Optimization for Machine Learning
CS-439

Lecture 10: Duality, Gradient-free, and Applications

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EPFL – `github.com/epfml/OptML_course`

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Chapter X.1

Duality

Duality

Given a function $f : \mathbb{R}^d \to \mathbb{R}^+$, define its **conjugate** $f^* : \mathbb{R}^d \to \mathbb{R}^+$ as

$$f^*(\mathbf{y}) := \max_{\mathbf{x}} \; \mathbf{x}^\top \mathbf{y} - f(\mathbf{x})$$

a.k.a. Legendre transform or Fenchel conjugate function. (Note $\mathbb{R}^+ := \mathbb{R} \cup \{+\infty\}$)

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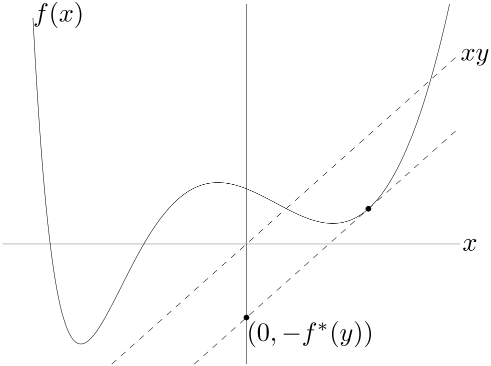

figure by Boyd & Vandenberghe

Figure: maximum gap between linear function $\mathbf{x}^\top \mathbf{y}$ and $f(\mathbf{x})$.

Properties

- f^* is always convex, even if f is not.
  Proof: point-wise maximum of convex (affine) functions in $\mathbf{y}$.
- Fenchel's inequality: for any $\mathbf{x}, \mathbf{y}$,

$$f(\mathbf{x}) + f^*(\mathbf{y}) \geq \mathbf{x}^\top \mathbf{y}$$

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- If f is closed and convex, then $f^{**} = f$.

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Exercise!

$$\begin{aligned} \mathbf{y} \in \partial f(\mathbf{x}) &\Leftrightarrow \mathbf{x} \in \partial f^*(\mathbf{y}) \\ &\Leftrightarrow f(\mathbf{x}) + f^*(\mathbf{y}) = \mathbf{x}^\top \mathbf{y} \end{aligned}$$

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Exercise!

- Separable functions: If $f(\mathbf{u}, \mathbf{v}) = f_1(\mathbf{u}) + f_2(\mathbf{v})$, then
$$f^*(\mathbf{w}, \mathbf{z}) = f_1^*(\mathbf{w}) + f_2^*(\mathbf{z})$$

Examples

- Recall: Indicator function of a set $C \subseteq \mathbb{R}^d$ is

$$\boldsymbol{\iota}_C(\mathbf{x}) := \begin{cases} 0 & \mathbf{x} \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

  If $f(\mathbf{x}) = \boldsymbol{\iota}_C(\mathbf{x})$, then its conjugate is

$$f^*(\mathbf{y}) = \max_{\mathbf{x} \in C} \mathbf{y}^\top \mathbf{x}$$

  called the support function of C.

Examples

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- Norm: if $f(\mathbf{x}) = \|\mathbf{x}\|$, then its conjugate is

$$f^*(\mathbf{y}) = \boldsymbol{\iota}_{\{\mathbf{z}:\|\mathbf{z}\|_* \leq 1\}}(\mathbf{y})$$

  (i.e. indicator of the dual norm ball) Note: The dual norm of $\|.\|$ is defined as $\|\mathbf{y}\|_* := \max_{\|\mathbf{x}\| \leq 1} \mathbf{y}^\top \mathbf{x}$. E.g. $\|.\|_1 \leftrightarrow \|.\|_\infty$.

Examples, cont

Generalized linear models

$$\min_{\mathbf{x} \in \mathbb{R}^d} \; f(A\mathbf{x}) + g(\mathbf{x})$$

reformulate

$$\min_{\mathbf{x} \in \mathbb{R}^d, \mathbf{w} \in \mathbb{R}^n} \; f(\mathbf{w}) + g(\mathbf{x}) \;\; \text{s.t.} \;\; \mathbf{w} = A\mathbf{x}$$

Examples, cont

Generalized linear models

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Lagrange dual function

$$\begin{aligned}\mathcal{L}(\mathbf{u}) &:= \min_{\mathbf{x}\in\mathbb{R}^d,\mathbf{w}\in\mathbb{R}^n} f(\mathbf{w}) + g(\mathbf{x}) + \mathbf{u}^\top(\mathbf{w} - A\mathbf{x}) \\ &= -\, f^*(-\mathbf{u}) - g^*(A^\top\mathbf{u})\end{aligned}$$

