Zero weight scalar fields

Borja Diez

Instituto de Ciencias Exactas y Naturales, Universidad Arturo Prat,
Avenida Playa Brava 3256, 1111346, Iquique, Chile.
Facultad de Ciencias, Universidad Arturo Prat,
Avenida Arturo Prat Chacón 2120, 1110939, Iquique, Chile.

Abstract

Personal compilation of some calculations related to zero-weight conformal scalar fields.

Please write to borjadiez1014@gmail.com *for corrections, typos, and literature suggestions.*

Contents

1 Building the equation of motion

Let us consider a scalar field with zero conformal weight that is, under an infinitesimal conformal transformation

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \qquad \delta_\omega \phi = 0 \tag{1}$$

Let us consider the most general second-order pseudoscalar constructed from the scalar field ϕ and its derivatives up to second order, together with the metric tensor and its associates curvature

$$\mathcal{E} = \sqrt{-g} E \left(\phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi, g_{\mu\nu}, R^{\alpha\beta}_{\mu\nu} \right) = 0 \tag{2}$$

The variation of this equation under infinitesimal conformal transformations reads,

$$\delta_\omega \mathcal{E} = \delta_\omega(\sqrt{-g}) E + \sqrt{-g} \delta_\omega E \tag{3}$$

$$= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta_\omega g^{\mu\nu} E + \sqrt{-g} \delta_\omega E \tag{4}$$

From $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$ and $\delta_\omega(g_{\mu\nu} g^{\mu\nu}) = 0$ implies that

$$g_{\mu\nu} \delta_\omega g^{\mu\nu} = -g^{\mu\nu} \delta_\omega g_{\mu\nu} \tag{5}$$

Furthermore, from (63) and (75), we have

$$\delta_\omega \mathcal{E} = \frac{\partial E}{\partial \phi} \cancelto{0}{\delta_\omega \phi} + \frac{\partial E}{\partial (\nabla_\mu \phi)} \cancelto{0}{\delta_\omega \nabla_\mu \phi} + \frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)} \delta_\omega (\nabla_\mu \nabla_\nu \phi) + \frac{\partial E}{\partial g^{\mu\nu}} \delta_\omega g^{\mu\nu} + P^{\mu\nu}_{\alpha\beta} \delta_\omega R^{\alpha\beta}_{\mu\nu} \tag{6}$$

where we have defined

$$P^{\mu\nu}_{\alpha\beta} := \frac{\partial E}{\partial R^{\alpha\beta}_{\mu\nu}}, \qquad P^\mu_\nu := P^{\alpha\mu}_{\alpha\nu} \tag{7}$$

So then

$$\begin{aligned}
\delta_\omega \mathcal{E} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta_\omega g^{\mu\nu} E + \sqrt{-g} \frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)} \delta_\omega (\nabla_\mu \nabla_\nu \phi) + \sqrt{-g} \frac{\partial E}{\partial g^{\mu\nu}} \delta_\omega g^{\mu\nu} + \sqrt{-g} P^{\mu\nu}_{\alpha\beta} \delta_\omega R^{\alpha\beta}_{\mu\nu} \\
&= \sqrt{-g} \frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)} \left(g_{\nu\beta} \nabla^\rho \phi \nabla_\rho \omega - 2 \nabla_{(\mu} \phi \nabla_{\nu)} \omega \right) + \sqrt{-g} \frac{\partial E}{\partial g^{\mu\nu}} \delta_\omega g^{\mu\nu} + \sqrt{-g} P^{\mu\nu}_{\alpha\beta} \delta_\omega R^{\alpha\beta}_{\mu\nu} \\
&- \frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta_\omega g^{\mu\nu} E \\
&= \sqrt{-g} \frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)} \left(g_{\nu\beta} \nabla^\rho \phi \nabla_\rho \omega - 2 \nabla_{(\mu} \phi \nabla_{\nu)} \omega \right) + \sqrt{-g} P^{\mu\nu}_{\alpha\beta} \delta_\omega R^{\alpha\beta}_{\mu\nu} - 2\omega g^{\mu\nu} \sqrt{-g} \left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} E \right)
\end{aligned}$$

Note that

$$\frac{\partial(\sqrt{-g} E)}{\partial g^{\mu\nu}} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} E + \sqrt{-g} \frac{\partial E}{\partial g^{\mu\nu}} \tag{8}$$

$$= \sqrt{-g} \left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} E \right) \tag{9}$$

