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Zero-weight scalar fields

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Abstract: Personal compilation of some calculations related to zero-weight conformal scalar fields.

Contents

1 Building the equation of motion

Let us consider a scalar field with zero conformal weight, that is, under an infinitesimal conformal transformation

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad \delta_\omega\phi = 0 \tag{1.1}$$

Let us consider the most general second order pseudoscalar constructed from the scalar field ϕ and its derivatives up to second order, together with the metric tensor and its associates curvature

$$\mathcal{E} = \sqrt{-g}E\left(\phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi, g_{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = 0 \tag{1.2}$$

The variation of this equation under infinitesimal conformal transformations reads,

$$\begin{align}\delta_\omega\mathcal{E} &= \delta_\omega(\sqrt{-g})E + \sqrt{-g}\delta_\omega E \tag{1.3}\\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\delta_\omega E \tag{1.4}\end{align}$$

From $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta_\omega(g_{\mu\nu}g^{\mu\nu}) = 0$ implies that

$$g_{\mu\nu}\delta_\omega g^{\mu\nu} = -g^{\mu\nu}\delta_\omega g_{\mu\nu} \tag{1.5}$$

Furthermore, from (A.3) and (A.15), we have

$$\delta_\omega\mathcal{E} = \frac{\partial E}{\partial\phi}\cancelto{0}{\delta_\omega\phi} + \frac{\partial E}{\partial(\nabla_\mu\phi)}\cancelto{0}{\delta_\omega\nabla_\mu\phi} + \frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\delta_\omega(\nabla_\mu\nabla_\nu\phi) + \frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} \tag{1.6}$$

where we have defined

$$P^{\mu\nu}_{\alpha\beta} := \frac{\partial E}{\partial R^{\alpha\beta}_{\mu\nu}}, \qquad P^{\mu}_{\nu} := P^{\alpha\mu}_{\alpha\nu} \tag{1.7}$$

So then

$$\begin{aligned}\delta_\omega\mathcal{E} &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\delta_\omega(\nabla_\mu\nabla_\nu\phi) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu}\\ &= \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_{(\mu}\phi\nabla_{\nu)}\omega\right) + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}}\delta_\omega g^{\mu\nu} + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta_\omega g^{\mu\nu}E\\ &= \sqrt{-g}\frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}\left(g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_{(\mu}\phi\nabla_{\nu)}\omega\right) + \sqrt{-g}P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu} - 2\omega g^{\mu\nu}\sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right)\end{aligned}$$

Note that

$$\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}E + \sqrt{-g}\frac{\partial E}{\partial g^{\mu\nu}} \tag{1.8}$$

$$= \sqrt{-g}\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) \tag{1.9}$$

By defining

$$A^{\mu\nu} := \frac{\partial E}{\partial(\nabla_\mu\nabla_\nu\phi)}, \qquad B_{\mu\nu} = -2\left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}E\right) = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}E)}{\partial g^{\mu\nu}} \tag{1.10}$$

we obtain

$$\delta_\omega\mathcal{E} = \sqrt{-g}A^{\mu\nu}\left(\gamma_{\mu\nu}\nabla^\rho\phi\nabla_\rho\omega - 2\nabla_\mu\phi\nabla_\nu\omega\right) + \sqrt{-g}P^{\mu\nu}\delta_\omega R^{\alpha\beta}_{\mu\nu} + \sqrt{-g}\omega B_{\mu\nu}g_{\mu\nu} \tag{1.11}$$

$$= \sqrt{-g}\left(A\nabla^\rho\phi\nabla_\rho\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + P^{\mu\nu}_{\alpha\beta}\delta_\omega R^{\alpha\beta}_{\mu\nu}\right) \tag{1.12}$$

Using

$$\delta_\omega R^\alpha{}_{\beta\mu\nu} = 2\nabla_{[\mu}\delta_\omega\Gamma^\alpha{}_{\nu]\beta} \tag{1.13}$$

we have

$$\delta_\omega\mathcal{E} = \sqrt{-g}\left(A\nabla^\rho\phi\nabla_\rho\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^\alpha{}_{\nu]\beta}\right) \tag{1.14}$$

$$= \sqrt{-g}\left(A\nabla^\nu\phi\nabla_\nu\omega - 2A^{\mu\nu}\nabla_\mu\phi\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^\alpha{}_{\nu]\beta}\right) \tag{1.15}$$

$$= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_{[\mu}\delta_\omega\Gamma^\alpha{}_{\nu]\beta}\right] \tag{1.16}$$

