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Zero-weight scalar fields

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Abstract: Personal compilation of some calculations related to zero-weight conformal scalar fields.

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1 Building the equation of motion

Let us consider a scalar field with zero conformal weight, that is, under an infinitesimal conformal transformation

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad \delta_\omega \phi = 0 \tag{1.1}$$

Let us consider the most general second order pseudoscalar constructed from the scalar field ϕ and its derivatives up to second order, together with the metric tensor and its associates curvature

$$\mathcal{E} = \sqrt{-g}E\left(\phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi, g_{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = 0 \tag{1.2}$$

The variation of this equation under infinitesimal conformal transformations reads,

$$\delta_\omega\mathcal{E} = \delta_\omega(\sqrt{-g})E + \sqrt{-g}\delta_\omega E \tag{1.3}$$

$$= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\delta_\omega E \tag{1.4}$$

From $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta_\omega(g_{\mu\nu}g^{\mu\nu}) = 0$ implies that

$$g_{\mu\nu}\delta_\omega g^{\mu\nu} = -g^{\mu\nu}\delta_\omega g_{\mu\nu} \tag{1.5}$$

Furthermore, from (A.3) and (A.15), we have

$$\delta_\omega\mathcal{E} = \frac{\partial E}{\partial\phi}\cancelto{0}{\delta_\omega\phi} + \frac{\partial E}{\partial(\nabla_\mu\phi)}\cancelto{0}{\delta_\omega\nabla_\mu\phi} + \frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\delta_\omega(\nabla_\mu\nabla_\nu\phi) + \frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \tag{1.6}$$

where we have defined

$$P^{\mu\nu}_{\alpha\beta} := \frac{\partial E}{\partial R^{\alpha\beta}_{\mu\nu}}, \qquad P^{\mu}_{\nu} := P^{\alpha\mu}_{\alpha\nu} \tag{1.7}$$

So then

$$\begin{aligned}
\delta_\omega \mathcal{E} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\delta_\omega(\nabla_\mu\nabla_\nu\phi) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \\
&= \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_{(\mu}\phi\nabla_{\nu)}\omega\right) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \\
&- \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E \\
&= \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_{(\mu}\phi\nabla_{\nu)}\omega\right) + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} - 2\omega g^{\mu\nu}\sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}} &(1.8)\\
&= \sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) &(1.9)
\end{aligned}$$

By defining

$$A^{\mu\nu} := \frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}, \qquad B_{\mu\nu} = -2\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} \tag{1.10}$$

we obtain

$$\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g}A^{\mu\nu}\left(\gamma_{\mu\nu}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_\mu\phi\nabla_\nu\omega\right) + \sqrt{-g}P^{\mu\nu}\delta_\omega R^{\alpha\beta}_{\mu\nu} + \sqrt{-g}\omega B_{\mu\nu}g_{\mu\nu} &(1.11)\\
&= \sqrt{-g}\left(A\nabla^\rho\phi\nabla_\rho\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu}\right) &(1.12)
\end{aligned}$$

Using

$$\delta_\omega R^{\alpha}{}_{\beta\mu\nu} = 2\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta} \tag{1.13}$$

we have

$$\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g}\left(A\nabla^\rho\phi\nabla_\rho\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta}\right) &(1.14)\\
&= \sqrt{-g}\left(A\nabla^\nu\phi\nabla_\nu\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta}\right) &(1.15)\\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^{\alpha}{}_{\nu]\beta}\right] &(1.16)
\end{aligned}$$

Noting that

$$\begin{aligned}
\delta_\omega\Gamma^{\alpha}{}_{\nu\beta} &= \frac{1}{2}g^{\alpha\lambda}\left[\nabla_\nu(\delta_\omega g_{\beta\lambda}) + \nabla_\beta(\delta_\omega g_{\nu\lambda}) - \nabla_\lambda(\delta_\omega g_{\nu\beta})\right] &(1.17)\\
&= g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + g_{\nu\lambda}\nabla_\beta\omega - g_{\nu\beta}\nabla_\lambda\omega\right) &(1.18)\\
&= g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega\right) &(1.19)
\end{aligned}$$

we obtain

$$
\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_\mu g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega\right)\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P^{\lambda\beta\mu\nu}\nabla_\mu\left(\cancel{g_{\beta\lambda}}\nabla_\nu\omega + 2g_{\nu\lambda}\nabla_\beta\omega\right)\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 4P^{\lambda\beta\mu\nu}\nabla_\mu g_{\nu\lambda}\nabla_\beta\omega\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B - 4P^{\mu\nu}\nabla_\mu\nabla_\nu\omega\right]
\end{aligned}
$$

Imposing $\delta_\omega\mathcal{E} = 0$ for all ω, we obtain the following conditions,

$$A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi = 0 \tag{1.20a}$$

$$B = 0 \tag{1.20b}$$

$$P^{\mu\nu} = 0 \tag{1.20c}$$

From (A.21),

$$\frac{\partial E}{\partial R^{\mu\nu}_{\alpha\beta}} = P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu}\hat{I}^{\beta]}_{\nu]} + J\delta^\alpha_{[\mu}\delta^\beta_{\nu]} \tag{1.21}$$

we notice that since $\hat{H}^{\alpha\beta}_{\mu\nu}$ is the traceless part of $H^{\alpha\beta}_{\mu\nu}$,

$$P^\beta_\nu = P^{\alpha\beta}_{\alpha\nu} = \delta^{[\alpha}_{[\alpha}\hat{I}^{\beta]}_{\nu]} + J\delta^\alpha_{[\alpha}\delta^\beta_{\nu]} \tag{1.22}$$

Since the variation with respect to the Weyl tensor does not contribute so the first trace (1.21), the contribution of the Riemann tensor to E has to be through its traceless part,

$$\mathcal{E} = \sqrt{-g}E\left(\phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi, g_{\mu\nu}, C^{\alpha\beta}_{\mu\nu}\right) = 0\,. \tag{1.23}$$

Example 1.1. *Let us consider the following action principle*

$$S[\phi, g] = \int \mathrm{d}^D x\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right)^{D/2} \tag{1.24}$$

$$= \int \mathrm{d}^D x\sqrt{-g}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{D/2} \tag{1.25}$$

$$= \int \mathrm{d}^D x\sqrt{-g}X^{D/2} \tag{1.26}$$

where we have defined $X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$.

Now, we must to find E. *Varying with respect to* ϕ,

$$\delta_\phi S = -\int \mathrm{d}^D x\sqrt{-g}\frac{D}{2}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\nabla_\mu\delta\phi \tag{1.27}$$

$$= \int \mathrm{d}^D x\sqrt{-g}\frac{D}{2}\nabla_\mu\left[\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\right]\delta\phi + \mathrm{b.t} \tag{1.28}$$

therefore,

$$E = \nabla_\mu \left[\left(-\frac{1}{2}(\nabla\phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \right] \tag{1.29}$$

$$= \frac{D-2}{2} \left(-\frac{1}{2}(\nabla\phi)^2 \right)^{\frac{D-4}{2}} (-1)\nabla^\alpha\phi\nabla_\mu\nabla_\alpha\phi\nabla^\mu\phi + \left(-\frac{1}{2}(\nabla\phi)^2 \right)^{\frac{D-2}{2}} \Box\phi \tag{1.30}$$

which can be rewritten as

$$E = X^{\frac{D-2}{2}}\Box\phi - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi \tag{1.31}$$

Let's see how (1.20a) *looks,*

$$A^{\mu\nu} = \frac{\partial}{\partial(\nabla_\mu\phi\nabla_\nu\phi)} \left[X^{\frac{D-2}{2}} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi \right] \tag{1.32}$$

$$= X^{\frac{D-2}{2}} g^{\mu\nu} - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi \tag{1.33}$$

and its trace yields

$$A = DX^{\frac{D-2}{2}} - \frac{D-2}{2}X^{\frac{D-4}{2}}(\nabla\phi)^2 \tag{1.34}$$

$$= DX^{\frac{D-2}{2}} + (D-2)X^{\frac{D-4}{2}}\left(-\frac{1}{2}(\nabla\phi)^2 \right) \tag{1.35}$$

$$= DX^{\frac{D-2}{2}} + (D-2)X^{\frac{D-2}{2}} \tag{1.36}$$

$$= 2(D-1)X^{\frac{D-2}{2}} \tag{1.37}$$

Pluggin into (1.20a)*,*

$$\begin{aligned}
A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi &= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}g^{\mu\nu}\nabla_\mu\phi + (D-2)X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\phi \\
&= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}\nabla^\nu\phi + (D-2)X^{\frac{D-4}{2}}(\nabla\phi)^2\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-4}{2}}X \right]\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-2}{2}}2 \right]\nabla^\nu\phi \\
&= 0 \qquad \checkmark
\end{aligned}$$

2 Building the auxiliar metric

We know the conditions that the most general second order equation of motion for the zero conformal weight scalar field must satisfy. They are given by (1.20). Now, the question is: how do we construct an auxiliary metric $\tilde{g}_{\mu\nu}$ such that $\delta_\omega\tilde{g}_{\mu\nu} = 0$?

