DAN OTIENO - EE 307 - Homework 5.

5.4 In a cylindrical conductor of radius 4 mm, the current density is $\mathbf{J} = 5e^{-10\rho}\mathbf{a}_z$ A/m^2. Find the current through the conductor.

$$I = \int \bar{J} \cdot d\bar{s} = 5\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} e^{-10\rho}\,\rho\, d\phi\, d\rho$$

$$= 5\int_{0}^{2\pi} d\phi \int_{\rho=0}^{a} \rho e^{-10\rho}\, d\rho \qquad ; \quad a = 4\text{ mm}$$

$$= 5(2\pi)\left[\frac{e^{-10\rho}}{100}(-10\rho-1)\right]\Bigg|_0^a$$

$$= \frac{\cancel{10}^{1}\pi}{\cancel{100}_{10}}\left[e^{-10a}(-10a-1) - 1(-0-1)\right]$$

$$= \frac{\pi}{10}\left[e^{-0.04}(-0.04-1)+1\right] = \frac{\pi}{10}(0.00078)$$

$$\approx \boxed{245\ \mu A}$$

5.20 In a slab of Teflon ($\varepsilon = 2.1\,\varepsilon_o$), $\mathbf{E} = 6\mathbf{a}_x + 12\mathbf{a}_y - 20\mathbf{a}_z$ V/m, find $\mathbf{D}$ and $\mathbf{P}$.

$$\bar{D} = \varepsilon_o \varepsilon_r E = 2.1 \times \left(\frac{10^{-9}}{36\pi}\right)(6, 12, -20)$$

$$\bar{D} = \boxed{0.1114 a_x + 0.2228 a_y - 0.3714 a_z \ \text{nC/m}^2}$$

$$\bar{P} = \chi_e \varepsilon_o E = 1.1 \times \left(\frac{10^{-9}}{36\pi}\right)(6, 12, -20)$$

$$\bar{P} = \boxed{0.0584 a_x + 0.1167 a_y - 0.1945 a_z \ \text{nC/m}^2}$$

5.31 If $\mathbf{J} = \frac{100}{\rho^2}\mathbf{a}_\rho$ A/m², find (a) the time rate of increase in the volume charge density, (b) the total current passing through surface defined by $\rho = 2, 0 < z < 1, 0 < \phi < 2\pi$.

a:

$$\nabla \cdot \bar{J} = \frac{1}{\rho}\frac{\delta}{\delta\rho}\left(\frac{100}{\rho}\right) = -\frac{100}{\rho^3}$$

$$-\frac{\delta \rho_v}{\delta t} = -\frac{100}{\rho^3} \quad \therefore \quad \frac{\delta \rho_v}{\delta t} = \boxed{\frac{100}{\rho^3}\ C/m^3 s}$$

b:

$$I = \int \bar{J} \cdot d\bar{s} = \iint \frac{100}{\rho^2}\rho\, d\phi\, dz \Big|_{\rho=2}$$

$$= \frac{100}{2}\int_0^{2\pi} d\phi \int_0^1 dz = 100\pi = \boxed{314.16\ A}$$

5.38 Let $z < 0$ be region 1 with dielectric constant $\varepsilon_{r1} = 4$, while $z > 0$ is region 2 with $\varepsilon_{r2} = 7.5$. Given that $\mathbf{E}_1 = 60\mathbf{a}_x - 100\mathbf{a}_y + 40\mathbf{a}_z$ V/m, (a) find $\mathbf{P}_1$, (b) calculate $\mathbf{D}_2$.

a: (Canvas instructions → find E_2 instead).

$$E_{1T} = 60a_x - 100a_y = E_{2T} \ ; \ E_{2N} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}(E_{1N})$$

$$E_{2N} = 4/7.5\,(40a_z) = 21.33\,\hat{a}_z$$

$$\therefore E_2 = \boxed{60\hat{a}_x - 100\hat{a}_y + 21.33\hat{a}_z}$$

b:

$$D_2 = \varepsilon_0 \varepsilon_{r2} E_2 = \left(\frac{10^{-9}}{36\pi}\right)(7.5)(60, -100, 21.33)$$

$$\therefore D_2 = \boxed{3.979\,a_x - 6.631\,\hat{a}_y + 1.414\,\hat{a}_z\ nC/m^2}$$

