DAN OTIENO - EE 307 - Homework 4.

4.1 Point charges $Q_1 = 5\,\mu C$ and $Q_2 = -4\,\mu C$ are placed at $(3, 2, 1)$ and $(-4, 0, 6)$, respectively. Determine the force on Q_1.

$$F_{Q_1} = \frac{(Q_1)(Q_2)\left[r_{Q_1} - r_{Q_2}\right]}{4\pi\varepsilon_0 \left|r_{Q_1} - r_{Q_2}\right|^3}$$

$$= \frac{(5 \times -4)(10^{-12})\left[(3,2,1) - (-4,0,6)\right]}{4\pi\varepsilon_0 \left|(3,2,1) - (-4,0,6)\right|^3}$$

$$= \frac{-20(10^{-12})\left[(7,2,-5)\right]}{4\pi \frac{10^{-9}}{36\pi} \left|(7,2,-5)\right|^3}$$

$$= \left[\left(\frac{-180(7,2,-5) \times 10^{-3}}{688.9}\right) N\right] \times 10^{3}\, mN$$

$$\approx \boxed{-1.829\, a_x - 0.5226\, a_y + 1.3064\, a_z}$$

4.2 Point charges Q_1 and Q_2 are, respectively, located at $(4, 0, -3)$ and $(2, 0, 1)$. If $Q_2 = 4$ nC, find Q_1 such that

(a) The **E** at $(5, 0, 6)$ has no z-component.

(b) The force on a test charge at $(5, 0, 6)$ has no x-component.

a

At E(5,0,6) - no z component.

$$R_1 = r_E - r_{Q_1} = (5,0,6) - (4,0,-3) = (1,0,9)$$
$$R_2 = r_E - r_{Q_2} = (5,0,6) - (2,0,1) = (3,0,5)$$

$$E(5,0,6) = \frac{Q_1 (1,0,9)}{4\pi\varepsilon_0 |R_1|^{3/2}} + \frac{Q_2 (3,0,5)}{4\pi\varepsilon_0 |R_2|^{3/2}}$$

If $E_z = 0$:

$$0 = \frac{9Q_1}{4\pi\varepsilon_0 (82)^{3/2}} + \frac{5Q_2}{4\pi\varepsilon_0 (34)^{3/2}}$$

$$\frac{9Q_1}{\cancel{4\pi\varepsilon_0}(82)^{3/2}} = -\frac{5Q_2}{\cancel{4\pi\varepsilon_0}(34)^{3/2}}$$

$$\frac{9Q_1}{(82)^{3/2}} = \frac{-5Q_2}{(34)^{3/2}}$$

$$Q_1 = -\frac{5Q_2}{9} \cdot \left(\frac{82}{34}\right)^{3/2} \text{ nC}$$

$$Q_1 = -\frac{5}{9}(4)(3.7454) \text{ nC}$$

$$Q_1 = \boxed{-8.323 \text{ nC}}$$

b:

The force on a test charge at (5,0,6) has no x-component:

$$F(5,0,6) = qE(5,0,6), \text{ if } F_x = 0;$$

$$\frac{qQ_1}{\cancel{4\pi\varepsilon_0}(82)^{3/2}} + \frac{3qQ_2}{\cancel{4\pi\varepsilon_0}(34)^{3/2}} = 0$$

$$Q_1 = -3Q_2\left(\frac{82}{34}\right)^{3/2} = -12(3.7454)\ nC$$

$$Q_1 = \boxed{-44.945\ nC}$$

4.5 Determine the total charge

(a) On line $0 < x < 5$ m if $\rho_L = 12x^2$ mC/m

(b) On the cylinder $\rho = 3$, $0 < z < 4$ m if $\rho_S = \rho z^2$ nC/m^2

(c) Within the sphere $r = 4$ m if $\rho_v = \dfrac{10}{r \sin\theta}$ C/m^3

a:

$$Q = \int \rho\, dl = \int_0^5 12x^2\, dx = 4x^3 \Big|_0^5$$

$$= 4(5^3)\ mC = 500\ mC = \boxed{0.5\ C}$$

b:

$$Q = \int \rho_s\, ds = \int_{z=0}^{4} \int_{\phi=0}^{2\pi} \rho z^2 \rho\, d\phi\, dz \Big|_{\rho=3} = 9(2\pi)\frac{z^3}{3}\Big|_0^4\ nC$$