Examples, cont

Generalized linear models

$$\min_{\mathbf{x}\in\mathbb{R}^d} \; f(A\mathbf{x}) + g(\mathbf{x})$$

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Dual problem

$$\max_{\mathbf{u}\in\mathbb{R}^n} \; \left[\mathcal{L}(\mathbf{u}) = -f^*(-\mathbf{u}) - g^*(A^\top\mathbf{u})\right].$$

Examples, cont

Lasso

$$\min_{\mathbf{x}\in\mathbb{R}^d} \ \tfrac{1}{2}\|A\mathbf{x}-\mathbf{b}\|^2 + \lambda\|\mathbf{x}\|_1$$

is an example, for $f(\mathbf{w}) := \frac{1}{2}\|\mathbf{w}-\mathbf{b}\|^2$ and $g(\mathbf{x}) := \lambda\|\mathbf{x}\|_1$.

Can compute $f^*(\mathbf{u}) = \frac{1}{2}\|\mathbf{b}\|^2 - \frac{1}{2}\|\mathbf{b}-\mathbf{u}\|^2$
and $g^*(\mathbf{v}) = \boldsymbol{\iota}_{\{\mathbf{z}:\|\mathbf{z}\|_\infty\le 1\}}(\mathbf{v}/\lambda)$,

Examples, cont

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and $g^*(\mathbf{v}) = \boldsymbol{\iota}_{\{\mathbf{z}:\|\mathbf{z}\|_\infty\le 1\}}(\mathbf{v}/\lambda)$,

so that the dual problem is

$$\begin{aligned}
&\max_{\mathbf{u}\in\mathbb{R}^n} \ -f^*(-\mathbf{u}) - g^*(A^\top\mathbf{u}).\\
\Leftrightarrow\quad &\max_{\mathbf{u}\in\mathbb{R}^n} \ -\tfrac{1}{2}\|\mathbf{b}\|^2 + \tfrac{1}{2}\|\mathbf{b}+\mathbf{u}\|^2 \ \text{ s.t. } \ \|A^\top\mathbf{u}/\lambda\|_\infty \le 1.\\
\Leftrightarrow\quad &\min_{\mathbf{u}\in\mathbb{R}^n} \ \|\mathbf{b}+\mathbf{u}\|^2 \ \text{ s.t. } \ \|A^\top\mathbf{u}\|_\infty \le \lambda.
\end{aligned}$$

Why Duality?

Similarly for least squares, ridge regression, SVM, logistic regression, elastic net, etc.

Advantages:

- Duality gap gives a **certificate** of current optimization quality

$$
\begin{gathered}
f(A\bar{\mathbf{x}}) + g(\bar{\mathbf{x}}) \\
\geq \min_{\mathbf{x}\in\mathbb{R}^d} \; f(A\mathbf{x}) + g(\mathbf{x}) \\
\geq \\
\max_{\mathbf{u}\in\mathbb{R}^n} \; -f^*(-\mathbf{u}) - g^*(A^\top\mathbf{u}) \\
\geq \; -f^*(-\bar{\mathbf{u}}) - g^*(A^\top\bar{\mathbf{u}})
\end{gathered}
$$

for any $\bar{\mathbf{x}}$,$\bar{\mathbf{u}}$.

- Stopping criterion
- Dual can in some cases be easier to solve

Chapter X.2

Zero-Order Optimization
$\Leftrightarrow$ Derivative-Free ..
$\Leftrightarrow$ Blackbox ..

Look mom no gradients!

Can we optimize $\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$ if without access to gradients?

meet the newest fanciest optimization algorithm,...
Random search

pick a random direction $\mathbf{d}_t \in \mathbb{R}^d$

$$\gamma := \operatorname*{argmin}_{\gamma \in \mathbb{R}} f(\mathbf{x}_t + \gamma \mathbf{d}_t) \qquad \text{(line-search)}$$

$$\mathbf{x}_{t+1} := \mathbf{x}_t + \gamma \mathbf{d}_t$$

Convergence rate for derivative-free random search

Converges same as gradient descent - up to a slow-down factor d.

Proof. Assume that f is a L-smooth convex, differentiable function. For any γ, by smoothness, we have:

$$f(\mathbf{x}_t + \gamma \mathbf{d}_t) \leq f(\mathbf{x}_t) + \gamma \langle \mathbf{d}_t, \nabla f(\mathbf{x}_t) \rangle + \frac{\gamma^2 L}{2} \|\mathbf{d}_t\|^2$$

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Minimizing the upper bound, there is a step size $\bar{\gamma}$ for which

$$f(\mathbf{x}_t + \bar{\gamma} \mathbf{d}_t) \leq f(\mathbf{x}_t) - \frac{1}{L} \left\langle \frac{\mathbf{d}_t}{\|\mathbf{d}_t\|^2}, \nabla f(\mathbf{x}_t) \right\rangle^2$$

