_\phi^\mu \delta_\phi(\tilde{\nabla}_\mu \phi) + E_\phi^{\mu\nu}\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right] \tag{3.7}$$

where

$$E_{\mu\nu} := -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}E)}{\partial \tilde{g}^{\mu\nu}} \tag{3.8}$$

Before proceeding, let's compute the variation w.r.t the scalar field ϕ of some quantities:

$$\delta_\phi \tilde{g}_{\mu\nu} = \delta_\phi(X g_{\mu\nu}) \tag{3.9}$$

$$= \delta_\phi\left(-\frac{1}{2}(\nabla\phi)^2 g_{\mu\nu}\right) \tag{3.10}$$

$$= -\nabla^\alpha\phi\nabla_\alpha\delta\phi g_{\mu\nu} \tag{3.11}$$

$$= -g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\delta\phi \tag{3.12}$$

$$= -g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.13}$$

$$= -g_{\mu\nu}Xg^{\alpha\beta}X^{-1}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.14}$$

$$= -\tilde{g}_{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.15}$$

$$\delta_\phi \tilde{g}^{\mu\nu} = \delta_\phi\left[\left(-\frac{1}{2}(\nabla\phi)^2\right)^{-1} g^{\mu\nu}\right] \tag{3.16}$$

$$= -\frac{1}{X^2}(-\nabla^\alpha\phi\nabla_\alpha\delta\phi)g^{\mu\nu} \tag{3.17}$$

$$= \frac{1}{X^2}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.18}$$

$$= X^{-1}g^{\mu\nu}X^{-1}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.19}$$

$$= \tilde{g}^{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.20}$$

$$\delta_\phi\sqrt{-\tilde{g}} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} \tag{3.21}$$

$$= -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.22}$$

$$= -\frac{D}{2}\sqrt{-\tilde{g}}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.23}$$

$$\delta_\phi(\tilde{\nabla}_\mu\phi) = \tilde{\nabla}_\mu\delta\phi \tag{3.24}$$

$$\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) = \delta_\phi(\tilde{\nabla}_\mu\partial_\nu\phi) \tag{3.25}$$

$$= \delta_\phi\left(\partial_\mu\partial_\nu\phi - \tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi\right) \tag{3.26}$$

$$= \partial_\mu\partial_\nu\delta\phi - \delta\phi\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi - \tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\delta\phi \tag{3.27}$$

$$= \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi - \tilde{\nabla}_\lambda\phi\delta\phi\tilde{\Gamma}^\lambda{}_{\mu\nu} \tag{3.28}$$

$$= \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi - \tilde{\nabla}_\lambda\phi\delta_\phi\tilde{\Gamma}^\lambda{}_{\mu\nu} \tag{3.29}$$

$$= \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi - \tilde{\nabla}_\lambda\phi\frac{1}{2}\tilde{g}^{\lambda\rho}\left(\tilde{\nabla}_\mu\delta_\phi\tilde{g}_{\nu\rho} + \tilde{\nabla}_\nu\delta_\phi\tilde{g}_{\mu\rho} - \tilde{\nabla}_\rho\delta_\phi\tilde{g}_{\mu\nu}\right) \tag{3.30}$$

$$= \tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi - \frac{1}{2}\tilde{\nabla}^\rho\phi\left(2\tilde{\nabla}_{(\mu}\delta_\phi\tilde{g}_{\nu)\rho} - \tilde{\nabla}_\rho\delta_\phi\tilde{g}_{\mu\nu}\right) \tag{3.31}$$

Therefore,

$$E^{\mu\nu}_\phi\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) = E^{\mu\nu}_\phi\left[\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi - \frac{1}{2}\tilde{\nabla}^\rho\phi\left(2\tilde{\nabla}_{(\mu}\delta_\phi\tilde{g}_{\nu)\rho} - \tilde{\nabla}_\rho\delta_\phi\tilde{g}_{\mu\nu}\right)\right] \tag{3.32}$$

$$= E^{\mu\nu}_\phi\left[\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi + \tilde{\nabla}^\rho\phi\tilde{\nabla}_\mu(\tilde{g}_{\nu\rho}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}\tilde{\nabla}^\rho\phi\tilde{\nabla}_\rho(\tilde{g}_{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi)\right] \tag{3.33}$$

$$= E^{\mu\nu}_\phi\left[\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\delta\phi + \tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu(\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi\tilde{\nabla}_\rho(\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi)\right] \tag{3.34}$$

So (3.7) becomes

$$\begin{aligned}
\delta_\phi\mathcal{E} = \sqrt{-\tilde{g}}\Big\{&E_\phi\delta\phi + E^\mu_\phi\tilde{\nabla}_\mu\delta\phi - \frac{1}{2}E_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha\delta\phi \\
&+E^{\mu\nu}_\phi\left[\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + \tilde{\nabla}_\nu\phi\tilde{\nabla}_\mu(\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi\tilde{\nabla}_\rho(\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi)\right]\Big\} \\
= \sqrt{-\tilde{g}}\Big\{&E_\phi\delta_\phi + \left(E^\mu_\phi\tilde{\nabla}_\mu - \frac{1}{2}E_{\rho\tau}\tilde{g}^{\rho\tau}\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + E^{\mu\nu}_\beta\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi \\
&+E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}E^{\mu\nu}_\phi\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi - \frac{1}{2}E^{\mu\nu}_\phi\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\Big\}
\end{aligned}$$

At this point it is convenient to introduce the following notation

$$\mathrm{tr}\,\tilde{E} := \tilde{g}^{\mu\nu}E_{\mu\nu}, \qquad \mathrm{tr}\,\tilde{E}_\phi := \tilde{g}_{\mu\nu}\tilde{E}^{\mu\nu}_\phi \tag{3.35}$$

Therefore,

$$\delta_\phi\mathcal{E} = \sqrt{-\tilde{g}}\left\{E_\phi\delta_\phi + \left(E^\mu_\phi\tilde{\nabla}_\mu - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}\,\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi\right.$$
$$\left.+E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}E^{\mu\nu}_\phi\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\right\}$$
$$= \sqrt{-\tilde{g}}\left\{E_\phi\delta_\phi + \left(E^\mu_\phi\tilde{\nabla}_\mu - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}\,\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi\right.$$
$$+E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}E^{\mu\nu}_\phi\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi$$
$$\left.-\frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\left[\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi + \tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\right]\right\}$$
$$= \sqrt{-\tilde{g}}\left\{E_\phi\delta\phi + \left(E^\mu_\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + \left[E^{\mu\nu}_\delta\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi) - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\right]\tilde{\nabla}_\delta\phi\right.$$
$$\left.+E^{\mu\nu}_\phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\right\}$$
$$= \sqrt{-\tilde{g}}\left\{E_\phi\delta\phi + \left[E^\mu_\phi\tilde{\nabla}_\mu - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}\tilde{\nabla}^\mu\phi + \left(E^{\alpha\beta}_\phi\tilde{\nabla}_\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\beta\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi\right]\tilde{\nabla}_\mu\delta\phi\right.$$
$$\left.+\left[E^{\mu\nu}_\phi + \left(E^{\mu\alpha}_\phi\tilde{\nabla}_\nu\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\right]\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi\right\}$$

Therefore, the Fréchet derivative is given by the following operator

$$D_\mathcal{E} = \sqrt{-\tilde{g}}\left[E_\phi + H^\mu\tilde{\nabla}_\mu + H^{(\mu\nu)}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right] \tag{3.36}$$

where

$$H^\mu := E^\mu_\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}\tilde{\nabla}^\mu\phi + \left(E^{\alpha\beta}_\phi\tilde{\nabla}_\beta\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi \tag{3.37}$$
$$H^{\mu\nu} := E^{\mu\nu}_\phi + \left(E^{\mu\alpha}_\phi\tilde{\nabla}_\alpha\phi - \frac{1}{2}\,\mathrm{tr}\,\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi \tag{3.38}$$