$$= \overset{3}{\cancel{9}}(2\pi)\left[\frac{4^3}{\cancel{3}}\right] = 1206.4\ nC$$

$$= \boxed{1.2064\ \mu C}$$

c:

$$Q = \int \rho_v \, dv = \iiint \frac{10}{r \sin\theta} r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 10 \int_0^{2\pi}\int_0^{\pi}\int_0^{4} d\phi \, d\theta \, r \, dr = 10(2\pi)(\pi)(8)$$

$$= \boxed{1{,}579.1 \text{ C}}$$

4.6 A cube is defined by $0 < x <$ a, $0 < y <$ a, and $0 < z <$ a. If it is charged with $\rho_v = \frac{\rho_o x}{a}$, where ρ_o is a constant, calculate the total charge in the cube.

$$Q = \int_V \rho_v \, dv \int_0^a\int_0^a\int_0^a \frac{\rho_o x}{a} \, dx \, dy \, dz = (a)(a)\rho_o \left[\frac{x^2}{2a} \Big|_0^a \right]$$

$$= a^2 \rho_o \left[\frac{a^{\cancel{2}}a}{2\cancel{a}} \right] = a^2 \rho_o \left(\frac{a}{2} \right)$$

$$= \boxed{\frac{a^3 \rho_o}{2}}$$

4.11 Line $0 < x < 1$ m is charged with density $12x^2$ nC/m. (a) Find the total charge. (b) Determine the electric field intensity at (0, 0, 1000 m).

a:

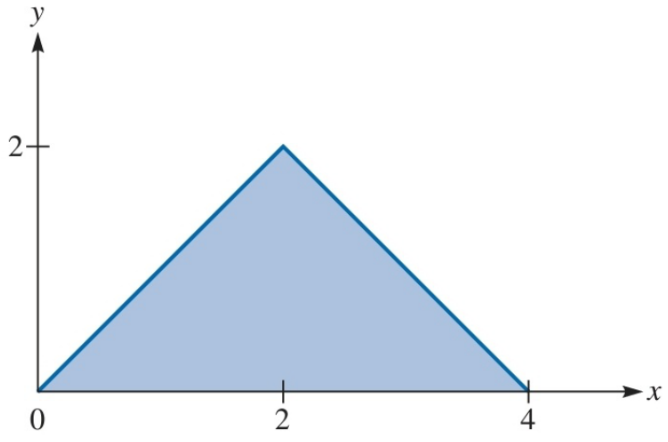
$$Q = \int_L \rho_L \, dl = \int_0^1 12x^2 \, dx = 4\left(x^3 \Big|_0^1\right) = \boxed{4 \text{nC}}$$

b:

$$E = \frac{Q}{4\pi\varepsilon_o r^2} a_r = \frac{4 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} (1000)^2} a_z = \frac{(4)(9)}{10^6} a_z$$

$$= \boxed{3.6 \times 10^{-6} a_z \text{ V/m}}$$

4.13 An annular disk of inner radius a and outer radius b is placed on the xy-plane and centered at the origin. If the disk carries uniform charge with density ρ_s, find $\mathbf{E}$ at $(0, 0, h)$.



$$E = \oint \frac{\rho_s \, dS}{4\pi \varepsilon_0 R^3} R \; ; \; R = \rho(-a_\rho) + h a_z \; ; \; dS = \rho \, d\phi \, d\rho$$

$$E = \frac{\rho_s}{4\pi\varepsilon_0} \oint \frac{\rho \, d\phi \, d\rho}{(\rho^2 + h^2)^{3/2}} (-\rho a_\rho + h a_z)$$

$$E = \frac{\rho_s h a_z}{4\pi\varepsilon_0} \int_a^b \rho (\rho^2 + h^2)^{-3/2} d\rho \int_0^{2\pi} d\phi$$

using u-sub ; $u = \rho^2 + h^2$, $du = 2\rho \, d\rho$

$$E = \frac{\rho_s h a_z}{4\pi\varepsilon_0} (2\pi) \int \frac{1}{2} u^{-3/2} du = \frac{\rho_s h a_z}{2\varepsilon_0} \frac{1/2 \, u^{-1/2}}{-1/2}$$

$$E = \frac{\rho_s h a_z}{2\varepsilon_0} \left[-\frac{1}{\sqrt{\rho^2 + h^2}} \Bigg|_a^b \right]$$

$$\boxed{E = \frac{\rho_s h}{4\pi\varepsilon_0} \left[\frac{1}{\sqrt{a^2 + h^2}} - \frac{1}{\sqrt{b^2 + h^2}} \right] a_z}$$

4.21 A ring placed along $y^2 + z^2 = 4$, $x = 0$ carries a uniform charge of 5 μC/m.

(a) Find **D** at $P(3, 0, 0)$.