By defining

$$A^{\mu\nu} := \frac{\partial E}{\partial (\nabla_\mu \nabla_\nu \phi)}, \qquad B_{\mu\nu} = -2 \left(\frac{\partial E}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu} E \right) = -\frac{2}{\sqrt{-g}} \frac{\partial (\sqrt{-g} E)}{\partial g^{\mu\nu}} \tag{10}$$

we obtain

$$\delta_\omega \mathcal{E} = \sqrt{-g} A^{\mu\nu} \left(g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \omega - 2\nabla_\mu \phi \nabla_\nu \omega \right) + \sqrt{-g} P^{\mu\nu} \delta_\omega R^{\alpha\beta}_{\mu\nu} + \sqrt{-g}\, \omega B_{\mu\nu} g_{\mu\nu} \tag{11}$$

$$= \sqrt{-g} \left(A \nabla^\rho \phi \nabla_\rho \omega - 2A^{\mu\nu} \nabla_\mu \phi \nabla_\nu \omega + \omega B + P^{\mu\nu}_{\alpha\beta} \delta_\omega R^{\alpha\beta}_{\mu\nu} \right) \tag{12}$$

Using

$$\delta_\omega R^\alpha{}_{\beta\mu\nu} = 2\nabla_{[\mu} \delta_\omega \Gamma^\alpha{}_{\nu]\beta} \tag{13}$$

we have

$$\delta_\omega \mathcal{E} = \sqrt{-g} \left(A \nabla^\rho \phi \nabla_\rho \omega - 2A^{\mu\nu} \nabla_\mu \phi \nabla_\nu \omega + \omega B + 2P_\alpha{}^{\beta\mu\nu} \nabla_{[\mu} \delta_\omega \Gamma^\alpha{}_{\nu]\beta} \right) \tag{14}$$

$$= \sqrt{-g} \left(A \nabla^\nu \phi \nabla_\nu \omega - 2A^{\mu\nu} \nabla_\mu \phi \nabla_\nu \omega + \omega B + 2P_\alpha{}^{\beta\mu\nu} \nabla_{[\mu} \delta_\omega \Gamma^\alpha{}_{\nu]\beta} \right) \tag{15}$$

$$= \sqrt{-g} \left[(A \nabla^\nu \phi - 2A^{\mu\nu} \nabla_\mu \phi) \nabla_\nu \omega + \omega B + 2P_\alpha{}^{\beta\mu\nu} \nabla_{[\mu} \delta_\omega \Gamma^\alpha{}_{\nu]\beta} \right] \tag{16}$$

Noting that

$$\delta_\omega \Gamma^\alpha{}_{\nu\beta} = \frac{1}{2} g^{\alpha\lambda} \left[\nabla_\nu (\delta_\omega g_{\beta\lambda}) + \nabla_\beta (\delta_\omega g_{\nu\lambda}) - \nabla_\lambda (\delta_\omega g_{\nu\beta}) \right] \tag{17}$$

$$= g^{\alpha\lambda} \left(g_{\beta\lambda} \nabla_\nu \omega + g_{\nu\lambda} \nabla_\beta \omega - g_{\nu\beta} \nabla_\lambda \omega \right) \tag{18}$$

$$= g^{\alpha\lambda} \left(g_{\beta\lambda} \nabla_\nu \omega + 2g_{\nu[\lambda} \nabla_{\beta]} \omega \right) \tag{19}$$

we obtain

$$\begin{aligned}
\delta_\omega \mathcal{E} &= \sqrt{-g} \left[(A \nabla^\nu \phi - 2A^{\mu\nu} \nabla_\mu \phi) \nabla_\nu \omega + \omega B + 2P_\alpha{}^{\beta\mu\nu} \nabla_\mu g^{\alpha\lambda} \left(g_{\beta\lambda} \nabla_\nu \omega + 2g_{\nu[\lambda} \nabla_{\beta]} \omega \right) \right] \\
&= \sqrt{-g} \left[(A \nabla^\nu \phi - 2A^{\mu\nu} \nabla_\mu \phi) \nabla_\nu \omega + \omega B + 2P^{\lambda\beta\mu\nu} \nabla_\mu \left(\cancel{g_{\beta\lambda}} \nabla_\nu \omega + 2g_{\nu\lambda} \nabla_\beta \omega \right) \right] \\
&= \sqrt{-g} \left[(A \nabla^\nu \phi - 2A^{\mu\nu} \nabla_\mu \phi) \nabla_\nu \omega + \omega B + 4P^{\lambda\beta\mu\nu} \nabla_\mu g_{\nu\lambda} \nabla_\beta \omega \right] \\
&= \sqrt{-g} \left[(A \nabla^\nu \phi - 2A^{\mu\nu} \nabla_\mu \phi) \nabla_\nu \omega + \omega B - 4P^{\mu\nu} \nabla_\mu \nabla_\nu \omega \right]
\end{aligned}$$