Remember that in the exponential frame for the scalar field, the conformal transformations look like

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \mathrm{e}^{2\omega}g_{\mu\nu}, \qquad \phi \to \bar{\phi} = \phi \tag{2.1}$$

Since the inverse metric transforms as

$$g^{\mu\nu} \to \bar{g}^{\mu\nu} = \mathrm{e}^{-2\omega} g^{\mu\nu} \tag{2.2}$$

the kinetic term for the scalar field, defined as

$$X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{2.3}$$

transforms as

$$\bar{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = -\frac{1}{2}\mathrm{e}^{-2\omega}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \mathrm{e}^{-2\omega}X \tag{2.4}$$

Thus, the auxiliary metric defined as

$$\tilde{g}_{\mu\nu} = X g_{\mu\nu} = -\frac{1}{2}(\nabla\phi)^2 g_{\mu\nu} \qquad \Longrightarrow \qquad \delta_\omega \tilde{g}_{\mu\nu} = 0 \tag{2.5}$$

is conformally invariant.

Let us consider a pseudoscalar built from the zero wight conformal scalar field and its derivatives up to second order, and the conformally invariant geometry,

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{R}^{\alpha\beta}_{\mu\nu}) \tag{2.6}$$

We notice that $\delta_\omega\mathcal{E} = 0$. Indeed

$$\delta_\omega\mathcal{E} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}E\cancel{\delta_\omega\tilde{g}_{\mu\nu}} + \sqrt{-\tilde{g}}\left(\frac{\partial E}{\partial\phi}\cancel{\delta_\omega\phi} + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\omega(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right.$$
$$\left. + \frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\cancel{\delta_\omega\tilde{g}^{\mu\nu}} + \tilde{P}^{\mu\nu}_{\alpha\beta}\cancel{\delta_\omega\tilde{R}^{\alpha\beta}_{\mu\nu}}\right)$$

but

$$\delta_\omega\tilde{\nabla}_\mu\phi = \delta_\omega\partial_\mu\phi = \partial_\mu\delta_\omega\phi = 0 \tag{2.7}$$

and

$$\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) = \delta_\omega(\tilde{\nabla}_\mu\partial_\nu\phi) \tag{2.8}$$
$$= \delta_\omega(\partial_\mu\partial_\nu\phi - \tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi) \tag{2.9}$$
$$= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi - \tilde{\Gamma}^\lambda_{\mu\nu}\partial_\lambda\delta_\omega\phi \tag{2.10}$$
$$= -\delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi \tag{2.11}$$
$$= 0 \tag{2.12}$$

since $\delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu} = 0$. Therefore,

$$\delta_\omega\mathcal{E} = 0\,. \tag{2.13}$$

Since from conformal invariance for zero weight scalar field, the $P^{\mu\nu} = 0$ condition implies that the explicit dependence of the Riemann tensor in the equation of motion is through the Weyl tensor, then, for a equation of motion built from the conformally invariant geometry we have

$$E = E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}) = 0 \tag{2.14}$$

3 Fréchet derivative

Now, let's try to find an action principle from which it emerges (2.14). To do that, we will use the formalism used in [1]. Hence, first we must to introduce the concept of Fréchet derivative.

Let $P[u] = P\left(x, u^{(n)}\right)$ be a differential function, i.e. that depends in the point x, the function u together with its derivatives $u^{(n)}$. Consider now its variation under a one-parameter family of functions. After interchanging the variation with derivatives (and without integrating by parts) we end with a differential operator acting on an arbitrary variation δu called the *Fréchet derivative* of P,

$$\delta P = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, P[u+\varepsilon\delta u]\bigg|_{\varepsilon=0} := D_P(\delta u) \tag{3.1}$$

Here, we consider the second order conformally invariant psudoscalar defined by

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}) \tag{3.2}$$

which is the natural quantity that could be derived from a convariant action. The role of he dependent function u is played by the zero weight conformally invariant scalar field ϕ, and hence the Fréchet derivative of $\mathcal{E}$ can be calculated from

$$D_{\mathcal{E}}(\delta\phi) = \delta_\phi\mathcal{E}\,. \tag{3.3}$$

From (3.2) and using that $\delta\phi = 0$, we have

$$\begin{aligned}
\delta_\phi\mathcal{E} &= -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left(\frac{\partial E}{\partial\phi}\delta\phi\frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\phi(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right.\\
&\qquad\left. + \frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\phi\tilde{g}^{\mu\nu} + \tilde{P}^{\mu\nu}_{\alpha\beta}\delta_\phi\tilde{C}^{\alpha\beta}_{\mu\nu}\right)\\
&= -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\phi\tilde{g}^{\mu\nu} + E_\phi\delta\phi + E^\mu_\phi\delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) + \tilde{P}^{\mu\nu}_{\alpha\beta}\cancel{\delta_\phi\tilde{C}^{\alpha\beta}_{\mu\nu}}\right]\\
&= -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\phi\tilde{g}^{\mu\nu} + E_\phi\delta\phi + E^\mu_\phi\delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right]
\end{aligned}$$

where we have defined

$$E_\phi := \frac{\partial E}{\partial\phi}, \qquad E^\mu_\phi := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}, \qquad E^{\mu\nu}_\phi := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} \tag{3.4}$$

in order to reduce some notation and we used the fact that the Weyl tensor is conformally invariant, $\tilde{C}^{\alpha\beta}_{\mu\nu} = C^{\alpha\beta}_{\mu\nu}$, so that $\delta_\phi C^{\alpha\beta}_{\mu\nu} = 0$. Then,

$$\delta_\phi\mathcal{E} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\phi\tilde{g}^{\mu\nu} + E_\phi\delta\phi + E^\mu_\phi\delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)\right] \tag{3.5}$$

Using (1.9), we can write

$$\delta_\phi \mathcal{E} = \sqrt{-\tilde{g}}\left[\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}E)}{\partial \tilde{g}^{\mu\nu}}\delta_\phi \tilde{g}^{\mu\nu} + E_\phi \delta\phi + E^\mu_\phi \delta_\phi(\tilde{\nabla}_\mu \phi) + E^{\mu\nu}_\phi \delta_\phi(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi)\right] \tag{3.6}$$

$$= \sqrt{-\tilde{g}}\left[-\frac{1}{2}E_{\mu\nu} + E_\phi \delta\phi + E^\mu_\phi \delta_\phi(\tilde{\nabla}_\mu \phi) + E^{\mu\nu}_\phi \delta_\phi(\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi)\right] \tag{3.7}$$

where

$$E_{\mu\nu} := -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}E)}{\partial \tilde{g}^{\mu\nu}} \tag{3.8}$$

Before proceeding, let's compute the variation w.r.t the scalar field ϕ of some quantities:

$$\delta_\phi \tilde{g}_{\mu\nu} = \delta_\phi(X g_{\mu\nu}) \tag{3.9}$$

$$= \delta_\phi\left(-\frac{1}{2}(\nabla\phi)^2 g_{\mu\nu}\right) \tag{3.10}$$

$$= -\nabla^\alpha\phi\nabla_\alpha\delta\phi g_{\mu\nu} \tag{3.11}$$

$$= -g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\delta\phi \tag{3.12}$$

$$= -g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.13}$$

$$= -g_{\mu\nu}Xg^{\alpha\beta}X^{-1}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.14}$$

$$= -\tilde{g}_{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.15}$$

$$\delta_\phi \tilde{g}^{\mu\nu} = \delta_\phi\left[\left(-\frac{1}{2}(\nabla\phi)^2\right)^{-1} g^{\mu\nu}\right] \tag{3.16}$$

$$= -\frac{1}{X^2}(-\nabla^\alpha\phi\nabla_\alpha\delta\phi)g^{\mu\nu} \tag{3.17}$$

$$= \frac{1}{X^2}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.18}$$

$$= X^{-1}g^{\mu\nu}X^{-1}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\delta\phi \tag{3.19}$$

$$= \tilde{g}^{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.20}$$

$$\delta_\phi\sqrt{-\tilde{g}} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} \tag{3.21}$$

$$= -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.22}$$

$$= -\frac{D}{2}\sqrt{-\tilde{g}}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\delta\phi \tag{3.23}$$

$$\delta_\phi(\tilde{\nabla}_\mu\phi) = \tilde{\nabla}_\mu\delta\phi \tag{3.24}$$

$$\delta_\phi(\tilde\nabla_\mu\tilde\nabla_\nu\phi) = \delta_\phi(\tilde\nabla_\mu\partial_\nu\phi) \tag{3.25}$$
$$= \delta_\phi\left(\partial_\mu\partial_\nu\phi - \tilde\Gamma^\lambda{}_{\mu\nu}\partial_\lambda\phi\right) \tag{3.26}$$
$$= \partial_\mu\partial_\nu\delta\phi - \delta\phi\tilde\Gamma^\lambda{}_{\mu\nu}\partial_\lambda\phi - \tilde\Gamma^\lambda{}_{\mu\nu}\partial_\lambda\delta\phi \tag{3.27}$$
$$= \tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi - \tilde\nabla_\lambda\phi\delta\phi\tilde\Gamma^\lambda{}_{\mu\nu} \tag{3.28}$$
$$= \tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi - \tilde\nabla_\lambda\phi\delta_\phi\tilde\Gamma^\lambda{}_{\mu\nu} \tag{3.29}$$
$$= \tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi - \tilde\nabla_\lambda\phi\frac{1}{2}\tilde g^{\lambda\rho}\left(\tilde\nabla_\mu\delta_\phi\tilde g_{\nu\rho} + \tilde\nabla_\nu\delta_\phi\tilde g_{\mu\rho} - \tilde\nabla_\rho\delta_\phi\tilde g_{\mu\nu}\right) \tag{3.30}$$
$$= \tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi - \frac{1}{2}\tilde\nabla^\rho\phi\left(2\tilde\nabla_{(\mu}\delta_\phi\tilde g_{\nu)\rho} - \tilde\nabla_\rho\delta_\phi\tilde g_{\mu\nu}\right) \tag{3.31}$$

