INDIAN INSTITUTE OF TECHNOLOGY

NUMERICAL ANALYSIS OF A LINEAR BLACK-SCHOLES MODEL

MA 202 PROJECT REPORT

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1. Problem Statement:

An option is a contract that gives the bearer the right to buy or sell the underlying assets at a predetermined price (this is called a premium) at or before the contract expires. A call option enables the holder to buy the stock, and a put option gives the user the right to sell. It is kind of a down-payment (but not in full) option for a future deal.

Trading stock options is a significant part of the financial markets. In the past, there didn't exist an efficient mathematical model to determine the value of an option. Thus, the Black Scholes model provides the analytical framework for options trading. The Black Scholes equation is a famous partial differential equation in financial mathematics. In this project, we discuss the solution methods for the Black Scholes model with European call and put options.

Here, we'll study the weighted average method using different weights for numerical approximations and solve the modified Black Scholes equation pricing option with a discrete dividend. We will use the delta-defining sequence of the generalized *Dirac-delta* function and apply the Mellin transformation to obtain an integral formula.

We will also discuss analytical solutions of the Black-Scholes equation using Fourier.

Transformation method for European options. We will discuss a finite-difference scheme to approximate the solutions.

2. Call and Put Options:

Before we explain the call and put option, first, we need to understand some of the basic terminologies that drive the options market system.

1. **Strike price:** A strike price, also known as the exercise price, is set before getting into the contract. This price is the price at which a stock is bought or sold when exercised.
2. **Maturity/ Expiration date:** It is a date at which the deal is to be exercised.
3. **Time Period:** It is the period after which the actual buying or the selling of the stock occurs.
4. **Stock Price or Exercise Price:** This is the price that holds the stock's current (present-day) value.
5. **Premium:** It is the value that needs to be paid when signing the for owning the right of the stock.

There are two types of styles present in the options market- the American and European styles. The fundamental difference between American and European styles is that American call and put options can be exercised before and on the expiration date. In comparison, the European call and put options can only be performed after maturity.
Our primary focus will be on the European style in this project as the American style becomes complex at later stages, which is beyond the scope of our understanding.

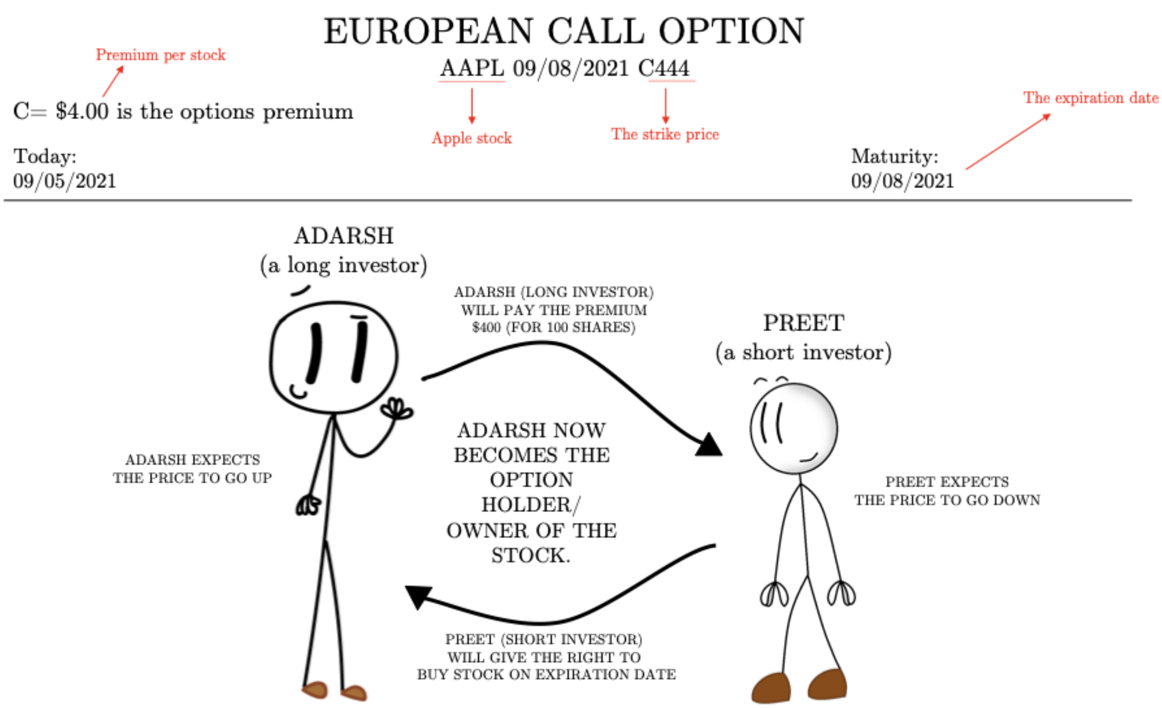
Call option:

A call option is known as a "call" because this option gives a right to the owner of the stock to "call the stock away" from the seller. It lets the owner decide the maximum buying price of the stock. In a call option, the owner has the right and not the obligation to buy a stock at the strike price. That means the owner of the option can or cannot exercise the option or buy the stock according to his wish. For example, if the stock is unprofitable, he can just let the option go worthless.

The best part about owning an option is that there is a definite amount to lose (the price paid for the option) and theoretically unlimited scope of profit.

Explaining the Call Option:

Here is an example that would enable you to understand the call option better.