Imposing $\delta_\omega \mathcal{E} = 0$ for all ω, we obtain the following conditions,

$$A \nabla^\nu \phi - 2A^{\mu\nu} \nabla_\mu \phi = 0 \tag{20a}$$

$$B = 0 \tag{20b}$$

$$P^{\mu\nu} = 0 \tag{20c}$$

From (81),

$$\frac{\partial E}{\partial R^{\mu\nu}_{\alpha\beta}} = P^{\alpha\beta}_{\mu\nu} = \hat{H}^{\alpha\beta}_{\mu\nu} + \delta^{[\alpha}_{[\mu} \hat{I}^{\beta]}_{\nu]} + J \delta^\alpha_{[\mu} \delta^\beta_{\nu]} \tag{21}$$

we notice that since $\hat{H}^{\alpha\beta}_{\mu\nu}$ is the traceless part of $H^{\alpha\beta}_{\mu\nu}$,

$$P^\beta_\nu = P^{\alpha\beta}_{\alpha\nu} = \delta^{[\alpha}_{[\alpha} \hat{I}^{\beta]}_{\nu]} + J \delta^\alpha_{[\alpha} \delta^\beta_{\nu]} \tag{22}$$

Since the variation with respect to the Weyl tensor does not contribute to the first trace (21), the contribution of the Riemann tensor to E has to be through its traceless part,

$$\mathcal{E} = \sqrt{-g} E \left(\phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi, g_{\mu\nu}, C^{\alpha\beta}_{\mu\nu} \right) = 0\,. \tag{23}$$

Example 1.1 *Let us consider the following action principle*

$$S[\phi, g] = \int \mathrm{d}^D x \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right)^{D/2} \tag{24}$$

$$= \int \mathrm{d}^D x \sqrt{-g} \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{D/2} \tag{25}$$

$$= \int \mathrm{d}^D x \sqrt{-g} X^{D/2} \tag{26}$$

where we have defined $X := -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$.

Now, we must to find E. *Varying with respect to* ϕ,

$$\delta_\phi S = - \int \mathrm{d}^D x \sqrt{-g} \frac{D}{2} \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \nabla_\mu \delta \phi \tag{27}$$

$$= \int \mathrm{d}^D x \sqrt{-g} \frac{D}{2} \nabla_\mu \left[\left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \right] \delta \phi + \mathrm{b.t} \tag{28}$$

therefore,

$$E = \nabla_\mu \left[\left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \nabla^\mu \phi \right] \tag{29}$$

$$= \frac{D-2}{2} \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-4}{2}} (-1) \nabla^\alpha \phi \nabla_\mu \nabla_\alpha \phi \nabla^\mu \phi + \left(-\frac{1}{2} (\nabla \phi)^2 \right)^{\frac{D-2}{2}} \Box \phi \tag{30}$$

which can be rewritten as

$$E = X^{\frac{D-2}{2}} \Box \phi - \frac{D-2}{2} X^{\frac{D-4}{2}} \nabla_\mu \nabla_\nu \phi \nabla^\mu \phi \nabla^\nu \phi \tag{31}$$

Let's see how (20a) *looks,*

$$A^{\mu\nu} = \frac{\partial}{\partial (\nabla_\mu \phi \nabla_\nu \phi)} \left[X^{\frac{D-2}{2}} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{D-2}{2} X^{\frac{D-4}{2}} \nabla_\mu \nabla_\nu \phi \nabla^\mu \phi \nabla^\nu \phi \right] \tag{32}$$

$$= X^{\frac{D-2}{2}} g^{\mu\nu} - \frac{D-2}{2} X^{\frac{D-4}{2}} \nabla^\mu \phi \nabla^\nu \phi \tag{33}$$

and its trace yields

$$A = D X^{\frac{D-2}{2}} - \frac{D-2}{2} X^{\frac{D-4}{2}} (\nabla \phi)^2 \tag{34}$$

$$= D X^{\frac{D-2}{2}} + (D-2) X^{\frac{D-4}{2}} \left(-\frac{1}{2} (\nabla \phi)^2 \right) \tag{35}$$