Therefore,

$$E_\phi^{\mu\nu}\delta_\phi(\tilde\nabla_\mu\tilde\nabla_\nu\phi) = E_\phi^{\mu\nu}\left[\tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi - \frac{1}{2}\tilde\nabla^\rho\phi\left(2\tilde\nabla_{(\mu}\delta_\phi\tilde g_{\nu)\rho} - \tilde\nabla_\rho\delta_\phi\tilde g_{\mu\nu}\right)\right] \tag{3.32}$$
$$= E_\phi^{\mu\nu}\left[\tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi + \tilde\nabla^\rho\phi\tilde\nabla_\mu(\tilde g_{\nu\rho}\tilde\nabla^\alpha\phi\tilde\nabla_\alpha\delta\phi) - \frac{1}{2}\tilde\nabla^\rho\phi\tilde\nabla_\rho(\tilde g_{\mu\nu}\tilde\nabla^\alpha\phi\tilde\nabla_\alpha\delta\phi)\right] \tag{3.33}$$
$$= E_\phi^{\mu\nu}\left[\tilde\nabla_\mu\phi\tilde\nabla_\nu\delta\phi + \tilde\nabla_\nu\phi\tilde\nabla_\mu(\tilde\nabla^\alpha\phi\tilde\nabla_\alpha\delta\phi) - \frac{1}{2}\tilde g_{\mu\nu}\tilde\nabla^\rho\phi\tilde\nabla_\rho(\tilde\nabla^\alpha\phi\tilde\nabla_\alpha\delta\phi)\right] \tag{3.34}$$

So (3.7) becomes

$$\begin{aligned}\delta_\phi\mathcal{E} = \sqrt{-\tilde g}&\left\{E_\phi\delta\phi + E_\phi^\mu\tilde\nabla_\mu\delta\phi - \frac{1}{2}E_{\mu\nu}\tilde g^{\mu\nu}\tilde\nabla^\alpha\tilde\nabla_\alpha\delta\phi\right.\\
&\left.+E_\phi^{\mu\nu}\left[\tilde\nabla_\mu\tilde\nabla_\nu\delta\phi + \tilde\nabla_\nu\phi\tilde\nabla_\mu(\tilde\nabla^\alpha\tilde\nabla_\alpha\delta\phi) - \frac{1}{2}\tilde g_{\mu\nu}\tilde\nabla^\rho\phi\tilde\nabla_\rho(\tilde\nabla^\alpha\phi\tilde\nabla_\alpha\delta\phi)\right]\right\}\\
= \sqrt{-\tilde g}&\left\{E_\phi\delta_\phi + \left(E_\phi^\mu\tilde\nabla_\mu - \frac{1}{2}E_{\rho\tau}\tilde g^{\rho\tau}\tilde\nabla^\mu\phi\right)\tilde\nabla_\mu\delta\phi + E_\beta^{\mu\nu}\tilde\nabla_\mu\tilde\nabla_\nu\delta\phi + E_\phi^{\mu\nu}\tilde\nabla_\nu\phi(\tilde\nabla_\mu\tilde\nabla^\alpha\phi)\tilde\nabla_\alpha\delta\phi\right.\\
&\left.+E_\phi^{\mu\nu}\tilde\nabla_\nu\phi\tilde\nabla^\alpha\phi(\tilde\nabla_\mu\tilde\nabla_\alpha\delta\phi) - \frac{1}{2}E_\phi^{\mu\nu}\tilde g_{\mu\nu}\tilde\nabla^\rho\phi(\tilde\nabla_\rho\tilde\nabla^\alpha\phi)\tilde\nabla_\alpha\delta\phi - \frac{1}{2}E_\phi^{\mu\nu}\tilde g_{\mu\nu}\tilde\nabla^\rho\phi\tilde\nabla^\alpha\phi(\tilde\nabla_\rho\tilde\nabla_\alpha\delta\phi)\right\}\end{aligned}$$

At this point it is convenient to introduce the following notation

$$\mathrm{tr}\,\tilde E := \tilde g^{\mu\nu}E_{\mu\nu}, \qquad \mathrm{tr}\,\tilde E_\phi := \tilde g_{\mu\nu}\tilde E_\phi^{\mu\nu} \tag{3.35}$$

Therefore,

$$\delta_\phi \mathcal{E} = \sqrt{-\tilde{g}}\left\{E_\phi \delta_\phi + \left(E^\mu_\phi \tilde{\nabla}_\mu - \frac{1}{2}\operatorname{tr}\tilde{E}\,\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi\right.$$
$$\left. + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}E^{\mu\nu}_\phi\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\right\}$$
$$= \sqrt{-\tilde{g}}\left\{E_\phi \delta_\phi + \left(E^\mu_\phi \tilde{\nabla}_\mu - \frac{1}{2}\operatorname{tr}\tilde{E}\,\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi\right.$$
$$+ E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}E^{\mu\nu}_\phi\tilde{g}_{\mu\nu}\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi$$
$$\left. - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\left[\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\tilde{\nabla}_\alpha\delta\phi + \tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\right]\right\}$$
$$= \sqrt{-\tilde{g}}\left\{E_\phi\delta\phi + \left(E^\mu_\phi - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}_\mu\delta\phi + \left[E^{\mu\nu}_\delta\tilde{\nabla}_\nu\phi(\tilde{\nabla}_\mu\tilde{\nabla}^\alpha\phi) - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\rho\phi(\tilde{\nabla}_\rho\tilde{\nabla}^\alpha\phi)\right]\tilde{\nabla}_\delta\phi\right.$$
$$\left. + E^{\mu\nu}_\phi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi + E^{\mu\nu}_\phi\tilde{\nabla}_\nu\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\alpha\delta\phi) - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\rho\phi\tilde{\nabla}^\alpha\phi(\tilde{\nabla}_\rho\tilde{\nabla}_\alpha\delta\phi)\right\}$$
$$= \sqrt{-\tilde{g}}\left\{E_\phi\delta\phi + \left[E^\mu_\phi\tilde{\nabla}_\mu - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^\mu\phi + \left(E^{\alpha\beta}_\phi\tilde{\nabla}_\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\beta\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi\right]\tilde{\nabla}_\mu\delta\phi\right.$$
$$\left. + \left[E^{\mu\nu}_\phi + \left(E^{\mu\alpha}_\phi\tilde{\nabla}_\nu\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\right]\tilde{\nabla}_\mu\tilde{\nabla}_\nu\delta\phi\right\}$$

Therefore, the Fréchet derivative is given by the following operator

$$D_{\mathcal{E}} = \sqrt{-\tilde{g}}\left[E_\phi + H^\mu\tilde{\nabla}_\mu + H^{(\mu\nu)}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right] \tag{3.36}$$

where

$$H^\mu := E^\mu_\phi - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^\mu\phi + \left(E^{\alpha\beta}_\phi\tilde{\nabla}_\beta\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi \tag{3.37}$$
$$H^{\mu\nu} := E^{\mu\nu}_\phi + \left(E^{\mu\alpha}_\phi\tilde{\nabla}_\alpha\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi \tag{3.38}$$

At this point let us remember that the adjoint of a differential operator O, denoted bu $O^\dagger$ satisfies

$$\int \mathrm{d}^D x A O(B) = \int \mathrm{d}^D x B O^\dagger(A) \tag{3.39}$$

for every pair of differential functions A and B, with equality achieved up to boundary terms. In order to ensure that the equations arise from an action principle, we need $D_{\mathcal{E}}$ to be self-adjoint

$$\int \mathrm{d}^D x A D_{\mathcal{E}}(B) = \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[A\left(E_\phi + H^\mu\tilde{\nabla}_\mu + H^{(\mu\nu)}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\right)B\right] \tag{3.40}$$
$$= \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[AE_\phi B + AH^\mu\tilde{\nabla}_\mu B + AH^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu B\right] \tag{3.41}$$
$$= \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[BE_\phi A - B\tilde{\nabla}_\mu(H^\mu A) - \tilde{\nabla}_\mu(AH^{\mu\nu})\tilde{\nabla}_\nu B\right] + \mathrm{b.t} \tag{3.42}$$
$$= \int \mathrm{d}^D x\sqrt{-\tilde{g}}\left[BE_\phi A - B\tilde{\nabla}_\mu(H^\mu A) + B\tilde{\nabla}_\mu\tilde{\nabla}_\nu(H^{\mu\nu}A)\right] + \mathrm{b.t} \tag{3.43}$$

Then,

$$D_{\mathcal{E}}^{\dagger}(A) = \sqrt{-\tilde{g}}\left[E_\phi A - \tilde{\nabla}_\mu(H^\mu A) + \tilde{\nabla}_\mu\tilde{\nabla}_\nu(H^{\mu\nu}A)\right] \tag{3.44}$$

$$= \sqrt{-\tilde{g}}\left[E_\phi A - A\tilde{\nabla}_\mu H^\mu - H^\mu\tilde{\nabla}_\mu A + \tilde{\nabla}_\nu(A\tilde{\nabla}_\mu H^{\mu\nu} + H^{\mu\nu}\tilde{\nabla}_\mu A)\right] \tag{3.45}$$