At this point let us remember that the adjoint of a differential operator O, denoted bu $O^\dagger$ satisfies

$$\int \mathrm{d}^D x A O(B) = \int \mathrm{d}^D x B O^\dagger(A) \tag{3.39}$$

for every pair of differential functions A and B, with equality achieved up to boundary terms. In order to ensure that the equations arise from an action principle, we need $D_\mathcal{E}$ to be self-adjoint

$$\int \mathrm{d}^D x A D_\mathcal{E}(B) = \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[A\left(E_\phi + H^\mu\tilde{\nabla}_\mu + H^{(\mu\nu)}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right)B\right] \tag{3.40}$$
$$= \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[AE_\phi B + AH^\mu\tilde{\nabla}_\mu B + AH^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu B\right] \tag{3.41}$$
$$= \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[BE_\phi A - B\tilde{\nabla}_\mu(H^\mu A) - \tilde{\nabla}_\mu(AH^{\mu\nu})\tilde{\nabla}_\nu B\right] + \mathrm{b.t} \tag{3.42}$$
$$= \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[BE_\phi A - B\tilde{\nabla}_\mu(H^\mu A) + B\tilde{\nabla}_\mu\tilde{\nabla}_\nu(H^{\mu\nu} A)\right] + \mathrm{b.t} \tag{3.43}$$

Then,

$$D_{\mathcal{E}}^{\dagger}(A) = \sqrt{-\tilde{g}}\left[E_\phi A - \tilde{\nabla}_\mu(H^\mu A) + \tilde{\nabla}_\mu\tilde{\nabla}_\nu(H^{\mu\nu}A)\right] \tag{3.44}$$

$$= \sqrt{-\tilde{g}}\left[E_\phi A - A\tilde{\nabla}_\mu H^\mu - H^\mu\tilde{\nabla}_\mu A + \tilde{\nabla}_\nu(A\tilde{\nabla}_\mu H^{\mu\nu} + H^{\mu\nu}\tilde{\nabla}_\mu A)\right] \tag{3.45}$$

But, in order to make appear $D_{\mathcal{E}}$ from (3.36), we can add an smart zero,

$$\begin{aligned}
D_{\mathcal{E}}^{\dagger}(A) &= \sqrt{-\tilde{g}}\left[E_\phi A - \tilde{\nabla}_\mu(H^\mu A) + \tilde{\nabla}_\mu\tilde{\nabla}_\nu(H^{\mu\nu}A)\right] + 2H^\mu\tilde{\nabla}_\mu A - 2H^\mu\tilde{\nabla}_\mu A \\
&= \sqrt{-\tilde{g}}\Big[E_\phi A - A\tilde{\nabla}_\mu H^\mu + H^\mu\tilde{\nabla}_\mu A + \tilde{\nabla}_\nu A\tilde{\nabla}_\mu H^{\mu\nu} + A\tilde{\nabla}_\mu\tilde{\nabla}_\nu H^{\mu\nu} \\
&\quad + \tilde{\nabla}_\nu H^{\mu\nu}\tilde{\nabla}_\mu A + H^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu A - 2H^\mu\tilde{\nabla}_\mu A\Big] \\
&= D_{\mathcal{E}}(A) + \sqrt{-\tilde{g}}\left[-A\tilde{\nabla}_\mu H^\mu + \tilde{\nabla}_\nu A\tilde{\nabla}_\mu H^{\mu\nu} + A\tilde{\nabla}_\mu\tilde{\nabla}_\nu H^{\mu\nu} + \tilde{\nabla}_\nu H^{\mu\nu}\tilde{\nabla}_\mu A - 2H^\mu\tilde{\nabla}_\mu A\right] \\
&= D_{\mathcal{E}}(A) + \sqrt{-\tilde{g}}\left[-A\tilde{\nabla}_\mu H^\mu + 2\tilde{\nabla}_\mu H^{\mu\nu}\tilde{\nabla}_\nu A + A\tilde{\nabla}_\mu\tilde{\nabla}_\nu H^{\mu\nu} - 2H^\mu\tilde{\nabla}_\mu A\right] \\
&= D_{\mathcal{E}}(A) + \sqrt{-\tilde{g}}\left[2J^\mu\tilde{\nabla}_\mu A + \tilde{\nabla}_\mu J^\mu A\right]
\end{aligned}$$

where we have defined

$$J^\mu := \tilde{\nabla}_\nu H^{(\nu\mu)} - H^\mu \tag{3.46}$$

Thus, for $D_{\mathcal{E}}$ to be self-adjoint, it must be fulfilled that

$$D_{\mathcal{E}}^{\dagger}(A) = D_{\mathcal{E}}(A) \Leftrightarrow J^\mu = 0 \tag{3.47}$$

and the Helmholtz condition is reduced to

$$\boxed{\tilde{\nabla}_\nu H^{(\nu\mu)} = H^\mu} \tag{3.48}$$

where

$$H^\mu := \left[E_\phi^\mu - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^\mu\phi + \left(E_\phi^{\alpha\beta}\tilde{\nabla}_\beta\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi\right] \tag{3.49}$$

$$H^{\mu\nu} := E_\phi^{\mu\nu} + \left(E_\phi^{\mu\alpha}\tilde{\nabla}_\alpha\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi \tag{3.50}$$

Helmholtz conditions imply that a symmetric second rank tensor depending up to second order in the scalar field ϕ and auxiliary metric $\tilde{g}_{\mu\nu}$ has a second order divergence. Horndeski characterized the most general tensor with these properties [2]. Consequently, $H^{\mu\nu}$ belongs to the Horndeski family built for the auxiliary metric $\tilde{g}_{\mu\nu}$.

3.1 Quantities written in the auxiliary frame

As an example, let's consider again the action principle

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^D x\sqrt{-g}X^{D/2} \tag{3.51}$$

Its equation is rewritten in the auxiliary frame as

$$\mathcal{E} = \sqrt{-g}E \tag{3.52}$$
$$= \sqrt{-g}\nabla_\mu \left(X^{\frac{D-2}{2}} \nabla^\mu \phi \right) \tag{3.53}$$
$$= \partial_\mu \left(\underbrace{\sqrt{-g}X^{\frac{D}{2}}}_{\sqrt{-\tilde{g}}} \underbrace{X^{-1}g^{\mu\nu}}_{\tilde{g}^{\mu\nu}} \partial_\nu \phi \right) \tag{3.54}$$
$$= \partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi) \tag{3.55}$$
$$= \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi \tag{3.56}$$
$$= \sqrt{-\tilde{g}}\tilde{\Box}\phi \tag{3.57}$$

which implies that

$$E = \tilde{\Box}\phi = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi \tag{3.58}$$

Thus

$$E_\phi^{\mu\nu} := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} = \tilde{g}^{\mu\nu} \tag{3.59}$$

So then

$$\begin{aligned} H^{\mu\nu} &:= E_\phi^{\mu\nu} + \left(E_\phi^{\alpha\mu}\tilde{\nabla}_\alpha\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi \right)\tilde{\nabla}^\nu\phi \\ &= \tilde{g}^{\mu\nu} + \left(\tilde{g}^{\alpha\mu}\tilde{\nabla}_\alpha\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{E}_\phi^{\lambda\rho}\tilde{\nabla}^\mu\phi \right)\tilde{\nabla}^\nu\phi \\ &= \tilde{g}^{\mu\nu} + \left(\tilde{\nabla}^\mu\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{g}^{\lambda\rho}\tilde{\nabla}^\mu\phi \right)\tilde{\nabla}^\nu\phi \\ &= \tilde{g}^{\mu\nu} \left(\tilde{\nabla}^\mu\phi - \frac{D}{2}\tilde{\nabla}^\mu\phi \right)\tilde{\nabla}^\nu\phi \\ &= \tilde{g}^{\mu\nu} - \frac{(D-2)}{2}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi \end{aligned} \tag{3.60}$$

Furthermore,

$$E_{\mu\nu} := -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}E)}{\partial\tilde{g}^{\mu\nu}} \tag{3.61}$$
$$= -\frac{2}{\sqrt{-\tilde{g}}}\left(-\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}E + \sqrt{-\tilde{g}}\frac{\partial E}{\partial\tilde{g}^{\mu\nu}} \right) \tag{3.62}$$
$$= -2\left(-\frac{1}{2}\tilde{g}_{\mu\nu}E + \tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi \right) \tag{3.63}$$
$$= -2\left(-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\Box}\phi + \tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi \right) \tag{3.64}$$
$$= \tilde{g}_{\mu\nu}\tilde{\Box}\phi - 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi \tag{3.65}$$

which implies

$$\operatorname{tr}\tilde{E} := \tilde{g}^{\mu\nu}E_{\mu\nu} = D\tilde{\Box}\phi - 2\tilde{\Box}\phi = (D-2)\tilde{\Box}\phi \tag{3.66}$$