$$= D X^{\frac{D-2}{2}} + (D-2) X^{\frac{D-2}{2}} \tag{36}$$

$$= 2(D-1) X^{\frac{D-2}{2}} \tag{37}$$

Pluggin into (20a),

$$
\begin{aligned}
A\nabla^\nu\phi - 2A^{\mu\nu}\nabla_\mu\phi &= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}g^{\mu\nu}\nabla_\mu\phi + (D-2)X^{\frac{D-4}{2}}\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\phi \\
&= 2(D-1)X^{\frac{D-2}{2}}\nabla^\nu\phi - 2X^{\frac{D-2}{2}}\nabla^\nu\phi + (D-2)X^{\frac{D-4}{2}}(\nabla\phi)^2\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-4}{2}}X\right]\nabla^\nu\phi \\
&= \left[2(D-2)X^{\frac{D-2}{2}} - 2(D-2)X^{\frac{D-2}{2}}2\right]\nabla^\nu\phi \\
&= 0 \qquad \checkmark
\end{aligned}
$$

We notice that the constraints (20) on E are an overdetermined linear PDE system with the following characteristic vector fields

$$Z = g^{\mu\nu}\frac{\partial}{\partial g^{\mu\nu}} - \frac{D}{2}E\frac{\partial}{\partial E}, \tag{38}$$

$$Z^\nu = \left(g_{\alpha\mu}g^{\beta\nu}\nabla_\beta\phi - 2\delta^\nu_\alpha\nabla_\mu\phi\right)\frac{\partial}{\partial(\nabla_\alpha\nabla_\mu\phi)} \tag{39}$$

Let's see the integrability of the system. To do that, we compute the commutators between the vector fields. It is straightforward to see that $[Z^\nu, Z^\lambda] = 0$. On the other hand

$$
\begin{aligned}
[Z, Z^\nu] &= ZZ^\nu - \cancel{Z^\nu Z} &(40)\\
&= \left[\left(g^{\rho\tau}\frac{\partial}{\partial g^{\rho\tau}} - \frac{DE}{2}\frac{\partial}{\partial E}\right), (g_{\alpha\mu}g^{\beta\nu}\phi_\beta - 2\delta^\nu_\alpha\phi_\mu)\frac{\partial}{\partial\phi_{\alpha\mu}}\right] &(41)\\
&= g^{\rho\tau}\frac{\partial}{\partial g^{\rho\tau}}\left(g_{\alpha\mu}g^{\beta\nu}\right)\phi_\beta\frac{\partial}{\partial\phi_{\alpha\mu}} &(42)
\end{aligned}
$$

For compute the derivative, we note that

$$\delta(g_{\alpha\mu}g^{\beta\nu}) = g^{\beta\nu}\delta g_{\alpha\mu} + g_{\alpha\mu}\delta g^{\beta\nu}. \tag{43}$$

Using that

$$\delta g_{\alpha\mu} = -g_{\alpha\rho}g_{\mu\tau}\delta g^{\rho\tau}, \tag{44}$$

we obtain

$$\delta(g_{\alpha\mu}g^{\beta\nu}) = -g^{\beta\nu}g_{\alpha\rho}g_{\mu\tau}\delta g^{\rho\tau} + g_{\alpha\mu}\delta^\beta_\rho\delta^\nu_\tau\delta g^{\rho\tau}. \tag{45}$$

Moreover,

$$\delta(g_{\alpha\mu}g^{\beta\nu}) = \frac{\partial}{\partial g^{\rho\tau}}\left(g_{\alpha\mu}g^{\beta\nu}\right)\delta g^{\rho\tau}, \tag{46}$$

which implies that

$$\frac{\partial}{\partial g^{\rho\tau}}\left(g_{\alpha\mu}g^{\beta\nu}\right) = -g^{\beta\nu}g_{\alpha(\rho}g_{\mu|\tau)} + g_{\alpha\mu}\delta^\beta_{(\rho}\delta^\nu_{\tau)}. \tag{47}$$

