Spinning A Wheel

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Abstract

In this paper, we model a cockatoo-wheel system with the cockatoo capable of moving along the radial and tangential axes over some fixed finite interval. We aim to find the optimal combination of radial and tangential movement that maximizes the effect of "pumping" on the system or the net acceleration over spatially periodic movements. It is proven that the radial position needs to be maximized at the top of each rotation and minimized at the bottom. It is also proven that the tangential position needs to move the bird closer to the edge on falling intervals and away from the edge on rising intervals. Finally, combining the effects of both components of motion, we find the maximum speed given a frictional force.

1 Introduction

The act of pumping a swing by systematically moving one's center mass throughout different points of the swinging motion is observed and done by many. In this paper, we model an altered form of this problem involving a cockatoo on a wheel with a mass on a wheel capable of moving radially and tangentially within certain boundaries. We mathematically prove the optimal trajectory of the bird that maximizes the effects of pumping by observing the effects of both the radial and tangential components of movements of the bird on the system. Specifically, we describe the optimal movement of this "bird" mass as what maximizes the angular momentum of the system during both alternating rotations (modeled as a pendulum swinging back and forth) and full rotations (modeled as the mass going all the way around a circular path). This is achieved by constructing and analyzing the properties of the equations of motion for each of these systems. Both radial and tangential components are then combined, allowing us to find the maximum speed of the system (limited by some frictional force proportional to angular velocity). Phase portraits are also plotted for each of the components as well as the combined model. Finally, we discuss the important results as well as the limitations of our model.

2 Radial Axis of Freedom

2.1 Equation of Motion

To model the effects of radial movements of the bird on the wheel, we represent the bird with an apparatus with the ability to move a point mass perpendicular to the circle over a finite interval d_0. The base is fixed of the rod is fixed to the wheel.

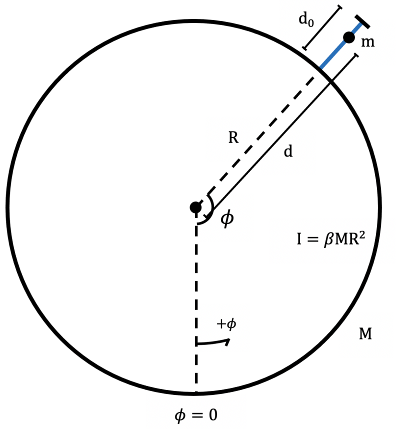


Figure 1: Mass with radial axis of freedom on wheel

The equations of motion are then written for this system by applying the differential relation between torque and angular momentum around the center of the wheel. The frictional force is modeled to be proportional to the angular velocity, allowing us to ignore effects of the normal force from movements of the mass on the frictional force.

$$\tau = \frac{dL}{dt} = \frac{d}{dt}\left(I_t\,\dot{\phi}\right) \tag{1}$$

Defining d' as the pseudo point mass for the wheel-mass system, such that the moment of inertia of the system can be computed as a point mass with distance d' from the center,

$$d' \equiv \sqrt{\frac{\beta MR^2 + md^2}{m+M}} \propto d \quad \text{and} \quad I_t = (m+M)d'^2$$

Notice that there exists a linear relation between d' and d, signifying that taking the extreme of one of these variables is identical to doing the same for its counterpart. We can finally write the equations of motion using equation (1) to be

$$\frac{d}{dt}\left((m+M)\,d'^2\,\dot{\phi}\right) = -mgd\sin\phi - b\dot{\phi} \tag{2}$$

We observe that all trajectories can be represented by two categories of motion: alternating rotations and full rotations. The reason this differentiation is made is because the definition of the "amplitude" of oscillations for these two categories are different and thus require different proofs (maximum angular position and initial angular velocity respectively). Nevertheless, the ideas used in both proofs are very similar. We first explore the general properties of our equation of motion for this system and then apply them to these two categories.

2.2 General Results

Suppose d satisfies the equation $d' = d'(t)$. Observing a time interval of motion $[t_0,\, t_0 + \Delta t]$, we can integrate equation (2) from t_0 to t where $t_0 \le t \le t_0 + \Delta t$, or equivalently apply the conservation of angular momentum to obtain

$$d'(t)^2\dot{\phi}(t) - d'(t_0)^2\dot{\phi}(t_0) = -\frac{1}{m+M}\int_{t_0}^{t}\left[mgd\sin\phi + b\dot{\phi}\right]dt \tag{3}$$

$$\dot{\phi}(t) - \frac{d'(t_0)^2}{d'(t)^2}\,\dot{\phi}(t_0) = -\frac{1}{d'(t)^2}\,\frac{1}{m+M}\int_{t_0}^{t}\left[mgd\sin\phi + b\dot{\phi}\right]dt$$

Equation (3) applies to any time interval $[t_0,\, t]$. Integrating again from t_0 to $t_0 + \Delta t$,

$$\phi(t)\big|_{t_0}^{t_0+\Delta t} - \int_{t_0}^{t_0+\Delta t}\frac{d'(t_0)^2}{d'(t)^2}\,\dot{\phi}(t_0)dt = -\int_{t_0}^{t_0+\Delta t}\frac{1}{d'(t)^2}\,\frac{1}{m+M}\left[\int_{t_0}^{t}\left(mgd\sin\phi + b\dot{\phi}\right)dt\right]dt \tag{4}$$

Equation (4) applies to any time interval $[t_0,\, t+\Delta t]$. While the above equation does hold for any $\Delta t > 0$, taking Δt to the infinitesimal limit yields conclusions that confirm the validity of the equation. We first observe that the inner integral corresponds to

$$\int_{t_0}^{t}\left(mgd\sin\phi + b\dot{\phi}\right) = L(t)\big|_{t_0}^{t} = O(\Delta t)$$

Where L represents the angular momentum, whose change on the interval $[t_0,\, t]$ is on the order of Δt. Moreover, a second integration resulting in the form of the right side of equation (4) will as a result be $O(\Delta t^2)$. Taking the limit $\Delta t \to 0$, the right term vanishes,

$$\phi(t_0 + \Delta t) - \phi(t_0) = \int_{t_0}^{t_0+\Delta t}\frac{d'(t_0)^2}{d'(t)^2}\,\dot{\phi}(t_0)dt$$

By the fundamental theorem of calculus, the integrand must be equivalent to

$$\dot{\phi}(t) = \frac{d'(t_0)^2}{d'(t)^2}\,\dot{\phi}(t_0) \tag{5}$$

This matches our expectations for conservation of angular momentum for infinitesimal time intervals. It proves essential for later numerical modeling of the trajectory of oscillations.

