Linear Time invariant systems

Self-study notes on EE120 Berkeley

June 24, 2022

1 Foundations in signals and systems

Definition 1 (Signals)**.** Signals is a function that depicts physical phenomena with usually one variable(often refers to time t), we denote a continous signal as $x(t)$ and a discrete signal as $x[n]$. Here are some examples of special discrete signals:

1. Unit pulse: $\delta[n] = \delta_{n=0}$;
2. Unit step: $u[n] = I_{n>0}$;

where $\delta_{n=0}$ represents the Kronecker's delta and $I_{n>0}$ represents the indicator function.

Notice that a useful proposition:

$$\boxed{\delta[n] = u[n] - u[n-1] \Longleftrightarrow u[n] = \sum_{k=0}^{\infty} \delta[n-k]}$$

Definition 2 (System)**.** A working function that transform signal $x(t) \mapsto y(t)$. A system is called **memoryless** if its outputs depend solely on the input given at that time. For the case of output only depends on the input before the given time, we call it a **causal** system. For system giving bounded outputs with bounded imputs, we say it is **stable.**

Example 1 (Categories of systems)**.**

An example for memoryless system is that consider a resistor, with input signal current $i(t)$ and output voltage $v(t)$ we have $v(t) = i(t)R$. However in the case of moving average filter $y[n] = 1/3\{x[n+1] + x[n] + x[n-1]\}$, since we require register to store information before the n-th sample, we know it has memory; and $y[n]$ requires the infomation of the $n+1$-th sample, hence it is not causal.

The model of accomulator is some time unstable since it sum up all signal from $x[n]$, may result in divergence.

Definition 3 (Linearity and time invariance)**.** We are very familiar with the definition of a linear map, just apply citeria to a given system. For time-invariant property, we have

$$x(t-T) \mapsto y(x-T)$$

for any shift T.

In a LTI system, we denote $h[n]$ as the output of unit impulse $\delta[n]$, we call $h[n]$ as unit impulse. Since for signal $x[n]$ we can rewrite it into the form of

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

applying linearity we have

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \equiv (x * h)[n]$$

where $(x * h)[n]$ denotes the **convolution** of signal $x[n]$ and $h[n]$.

2 Properties of convolutions

2.1 Unit pulse as identity

$$(x * \delta)[n] = x[n].$$

Proof: Consider that

$$(x * \delta)[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] = x[n]$$

Remarks: The proof might look like I am performing circular reasoning, since we do use the formula $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$ to derive the convolution operation! However, you should get the operation first from some observations of normal example to derive its generalized form, without the definition of convolution we can get $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$ as well! So there is no worry about circular reasoning.

2.2 Shift impulse property

$$x[n] * \delta[n-T] = x[n-T]$$

2.3 Commutativity

$$x * h = h * x$$

2.4 Distributivity

$$x * (h_1 + h_2) = x * h_1 + x * h_2$$

2.5 Associativity

$$x * (h_1 * h_2) = (x * h_1) * h_2$$

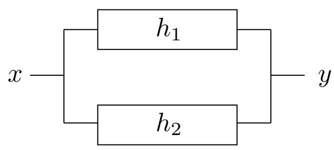
Proof. Expanding the LHS we have

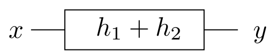
$$\begin{aligned}
x * (h_1 * h_2) &= \sum_{k=-\infty}^{+\infty} x[k] \sum_{r=-\infty}^{+\infty} h_1[r]h_2[k-r] \\
&= \sum_{k=-\infty}^{+\infty} \sum_{r=-\infty}^{+\infty} x[k]h_1[r]h_2[k-r] \\
&= [s = n + r - k]m = n - k \sum_{s=-\infty}^{+\infty} h_2[k-s] \sum_{m=-\infty}^{+\infty} x[m]h_1[s-m] \\
&= h_2 * (x * h_1) = (x * h_1) * h_2
\end{aligned}$$

□

3 From convolution to LTI systems

First example we need to check is the parallel combination of two LTI, by distributivity we yield that two LTI in parallel is equal to sum two maps up.

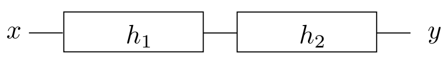


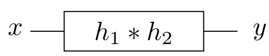


And then the deduction of two LTI in series may be easy to check since

$$(x * h_1) * h_2 = x * (h_1 * h_2)$$

due to the associativity.





4 Determining causality and stability

Theorem 1 (Causality in discrete LTI)**.** A discrete time LTI is causal if and only if

$$h[n] = 0, n < 0$$

Consider $y[n] = 1/3\{x[n+1] + x[n] + x[n-1]\}$ as the moving average, we say it is not causal since $h[-1] \neq 0 = 1/3$.