Finally,

$$
\begin{aligned}
[Z, Z^{\nu}] &= g^{\rho\tau} \frac{\partial}{\partial g^{\rho\tau}} \left(g_{\alpha\mu} g^{\beta\nu} \right) \phi_{\beta} \frac{\partial}{\partial \phi_{\alpha\mu}} && (48)\\
&= g^{\rho\tau} (g_{\alpha\mu} \delta^{\beta}_{\rho} \delta^{\nu}_{\tau} - g^{\beta\nu} g_{\alpha\rho} g_{\mu\tau}) \phi_{\beta} \frac{\partial}{\partial \phi_{\alpha\mu}} && (49)\\
&= (g^{\beta\nu} g_{\alpha\mu} - g^{\beta\nu} \delta^{\tau}_{\alpha} g_{\mu\tau}) \phi_{\beta} \frac{\partial}{\partial \phi_{\alpha\mu}} && (50)\\
&= (g^{\beta\nu} g_{\alpha\mu} - g^{\beta\nu} g_{\alpha\mu}) \phi_{\beta} \frac{\partial}{\partial \phi_{\alpha\mu}} && (51)\\
&= 0. && (52)
\end{aligned}
$$

Then, the system is integrable.

The characteristic system of Z is

$$
\frac{\mathrm{d}g^{11}}{g^{11}} = \cdots = \frac{\mathrm{d}g^{\alpha\beta}}{g^{\alpha\beta}} = \cdots = \frac{\mathrm{d}g^{DD}}{g^{DD}} = \frac{\mathrm{d}E}{\frac{DE}{2}}. \tag{53}
$$

For α, β fixed, a combination of the characteristic equations can also be incorporated, giving

$$
\frac{\mathrm{d}g^{\alpha\beta}}{g^{\alpha\beta}} = \frac{C_{\mu\nu} \mathrm{d}g^{\mu\nu}}{C_{\mu\nu} g^{\mu\nu}} \implies \mathrm{d}\left(\ln g^{\alpha\beta} \right) = \mathrm{d}\left(\ln C_{\mu\nu} g^{\mu\nu} \right), \tag{54}
$$

where $C_{\mu\nu}$ is an arbitrary tensor which does not depend on $g_{\mu\nu}$. By integrating and exponentiating both sides, we obtain the invariant

$$
\Omega^{\alpha\beta} = \frac{g^{\alpha\beta}}{C_{\mu\nu} g^{\mu\nu}}. \tag{55}
$$

Choosing $C_{\mu\nu} = -\frac{1}{2} \phi_{\mu} \phi_{\nu}$ we recover the auxilliary metric

$$
\Omega^{\alpha\beta} = \frac{g^{\alpha\beta}}{X} = \tilde{g}^{\alpha\beta}. \tag{56}
$$

They corresponds to $\frac{D(D+1)}{2} - 1$ invariants, since their symmetry and the condition $C_{\mu\nu} \Omega^{\mu\nu} = 1$. The general solution to $Z(E) = 0$ can be obtained from

$$
\frac{\mathrm{d}X}{X} = \frac{\mathrm{d}E}{\frac{DE}{2}} \implies \ln X = \frac{2}{D} \ln E, \tag{57}
$$

giving

$$
E = X^{\frac{D}{2}} \tilde{E}(\phi, \phi_{\mu}, \phi_{\mu\nu}, \tilde{g}^{\mu\nu}, \tilde{C}^{\alpha\beta}_{\mu\nu}). \tag{58}
$$

Now, we have to find the general solution to $Z^{\nu}(E) = 0$. First, we notice that

$$
\begin{aligned}
\tilde{\phi}_{\mu\nu} &:= \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} \phi && (59)\\
&= \partial_{\mu} \partial_{\nu} \phi - \tilde{\Gamma}^{\alpha}{}_{\mu\nu} \partial_{\alpha} \phi && (60)
\end{aligned}
$$

A Some useful calculations

A.1 Variation of the Christoffel symbols

The Christoffel symbols in terms of the metric are given by

$$\Gamma^{\lambda}{}_{\mu\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right)$$

Varying both sides, we have

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= \frac{1}{2}\delta g^{\lambda\rho} \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= -\frac{1}{2} g^{\lambda\sigma} g^{\rho\tau} (\delta g_{\sigma\tau}) \left(\partial_\mu g_{\beta\rho} + \partial_\beta g_{\mu\rho} - \partial_\rho g_{\mu\beta} \right) + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= -g^{\lambda\sigma} (\delta g_{\sigma\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right)
\end{aligned}$$