5.39 Region 1 is $x < 0$ with, $\boldsymbol{\varepsilon}_1 = 4\boldsymbol{\varepsilon}_o$, while region 2 is $x > 0$ with $\boldsymbol{\varepsilon} = 2\boldsymbol{\varepsilon}_o$. If $\mathbf{E}_2 = 6\mathbf{a}_x - 10\mathbf{a}_y + 8\mathbf{a}_z$ V/m, (a) find $\mathbf{P}_1$, and $\mathbf{P}_2$, (b) calculate the energy densities in both regions.

a: (Canvas instructions → find E_1 & P_2).

$$E_{2T} = -10\hat{a}_y + 8\hat{a}_z = E_{1T} \;;\quad E_{1N} = \frac{\varepsilon}{\varepsilon_1}(E_{2N})$$

$$E_{1N} = \frac{2\varepsilon_0}{4\varepsilon_0}(6\hat{a}_x) = 3\hat{a}_x$$

$$\therefore E_1 = \boxed{3\hat{a}_x - 10\hat{a}_y + 8\hat{a}_z}$$

$$P_2 = \chi_{e_2}\varepsilon_0 E_2 = (1)\left(\frac{10^{-9}}{36\pi}\right)(6,-10,8)$$

$$P_2 = \boxed{53.05\hat{a}_x - 88.42\hat{a}_y + 70.74\hat{a}_z \ \text{nC/m}^2}$$

b:

$$W_1 = \frac{1}{2}\varepsilon_1 E_1^2 = \frac{1}{2}(4\varepsilon_0)(3,-10,8)^2$$

$$= \frac{1}{2}(4\varepsilon_0)(9+100+64) = 2\varepsilon_0(173)$$

$$= 346\varepsilon_0 = 346\left(\frac{10^{-9}}{36\pi}\right) \approx \boxed{3.0593 \ \text{nJ/m}^2}$$

$$W_2 = \frac{1}{2}\varepsilon_2 E_2^2 = \frac{1}{2}(2\varepsilon_0)(6,-10,8)^2$$

$$= \frac{1}{2}(2\varepsilon_0)(36+100+64) = \varepsilon_0(200)$$

$$= 200\varepsilon_0 = 200\left(\frac{10^{-9}}{36\pi}\right) \approx \boxed{1.7684 \ \text{nJ/m}^2}$$