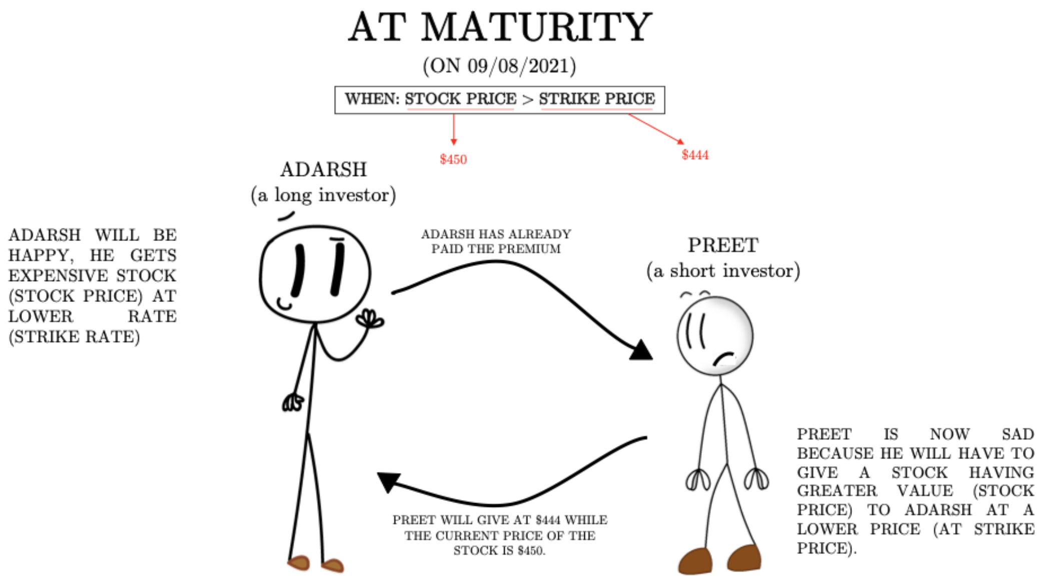
Here,

a. Premium- $4.44 per stock.
b. Strike price- The strike price at which the buying or selling will take place is $444 per stock.
c. Expiration Date- The expiration date at which the buying or selling of the stock will take place is 09/08/2021.
d. Time period- The time period for the trading exercise is 3 months.

Scenario-1:

Now, let's say that at the time of maturity (after three months).

a. Stock Price- The price of the stock at 09/08/2021 is $450 per stock.
b. Strike Price- The strike price remains the same, i.e., $444 per stock.

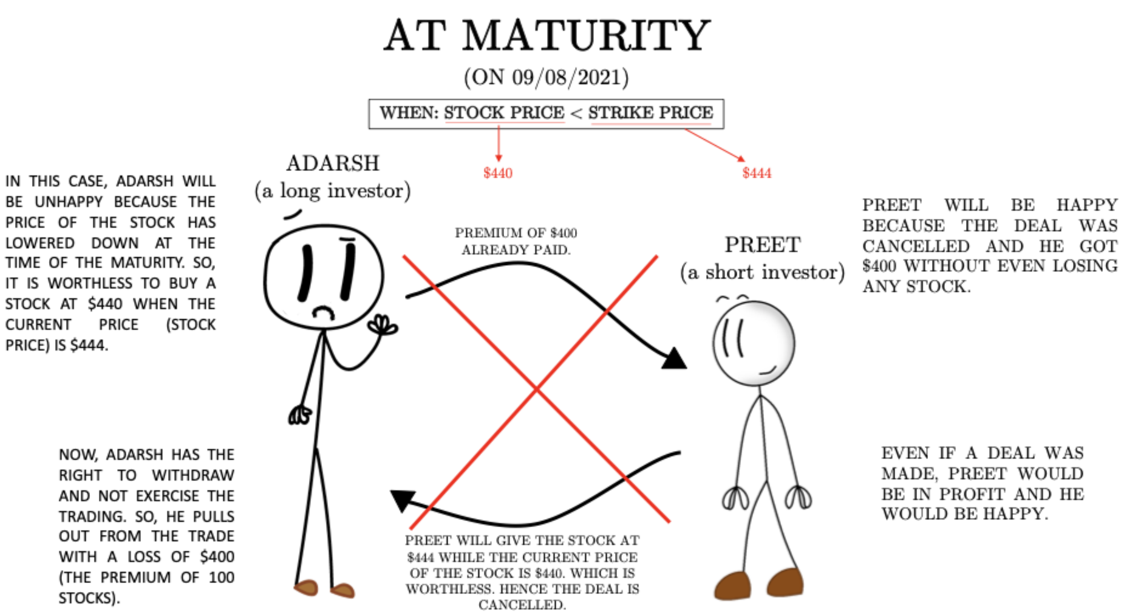


Scenario-2:

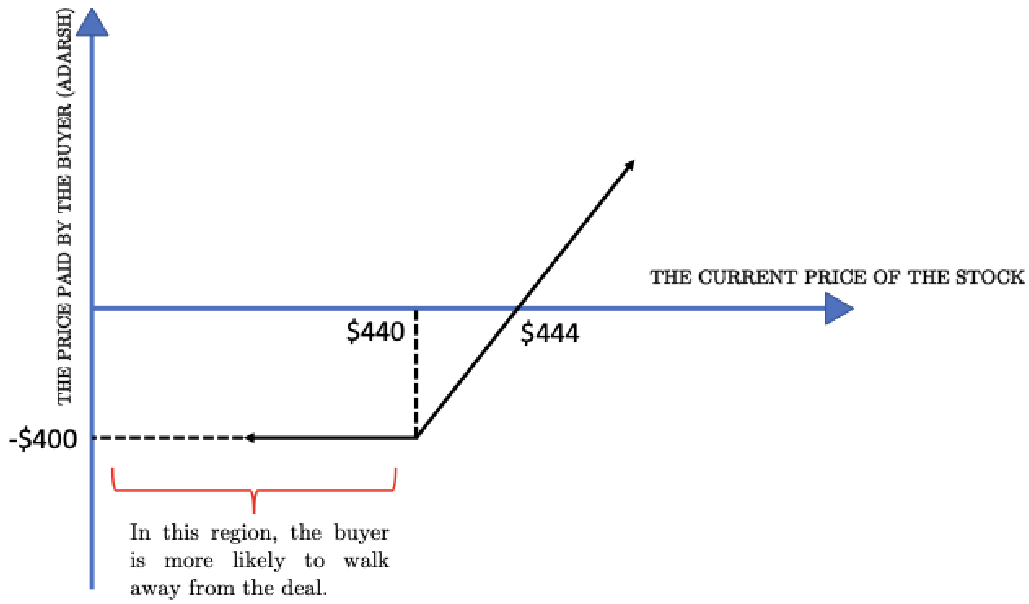
Now, let's say that at the time of maturity (after three months).

a. Stock Price- The price of the stock at 09/08/2021 is $440 per stock.

b. Strike Price- The strike price remains the same, i.e., $444 per stock.



In the above example, we have seen that Adarsh gives a premium to Preet so as to take the right of the stock three months later. That means, Adarsh has the option to 'call' if he does not wish to continue buying of the stock depending upon the stock price (exercise price).



