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Nonlinear Electrodynamics

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Abstract: Personal notes on nonlinear electrodynamics.

Contents

1 Nonlinear Electrodynamics

As is defined in the Plebanski book [XX], nonlinear electrodynamics are described by the following action principle

$$S_{\mathrm{E}}[g,A,P]=-\frac{1}{4\pi}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}F_{\mu\nu}P^{\mu\nu}-\mathscr{H}(\mathscr{P},\mathscr{Q})\right) \tag{1.1}$$

which depends on the metric $g_{\mu\nu}$, the gauge potential A_μ, and the antisymmetric tensor $P_{\mu\nu}$. Here the structural function $\mathscr{H}$ describes the precise nonlinear electrodynamics and depends, in general, on the two Lorentz scalars that can be constructed with $P^{\mu\nu}$ [XX]. As usual, the field strength is related to the gauge potential as $F=\mathrm{d}A$, ensuring the Faraday equations

$$\mathrm{d}F=0. \tag{1.2}$$

On the other hand, the variation of action (1.1) with respect to the gauge potential leads to the Maxwell equations

$$\mathrm{d}\star P=0, \tag{1.3}$$

where $\star$ stands for the Hodge dual, whereas varying (1.1) with respect to the antisymmetric tensor $P^{\mu\nu}$ yields the constitutive relations

$$F_{\mu\nu}=\frac{\partial\mathscr{H}}{\partial\mathscr{P}}P_{\mu\nu}+\frac{\partial\mathscr{H}}{\partial\mathscr{Q}}\star P_{\mu\nu}. \tag{1.4}$$

Notice that Maxwell electrodynamics is recovered for $\mathscr{H}=\mathscr{P}$, giving linear constitutive relations. Lastly, the corresponding energy-momentum tensor reads

$$4\pi T^{\mathrm{E}}_{\mu\nu}=F_{\mu\alpha}P_{\nu}{}^{\alpha}-g_{\mu\nu}\left(\frac{1}{2}F_{\alpha\beta}P^{\alpha\beta}-\mathscr{H}\right). \tag{1.5}$$

The main motivation for the action principle (1) is that now Maxwell equations (3) remain linear as the Fara- day ones (2) while the nonlinearity is encoded into the constitutive relations (4). Consequently, Maxwell equa- tions (3) can be now understood just like the Faraday ones (2), i.e., implying the local existence of a vector po- tential P = dA. Therefore, from the point of view of the action principle (1), a solution to nonlinear electro- dynamics can be understood as a pair of vector poten- tials A and A compatible with the constitutive relations (4). Additionally, since in four dimensions both Faraday (2) and Maxwell (3) equations define conservation laws, there are conserved quantities related to them defined by the following integrals

$$p = \frac{1}{4\pi}\int_{\partial\Sigma} F, \qquad q = \frac{1}{4\pi}\int_{\partial\Sigma} \star P, \tag{1.6}$$

where the integration is taken at the boundary of con- stant time hypersurfaces Σ; obviously, these are noth- ing other than the magnetic and electric charges, respec- tively.

After this brief and useful introduction, and in order to prepare for what follows, we review a strategy that has proved to be fruitful when nonlinear electrodynamics is considered in General Relativity [XX]. This strategy sim- ply consists of working in a null tetrad of the spacetime metric

$$g = 2e^1 \otimes_s e^2 + 2e^3 \otimes_s e^4, \tag{1.7}$$

aligned along the common eigenvectors of the electro- magnetic fields, i.e.

$$F + i \star P = (D + iB)e^1 \wedge e^2 + (E + iH)e^3 \wedge e^4\,. \tag{1.8}$$

References