

SHORT-CUT MATH



GERARD W. KELLY

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Gerard W. Kelly

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Table of Contents

DOVER BOOKS ON MATHEMATICS

Title Page

Copyright Page

MATH AS YOU LIKE IT

INTRODUCTION

I. FUNDAMENTALS OF SHORT-CUT METHODS

II. ADDITION

III. SUBTRACTION

IV. MULTIPLICATION

V. DIVISION

VI. FRACTIONS, MIXED NUMBERS, AND PERCENTAGES

INDEX

A CATALOG OF SELECTED DOVER BOOKS IN ALL FIELDS OF
INTEREST

MATH AS YOU LIKE IT

Do you yearn for multiplication
You can do without frustration?
Does it please you not to carry in your head?
Do you wish some court decision
Would abolish long division?
Do you look upon remainders with some dread?

Does the borrow in subtraction
Often drive you to distraction?
Do you dream of doing fractions in a breeze?
With each answer must you fidget
To be sure of ev'ry digit?
Wouldn't you rather do it all with speed and ease?

INTRODUCTION

This book contains about 150 short-cuts that provide faster, easier ways to add, subtract, multiply and divide. Anyone who uses numbers and does calculations can benefit from these short methods and even enjoy using them. You can apply them at school, home, shop, or office to save time and trouble. Short-cuts can help to improve your math and provide the urge to learn more or to review the math you have forgotten. Or you can use short-cuts simply because they are fascinating and make it fun to work with numbers.

The short-cuts in this book require no special ability in mathematics. If you can do ordinary arithmetic, you will have no trouble in using these methods. There is nothing to “unlearn,” no foreign devices, strange systems, or difficult operations that would require long study and practice. Furthermore, the book uses only a minimum of technical terms, so you don’t have to relearn what an augend, a minuend, or a multiplicand is, in case you have forgotten your school definitions.

To get the most out of this book, go through it briefly to see what it covers and how it is organized. After that, study whatever methods are of special interest to *you*. Naturally, you will not be able to use, or even to remember, every short-cut (to remember a short-cut is usually the hardest part of it). The best approach is to concentrate on the methods you can use in your daily work or the ones you would like to learn just for the fun of knowing them.

Examples are given under each short-cut to demonstrate how it works. However, there are no long drills and exercises. You will still need to practice any short-cut you want to put to work but you will probably find it more satisfying to make up your own problems, keeping in mind that the best way to develop speed, ease, and accuracy with any method is to use it regularly.

Among the features used throughout the book are “Proving the answer” and “Why it works.” Proof of the answer is given to demonstrate that the short-cut really arrives at the same answer as ordinary arithmetic, a result often so surprising that you may find it hard to believe. Proving the answer also helps to show how much easier the short-cut is and to encourage the good habit of

checking results.

“Why it works” is included under many short-cuts where a short, simple explanation can be devised. This aids in understanding the method and in remembering it, too. Even if you forget the details of a short-cut, you can often reconstruct the method by recalling the “why” of it and working out an example with small numbers. This will give you a simple pattern to follow and will make the short-cut easier to remember and use.

Under each short-cut in this book, either a rapid “Solution” or a longer “Solution by steps” is worked out. The steps are used where needed for additional clarity and are followed by a “Short way” indicating how the method would actually be used.

Also, after a group of related methods, a summary or table is given so you can see the methods at a glance, and as a further help in remembering them.

As you work with short-cuts, you will probably see other ways of calculating faster and easier. Discovering methods of your own can add new excitement to your journey through this corner of the magic world of numbers.

The short-cuts, of course, are not intended to supplant the pocket calculator. Rather, there’s a place for both. For long, complicated computations the electronic marvel has no rival, but for simpler problems a short-cut or rapid mental calculation can often give the answer faster than you can reach for the machine. And don’t forget those old reliables, pencil and paper. They’re inexpensive, easily carried, never need a battery, and give you a record of what you did so the work can be checked. Each of these has advantages of its own, so choose whichever method or combination of methods best suits your need and the problem to be done.