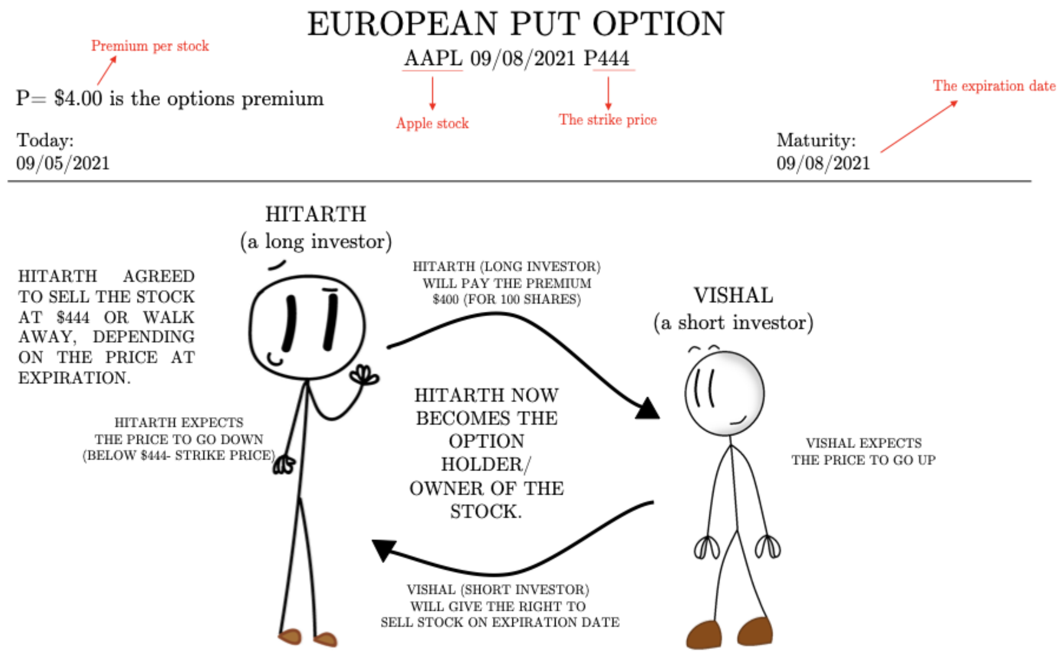
From the above example, we can say that the best part about owning an option is that there is a definite amount to lose (the premium price paid for the option) and theoretically unlimited scope of profit. In the graph, we can see that Adarsh's maximum loss will have to bear is $400 if he walks away from the deal, and there can be unlimited profit.

Put option:

A put option is known as a "put" because this option gives a right to the owner of the stock to "put the stock away" from the buyer. It lets the owner decide the minimum selling price of a stock. In a put option, the owner has the right and not the obligation to sell a stock at the strike price. That means the owner of the option can or cannot exercise the option or sell the stock according to his wish. For example, if the stock is unprofitable, he can just let the option go worthless.

Explaining the Put Option:

Here is an example that would enable you to understand the put option better.



Here,

a. Premium- $4.44 per stock.
b. Strike price- The strike price at which the buying or selling will take place is $444 per stock.
c. Expiration Date- The expiration date at which the buying or selling of the stock will take place is 09/08/2021.
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Scenario-1:

Now, let's say that at the time of maturity (after three months).

a. Stock Price- The price of the stock at 09/08/2021 is $440 per stock.
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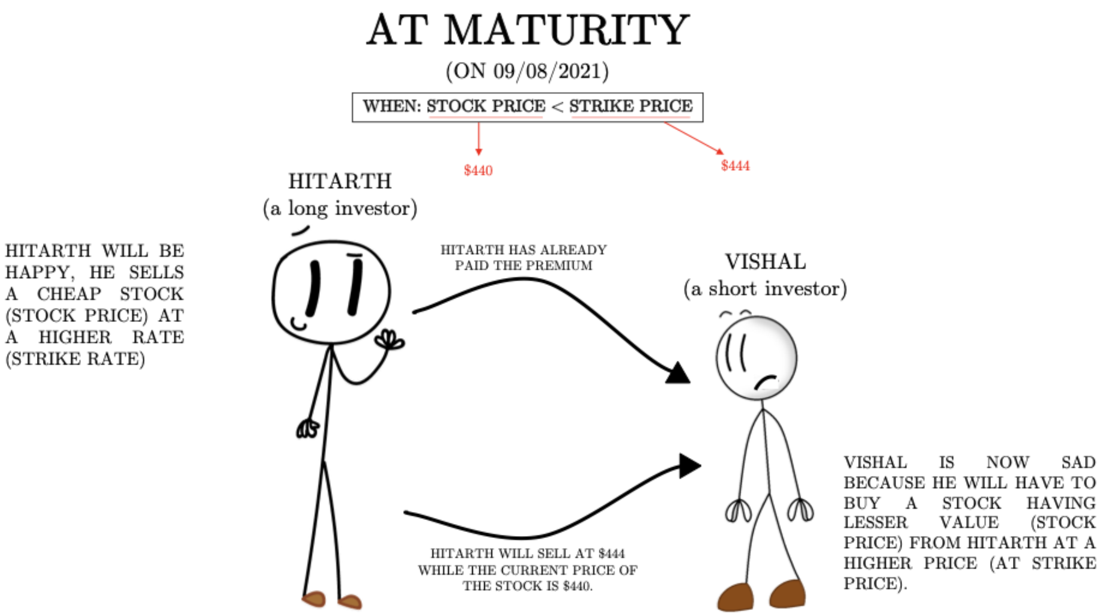
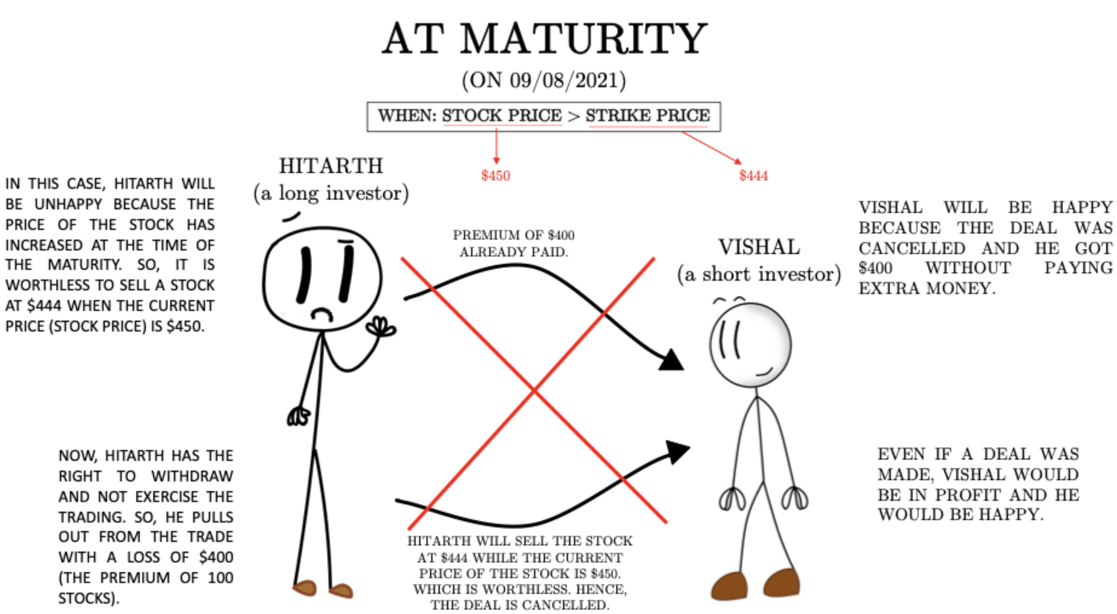