Noting that

$$\delta_\omega\Gamma^\alpha{}_{\nu\beta} = \frac{1}{2}g^{\alpha\lambda}\left[\nabla_\nu(\delta_\omega g_{\beta\lambda}) + \nabla_\beta(\delta_\omega g_{\nu\lambda}) - \nabla_\lambda(\delta_\omega g_{\nu\beta})\right] \tag{1.17}$$

$$= g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + g_{\nu\lambda}\nabla_\beta\omega - g_{\nu\beta}\nabla_\lambda\omega\right) \tag{1.18}$$

$$= g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega\right) \tag{1.19}$$

we obtain

$$\begin{aligned}
\delta_\omega\mathcal{E} &= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P_\alpha{}^{\beta\mu\nu}\nabla_\mu g^{\alpha\lambda}\left(g_{\beta\lambda}\nabla_\nu\omega + 2g_{\nu[\lambda}\nabla_{\beta]}\omega\right)\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 2P^{\lambda\beta\mu\nu}\nabla_\mu\left(\cancel{g_{\beta\lambda}}\nabla_\nu\omega + 2g_{\nu\lambda}\nabla_\beta\omega\right)\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B + 4P^{\lambda\beta\mu\nu}\nabla_\mu g_{\nu\lambda}\nabla_\beta\omega\right] \\
&= \sqrt{-g}\left[(A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi)\nabla_\nu\omega + \omega B - 4P^{\mu\nu}\nabla_\mu\nabla_\nu\omega\right]
\end{aligned}$$

Imposing $\delta_\omega\mathcal{E} = 0$ for all ω, we obtain the following conditions,

$$A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi = 0 \tag{1.20a}$$

$$B = 0 \tag{1.20b}$$

$$P^{\mu\nu} = 0 \tag{1.20c}$$

From (A.21),

$$\frac{\partial E}{\partial R^{\mu\nu}_{\alpha\beta}} = P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu}\hat{I}^{\beta]}_{\nu]} + J\delta^\alpha_{[\mu}\delta^\beta_{\nu]} \tag{1.21}$$

we notice that since $\hat{H}^{\alpha\beta}_{\mu\nu}$ is the traceless part of $H^{\alpha\beta}_{\mu\nu}$,

$$P^\beta_\nu = P^{\alpha\beta}_{\alpha\nu} = \delta^{[\alpha}_{[\alpha}\hat{I}^{\beta]}_{\nu]} + J\delta^\alpha_{[\alpha}\delta^\beta_{\nu]} \tag{1.22}$$

Since the variation with respect to the Weyl tensor does not contribute so the first trace (1.21), the contribution of the Riemann tensor to E has to be through its traceless part,

$$\mathcal{E} = \sqrt{-g}E\left(\phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi, g_{\mu\nu}, C^{\alpha\beta}_{\mu\nu}\right) = 0\,. \tag{1.23}$$

Example 1.1. *Let us consider the following action principle*

$$S[\phi, g] = \int \mathrm{d}^D x\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right)^{D/2} \tag{1.24}$$

$$= \int \mathrm{d}^D x\sqrt{-g}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{D/2} \tag{1.25}$$

$$= \int \mathrm{d}^D x\sqrt{-g}X^{D/2} \tag{1.26}$$

where we have defined $X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$.

Now, we must to find E. *Varying with respect to* ϕ,

$$\delta_\phi S = -\int \mathrm{d}^D x\sqrt{-g}\frac{D}{2}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\nabla_\mu\delta\phi \tag{1.27}$$

$$= \int \mathrm{d}^D x\sqrt{-g}\frac{D}{2}\nabla_\mu\left[\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\right]\delta\phi + \mathrm{b.t} \tag{1.28}$$

therefore,

$$E = \nabla_\mu\left[\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\nabla^\mu\phi\right] \tag{1.29}$$

$$= \frac{D-2}{2}\left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-4}{2}}(-1)\nabla^\alpha\phi\nabla_\mu\nabla_\alpha\phi\nabla^\mu\phi + \left(-\frac{1}{2}(\nabla\phi)^2\right)^{\frac{D-2}{2}}\Box\phi \tag{1.30}$$

which can be rewritten as

$$E = X^{\frac{D-2}{2}}\Box\phi - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi \tag{1.31}$$