2.3 Category 1: Alternating Rotations

This category of motion is characterized by a mass spinning from side to side, with insufficient speed to revolve all the way around the circle. The initial conditions are in the form of $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$. The amplitude of this category of oscillations is defined to be the value of ϕ_0 with every new cycle. Thus, our goal is to find the function $d(t)$ such that the value of ϕ_0 for the next cycle is maximized. To achieve this, we divide the motion of the mass into two phases: falling and rising (encompasses entire motion by trajectory).

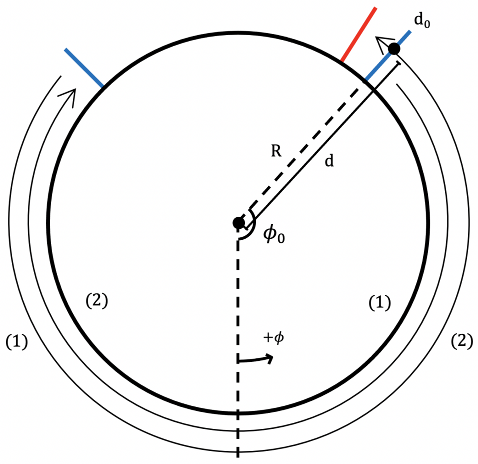


Figure 2: Alternating Rotation

2.3.1 Phase 1: Falling

The first phase of motion occurs on the angular interval $[\pm\phi_0,\, 0]$. Defining the initial instant to be $t_0 = 0$ and the final instant t_f to satisfy $\phi(t_f) = 0$, we can write the time boundary conditions:

$$\phi(0) = \phi_0 \qquad\qquad \dot{\phi}(0) = 0$$
$$\phi(t_f) = 0 \qquad\qquad \dot{\phi}(t_f) = \dot{\phi}_f$$

To maximize ϕ_0 on the next cycle, we must maximize the total angular momentum at the end of the interval. This can be confirmed by observing the first term in equation (7), as the initial momentum of phase 2 corresponds to the final momentum of phase 1. This means that $|d'(t_f)^2\dot{\phi}_f|$ should be maximized with a sign opposite of ϕ_0 by an optimal function $d(t)$. We begin by using equation (3) with over interval $[0,\, t_f]$.

$$d'(t_f)^2\,\dot{\phi}_f = -\frac{1}{m+M}\int_0^{t_f}\Big(mgd(t)\sin\phi + b\dot{\phi}\Big)\,dt \tag{6}$$

The second term on the left has vanished because of the boundary condition $\dot{\phi}(0) = 0$. An inspection of the signs of equation confirms that total angular momentum has the opposite sign as the sign of the angular interval due to the $\sin\phi$ term in the right integral. Thus, to maximize final angular momentum,

$$d(t) = d_{max} = R + d_0 \;\text{ for }\; [t = (0,\, t_f) \;\text{ or }\; \phi = (\pm\phi_0,\, 0)]$$

2.3.2 Phase 2: Rising

The second phase of motion occurs on the angular interval $[0,\, \pm\phi_f]$. Once again defining the initial instant to be $t_0 = 0$ and the final instant t_f to satisfy $\dot{\phi}(t_f) = 0$, we can write the time boundary conditions:

$$\phi(0) = 0 \qquad\qquad \dot{\phi}(0) = \phi_0$$
$$\phi(t_f) = \phi_f \qquad\qquad \dot{\phi}(t_f) = 0$$

To maximize ϕ_f, we use equation (4) over interval $[0, t_f]$.

$$\phi_f - \int_0^{t_f} \frac{d'(0)^2}{d'(t)^2}\,\dot{\phi}_0\,dt = -\int_0^{t_f} \frac{1}{d'(t)^2}\,\frac{1}{m+M}\left[\int_0^t \left(mgd\sin\phi + b\dot{\phi}\right)dt\right]dt$$

The lower limit of the first term $\phi(0)$ disappears due to boundary conditions. Simplifying,

$$\phi_f = \int_0^{t_f} \frac{1}{d'(t)^2}\left[d'(0)^2\,\dot{\phi}_0 - \frac{1}{m+M}\int_0^t \left(mgd(t)\sin\phi + b\dot{\phi}\right)dt\right]dt \tag{7}$$

We once again inspect the signs of equation, seeing that both the term related to initial angular momentum and the term related to the combined effects of the gravitational and frictional forces are of the same sign as ϕ_f. Thus, to maximize $|\phi_f|$, we maximize the magnitude of both terms within the integral. The magnitude of the first term is already maximized by our resulting trajectory from phase 1. On the other hand, in order to maximize the magnitude of the second term, $d'(t)$ or equivalently $d(t)$ must be minimized for the entire interval:

$$d(t) = d_{min} = R \;\text{ for }\; [t = (0,\, t_f) \;\text{ or }\; \phi = (0,\, \pm\phi_0)]$$

2.3.3 Combined Trajectory

Combining the results from phase 1 and phase 2 results in a continuous trajectory

$$\begin{aligned} d(t) = d_{max} &= R + d_0 \;\text{ for }\; \phi = (\pm\phi_0,\, 0) \\ d(t) = d_{min} &= R \qquad\;\;\text{ for }\; \phi = (0,\, \mp\phi_0) \end{aligned}$$