Scenario-2:

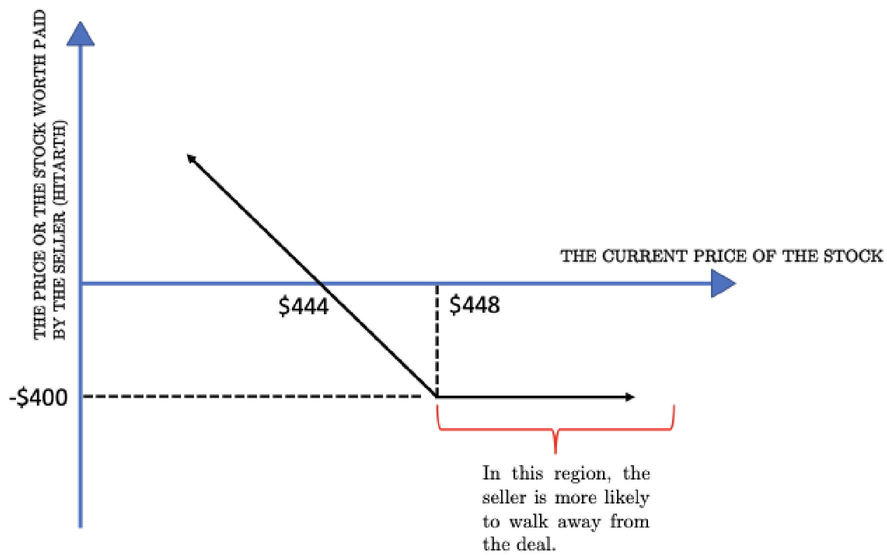
Now, let's say that at the time of maturity (after three months).

a. Stock Price- The price of the stock at 09/08/2021 is $440 per stock.

b. Strike Price- The strike price remains the same, i.e., $444 per stock.



In the above example, we have seen that Hitarth gives a premium to Vishal and Hitarth has the right to sell or not to sell the stock after maturity. That means, Hitarth has the option to 'put' the option if he does not wish to continue the selling depending upon the stock price (exercise price).



From the above example, we can say that the best part about owning an option is that there is a definite amount to lose (the premium price paid for the option). The graph shows that Hitarth's maximum loss will have to bear $400 if he walks away from the deal.

3. Physical Model

The Black Scholes Merton equation is a partial differential equation developed by three genius economists Fischer Black, Myron Scholes, and Robert Merton. This equation is used to price European calls and put options on an underlying stock paying no dividends.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Here,

$V = V(S, t)$ is the pay-off function,

$S = S(t)$ the stock price with $S = S(t) \geq 0$,

t = time with $t \in (0; T)$ for T is the time of maturity,

r = Risk-free interest rate,

σ = Volatility Condition,

The equation has some underlying assumptions,

1. Stock prices follow a stochastic process, meaning that we cannot use historical prices to predict future movement.
2. There is no dividend, and transactions have cost.
3. The risk-free rate and volatility of the stock price are known and constant.

To understand how the equation was derived, we have to understand the major components behind the model:

1. Geometric Brownian Motion - Initially there's a generalized wiener process called Arithmetic Brownian Motion

$$dS = \mu dt + \sigma dz$$

   that shows changes in the stock price in two terms first the drift term (μdt) which shows expected return of the stock in a given short time, second volatility in stock price given dz comes from this formula,

$$dz = \varepsilon\sqrt{dt}$$

   Where ε= random variable(μ= 0, variance=1).

   Since Arithmetic Brownian Motion has a big disadvantage that the volatility can be negative, thus we employ Geometric Brownian Motion

$$dS = \mu S dt + \sigma S dz$$

   By adding S into each term, the equation comprehensively accounts for the compounding effect and also has absorbing barriers which mean that S will become 0 due to the company's bankruptcy, there will be no further change in S at all.

2. ITO's lemma - It represents a change in the value of an option as a function of stock price

$$dV = \frac{dV}{dS}dS + \frac{dV}{dt}dt + \frac{1}{2}\frac{d^2V}{dS^2}dS^2 + \frac{1}{2}\frac{d^2V}{dt^2}dt^2 + \frac{d^2V}{dtdS}dtdS$$

3. Delta Hedge Portfolio - It consists of two terms, a long position in call and a short position in stocks

$$\Pi = V - \Delta S$$

Further employing complex calculations of differentiation and integration(in some cases), we derive terms and then substitute them into the equations and get the final equation of the Black-Scholes Merton model for call and put option that is applicable on European options.

We have not shown the process of calculation, as it is too complex and also not relevant to the paper.

There are many successful tries at transforming the classical BlackScholes PDE, by singly taking under consideration the impact of transaction prices, price slippage, and market illiquidity on options costs.

We will emphasize the use of numerical methods for PDEs, particularly finite difference methods. The Black-Scholes PDE is equivalent to the classical heat equation, and we will use that connection to present the basic theoretical properties of its solutions. We will then see how the solution of the heat equation is used in solving the equation of the Linear Black-Scholes model.

4. Assumptions

We need to do a numerical analysis of the Black-Scholes Model. So we need to develop a mathematical model for it. We use the following assumptions to simplify the problem:

1. The value of the risk-free interest rate remains constant over the period of time.
2. The value of the standard deviation of log returns (σ)remains constant over the period of time.
3. We assume the European call and put options to simplify the problem as it can only be exercised only after maturity, unlike American options that can be exercised before maturity.
4. The value of the stock price is converted into heat variables for which the boundary conditions can go into negative values but we have assumed that they remain positive since the value of the stock cannot be negative.

5. Converting Black-Scholes Model into Heat Equation

We will convert our variables into those used for heat equations. This converts the solutions of our Black Scholes equation into that of the classical heat equation, the solutions of which we know how to easily find out.

Here are the steps we will follow to execute the same:

Step 1:

We have that, in terms of y = log(s) the terms $s \cdot \frac{\partial}{\partial s}$ and $s^2 \cdot \frac{\partial^2}{\partial s^2}$ convert to constant multiples of $\frac{\partial}{\partial y}$ and $\frac{\partial^2}{\partial y^2}$. Specifically:

$$v(s, t) = f(y, t) = f(\log(s), t)$$

For some f(y,t).