Let's see how (1.20a) *looks,*

$$A^{\mu\nu} = \frac{\partial}{\partial(\nabla_\mu\phi\nabla_\nu\phi)}\left[X^{\frac{D-2}{2}}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi\right] \tag{1.32}$$

$$= X^{\frac{D-2}{2}}g^{\mu\nu} - \frac{D-2}{2}X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi \tag{1.33}$$

and its trace yields

$$A = DX^{\frac{D-2}{2}} - \frac{D-2}{2}X^{\frac{D-4}{2}}(\nabla\phi)^2 \tag{1.34}$$

$$= DX^{\frac{D-2}{2}} + (D-2)X^{\frac{D-4}{2}}\left(-\frac{1}{2}(\nabla\phi)^2\right) \tag{1.35}$$

$$= DX^{\frac{D-2}{2}} + (D-2)X^{\frac{D-2}{2}} \tag{1.36}$$

$$= 2(D-1)X^{\frac{D-2}{2}} \tag{1.37}$$

Pluggin into (1.20a),

$$\begin{aligned}
A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi &= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}g^{\mu\nu}\nabla_\mu\phi + (D-2)X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\phi \\
&= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}\nabla^\nu\phi + (D-2)X^{\frac{D-4}{2}}(\nabla\phi)^2\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-4}{2}}X\right]\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-2}{2}}2\right]\nabla^\nu\phi \\
&= 0 \qquad \checkmark
\end{aligned}$$

2 Building the auxiliar metric

We know the conditions that the most general second orden equation of motion for the zero conformal weight scalar field must satisfy. They are given by (1.20). Now, the question is: how do we construct an auxiliary metric $\tilde{g}_{\mu\nu}$ such that $\delta_\omega\tilde{g}_{\mu\nu} = 0$?

Remember that in the exponential frame for the scalar field, the conformal transformations look like

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \mathrm{e}^{2\omega}g_{\mu\nu}, \qquad \phi \to \bar{\phi} = \phi \tag{2.1}$$

Since the inverse metric transforms as

$$g^{\mu\nu} \to \bar{g}^{\mu\nu} = \mathrm{e}^{-2\omega}g^{\mu\nu} \tag{2.2}$$

the kinetic term for the scalar field, defined as

$$X := -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{2.3}$$

transforms as

$$\bar{X} = -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = -\frac{1}{2}\mathrm{e}^{-2\omega}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \mathrm{e}^{-2\omega}X \tag{2.4}$$

Thus, the auxiliary metric defined as

$$\tilde{g}_{\mu\nu} = Xg_{\mu\nu} = -\frac{1}{2}(\nabla\phi)^2 g_{\mu\nu} \qquad \Longrightarrow \qquad \delta_\omega\tilde{g}_{\mu\nu} = 0 \tag{2.5}$$

is conformally invariant.

Let us consider a pseudoscalar built from the zero wight conformal scalar field and its derivatives up to second order, and the conformally invariant geometry,

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{R}^{\alpha\beta}_{\mu\nu}) \tag{2.6}$$

We notice that $\delta_\omega\mathcal{E} = 0$. Indeed

$$\begin{aligned}
\delta_\omega\mathcal{E} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}E\cancel{\delta_\omega\tilde{g}_{\mu\nu}} + \sqrt{-\tilde{g}}\Bigg(&\frac{\partial E}{\partial\phi}\cancel{\delta_\omega\phi} + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\omega(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) \\
&+ \frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\cancel{\delta_\omega\tilde{g}^{\mu\nu}} + \tilde{P}^{\mu\nu}_{\alpha\beta}\cancel{\delta_\omega\tilde{R}^{\alpha\beta}_{\mu\nu}}\Bigg)
\end{aligned}$$

but

$$\delta_\omega\tilde{\nabla}_\mu\phi = \delta_\omega\partial_\mu\phi = \partial_\mu\delta_\omega\phi = 0 \tag{2.7}$$

and

$$\delta_\omega(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) = \delta_\omega(\tilde{\nabla}_\mu\partial_\nu\phi) \tag{2.8}$$
$$= \delta_\omega(\partial_\mu\partial_\nu\phi - \tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi) \tag{2.9}$$
$$= \partial_\mu\partial_\nu\delta_\omega\phi - \delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi - \tilde{\Gamma}^\lambda_{\mu\nu}\partial_\lambda\delta_\omega\phi \tag{2.10}$$
$$= -\delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu}\partial_\lambda\phi \tag{2.11}$$
$$= 0 \tag{2.12}$$