Plotting these results gives the graphs (symmetry applied for full picture):

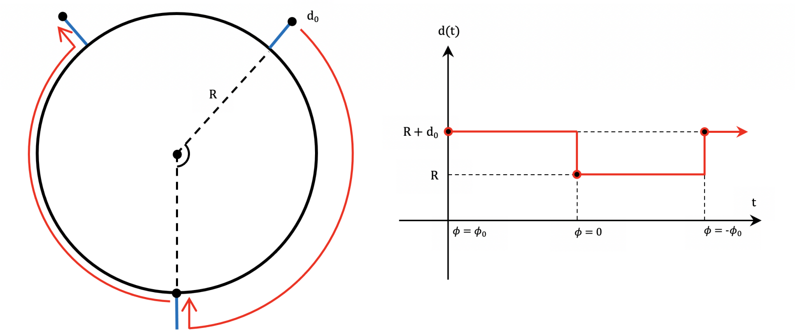


Figure 3: Optimal Trajectory of mass with radial degree of freedom

2.4 Category 2: Full Rotations

The second category of motion is characterized by a mass spinning in the same direction at all times with an increased angular velocity every new cycle. The initial conditions are in the form of $\phi(0) = \pi$ and $\dot{\phi}(0) = \phi_0$. There exists a transitional cycle of rotational motion between category 1 and category 2 at which the mass gains just enough speed to rotate all the way around. We can define the amplitude of this category of "oscillations" as the value of $\dot{\phi}_0$ with every new cycle.

2.4.1 Finding the trajectory

Once again, we divide oscillations into a falling phase and a rising phase. The following form of equation (3) over the interval $[0,\, t_f]$ is utilized for both phases.

$$d'(t_f)^2\dot{\phi}_f - d'(0)^2\,\dot{\phi}_0 = -\frac{1}{m+M}\int_0^t \left(mgd(t)\sin\phi + b\dot{\phi}\right)dt$$

Defining t_f as the time required for a full rotation and ΔL as the change in momentum during that period,

$$\Delta L = -\int_0^t \left(mgd(t)\sin\phi + b\dot{\phi}\right)dt$$

Suppose the system was rotating in the clockwise direction, hence $\dot{\phi} < 0$, we seek to minimize ΔL. This corresponds to maximizing d on the interval $[\pi,\, 0]$ and minimizing d on the interval $[0,\, -\pi]$. By symmetry arguments, the opposite applies for a positive initial angular velocity. Thus, assuming rotations are clockwise (symmetry applies for counterclockwise motion), combining our previous solutions results in a continuous trajectory

$$\begin{aligned} d(t) &= d_{max} = R + d_0 && \text{for } \phi = (\pm\pi,\, 0) \\ d(t) &= d_{min} = R && \text{for } \phi = (0,\, \mp\pi) \end{aligned}$$

Which is graphically represented by

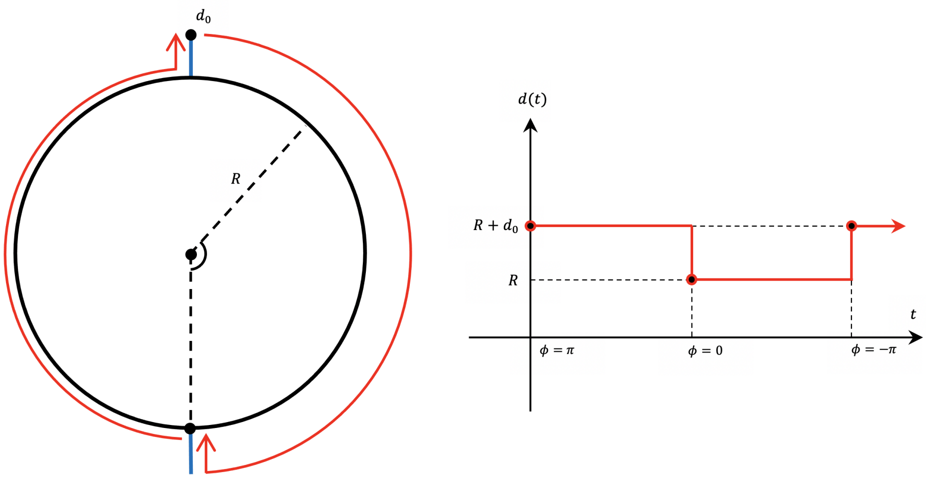


Figure 4: Optimal trajectory for mass

2.5 Numerical Modeling

Using the optimal functions of $d(t)$ as obtained in section 2.3.3 and 2.4.1, we can plot the trajectory of a mass given any initial condition. This trajectory will be portrayed by two separate plots: a phase portrait or parametric plot of $\dot{\phi}(\phi)$ and a plot of both $\dot{\phi}(t)$ and $\phi(t)$. Both plots are obtained by inputting equation (2) with the optimal functions of d into a differential equation solver.

The values used for the system are:

$$m = 0.5 \qquad \beta = 0.9 \qquad d_0 = 0.15$$
$$M = 1.2 \qquad R = 0.4 \qquad b = 0.0525$$

Where d_0 represents the allowed limited motion along both the radial and tangential directions. The value of d_0 is 0.15m and applies to both directions because the height of the cockatoo is 0.30m and allowed movement of the center of mass is half of that. Specifically, $R \leq d_r \leq R + d_0$ and $-d_0 \leq d_t \leq d_0$.