It is also clear to see that

$$E^{\mu}_{\phi} := \frac{\partial E}{\partial(\tilde{\nabla}_{\mu}\phi)} = 0 \tag{3.67}$$

So then

$$\begin{aligned}
H^{\mu} &:= \left[E^{\mu}_{\phi} - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^{\mu}\phi + \left(E^{\alpha\beta}_{\phi}\tilde{\nabla}_{\beta}\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_{\phi}\tilde{\nabla}^{\alpha}\phi\right)\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\mu}\phi\right] && (3.68)\\
&= -\frac{1}{2}(D-2)\tilde{\square}\phi\tilde{\nabla}^{\mu}\phi + \left(\tilde{g}^{\alpha\beta}\tilde{\nabla}_{\beta}\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{E}^{\lambda\rho}_{\phi}\tilde{\nabla}^{\alpha}\phi\right)\tilde{\nabla}_{\alpha}\tilde{\nabla}_{\mu}\phi && (3.69)\\
&= -\frac{(D-2)}{2}\tilde{\square}\phi\tilde{\nabla}^{\mu}\phi + \left(\tilde{\nabla}^{\alpha}\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{g}^{\lambda\rho}\tilde{\nabla}^{\alpha}\phi\right)\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\mu}\phi && (3.70)\\
&= -\frac{(D-2)}{2}\tilde{\square}\phi\tilde{\nabla}^{\mu}\phi + \left(\tilde{\nabla}^{\alpha}\phi - \frac{D}{2}\tilde{\nabla}^{\alpha}\phi\right)\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\mu}\phi && (3.71)\\
&= -\frac{(D-2)}{2}\tilde{\square}\phi\tilde{\nabla}^{\mu}\phi - \frac{(D-2)}{2}\tilde{\nabla}^{\alpha}\phi\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\mu}\phi && (3.72)\\
&= -\frac{(D-2)}{2}\left(\tilde{\square}\phi\tilde{\nabla}^{\mu}\phi + \tilde{\nabla}^{\alpha}\phi\tilde{\nabla}_{\alpha}\tilde{\nabla}^{\mu}\phi\right) && (3.73)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\tilde{\nabla}_{\mu}H^{\mu\nu} &= \tilde{\nabla}_{\mu}\left(\tilde{g}^{\mu\nu} - \frac{(D-2)}{2}\tilde{\nabla}^{\mu}\phi\tilde{\nabla}^{\nu}\phi\right) && (3.74)\\
&= -\frac{(D-2)}{2}\tilde{\nabla}_{\mu}\left(\tilde{\nabla}^{\mu}\phi\tilde{\nabla}^{\nu}\phi\right) && (3.75)\\
&= -\frac{(D-2)}{2}\left(\tilde{\square}\phi\tilde{\nabla}^{\nu}\phi + \tilde{\nabla}^{\mu}\phi\tilde{\nabla}_{\mu}\tilde{\nabla}^{\nu}\phi\right) && (3.76)\\
&= H^{\nu} && (3.77)
\end{aligned}$$

That is to say, the equations of motion coming from (3.51), writen in the auxiliar frame, satisfy the Hemholtz conditions (3.48).

4 Horndeski theorem

Horndeski theorem says that in a space of dimension four, the most general symmetric contravariant tensor density of the form

$$A^{ab} = A^{ab}(g_{ij}, \partial_h g_{ij}, \partial_h\partial_k g_{ij}, \phi, \partial_h\phi, \partial_h\partial_k\phi) \tag{4.1}$$

which is such that $\nabla_a A^{ab}$ is at most of second-order in the derivatives of both g_{ij} and ϕ is given by

$$\begin{aligned}
A^{ab} = \sqrt{-g}\Big\{&K_1\delta^{acde}_{fhjk}g^{fb}\nabla^h\nabla_c\phi R^{jk}_{de} + K_2\delta^{acd}_{efh}g^{eb}R^{fh}_{cd}\\
&+K_3\delta^{acde}_{fhjk}g^{fb}\nabla_c\phi\nabla^h\phi R^{jk}_{de} + K_4\delta^{acde}_{fhjk}g^{fb}\nabla_h\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi\\
&+K_5\delta^{acd}_{efh}g^{eb}\nabla^f\nabla_c\phi\nabla^h\nabla_d\phi + K_6\delta^{acde}_{fhjk}g^{fb}\nabla_c\phi\nabla^h\phi\nabla^j\nabla_f\phi\nabla^k\nabla_e\phi\\
&+K_7\delta^{ac}_{de}g^{db}\nabla^e\nabla_c\phi + K_8\delta^{acd}_{efh}g^{eb}\nabla_c\phi\nabla^f\phi\nabla^h\nabla_c\nabla_d\phi + K_9 g^{ab} + K_{10}\nabla^a\phi\nabla^b\phi\Big\}
\end{aligned} \tag{4.2}$$

where K_i are arbitrary differentiable functions of ϕ and $\partial_i\phi$ [2].

Note that the dependence on the scalar curvature only appears in the first three terms of (4.2) with all indices contracted with the Kronecker delta, then, from its irreducible decomposition

$$R^{ab}_{mn} = C^{ab}_{mn} + 2\delta^{[a}_{[m}S^{b]}_{n]} + \frac{1}{6}\delta^{a}_{[m}\delta^{b}_{n]}R \tag{4.3}$$

only the terms proportional to the trace of the Ricci tensor survive.

Now, using the identities

$$\delta^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_p}\delta^{\nu_1}_{\mu_1}\cdots\delta^{\nu_k}_{\mu_k} = \frac{(D-p+k)!}{(D-p)!}\delta^{\mu_{k+1}\ldots\mu_p}_{\nu_{k+1}\ldots\nu_p} \tag{4.4}$$

and

$$\delta^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_p} = p!\delta^{\mu_1}_{[\nu_1}\ldots\delta^{\mu_p}_{\nu_p]} = p!\delta^{[\mu_1}_{\nu_1}\ldots\delta^{\mu_p]}_{\nu_p} \tag{4.5}$$

let's see how the first three terms of (4.2) look like:

$$K_1\delta^{acde}_{fhjk}g^{fb}\nabla^h\nabla_c\phi R^{jk}_{de} = \frac{K_1}{6}\delta^{acde}_{fhjk}g^{fb}\nabla^h\nabla_c\phi\delta^j_d\delta^k_e R \tag{4.6}$$

$$= \frac{K_1}{6}\frac{(4-4+2)!}{(4-4)!}\delta^{ac}_{fh}g^{fb}\nabla^h\nabla_c\phi R \tag{4.7}$$

$$= \frac{K_1}{3}\delta^{ac}_{fh}g^{fb}\nabla^h\nabla_c\phi R \tag{4.8}$$

$$K_2\delta^{acd}_{efh}g^{eb}R^{fh}_{cd} = \frac{K_2}{6}\delta^{acd}_{efh}g^{eb}\delta^c_e\delta^h_d R \tag{4.9}$$

$$= \frac{K_2}{6}\frac{(4-3+2)!}{(4-3)!}\delta^a_e g^{eb}R \tag{4.10}$$

$$= K_2 g^{ab}R \tag{4.11}$$

$$K_3\delta^{acde}_{fhjk}g^{fb}\nabla_c\phi\nabla^h\phi R^{jk}_{de} = \frac{K_3}{6}\delta^{acde}_{fhjk}g^{fb}\nabla_c\phi\nabla^h\phi\delta^j_d\delta^k_e R \tag{4.12}$$

$$= \frac{K_3}{6}\frac{(4-4+2)!}{(4-4)!}\delta^{ac}_{fh}g^{fb}\nabla_c\phi\nabla^h\phi R \tag{4.13}$$

$$= \frac{K_3}{3}\delta^{ec}_{fh}g^{fb}\nabla_c\phi\nabla^h\phi R \tag{4.14}$$

Adding these terms, we have

$$\begin{aligned}
[(4.8)+(4.11)+(4.14)] &= \frac{1}{3}\delta^{ac}_{fh}g^{fb}\left(K_1\nabla^h\nabla_c\phi+K_3\nabla_c\phi\nabla^h\phi\right)R+K_2g^{ab}R \\
&= \frac{1}{3}\delta^a_f\delta^c_h g^{fb}(K_1\nabla^h\nabla_c\phi+K_3\nabla_c\phi\nabla^h\phi)-\frac{1}{3}\delta^a_h\delta^c_f g^{fb}(K_1\nabla^h\nabla_c\phi+K_3\nabla_c\phi\nabla^h\phi) \\
&\quad +K_2g^{ab}R \\
&= \frac{1}{3}g^{ab}(K_1\Box\phi+K_3\nabla_c\phi\nabla^c\phi)R-\frac{1}{3}g^{cb}(K_1\nabla^a\nabla_c\phi+K_3\nabla_c\phi\nabla^a\phi)R+K_2g^{ab}R \\
&= \frac{1}{3}g^{ab}(K_1\Box\phi+K_3\nabla_c\phi\nabla^c\phi)R-\frac{1}{3}(K_1\nabla^a\nabla^b\phi+K_3\nabla^a\phi\nabla^b\phi)R+K_2g^{ab}R \\
&= \frac{1}{3}g^{ab}(K_1\Box\phi-2K_3X)R-\frac{1}{3}(K_1\nabla^a\nabla^b\phi+K_3\nabla^a\phi\nabla^b\phi)R+K_2g^{ab}R \\
&= \frac{1}{3}g^{ab}R(K_1\Box\phi+3K_2-2K_3X)-\frac{1}{3}(K_1\nabla^a\nabla^b\phi+K_3\nabla^a\phi\nabla^b\phi)R
\end{aligned}$$