since $\delta_\omega\tilde{\Gamma}^\lambda{}_{\mu\nu} = 0$. Therefore,

$$\delta_\omega\mathcal{E} = 0\,. \tag{2.13}$$

Since from conformal invariance for zero weight scalar field, the $P^{\mu\nu} = 0$ condition implies that the explicit dependence of the Riemann tensor in the equation of motion is through the Weyl tensor, then, for a equation of motion built from the conformally invariant geometry we have

$$E = E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}) = 0 \tag{2.14}$$

[Is this last true?]

3 Fréchet derivative

Now, let's try to find an action principle from which it emerges (2.14). To do that, we will use the formalism used in [1]. Hence, first we must to introduce the concept of Fréchet derivative.

Let $P[u] = P\left(x, u^{(n)}\right)$ be a differential function, i.e. that depends in the point x, the function u together with its derivatives $u^{(n)}$. Consider noew its variation under a one-parameter family of functions. After interchanging the variation with derivatives (and without integrating by parts) we end with a differential operator acting on an arbitrary variation δu called the *Fréchet derivative* of P,

$$\delta P = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, P[u + \varepsilon\delta u]\bigg|_{\varepsilon=0} := D_P(\delta u) \tag{3.1}$$

Here, we consider the scond order conformally invariant psudoscalar defined by

$$\mathcal{E} = \sqrt{-\tilde{g}}E(\phi, \tilde{\nabla}_\mu\phi, \tilde{\nabla}_\mu\nabla_\nu\phi, \tilde{g}^{\mu\nu}, \tilde{R}^{\alpha\beta}_{\mu\nu}) \tag{3.2}$$

which is the natural quantity that could be derived from a convariant action. The role of he dependet function u is played by the zero weight conformally invariant sacalr field ϕ, and hence the Fréchet derivative of $\mathcal{E}$ can be calculated from

$$D_{\mathcal{E}}(\delta\phi) = \delta_\phi\mathcal{E}\,. \tag{3.3}$$

From (3.2) and using that $\delta\phi = 0$, we have

$$\begin{aligned}\delta_\phi\mathcal{E} =& -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left(\frac{\partial E}{\partial\phi}\delta\phi\frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}\delta_\phi(\tilde{\nabla}_\mu\phi) + \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)}\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) + \frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\phi\tilde{g}^{\mu\nu} + \tilde{P}^{\mu\nu}_{\alpha\beta}\delta_\phi\tilde{R}^{\alpha\beta}_{\mu\nu}\right)\\ =& -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{\mu\nu}\delta_\phi\tilde{g}^{\mu\nu} + \sqrt{-\tilde{g}}\left[\frac{\partial E}{\partial\tilde{g}^{\mu\nu}}\delta_\phi\tilde{g}^{\mu\nu} + E_\phi\delta\phi + E^\mu_\phi\delta_\phi(\tilde{\nabla}_\mu\phi) + E^{\mu\nu}_\phi\delta_\phi(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi) + \tilde{P}^{\mu\nu}_{\alpha\beta}\delta_\phi\tilde{R}^{\alpha\beta}_{\mu\nu}\right]\end{aligned}$$

where we have defined

$$E_\phi := \frac{\partial E}{\partial\phi}, \qquad E^\mu_\phi := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\phi)}, \qquad E^{\mu\nu}_\phi := \frac{\partial E}{\partial(\tilde{\nabla}_\mu\tilde{\nabla}_\nu\phi)} \tag{3.4}$$

in order to reduce some notation.