If you use a calculator, short-cuts can provide an easy way to estimate the answer in advance. This should be done to guard against large errors since pushing some wrong keys can send the answer into Never-Never Land; and since the machine leaves no record of entries, there’s no way of telling what *keys* you *did* push. An estimate helps to warn of big errors that might otherwise go unnoticed.

And, of course, a calculator can’t run itself. It needs you to tell it what to do and this requires a knowledge of arithmetic. It is hoped that this book will assist toward a better understanding of the basics involved and a deeper appreciation of the wonders of working with numbers.

I. FUNDAMENTALS OF SHORT-CUT METHODS

1-1. Basic Ways of Simplifying Calculations

Short-cut methods are based upon the principle of changing “difficult” numbers and processes into easier ones.. For example, to find the sum of 29 and 36, you can add 1 to 29 and subtract 1 from 36 so that the problem becomes $30+35$; you can see at once that the answer is 65. Adding and subtracting 1 simplified the problem, but did not change the answer. This principle of “equivalency” as used in short-cuts enables you to reach the right answer faster and easier.

You can apply the same principle to multiplying. Take 28×15 , for instance. If you halve 28 ($= 14$) and double 15 ($= 30$), you change the problem to 14×30 which, as you can see, is an easy 420 — the same answer you will get by multiplying 28 by 15.

It’s not always the size of a number that makes it difficult to handle but the KIND of number it is. You know, of course, that it’s much easier to multiply by 10 than by 9 even though 10 is the larger number. However, the “difficult” number 9 can be changed into two “easy” numbers, 10 minus 1, which equal 9. Therefore, to multiply a number by 9, you can multiply by 10 (simply add a zero to the number) and subtract the number, thus: $37 \times 9 = 370 - 37 = 333$. The $370 - 37$ is, of course, $37 \times (10 - 1)$ instead of 37×9 . Either way, the answer is the same.

Incidentally, the easiest numbers to handle are 0, 1, 10, and 2, although 100, 1,000, etc., are also easy to get along with; so are other “zero numbers,” such as 20, 30, etc. If you can change a difficult number to one of these, you are on your way to an easier solution.

In many cases, you will find it faster and easier to perform two or three simple operations rather than a single more difficult one, like taking the stairs to the

next floor rather than trying to make it in one big jump. To multiply 64 by 25, for instance, you can divide 64 by 4 (= 16) and multiply by 100 (just add two zeros), and there's your answer: 1,600. This works because dividing by 4 and multiplying by 100 is the same as multiplying by $\frac{100}{4}$ which equals 25. Again, the method has been simplified but the answer is not changed because 25 and $\frac{100}{4}$ are equivalent.

Another basic way of simplifying a calculation is to break up the numbers into easier parts. For example, to divide by 16, you can divide first by 2 and then by 8, or divide twice by 4, since 2×8 and 4×4 are each equivalent to 16. Try dividing 16 into a number like 224. You can divide the number in half (= 112) and divide this by 8 (= 14), or you can divide 224 by 4 (= 56) and divide again by 4 (= 14). Dividing 224 by 16 will, of course, give the same answer, 14, because all three ways of dividing are equivalent.

A number may be broken up into other parts, too, such as into units, tens, hundreds, etc. These smaller parts can make the number much easier to handle. The number 613, for instance, can be broken up into $600 + 10 + 3$. If you want to multiply 613 by another number, such as by 12, you can multiply each part and then add the partial results to get the answer, thus:

$$\begin{array}{r} 600 \times 12 = 7,200 \\ 10 \times 12 = 120 \\ \underline{3 \times 12 = 36} \\ 613 \times 12 = 7,356 \end{array}$$

Besides such methods as the foregoing, short-cuts are also based upon eliminating or easing processes that most people find difficult, such as "carrying" in addition and multiplication, "borrowing" in subtraction, the intricacies of "long" division, and the need to remember results when doing problems mentally.

1-2. Results Are What You Want

In doing a problem, it's better to think only of the *results* of each step and to omit any unnecessary details of the step itself. For example, when you see $36 \div 9$, or $9 \overline{)36}$, think immediately of 4. Don't go through the details of the process such as by saying "36 divided by 9 goes 4 times." Or when adding, like $5 + 11 + 8$, you just slow yourself up if you say "5 plus 11 makes 16, plus 8 makes 24."