Since conformal invariance only allows those second order tensors independent S^a_b and R and here are all the terms that depend on the curvature in (4.2), we have

$$0=\frac{\partial}{\partial R}\left(\frac{A^{ab}}{\sqrt{-g}}\right)=\frac{1}{3}g^{ab}(K_1\Box\phi+3K_2-2K_3X)-\frac{1}{3}(K_1\nabla^a\nabla^b\phi+K_3\nabla^a\phi\nabla^b\phi) \tag{4.15}$$

Taking the trace,

$$\begin{aligned}
0=g_{ab}\frac{\partial}{\partial R}\left(\frac{A^{ab}}{\sqrt{-g}}\right) &= \frac{4}{3}(K_1\Box\phi+3K_2-2K_3X)-\frac{1}{3}(K_1\Box\phi-2K_3X) \\
&= K_1\Box\phi+4K_2-2K_3X
\end{aligned} \tag{4.16}$$

Since $K_i=K_i(\phi,\partial_a\phi)$ are independent of second scalar field derivatives we have

$$K_1=0 \qquad \text{and} \qquad K_2=\frac{1}{2}K_3X \tag{4.17}$$

Plugging into (4.16),

$$0=g_{ab}\frac{\partial}{\partial R}\left(\frac{A^{ab}}{\sqrt{-g}}\right)=\frac{1}{3}g^{ab}\left(\frac{3}{2}K_3X-2K_3X\right)-\frac{1}{3}K_3(\nabla^a\phi\nabla^b\phi) \tag{4.18}$$

$$=-\frac{1}{6}g^{ab}K_3X-\frac{1}{3}K_3\nabla^a\phi\nabla^b\phi \tag{4.19}$$

$$=-\frac{K_3}{3}\left(\nabla^a\phi\nabla^b\phi+\frac{1}{2}g^{ab}X\right) \tag{4.20}$$

Now, using the fact that given any scalar field there always exists a vector field Y^a for which [2]

$$Y^a\nabla_a\phi=0 \qquad \text{and} \qquad Y^aY_a\neq 0 \tag{4.21}$$

we can multiply (4.20) by Y_aY_b,

$$0=-\frac{K_3}{3}\left(\cancel{Y_a\nabla^a\phi Y_b\nabla^b\phi}+\frac{1}{2}Y_aY^ag^{ab}X\right) \tag{4.22}$$

$$=-\frac{K_3}{3}\frac{1}{2}Y^aY_aX \tag{4.23}$$

but, $X \neq 0$, so then $K_3 = 0$. In summary we have

$$K_1 = K_2 = K_3 = 0\,. \tag{4.24}$$

Let's consider now the associated divergence, calculated by Horndeski as [2]

$$\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= K_1' \delta^{acde}_{fhjk} \nabla^f \phi \nabla^h \nabla_c \phi R^{jk}_{de} + 2\dot{K}_2 \delta^{acd}_{efh} \nabla_p \phi \nabla^e \nabla^p \phi R^{fh}_{cd} \\
&+ K_3 \delta^{acde}_{fhjk} \nabla^h \phi \nabla^f \nabla_c \phi R^{jk}_{de} + K_5 \delta^{acd}_{efh} \nabla^m \phi R^{fe}_{mc} \nabla^h \nabla_d \phi \\
&+ 2\dot{K}_1 \delta^{acde}_{fhjk} \nabla_p \phi \nabla^f \nabla^p \phi \nabla^h \nabla_c \phi R^{jk}_{de} + \frac{3}{2} K_4 \delta^{acde}_{fhjk} \nabla^m R^{hf}_{mc} \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi \\
&+ \frac{1}{2} K_1 \delta^{acde}_{fhjk} \nabla^m \phi R^{hf}_{mc} R^{jk}_{de} + K_2' \delta^{acd}_{efh} \nabla^e \phi R^{fh}_{cd} + \frac{1}{2} K_7 \delta^{ac}_{de} \nabla^m \phi R^{ed}_{mc} \\
&+ \frac{1}{2} K_8 \delta^{acd}_{efh} \nabla_c \phi \nabla^f \phi \nabla^m \phi R^{he}_{md} + 2\dot{K}_3 \delta^{acde}_{fhjk} \nabla_p \phi \nabla^f \nabla^p \phi \nabla_c \phi R^{jk}_{de} \\
&+ K_6 \delta^{acde}_{fhjk} \nabla_c \phi \nabla^h \phi \nabla^m \phi R^{jf}_{md} \nabla^k \nabla_e \phi + K_4' \delta^{acde}_{fhjk} \nabla^f \phi \nabla^h \nabla_c \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi \\
&+ 2\dot{K}_5 \delta^{acd}_{efh} \nabla_p \phi \nabla^e \nabla^p \phi \nabla^f \nabla_c \phi \nabla^h \nabla_d \phi + K_6 \delta^{acde}_{fhjk} \nabla^h \phi \nabla^f \nabla_c \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi \\
&+ 2\dot{K}_8 \delta^{acd}_{efh} \nabla_p \phi \nabla^e \nabla^p \phi \nabla_c \phi \nabla^f \phi \nabla^h \nabla_d \phi + 2\dot{K}_6 \delta^{acde}_{fhjk} \nabla_p \phi \nabla^f \nabla^p \phi \nabla_c \phi \nabla^h \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi \\
&+ 2\dot{K}_4 \delta^{acde}_{fhjk} \nabla_p \phi \nabla^f \nabla^p \phi \nabla^h \nabla_c \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi + K_5' \delta^{acd}_{efh} \nabla^e \phi \nabla^f \nabla_c \phi \nabla^h \nabla_d \phi \\
&+ 2\dot{K}_7 \delta^{ac}_{de} \nabla_p \phi \nabla^d \nabla^p \phi \nabla^e \nabla_c \phi + K_8 \delta^{acd}_{efh} \nabla^f \phi \nabla^e \nabla_c \phi \nabla^h \nabla_d \phi + (2\dot{K}_9 + K_{10}) \nabla_b \phi \nabla^b \nabla^a \phi \\
&+ K_7' \delta^{ac}_{de} \nabla^d \phi \nabla^e \nabla_c \phi + \nabla^a \phi (K_9' + \rho K_{10}' + 2\dot{K}_{10} \nabla^b \phi \nabla^c \phi \nabla_c \nabla_b \phi + K_{10} \nabla^c \nabla_c \phi)
\end{aligned} \tag{4.25}$$

where a prime denotes a partial derivative with respect to ϕ and a dot denotes a partial derivative with respect to ρ. Pluggin (4.24) into (4.25) we obtain