Before proceeding, let's compute the variation w.r.t the scalar field ϕ of some quantities:

$$\begin{aligned}
\delta_\phi \tilde{g}_{\mu\nu} &= \delta_\phi (X g_{\mu\nu}) && (3.5)\\
&= \delta_\phi \left(-\frac{1}{2} (\nabla\phi)^2 g_{\mu\nu} \right) && (3.6)\\
&= -\nabla^\alpha \phi \nabla_\alpha \delta\phi g_{\mu\nu} && (3.7)\\
&= -g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \delta\phi && (3.8)\\
&= -g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \delta\phi && (3.9)\\
&= -g_{\mu\nu} X g^{\alpha\beta} X^{-1} \partial_\alpha \phi \partial_\beta \delta\phi && (3.10)\\
&= -\tilde{g}_{\mu\nu} \tilde{\nabla}^\alpha \phi \tilde{\nabla}_\alpha \delta\phi && (3.11)
\end{aligned}$$

$$\begin{aligned}
\delta_\phi \tilde{g}^{\mu\nu} &= \delta_\phi \left[\left(-\frac{1}{2} (\nabla\phi)^2 \right)^{-1} g^{\mu\nu} \right] && (3.12)\\
&= -\frac{1}{X^2} (-\nabla^\alpha \phi \nabla_\alpha \delta\phi) g^{\mu\nu} && (3.13)\\
&= \frac{1}{X^2} g^{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \delta\phi && (3.14)\\
&= X^{-1} g^{\mu\nu} X^{-1} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \delta\phi && (3.15)\\
&= \tilde{g}^{\mu\nu} \tilde{\nabla}^\alpha \phi \tilde{\nabla}_\alpha \delta\phi && (3.16)
\end{aligned}$$

$$\begin{aligned}
\delta_\phi \sqrt{-\tilde{g}} &= -\frac{1}{2} \sqrt{-\tilde{g}} \tilde{g}_{\mu\nu} \delta_\phi \tilde{g}^{\mu\nu} && (3.17)\\
&= -\frac{1}{2} \sqrt{-\tilde{g}} \tilde{g}_{\mu\nu} \tilde{g}^{\mu\nu} \tilde{\nabla}^\alpha \phi \tilde{\nabla}_\alpha \delta\phi && (3.18)\\
&= -\frac{D}{2} \sqrt{-\tilde{g}} \tilde{\nabla}^\alpha \phi \tilde{\nabla}_\alpha \delta\phi && (3.19)
\end{aligned}$$

A Some useful calculations

A.1 Variation of the Christoffel symbols

The Christoffel symbols in terms of the metric are given by

$$\Gamma^{\lambda}{}_{\mu\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta}\right)$$

Varying both sides, we have

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= \frac{1}{2}\delta g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta}\right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta}\right) \\
&= -\frac{1}{2} g^{\lambda\sigma} g^{\rho\tau} (\delta g_{\sigma\tau}) \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta}\right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta}\right) \\
&= -g^{\lambda\sigma} (\delta g_{\sigma\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta}\right)
\end{aligned}$$

Changing the dumb indice σ by ρ,

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= -g^{\lambda\rho} (\delta g_{\rho\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta}\right) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} - 2\delta g_{\rho\tau} \Gamma^{\tau}{}_{\mu\beta}\right) \\
&= \frac{1}{2} g^{\lambda\rho} \big(\partial_\mu \delta g_{\beta\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\mu} \delta g_{\tau\beta} + \partial_\beta \delta g_{\mu\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\beta} \delta g_{\tau\mu} \\
&\quad -\partial_\rho \delta g \mu\beta + \Gamma^{\tau}{}_{\mu\rho} \delta g_{\tau\beta} + \Gamma^{\tau}{}_{\beta\rho} \delta g_{\mu\tau}\big) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\nabla_\mu \delta g_{\beta\rho} + \nabla_\beta \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\beta}\right)
\end{aligned} \tag{A.1}$$

A.2 Variation of the Riemann tensor

The Riemann tensor is given by

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} - \Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu}$$

Varying both sides,

$$\begin{aligned}
\delta R^{\rho}{}_{\lambda\mu\nu} &= \partial_\mu \delta\Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\lambda\mu} + \delta\Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\lambda\nu} - \delta\Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\lambda\mu} \\
&= \partial_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\nu\lambda} - \Gamma^{\tau}{}_{\mu\lambda} \delta\Gamma^{\rho}{}_{\tau\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\tau}{}_{\nu\lambda} \delta\Gamma^{\rho}{}_{\tau\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\mu\lambda}
\end{aligned}$$

Adding a convenient zero of the form $\Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda} - \Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda}$, and using the fact that $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor, we have

$$\delta R^{\rho}{}_{\lambda\mu\nu} = \nabla_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} = 2\nabla_{[\mu} \delta\Gamma^{\rho}{}_{\nu]\lambda} \tag{A.2}$$