Suppose the system is described by the set of initial conditions:

$$\phi(0) = \phi_0 \qquad \dot{\phi}(0) = 0$$

The above initial conditions represent a mass swinging from side to side. However, it will soon be trivial that all trajectories of initial condition of this form with the exception of $\phi_0 = 0$ will evolve into full rotations. For computational purposes, we plot falling phases and rising phases separately, using equation (5) to calculate changes in angular velocity for instantaneous changes in d. It is worthwhile to mention that movements in d at the stationary points of oscillation do not change the angular velocity. Setting $\phi_0 = 5\pi/8$ and plotting the parametric graph of $\dot{\phi}(\phi)$,

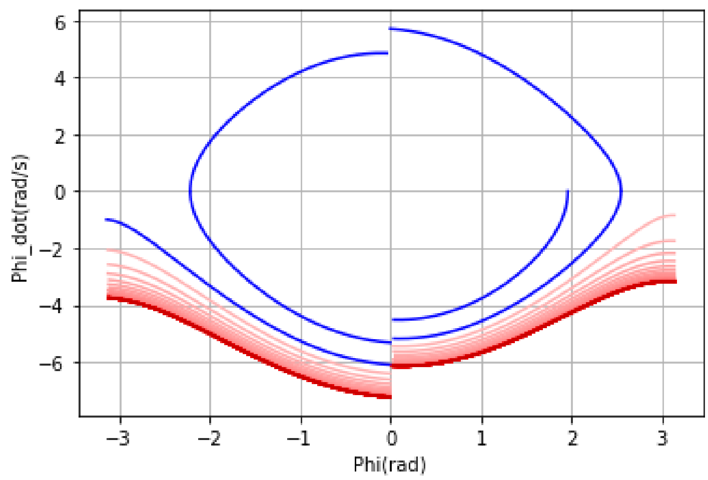


Figure 5: $\dot{\phi}(\phi)$ for radially moving mass

In the above graph, alternating rotations are depicted by blue curves whereas full rotations are depicted by red curves. Full rotations are graphed so positions beyond $|\pi|$ are looped back, conserving the angular velocity but merely teleported to the other side. We observe that the outward jumps occur at $\phi = 0$, corresponding to the instantaneous increases in the magnitude of angular velocities at the bottom of oscillations as a result of the decreasing of d. Furthermore, as the amplitude of oscillation increases (synonymous to size of circle), the system will at some point diverge or be able to rotate all the way around. This point or specifically the final blue curve that ends at $\pm\pi$ represents the shift from alternating rotations to full rotations. Moreover, the sign of the final position of that blue curve is identical to the velocity for the rest of the time, as future acceleration will always happen in that direction. Plotting the graph of $\dot{\phi}(t)$ and $\phi(t)$ using the same initial conditions,

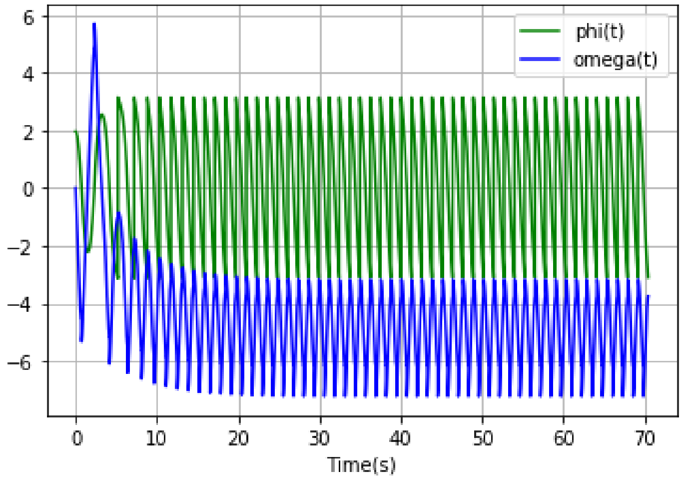


Figure 6: $\dot{\phi}(t)$ and $\phi(t)$ for radially moving mass

We first see that the cycle in which $\dot{\phi}$ stays negative indicates the transitional cycle between alternative rotations and full rotations, confirming our previous results. We then see that the period of oscillation for both curves steadily decreasingly, indicating faster angular velocities. Finally, one can observe that values of $\dot{\phi}$ converge to some fixed range of values as time increases, indicating the moment where the change in angular momentum over one revolution becomes 0 due to the counteracting frictional force. The outer bound of this range of $\dot{\phi}$ represents the maximum speed at which the bird will travel at for purely radial movements, evaluating to

$$|\dot{\phi}_{r,\,max}| = 7.231\ \frac{\text{rad}}{\text{s}}$$

Moreover, the minimum period experienced by the system evaluates to

$$|T_{r,\,min}| = 1.276\ \text{s}$$

3 Tangential Axis of Freedom

3.1 Equation of Motion

To model the effects of tangential movements of the bird on the wheel, we represent the bird with an apparatus with the ability to move a point mass parallel to the circle over an interval d_0. We assume d_0 to be much smaller than R.

The equations of motion are then written for this system using equation (1). There are two contributions to the net torque: gravitational and frictional. Furthermore, this assumption allows us to consider every part rod as being the same distance from the center. This simplifies the bird to merely a point mass a distance R away from the center at all times. This matches the very definition of tangential movement. The frictional force is once again modeled as being proportional to the angular velocity of the system. It is worthwhile to mention that the inertial force from the acceleration in d is an internal force and doesn't affect total angular momentum.