But, in order to make appear $D_{\mathcal{E}}$ from (3.36), we can add an smart zero,

$$\begin{aligned}
D_{\mathcal{E}}^{\dagger}(A) &= \sqrt{-\tilde{g}}\left[E_\phi A - \tilde{\nabla}_\mu(H^\mu A) + \tilde{\nabla}_\mu\tilde{\nabla}_\nu(H^{\mu\nu}A)\right] + 2H^\mu\tilde{\nabla}_\mu A - 2H^\mu\tilde{\nabla}_\mu A \\
&= \sqrt{-\tilde{g}}\Big[E_\phi A - A\tilde{\nabla}_\mu H^\mu + H^\mu\tilde{\nabla}_\mu A + \tilde{\nabla}_\nu A\tilde{\nabla}_\mu H^{\mu\nu} + A\tilde{\nabla}_\mu\tilde{\nabla}_\nu H^{\mu\nu} \\
&\quad + \tilde{\nabla}_\nu H^{\mu\nu}\tilde{\nabla}_\mu A + H^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu A - 2H^\mu\tilde{\nabla}_\mu A\Big] \\
&= D_{\mathcal{E}}(A) + \sqrt{-\tilde{g}}\left[-A\tilde{\nabla}_\mu H^\mu + \tilde{\nabla}_\nu A\tilde{\nabla}_\mu H^{\mu\nu} + A\tilde{\nabla}_\mu\tilde{\nabla}_\nu H^{\mu\nu} + \tilde{\nabla}_\nu H^{\mu\nu}\tilde{\nabla}_\mu A - 2H^\mu\tilde{\nabla}_\mu A\right] \\
&= D_{\mathcal{E}}(A) + \sqrt{-\tilde{g}}\left[-A\tilde{\nabla}_\mu H^\mu + 2\tilde{\nabla}_\mu H^{\mu\nu}\tilde{\nabla}_\nu A + A\tilde{\nabla}_\mu\tilde{\nabla}_\nu H^{\mu\nu} - 2H^\mu\tilde{\nabla}_\mu A\right] \\
&= D_{\mathcal{E}}(A) + \sqrt{-\tilde{g}}\left[2J^\mu\tilde{\nabla}_\mu A + \tilde{\nabla}_\mu J^\mu A\right]
\end{aligned}$$

where we have defined

$$J^\mu := \tilde{\nabla}_\nu H^{(\nu\mu)} - H^\mu \tag{3.46}$$

Thus, for $D_{\mathcal{E}}$ to be self-adjoint, it must be fulfilled that

$$D_{\mathcal{E}}^{\dagger}(A) = D_{\mathcal{E}}(A) \Leftrightarrow J^\mu = 0 \tag{3.47}$$

and the Helmholtz condition is reduced to

$$\boxed{\tilde{\nabla}_\nu H^{(\nu\mu)} = H^\mu} \tag{3.48}$$

where

$$H^\mu := \left[E_\phi^\mu - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^\mu\phi + \left(E_\phi^{\alpha\beta}\tilde{\nabla}_\beta\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi\right] \tag{3.49}$$

$$H^{\mu\nu} := E_\phi^{\mu\nu} + \left(E_\phi^{\mu\alpha}\tilde{\nabla}_\alpha\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi \tag{3.50}$$

Helmholtz conditions implies that a symmetric second rank tensor depending up to second order in the scalar field ϕ and auxiliar metric $\tilde{g}_{\mu\nu}$ has a second order divergence. The most general tensor with these properties was characterized by Horndeski [2]. Consequently, $H^{\mu\nu}$ belongs to the Horndeski family builded for the auxiliar metric $\tilde{g}_{\mu\nu}$.

3.1 Quantities written in the auxiliar frame

As an example, let's consider again the action principle

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^D x\sqrt{-g}X^{D/2} \tag{3.51}$$

Its equation is rewritten in the auxiliar frame as

$$\begin{aligned}
\mathcal{E} &= \sqrt{-g}E && (3.52)\\
&= \sqrt{-g}\nabla_\mu \left(X^{\frac{D-2}{2}}\nabla^\mu\phi\right) && (3.53)\\
&= \partial_\mu \left(\underbrace{\sqrt{-g}X^{\frac{D}{2}}}_{\sqrt{-\tilde{g}}}\underbrace{X^{-1}g^{\mu\nu}}_{\tilde{g}^{\mu\nu}}\partial_\nu\phi\right) && (3.54)\\
&= \partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi) && (3.55)\\
&= \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi && (3.56)\\
&= \sqrt{-\tilde{g}}\tilde{\Box}\phi && (3.57)
\end{aligned}$$

which implies that

$$E = \tilde{\Box}\phi = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi \tag{3.58}$$

Thus

$$E_\phi^{\mu\nu} := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} = \tilde{g}^{\mu\nu} \tag{3.59}$$

So then

$$\begin{aligned}
H^{\mu\nu} &:= E_\phi^{\mu\nu} + \left(E_\phi^{\alpha\mu}\tilde{\nabla}_\alpha\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\\
&= \tilde{g}^{\mu\nu} + \left(\tilde{g}^{\alpha\mu}\tilde{\nabla}_\alpha\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{E}_\phi^{\lambda\rho}\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\\
&= \tilde{g}^{\mu\nu} + \left(\tilde{\nabla}^\mu\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{g}^{\lambda\rho}\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\\
&= \tilde{g}^{\mu\nu}\left(\tilde{\nabla}^\mu\phi - \frac{D}{2}\tilde{\nabla}^\mu\phi\right)\tilde{\nabla}^\nu\phi\\
&= \tilde{g}^{\mu\nu} - \frac{(D-2)}{2}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi && (3.60)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E_{\mu\nu} &:= -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}E)}{\partial\tilde{g}^{\mu\nu}} && (3.61)\\
&= -\frac{2}{\sqrt{-\tilde{g}}}\left(-\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}E + \sqrt{-\tilde{g}}\frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\right) && (3.62)\\
&= -2\left(-\frac{1}{2}\tilde{g}_{\mu\nu}E + \tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi\right) && (3.63)\\
&= -2\left(-\frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\Box}\phi + \tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi\right) && (3.64)\\
&= \tilde{g}_{\mu\nu}\tilde{\Box}\phi - 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi && (3.65)
\end{aligned}$$

which implies

$$\operatorname{tr}\tilde{E} := \tilde{g}^{\mu\nu}E_{\mu\nu} = D\tilde{\Box}\phi - 2\tilde{\Box}\phi = (D-2)\tilde{\Box}\phi \tag{3.66}$$

It is also clear to see that

$$E_\phi^\mu := \frac{\partial E}{\partial(\tilde{\nabla}_\mu \phi)} = 0 \tag{3.67}$$

So then

$$\begin{aligned}
H^\mu &:= \left[E_\phi^\mu - \frac{1}{2}\operatorname{tr}\tilde{E}\tilde{\nabla}^\mu\phi + \left(E_\phi^{\alpha\beta}\tilde{\nabla}_\beta\phi - \frac{1}{2}\operatorname{tr}\tilde{E}_\phi\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi\right] && (3.68)\\
&= -\frac{1}{2}(D-2)\tilde{\Box}\phi\tilde{\nabla}^\mu\phi + \left(\tilde{g}^{\alpha\beta}\tilde{\nabla}_\beta\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{E}_\phi^{\lambda\rho}\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}_\mu\phi && (3.69)\\
&= -\frac{(D-2)}{2}\tilde{\Box}\phi\tilde{\nabla}^\mu\phi + \left(\tilde{\nabla}^\alpha\phi - \frac{1}{2}\tilde{g}_{\lambda\rho}\tilde{g}^{\lambda\rho}\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi && (3.70)\\
&= -\frac{(D-2)}{2}\tilde{\Box}\phi\tilde{\nabla}^\mu\phi + \left(\tilde{\nabla}^\alpha\phi - \frac{D}{2}\tilde{\nabla}^\alpha\phi\right)\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi && (3.71)\\
&= -\frac{(D-2)}{2}\tilde{\Box}\phi\tilde{\nabla}^\mu\phi - \frac{(D-2)}{2}\tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi && (3.72)\\
&= -\frac{(D-2)}{2}\left(\tilde{\Box}\phi\tilde{\nabla}^\mu\phi + \tilde{\nabla}^\alpha\phi\tilde{\nabla}_\alpha\tilde{\nabla}^\mu\phi\right) && (3.73)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\tilde{\nabla}_\mu H^{\mu\nu} &= \tilde{\nabla}_\mu\left(\tilde{g}^{\mu\nu} - \frac{(D-2)}{2}\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi\right) && (3.74)\\
&= -\frac{(D-2)}{2}\tilde{\nabla}_\mu\left(\tilde{\nabla}^\mu\phi\tilde{\nabla}^\nu\phi\right) && (3.75)\\
&= -\frac{(D-2)}{2}\left(\tilde{\Box}\phi\tilde{\nabla}^\nu\phi + \tilde{\nabla}^\mu\phi\tilde{\nabla}_\mu\tilde{\nabla}^\nu\phi\right) && (3.76)\\
&= H^\nu && (3.77)
\end{aligned}$$

That is to say, the equations of motion coming from (3.51), writen in the auxiliar frame, satisfy the Hemholtz conditions (3.48).