$$\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= K_5 \delta^{acd}_{efh} \nabla^m \phi R^{fe}_{mc} \nabla^h \nabla_d \phi + \frac{3}{2} K_4 \delta^{acde}_{fhjk} \nabla^m R^{hf}_{mc} \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi + \frac{1}{2} K_7 \delta^{ac}_{de} \nabla^m \phi R^{ed}_{mc} \\
&+ \frac{1}{2} K_8 \delta^{acd}_{efh} \nabla_c \phi \nabla^f \phi \nabla^m \phi R^{he}_{md} + K_6 \delta^{acde}_{fhjk} \nabla_c \phi \nabla^h \phi \nabla^m \phi R^{jf}_{md} \nabla^k \nabla_e \phi \\
&+ K_4' \delta^{acde}_{fhjk} \nabla^f \phi \nabla^h \nabla_c \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi + 2\dot{K}_5 \delta^{acd}_{efh} \nabla_p \phi \nabla^e \nabla^p \phi \nabla^f \nabla_c \phi \nabla^h \nabla_d \phi \\
&+ K_6 \delta^{acde}_{fhjk} \nabla^h \phi \nabla^f \nabla_c \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi + 2\dot{K}_8 \delta^{acd}_{efh} \nabla_p \phi \nabla^e \nabla^p \phi \nabla_c \phi \nabla^f \phi \nabla^h \nabla_d \phi \\
&+ 2\dot{K}_6 \delta^{acde}_{fhjk} \nabla_p \phi \nabla^f \nabla^p \phi \nabla_c \phi \nabla^h \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi \\
&+ 2\dot{K}_4 \delta^{acde}_{fhjk} \nabla_p \phi \nabla^f \nabla^p \phi \nabla^h \nabla_c \phi \nabla^j \nabla_d \phi \nabla^k \nabla_e \phi + K_5' \delta^{acd}_{efh} \nabla^e \phi \nabla^f \nabla_c \phi \nabla^h \nabla_d \phi \\
&+ 2\dot{K}_7 \delta^{ac}_{de} \nabla_p \phi \nabla^d \nabla^p \phi \nabla^e \nabla_c \phi + K_8 \delta^{acd}_{efh} \nabla^f \phi \nabla^e \nabla_c \phi \nabla^h \nabla_d \phi + (2\dot{K}_9 + K_{10}) \nabla_b \phi \nabla^b \nabla^a \phi \\
&+ K_7' \delta^{ac}_{de} \nabla^d \phi \nabla^e \nabla_c \phi + \nabla^a \phi (K_9' + \rho K_{10}' + 2\dot{K}_{10} \nabla^b \phi \nabla^c \phi \nabla_c \nabla_b \phi + K_{10} \nabla^c \nabla_c \phi)
\end{aligned}$$

The scalar curvature dependence would be through those terms with the Riemann tensor contracted totally with the Kronecker delta

$$\begin{aligned}\frac{6}{R}\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= K_5\delta^{acd}_{efh}\delta^f_m\delta^e_c\nabla^m\phi\nabla^h\nabla_d\phi + \frac{3}{2}K_4\delta^{acde}_{fhjk}\delta^h_m\delta^f_c\nabla^m\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + \frac{1}{2}K_7\delta^{ac}_{de}\delta^e_m\delta^d_c\nabla^m\phi \\ &\quad + \frac{1}{2}K_8\delta^{acd}_{efh}\delta^h_m\delta^e_d\nabla_c\phi\nabla^f\phi\nabla^m\phi + K_6\delta^{acde}_{fhjk}\delta^j_m\delta^f_d\nabla_c\phi\nabla^h\phi\nabla^m\phi\nabla^k\nabla_e\phi \\ &= -2K_5\delta^{ad}_{fh}\nabla^f\phi\nabla^h\nabla_d\phi - \frac{3}{2}K_4\delta^{ade}_{hjk}\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi - \frac{3}{2}K_7\nabla^a\phi \\ &\quad + K_8\delta^{ac}_{fh}\nabla_c\phi\nabla^f\phi\nabla^h\phi + K_6\delta^{ace}_{hjk}\nabla_c\phi\nabla^h\phi\nabla^j\phi\nabla^k\nabla_e\phi\end{aligned}$$

Note that due to symmetry in the covariant derivative indices contracted with the anti-symmetric Kronecker delta, the last two terms vanish. Therefore,

$$\begin{aligned}\frac{6}{R}\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= -2K_5\delta^{ad}_{fh}\nabla^f\phi\nabla^h\nabla_d\phi - \frac{3}{2}K_4\delta^{ade}_{hjk}\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi - \frac{3}{2}K_7\nabla^a\phi \\ &= -2K_5\left(\delta^a_f\delta^d_h - \delta^d_f\delta^a_h\right)\nabla^f\phi\nabla^h\nabla_d\phi - \frac{3}{2}K_7\nabla^a\phi \\ &\quad - \frac{3}{2}K_4 3!\left(\delta^a_h\delta^d_j\delta^e_k + \delta^a_j\delta^d_k\delta^e_h + \delta^a_k\delta^d_h\delta^e_j - \delta^a_j\delta^d_h\delta^e_k - \delta^a_h\delta^d_k\delta^e_j - \delta^a_j\delta^d_k\delta^e_h\right)\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\ &= -2K_5(\nabla^a\Box\phi - \nabla^d\phi\nabla^a\nabla_d\phi) - \frac{3}{2}K_7\nabla^a\phi \\ &\quad - 9K_4\Big[\nabla^a(\Box\phi)^2 + \nabla^e\phi\nabla^a\nabla_d\phi\nabla^d\nabla_e\phi + \nabla^d\phi\nabla^e\nabla_d\phi\nabla^a\nabla_e\phi - \nabla^d\phi\nabla^a\nabla_d\phi\Box\phi \\ &\qquad -\nabla^a\phi\nabla^e\nabla_d\phi\nabla^d\nabla_e\phi - \nabla^d\phi\nabla^a\nabla_d\phi\Box\phi\Big]\end{aligned}$$

Introducing the following notation,

$$\phi^a := \nabla^a\phi, \quad \phi^{ab} := \nabla^a\nabla^b\phi, \quad X^a := \nabla^a X = -\nabla^b\nabla^a\nabla_b\phi = -\phi^b\nabla^a\phi_b \tag{4.26}$$

we have

$$\begin{aligned}\frac{6}{R}\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= -2K_5(\phi^a\Box\phi + X^a) - \frac{3}{2}K_7\phi^a \\ &\quad - 9K_4\left[\phi^a(\Box\phi)^2 - \nabla^a\phi_d X^d - X^e\nabla^a\phi_e + X^a\Box\phi - \phi^a\phi_{ed}\phi^{ed} + X^a\Box\phi\right] \\ &= -2K_5(\phi^a\Box\phi + X^a) - \frac{3}{2}K_7\phi^a \\ &\quad - 9K_4\left[\phi^a(\Box\phi)^2 - \phi^a\phi_{ed}\phi^{ed} + 2X^a\Box\phi - 2X^d\nabla^a\phi_d\right]\end{aligned}$$

The partial derivative with respect to the scalar curvature is given by

$$\frac{\partial}{\partial R}\left(\frac{6\nabla_b A^{ab}}{\sqrt{-g}}\right) = -2K_5(\phi^a\Box\phi + X^a) - \frac{3}{2}K_7\phi^a \tag{4.27}$$

$$- 9K_4\left[\phi^a(\Box\phi)^2 - \phi^a\phi_{ed}\phi^{ed} + 2X^a\Box\phi - 2X^d\nabla^a\phi_d\right] \tag{4.28}$$

[Check factors]

The above must vanish for any value of the second derivative. Since the terms with the same degree have a common coefficient, such coefficients must vanish independently. Hence

$$K_4 = K_5 = K_7 = 0 \tag{4.29}$$

In this way, the divergence of A^{ab} is reduced to

$$\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= \frac{1}{2}K_8\delta^{acd}_{efh}\nabla_c\phi\nabla^f\phi\nabla^m\phi R^{he}_{md} + K_6\delta^{acde}_{fhjk}\nabla_c\phi\nabla^h\phi\nabla^m\phi R^{jf}_{md}\nabla^k\nabla_e\phi \\
&+ K_6\delta^{acde}_{fhjk}\nabla^h\phi\nabla^f\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + 2\dot{K}_8\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi\nabla_c\phi\nabla^f\phi\nabla^h\nabla_d\phi \\
&+ 2\dot{K}_6\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla_c\phi\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ K_8\delta^{acd}_{efh}\nabla^f\phi\nabla^e\nabla_c\phi\nabla^h\nabla_d\phi + (2\dot{K}_9 + K_{10})\nabla_b\phi\nabla^b\nabla^a\phi \\
&+ \nabla^a\phi(K_9' + \rho K_{10}' + 2\dot{K}_{10}\nabla^b\phi\nabla^c\phi\nabla_c\nabla_b\phi + K_{10}\nabla^c\nabla_c\phi)
\end{aligned}$$