Changing the dumb indice σ by ρ,

$$\begin{aligned}
\delta\Gamma^{\lambda}{}_{\mu\beta} &= -g^{\lambda\rho} (\delta g_{\rho\tau}) \Gamma^{\tau}{}_{\mu\beta} + \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu \delta g_{\beta\rho} + \partial_\beta \delta g_{\mu\rho} - \partial_\rho \delta g_{\mu\beta} - 2\delta g_{\rho\tau} \Gamma^{\tau}{}_{\mu\beta} \right) \\
&= \frac{1}{2} g^{\lambda\rho} \Big(\partial_\mu \delta g_{\beta\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\mu} \delta g_{\tau\beta} + \partial_\beta \delta g_{\mu\rho} - \Gamma^{\tau}{}_{\mu\beta} \delta g_{\rho\tau} - \Gamma^{\tau}{}_{\rho\beta} \delta g_{\tau\mu} \\
&\quad - \partial_\rho \delta g \mu\beta + \Gamma^{\tau}{}_{\mu\rho} \delta g_{\tau\beta} + \Gamma^{\tau}{}_{\beta\rho} \delta g_{\mu\tau} \Big) \\
&= \frac{1}{2} g^{\lambda\rho} \left(\nabla_\mu \delta g_{\beta\rho} + \nabla_\beta \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\beta} \right)
\end{aligned} \tag{61}$$

A.2 Variation of the Riemann tensor

The Riemann tensor is given by

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} + \Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} - \Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu}$$

Varying both sides,

$$\begin{aligned}
\delta R^{\rho}{}_{\lambda\mu\nu} &= \partial_\mu \delta\Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\lambda\mu} + \delta\Gamma^{\rho}{}_{\tau\mu} \Gamma^{\tau}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\lambda\nu} - \delta\Gamma^{\rho}{}_{\tau\nu} \Gamma^{\tau}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\lambda\mu} \\
&= \partial_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} + \Gamma^{\rho}{}_{\tau\mu} \delta\Gamma^{\tau}{}_{\nu\lambda} - \Gamma^{\tau}{}_{\mu\lambda} \delta\Gamma^{\rho}{}_{\tau\nu} - \partial_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\tau}{}_{\nu\lambda} \delta\Gamma^{\rho}{}_{\tau\mu} - \Gamma^{\rho}{}_{\tau\nu} \delta\Gamma^{\tau}{}_{\mu\lambda}
\end{aligned}$$

Adding a convenient zero of the form $\Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda} - \Gamma^{\tau}{}_{\mu\nu} \delta\Gamma^{\rho}{}_{\tau\lambda}$, and using the fact that $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor, we have

$$\delta R^{\rho}{}_{\lambda\mu\nu} = \nabla_\mu \delta\Gamma^{\rho}{}_{\nu\lambda} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\lambda} = 2\nabla_{[\mu} \delta\Gamma^{\rho}{}_{\nu]\lambda} \tag{62}$$

A.3 Variation of derivatives of ϕ w.r.t ω

Let's compute the infinitesimal variations of the covariant derivatives of the scalar field. First, let us calculate $\delta_\omega \nabla_\mu \phi$:

$$\delta_\omega \nabla_\mu \phi = \nabla_\mu \delta_\omega \phi = 0 \tag{63}$$

Now, let's compute $\delta_\omega(\nabla_\mu \nabla_\nu \phi)$:

$$\begin{aligned}
\delta_\omega(\nabla_\mu \nabla_\nu \phi) &= \delta_\omega \nabla_\mu(\partial_\nu \phi) && (64)\\
&= \delta_\omega(\partial_\mu \partial_\nu \phi - \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \phi) && (65)\\
&= \partial_\mu \partial_\nu \delta_\omega \phi - \delta_\omega \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \phi - \Gamma^\lambda{}_{\nu\mu}\delta_\omega \partial_\lambda \phi && (66)\\
&= \partial_\mu \partial_\nu \delta_\omega \phi - \delta_\omega \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \phi - \Gamma^\lambda{}_{\nu\mu}\partial_\lambda \delta_\omega \phi && (67)\\
&= -\partial_\lambda \phi \delta_\omega \Gamma^\lambda{}_{\nu\mu} && (68)
\end{aligned}$$

Using that the variation of the Christoffel connection is

$$\delta \Gamma^\lambda{}_{\mu\beta} = \frac{1}{2} g^{\lambda\rho}(\nabla_\mu \delta g_{\beta\rho} + \nabla_\beta \delta g_{\mu\rho} - \nabla_\rho \delta g_{\mu\beta}) \tag{69}$$

we have

$$\delta_\omega(\nabla_\mu \nabla_\nu \phi) = -\partial_\lambda \phi \frac{1}{2} g^{\lambda\rho}(\nabla_\nu \delta_\omega g_{\mu\rho} + \nabla_\mu \delta_\omega g_{\nu\rho} - \nabla_\rho \delta_\omega g_{\nu\beta}) \tag{70}$$

but $\delta_\omega g_{\nu\mu} = 2\omega g_{\nu\mu}$, so then

$$\delta_\omega(\nabla_\mu \nabla_\nu \phi) = -\partial_\lambda \phi g^{\lambda\rho}\left[\nabla_\nu(\omega g_{\mu\rho}) + \nabla_\mu(\omega g_{\nu\rho}) - \nabla_\rho(\omega g_{\nu\beta})\right] \tag{71}$$