Instead, just think: “16, 24.”

Omit the words of a process as much as possible and concentrate on the results. You can work problems faster without the words.

Numbers and calculating are like the alphabet and reading. When you see a word, you don’t stop to spell out the letters that make it up; you recognize the word at once in its entirety. Nor do you stop to consider each word separately; you read words together and think of what they mean as a whole. Doing the same with numbers and calculations will enable you to solve problems faster and easier.

1-3. Combining Mental and Written Math for Best Results

Short-cuts are not just a way of solving problems in your mind and in a flash, although they *can* help you do this. Taking the time to jot things down can often speed them up. Consider the following:

Calculating involves the mind in two processes: (1) thinking out each step in the problem, and (2) remembering the results of each step for use in succeeding steps. Naturally, the longer the problem and the more complex, the more you have to remember if you do it entirely by mental math, and the more chances there are of making a mistake.

On the other hand, if you work out the whole problem on paper, you don’t have to keep remembering the results of steps and you can concentrate on the solution. Writing it down may take longer but it reduces the likelihood of error, and this can save time in the long run.

You can often achieve the maximum of speed with accuracy by using both the mental and the written methods. You do this by writing down the *results* of the various steps but not the details of the steps. For example, to multiply 49,837 by 11, there is no need to write out the whole problem or to struggle with it as a mental exercise; just write the partial products, then add to get the answer, thus:

$$\begin{array}{r} 49837 \\ 49837 \\ \hline 548207 \end{array}$$

Or, if you want to divide 2,184 by 24, you might divide by 3 and write down

728 so you don't have to remember it; then divide by 8 to get the answer, 91.

Some short-cuts in this book will enable you to solve problems mentally, and even on sight; others will work better if you write down partial results by steps; still other short-cuts will serve you best if you do the problem entirely on paper. It's up to you to determine how much to carry in your head and how much to write down in order to achieve both speed and accuracy. While speed is desirable, a short-cut to the wrong answer is obviously useless. It pays to take a longer route, if necessary, to reach the right answer the first time.

1-4. Approximating and Rounding Off

Before working out a problem, determine how accurate your answer must be. A good estimate or approximation may be all you need and can save you a lot of needless figuring. Approximating depends upon determining the "significant figures" of the numbers involved — the figures that are important to your estimate.

For example, if an item is priced at 2 for \$1.97, you can't go far wrong in figuring that 3 will cost about \$3.00. Or if you want to paint some walls measured at 2,386 square feet, you might use 2,500 in estimating how much paint you will need. If a gallon of paint covers 500 square feet, you have your answer faster than you can reach for a pencil.

To arrive at the significant figures, "round off" a number to whatever extent may be required to suit your purpose and drop all figures (digits) to the right of this. If the first digit dropped on the right is 5 or more, add 1 to the last digit remaining; otherwise let the last digit stand as it is. The following examples will show how this is done.

Example (1): Round off 6,437:

To the nearest thousand: 6,000

To the nearest hundred: 6,400

To the nearest ten: 6,440

Example (2): Round off 83.652:

To the nearest unit: 84

To the nearest tenth: 83.7

To the nearest hundredth: 83.65

Ways for rounding off numbers to approximate or estimate the answer in addition, multiplication, and division are given in the chapters dealing with these operations.

1-5. Achieving Accuracy with Speed

Some short-cuts promote both speed and accuracy because they enable you to see the correct answer at once. However, where several steps are involved, and especially with large numbers, there is much more chance of error. As previously suggested, you will often save time in reaching the right answer if you write down at least some of the results as you go.

You can further ensure accuracy by estimating your answer in advance and then checking the answer after you get it, as discussed next.

Estimating the result in advance:

Estimating the answer before doing any problem helps to avoid big mistakes, such as omitting a decimal point or putting it in the wrong place, or writing 14,444 instead of 1,444, etc. An estimate can catch errors that make the answer much larger or smaller than it should be. You can estimate your answer by rounding off numbers or by using a short-cut, as covered in later chapters.