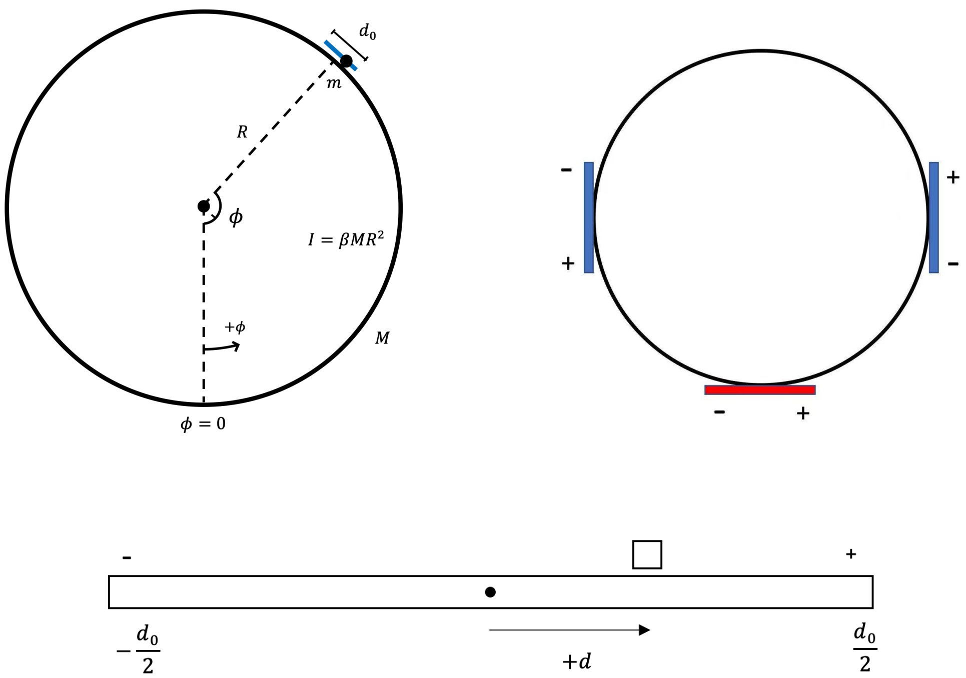


Figure 7: Mass with tangential degree of freedom on wheel

The equations of motion will be given by

$$\frac{d}{dt}\left(\left(m+\beta M\right)R^2\dot{\phi}\right) = -mgR\sin\left(\phi+\frac{d}{R}\right) - b\dot{\phi} \tag{8}$$

Similarly, the motion of this system can be divided into trajectories with two categories of initial conditions: alternating rotation and full rotations. We first explore the general properties of our equation of motion for this system and then apply them to these two categories.

3.2 General Results

We first define a time interval $[t_0,\, t_0+\Delta t]$ and t where $t_0 \leq t \leq t_0+\Delta t$. Integrating equation (8) from t_0 to t while eliminating the last term,

$$\dot{\phi}(t) - \dot{\phi}(t_0) = -\frac{1}{R^2}\frac{1}{m+\beta M}\int_{t_0}^{t}\left[mgR\sin\left(\phi+\frac{d}{R}\right) - b\dot{\phi}\right]dt \tag{9}$$

Equation (9) applies to any interval $[t_0,\, t]$. Integrating again from t_0 to $t_0+\Delta t$,

$$\phi(t)|_{t_0}^{t_0+\Delta t} - \int\limits_{t_0}^{t_0+\Delta t}\dot{\phi}(t_0)dt = -\int\limits_{t_0}^{t_0+\Delta t}\frac{1}{R^2}\frac{1}{m+\beta M}\left(\int_{t_0}^{t}\left[mgR\sin\left(\phi+\frac{d}{R}\right) + b\dot{\phi}\right]dt\right)dt \tag{10}$$

Equation (10) applies to any interval $[t_0,\ t_0+\Delta t]$. Moreover, taking equation (10) to the infinitesimal limit, the righter term vanishes, resulting in the key result

$$\phi(t) = \phi(t_0) \qquad \dot{\phi}(t) = \dot{\phi}(t_0) \tag{11}$$

This indicates that instantaneous movements in d effect neither the angular position nor the angular velocity of the system. Equations (9) and (10) are going to be used in proving the optimal function of d that maximizes the "amplitude" of oscillations, while equation (11) is used in the numerical modeling process.

3.3 Category 1: Alternating Rotations

We analyze alternating rotations for tangential movement in an identical manner with that for radial movement. Further dividing alternating rotations into a falling phase and a rising phase and applying equations (8) (to maximize final angular momentum) and (9) (to maximize final angular position) respectively and defining ϕ_0 and ϕ_f are angular magnitudes, we arrive at the results

If the interval motion is given by $[\phi_0, -\phi_f]$,

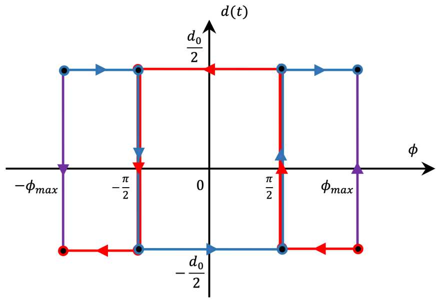
$$\begin{aligned} d(t) = d_{min} &= -\frac{d_0}{2} \quad \text{for} \quad \left(\phi_0,\ \frac{\pi}{2}\right) \\ d(t) = d_{max} &= \frac{d_0}{2} \quad \text{for} \quad \left(\frac{\pi}{2},\ -\frac{\pi}{2}\right) \\ d(t) = d_{min} &= -\frac{d_0}{2} \quad \text{for} \quad \left(-\frac{\pi}{2},\ -\phi_f\right) \end{aligned}$$

If the interval motion is given by $[-\phi_0, \phi_f]$,

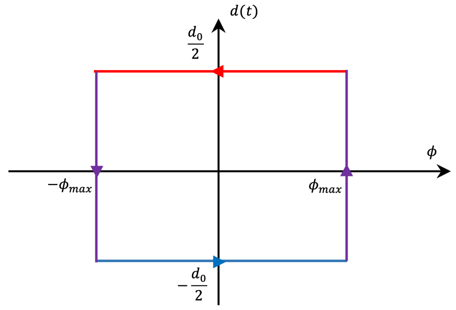
$$\begin{aligned} d(t) = d_{max} &= \frac{d_0}{2} \quad \text{for} \quad \left(-\phi_0,\ -\frac{\pi}{2}\right) \\ d(t) = d_{min} &= -\frac{d_0}{2} \quad \text{for} \quad \left(-\frac{\pi}{2},\ \frac{\pi}{2}\right) \\ d(t) = d_{max} &= \frac{d_0}{2} \quad \text{for} \quad \left(\frac{\pi}{2},\ \phi_f\right) \end{aligned}$$