Now, we apply the chain rule to get:

$$sv_s(s,t) = s \cdot \frac{1}{s} f_y(\log(s),t) = f_y(y,t)$$

$$s^2 v_{ss}(s,t) = s^2 \left[\frac{-1}{s^2} f_y(\log(s),t) + \frac{1}{s^2} f_{yy}(\log(s),t) \right] = f_{yy}(y,t) - f_y(y,t).$$

The time derivative on the other hand remains unchanged:

$$v_t(s,t) = \partial_t[f(\log(s),t)] = f_t(y,t).$$

Now, if we make these substitutions in our Black Scholes Equation, we obtain:

$$\frac{1}{2}\sigma^2 s^2 v_{ss} + rsv_s - rv + v_t = \frac{1}{2}\sigma^2 f_{yy} + (r - \frac{1}{2}\sigma^2) f_y - rf + f_t,$$

We evaluate the left and right sides at (s,t) and (y,t) respectively. Thus, in terms of the new variable y = log(s), we find that $v(s,t) = f(y,t)$ solves our Black Scholes Equation if and only if f were to solve this equation:

$$\frac{1}{2}\sigma^2 f_{yy} + (r - \frac{1}{2}\sigma^2) f_y + f_t - rf = 0. \quad \text{……… (1)}$$

So, we now have an equation with constant coefficients as opposed to the variable of the Black Scholes Equation. (Observation: $0 < s < \infty$ corresponds to $-\infty < y < \infty$ {parameters}).

Step 2:

Now, we have to convert equation (1) into a heat equation so that we can solve it further. Lets first eliminate the first-order derivative f_y:

$$f(y,t) = e^{c_1 y} g(y,t),$$

The constant c_1 here must be chosen cautiously.

After substituting this expression into equation (1) we get:

$$e^{c_1 y} \left[\frac{1}{2}\sigma^2 g_{yy} + (c_1\sigma^2 + r - \frac{1}{2}\sigma^2) g_y + g_t + (c_1 - 1)(r + c_1 \frac{1}{2}\sigma^2) g \right] = 0.$$

To eliminate the term g_y from the equation, we make the following selection for the value of c_1:

$$c_1 = \frac{1}{2} - \frac{r}{\sigma^2}$$

We thus get our expression as:

$$\frac{1}{2}\sigma^2 g_{yy} + g_t + (c_1 - 1)(r + c_1 \frac{1}{2}\sigma^2)g = 0. \quad \text{....... (2)}$$

Step 3:

We change another dependent variable and eliminate the term g that is undifferentiable:

$$g(y,t) = e^{c_2 t} h(y,t).$$

Thus we get:

$$e^{c_2 t}\left[\frac{1}{2}\sigma^2 h_{yy} + h_t + c_2 h + (c_1 - 1)(r + c_1 \frac{1}{2}\sigma^2)h\right] = 0.$$

Now to simplify this further, we take our c_2 as:

$$c_2 = -(c_1 - 1)(r + c_1 \frac{1}{2}\sigma^2) = \frac{(\sigma^2 + 2r)^2}{8\sigma^2}.$$

Our Equation (2) thus becomes this simple equation:

$$\frac{1}{2}\sigma^2 h_{yy} + h_t = 0. \quad \text{...... (3)}$$

Step 4:

Finally, we will scale out the $\frac{1}{2} \cdot \sigma^2$ and reverse the sign on the time derivative. We will use:

$x = \frac{\sqrt{2y}}{\sigma}$ and $\tau = T - t$ and,

$$h(y,t) = e^{-c_2 T} u(x,\tau) = e^{-c_2 T} u(\sqrt{2}y/\sigma, T - t)$$

We now find that equation (3) is equivalent to the heat equation:

$$u_{xx}(x,\tau) - u_\tau(x,\tau) = 0. \quad \text{..... (4)}$$

Putting the steps together, we get:

$$\begin{aligned} v(s,t) &= f(y,t) \\ &= e^{c_1 y} g(y,t) \\ &= e^{c_1 y + c_2 t} h(y,t) \\ &= e^{c_1 y + c_2 t} e^{-c_2 T} u(x,\tau) \\ &= e^{(c_1\sigma/\sqrt{2})x - c_2\tau} u(x,\tau). \end{aligned}$$

For the heat equation, we will use τ as our time variable instead of the t in the Black Scholes Equation. So the term $e^{-c_2 T}$ is used to get all the t occurring in the combination,

$$T - t = \tau.$$

To conveniently refer to this change of variables, we will use the following parameters:

$$\mu = c_1\sigma/\sqrt{2} \text{ and } \nu = c_2.$$

After Simplification, we get the following group of equations:

$$\begin{aligned} v(s,t) &= e^{\mu x - \nu\tau} u(x,\tau), \quad \text{where} \\ s &= e^{\sigma x/\sqrt{2}} \\ t &= T - \tau \\ \mu &= \frac{1}{\sqrt{2}}\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right) \\ \nu &= r + \mu^2. \end{aligned} \quad \text{.... (5)}$$

Financial Variables: $v,\ s$ and t

Corresponding Heat Equation Variables: $u,\ x,$ and τ

These formulas establish a one-one correspondence between

$$v(s,t)\ :\ (0,\infty)\times(-\infty,T]\ \to\ \mathbb{R}$$ and

$$u(x,\tau):(-\infty,\infty)\times[0,\infty)\to\mathbb{R}.$$

Along with this property:

$$\mathcal{B}v(s,t)+\partial_t v(s,t)=0 \text{ if and only if } \partial_x^2 u(x,\tau)-\partial_\tau u(x,\tau)=0.$$

We start with values of u for $\tau = 0$ given by some initial value function:

$$u(x,0)=\psi(x),$$

and solve for u as τ moves ahead in time.

We start with the values given at t = T by a terminal value function,

$$v(s,T)=\phi(s),$$

for the Black Scholes Equation. We begin with $\boldsymbol{v}$ as t moves behind in time.

We also have the following relations connection functions:

$$\phi(s)=v(s,T)=e^{\mu x}u(x,0)=e^{\mu x}\psi(x).$$

Now let $u^{(1)}$ and $u^{(2)}$ be the solutions of equation (4). Then any linear combination of these must also be one of the solutions. The combining of equations in this manner is called the Superposition *Principle*.

$$u(x,\tau)=c_1u^{(1)}(x,\tau)+c_2u^{(2)}(x,\tau),$$

Then,

$$\begin{aligned} u_{xx}-u_\tau &= (c_1u_{xx}^{(1)}+c_2u_{xx}^{(2)})-(c_1u_t^{(1)}+c_2u_t^{(2)}) \\ &= c_1(u_{xx}^{(1)}-u_\tau^{(1)})+c_2(u_{xx}^{(2)}-u_\tau^{(2)}) \\ &= 0. \end{aligned}$$

The heat equation has multiple solutions.

