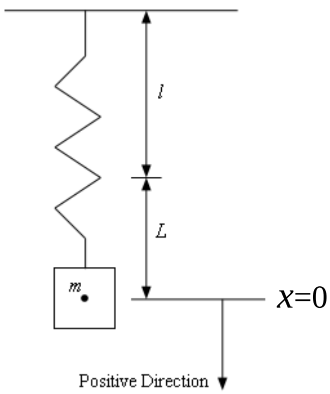
Department of Electrical and Computer Engineering
The University of Alabama in Huntsville

CPE 381: Fundamentals of Signals and Systems for Computer Engineers

Homework #3 Solution

1. Write differential equation describing displacement x of suspended weight m on spring with elastic constant k.



In equilibrium, elastic forces is proportional to the displacement L:

$$F = kL, \qquad k = \frac{F}{L} = \frac{mg}{L}$$

At any time, sum of all forces is equal to zero

$$m\ddot{x} + kx = 0$$

Example: A 1 kg weight is hung on the end of a vertically suspended spring, thereby stretching the spring *L*=10 cm. The weight is raised 5 cm above its equilibrium position and released from rest at time *t*=0. Find the displacement x of the weight from its equilibrium position at time *t.* Use g=10m/s^2.

$$F = kL, \qquad k = \frac{F}{L} = \frac{mg}{L} = \frac{1[kg]\,10\left[\frac{m}{s^2}\right]}{0.1[m]} = 100\left[\frac{kg}{s^2}\right]$$

At any time, sum of all forces is equal to zero

$$m\ddot{x} + kx = 0$$

With initial conditions

$$x(0) = -0.05[m] \;\; \dot{x}(0) = 0$$

By using Laplace transform

$$\mathcal{L}(m\ddot{x} + kx) = s^2X(s) - sx(0) - \dot{x}(0) + kX(s) = 0$$
$$(s^2 + 100)X(s) = -0.05s$$
$$X(s) = \frac{-0.05s}{s^2 + 100}$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = -0.05\cos(10t)$$

2. Use Matlab symbolic computation to find the Laplace transform of a real exponential

$$x(t) = 4e^{-2t}\cos(8t)\,u(t)$$

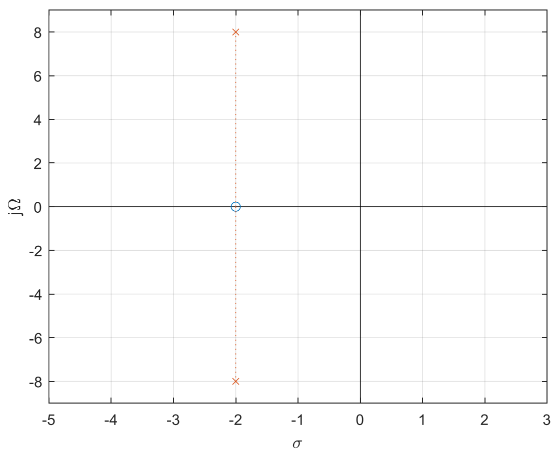
Plot the signal and the poles and zeros of their Laplace transform.

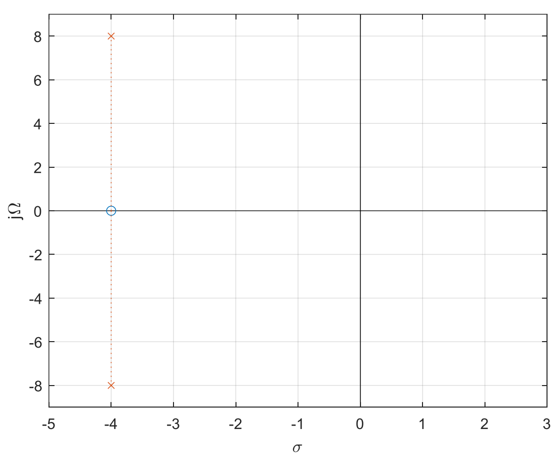
Repeat the analysis and plot the results for $x(t) = 4e^{-4t}\cos(8t)\,u(t)$

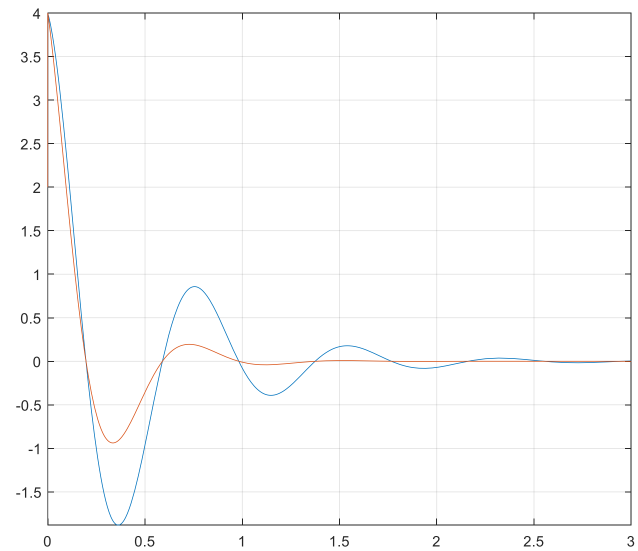
Discuss the changes in the s plane and describe their effect on function in time domain.

```
syms t x1 x2
x1=4*exp(-2*t)*cos(8*t)*heaviside(t);
x2=4*exp(-4*t)*cos(8*t)*heaviside(t);
X1=laplace(x1)
% X1 = (4*(s + 2))/((s + 2)^2 + 64)
% X1 = (4*s+8)/(s^2+4*s+68)
% plot using splane
splane([4 8],[1 4 68])

X2=laplace(x2)
% X2 = (4*(s + 4))/((s + 4)^2 + 64)
% X2 = (4*s+16)/(s^2+8*s+80)
figure
% plot
splane([4 16],[1 8 80])
```







Zeros and poles of X1 and X2	Output y1 and y2

Discuss the changes in the s plane and describe their effect on function in time domain

Zeros and poles shifted to the left (larger absolute values of σ); consequently, signal in time domain is more attenuated (damped).