These results are derived by trying to maximize or minimize the sine term within the inner integral of equations (9) and (10). Moreover, they apply for all values of $|\phi_0|$ and $|\phi_f|$, both larger and smaller than $\pi/2$. The above results can be summarized by a d function aiming to move the mass closer to the two sides or $|\phi| = \pi/2$ on falling intervals and away on rising intervals. This can be explained qualitatively because the system aims to maximize the gravitational impulse on the wheel and achieves this by maximizing the gravitational force on falling intervals and minimizing it on rising intervals. Furthermore, it is crucial to mention that d is asymmetric because of how function d was defined on the tangential rod: positive to the right and vice versa, making the sign opposite at identical heights on each side. However, the position d is geometrically symmetric. As a result, it the mass will move downwards above the horizontal and upwards below the horizontal.

Plotting these results for both $|\phi_0| > \pi/2$ and $|\phi_0| < \pi/2$,



(a) $\phi_{max} > \pi/2$



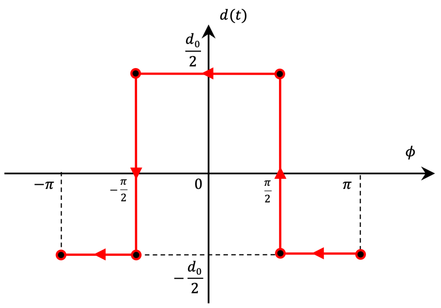
(b) $\phi_{max} < \pi/2$

Figure 8: Blue: $[-\phi_{max}, \phi_{max}]$ Red: $[\phi_{max}, -\phi_{max}]$

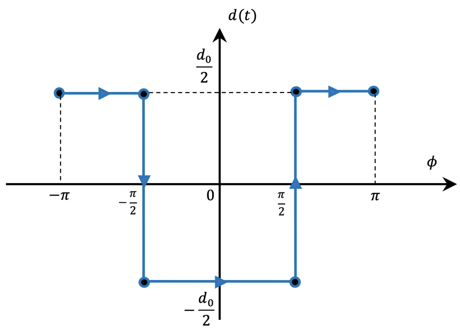
3.4 Category 2: Full Rotations

Full rotations of the system for tangential movement is also analyzed in an identical manner with radial movement. The torque of the system is integrated over one full revolution and the magnitude of the change in momentum is maximized by using equation (8), resulting in identical results as those featured in 3.3 except that the sign of the initial velocity will fix the optimal d function to one of the two sets.

Plotting these results for both $\dot{\phi}_0 > 0$ and $\dot{\phi}_0 < 0$,



(a) $\dot{\phi}_0 < 0$



(b) $\dot{\phi}_0 > 0$

Figure 9: Blue: $[-\pi, \pi]$ Red: $[\pi, -\pi]$

3.5 Numerical Modeling

Once again, using the optimal functions of $d(t)$ as obtained in sections 3.3 and 3.4 as well as the set of constant values defined in section 2.5, we can plot the trajectory of a mass given any initial condition. Plots for tangential movements are obtained by inputting equation (8) with the optimal functions of d into a differential equation solver.

Suppose the system is described by the set of initial conditions:

$$\phi(0) = \phi_0 \qquad\qquad \dot{\phi}(0) = 0$$

For computational purposes, we divide the motion into several phases, marked by instantaneous movements of d in between. From equation (11), we see that instantaneous changes in d affect neither the angular position nor the angular velocity of the system. Setting $\phi_0 = \pi/4$ and plotting the parametric graph of $\dot{\phi}(\phi)$,

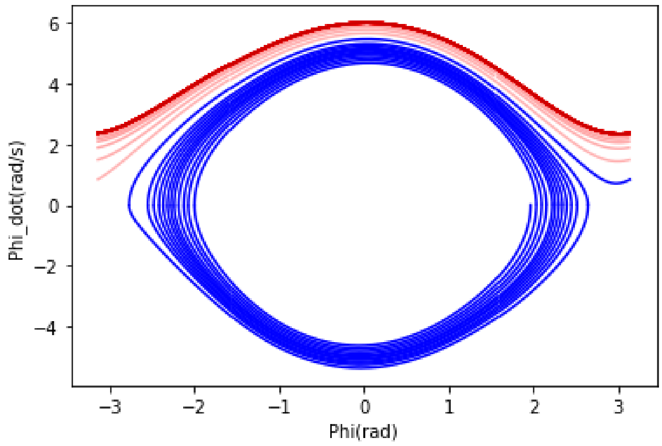


Figure 10: $\dot{\phi}(t)$ and $\phi(t)$ for tangentially moving mass

Unlike the phase portrait for the system with a radial axis of freedom, the above graph does not contain jumps. Instead, our tangential system increases its angular momentum gradually over one cycle as evidenced by a continuous expansion of the circle. This can be attributed to how movements in d in the tangential direction can be treated as internal and thus do not affect the system. Once again, the trajectory will at some point diverge from alternating rotations and evolve into full rotations as marked by the shift from blue to red curves. Plotting the graph of $\dot{\phi}(t)$ and $\phi(t)$ using the same initial conditions,