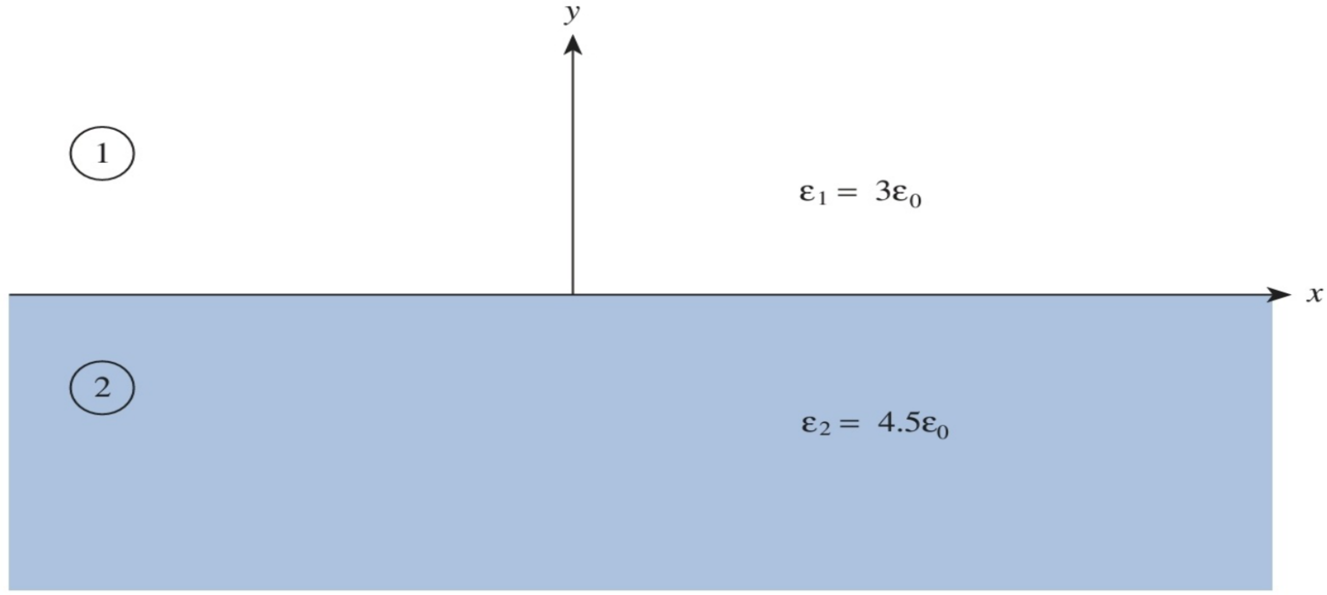

FIGURE 5.22 For Problem 5.42.

5.42 Given that $\mathbf{E}_1 = 10\mathbf{a}_x - 6\mathbf{a}_y + 12\mathbf{a}_z$ V/m in Figure 5.22, find (a) $\mathbf{P}_1$, (b) $\mathbf{E}_2$ and the angle $\mathbf{E}_2$ makes with the y-axis, (c) the energy density in each region.

a:

$$P_1 = \varepsilon_0 \chi_{e1} E_1$$

$$= (2)\left(\frac{10^{-9}}{36\pi}\right)(10, -6, 12)$$

$$= \boxed{0.1768\,\hat{a}_x - 0.1061\,\hat{a}_y + 0.2122\,\hat{a}_z \ \text{nC/m}^2}$$

b:

$$E_{2N} = \frac{\varepsilon_1}{\varepsilon_2} E_{1N} = \frac{3\cancel{\varepsilon_0}}{4.5\cancel{\varepsilon_0}}(-6\hat{a}_z) = 0.67(-6) = -4\hat{a}_z$$

$$\therefore\ E_2 = 10\hat{a}_x - 4\hat{a}_y + 12\hat{a}_z \ \text{V/m}.$$

$$\tan\theta_2 = \frac{E_{2T}}{E_{2N}} = \frac{\sqrt{100+144}}{4} = \frac{15.62}{4} = 3.905$$

$$\theta = \tan^{-1}(3.905) \approx \boxed{75.64^\circ}$$

c:

$$W_{E1} = \frac{1}{2}\varepsilon_1 |E_1|^2 = \frac{1}{\cancel{2}}\cancel{(3)}\left(\frac{10^{-9}}{\cancel{36}\pi}\right)(10^2+6^2+12^2)$$

$\cancel{280}$ 140, 12

$$= \left(\frac{10^{-9}}{12\pi}\right)(140) \approx \boxed{3.714\ nJ/m^3}$$

$$W_{E2} = \frac{1}{2}\varepsilon_2 |E_2|^2 = \frac{1}{\cancel{2}}(4.5)\left(\frac{10^{-9}}{36\pi}\right)(10^2+4^2+12^2)$$

$\cancel{260}$ 130

$$= 4.5\left(\frac{10^{-9}}{36\pi}\right)(130) \approx \boxed{5.173\ nJ/m^3}$$

5.43 Two homogeneous dielectric regions 1 ($\rho \le 4$ cm) and 2 ($\rho \ge 4$ cm) have dielectric constants 3.5 and 1.5, respectively. If $\mathbf{D}_2 = 12\mathbf{a}_\rho - 6\mathbf{a}_\phi + 9\mathbf{a}_z$ nC/m², calculate (a) $\mathbf{E}_1$ and $\mathbf{D}_1$, (b) $\mathbf{P}_2$ and ρ_{pv2}, (c) the energy density for each region.