Theorem 2 (Stability in discrete LTI)**.** A discrete time LTI is stable if and only if

$$\sum_{k=-\infty}^{\infty} |h[k]| < +\infty$$

5 Continous time convolution

All discussion about convolution in discrete time we rely on a very import function $\delta[n]$, now let see the behavior of it in the continous case.

$$\delta(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} I_{t \in [0,\Delta]}$$

Definition 4. Convolution integral

For imput signal $x(t)$ and impulse response of LTI $h(t)$, we have the output signal $y(t)$ to be

$$y(t) = \int_{-\infty}^{+\infty} x(t)h(\tau - t)\mathrm{d}\tau.$$

We notice that the integral form of convolution can actually derive from **sampling**:

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_{\Delta}(t - k\Delta).$$

Applying the LTI system we transfrom all $\delta_{\Delta}(t - k\Delta)$ to $h_{\Delta}(t - k\Delta)$ we have

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_{\Delta}(t - k\Delta) \Leftrightarrow \int_{-\infty}^{+\infty} x(t)h(\tau - t)\mathrm{d}\tau.$$

Above deduction we use the Riemman sum's defintion of integrals. Since the convolution integral possesses similar properties to the discrete case, we omit some introduction of them.

6 Complex Exponentials

We usually complex expontentials to express the two dimensional signals as they as spining around a given axis

- continous case:

$$x(t) = \mathrm{e}^{st}, s \in \mathbb{C} \to x(t) = \underbrace{\mathrm{e}^{\sigma t}}_{\text{envelope}} \cdot \underbrace{\mathrm{e}^{j\omega t}}_{\text{periodic}}$$

- discrete case:

$$x[n] = z^n, z \in \mathbb{C} \to x[n] = r^n \mathrm{e}^{j\omega n}$$

In order to introduce the definition of system function, we shall first calculate the following result:

The response of a LTI system to a complex exponential is the same complex exponential scaled by a constant.

- continous case

$$\mathrm{e}^{st} \to \boxed{h(t)} \to y(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{s(\tau - t)} h(\tau) \mathrm{d}\tau = \mathrm{e}^{st} \underbrace{\int_{-\infty}^{+\infty} \mathrm{e}^{s\tau} h(\tau) \mathrm{d}\tau}_{H(s)}$$

- discrete case

$$z^n \to \boxed{h[n]} \to y[n] = \sum_{k=-\infty}^{+\infty} r^{n-k} \mathrm{e}^{j\omega(n-k)} h[k] = r^n \mathrm{e}^{j\omega n} \underbrace{\sum_{k=-\infty}^{+\infty} r^{-k} \mathrm{e}^{-j\omega k} h[k]}_{H(z)}$$

Definition 5 (system functions)**.** Functions denoted $H(s)$ and $H(z)$ above is called the system function of LTI.

Definition 6 (frequency response of LTI systems)**.** Let $s = j\omega$ and $z = \mathrm{e}^{j\omega}$ we have

$$H(j\omega) = \int_{-\infty}^{+\infty} \mathrm{e}^{j\omega\tau} h(\tau) \mathrm{d}\tau, H[\mathrm{e}^{j\omega}] = \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-j\omega k} h[k].$$

7 Filtering signals

Consider a paradigm for low-pass filter: the moving average system

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]$$

We shall calculate the frequency response of this system, we know that the system function $H[n] \equiv \mathbf{1}_{-M \leq n \leq M}$ we have

$$H[\mathrm{e}^{j\omega}] = \frac{1}{2M+1} \sum_{k=-M}^{M} \mathrm{e}^{j\omega k} = \frac{\mathrm{e}^{j\omega M}}{2M+1} \frac{1 - \mathrm{e}^{-j\omega(2M+1)}}{1 - \mathrm{e}^{-j\omega}}, \omega^2 \neq 0$$

Notice that we have $\mathrm{e}^{j\omega M} = z^M$, we rewrite above formula into

$$\begin{aligned} H[\mathrm{e}^{j\omega}] &= \frac{z^M}{2M+1}\frac{1-z^{-(2M+1)}}{1-z^{-1}} \\ &= \frac{1}{2M+1}\frac{z^{M+1/2}-z^{-(M+1/2)}}{z^{1/2}-z^{-1/2}}, \omega^2 \neq 0 \\ &= \frac{1}{2M+1}\frac{\sin(2M+1/2)\omega}{\sin(\omega/2)} \end{aligned}$$

hence we have low frequency like $\omega = 0$ can pass the filter with no change in signal due to $H[\mathrm{e}^{j\cdot 0}] = 1$ and high frequency like $\omega = \pi$ cannot pass(or be attenuated) due to $H[\mathrm{e}^{j\cdot\pi}] = \frac{1}{2M+1}(-1)^n$, we will find the amptitude of such signals smaller than the original.

LTI system designed such that frequency response is zero or close to zero for frequencies to be eliminated. To be clear, if we enlarge the quantity of M like approaching infinity we will find that signal with freqency $\omega = \pi$ will be eliminate finally. That's how a filter works.

8 FIR and IIR

From above example about a low-pass filter we reveal some basics about a system with finite or infinite response, now we are going to generalize them.

Definition 7 (FIR and IIR)**.** A causal **finite impulse response** (FIR) system has the form:

$$y[n] = \sum_{k=0}^{M} a_k x[n-k], h[n] = \sum_{k=0}^{M} a_k \delta[n-k]$$

where a_k are the coefficients. Notice that a **FIR is always stable** since finite sum of terms $h[n]$ is still bounded. An **infinite impulse response**(IIR) system, like an accumulator, holds infinite duration of $h[n]$, we usually use difference equations to describle them(differential equations in continous)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

or a normalized form like

$$y[n] = -c_1 y[n-1] - c_2 y[n-2] - \cdots - c_N y[n-N] + \sum_{k=0}^{M} d_k x[n-k]$$