DAN OTIENO - EE 307 - Homework 7.

7.1 (a) State Biot–Savart's law.

(b) The y- and z-axes, respectively, carry filamentary currents 10 A along $\mathbf{a}_y$ and 20 A along $-\mathbf{a}_z$. Find $\mathbf{H}$ at $(-3, 4, 5)$.

a:

Bio-Savart's law:

$$dH = \frac{I\,dl \times a_R}{4\pi R^2} = \frac{I\,dl \times R}{4\pi R^3}$$

where $R = |R|$, and dH = differential magnetic field intensity produced at a point P due to the differential current element $I\,dl$.

b:

If $H = H_y + H_z$ and if $H_z = \frac{I}{2\pi\rho} a_\phi$, $\rho = \sqrt{25} = 5$

$$a_\phi = -a_z \frac{(-3a_x + 4a_y)}{5} = \frac{(3a_y - 4a_x)}{5}$$

$$H_z = \frac{20}{2\pi(25)}(4a_x + 3a_y) = 0.509\,a_x + 0.382\,a_y$$

If $H_y = \frac{I}{2\pi\rho} a_\phi$, $\rho = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$

$$a_\phi = a_y \frac{(-3a_x + 5a_z)}{\sqrt{34}} = \frac{3a_z + 5a_z}{\sqrt{34}}$$

$$H_y = \frac{10}{2\pi(34)}(5a_x + 3a_z) = 0.234\,a_x + 0.140\,a_z$$

$$\therefore \boxed{H = 0.743\,a_x + 0.382\,a_y + 0.140\,a_z}$$

7.3 Two infinitely long wires, placed parallel to the z-axis, carry currents 10 A in opposite directions as shown in Figure 7.28. Find **H** at point *P*.

$$\bar{H} = \bar{H}_1 + \bar{H}_2 \;;\; H_1 = I/2\pi\rho\, a_\phi \text{ where } \rho = 5.$$

$$\therefore \bar{H}_1 = \frac{10}{2\pi(5)}\hat{a}_y = \hat{a}_y/\pi$$

$$\bar{H}_2 = I/2\pi\rho\, a_\phi, \text{ where } \rho = 5\sqrt{2}$$

$$a_\rho = \frac{5a_x - 5a_y}{5\sqrt{2}} = \frac{a_x - a_y}{\sqrt{2}}$$

$$a_\phi = -a_z \times \left(\frac{a_x - a_y}{\sqrt{2}}\right) = \frac{-\hat{a}_x - \hat{a}_y}{\sqrt{2}}$$

$$\bar{H}_2 = \frac{10}{2\pi\, 5\sqrt{2}}\left(\frac{-a_x - a_y}{\sqrt{2}}\right) = \frac{1}{2\pi}(-a_x - a_y)$$

$$\text{Total } \bar{H} = \frac{\hat{a}_y}{\pi} + \frac{1}{2\pi}(-\hat{a}_x - \hat{a}_y) \approx \boxed{-0.16\hat{a}_x + 0.16\hat{a}_y}$$

7.6 Consider *AB* in Figure 7.29 as part of an electric circuit. Find **H** at the origin due to *AB*.

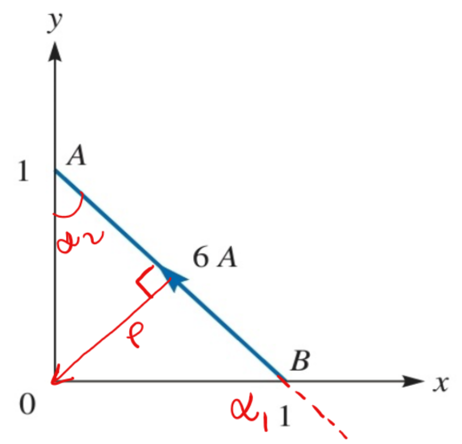

FIGURE 7.29 For Problem 7.6.

$$H = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,a_\phi$$

$$\alpha_1 = 135^\circ;\quad \alpha_2 = 45^\circ;\quad \rho = \frac{\sqrt{2}}{2}$$

$$a_\phi = \alpha_l \times \alpha_\rho = \left[\frac{-a_x + a_y}{\sqrt{2}}\right] \times \left[\frac{-a_x - a_y}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\begin{vmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = a_z$$

$$H = \frac{\overset{3}{\cancel{6}}}{\underset{1}{\cancel{2}}\cancel{4}\pi\frac{\sqrt{2}}{\underset{1}{\cancel{2}}}}(\cos 45^\circ - \cos 135^\circ)\,a_z$$

$$= \frac{3}{\sqrt{2}\pi}(1.414) \approx \boxed{0.9548\ \text{A/m}}$$

7.20 Current sheets of $20\mathbf{a}_x$ A/m and $-20\mathbf{a}_x$ A/m are located at $y = 1$ and $y = -1$, respectively. Find $\mathbf{H}$ in region $-1 < y < 1$.