Using the metric compatibility condition $\nabla_\mu g_{\alpha\beta} = 0$ and $\nabla_\alpha \phi = \partial_\alpha \phi$, we obtain

$$\begin{aligned}
\delta_\omega(\nabla_\mu \nabla_\nu \phi) &= -\partial_\lambda \phi g^{\lambda\rho}(g_{\mu\rho}\nabla_\nu \omega + g_{\nu\rho}\nabla_\mu \omega - g_{\nu\beta}\nabla_\rho \omega) && (72)\\
&= -\partial^\rho \phi(g_{\mu\rho}\nabla_\nu \omega + g_{\nu\rho}\nabla_\mu \omega - g_{\nu\beta}\nabla_\rho \omega) && (73)\\
&= -\nabla_\mu \phi \nabla_\nu \omega - \nabla_\nu \phi \nabla_\mu \omega + g_{\nu\beta}\nabla^\rho \phi \nabla_\rho \omega && (74)\\
&= -2\nabla_{(\mu}\phi \nabla_{\nu)}\omega + g_{\nu\beta}\nabla^\rho \phi \nabla_\rho \omega && (75)
\end{aligned}$$

A.4 Variation of E w.r.t Riemann tensor

In order to see what (20c) implies, let us split the dependence of the Riemann tensor in terms of its traceless part $C^{\alpha\beta}_{\mu\nu}$, the traceless part of the Ricci tensor S^α_β, and the scalar curvature R. So we have

$$E\left(g^{\mu\nu}, R^{\alpha\beta}_{\mu\nu}\right) = E\left(g^{\mu\nu}, C^{\alpha\beta}_{\mu\nu}, S^\alpha_\beta, R\right)$$

The variation w.r.t the Riemann tensor yields

$$\begin{aligned}
\delta_{\text{Riem}} E &= P^{\alpha\beta}_{\mu\nu}\delta R^{\mu\nu}_{\alpha\beta}\\
&= H^{\alpha\beta}_{\mu\nu}\delta_{\text{Riem}} C^{\mu\nu}_{\alpha\beta} + I^\alpha_\beta \delta_{\text{Riem}} S^\beta_\alpha + J\delta_{\text{Riem}} R && (76)
\end{aligned}$$

where

$$H^{\alpha\beta}_{\mu\nu} \equiv \frac{\partial E}{\partial C^{\mu\nu}_{\alpha\beta}}, \qquad I^\alpha_\beta \equiv \frac{\partial E}{\partial S^\beta_\alpha} \qquad \text{y} \qquad J \equiv \frac{\partial E}{\partial R}$$

Since $P^{\mu\nu}_{\alpha\beta}$ has the same algebraic symmetries as the Riemann tensor, its traceless part is given by

$$\hat{P}^{\mu\nu}_{\alpha\beta} = P^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}\delta^{[\mu}_{[\alpha}P^{\nu]}_{\beta]} + \frac{2}{(D-2)(D-1)}P\delta^\mu_{[\alpha}\delta^\nu_{\beta]} \tag{77}$$

Let us note that

$$\begin{aligned} J\delta_{\text{Riem}} R &= J\delta_{\text{Riem}}\left(R^{\alpha\beta}_{\mu\nu}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\right) \\ &= J\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}R^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{78}$$

Writing S^{α}_{β} in terms of the Riemann,

$$\begin{aligned} S^{\beta}_{\nu} &= R^{\beta}_{\nu} - \frac{1}{D}R\delta^{\beta}_{\nu} \\ &= R^{\alpha\beta}_{\mu\nu}\delta^{\mu}_{\alpha} - \frac{1}{D}\delta^{\beta}_{\nu}Rs^{\alpha\gamma}_{\mu\lambda}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma} \end{aligned}$$

then,

$$\delta_{\text{Riem}}\tilde{S}^{\beta}_{\nu} = \delta^{\mu}_{\alpha}\delta\tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D}\delta^{\beta}_{\nu}\delta\tilde{R}^{\alpha\gamma}_{\mu\lambda}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}$$