(b) If two identical point charges Q are placed at $(0, -3, 0)$ and $(0, 3, 0)$ in addition to the ring, find the value of Q such that $\mathbf{D} = 0$ at P.

a:

$$\bar{D} = \int_0^{2\pi} \frac{\rho_\ell \, dl \, \bar{R}}{4\pi R^3}$$

$$\rho_\ell = 5 \times 10^{-6}; \quad dl = \rho \, d\phi = 2 \, d\phi; \quad \bar{R} = -2\hat{a}_\rho + 3\hat{a}_x$$

$$R^3 = (13)^{3/2} = 13\sqrt{13}.$$

$$\therefore \bar{D} = \int_0^{2\pi} \frac{5 \times 10^{-6} \, 2 \, d\phi \, (\cancel{-2\hat{a}_\rho} + 3\hat{a}_x)}{4\pi \, 13\sqrt{13}}$$

$$= \frac{5 \times 10^{-6} (2)(3\hat{a}_x) \, 2\pi}{4\pi \, 13\sqrt{13}} = \boxed{320 \times 10^{-9} \hat{a}_x \ C/m^2}$$

b:

$$D_P = 0 = 320 \times 10^{-9} \hat{a}_x + \frac{Q[(3,0,0) - (0,-3,0)]}{4\pi \, 18\sqrt{18}}$$

$$+ \frac{Q[(3,0,0) - (0,3,0)]}{4\pi \, 18\sqrt{18}}$$

$$= 320 \times 10^{-9} \hat{a}_x + \frac{Q(3,3,0)}{4\pi \, 18\sqrt{18}} + \frac{Q(3,-3,0)}{4\pi \, 18\sqrt{18}} = 0$$

$$= 320 \times 10^{-9} \hat{a}_x + \frac{Q(6,0,0)}{4\pi \, 18\sqrt{18}} = 0$$

$$\Rightarrow -320 \times 10^{-9} = \frac{6Q}{4\pi \, 18\sqrt{18}}$$

$$\therefore \boxed{Q = -51.18 \times 10^{-6} \ C}$$

4.25 Determine the charge density due to each of the following electric flux densities:

(a) $\mathbf{D} = 8xy\mathbf{a}_x + 4x^2\mathbf{a}_y$ C/m^2

(b) $\mathbf{D} = 4\rho \sin\phi\, \mathbf{a}_\rho + 2\rho \cos\phi\, \mathbf{a}_\phi + 2z^2\mathbf{a}_z$ C/m^2

(c) $\mathbf{D} = \dfrac{2\cos\theta}{r^3}\mathbf{a}_r + \dfrac{\sin\theta}{r^3}\mathbf{a}_\theta$ C/m^2

a:

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= 0 + 8 = \boxed{8}$$

b:

$$\rho_v = \nabla \cdot D = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$= 8\sin\phi - 2\sin\phi + 4z$$

$$= \boxed{6\sin\phi + 4z \ \ C/m^3}$$

c:

$$\rho_v = \nabla \cdot D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\theta) + 0$$

$$= -\frac{2}{r^4}\cos\theta + \frac{2\cos\theta}{r^4} = \boxed{0}$$

4.29 Let $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and find

(a) The volume charge density ρ_v.

(b) The flux through surface $0 < x < 1, 0 < z < 1, y = 1$.

(c) The total charge contained in the region $0 < x < 1, 0 < y < 1, 0 < z < 1$.