A.3 Variation of derivatives of ϕ w.r.t ω

Let's compute the infinitesimal variations of the covariant derivatives of the scalar field. First, let us calculate $\delta_\omega \nabla_\mu \phi$:

$$\delta_\omega \nabla_\mu \phi = \nabla_\mu \delta_\omega \phi = 0 \tag{A.3}$$

Now, let's compute $\delta_\omega(\nabla_\mu \nabla_\nu \phi)$:

$$\delta_\omega(\nabla_\mu \nabla_\nu \phi) = \delta_\omega \nabla_\mu (\partial_\nu \phi) \tag{A.4}$$

$$= \delta_\omega(\partial_\mu \partial_\nu \phi - \Gamma^{\lambda}{}_{\nu\mu} \partial_\lambda \phi) \tag{A.5}$$

$$= \partial_\mu \partial_\nu \delta_\omega \phi - \delta_\omega \Gamma^{\lambda}{}_{\nu\mu} \partial_\lambda \phi - \Gamma^{\lambda}{}_{\nu\mu} \delta_\omega \partial_\lambda \phi \tag{A.6}$$

$$= \partial_\mu \partial_\nu \delta_\omega \phi - \delta_\omega \Gamma^{\lambda}{}_{\nu\mu} \partial_\lambda \phi - \Gamma^{\lambda}{}_{\nu\mu} \partial_\lambda \delta_\omega \phi \tag{A.7}$$

$$= -\partial_\lambda \phi \delta_\omega \Gamma^{\lambda}{}_{\nu\mu} \tag{A.8}$$

Using that the variation of the Christoffel connection is

$$\delta\Gamma^{\lambda}{}_{\mu\beta} = \frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\beta\rho} + \nabla_\beta\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\beta}) \tag{A.9}$$

we have

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi\frac{1}{2}g^{\lambda\rho}(\nabla_\nu\delta_\omega g_{\mu\rho} + \nabla_\mu\delta_\omega g_{\nu\rho} - \nabla_\rho\delta_\omega g_{\nu\beta}) \tag{A.10}$$

but $\delta_\omega g_{\nu\mu} = 2\omega g_{\nu\mu}$, so then

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi g^{\lambda\rho}\left[\nabla_\nu(\omega g_{\mu\rho}) + \nabla_\mu(\omega g_{\nu\rho}) - \nabla_\rho(\omega g_{\nu\beta})\right] \tag{A.11}$$

Using the metric compatibility condition $\nabla_\mu g_{\alpha\beta} = 0$ and $\nabla_\alpha\phi = \partial_\alpha\phi$, we obtain

$$\delta_\omega(\nabla_\mu\nabla_\nu\phi) = -\partial_\lambda\phi g^{\lambda\rho}(g_{\mu\rho}\nabla_\nu\omega + g_{\nu\rho}\nabla_\mu\omega - g_{\nu\beta}\nabla_\rho\omega) \tag{A.12}$$
$$= -\partial^\rho\phi(g_{\mu\rho}\nabla_\nu\omega + g_{\nu\rho}\nabla_\mu\omega - g_{\nu\beta}\nabla_\rho\omega) \tag{A.13}$$
$$= -\nabla_\mu\phi\nabla_\nu\omega - \nabla_\nu\phi\nabla_\mu\omega + g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega \tag{A.14}$$
$$= -2\nabla_{(\mu}\phi\nabla_{\nu)}\omega + g_{\nu\beta}\nabla^\rho\phi\nabla_\rho\omega \tag{A.15}$$

A.4 Variation of E w.r.t Riemann tensor

In order to see what (1.20c) implies, let us split the dependence of the Riemann tensor in terms of its traceless part $C^{\alpha\beta}_{\mu\nu}$, the traceless part of the Ricci tensor S^α_β, and the scalar curvature R. So we have

$$E\left(g^{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = E\left(g^{\mu\nu}, C^{\alpha\beta}_{\mu\nu}, S^\alpha_\beta, R\right)$$

The variation w.r.t the Riemann tensor yields

$$\begin{aligned}\delta_{\mathrm{Riem}}E &= P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta}\\ &= H^{\alpha\beta}_{\mu\nu}\delta_{\mathrm{Riem}}C^{\mu\nu}_{\alpha\beta} + I^\alpha_\beta\delta_{\mathrm{Riem}}S^\beta_\alpha + J\delta_{\mathrm{Riem}}R\end{aligned} \tag{A.16}$$

where

$$H^{\alpha\beta}_{\mu\nu} \equiv \frac{\partial E}{\partial C^{\mu\nu}_{\alpha\beta}}, \qquad I^\alpha_\beta \equiv \frac{\partial E}{\partial S^\beta_\alpha} \qquad \text{y} \qquad J \equiv \frac{\partial E}{\partial R}$$