a:

$$D_{1N} = D_{2N} = 12\hat{a}_\rho \ ; \quad D_{2T} = -6\hat{a}_\phi + 9\hat{a}_z$$

$$D_{1T} = \frac{\varepsilon_1}{\varepsilon_2} D_{2T} = \frac{3.5\varepsilon_0}{1.5\varepsilon_0}(-6\hat{a}_\phi + 9\hat{a}_z) = -14\hat{a}_\phi + 21\hat{a}_z$$

$$\therefore \boxed{D_1 = 12\hat{a}_\rho - 14\hat{a}_\phi + 21\hat{a}_z \ nC/m^2}$$

$$E_1 = \frac{D_1}{\varepsilon_1} = \frac{(12, -14, 21) \times \cancel{10^{-9}}}{3.5 \times \frac{\cancel{10^{-9}}}{36\pi}}$$

$$= \left(12 \times \frac{36\pi}{3.5}\right)\hat{a}_\rho - \left(14 \times \frac{36\pi}{3.5}\right)\hat{a}_\phi + \left(21 \times \frac{36\pi}{3.5}\right)\hat{a}_z$$

$$\boxed{E_1 = 387.76\hat{a}_\rho - 452.39\hat{a}_\phi + 678.58\hat{a}_z}$$

b:

$$P_2 = \varepsilon_o \chi_{e2} E_2 = 0.5\varepsilon_o \frac{D_2}{\varepsilon_2} = \frac{0.5\varepsilon_o(12,-6,9)}{1.5\varepsilon_o}$$

$$= \boxed{4\hat{a}_\rho - 2\hat{a}_\phi + 3a_z \ nC/m^2} \qquad \rho_{pv2} = \boxed{\nabla\cdot P_2 = 0}$$

c:

$$W_{E1} = \frac{1}{2} D_1 \cdot E_1 = \frac{1}{2}\frac{D_1\cdot D_1}{\varepsilon_o \varepsilon_1}$$

$$= \frac{1}{2}\left(\frac{(144+196+441)\times 10^{-18}}{3.5\times \frac{10^{-9}}{36\pi}}\right) \approx \boxed{12.62\ \mu J/m^2}$$

$$W_{E2} = \frac{1}{2}\frac{D_2\cdot D_2}{\varepsilon_o \varepsilon_{r2}} = \frac{1}{2}\left(\frac{(144+36+81)\times 10^{-18}}{1.5\times \frac{10^{-9}}{36\pi}}\right)$$

$$\approx \boxed{9.84\ \mu J/m^2}$$

5.45 A dielectric sphere $\boldsymbol{\varepsilon}_1 = 2\boldsymbol{\varepsilon}_o$ is buried in a medium with $\boldsymbol{\varepsilon}_2 = 6\boldsymbol{\varepsilon}_o$. Given that $\mathbf{E}_2 = 10\sin\theta\mathbf{a}_r + 5\cos\theta\mathbf{a}_\theta$ in the medium, calculate **E1** and **D1** in the dielectric sphere.

$$E_{1T} = E_{2T} = 5\cos\theta\,\hat{a}_\theta \ ; \ D_{1N} = D_{2N}$$

$$E_{1N} = \frac{\varepsilon_2}{\varepsilon_1} E_{2N} = \frac{\overset{3}{\cancel{6}}\cancel{\varepsilon_o}}{\cancel{2}\cancel{\varepsilon_o}}(10\sin\theta)\hat{a}_r = 30\sin\theta\,\hat{a}_r$$

$$E_1 = E_{1T} + E_{1N} = \boxed{30\sin\theta\,\hat{a}_r + 5\cos\theta\,\hat{a}_\theta}$$

$$D_1 = \varepsilon_1 E_1 = 2\varepsilon_o E_1 = \varepsilon_o\left[2(30\sin\theta\,\hat{a}_r + 5\cos\theta\,\hat{a}_\theta)\right]$$

$$D_1 = \boxed{\varepsilon_o(60\sin\theta\,\hat{a}_r + 10\cos\theta\,\hat{a}_\theta)}$$

5.47 At a point on a conducting surface, $\mathbf{E} = 30\mathbf{a}_x - 40\mathbf{a}_y + 20\mathbf{a}_z$ mV/m. Calculate the surface charge density at that point.

$$\rho_s = D_N = \varepsilon_0 E$$

$$= \left(\frac{10^{-9}}{36\pi}\right)\sqrt{30^2+40^2+20^2}\,(10^{-3})$$

$$= \left(\frac{10^{-9}}{36\pi}\right)\sqrt{2900}\,(10^{-3})$$

$$= \frac{\sqrt{2900}}{36\pi} \times 10^{-13} = \left(\frac{\sqrt{2900}}{36\pi}\right) \text{pC/m}^2$$

$$\boxed{\rho_s \approx 0.476}$$