Let's see the dependence of the traceless Ricci tensor S^a_b in the expression above. Using (4.3) we have

$$\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= \frac{1}{2}K_8\delta^{acd}_{efh}\phi_c\phi^f\phi^m R^{he}_{md} + K_6\delta^{acde}_{fhjk}\phi_c\phi^h\phi^m\phi^k_e R^{jf}_{md} \\
&= \frac{1}{2}2K_8\delta^{acd}_{efh}\delta^{[h}_{[m}S^{e]}_{d]}\phi_c\phi^f\phi^m + 2K_6\delta^{acde}_{fhjk}\delta^{[j}_{[m}S^{f]}_{d]}\phi_c\phi^h\phi^m\phi^k_e \\
&= \frac{K_8}{2}\delta^{acd}_{efh}\left(\delta^h_m S^e_d - \delta^h_d S^e_m\right)\phi_c\phi^f\phi^m + K_6\delta^{acde}_{fhjk}\left(\delta^j_m S^f_d - \delta^j_d S^f_m\right)\phi_c\phi^h\phi^m\phi^k_e \\
&= \frac{K_8}{2}\left(\underbrace{\delta^{acd}_{efh}S^e_d\phi_c\phi^{(f}\phi^{h)}}_{=0} - 2\delta^{ac}_{ef}S^e_m\phi_c\phi^f\phi^m\right) + K_6\left(\underbrace{\delta^{acde}_{fhjk}S^f_d\phi_c\phi^{(h}\phi^{j)}\phi^k_e}_{=0} - \delta^{ace}_{fhk}S^f_m\phi_c\phi^h\phi^m\phi^k_e\right) \\
&= -K_8\delta^{ac}_{ef}S^e_m\phi_c\phi^f\phi^m - K_6\delta^{ace}_{fhk}S^f_m\phi_c\phi^h\phi^m\phi^k_e \\
&= -2K_8\delta^a_{[e}\delta^c_{f]}S^e_m\phi_c\phi^f\phi^m - 6K_6\delta^a_{[f}\delta^c_h\delta^e_{k]}S^f_m\phi_c\phi^h\phi^m\phi^k_e
\end{aligned}$$

Since both terms have different degrees in the second derivative, they must vanish independently, that is

$$K_6 = K_8 = 0 \tag{4.30}$$

In summary, if one demands that only curvature couplings occur through the Weyl tensor, then there is no nonminimal coupling at all, and the second-order tensor becomes, in fact, of first-order,

$$A^{ab} = \sqrt{-g}\left\{K_9 g^{ab} + K_{10}\nabla^a\phi\nabla^b\phi\right\} \tag{4.31}$$

with second-order divergence given by

$$\nabla_b A^{ab} = \sqrt{-g}\Big\{(2\dot{K}_9 + K_{10})\nabla_b\phi\nabla^b\nabla^a\phi \tag{4.32}$$

$$+\nabla^a\phi(K_9' + \rho K_{10}' + 2\dot{K}_{10}\nabla^b\phi\nabla^c\phi\nabla_c\nabla_b\phi + K_{10}\nabla^c\nabla_c\phi)\Big\} \tag{4.33}$$

Another restriction in the allowed terms is given by the fact that

$$\tilde{X} = -\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi = -\frac{1}{2}\frac{1}{X}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = \frac{1}{X}X = 1 \tag{4.34}$$

In this way, the symmetric second rank tensor depending up to second order in the scalar field ϕ and auxiliary metric $\tilde{g}^{\mu\nu}$ such that a second order divergence is given by

$$H^{\mu\nu} = \tilde{K}_9\tilde{g}^{\mu\nu} + \tilde{K}_{10}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi \tag{4.35}$$

Taking its divergence,

$$\begin{aligned}\nabla_\nu H^{(\mu\nu)} &= \partial_\phi\tilde{K}_9\tilde{\nabla}^\mu\phi + \partial_\phi\tilde{K}_{10}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi\tilde{\nabla}_\nu\phi + \tilde{K}_{10}\tilde{\nabla}^\nu\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi + \tilde{K}_{10}\tilde{\nabla}^\mu\phi\tilde{\Box}\phi & (4.36)\\ &= \partial_\phi\tilde{K}_9\tilde{\nabla}^\mu\phi + \tilde{K}_{10}\tilde{\Box}\phi\tilde{\nabla}^\mu\phi - 2\partial_\phi\tilde{K}_{10}\tilde{X}\tilde{\nabla}^\mu\phi - \tilde{K}_{10}\tilde{\nabla}^\mu\phi & (4.37)\end{aligned}$$

but $\tilde{X} = 1$, so then

$$\begin{aligned}H^\mu = \nabla_\nu H^{(\mu\nu)} &= \partial\phi\tilde{K}_9 + \tilde{K}_{10}\tilde{\Box}\phi\tilde{\nabla}^\mu\phi - 2\partial_\phi\tilde{K}_{10}\tilde{\nabla}^\mu\phi & (4.38)\\ &= (\partial_\phi\tilde{K}_9 + \tilde{K}_{10}\Box\phi - 2\partial_\phi\tilde{K}_{10})\tilde{\nabla}^\mu\phi & (4.39)\end{aligned}$$

In summary:

$$\begin{aligned}H^{\mu\nu} &= \tilde{K}_9\tilde{g}^{\mu\nu} + \tilde{K}_{10}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi & (4.40)\\ H^\mu &= (\partial_\phi\tilde{K}_9 + \tilde{K}_{10}\Box\phi - 2\partial_\phi\tilde{K}_{10})\tilde{\nabla}^\mu\phi & (4.41)\\ K_i &= K_i(\phi,\tilde{X})\Big|_{\tilde{X}=1} & (4.42)\end{aligned}$$

We can decompose $\partial E\big/\partial(\tilde{\nabla}_\nu\tilde{\nabla}_\nu\phi)$ in its trace and traceless part,

$$\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi = \mathrm{TL}_{\mu\nu} + \frac{1}{D}\tilde{g}^{\mu\nu}\tilde{\Box}\phi \tag{4.43}$$

where the $\mathrm{TL}_{\mu\nu}$ is the traceless part,

$$\mathrm{TL}^\mu_\nu = \tilde{\nabla}^\mu\tilde{\nabla}_\nu\phi - \frac{1}{D}\delta^\mu_\nu\tilde{\Box}\phi \tag{4.44}$$

Therefore, we have

$$E^{\mu\nu}_\phi = \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} = \frac{\partial E}{\partial(\mathrm{TL}^\alpha_\beta)}\frac{\partial(\mathrm{TL}^\alpha_\beta)}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} + \frac{\partial E}{\partial(\tilde{\Box}\phi)}\frac{\partial(\tilde{\Box}\phi)}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} \tag{4.45}$$

but

$$\begin{aligned}\mathrm{TL}_{\alpha\beta} &= \tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi - \frac{1}{D}\tilde{g}_{\alpha\beta}\tilde{\Box}\phi & (4.46)\\ \Longrightarrow\ \mathrm{TL}^\alpha_\beta &= \tilde{\nabla}^\alpha\tilde{\nabla}_\beta\phi - \frac{1}{D}\delta^\alpha_\beta\tilde{\Box}\phi & (4.47)\\ &= \tilde{g}^{\mu\alpha}\tilde{\nabla}_\mu\tilde{\nabla}_\beta\phi - \frac{1}{D}\delta^\alpha_\beta\tilde{g}^{\mu\nu}\tilde{\nabla}_m\tilde{\nabla}_\nu\phi & (4.48)\\ \Longrightarrow\ \frac{\partial(\mathrm{TL}^\alpha_\beta)}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} &= \tilde{g}^{\mu\alpha}\delta^\nu_\beta - \frac{1}{D}\delta^\alpha_\beta\tilde{g}^{\mu\nu} & (4.49)\end{aligned}$$

and

$$\frac{\partial(\tilde{\Box}\phi)}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} = \frac{\partial}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\left(\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi\right) = \tilde{g}^{\mu\nu} \tag{4.50}$$

Therefore

$$\begin{aligned}
E^{\mu\nu}_\phi &= \frac{\partial E}{\partial(\mathrm{TL}^\alpha_\beta)}\left(\tilde{g}^{\mu\alpha}\delta^\nu_\beta - \frac{1}{D}\delta^\alpha_\beta\tilde{g}^{\mu\nu}\right) + \frac{\partial E}{\partial(\tilde{\Box}\phi)}\tilde{g}^{\mu\nu} && (4.51)\\
&= \mathbb{A}^{\mu\nu} - \frac{1}{D}\mathbb{A}\tilde{g}^{\mu\nu} + \frac{\partial E}{\partial(\tilde{\Box}\phi)}\tilde{g}^{\mu\nu} && (4.52)\\
&= \hat{\mathbb{A}}^{\mu\nu} + \frac{\partial E}{\partial(\tilde{\Box}\phi)}\tilde{g}^{\mu\nu} && (4.53)
\end{aligned}$$

where we have defined $\mathbb{A}^{\mu\nu} := \frac{\partial E}{\partial(\mathrm{TL}_{\mu\nu})}$ and $\hat{\mathbb{A}}$ is its traceless part.