Since $P^{\mu\nu}_{\alpha\beta}$ has the same algebraic symmetries as the Riemann tensor, its traceless part is given by

$$\hat{P}^{\mu\nu}_{\alpha\beta} = P^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}\delta^{[\mu}_{[\alpha}P^{\nu]}_{\beta]} + \frac{2}{(D-2)(D-1)}P\delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta]} \tag{A.17}$$

Let us note that

$$\begin{aligned}J\delta_{\mathrm{Riem}}R &= J\delta_{\mathrm{Riem}}\left(R^{\alpha\beta}_{\mu\nu}\delta^\mu_\alpha\delta^\nu_\beta\right)\\ &= J\delta^\mu_\alpha\delta^\nu_\beta R^{\alpha\beta}_{\mu\nu}\end{aligned} \tag{A.18}$$

Writing S^α_β in terms of the Riemann,

$$\begin{aligned}S^\beta_\nu &= R^\beta_\nu - \frac{1}{D}R\delta^\beta_\nu\\ &= R^{\alpha\beta}_{\mu\nu}\delta^\mu_\alpha - \frac{1}{D}\delta^\beta_\nu Rs^{\alpha\gamma}_{\mu\lambda}\delta^\mu_\alpha\delta^\lambda_\gamma\end{aligned}$$

then,

$$\delta_{\mathrm{Riem}}\tilde{S}^\beta_\nu = \delta^\mu_\alpha\delta\tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D}\delta^\beta_\nu\delta\tilde{R}^{\alpha\gamma}_{\mu\lambda}\delta^\mu_\alpha\delta^\lambda_\gamma$$

Hence,

$$
\begin{aligned}
I^{\nu}_{\beta}\delta_{\text{Riem}}S^{\beta}_{\nu} &= I^{\nu}_{\beta}\delta^{\mu}_{\alpha}\delta R^{\alpha\beta}_{\mu\nu} - \frac{1}{D}I^{\nu}_{\beta}\delta^{\beta}_{\nu}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}\delta R^{\alpha\gamma}_{\mu\lambda} \\
&= I^{\nu}_{\beta}\delta^{\mu}_{\alpha}\delta\tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D}I\delta^{\mu}_{\alpha}\delta R^{\alpha\gamma}_{\mu\lambda} \\
&= \delta^{\mu}_{\alpha}\delta R^{\alpha\beta}_{\mu\nu}\left(I^{\nu}_{\beta} - \frac{1}{D}I\delta^{\nu}_{\beta}\right) \\
&= \delta^{\mu}_{\alpha}\hat{I}\delta R^{\alpha\beta}_{\mu\nu}
\end{aligned} \tag{A.19}
$$

Finally, let us write the Weyl tensor in terms of the Riemann,

$$
\begin{aligned}
\tilde{C}^{\alpha\beta}_{\mu\nu} &= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{[\alpha}_{[\mu}\tilde{R}^{\beta]}_{\nu]} + \frac{2}{(D-1)(D-2)}\tilde{R}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]} \\
&= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\tilde{R}^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\tilde{R}^{\rho\sigma}_{\tau\lambda}
\end{aligned}
$$

Varying with respect to $R^{\alpha\beta}_{\mu\nu}$,

$$
\delta_{\text{Riem}}C^{\alpha\beta}_{\mu\nu} = \delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\delta R^{\rho\sigma}_{\tau\lambda}
$$