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The step size we actually took (based on f directly) can only be better:

$$f(\mathbf{x}_t + \gamma \mathbf{d}_t) \leq f(\mathbf{x}_t + \bar{\gamma} \mathbf{d}_t) \ .$$

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Taking expectations, and using the Lemma $\mathbb{E}_{\mathbf{r}}(\mathbf{r}^\top \mathbf{g})^2 = \frac{1}{d} \|\mathbf{g}\|^2$ for $\mathbf{r} \sim$ sphere $\subseteq \mathbb{R}^d$:

$$\mathbb{E}[f(\mathbf{x}_t + \gamma \mathbf{d}_t)] \le \mathbb{E}[f(\mathbf{x}_t)] - \frac{1}{Ld} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \ .$$

Convergence rate for derivative-free random search

Same as what we obtained for gradient descent,
now with an **extra factor of d**. d can be huge!!!

Can do the same for different function classes, as before

- For convex functions, we get a rate of $\mathcal{O}(dL/\varepsilon)$.
- For strongly convex, we get $\mathcal{O}(dL\log(1/\varepsilon))$.

Always d times the complexity of gradient descent on the function class.

credits to Moritz Hardt

Applications for derivative-free random search

Applications

- competitive method for **Reinforcement learning**
- memory and communication advantages: never need to store a gradient
- hyperparameter optimization, and other difficult e.g. discrete optimization problems
- can be improved to learn a second-order model of the function, during optimization [Stich PhD thesis, 2014]

Reinforcement learning

$$\mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t, \mathbf{e}_t)\,.$$

where $\mathbf{s}_t$ is the state of the system, $\mathbf{a}_t$ is the control action, and $\mathbf{e}_t$ is some random noise. We assume that f is fixed, but unknown.

We search for a control 'policy'

$$\mathbf{a}_t := \pi(\mathbf{a}_1, \ldots, \mathbf{a}_{t-1}, \mathbf{s}_0, \ldots, \mathbf{s}_t)\,.$$

which takes a trajectory of the dynamical system and outputs a new control action.
Want to maximize overall reward

$$\begin{aligned} \max_{\mathbf{a}_t} \ & \mathbb{E}_{\mathbf{e}_t}\Big[\sum_{t=0}^{N} R_t(\mathbf{s}_t, \mathbf{a}_t)\Big] \\ \text{s.t.} \ & \mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t, \mathbf{e}_t) \\ & (\mathbf{s}_0 \text{ given}) \end{aligned}$$

Examples: Simulations, Games (e.g. Atari), Alpha Go

Chapter X.3

Adaptive & other SGD Methods

Adagrad

Adagrad is an adaptive variant of SGD

pick a stochastic gradient $\mathbf{g}_t$

$$\text{update } [G_t]_i := \sum_{s=0}^{t} ([\mathbf{g}_s]_i)^2 \qquad \forall i$$

$$[\mathbf{x}_{t+1}]_i := [\mathbf{x}_t]_i - \frac{\gamma}{\sqrt{[G_t]_i}} [\mathbf{g}_t]_i \qquad \forall i$$

(standard choice of $\mathbf{g}_t := \nabla f_j(\mathbf{x}_t)$ for sum-structured objective functions $f = \sum_j f_j$)

- chooses an adaptive, coordinate-wise learning rate
- strong performance in practice
- Variants: Adadelta, Adam, RMSprop

Adam

Adam is a momentum variant of Adagrad

pick a stochastic gradient $\mathbf{g}_t$

$$\mathbf{m}_t := \beta_1 \mathbf{m}_{t-1} + (1-\beta_1)\mathbf{g}_t \qquad \text{(momentum term)}$$

$$[\mathbf{v}_t]_i := \beta_2 [\mathbf{v}_{t-1}]_i + (1-\beta_2)([\mathbf{g}_s]_i)^2 \quad \forall i \qquad \text{(2nd-order statistics)}$$

$$[\mathbf{x}_{t+1}]_i := [\mathbf{x}_t]_i - \frac{\gamma}{\sqrt{[\mathbf{v}_t]_i}} [\mathbf{m}_t]_i \quad \forall i$$

- faster forgetting of older weights
- momentum from previous gradients (see acceleration, lecture 6)
- (simplified version, without correction for initialization of $\mathbf{m}_0$,$\mathbf{v}_0$)
- strong performance in practice, e.g. for self-attention networks

SignSGD

Only use the sign (one bit) of each gradient entry:
SignSGD is a communication efficient variant of SGD.

$$\begin{aligned} &\text{pick a stochastic gradient } \mathbf{g}_t \\ &[\mathbf{x}_{t+1}]_i := [\mathbf{x}_t]_i - \gamma_t \, sign([\mathbf{g}_t]_i) \qquad \forall i \end{aligned}$$

(with possible rescaling of γ_t with $\|\mathbf{g}_t\|_1$)

- communication efficient for distributed training
- convergence issues