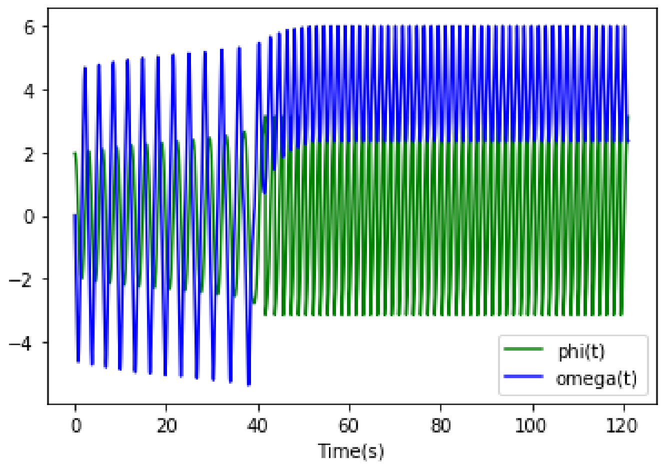


Figure 11: $\dot{\phi}(t)$ and $\phi(t)$ for tangentially moving mass

We find an extremely similar graph as the one featured for radial movements. However, the maximum speed at which the bird will travel at for purely tangential movements will evaluate to

$$|\dot{\phi}_{t,\,max}| = 5.982\ \frac{\text{rad}}{\text{s}}$$

And the minimum period experienced by the system evaluates to

$$|T_{t,\,min}| = 1.582\ \text{s}$$

4 Hybrid Model

After exploring the effects of radial and tangential movements on the motion of wheel, the optimal functions of d_r and d_t respectively can be combined to create the ideal model. For this system, the equation of motion is

$$\frac{d}{dt}\left((m+M)\,d'^2\dot{\phi}\right) = -mgd_r\,\sin\left(\phi + \frac{d_t}{R}\right) - b\dot{\phi} \tag{12}$$

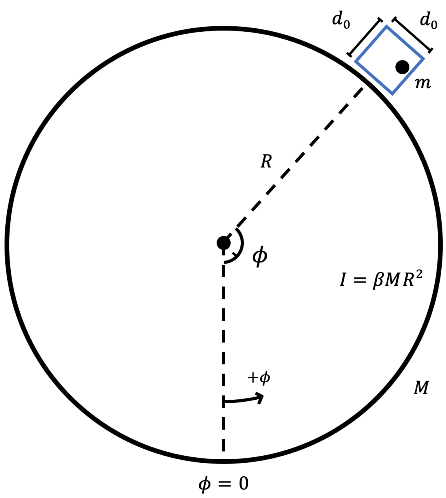


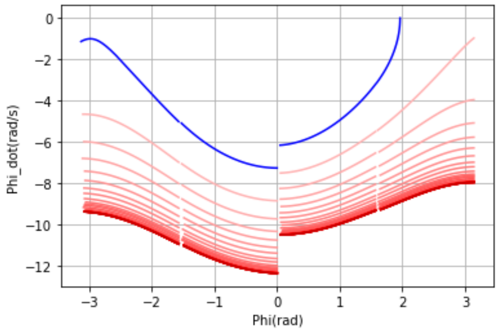
Figure 12: Mass with both radial and tangential degree of freedom on wheel

The combined system can be visually represented by the diagram above, with the bird denoted by a point mass m capable of moving anywhere within the blue square of side length d_0. Inputting the equation (12) with the optimal functions of d_r and d_t as solved previously as well as the same initial conditions as used in 2.5 and 3.5,

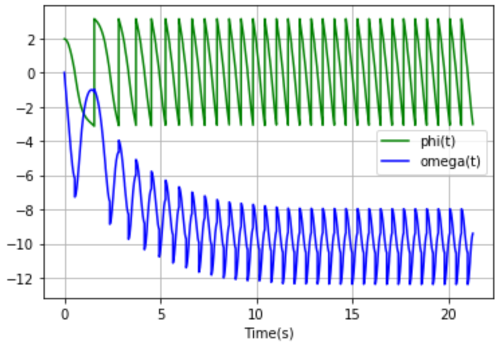
$$\phi(0) = \phi_0 \qquad\qquad \dot{\phi}(0) = 0$$

Trajectories for varying values of b (under-damped, critically damped, and over-damped) are then plotted with the parametric plot $\dot{\phi}(\phi)$ as well as $\dot{\phi}(t)$ and $\phi(t)$.

For an under-damped system or $b = 0.0525$ as used previously for radial and tangential movements of the mass, oscillations starting from alternating rotations will eventually evolve into full rotations and finally converging to some range of angular velocities.



(a) $\dot{\phi}(\phi)$

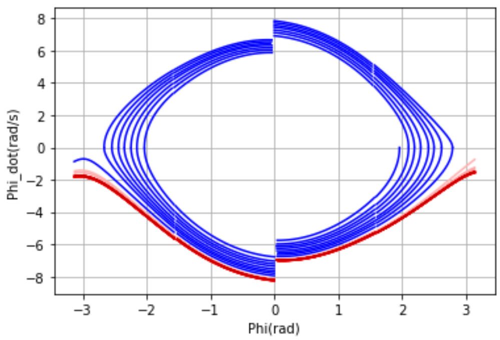


(b) $\dot{\phi}(t)$ and $\phi(t)$

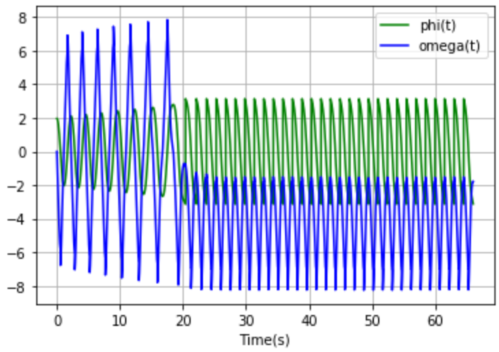
The maximum angular speed and minimum period achieved by the under-damped hybrid system ($b = 0.525$) as observed in the graphs above are

$$|\dot{\phi}_{t,\,max}| = 12.385\ \frac{\text{rad}}{\text{s}} \qquad\qquad |T_{h,\,min}| = 0.622\ \text{s}$$

For a critically damped system or $b \approx 0.105$ as computationally obtained, oscillations starting from alternating rotations will gain exactly enough angular momentum to evolve into full rotations.