Now, let's decompose the two-rank Helmholtz conditions in its trace and traceless part,

$$\begin{aligned}
\tilde{g}_{\mu\nu}H^{\mu\nu} &= \tilde{g}_{\mu\nu}\left[E^{\mu\nu}_\phi + \left(E^{\mu\alpha}_\phi\tilde{\nabla}_\alpha\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\right] && (4.54)\\
&= \operatorname{tr}\tilde{E}_\phi + E^{\alpha\mu}_\phi\tilde{\nabla}_\alpha\tilde{\nabla}_\mu\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\tilde{\nabla}_\mu\phi && (4.55)\\
&= \operatorname{tr}\tilde{E}_\phi + E^{\alpha\mu}_\phi\tilde{\nabla}_\alpha\phi\tilde{\nabla}_\mu\phi + \operatorname{tr}\tilde{E}_\phi\tilde{X} && (4.56)\\
&= 2\operatorname{tr}\tilde{E}_\phi + E^{\alpha\mu}_\phi\tilde{\nabla}_\alpha\phi\tilde{\nabla}_\mu\phi && (4.57)\\
&= (2\tilde{g}_{\alpha\mu} + \tilde{\nabla}_\alpha\phi\tilde{\nabla}_\mu\phi)E^{\alpha\mu}_\phi && (4.58)\\
&= (2\tilde{g}_{\alpha\mu} + \tilde{\nabla}_\alpha\phi\tilde{\nabla}_\mu\phi)\left(\hat{\mathbb{A}}^{\mu\nu} + \frac{\partial E}{\partial(\tilde{\Box}\phi)}\tilde{g}^{\mu\nu}\right) && (4.59)\\
&= \cancel{2\tilde{g}_{\alpha\mu}\hat{\mathbb{A}}^{\alpha\mu}} + 8\frac{\partial E}{\partial(\tilde{\Box}\phi)} + \hat{\mathbb{A}}^{\alpha\mu}\tilde{\nabla}_\alpha\phi\tilde{\nabla}_\mu\phi + \frac{\partial E}{\partial(\tilde{\Box}\phi)}\left(\tilde{\nabla}\phi\right)^2 && (4.60)\\
&= \frac{\partial E}{\partial(\tilde{\Box}\phi)}\left(8 - 2\tilde{X}\right) + \hat{\mathbb{A}}^{\alpha\mu}\tilde{\nabla}_\alpha\phi\tilde{\nabla}_\mu\phi && (4.61)\\
&= 6\frac{\partial E}{\partial(\tilde{\Box}\phi)} + \hat{\mathbb{A}}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi && (4.62)\\
&= 6\frac{\partial E}{\partial(\tilde{\Box}\phi)} + \mathbb{A}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - \frac{1}{4}\mathbb{A}(\tilde{\nabla}\phi)^2 && (4.63)\\
&= 6\frac{\partial E}{\partial(\tilde{\Box}\phi)} + \mathbb{A}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \frac{1}{2}\mathbb{A}\tilde{X} && (4.64)\\
&= 6\frac{\partial E}{\partial(\tilde{\Box}\phi)} + \left(\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \frac{1}{2}\tilde{g}_{\mu\nu}\right)\frac{\partial E}{\partial(\mathrm{TL}_{\mu\nu})} && (4.65)
\end{aligned}$$

but also

$$\begin{aligned}
\tilde{g}_{\mu\nu}H^{\mu\nu} &= \tilde{g}_{\mu\nu}(\tilde{K}_9\tilde{g}^{\mu\nu} + \tilde{K}_{10}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi) && (4.66)\\
&= 4\tilde{K}_9 + \tilde{K}_{10}(\tilde{\nabla}\phi)^2 && (4.67)\\
&= 4\tilde{K}_9 - 2\tilde{K}_{10}\tilde{X} && (4.68)\\
&= 2(2\tilde{K}_9 - \tilde{K}_{10}) && (4.69)
\end{aligned}$$

Thus

$$6\frac{\partial E}{\partial(\tilde{\Box}\phi)} + \left(\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \frac{1}{2}\tilde{g}_{\mu\nu}\right)\frac{\partial E}{\partial(\mathrm{TL}_{\mu\nu})} = 2(2\tilde{K}_9 - \tilde{K}_{10}) \tag{4.70}$$

A Some useful calculations

A.1 Variation of the Christoffel symbols

The Christoffel symbols in terms of the metric are given by

$$\Gamma^{\lambda}{}_{\mu\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right)$$

Varying both sides, we have

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= \frac{1}{2}\delta g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= -\frac{1}{2} g^{\lambda\sigma} g^{\rho\tau} (\delta g_{\sigma\tau}) \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= -g^{\lambda\sigma} (\delta g_{\sigma\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right)
\end{aligned}$$

Changing the dumb indice σ by ρ,

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= -g^{\lambda\rho} (\delta g_{\rho\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} - 2\delta g_{\rho\tau} \Gamma^{\tau}{}_{\mu\beta} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \big(\partial_\mu \delta g_{\beta\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\mu} \delta g_{\tau\beta} + \partial_\beta \delta g_{\mu\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\beta} \delta g_{\tau\mu} \\
&\quad - \partial_\rho \delta g \mu\beta + \Gamma^{\tau}{}_{\mu\rho} \delta g_{\tau\beta} + \Gamma^{\tau}{}_{\beta\rho} \delta g_{\mu\tau} \big) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\nabla_\mu \delta g_{\beta\rho} + \nabla_\beta \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\beta} \right)
\end{aligned} \tag{A.1}$$

A.2 Variation of the Riemann tensor

The Riemann tensor is given by

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} - \Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu}$$

Varying both sides,

$$\begin{aligned}
\delta R^{\rho}{}_{\lambda\mu\nu} &= \partial_\mu \delta\Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\lambda\mu} + \delta\Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\lambda\nu} - \delta\Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\lambda\mu} \\
&= \partial_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\nu\lambda} - \Gamma^{\tau}{}_{\mu\lambda} \delta\Gamma^{\rho}{}_{\tau\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\tau}{}_{\nu\lambda} \delta\Gamma^{\rho}{}_{\tau\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\mu\lambda}
\end{aligned}$$

Adding a convenient zero of the form $\Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda} - \Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda}$, and using the fact that $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor, we have

$$\delta R^{\rho}{}_{\lambda\mu\nu} = \nabla_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} = 2\nabla_{[\mu} \delta\Gamma^{\rho}{}_{\nu]\lambda} \tag{A.2}$$

A.3 Variation of derivatives of ϕ w.r.t ω

Let's compute the infinitesimal variations of the covariant derivatives of the scalar field. First, let us calculate $\delta_\omega \nabla_\mu \phi$:

$$\delta_\omega \nabla_\mu \phi = \nabla_\mu \delta_\omega \phi = 0 \tag{A.3}$$

Now, let's compute $\delta_\omega(\nabla_\mu\nabla_\nu\phi)$:

$$\begin{aligned}\delta_\omega(\nabla_\mu\nabla_\nu\phi) &= \delta_\omega\nabla_\mu(\partial_\nu\phi) && \text{(A.4)}\\ &= \delta_\omega(\partial_\mu\partial_\nu\phi - \Gamma^\lambda{}_{\nu\mu}\partial_\lambda\phi) && \text{(A.5)}\\ &= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\Gamma^\lambda{}_{\nu\mu}\partial_\lambda\phi - \Gamma^\lambda{}_{\nu\mu}\delta_\omega\partial_\lambda\phi && \text{(A.6)}\\ &= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\Gamma^\lambda{}_{\nu\mu}\partial_\lambda\phi - \Gamma^\lambda{}_{\nu\mu}\partial_\lambda\delta_\omega\phi && \text{(A.7)}\\ &= -\partial_\lambda\phi\delta_\omega\Gamma^\lambda{}_{\nu\mu} && \text{(A.8)}\end{aligned}$$