Checking your answer:

You can check your answer by doing the problem again, either in reverse order or by some different method. Doing the problem in a different way helps to avoid the repetition of a mistake that may be habitual.

Various ways for checking your answer in addition, subtraction, multiplication, and division are given in the chapters dealing with these operations. These checks include “casting out 9’s” which can be used to test all

operations. The process of casting out 9's is explained in 1-6 (the next section).

However, no method of checking gives absolute proof that your answer is correct. There is always the chance that you might make a mistake in the check and confirm a wrong answer. However, if your answer does check out, you can usually assume it is correct.

Another essential for achieving accuracy with speed is to keep numbers and decimal points properly placed in every problem. The careful alignment of digits by units, tens, hundreds, etc., will help to attain fast, accurate results.

And finally, accuracy and speed also require that you be reasonably well skilled in basic arithmetic and can quickly recognize that $9 + 8 = 17$, $16 - 5 = 11$, $8 \times 6 = 48$, $56 \div 7 = 8$, etc. If you must stop to work out problems like these, you will need to review and practice your "tables."

1-6. Testing Answers by Casting Out 9's

This test is based upon a remarkable property of 9. When any number is divided by 9, the remainder will equal the sum of the number's digits, or the sum of the number's digits after "casting out" 9's. This may sound strange but the idea itself is easy to understand, as the following examples will show.

Example (1): $16 \div 9 = 1$, with 7 remainder. The digit sum of 16 ($1 + 6$) is also 7, the same as the remainder after dividing 16 by 9.

Example (2): $24 \div 9 = 2$, with 6 remainder. The digit sum of 24 ($2+4$) is also 6.

Example (3): $38 \div 9 = 4$, with 2 remainder. Of course, the digit sum of 38 is not 2 but 11. However, after casting out 9, such as by subtracting 9 from 11, the remainder does become 2, the same as the remainder after dividing 38 by 9.

Remainders obtained by casting out 9's can be used to provide "check numbers" for testing your answers in addition, subtraction, multiplication, and division.

You can use any of four methods to cast out 9's and obtain the remainders, as shown in the following examples. The same numbers are used in each example to demonstrate that the remainders obtained are the same for all four methods.

A. Casting out 9's by dividing:

Divide the number by 9 to obtain the remainder.

<i>Number</i>	<i>Remainder</i>
---------------	------------------

46	1
----	---

85	4
----	---

198	0
-----	---

6,368	5
-------	---

B. Casting out 9's by adding:

Add the number's digits, and add the digits in the sum if there is more than one digit in the sum, to obtain a one-digit remainder. If the remainder is 9 or a number evenly divisible by 9 (18, 27, 36, etc.), count the remainder as zero.

<i>Number</i>	<i>Remainder</i>
---------------	------------------

46: $4 + 6 = 10$; $1 + 0 =$	
------------------------------	--

85: $8 + 5 = 13$; $1 + 3 =$	
------------------------------	--

198: $1 + 9 + 8 = 18$; $1 + 8 = 9$;	
---------------------------------------	--

6,368: $6 + 3 + 6 + 8 = 23$; $2 + 3 =$	
---	--

C. Casting out 9's by subtracting:

Add the number's digits and subtract 9 each time the sum equals or exceeds 9. Do this until the final remainder becomes less than 9. Or, you can first get the digit sum and then keep subtracting 9 until the remainder becomes less than 9.

<i>Number</i>	<i>Remainder</i>
46: $4 + 6 = 10; - 9 =$	1
85: $8 + 5 = 13; - 9 =$	4
198: $1 + 9 = 10; - 9 = 1; + 8 = 9; - 9 =$	0
also: 198: $1 + 9 + 8 = 18; - 9 = 9; - 9 =$	0
6,368: $6 + 3 = 9; - 9 = 0; + 6 + 8 = 14; - 9 =$	5
also: 6,368: $6 + 3 + 6 + 8 = 23; - 9 = 14; - 9 =$	5

D. Casting out 9's by omitting them:

Add the number's digits but do not include 9's or combinations equal to 9 (6 and 3; 5 and 4; 1, 3, and 5, etc.). Omitting 9's and 9-combinations saves adding them up and subtracting 9's later. If the sum comes to 9 or more, cast out 9's to obtain the remainder.