4 Horndeski theorem

Horndeski theorem says that in a space of dimension four, the most general symmetric contravariant tensor density of the form

$$A^{ab} = A^{ab}(g_{ij}, \partial_h g_{ij}, \partial_h\partial_k g_{ij}, \phi, \partial_h\phi, \partial_h\partial_k\phi) \tag{4.1}$$

which is such that $\nabla_a A^{ab}$ is at most of second-order in the derivatives of both g_{ij} and ϕ is given by

$$\begin{aligned}
A^{ab} = \sqrt{-g}\Big\{&K_1\delta^{acde}_{fhjk}g^{fb}\nabla^h\nabla_c\phi R^{jk}_{de} + K_2\delta^{acd}_{efh}g^{eb}R^{fh}_{cd}\\
&+K_3\delta^{acde}_{fhjk}g^{fb}\nabla_c\phi\nabla^h\phi R^{jk}_{de} + K_4\delta^{acde}_{fhjk}g^{fb}\nabla_h\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi\\
&+K_5\delta^{acd}_{efh}g^{eb}\nabla^f\nabla_c\phi\nabla^h\nabla_d\phi + K_6\delta^{acde}_{fhjk}g^{fb}\nabla_c\phi\nabla^h\phi\nabla^j\nabla_f\phi\nabla^k\nabla_e\phi\\
&+K_7\delta^{ac}_{de}g^{db}\nabla^e\nabla_c\phi + K_8\delta^{acd}_{efh}g^{eb}\nabla_c\phi\nabla^f\phi\nabla^h\nabla_c\nabla_d\phi + K_9 g^{ab} + K_{10}\nabla^a\phi\nabla^b\phi\Big\}
\end{aligned} \tag{4.2}$$

where K_i are arbitrary differentiable functions of ϕ and $\partial_i \phi$ [2].

Note that the dependece on the Riemann tensor only appears in the first three terms of (4.2) with all indices contracted with the Kronecker delta, then, from its irreducible decomposition

$$R^{ab}_{mn} = C^{ab}_{mn} + 2\delta^{[a}_{[m} S^{b]}_{n]} + \frac{1}{6}\delta^{a}_{[m}\delta^{b}_{n]} R \tag{4.3}$$

only the term proportional to the trace of the Ricci tensor survives.

Now, using the identities

$$\delta^{\mu_1 \ldots \mu_s\, \mu_{s+1} \ldots \mu_p}_{\nu_1 \ldots \nu_s\, \mu_{s+1} \ldots \mu_p} = \frac{(D-s)!}{(D-p)!} \delta^{\mu_1 \ldots \mu_s}_{\nu_1 \ldots \nu_s}\,, \tag{4.4}$$

and

$$\delta^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p} = p! \delta^{\mu_1}_{[\nu_1} \ldots \delta^{\mu_p}_{\nu_p]} = p! \delta^{[\mu_1}_{\nu_1} \ldots \delta^{\mu_p]}_{\nu_p} \tag{4.5}$$

let's see how the first three terms of (4.2) look like:

$$\begin{aligned} K_1 \delta^{acde}_{fhjk} g^{fb} \nabla^h \nabla_c \phi R^{jk}_{de} &= \frac{K_1}{6} \delta^{acde}_{fhjk} g^{fb} \nabla^h \nabla_c \phi \delta^j_d \delta^k_e R &(4.6)\\ &= \frac{K_1}{6} \frac{(4-4+2)!}{(4-4)!} \delta^{ac}_{fh} g^{fb} \nabla^h \nabla_c \phi R &(4.7)\\ &= \frac{K_1}{3} \delta^{ac}_{fh} g^{fb} \nabla^h \nabla_c \phi R &(4.8) \end{aligned}$$

$$\begin{aligned} K_2 \delta^{acd}_{efh} g^{eb} R^{fh}_{cd} &= \frac{K_2}{6} \delta^{acd}_{efh} g^{eb} \delta^c_e \delta^h_d R &(4.9)\\ &= \frac{K_2}{6} \frac{(4-3+2)!}{(4-3)!} \delta^a_e g^{eb} R &(4.10)\\ &= K_2 g^{ab} R &(4.11) \end{aligned}$$

$$\begin{aligned} K_3 \delta^{acde}_{fhjk} g^{fb} \nabla_c \phi \nabla^h \phi R^{jk}_{de} &= \frac{K_3}{6} \delta^{acde}_{fhjk} g^{fb} \nabla_c \phi \nabla^h \phi \delta^j_d \delta^k_e R &(4.12)\\ &= \frac{K_3}{6} \frac{(4-4+2)!}{(4-4)!} \delta^{ac}_{fh} g^{fb} \nabla_c \phi \nabla^h \phi R &(4.13)\\ &= \frac{K_3}{3} \delta^{ec}_{fh} g^{fb} \nabla_c \phi \nabla^h \phi R &(4.14) \end{aligned}$$

Adding these terms, we have

$$
\begin{aligned}
[(4.8)+(4.11)+(4.14)] &= \frac{1}{3}\delta^{ac}_{fh}g^{fb}\left(K_1\nabla^h\nabla_c\phi + K_3\nabla_c\phi\nabla^h\phi\right)R + K_2 g^{ab}R \\
&= \frac{1}{3}\delta^a_f\delta^c_h g^{fb}(K_1\nabla^h\nabla_c\phi + K_3\nabla_c\phi\nabla^h\phi) - \frac{1}{3}\delta^a_h\delta^c_f g^{fb}(K_1\nabla^h\nabla_c\phi + K_3\nabla_c\phi\nabla^h\phi) \\
&\quad + K_2 g^{ab}R \\
&= \frac{1}{3}g^{ab}(K_1\Box\phi + K_3\nabla_c\phi\nabla^c\phi)R - \frac{1}{3}g^{cb}(K_1\nabla^a\nabla_c\phi + K_3\nabla_c\phi\nabla^a\phi)R + K_2 g^{ab}R \\
&= \frac{1}{3}g^{ab}(K_1\Box\phi + K_3\nabla_c\phi\nabla^c\phi)R - \frac{1}{3}(K_1\nabla^a\nabla^b\phi + K_3\nabla^a\phi\nabla^b\phi)R + K_2 g^{ab}R \\
&= \frac{1}{3}g^{ab}(K_1\Box\phi - 2K_3X)R - \frac{1}{3}(K_1\nabla^a\nabla^b\phi + K_3\nabla^a\phi\nabla^b\phi)R + K_2 g^{ab}R \\
&= \frac{1}{3}g^{ab}R(K_1\Box\phi + 3K_2 - 2K_3X) - \frac{1}{3}(K_1\nabla^a\nabla^b\phi + K_3\nabla^a\phi\nabla^b\phi)R
\end{aligned}
$$

Since conformal invariance only allows those second order tensors independent S^a_b and R and here are all the terms that depend on the curvature in (4.2), we have

$$
0 = \frac{\partial}{\partial R}\left(\frac{A^{ab}}{\sqrt{-g}}\right) = \frac{1}{3}g^{ab}(K_1\Box\phi + 3K_2 - 2K_3X) - \frac{1}{3}(K_1\nabla^a\nabla^b\phi + K_3\nabla^a\phi\nabla^b\phi) \tag{4.15}
$$

Taking the trace,

$$
\begin{aligned}
0 = g_{ab}\frac{\partial}{\partial R}\left(\frac{A^{ab}}{\sqrt{-g}}\right) &= \frac{4}{3}(K_1\Box\phi + 3K_2 - 2K_3X) - \frac{1}{3}(K_1\Box\phi - 2K_3X) \\
&= K_1\Box\phi + 4K_2 - 2K_3X
\end{aligned} \tag{4.16}
$$

Since $K_i = K_i(\phi, \partial_a\phi)$ are independent of second scalar field derivatives we have

$$
K_1 = 0 \qquad \text{and} \qquad K_2 = \frac{1}{2}K_3X \tag{4.17}
$$

Plugging into (4.16),

$$
0 = g_{ab}\frac{\partial}{\partial R}\left(\frac{A^{ab}}{\sqrt{-g}}\right) = \frac{1}{3}g^{ab}\left(\frac{3}{2}K_3X - 2K_3X\right) - \frac{1}{3}K_3(\nabla^a\phi\nabla^b\phi) \tag{4.18}
$$

$$
= -\frac{1}{6}g^{ab}K_3X - \frac{1}{3}K_3\nabla^a\phi\nabla^b\phi \tag{4.19}
$$

$$
= -\frac{K_3}{3}\left(\nabla^a\phi\nabla^b\phi + \frac{1}{2}g^{ab}X\right) \tag{4.20}
$$