6. Boundary Conditions

Hence the equations (3) and heat equations are to be used to get to the solution, along with source terms define a system of equations that represent the problem at hand. Now, to solve these differential equations, we need boundary conditions for each.

In our original equation, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

$S = S(t)$ the stock price with $S = S(t) \geq 0$,

t = time with $t \in (0;T)$ for T is the time of maturity,

We refer $v, s\ and\ t\ as\ financial\ variables$, and $u, x\ and\ \tau$ as corresponding heat variables. The equation 4 establish a one-to-one conversion between functions $v(s,t): (0,\infty) \times (-\infty, T] \rightarrow \Re$ and functions $u(x,\tau): (-\infty,\infty) \times [0,\infty) \rightarrow \Re$

From above the Boundary Conditions for the given heat equation becomes

$$u_{xx}(x,\tau) - u_\tau(x,\tau) = 0; \quad -\infty < x < \infty, \tau \geq 0$$

The initial and boundary conditions for standard European Put-Call options in financial variables are given below:-

European Call Option

The solution to the Black-Scholes equation (1) is the value V(S,t) of the European Call option, on $0 \leq S < \infty\ ;\ 0 \leq t \leq T$. The boundary and terminal conditions are as follows

$$V(0,t) = 0 \qquad for\ \ 0 \leq t \leq T,$$

$$V(S,t) \sim S - Ke^{-r(T-t)} \qquad as\ \ S \rightarrow \infty,$$

$$V(S,T) = max(S-K, 0) \qquad for\ 0 \leq S < \infty$$

European Put Option

European Put option is the reciprocal of the European Call option and the boundary and terminal conditions are

$$V(0,t) = Ke^{r(T-t)} \qquad for\ 0 \le t \le T,$$
$$V(S,t) \sim 0 \qquad as\ S \to \infty,$$
$$V(S,T) = max(K-S,0) \qquad for\ 0 \le S < \infty$$

The Initial and Boundary Conditions of Standard European Put-Call Options in heat Variables is given below,

For European Call Options,

$$u(x,0) = max(e^{(\lambda+1)x} - e^{\lambda x}, 0) \qquad as\ x \in \Re,$$
$$u(x,\tau) = 0 \qquad as\ x \to -\infty\ ,$$
$$u(x,\tau) = e^{(\lambda+1)x+v^2\tau} - e^{\lambda x+\lambda^2\tau} \qquad as\ x \to \infty$$

……(i)

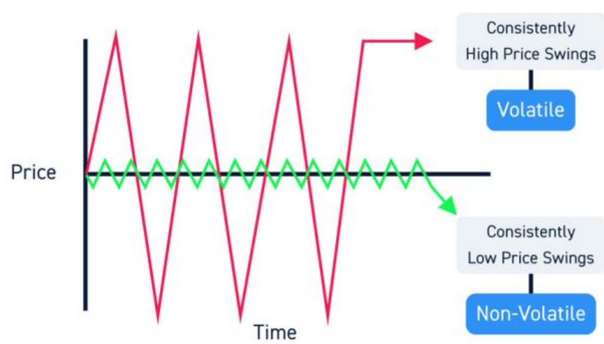
For European Put Options,

$$u(x,0) = max(e^{\lambda x} - e^{(\lambda+1)x}, 0) \qquad as\ x \in \Re,$$
$$u(x,\tau) = e^{\lambda x+\lambda^2\tau} \qquad as\ x \to -\infty$$
$$u(x,\tau) = 0 \qquad as\ x \to \infty\ ,$$

…….(ii)

7. Parameters

1. K is the exercise price. This typically remains constant as we have set a limit at which we will exercise our option.
2. r is the risk-free interest rate. This is used to discount the exercise price back to today. Taking into account factors such as inflation, the price of something will always increase, but that does not mean that its value will increase. So we need to discount the future price of the stock to know its true value in the present conditions when we perform calculations. We assume this to be constant.
3. T is the time to exercise. We can only exercise our option at the end of the time period T in European options.
4. The range in which the stock price varies.
5. $\sigma$is the standard deviation of log returns or in simple terms the volatility of the stock. You can see in the figure that red stock is very dispersed and had a very high deviation from the mean. In other words, it is more volatile and has a higher standard deviation. Whereas the blue stock is more or less constant. Thus, options of stock with higher volatility will tend to produce more returns than a non-volatile stock. This is also assumed to be constant.



"The Importance of Liquidity and Volatility for Traders." 2020. Centerpointsecurities.Com. October 30, 2020. https://centerpointsecurities.com/liquidity-volatility-day-trading/

8. Solution Methodology

To solve this problem, we employ the following method. We first convert the Black-Scholes model into a linear heat equation in 1 dimension. We achieve this by some smart substitutions. Then we get a linear second-order differential equation. Now we have a boundary value problem. We initially have the boundary conditions and initial value in financial terms. So we convert those variables into variables of heat equation by the same substitutions used to convert the Black-Scholes model into heat equation. Then we use numerical methods to solve the heat equation with new boundary conditions and initial value. We employ the method of explicit finite differences to solve the problem. Upon solving we get the value of u which could then be converted back into financial terms.

9. Analytical Solution

We will Discuss the Analytical Solution for the Linear Black-Scholes Model Equation Using Fourier Transformation for European Options.

We assume some variables so that it is easy to solve the complex differential equations involved. Let

$$S = Ke^{x}, t = T - \frac{r}{\sigma^2/2}, \text{ and } v(x, \tau) = \frac{1}{K} V(S, t)$$

Now substituting these variables into the major equation of the Black-Scholes Model, and later simplifying the equation we get,

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v$$

Letting $\frac{2r}{\sigma^2} = \theta$ the upper equation can be written as

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (\theta - 1)\frac{\partial v}{\partial x} - \theta v$$

Now consider $\lambda = \frac{1}{2}(\theta - 1), v = \frac{1}{2}(\theta + 1) = \lambda + 1\ to\ get\ v^2 = \lambda^2 + \theta$ and than $v(x,t) = e^{-\lambda x - v^2\tau}u(x,t)$. Solving these equations in the differential form to finally get to the heat equation that is

$$u_\tau = u_{xx}$$

Now using the Initial and Boundary Conditions for European Put and Call options respectively,

$$u(x,0) = max(e^{(\lambda+1)x} - e^{\lambda x}, 0) \qquad as\ x \in \Re,$$

$$u(x,\tau) = 0 \qquad as\ x \to -\infty,$$

$$u(x,\tau) = e^{(\lambda+1)x + v^2\tau} - e^{\lambda x + \lambda^2\tau} \qquad as\ x \to \infty$$