3. Consider a second order differential equation,

$$\frac{d^2y(t)}{dt}+3\frac{dy(t)}{dt}+2y(t)=x(t)$$

with initial conditions $y(0) = 1$ and $\frac{dy(t)}{dt}|_{t=0} = 0$ and x(t) = u(t).

- Find the complete response y(t)
- Find the steady state response and the transient response.

The Laplace transform of the differential equation gives

$$[s^2Y(s) - sy(0) - \frac{dy(t)}{dt}|_{t=0}] + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$
$$Y(s)(s^2 + 3s + 2) - (s + 3) = X(s)$$

so we have that

$$\begin{aligned} Y(s) &= \frac{X(s)}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)} \\ &= \frac{1+3s+s^2}{s(s+1)(s+2)} = \frac{B_1}{s} + \frac{B_2}{s+1} + \frac{B_3}{s+2} \end{aligned}$$

We find B_1 = 0.5, B_2 = 1, and B_3 = - 0.5.

therefore:

$$y(t) = [0.5 + e^{-t} - 0.5e^{-2t}]\, u(t)$$

steady state response is

$$y(t) = 0.5\, u(t)$$

and transient response is

$$y(t) = [e^{-t} - 0.5e^{-2t}]\, u(t)$$

4. (15 points) General solution:

$$Y(s) = \big(X(s) - G(s)\,Y(s)\big)F(s) = \frac{F(s)}{1 + F(s)G(s)}X(s)$$

$$H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

In this particular case system output is:

$$Y(s) = \big(X(s) - KY(s)\big)H(s) = X(s)H(s) - KH(s)Y(s)$$

and

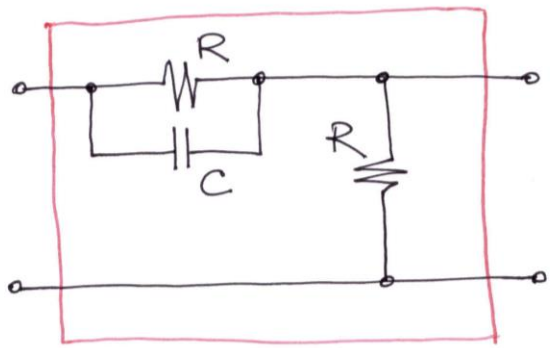
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s-2}}{1 + K\frac{1}{s-2}} = \frac{1}{s - 2 + K}$$

In order to have the pole in the left-hand s-plane we need K – 2 > 0 → K > 2

For example, K = 3 → pole at s = -1 and impulse response

$$h(t) = e^{-t}u(t)$$

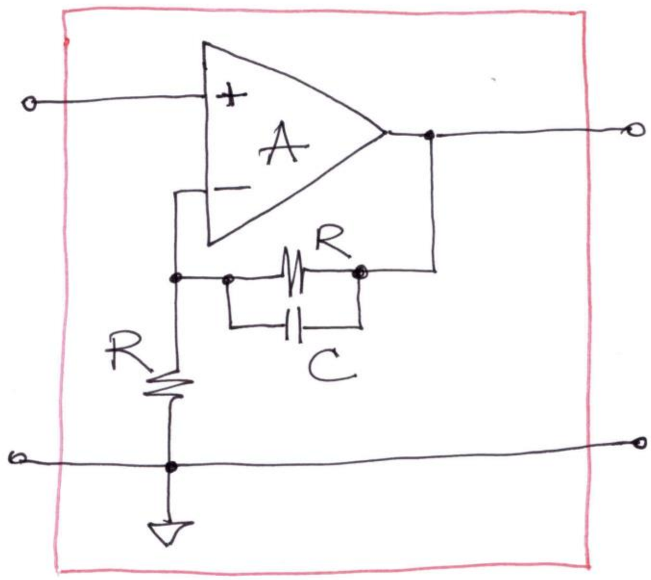
5. a) What is the transfer function of the following circuit:



$$H(s) = \frac{R}{R+R\,||\,\frac{1}{Cs}} = \frac{R}{R+\frac{R}{RCs+1}} = \frac{RCs+1}{RCs+2} = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$$

b) What is the transfer function of the following
Hints:

- you can use solutions of problem #5 and #6a
- to simplify the result you can assume that A → ∞



Since

$$H(s) = \frac{F(s)}{1+F(s)\cdot G(s)}$$

$F(s) = A$ and $G(s) = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$

$$H(s) = \frac{A}{1+A\left(\frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}\right)} \quad for\ A \to \infty\ H(s) = \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}}$$

c) Find and plot the unit-step response *s*(t) of the system?

$$S(s) = \frac{1}{s} \cdot \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}} = \frac{A}{s} + \frac{B}{s+\frac{1}{RC}} = \frac{2}{s} - \frac{1}{s+\frac{1}{RC}} =$$

$$s(t) = (2 - e^{-\frac{t}{RC}}) \cdot u(t)$$