Now, using the fact that given any scalar field there always exists a vector field Y^a for which [2]

$$
Y^a\nabla_a\phi = 0 \qquad \text{and} \qquad Y^aY_a \neq 0 \tag{4.21}
$$

we can multiply (4.20) by Y_aY_b,

$$
0 = -\frac{K_3}{3}\left(\cancel{Y_a\nabla^a\phi Y_b\nabla^b\phi} + \frac{1}{2}Y_aY^a g^{ab}X\right) \tag{4.22}
$$

$$
= -\frac{K_3}{3}\frac{1}{2}Y^aY_aX \tag{4.23}
$$

but, $X \neq 0$, so then $K_3 = 0$. In summary we have

$$K_1 = K_2 = K_3 = 0\,. \tag{4.24}$$

Let's consider now the associated divergence, calculated by Horndeski as [2]

$$\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= K_1'\delta^{acde}_{fhjk}\nabla^f\phi\nabla^h\nabla_c\phi R^{jk}_{de} + 2\dot{K}_2\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi R^{fh}_{cd} \\
&+ K_3\delta^{acde}_{fhjk}\nabla^h\phi\nabla^f\nabla_c\phi R^{jk}_{de} + K_5\delta^{acd}_{efh}\nabla^m\phi R^{fe}_{mc}\nabla^h\nabla_d\phi \\
&+ 2\dot{K}_1\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla^h\nabla_c\phi R^{jk}_{de} + \frac{3}{2}K_4\delta^{acde}_{fhjk}\nabla^m R^{hf}_{mc}\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ \frac{1}{2}K_1\delta^{acde}_{fhjk}\nabla^m\phi R^{hf}_{mc}R^{jk}_{de} + K_2'\delta^{acd}_{efh}\nabla^e\phi R^{fh}_{cd} + \frac{1}{2}K_7\delta^{ac}_{de}\nabla^m\phi R^{ed}_{mc} \\
&+ \frac{1}{2}K_8\delta^{acd}_{efh}\nabla_c\phi\nabla^f\phi\nabla^m\phi R^{he}_{md} + 2\dot{K}_3\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla_c\phi R^{jk}_{de} \\
&+ K_6\delta^{acde}_{fhjk}\nabla_c\phi\nabla^h\phi\nabla^m\phi R^{jf}_{md}\nabla^k\nabla_e\phi + K_4'\delta^{acde}_{fhjk}\nabla^f\phi\nabla^h\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ 2\dot{K}_5\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi\nabla^f\nabla_c\phi\nabla^h\nabla_d\phi + K_6\delta^{acde}_{fhjk}\nabla^h\phi\nabla^f\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ 2\dot{K}_8\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi\nabla_c\phi\nabla^f\phi\nabla^h\nabla_d\phi + 2\dot{K}_6\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla_c\phi\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ 2\dot{K}_4\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla^h\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + K_5'\delta^{acd}_{efh}\nabla^e\phi\nabla^f\nabla_c\phi\nabla^h\nabla_d\phi \\
&+ 2\dot{K}_7\delta^{ac}_{de}\nabla_p\phi\nabla^d\nabla^p\phi\nabla^e\nabla_c\phi + K_8\delta^{acd}_{efh}\nabla^f\phi\nabla^e\nabla_c\phi\nabla^h\nabla_d\phi + (2\dot{K}_9 + K_{10})\nabla_b\phi\nabla^b\nabla^a\phi \\
&+ K_7'\delta^{ac}_{de}\nabla^d\phi\nabla^e\nabla_c\phi + \nabla^a\phi(K_9' + \rho K_{10}' + 2\dot{K}_{10}\nabla^b\phi\nabla^c\phi\nabla_c\nabla_b\phi + K_{10}\nabla^c\nabla_c\phi)
\end{aligned} \tag{4.25}$$

where a prime denotes a partial derivative with respect to ϕ and a dot denotes a partial derivative with respect to ρ. Pluggin (4.24) into (4.25) we obtain

$$\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= K_5\delta^{acd}_{efh}\nabla^m\phi R^{fe}_{mc}\nabla^h\nabla_d\phi + \frac{3}{2}K_4\delta^{acde}_{fhjk}\nabla^m R^{hf}_{mc}\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + \frac{1}{2}K_7\delta^{ac}_{de}\nabla^m\phi R^{ed}_{mc} \\
&+ \frac{1}{2}K_8\delta^{acd}_{efh}\nabla_c\phi\nabla^f\phi\nabla^m\phi R^{he}_{md} + K_6\delta^{acde}_{fhjk}\nabla_c\phi\nabla^h\phi\nabla^m\phi R^{jf}_{md}\nabla^k\nabla_e\phi \\
&+ K_4'\delta^{acde}_{fhjk}\nabla^f\phi\nabla^h\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + 2\dot{K}_5\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi\nabla^f\nabla_c\phi\nabla^h\nabla_d\phi \\
&+ K_6\delta^{acde}_{fhjk}\nabla^h\phi\nabla^f\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + 2\dot{K}_8\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi\nabla_c\phi\nabla^f\phi\nabla^h\nabla_d\phi \\
&+ 2\dot{K}_6\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla_c\phi\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ 2\dot{K}_4\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla^h\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + K_5'\delta^{acd}_{efh}\nabla^e\phi\nabla^f\nabla_c\phi\nabla^h\nabla_d\phi \\
&+ 2\dot{K}_7\delta^{ac}_{de}\nabla_p\phi\nabla^d\nabla^p\phi\nabla^e\nabla_c\phi + K_8\delta^{acd}_{efh}\nabla^f\phi\nabla^e\nabla_c\phi\nabla^h\nabla_d\phi + (2\dot{K}_9 + K_{10})\nabla_b\phi\nabla^b\nabla^a\phi \\
&+ K_7'\delta^{ac}_{de}\nabla^d\phi\nabla^e\nabla_c\phi + \nabla^a\phi(K_9' + \rho K_{10}' + 2\dot{K}_{10}\nabla^b\phi\nabla^c\phi\nabla_c\nabla_b\phi + K_{10}\nabla^c\nabla_c\phi)
\end{aligned}$$

The scalar curvature dependence would be through those terms with the Riemann tensor contracted totally with the Kronecker delta

$$
\begin{aligned}
\frac{6}{R}\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= K_5 \delta^{acd}_{efh}\delta^f_m\delta^e_c \nabla^m\phi\nabla^h\nabla_d\phi + \frac{3}{2}K_4\delta^{acde}_{fhjk}\delta^h_m\delta^f_c\nabla^m\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + \frac{1}{2}K_7\delta^{ac}_{de}\delta^e_m\delta^d_c\nabla^m\phi \\
&\quad + \frac{1}{2}K_8\delta^{acd}_{efh}\delta^h_m\delta^e_d\nabla_c\phi\nabla^f\phi\nabla^m\phi + K_6\delta^{acde}_{fhjk}\delta^j_m\delta^f_d\nabla_c\phi\nabla^h\phi\nabla^m\phi\nabla^k\nabla_e\phi \\
&= -2K_5\delta^{ad}_{fh}\nabla^f\phi\nabla^h\nabla_d\phi - \frac{3}{2}K_4\delta^{ade}_{hjk}\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi - \frac{3}{2}K_7\nabla^a\phi \\
&\quad + K_8\delta^{ac}_{fh}\nabla_c\phi\nabla^f\phi\nabla^h\phi + K_6\delta^{ace}_{hjk}\nabla_c\phi\nabla^h\phi\nabla^j\phi\nabla^k\nabla_e\phi
\end{aligned}
$$

Note that due to symmetry in the covariant derivative indices contracted with the anti-symmetric Kronecker delta, the last two terms vanish. Therefore,

$$
\begin{aligned}
\frac{6}{R}\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= -2K_5\delta^{ad}_{fh}\nabla^f\phi\nabla^h\nabla_d\phi - \frac{3}{2}K_4\delta^{ade}_{hjk}\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi - \frac{3}{2}K_7\nabla^a\phi \\
&= -2K_5\left(\delta^a_f\delta^d_h - \delta^d_f\delta^a_h\right)\nabla^f\phi\nabla^h\nabla_d\phi - \frac{3}{2}K_7\nabla^a\phi \\
&\quad - \frac{3}{2}K_4 3!\left(\delta^a_h\delta^d_j\delta^e_k + \delta^a_j\delta^d_k\delta^e_h + \delta^a_k\delta^d_h\delta^e_j - \delta^a_j\delta^d_h\delta^e_k - \delta^a_h\delta^d_k\delta^e_j - \delta^a_j\delta^d_k\delta^e_h\right)\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&= -2K_5(\nabla^a\Box\phi - \nabla^d\phi\nabla^a\nabla_d\phi) - \frac{3}{2}K_7\nabla^a\phi \\
&\quad - 9K_4\Big[\nabla^a(\Box\phi)^2 + \nabla^e\phi\nabla^a\nabla_d\phi\nabla^d\nabla_e\phi + \nabla^d\phi\nabla^e\nabla_d\phi\nabla^a\nabla_e\phi - \nabla^d\phi\nabla^a\nabla_d\phi\Box\phi \\
&\qquad -\nabla^a\phi\nabla^e\nabla_d\phi\nabla^d\nabla_e\phi - \nabla^d\phi\nabla^a\nabla_d\phi\Box\phi\Big]
\end{aligned}
$$