And,

$$u(x,0) = max(e^{\lambda x} - e^{(\lambda+1)x}, 0) \qquad as\ x \in \Re,$$

$$u(x,\tau) = e^{\lambda x + \lambda^2\tau} \qquad as\ x \to -\infty$$

$$u(x,\tau) = 0 \qquad as\ x \to \infty,$$

The Black Scholes reduces to Heat Equation

We apply Fourier Transformation in the Heat Equation we have,

$$\frac{\partial}{\partial\tau}\hat{u}(\xi,\tau) = (i\xi)^2\hat{u}(\xi,\tau),$$

where

$$\hat{u}(\xi,\tau) = \int_{-\infty}^{\infty} u(x,\tau)e^{2\pi i x\xi}\,dx.$$

Solving For **Call Options** we get

Here we are not showing the complex calculation, as it is not relevant to the report. We will introduce two new variables d1 and d2 and then convert the variables of the heat equation back into the Original Black-Scholes Variables.

Thus, the Final Equation we get is

$$V_c(S,t) = S\phi(d1) - ke^{-r(T-t)}\phi(d2)$$

Where $\phi(d)$is the standard Normal Distribution of the function is d. Setting the appropriate parameters of K,σ,dt,r, and T we get the True Value of Data in the Code written

Solving For **Put Options** we get

A similar type of calculation for the Put-Option is done, The Final Equation for the Put Option is

$$V_p(S,t) = Ke^{-r(T-t)}\phi(-d2) - S\phi(-d1)$$

10. Numerical Solution

We have the model as,

$$u_\tau = u_{xx} \; where \; x \in R \,, 0 \leq \tau \leq T$$

With the initial and boundary conditions (for call options) as,

$$V(0,t) = 0 \; for \; 0 \leq t \leq T,$$

$$V(S,t) = S - Ke^{-r(T-t)} \; as \; S \to \infty$$

$$V(S,T) = max(S-K,0) \; for \; 0 \leq S < \infty$$

We then convert these conditions into heat variables and solve the heat equation using an **Explicit Finite Difference.**

We approximate,

$$u_{xx}(x_n,\tau_m) = \frac{u^m{}_{n+1} - 2u^m{}_n + u^m{}_{n-1}}{\Delta x^2}$$

$$u_t(x_n,\tau_m) = \frac{u_n{}^{m+1} - u_n{}^m}{\Delta\tau}$$

Therefore, our equation becomes,

$$\frac{u^m{}_{n+1} - 2u^m{}_n + u^m{}_{n-1}}{\Delta x^2} = \frac{u_n{}^{m+1} - u_n{}^m}{\Delta\tau}$$

$$u^{m+1}{}_n = \alpha u^m{}_{n+1} + (1-2\alpha)u^m{}_n + \alpha u^m{}_{n-1} \; where \; \alpha = \frac{\Delta\tau}{\Delta x^2}$$

Therefore, now if we know the value of 3 points, $u^m{}_{n+1}$, $u^m{}_n$, $u^m{}_{n-1}$ we can calculate $u^{m+1}{}_n$.

We know the initial condition so we start from there, $u^0{}_n = 0$, and proceed forwards.

We use the boundary conditions,

$$u^m{}_0 = u^L(\tau_m) \; and \; u^m{}_{N+1} = u^H(\tau_m)$$

here we get $u^0{}_n$, $u^L(\tau_m)$, and $u^H(\tau_m)$by converting the boundary conditions into heat variables using the conversion formula (equation 5).

For the put options we have initial and boundary conditions as,

$$V(0,t) \; = \; Ke^{-r(T-t)} \; for \; 0 \leq t \leq T,$$

$$V(S,t) \; = 0 \; as \; S \to \infty$$

$$V(S,T) \; = \; max(K - S, 0) \; for \; 0 \leq S < \infty$$

We can apply the same method as call options for the put options.

Here we are not rigid with the explicit method, we can also use the implicit method, the Crank-Nicolson method, or the LU decomposition method.

11. Algorithm Used

1. Take the limits of stock price and convert them into the distance(heat variable) to get the length of the rod.
2. Define the input grid and other parameters.
3. Initialize the vector u with the initial condition. We only initialize the vector from x_1 to x_N and leave out x_0 and x_{N+1}. We will fill them later in accordance with the boundary conditions.
4. Use the explicit method to traverse through time and update the vector u.
5. Iterate for the whole time duration.
6. After the loop completes, add the boundary conditions back into u as we skipped them while initializing the vector u.
7. Repeat the same with new boundary conditions for the put options.
8. https://github.com/HitarthGandhi/MA202_FinalReport - Link to the MATLAB code.

12. Results and Discussions

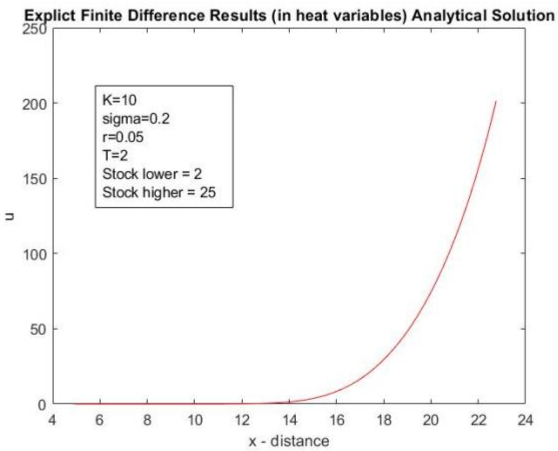


Figure 1 - Call option Analytical for value set 1.

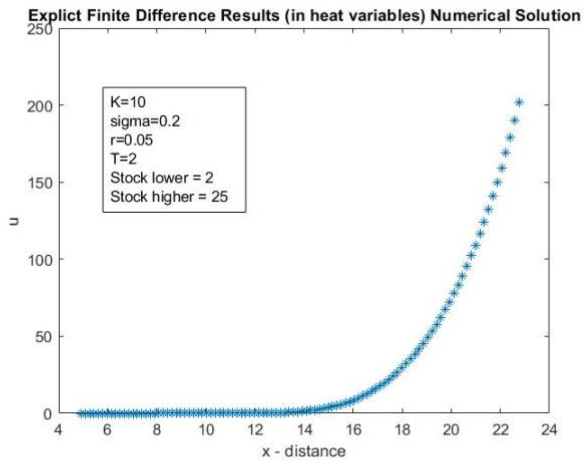


Figure 2 - Call option numerical for value set 1.