(a) $\dot{\phi}(\phi)$

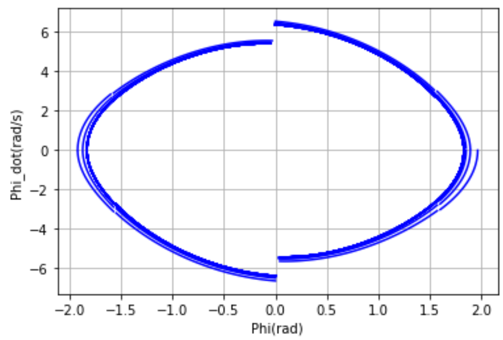


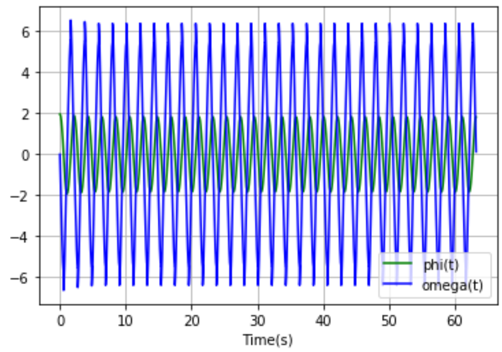
(b) $\dot{\phi}(t)$ and $\phi(t)$

The maximum angular speed and minimum period achieved by the critically damped hybrid system ($b = 0.105$) as observed in the graphs above are

$$|\dot{\phi}_{h,\,max}| = 8.262\ \frac{\text{rad}}{\text{s}} \qquad\qquad |T_{h,\,min}| = 1.529\ \text{s}$$

For an over-damped system or $b = 0.12$ (arbitrarily selected as long as greater than critical damping coefficient), oscillations starting from alternating rotations will never gain enough angular momentum to evolve into full rotations, forever rotating back and forth.





(a) $\dot{\phi}(\phi)$

(b) $\dot{\phi}(t)$ and $\phi(t)$

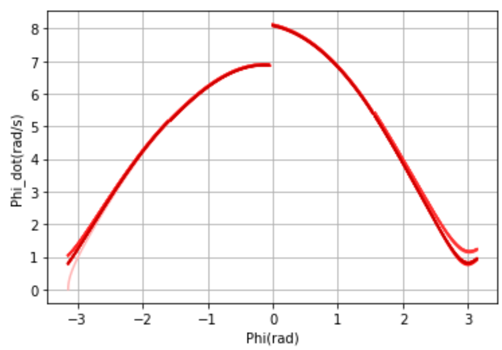
The maximum angular speed and minimum period achieved by the over-damped hybrid system ($b = 0.12$) as observed in the graphs above are

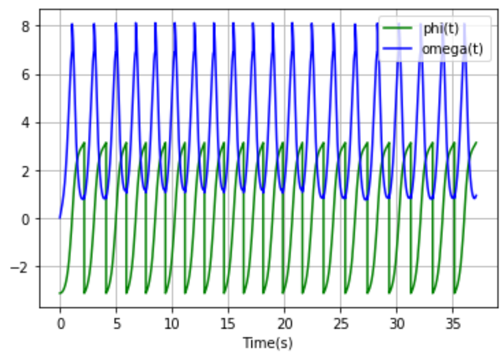
$$|\dot{\phi}_{h,\,max}| = 1.883\ \frac{\text{rad}}{\text{s}} \qquad\qquad |T_{h,\,min}| = 2.105\ \text{s}$$

Finally, using initial conditions that more accurately depict the motion of the cockatoo within the video [1], or a slightly moving bird at the top of the wheel:

$$\phi(0) = -\pi \qquad\qquad \dot{\phi}(0) = 0.001$$

Adopting $b = 0.11$ as a realistic damping value for the wheel justified by the observed motion of the bird within the video,





(a) $\dot{\phi}(\phi)$

(b) $\dot{\phi}(t)$ and $\phi(t)$

The maximum angular speed and minimum period achieved by the final model ($b = 0.11$) as observed in the graphs above are

$$|\dot{\phi}_{h,\,max}| = 8.127\ \frac{\text{rad}}{\text{s}} \qquad\qquad |T_{h,\,min}| = 1.960\ \text{s}$$

5 Limitations of Our Model

In the process of modeling the bird-wheel system, we have made several assumptions that only hold true in the most ideal case.

First, we assumed that the bird can be moved instantaneously between 2 points, which is not achievable in real life. To address this, a finite non-zero time interval depicting the minimal time required for the bird to move distances of d_0 can be introduced into the equations. While this does not affect the overall shape of the optimal function d for both radial and tangential, it will decrease the efficiency of our accelerating process.

Second, the frictional drag on the system is modeled to be purely proportional to the angular velocity of the wheel. In reality, frictional drag could also depend on the magnitude of the net normal force exerted by the center on the system. The inclusion of such a term will mean that extreme movements of the bird in the radial direction will not be viable as that produces a very large frictional force. Similarly, this will not affect the overall behavior of our solution for radial movements but there will now exist an optimal time interval Δt for transitions from one end to the other.

6 Conclusion

The results of this study showed that theoretically, the most optimal combination of radial and tangential motion of a "bird" mass on a wheel was a superposition of the most optimal motion of each of the components; Radial extension occurred at the highest point on a swing and radial contraction occurred at the lowest point; tangential motion occurred at the side-most extremes, moving upwards. These results corroborate our expectations that the energy used by the motors moving the mass should move upwards in all situations to directly translate into gravitational potential energy. Due to the existence of a frictional force proportional to angular velocity, there will exist a maximum angular velocity.

References

[1] Cockatoo Loves Going Around In Circles, https://www.youtube.com/watch?v=F1P6IWPOJl8, Last accessed 28 July 2020.