Using that the variation of the Christoffel connection is

$$\delta\Gamma^\lambda{}_{\mu\beta} = \frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\beta\rho} + \nabla_\beta\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\beta}) \tag{A.9}$$

we have

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi\frac{1}{2}g^{\lambda\rho}(\nabla_\nu\delta_\omega g_{\mu\rho} + \nabla_\mu\delta_\omega g_{\nu\rho} - \nabla_\rho\delta_\omega g_{\nu\beta}) \tag{A.10}$$

but $\delta_\omega g_{\nu\mu} = 2\omega g_{\nu\mu}$, so then

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi g^{\lambda\rho}\left[\nabla_\nu(\omega g_{\mu\rho}) + \nabla_\mu(\omega g_{\nu\rho}) - \nabla_\rho(\omega g_{\nu\beta})\right] \tag{A.11}$$

Using the metric compatibility condition $\nabla_\mu g_{\alpha\beta} = 0$ and $\nabla_\alpha\phi = \partial_\alpha\phi$, we obtain

$$\begin{aligned}\delta_\omega(\nabla_\mu\nabla_\nu\phi) &= -\partial_\lambda\phi g^{\lambda\rho}(g_{\mu\rho}\nabla_\nu\omega + g_{\nu\rho}\nabla_\mu\omega - g_{\nu\beta}\nabla_\rho\omega) && \text{(A.12)}\\ &= -\partial^\rho\phi(g_{\mu\rho}\nabla_\nu\omega + g_{\nu\rho}\nabla_\mu\omega - g_{\nu\beta}\nabla_\rho\omega) && \text{(A.13)}\\ &= -\nabla_\mu\phi\nabla_\nu\omega - \nabla_\nu\phi\nabla_\mu\omega + g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega && \text{(A.14)}\\ &= -2\nabla_{(\mu}\phi\nabla_{\nu)}\omega + g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega && \text{(A.15)}\end{aligned}$$

A.4 Variation of E w.r.t Riemann tensor

In order to see what (1.20c) implies, let us split the dependence of the Riemann tensor in terms of its traceless part $C^{\alpha\beta}_{\mu\nu}$, the traceless part of the Ricci tensor S^α_β, and the scalar curvature R. So we have

$$E\left(g^{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = E\left(g^{\mu\nu}, C^{\alpha\beta}_{\mu\nu}, S^\alpha_\beta, R\right)$$

The variation w.r.t the Riemann tensor yields

$$\begin{aligned}\delta_{\text{Riem}}E &= P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta}\\ &= H^{\alpha\beta}_{\mu\nu}\delta_{\text{Riem}}C^{\mu\nu}_{\alpha\beta} + I^\alpha_\beta\delta_{\text{Riem}}S^\beta_\alpha + J\delta_{\text{Riem}}R && \text{(A.16)}\end{aligned}$$

where

$$H^{\alpha\beta}_{\mu\nu} \equiv \frac{\partial E}{\partial C^{\mu\nu}_{\alpha\beta}}, \qquad I^\alpha_\beta \equiv \frac{\partial E}{\partial S^\beta_\alpha} \qquad \text{y} \qquad J \equiv \frac{\partial E}{\partial R}$$

Since $P^{\mu\nu}_{\alpha\beta}$ has the same algebraic symmetries as the Riemann tensor, its traceless part is given by

$$\hat{P}^{\mu\nu}_{\alpha\beta} = P^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}\delta^{[\mu}_{[\alpha}P^{\nu]}_{\beta]} + \frac{2}{(D-2)(D-1)}P\delta^\mu_{[\alpha}\delta^\nu_{\beta]} \tag{A.17}$$

Let us note that

$$
\begin{aligned}
J\delta_{\text{Riem}}R &= J\delta_{\text{Riem}}\left(R^{\alpha\beta}_{\mu\nu}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\right) \\
&= J\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}R^{\alpha\beta}_{\mu\nu}
\end{aligned} \tag{A.18}
$$

Writing S^{α}_{β} in terms of the Riemann,

$$
\begin{aligned}
S^{\beta}_{\nu} &= R^{\beta}_{\nu} - \frac{1}{D}R\delta^{\beta}_{\nu} \\
&= R^{\alpha\beta}_{\mu\nu}\delta^{\mu}_{\alpha} - \frac{1}{D}\delta^{\beta}_{\nu}Rs^{\alpha\gamma}_{\mu\lambda}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}
\end{aligned}
$$

then,

$$
\delta_{\text{Riem}}\tilde{S}^{\beta}_{\nu} = \delta^{\mu}_{\alpha}\delta\tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D}\delta^{\beta}_{\nu}\delta\tilde{R}^{\alpha\gamma}_{\mu\lambda}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}
$$

Hence,

$$
\begin{aligned}
I^{\nu}_{\beta}\delta_{\text{Riem}}S^{\beta}_{\nu} &= I^{\nu}_{\beta}\delta^{\mu}_{\alpha}\delta R^{\alpha\beta}_{\mu\nu} - \frac{1}{D}I^{\nu}_{\beta}\delta^{\beta}_{\nu}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}\delta R^{\alpha\gamma}_{\mu\lambda} \\
&= I^{\nu}_{\beta}\delta^{\mu}_{\alpha}\delta\tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D}I\delta^{\mu}_{\alpha}\delta R^{\alpha\gamma}_{\mu\lambda} \\
&= \delta^{\mu}_{\alpha}\delta R^{\alpha\beta}_{\mu\nu}\left(I^{\nu}_{\beta} - \frac{1}{D}I\delta^{\nu}_{\beta}\right) \\
&= \delta^{\mu}_{\alpha}\hat{I}\delta R^{\alpha\beta}_{\mu\nu}
\end{aligned} \tag{A.19}
$$

Finally, let us write the Weyl tensor in terms of the Riemann,

$$
\begin{aligned}
\tilde{C}^{\alpha\beta}_{\mu\nu} &= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{[\alpha}_{[\mu}\tilde{R}^{\beta]}_{\nu]} + \frac{2}{(D-1)(D-2)}\tilde{R}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]} \\
&= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\tilde{R}^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\tilde{R}^{\rho\sigma}_{\tau\lambda}
\end{aligned}
$$

Varying with respect to $R^{\alpha\beta}_{\mu\nu}$,

$$
\delta_{\text{Riem}}C^{\alpha\beta}_{\mu\nu} = \delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\delta R^{\rho\sigma}_{\tau\lambda}
$$

Then,

$$
\begin{aligned}
H^{\mu\nu}_{\alpha\beta}\delta_{\text{Riem}}C^{\alpha\beta}_{\mu\nu} &= H^{\mu\nu}_{\alpha\beta}\left[\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\delta R^{\rho\sigma}_{\tau\lambda}\right] \\
&= H^{\mu\nu}_{\alpha\beta}\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}H^{\lambda\nu}_{\gamma\beta}\delta^{\mu}_{\alpha}\delta^{\gamma}_{\lambda}\delta R^{\beta\alpha}_{\nu\mu} + \frac{2}{(D-1)(D-2)}H^{\tau\lambda}_{\rho\sigma}\delta^{\rho}_{\tau}\delta^{\sigma}_{\lambda}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta R^{\alpha\beta}_{\mu\nu} \\
&= \delta R^{\alpha\beta}_{\mu\nu}\left[H^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}H^{\nu}_{\beta}\delta^{\mu}_{\alpha} + \frac{2}{(D-1)(D-2)}H\right] \\
&= \hat{H}^{\mu\nu}_{\alpha\beta}\delta R^{\alpha\beta}_{\mu\nu}
\end{aligned} \tag{A.20}
$$

where the indices have been renamed in a convenient way and has been used the fact that $H^{\alpha\beta}_{\mu\nu}$ has the same algebraic symmetries as the Riemann tensor.

In this way, plugging (A.18), (A.19) and (A.20) into (A.16), we obtain

$$P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta} = \hat{H}^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta} + \delta^{\alpha}_{\mu}\hat{I}^{\beta}_{\nu}\delta R^{\mu\nu}_{\alpha\beta} + J\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta R^{\mu\nu}_{\alpha\beta}$$

Hence,

$$P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu}\hat{I}^{\beta]}_{\nu]} + J\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \tag{A.21}$$

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