Hence,

$$\begin{aligned} I^{\nu}_{\beta}\delta_{\text{Riem}}S^{\beta}_{\nu} &= I^{\nu}_{\beta}\delta^{\mu}_{\alpha}\delta R^{\alpha\beta}_{\mu\nu} - \frac{1}{D}I^{\nu}_{\beta}\delta^{\beta}_{\nu}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}\delta R^{\alpha\gamma}_{\mu\lambda} \\ &= I^{\nu}_{\beta}\delta^{\mu}_{\alpha}\delta\tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{1}{D}I\delta^{\mu}_{\alpha}\delta R^{\alpha\gamma}_{\mu\lambda} \\ &= \delta^{\mu}_{\alpha}\delta R^{\alpha\beta}_{\mu\nu}\left(I^{\nu}_{\beta} - \frac{1}{D}I\delta^{\nu}_{\beta}\right) \\ &= \delta^{\mu}_{\alpha}\hat{I}\delta R^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{79}$$

Finally, let us write the Weyl tensor in terms of the Riemann,

$$\begin{aligned} \tilde{C}^{\alpha\beta}_{\mu\nu} &= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{[\alpha}_{[\mu}\tilde{R}^{\beta]}_{\nu]} + \frac{2}{(D-1)(D-2)}\tilde{R}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]} \\ &= \tilde{R}^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\tilde{R}^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{[\alpha}_{[\mu}\delta^{\beta]}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\tilde{R}^{\rho\sigma}_{\tau\lambda} \end{aligned}$$

Varying with respect to $R^{\alpha\beta}_{\mu\nu}$,

$$\delta_{\text{Riem}}C^{\alpha\beta}_{\mu\nu} = \delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\delta R^{\rho\sigma}_{\tau\lambda}$$

Then,

$$\begin{aligned} H^{\mu\nu}_{\alpha\beta}\delta_{\text{Riem}}C^{\alpha\beta}_{\mu\nu} &= H^{\mu\nu}_{\alpha\beta}\left[\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}\delta^{\lambda}_{\gamma}\delta^{[\alpha}_{[\mu}\delta R^{\beta]\gamma}_{\nu]\lambda} + \frac{2}{(D-1)(D-2)}\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]}\delta^{\tau}_{\rho}\delta^{\lambda}_{\sigma}\delta R^{\rho\sigma}_{\tau\lambda}\right] \\ &= H^{\mu\nu}_{\alpha\beta}\delta R^{\alpha\beta}_{\mu\nu} - \frac{4}{D-2}H^{\lambda\nu}_{\gamma\beta}\delta^{\mu}_{\alpha}\delta^{\gamma}_{\lambda}\delta R^{\beta\alpha}_{\nu\mu} + \frac{2}{(D-1)(D-2)}H^{\tau\lambda}_{\rho\sigma}\delta^{\rho}_{\tau}\delta^{\sigma}_{\lambda}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta R^{\alpha\beta}_{\mu\nu} \\ &= \delta R^{\alpha\beta}_{\mu\nu}\left[H^{\mu\nu}_{\alpha\beta} - \frac{4}{D-2}H^{\nu}_{\beta}\delta^{\mu}_{\alpha} + \frac{2}{(D-1)(D-2)}H\right] \\ &= \hat{H}^{\mu\nu}_{\alpha\beta}\delta R^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{80}$$

where the indices have been renamed in a convenient way and has been used the fact that $H^{\alpha\beta}_{\mu\nu}$ has the same algebraic symmetries as the Riemann tensor.

In this way, plugging (78), (79) and (80) into (76), we obtain

$$P_{\mu\nu}^{\alpha\beta}\delta R_{\alpha\beta}^{\mu\nu} = \hat{H}_{\mu\nu}^{\alpha\beta}\delta R_{\alpha\beta}^{\mu\nu} + \delta_{\mu}^{\alpha}\hat{I}_{\nu}^{\beta}\delta R_{\alpha\beta}^{\mu\nu} + J\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta}\delta R_{\alpha\beta}^{\mu\nu}$$

Hence,

$$P_{\mu\nu}^{\alpha\beta} = \hat{H}_{\mu\nu}^{\alpha\beta} + \delta_{[\mu}^{[\alpha}\hat{I}_{\nu]}^{\beta]} + J\delta_{[\mu}^{\alpha}\delta_{\nu]}^{\beta} \tag{81}$$

References