a:

$$\rho_v = \nabla \cdot D = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= \boxed{2y \text{ C/m}^3}$$

b:

$$\text{Flux} = \int D \cdot d\bar{s} = \iint x^2 \, dx \, dz \Big|_{y=1}$$

$$= \int_0^1 x^2 \, dx \int_0^1 dz$$

$$= \left[\frac{x^3}{3}\right]_0^1 \left[z\right]_0^1 = \left[\frac{1}{3} - 0\right][1] = \boxed{\frac{1}{3} \text{ C}}$$

c:

$$Q = \int_v \rho_v \, dv = \iiint 2y \, dx \, dy \, dz$$

$$= 2 \int_0^1 dx \int_0^1 y \, dy \int_0^1 dz \quad \text{(no xs or zs)}$$

$$= 2 \left[x\right]_0^1 \left[y^2/2\right]_0^1 \left[z\right]_0^1$$

$$= 2 [1] [1/2] [1] = \boxed{1 \text{ C}}$$

4.35 Two point charges $Q = 2$ nC and $Q = -4$ nC are located at $(1, 0, 3)$ and $(-2, 1, 5)$, respectively. Determine the potential at $P(1, -2, 3)$.

$$V_P = \frac{Q_1}{4\pi\varepsilon_0 r_1} + \frac{Q_2}{4\pi\varepsilon_0 r_2}$$

$$= \frac{10^{-9}}{\cancel{4\pi}\left(\frac{10^{-9}}{\cancel{36\pi}}\right)} \left[\frac{2}{|(1,-2,3)-(1,0,3)|} - \frac{4}{|(1,-2,3)-(-2,1,5)|}\right]$$

9

$$= 9\left[\frac{2}{2} - \frac{4}{\sqrt{22}}\right] = 9(0.1472) \approx \boxed{1.3248}$$

4.40 $V = x^2y(z + 3)$ V. Find

(a) **E** at $(3, 4, -6)$

(b) the charge within the cube $0 < x < 1, 0 < y < 1, 0 < z < 1$.

a:

$$E = -\left(\frac{\delta V}{\delta x}\hat{a}_x + \frac{\delta V}{\delta y}\hat{a}_y + \frac{\delta V}{\delta z}\hat{a}_z\right)$$

$$= -2xy(z+3)\hat{a}_x - x^2(z+3)\hat{a}_y - x^2y\hat{a}_z$$

$$= [-2(3)(4)(-3)\hat{a}_x] - [9(-3)\hat{a}_y] - [9(4)\hat{a}_z]$$

$$= \boxed{72\hat{a}_x + 27\hat{a}_y - 36\hat{a}_z \ \text{V/m}}$$

b:

$$\rho_v = \nabla \cdot D = \varepsilon_0 \nabla \cdot E = -\varepsilon_0(2y)(z+3)$$

$$\Psi = \int \rho_v \, dv = -2\varepsilon_0 \iiint y(z+3)\, dx\, dy\, dz$$

$$= -2\varepsilon_0 \int_0^1 dx \int_0^1 y\, dy \int_0^1 (z+3)\, dz$$

$$= -2\varepsilon_0 \left[x\right]_0^1 \left[y^2/2\right]_0^1 \left[z^2/2 + 3z\right]_0^1$$

$$= -2\varepsilon_0 [1] \left[1/2\right] \left[1/2 + 3\right]$$

$$= -2\varepsilon_0 [1] [1/2] [3.5] \approx \boxed{-30.99 \times 10^{-12}\ C}$$

4.43 If $\mathbf{D} = 4x\mathbf{a}_x - 10y^2\mathbf{a}_y + z^2\mathbf{a}_z$ C/m², find the charge density at $P(1, 2, 3)$.

$$\rho_v = \nabla \cdot D = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} = 4 - 20y + 2z$$

At $P(1,2,3)$, $\rho_v = 4 - 20(2) + 2(3)$

$$= 4 - 40 + 6 = \boxed{-30\ C/m^2}$$

4.46 The electric field intensity in free space is given by

$$\mathbf{E} = 2\,xyz\mathbf{a}_x + x^2z\mathbf{a}_y + x^2y\mathbf{a}_z \text{ V/m}$$

Calculate the amount of work necessary to move a 2 μC charge from $(2, 1, -1)$ to $(5, 1, 2)$.

$$W = -Q\int E \cdot d\bar{l}\,;\quad \frac{W}{-Q} = \int_{x=2}^{5} 2xyz\, dx \Bigg|_{\substack{z=-1\\ y=1}} + \int_{z=-1}^{2} x^2 y\, dz \Bigg|_{\substack{x=5\\ y=1}}$$

$$\frac{W}{-Q} = 2[1][-1]\frac{x^2}{2}\Bigg|_2^5 + [5]^2[1]z\Bigg|_{-1}^{2} W =$$

$$\frac{W}{-Q} = -21 + 75 = 54, \therefore W = -54Q = \boxed{-0.000108\ J}$$