$$H = \sum \tfrac{1}{2} K \times a_n$$

$$= \frac{1}{2}(20a_x) \times (-a_y) + \frac{1}{2}(-20a_x) \times a_y$$

$$= 10(-a_z) - 10(a_z)$$

$$= \boxed{-20a_z\ \text{A/m}}$$

7.23 An infinitely long cylindrical conductor of radius a is placed along the z-axis. If the current density is $\mathbf{J} = \frac{J_o}{\rho}\mathbf{a}_z$, where J_o is constant, find $\mathbf{H}$ everywhere.

$$\int_L H \cdot dl = I_{enc} = \int J \cdot ds \quad \text{where } 0 < \rho < a$$

$$H_\phi \, 2\pi\rho = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\rho} \frac{J_o}{\rho} \rho \, d\phi \, d\rho = J_o \, 2\pi\rho \; ; \; H_\phi = J_o$$

$$\int H \cdot dl = \int J \cdot ds = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{J_o}{\rho} \rho \, d\phi \, d\rho \; ; \quad \rho > a$$

$$H_\rho \, 2\pi\rho = J_o \, 2\pi a \; ; \quad H_\phi = \frac{J_o a}{\rho}$$

$$\therefore \boxed{H_\phi = \begin{cases} J_o, & 0 < \rho < a \\ \frac{J_o a}{\rho}, & \rho > a \end{cases}}$$

7.24 Let $\mathbf{H} = y^2\mathbf{a}_x + x^2\mathbf{a}_y$ A/m. (a) Find $\mathbf{J}$. (b) Determine the current through the strip $z = 1, 0 < x < 2, 1 < y < 5$.

a:

$$J = \nabla \times H = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x^2 & 0 \end{vmatrix} \; ; \; \boxed{J = 2(x-y)\, a_z}$$

b:

$$I = \int_S J \cdot ds = \iint (2x - 2y)\, dx\, dy$$

$$= 2\int_1^5 dy \int_0^2 x\, dx - 2\int_0^2 dx \int_1^5 y\, dy = 2(4)\left[\frac{x^2}{2}\right]_0^2 - 2(2)\left[\frac{y^2}{2}\right]_1^5$$

$$I = 4(4) - 2(25-1) = 16 - 48 = \boxed{-32 \text{ A}}$$

7.34 In free space, the magnetic flux density is

$$\mathbf{B} = y^2\mathbf{a}_x + z^2\mathbf{a}_y + x^2\mathbf{a}_z \text{ Wb/m}^2$$

(a) Show that **B** is a magnetic field
(b) Find the magnetic flux through $x = 1, 0 < y < 1, 1 < z < 4$.
(c) Calculate **J**.
(d) Determine the total magnetic flux through the surface of a cube defined by $0 < x < 2, 0 < y < 2, 0 < z < 2$.

a:

$$\nabla \cdot B = \frac{\delta B_x}{\delta x} + \frac{\delta B_y}{\delta y} + \frac{\delta B_z}{\delta z} = 0$$

b:

$$ds = dy\,dz\,a_x$$

$$\Psi = \int B \cdot ds = \int_{z=1}^{4}\int_{y=0}^{1} y^2\,dy\,dz = \frac{y^3}{3}\Big|_0^1 \; z\Big|_1^4 = \boxed{1\text{ Wb}}$$

c:

$$J = \nabla \times H = \nabla \times \frac{B}{\mu_0} \; ; \quad \nabla \times B = \begin{vmatrix} \delta/\delta x & \delta/\delta y & \delta/\delta z \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$$\nabla \times B = -2z a_x - 2x a_y - 2y a_z$$

$$\therefore \boxed{J = -\frac{2}{\mu_0}(z a_x + x a_y + y a_z)\text{ A/m}^2}$$

d:

$$\nabla \cdot B = 0$$

$$\text{so } \Psi = \oint B \cdot ds = \int_v \nabla \cdot B\,dv = \boxed{0}$$

7.43 In free space, $\mathbf{A} = 10 \sin \pi y \mathbf{a}_x + (4 + \cos \pi x)\mathbf{a}_z$ Wb/m. Find **H** and **J**.

$$B = \mu_o H = \nabla \times A = \begin{vmatrix} \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 10 \sin \pi y & 0 & 4 + \cos \pi x \end{vmatrix}$$

$$= \pi \sin \pi x \, a_y - 10 \pi \cos \pi y \, a_z$$

$$\therefore \boxed{H = \frac{\pi}{\mu_o}\left(\sin \pi x \, a_y - 10 \cos \pi y \, a_z\right)}$$

$$J = \nabla \times H = \frac{\pi}{\mu_o} \begin{vmatrix} \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 0 & \sin \pi x & -10 \cos \pi y \end{vmatrix}$$

$$= \pi/\mu_o \left(10\pi \sin \pi y \, a_x + \pi \cos \pi x \, a_z\right)$$

$$\therefore \boxed{J = \frac{\pi^2}{\mu_o}\left(10 \sin \pi y \, a_x + \cos \pi x \, a_z\right)}$$