Introducing the following notation,

$$
\phi^a := \nabla^a\phi, \quad \phi^{ab} := \nabla^a\nabla^b\phi, \quad X^a := \nabla^a X = -\nabla^b\nabla^a\nabla_b\phi = -\phi^b\nabla^a\phi_b \tag{4.26}
$$

we have

$$
\begin{aligned}
\frac{6}{R}\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= -2K_5(\phi^a\Box\phi + X^a) - \frac{3}{2}K_7\phi^a \\
&\quad - 9K_4\left[\phi^a(\Box\phi)^2 - \nabla^a\phi_d X^d - X^e\nabla^a\phi_e + X^a\Box\phi - \phi^a\phi_{ed}\phi^{ed} + X^a\Box\phi\right] \\
&= -2K_5(\phi^a\Box\phi + X^a) - \frac{3}{2}K_7\phi^a \\
&\quad - 9K_4\left[\phi^a(\Box\phi)^2 - \phi^a\phi_{ed}\phi^{ed} + 2X^a\Box\phi - 2X^d\nabla^a\phi_d\right]
\end{aligned}
$$

The partial derivative with respect to the scalar curvature is given by

$$
\frac{\partial}{\partial R}\left(\frac{6\nabla_b A^{ab}}{\sqrt{-g}}\right) = -2K_5(\phi^a\Box\phi + X^a) - \frac{3}{2}K_7\phi^a \tag{4.27}
$$

$$
- 9K_4\left[\phi^a(\Box\phi)^2 - \phi^a\phi_{ed}\phi^{ed} + 2X^a\Box\phi - 2X^d\nabla^a\phi_d\right] \tag{4.28}
$$

[Check factors]

The above must vanish for any value of the second derivative. Since the terms with the same degree have a common coefficient, such coefficients must vanish independently. Hence

$$
K_4 = K_5 = K_7 = 0 \tag{4.29}
$$

In this way, the divergence of A^{ab} is reduced to

$$
\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} = &\frac{1}{2}K_8\delta^{acd}_{efh}\nabla_c\phi\nabla^f\phi\nabla^m\phi R^{he}_{md} + K_6\delta^{acde}_{fhjk}\nabla_c\phi\nabla^h\phi\nabla^m\phi R^{jf}_{md}\nabla^k\nabla_e\phi \\
&+ K_6\delta^{acde}_{fhjk}\nabla^h\phi\nabla^f\nabla_c\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi + 2\dot{K}_8\delta^{acd}_{efh}\nabla_p\phi\nabla^e\nabla^p\phi\nabla_c\phi\nabla^f\phi\nabla^h\nabla_d\phi \\
&+ 2\dot{K}_6\delta^{acde}_{fhjk}\nabla_p\phi\nabla^f\nabla^p\phi\nabla_c\phi\nabla^h\phi\nabla^j\nabla_d\phi\nabla^k\nabla_e\phi \\
&+ K_8\delta^{acd}_{efh}\nabla^f\phi\nabla^e\nabla_c\phi\nabla^h\nabla_d\phi + (2\dot{K}_9 + K_{10})\nabla_b\phi\nabla^b\nabla^a\phi \\
&+ \nabla^a\phi(K_9' + \rho K_{10}' + 2\dot{K}_{10}\nabla^b\phi\nabla^c\phi\nabla_c\nabla_b\phi + K_{10}\nabla^c\nabla_c\phi)
\end{aligned}
$$

Now, let's see the dependence of the traceless Ricci tensor S^a_b in the expression above. Using (4.3) we have

$$
\begin{aligned}
\frac{\nabla_b A^{ab}}{\sqrt{-g}} &= \frac{1}{2}K_8\delta^{acd}_{efh}\phi_c\phi^f\phi^m R^{he}_{md} + K_6\delta^{acde}_{fhjk}\phi_c\phi^h\phi^m\phi^k_e R^{jf}_{md} \\
&= \frac{1}{2}2K_8\delta^{acd}_{efh}\delta^{[h}_{[m}S^{e]}_{d]}\phi_c\phi^f\phi^m + 2K_6\delta^{acde}_{fhjk}\delta^{[j}_{[m}S^{f]}_{d]}\phi_c\phi^h\phi^m\phi^k_e \\
&= \frac{K_8}{2}\delta^{acd}_{efh}\left(\delta^h_m S^e_d - \delta^h_d S^e_m\right)\phi_c\phi^f\phi^m + K_6\delta^{acde}_{fhjk}\left(\delta^j_m S^f_d - \delta^j_d S^f_m\right)\phi_c\phi^h\phi^m\phi^k_e \\
&= \frac{K_8}{2}\left(\underbrace{\delta^{acd}_{efh}S^e_d\phi_c\phi^{(f}\phi^{h)}}_{=0} - 2\delta^{ac}_{ef}S^e_m\phi_c\phi^f\phi^m\right) + K_6\left(\underbrace{\delta^{acde}_{fhjk}S^f_d\phi_c\phi^{(h}\phi^{j)}\phi^k_e}_{=0} - \delta^{ace}_{fhk}S^f_m\phi_c\phi^h\phi^m\phi^k_e\right) \\
&= -K_8\delta^{ac}_{ef}S^e_m\phi_c\phi^f\phi^m - K_6\delta^{ace}_{fhk}S^f_m\phi_c\phi^h\phi^m\phi^k_e \\
&= -2K_8\delta^a_{[e}\delta^c_{f]}S^e_m\phi_c\phi^f\phi^m - 6K_6\delta^a_{[f}\delta^c_h\delta^e_{k]}S^f_m\phi_c\phi^h\phi^m\phi^k_e
\end{aligned}
$$

Since both terms have different degree in the second derivative, they must vanish independently, that is

$$K_6 = K_8 = 0 \tag{4.30}$$

In summary, if one demands that only curvature couplings occurs through the Weyl tensor, then there is no nonminimal coupling at all, and the second order tensor becomes, in fact, of first order

$$A^{ab} = \sqrt{-g}\left\{K_9 g^{ab} + K_{10}\nabla^a\phi\nabla^b\phi\right\} \tag{4.31}$$

with second order divergence given by

$$\nabla_b A^{ab} = \sqrt{-g}\Big\{(2\dot{K}_9 + K_{10})\nabla_b\phi\nabla^b\nabla^a\phi \tag{4.32}$$

$$+\nabla^a\phi(K_9' + \rho K_{10}' + 2\dot{K}_{10}\nabla^b\phi\nabla^c\phi\nabla_c\nabla_b\phi + K_{10}\nabla^c\nabla_c\phi)\Big\} \tag{4.33}$$

A Some useful calculations

A.1 Variation of the Christoffel symbols

The Christoffel symbols in terms of the metric are given by

$$\Gamma^{\lambda}{}_{\mu\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right)$$

Varying both sides, we have

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= \frac{1}{2}\delta g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= -\frac{1}{2} g^{\lambda\sigma} g^{\rho\tau} (\delta g_{\sigma\tau}) \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= -g^{\lambda\sigma}(\delta g_{\sigma\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right)
\end{aligned}$$

Changing the dumb indice σ by ρ,

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= -g^{\lambda\rho}(\delta g_{\rho\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} - 2\delta g_{\rho\tau} \Gamma^{\tau}{}_{\mu\beta} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \big(\partial_\mu \delta g_{\beta\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\mu} \delta g_{\tau\beta} + \partial_\beta \delta g_{\mu\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\beta} \delta g_{\tau\mu} \\
&\quad - \partial_\rho \delta g \mu\beta + \Gamma^{\tau}{}_{\mu\rho} \delta g_{\tau\beta} + \Gamma^{\tau}{}_{\beta\rho} \delta g_{\mu\tau} \big) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\nabla_\mu \delta g_{\beta\rho} + \nabla_\beta \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\beta} \right) \qquad \text{(A.1)}
\end{aligned}$$

A.2 Variation of the Riemann tensor

The Riemann tensor is given by

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} - \Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu}$$

Varying both sides,

$$\begin{aligned}
\delta R^{\rho}{}_{\lambda\mu\nu} &= \partial_\mu \delta\Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\lambda\mu} + \delta\Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\lambda\nu} - \delta\Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\lambda\mu} \\
&= \partial_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\nu\lambda} - \Gamma^{\tau}{}_{\mu\lambda} \delta\Gamma^{\rho}{}_{\tau\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\tau}{}_{\nu\lambda} \delta\Gamma^{\rho}{}_{\tau\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\mu\lambda}
\end{aligned}$$

Adding a convenient zero of the form $\Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda} - \Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda}$, and using the fact that $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor, we have

$$\delta R^{\rho}{}_{\lambda\mu\nu} = \nabla_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} = 2\nabla_{[\mu} \delta\Gamma^{\rho}{}_{\nu]\lambda} \qquad \text{(A.2)}$$

A.3 Variation of derivatives of ϕ w.r.t ω

Let's compute the infinitesimal variations of the covariant derivatives of the scalar field. First, let us calculate $\delta_\omega \nabla_\mu \phi$:

$$\delta_\omega \nabla_\mu \phi = \nabla_\mu \delta_\omega \phi = 0 \qquad \text{(A.3)}$$

Now, let's compute $\delta_\omega(\nabla_\mu\nabla_\nu\phi)$:

$$\begin{aligned}
\delta_\omega(\nabla_\mu\nabla_\nu\phi) &= \delta_\omega\nabla_\mu(\partial_\nu\phi) && \text{(A.4)}\\
&= \delta_\omega(\partial_\mu\partial_\nu\phi - \Gamma^\lambda{}_{\nu\mu}\partial_\lambda\phi) && \text{(A.5)}\\
&= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\Gamma^\lambda{}_{\nu\mu}\partial_\lambda\phi - \Gamma^\lambda{}_{\nu\mu}\delta_\omega\partial_\lambda\phi && \text{(A.6)}\\
&= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\Gamma^\lambda{}_{\nu\mu}\partial_\lambda\phi - \Gamma^\lambda{}_{\nu\mu}\partial_\lambda\delta_\omega\phi && \text{(A.7)}\\
&= -\partial_\lambda\phi\delta_\omega\Gamma^\lambda{}_{\nu\mu} && \text{(A.8)}
\end{aligned}$$