Then,

$$
\begin{aligned}
H^{\mu\nu}_{\alpha\beta}\delta_{\text{Riem}}C^{\alpha\beta}_{\mu\nu} &= H^{\mu\nu}_{\alpha\beta}\left[\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\delta R^{\rho\sigma}_{\tau\lambda}\right] \\
&= H^{\mu\nu}_{\alpha\beta}\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}H^{\lambda\nu}_{\gamma\beta}\delta^{\mu}_{\alpha}\delta^{\gamma}_{\lambda}\delta R^{\beta\alpha}_{\nu\mu} + \frac{2}{(D-1)(D-2)}H^{\tau\lambda}_{\rho\sigma}\delta^{\rho}_{\tau}\delta^{\sigma}_{\lambda}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta R^{\alpha\beta}_{\mu\nu} \\
&= \delta R^{\alpha\beta}_{\mu\nu}\left[H^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}H^{\nu}_{\beta}\delta^{\mu}_{\alpha} + \frac{2}{(D-1)(D-2)}H\right] \\
&= \hat{H}^{\mu\nu}_{\alpha\beta}\delta R^{\alpha\beta}_{\mu\nu}
\end{aligned} \tag{A.20}
$$

where the indices have been renamed in a convenient way and has been used the fact that $H^{\alpha\beta}_{\mu\nu}$ has the same algebraic symmetries as the Riemann tensor.

In this way, plugging (A.18), (A.19) and (A.20) into (A.16), we obtain

$$
P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta} = \hat{H}^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta} + \delta^{\alpha}_{\mu}\hat{I}^{\beta}_{\nu}\delta R^{\mu\nu}_{\alpha\beta} + J\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta R^{\mu\nu}_{\alpha\beta}
$$

Hence,

$$
P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu}\hat{I}^{\beta]}_{\nu]} + J\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} \tag{A.21}
$$

A.5 Christoffel of the conformally invariant geometry

$$
\begin{aligned}
\tilde{\Gamma}^{\lambda}{}_{\mu\nu} &= \frac{1}{2}\tilde{g}^{\lambda\rho}\left(\partial_{\mu}\tilde{g}_{\nu\rho} + \partial_{\nu}\tilde{g}_{\mu\rho} - \partial_{\rho}\tilde{g}_{\mu\nu}\right) && \text{(A.22)} \\
&= \frac{1}{2}\frac{1}{X}g^{\lambda\rho}\left[\partial_{\mu}(Xg_{\nu\rho}) + \partial_{\nu}(Xg_{\mu\rho}) - \partial_{\rho}(Xg_{\mu\nu})\right] && \text{(A.23)} \\
&= \frac{1}{2}\frac{1}{X}g^{\lambda\rho}\left[g_{\nu\rho}\partial_{\mu}X + X\partial_{\mu}g_{\nu\rho} + g_{\mu\rho}\partial_{\nu}X + X\partial_{\nu}g_{\mu\rho} - g_{\mu\nu}\partial_{\rho}X - X\partial_{\rho}g_{\mu\nu}\right] && \text{(A.24)} \\
&= \Gamma^{\lambda}{}_{\mu\nu} + \frac{1}{2}\frac{1}{X}g^{\lambda\rho}\left(g_{\nu\rho}\partial_{\mu}X + g_{\mu\rho}\partial_{\nu}X - g_{\mu\nu}\partial_{\rho}X\right) && \text{(A.25)} \\
&= \Gamma^{\lambda}{}_{\mu\nu} + \frac{1}{2}\frac{1}{X}g^{\lambda\rho}\left(-g_{\nu\rho}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\mu}\phi - -g_{\mu\rho}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\nu}\phi + g_{\mu\nu}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\rho}\phi\right) && \text{(A.26)} \\
&= \Gamma^{\lambda}{}_{\mu\nu} + \frac{1}{2}\frac{1}{X}\left(-\delta^{\lambda}_{\nu}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\mu}\phi - \delta^{\lambda}_{\mu}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\nu}\phi + g_{\mu\nu}\partial^{\alpha}\phi\partial_{\alpha}\partial^{\lambda}\phi\right) && \text{(A.27)} \\
&= \Gamma^{\lambda}{}_{\mu\nu} + G^{\lambda}{}_{\mu\nu} && \text{(A.28)}
\end{aligned}
$$

with

$$G^{\lambda}{}_{\mu\nu} := \frac{1}{2}\frac{1}{X}\left(-\delta^{\lambda}_{\nu}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\mu}\phi - \delta^{\lambda}_{\mu}\partial^{\alpha}\phi\partial_{\alpha}\partial_{\nu}\phi + g_{\mu\nu}\partial^{\alpha}\phi\partial_{\alpha}\partial^{\lambda}\phi\right) \tag{A.29}$$

References

[1] E. Ayón-Beato and M. Hassaine, *Non-Noetherian conformal scalar fields*, *Annals Phys.* **460** (2024) 169567 [2305.09806].