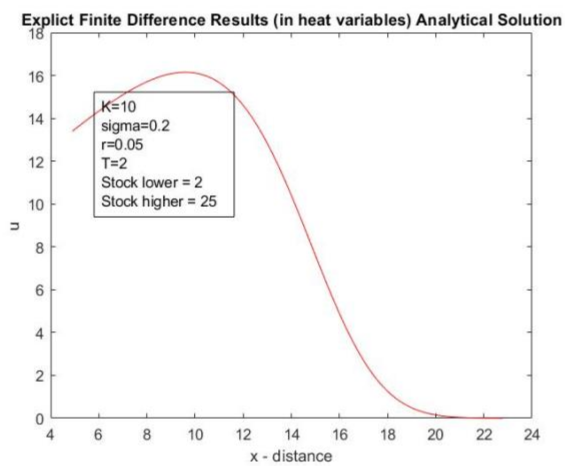


Figure 3 - Put option Analytical for value set 1.

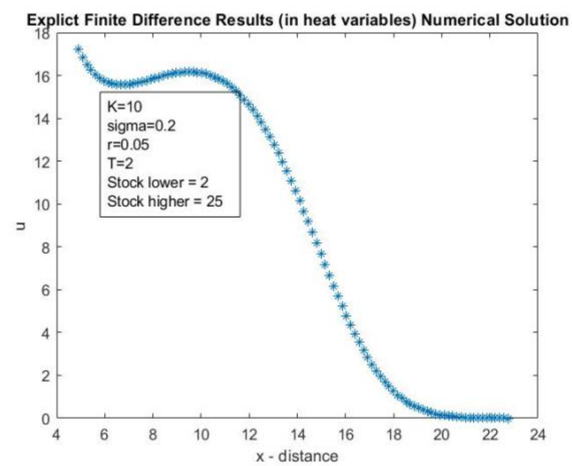


Figure 4 - Put option numerical for value set 1.

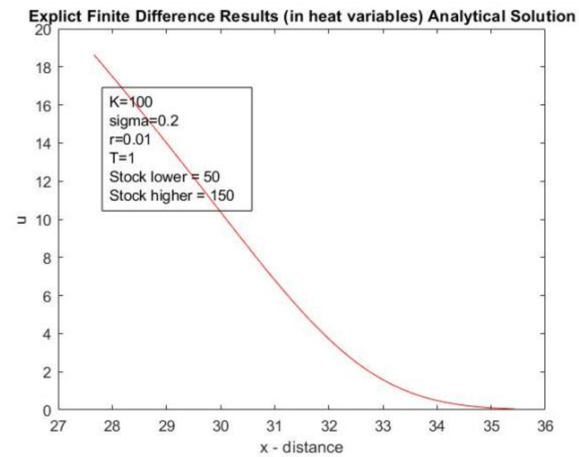


Figure 5 - Put option Analytical for value set 2.

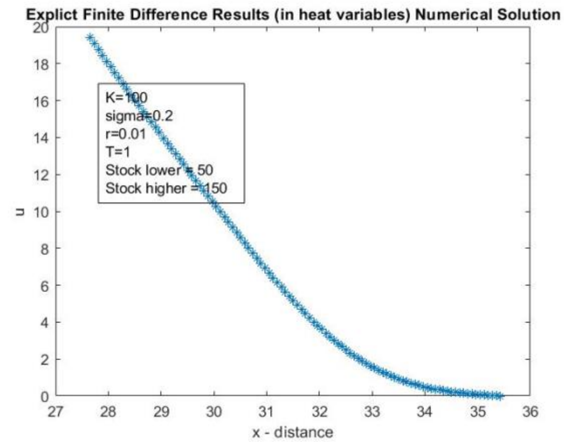


Figure 6 - Put option numerical for value set 2.

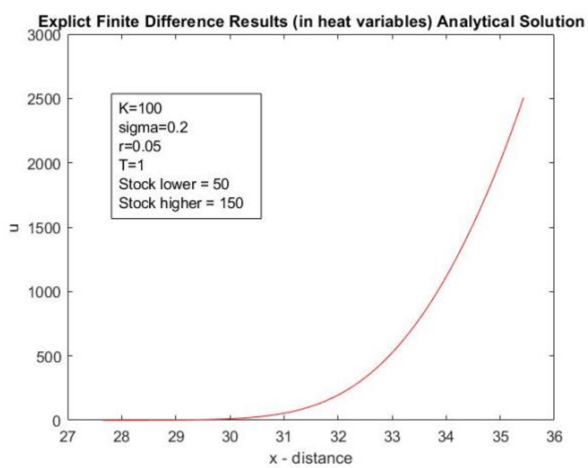


Figure 7- Call option Analytical for value set 3.

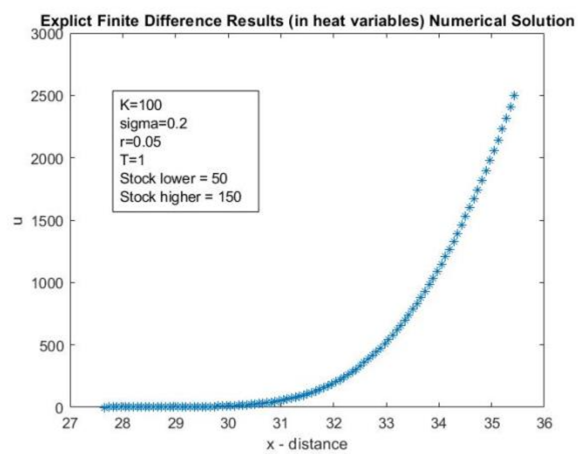


Figure 8 - Call option numerical for value set 3.

Inferences -

- Here the Figures on the Left side(Fig. 1,3,5,7) with continuous lines shows the solutions for Call and Put Options solved Analytically using **Fourier Transformation**.
- The Figures on Right Side(Fig. 2,4,6,8) graphs plotted with Asterisk shows the solutions for Call and Put Options solved Numerically using the **Explicit Finite Difference Method**.
- For the First Set of values, the Call Options are plotted Analytically(Fig 1.) and Numerically(Fig 2.) With M=200 N=100 where N is the No.of Distant steps in the range of X and M is the No. of distant steps in range of T, there is an absolute error of **0.00291943**.
- For the First Set of values, the Put Options are plotted Analytically(Fig 1.) and Numerically(Fig 2.) With M=200 N=100, there is an absolute error of **3.80826**.

Since the error in both the Call and Put Options are small, the Numerically Calculated solution is Correct and Appropriate for Predicting the Prices of European Stock Options using the Linear Model of Black Scholes Equation.

13. References

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