7.45 For a current distribution in free space,

$$\mathbf{A} = (2x^2y + yz)\mathbf{a}_x + (xy^2 - xz^3)\mathbf{a}_y - (6xyz - 2x^2y^2)\mathbf{a}_z \text{ Wb/m}$$

(a) Calculate **B**.

(b) Find the magnetic flux through a loop described by $x = 1, 0 < y < 2, 0 < z < 2$.

(c) Show that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

a:

$$B = \nabla \times A = \begin{vmatrix} \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 2x^2y + yz & xy^2 - xz^3 & -6xyz - 2x^2y^2 \end{vmatrix}$$

$$= \boxed{(-6xz + 4x^2y + 3xz^2)a_x + (y + 6yz - 4xy^2)a_y + (y^2 - z^3 - 2x^2 - z)a_z \ \text{Wb/m}^2}$$

b:

$$\Psi = \int_{z=0}^{2}\int_{y=0}^{2}(-6x^{z} + 4x^2y + 3xz^2)\,dy\,dz \Big|_{x=1}$$

$$= \iint_0 (-6xz)\,dy\,dz + 4\iint_0 x^2y\,dy\,dz + 3\iint_0 xz^2\,dy\,dz$$

$$= -6\int_0^2 dz\int_0^2 dy + 4\int_0^2 dz\int_0^2 y\,dy + 3\int_0^2 dy\int_0^2 z^2\,dz$$

$$= -6(2)(2) + 4(2)\left[\frac{y^2}{2}\Big|_0^2\right] + 3(2)\left[\frac{z^3}{3}\Big|_0^2\right]$$

$$= -24 + 8[2] + \cancel{6}^{2}[8/\cancel{3}_{1}]$$

$$\Psi = -24 + 16 + 16 = -24 + 32 = \boxed{8\text{ Wb}}$$

c:

$$\nabla\cdot A = \frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}$$

$$= 4xy + 2xy - 6xy = 0$$

$$\nabla\cdot B = -\cancel{6z} + \cancel{8xy} + \cancel{3z^3} + \cancel{6z} - \cancel{8xy} + \cancel{1} - \cancel{3z^3} - \cancel{1}$$

$$= 0$$

$$\therefore\ \nabla\cdot B = \nabla\cdot(\nabla\times A) = 0.$$

7.47 The magnetic vector potential of a current distribution in free space is given by

$$\mathbf{A} = 15e^{-\rho} \sin\phi \, \mathbf{a}_z \text{ Wb/m}$$

Find $\mathbf{H}$ at $(3, \pi/4, -10)$. Calculate the flux through $\rho = 5, 0 \leq \phi \leq \pi/2, 0 \leq z \leq 10$.

$$B = \nabla \times A = \frac{1}{\rho}\frac{\delta A_z}{\delta \phi} a_\rho - \frac{\delta A_z}{\delta \rho} a_\phi$$

$$= \frac{15}{\rho} e^{-\rho} \cos\phi \, a_\rho + 15 e^{-\rho} \sin\phi \, a_\phi$$

$$B\left(3, \frac{\pi}{4}, -10\right) = \frac{\cancel{15}^{5}}{\cancel{3}_{1}} e^{-3} \frac{1}{\sqrt{2}} a_\rho + 15 e^{-3} \frac{1}{\sqrt{2}} a_\phi$$

$$H = \frac{B}{\mu_0} = \frac{10^7}{4\pi} \frac{15}{\sqrt{2}} e^{-3} \left(\frac{1}{3} a_\rho + a_\phi\right)$$

$$\boxed{H = (14 a_\rho + 42 a_\phi)(10^4) \text{ A/m}}$$

$$\Psi = \int B \cdot ds = \iint \frac{15}{\rho} e^{-\rho} \cos\phi \, \rho \, d\phi \, dz$$

$$= 15z \Big|_0^{10} (\sin\phi) \Big|_0^{\pi/2} e^{-5}$$

$$= [15(10)] \cdot [(1)-(0)] \cdot [e^{-5}] = 150 e^{-5}$$

$$\therefore \boxed{\Psi \approx 1.011}$$