Using that the variation of the Christoffel connection is

$$\delta\Gamma^\lambda{}_{\mu\beta} = \frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\beta\rho} + \nabla_\beta\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\beta}) \tag{A.9}$$

we have

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi\frac{1}{2}g^{\lambda\rho}(\nabla_\nu\delta_\omega g_{\mu\rho} + \nabla_\mu\delta_\omega g_{\nu\rho} - \nabla_\rho\delta_\omega g_{\nu\beta}) \tag{A.10}$$

but $\delta_\omega g_{\nu\mu} = 2\omega g_{\nu\mu}$, so then

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi g^{\lambda\rho}\left[\nabla_\nu(\omega g_{\mu\rho}) + \nabla_\mu(\omega g_{\nu\rho}) - \nabla_\rho(\omega g_{\nu\beta})\right] \tag{A.11}$$

Using the metric compatibility condition $\nabla_\mu g_{\alpha\beta} = 0$ and $\nabla_\alpha\phi = \partial_\alpha\phi$, we obtain

$$\begin{aligned}
\delta_\omega(\nabla_\mu\nabla_\nu\phi) &= -\partial_\lambda\phi g^{\lambda\rho}(g_{\mu\rho}\nabla_\nu\omega + g_{\nu\rho}\nabla_\mu\omega - g_{\nu\beta}\nabla_\rho\omega) && \text{(A.12)}\\
&= -\partial^\rho\phi(g_{\mu\rho}\nabla_\nu\omega + g_{\nu\rho}\nabla_\mu\omega - g_{\nu\beta}\nabla_\rho\omega) && \text{(A.13)}\\
&= -\nabla_\mu\phi\nabla_\nu\omega - \nabla_\nu\phi\nabla_\mu\omega + g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega && \text{(A.14)}\\
&= -2\nabla_{(\mu}\phi\nabla_{\nu)}\omega + g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega && \text{(A.15)}
\end{aligned}$$

A.4 Variation of E w.r.t Riemann tensor

In order to see what (1.20c) implies, let us split the dependence of the Riemann tensor in terms of its traceless part $C^{\alpha\beta}_{\mu\nu}$, the traceless part of the Ricci tensor S^α_β, and the scalar curvature R. So we have

$$E\left(g^{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = E\left(g^{\mu\nu}, C^{\alpha\beta}_{\mu\nu}, S^\alpha_\beta, R\right)$$

The variation w.r.t the Riemann tensor yields

$$\begin{aligned}
\delta_{\text{Riem}}E &= P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta}\\
&= H^{\alpha\beta}_{\mu\nu}\delta_{\text{Riem}}C^{\mu\nu}_{\alpha\beta} + I^\alpha_\beta\delta_{\text{Riem}}S^\beta_\alpha + J\delta_{\text{Riem}}R && \text{(A.16)}
\end{aligned}$$

where

$$H^{\alpha\beta}_{\mu\nu} \equiv \frac{\partial E}{\partial C^{\mu\nu}_{\alpha\beta}}, \qquad I^\alpha_\beta \equiv \frac{\partial E}{\partial S^\beta_\alpha} \qquad \text{y} \qquad J \equiv \frac{\partial E}{\partial R}$$

Since $P^{\mu\nu}_{\alpha\beta}$ has the same algebraic symmetries as the Riemann tensor, its traceless part is given by

$$\hat{P}^{\mu\nu}_{\alpha\beta} = P^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}\delta^{[\mu}_{[\alpha}P^{\nu]}_{\beta]} + \frac{2}{(D-2)(D-1)}P\delta^\mu_{[\alpha}\delta^\nu_{\beta]} \tag{A.17}$$

Let us note that

$$\begin{aligned} J\delta_{\text{Riem}} R &= J\delta_{\text{Riem}} \left(R^{\alpha\beta}_{\mu\nu} \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} \right) \\ &= J\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} R^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{A.18}$$

Writing S^{α}_{β} in terms of the Riemann,

$$\begin{aligned} S^{\beta}_{\nu} &= R^{\beta}_{\nu} - \frac{1}{D} R \delta^{\beta}_{\nu} \\ &= R^{\alpha\beta}_{\mu\nu} \delta^{\mu}_{\alpha} - \frac{1}{D} \delta^{\beta}_{\nu} Rs^{\alpha\gamma}_{\mu\lambda} \delta^{\mu}_{\alpha} \delta^{\lambda}_{\gamma} \end{aligned}$$

then,

$$\delta_{\text{Riem}} \tilde{S}^{\beta}_{\nu} = \delta^{\mu}_{\alpha} \delta \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D} \delta^{\beta}_{\nu} \delta \tilde{R}^{\alpha\gamma}_{\mu\lambda} \delta^{\mu}_{\alpha} \delta^{\lambda}_{\gamma}$$

Hence,

$$\begin{aligned} I^{\nu}_{\beta} \delta_{\text{Riem}} S^{\beta}_{\nu} &= I^{\nu}_{\beta} \delta^{\mu}_{\alpha} \delta R^{\alpha\beta}_{\mu\nu} - \frac{1}{D} I^{\nu}_{\beta} \delta^{\beta}_{\nu} \delta^{\mu}_{\alpha} \delta^{\lambda}_{\gamma} \delta R^{\alpha\gamma}_{\mu\lambda} \\ &= I^{\nu}_{\beta} \delta^{\mu}_{\alpha} \delta \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D} I \delta^{\mu}_{\alpha} \delta R^{\alpha\gamma}_{\mu\lambda} \\ &= \delta^{\mu}_{\alpha} \delta R^{\alpha\beta}_{\mu\nu} \left(I^{\nu}_{\beta} - \frac{1}{D} I \delta^{\nu}_{\beta} \right) \\ &= \delta^{\mu}_{\alpha} \hat{I} \delta R^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{A.19}$$

Finally, let us write the Weyl tensor in terms of the Riemann,

$$\begin{aligned} \tilde{C}^{\alpha\beta}_{\mu\nu} &= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2} \delta^{[\alpha}_{[\mu} \tilde{R}^{\beta]}_{\nu]} + \frac{2}{(D-1)(D-2)} \tilde{R} \delta^{[\alpha}_{[\mu} \delta^{\beta]}_{\nu]} \\ &= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2} \delta^{\lambda}_{\gamma} \delta^{[\alpha}_{[\mu} \tilde{R}^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)} \delta^{[\alpha}_{[\mu} \delta^{\beta]}_{\nu]} \delta^{\tau}_{\rho} \delta^{\lambda}_{\sigma} \tilde{R}^{\rho\sigma}_{\tau\lambda} \end{aligned}$$

Varying with respect to $R^{\alpha\beta}_{\mu\nu}$,

$$\delta_{\text{Riem}} C^{\alpha\beta}_{\mu\nu} = \delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2} \delta^{\lambda}_{\gamma} \delta^{[\alpha}_{[\mu} \delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)} \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu]} \delta^{\tau}_{\rho} \delta^{\lambda}_{\sigma} \delta R^{\rho\sigma}_{\tau\lambda}$$

Then,

$$\begin{aligned} H^{\mu\nu}_{\alpha\beta} \delta_{\text{Riem}} C^{\alpha\beta}_{\mu\nu} &= H^{\mu\nu}_{\alpha\beta} \left[\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2} \delta^{\lambda}_{\gamma} \delta^{[\alpha}_{[\mu} \delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)} \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu]} \delta^{\tau}_{\rho} \delta^{\lambda}_{\sigma} \delta R^{\rho\sigma}_{\tau\lambda} \right] \\ &= H^{\mu\nu}_{\alpha\beta} \delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2} H^{\lambda\nu}_{\gamma\beta} \delta^{\mu}_{\alpha} \delta^{\gamma}_{\lambda} \delta R^{\beta\alpha}_{\nu\mu} + \frac{2}{(D-1)(D-2)} H^{\tau\lambda}_{\rho\sigma} \delta^{\rho}_{\tau} \delta^{\sigma}_{\lambda} \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} \delta R^{\alpha\beta}_{\mu\nu} \\ &= \delta R^{\alpha\beta}_{\mu\nu} \left[H^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2} H^{\nu}_{\beta} \delta^{\mu}_{\alpha} + \frac{2}{(D-1)(D-2)} H \right] \\ &= \hat{H}^{\mu\nu}_{\alpha\beta} \delta R^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{A.20}$$

where the indices have been renamed in a convenient way and has been used the fact that $H^{\alpha\beta}_{\mu\nu}$ has the same algebraic symmetries as the Riemann tensor.

In this way, plugging (A.18), (A.19) and (A.20) into (A.16), we obtain

$$P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta} = \hat{H}^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta} + \delta^{\alpha}_{\mu}\hat{I}^{\beta}_{\nu}\delta R^{\mu\nu}_{\alpha\beta} + J\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta R^{\mu\nu}_{\alpha\beta}$$

Hence,

$$P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu}\hat{I}^{\beta]}_{\nu]} + J\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \tag{A.21}$$

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