<i>Number</i>	<i>Remainder</i>
46: no 9 to omit; $4 + 6 = 10; - 9 =$	1
85: no 9 to omit; $8 + 5 = 13; - 9 =$	4
198: omit $1 + 8$ and omit 9; $=$	0
6,368: omit $6 + 3$; $6 + 8 = 14; - 9 =$	5

Did you notice that the remainders were the same for all four methods?

Omitting 9's is the easiest way to cast them out, especially with large numbers, since it reduces the adding and subtracting, and requires no division. In a number like 2,756,942, for instance, the 2 and 7, 5 and 4, and the 9 can be omitted, leaving only 6 and 2 to be added, giving a remainder of 8. Dividing 2,756,942 by 9 will, of course, also give a remainder of 8, as shown:

$$\begin{array}{r} 9 \overline{) 2,756,942} \\ \underline{306,326} \\ 8 \end{array}$$

Now you can apply casting out 9's to quickly check the results of adding

How you can apply casting out 9's to quickly check the results of adding, multiplying, and dividing is shown at the end of the chapters dealing with these operations. The method does not give positive proof that your answer is right (no method does) but if your answer does not meet the test of casting out 9's, you can be sure that there is an error somewhere, either in doing the calculation, or in making the check, or in both.

II. ADDITION

2-1. Basic Principles

Numbers may be added in any order. For example, $7 + 4 + 3 = 14$; $3 + 4 + 7 = 14$; $4 + 3 + 7 = 14$, and so on.

You can take advantage of this principle by adding numbers in whatever order is easiest for you. In getting the sum of $99 + 83 + 1$, for instance, you can first add 99 and 1 to make 100, and then add 83 to get 183. Obviously, this is easier than adding 99 and 83, and then 1.

You can also simplify addition by changing difficult numbers into easier ones. For example, to add 39 and 47:

(a) Simplify 39 into a number ending in zero: $39 + 1 = 40$

(b) Add: $40 + 47 = 87$

(c) Subtract the 1 you added: $87 - 1 = 86$, answer

Or you can do it this way:

$$(39 + 1) + (47 - 1) = 40 + 46 = 86$$

By adding an amount to one number and subtracting an equal amount from a second number, the sum of the two numbers remains the same.

As mentioned in Chapter I, you can do problems faster if you cut down on words as much as possible. In adding $9 + 4 + 7$, for example, you would not say “9 and 4 make 13, and 7 makes 20.” Instead, you “think” the addition without the words: 13, 20. You would do the same when adding a column.

$$\begin{array}{r}
 5 \\
 9 \\
 4 \\
 3 \\
 7 \\
 \underline{8} \\
 36
 \end{array}$$

Omit the words and think only of the result of each step: 15, 18, 22, 31, 36.

You should be able to “see” the sum of two digits at once, without stopping to count, and this means that you must be on good terms with your addition table, at least up to $9 + 9$.

2-2. Adding by 10-Groups

Adding by 10 is easy, so look for combinations that add to 10.

$$\begin{array}{r}
 4 \\
 \left(\begin{array}{c} 5 \\ 6 \end{array} \right) \\
 5 \\
 \left(\begin{array}{c} 8 \\ 2 \end{array} \right) \\
 \underline{\quad} \\
 30
 \end{array}$$

Notice the three pairs of numbers that make 10-groups: 2, 8; 5, 5; and 6, 4. These can be added quickly in three steps: 10, 20, 30.

Pairs of numbers that add to 10:
1,9; 2,8; 3,7; 4,6; 5,5 5

Trios of numbers that add to 10:

1, 1, 8; 1, 2, 7; 1, 3, 6; 1, 4, 5

2, 2, 6; 2, 3, 5; 2, 4, 4

3, 3, 4

Since numbers may be added in any order, it makes no difference whether a pair appears as 7, 3, or 3, 7; or a trio as 1, 3, 6, or 3, 6, 1, or 6, 1, 3, and so on. They all make an easy 10 to add.

