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Skyrmions on deformed $\mathbb{CP}^2$

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ABSTRACT: Abstract...

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1 Conventions

We define the gauge connection as $R_\mu = R^i_\mu t_i$ where $t_i = -\frac{i}{2}\tau_i$ are the generators of $SU(2)$ with τ_i being the Pauli matrices, which satisfy

$$\tau_i \tau_j = \delta_{ij} \mathrm{I}_{2\times 2} + i\epsilon_{ijk}\tau_k. \tag{1.1}$$

From this, one can show that

$$\mathrm{Tr}(t_i t_j) = -\frac{1}{2}\delta_{ij} \qquad \text{and} \qquad [t_i, t_j] = \epsilon_{ijk} t_k. \tag{1.2}$$

The Einstein-Skyrme action principle is given by

$$S = S_G + S_{\text{Skyrme}}, \tag{1.3}$$

where

$$S_G = \kappa \int \mathrm{d}^4x \sqrt{-g}(R - 2\Lambda), \tag{1.4}$$

$$S_{\text{Skyrme}} = \frac{K}{2} \int \mathrm{d}^4x \sqrt{-g}\, \mathrm{Tr}\left(\frac{1}{2} R^\mu R_\mu + \frac{\lambda}{16} F_{\mu\nu} F^{\mu\nu} \right). \tag{1.5}$$

Here $\kappa = (16\pi G)^{-1}$, G is the Newton gravitational constant and the parameters K and λ are fixed experimentaly.

R_μ y $F_{\mu\nu}$ are defined by

$$R_\mu := U^{-1}\nabla_\mu U, \tag{1.6}$$

$$F_{\mu\nu} := [R_\mu, R_\nu]. \tag{1.7}$$

The Einstein equations associated with (1.3) are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}, \tag{1.8}$$

where

$$T_{\mu\nu} = -\frac{K}{2}\, \mathrm{Tr}\left[\left(R_\mu R_\nu - \frac{1}{2} g_{\mu\nu} R^\alpha R_\alpha \right) + \frac{\lambda}{4} \left(F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right]. \tag{1.9}$$

Furthermore, the Skyrme field equations are given by

$$\nabla^\mu R_\mu + \frac{\lambda}{4} \nabla^\mu [R^\nu, F_{\mu\nu}] = 0. \tag{1.10}$$

It is convenient to define the Maurier-Cartan left-invariant forms of $SU(2)$ as,

$$\sigma_1 = \cos\psi \mathrm{d}\theta + \sin\theta \sin\psi \mathrm{d}\phi, \tag{1.11}$$
$$\sigma_2 = -\sin\psi \mathrm{d}\theta + \sin\theta \cos\psi \mathrm{d}\phi, \tag{1.12}$$
$$\sigma_3 = \mathrm{d}\psi + \cos\theta \mathrm{d}\phi, \tag{1.13}$$

which satisfy the relation $\mathrm{d}\sigma_i + \frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k = 0$.

The line element of $\mathbb{CP}^2$ can be written in terms of these forms as

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{(1+\frac{\Lambda}{6}r^2)^2} + \frac{r^2}{4}\frac{\sigma_3^2}{(1+\frac{\Lambda}{6}r^2)^2} + \frac{r^2}{4}\frac{(\sigma_1^2+\sigma_2^2)}{(1+\frac{\Lambda}{6}r^2)}, \tag{1.14}$$

where $0 \leq r < \infty, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$, and $0 \leq \psi < 4\pi$. This metric satisfy

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{1.15}$$

2 Equations to solve

To solve the Skyrme equations, we consider the ansatz

$$R = R^i_\mu t_i \mathrm{d}x^\mu = \sum_{i=1}^{3} \sigma^i t_i, \tag{2.1}$$

which satisfy automatically (1.10).

In order tho solve the Einstein equations (1.8) we consider as a metric ansatz, a deformed $\mathbb{CP}^2$ of the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{f(r)(1+ar^2)^2} + f(r)h(r)\frac{r^2}{4}\frac{\sigma_3^2}{(1+ar^2)^2} + \frac{r^2}{4}\frac{(\sigma_1^2+\sigma_2^2)}{(1+ar^2)}, \tag{2.2}$$

where a is a constant.

Performing some manipulations, the system is reduced to two first order equations for $f(r)$ and $h(r)$,

$$h(r)' = -\frac{(ar^2+1)^2 K\lambda}{\kappa r^3 f(r)^2} + \frac{2h(r)(h(r)-1)}{r(ar^2+1)} - \frac{K(ar^2+1)}{2r\kappa f(r)^2} \tag{2.3}$$

$$f(r)' = \frac{\lambda K}{2\kappa r^3}\left(\frac{(ar^2+1)^2}{h(r)f(r)} - \frac{(ar^2+1)}{2}\right) - \frac{f(r)(1+3h(r)-2ar^2)}{r(ar^2+1)} + \frac{K(ar^2+1)}{4\kappa r h(r) f(r)} \tag{2.4}$$

$$- \frac{Kar^2 + 2\Lambda\kappa r^2 - 8a\kappa r^2 + K - 8\kappa}{2\kappa r(ar^2+1)} \tag{2.5}$$

References