Of course, you can extend the same idea to adding by 20-groups : 2, 9, 9; 3, 8, 9; 4, 7, 9; 4, 8, 8, etc. By recognizing such combinations and converting them to a single number ending in zero, addition is made easier.

2-3. Reaching 10-Levels

In adding, look for opportunities to reach a 10-level (20, 30, 40, etc.). For instance, if you have added up to 36, look for a 4, or a 3 and a 1, or a pair of 2's, to make an easy 40; or if your addition is at 62, look for an 8 or some 8-combination that will take you to 70.

Because a 10-level number ends in zero, you will find it easier to remember the sum as you go and easier to add the next number.

2-4. Adding by Other Combinations of Numbers

Besides combinations that make 10 or 20, you can add other numbers by pairs or trios as though they were single numbers, such as adding 4 and 3 as 7, 5 and 1 as 6, 1, 3, and 5 as 9, etc. If you learn to “see” these groups as a single number, you will be able to add faster and easier.

21
14
33
47
96
83
294

Starting from the bottom, addition by groups could go like this: 9, 19, 24; write 4, carry 2; 10, 23, 29.

2-5. Adding by Multiplying

Multiplication is simply a rapid way of adding the same number over and over. For example, 7×4 is the same as adding 7 four times: $7 + 7 + 7 + 7$. The result comes to 28 either way.

If a number is repeated, especially if repeated often, you can use multiplication to speed up your addition. After multiplying, add the other numbers.

$$\begin{array}{r} 8 \\ 7 \\ 5 \quad 8 \times 3 = 24; 30, 42 \\ 8 \\ 6 \\ \hline 8 \\ 42 \end{array}$$

2-6. Adding Two or More Columns

In writing columns of numbers for addition, line up the columns carefully so that units are under units, tens under tens, and so on. If decimal points are used, keep these in a straight column, too. An orderly arrangement helps to promote both speed and accuracy.

As you get the sum of each column, write the “carry” as a small number above the next column on the left.

$$\begin{array}{r}
 32 \\
 346 \\
 521 \\
 789 \\
 640 \\
 375 \\
 \underline{246} \\
 2,917
 \end{array}$$

The sum of the units column is 27; write 7, carry 2; the sum of the tens column is 31; write 1, carry 3.

You may prefer to write the carry above and to the side of the numbers in the answer, such as 2,917. Either way, writing the carry relieves your mind of remembering what it is and makes it easier to locate an error if a check shows that the answer is wrong. Since the written carry lets you see the total for each column, you can check each column separately. Furthermore, if you are interrupted while adding, you will not have to re-add any column where the carry is shown.

2-7. Adding Long Columns

In a long column, you can add the numbers by groups to get sub-totals and then add the sub-totals to get the sum total. This gives your mind a chance to rest, makes it easier to check your answer, and enables you to locate a mistake more readily in case there is one. As a further help, you can write the carry for each group in the sub-totals.

4,382		
2,654		
7,655		
9,872	231	
<u>6,341</u>	30,904	Sub-total
9,534		
7,228		
5,287		
6,634	112	
<u>9,110</u>	<u>37,793</u>	Sub-total
	68,697	Sum total

2-8. No-Carry Addition

You can avoid carrying in addition by writing the total for each column, then adding the column-totals to obtain the sum total.

6,274
4,633
7,452
3,516
8,775
<u>1,942</u>
22
27
3,3
<u>29</u>
32,592

Notice that the column-totals 22, 27, 33, and 29 are lined up in their proper places. They are actually 22, 270, 3,300, and 29,000.

$$\begin{array}{r}
 6,274 \\
 4,633 \\
 7,452 \\
 3,516 \\
 8,775 \\
 \underline{1,942} \\
 29,000 \\
 3,300 \\
 270 \\
 \underline{22} \\
 32,592
 \end{array}$$

You can also add the columns from left to right. The zeros in the column-totals are not necessary but are shown to indicate the real values of these totals.

Left-to-right addition can be useful in making estimates since you can stop adding after any column that suits the accuracy you need. In the example just given, the sum of the first column on the left shows that the total is at least 29,000; adding the next column brings the total to at least 32,300. If this is close enough for your purpose, you can skip the addition of the next two columns.

2-9. Rounding Off Numbers for Addition

An even better way to estimate a sum is to round off the numbers. For example, suppose you want to estimate the sum of the following numbers to the nearest thousand.

$$\begin{array}{r}
 46,274 \\
 34,633 \\
 57,452 \\
 13,516 \\
 88,775 \\
 \underline{41,942} \\
 282,592
 \end{array}$$

To round off each number to the nearest thousand, add 1 to the thousand digit if the hundred digit is 5 or more; otherwise, let the thousand digit stand as it is. The rounded numbers would then be:

$$\begin{array}{r}
 46 \\
 35 \\
 57 \\
 14 \\
 89 \\
 \underline{42} \\
 283 \text{ (thousand)}
 \end{array}$$

The sum of the rounded numbers is only 408 greater than the sum of the original numbers.

2-10. Rounding Off Dollars and Cents for Addition

In adding sums of money, you can approximate the total by rounding off to the nearest dollar and dropping the cents. Cents of 50 or more are counted as a dollar and added to the dollar unit; 49 cents or less are dropped.

Exact Amounts Rounded Amounts

1 112 1	
\$5,621.42	\$5,621
3,842.59	3,843
8,034.91	8,035
<u>7,156.21</u>	<u>7,156</u>
\$24,655.13	\$24,655

This method saves writing and adding two columns of figures. As you can see, the adding and subtracting of cents tends to keep the total about the same, and as the dollar amounts become larger, the importance or significance of the cents becomes less anyway.

2-11. Adding a Regular Series of Numbers

Example (1): What is the sum of all the numbers from 1 to 15?

Solution by steps:

(a) Add the first and last numbers of the series: $1 + 15 = 16$

(b) Divide by 2 to get the average number: $16 \div 2 = 8$

(c) Multiply the average by the number of numbers in the series: $8 \times 15 = 120$,
answer

Short way:

$$\frac{1+15}{2} = 8; \times 15 = 120$$

If you add all the numbers from 1 to 15, you will see for yourself that their sum is 120.

Example (2): What is the sum of all the numbers from 20 to 29?

Solution: $\frac{20 + 29}{2} = 24\frac{1}{2}; \times 10 = 245$, answer

Proving the answer: 20

21

22

23

24 Notice that there are ten numbers

25 (not nine) from 20 to 29, inclusive.

26

27

28

29

245

Example (3): Find the sum of all the numbers from 1 to 99.

Solution: $\frac{1 + 99}{2} = 50; \times 99 = 4,950$, answer

2-12. Add-and-Subtract Method of Simplifying Numbers

You may be able to simplify an addition by an easy addition and subtraction before you add the numbers.

Example: $97 + 237$

Solution by steps:

(a) Add 3 to 97 to make it an easy 100

(b) $100 + 237 = 337$

(c) Subtract the 3 you added: $337 - 3 = 334$, answer

Alternate solution:

(a) $97 + 3 = 100$

(b) $237 - 3 = 234$

(c) $100 + 234 = 334$

Short way: $97 + 3 = 100; + (237 - 3) = 334$

Proving the answer:

$$\begin{array}{r} 97 \\ + 237 \\ \hline 334 \end{array}$$

Why it works: In any addition, adding and subtracting the same amount does not change the sum. Try this with any numbers you please; for example, $6+3=9$. Now add 2 and subtract 2: $(6 + 2) + (3 - 2) = 8 + 1 = 9$. Or add and subtract 3: $(6+3) + (3 - 3) = 9 + 0 = 9$.

You can apply this same idea to add difficult numbers like 9, 99, 999, etc. Simply add 1 to change the number to an easy 10, 100, 1,000, etc., and subtract 1 from the sum or from the other number.

Example: $13,862 + 999$

Solution:

$$999 + 1 = 1,000; + 13,862 = 14,862; - 1 = 14,861, \text{ answer}$$

2-13. Splitting Numbers into Easy Parts

If numbers contain two or more digits, splitting the numbers into smaller parts can make them easier to add.

Example (1): $76 + 57$

Solution: Think of 76 as $(70 + 6)$ and 57 as $(50 + 7)$, then add by parts: $70 + 50 = 120$; $+ 6 + 7 = 133$, answer

Example (2): $6,614 + 4,238$

Solution:

$$\begin{array}{rcl} 6,614 & = & 6,600 + 14 \\ 4,238 & = & \underline{4,200} + \underline{38} \\ & & 10,800 + 52 = 10,852, \text{ answer} \end{array}$$

Or you might find it easier to do it this way: $6,000 + 4,000 = 10,000$; $+ 600 + 200 = 10,800$; $+ 10 + 30 = 10,840$; $+ 4 + 8 = 10,852$

2-14. Thinking of Numbers as “Dollars” and “Cents”

This is simply another way of splitting numbers into smaller parts for easier addition.

Example: $962 + 837$

Solution: Think of the numbers as \$9.62 and \$8.37, then add the dollars and cents separately:

$\$9 + \$8 = \$17$; $62¢ + 37¢ = 99¢$;
total: \$17.99, which, of course, is 1,799

Checking Your Answer in Addition

No method of checking addition can tell for sure that your answer is right. You might make the same error in the check as you made in the original addition. There are such things as habitual mistakes that are repeated when adding certain numbers, like mistakes in regularly misspelling certain words.

Keep in mind that the significance or size of an error increases as the numbers go from right to left — from units, to tens, to hundreds, and so on. In adding money, for instance, an error of 5 in the cents column would only put you in or out a nickel, but the same error four columns to the left would be a 500-dollar mistake!

2-15. Checking Your Answer by Adding Again in Reverse

You can check a sum by adding again in a different order. If you add from the bottom up the first time, add from the top down as a check. If you get the same answer both ways, you may assume that you are right.

2-16. Checking Addition by the No-Carry Method

If you add in the regular way the first time, you can check your answer by

By adding in the regular way the first time, you can check your answer by getting sums by columns and then adding these to obtain the total. You can also reverse whatever order you used the first time: top to bottom or bottom to top; right to left or left to right.

	<i>Sums by Columns Right to Left</i>	<i>Sums by Columns Left to Right</i>
322		
6,274		
4,633		
7,452	22	29
3,516	27	33
8,775	33	27
1,942	29	22
32,592	<u>32592</u>	<u>32592</u>

2-17. Checking Addition by Casting Out 9's

Various ways to cast out 9's are shown in 1-6, Chapter I. How to apply this method to checking addition is discussed here. The longer the addition and the larger the numbers, the more time you will save by checking your answer by casting out 9's. ,

The remainders obtained after casting out 9's are used as check numbers for testing the answer in addition, as follows:

<i>Example (1):</i>	251	Remainder:	8
	672	„	: 6
	415	„	: 1
	<u>982</u>	„	: <u>1</u>
	2,320	Total:	16: 7 Remainder
	Remainder of answer (2,320): 7		

Since the two remainders (7) are equal, you may assume that the answer is correct.

Example (2):

729	Remainder:	0
436	„	: 4
184	„	: .4
<u>655</u>	„	: <u>7</u>
2,004	Total:	15: 6 Remainder
Remainder of answer (2,004): 6		

Remainders are equal; the answer checks.

Casting out 9's does not give positive proof that the answer is correct but if the answer does *not* check, there is definitely an error in the addition, in the check, or in both.

III. SUBTRACTION

3-1. Introduction

Subtraction is the reverse of addition; unlike addition, it cannot be done in any order. $5+3=8$ and $3 + 5 = 8$ but obviously $5 - 3$ and $3 - 5$ are not the same.

“Borrowing” is usually regarded as the main difficulty in subtraction. There are some ways to eliminate or reduce this but there are not as many short-cuts in subtraction as in the other operations.

Subtraction shows the *difference* or *remainder* after taking one number from another, and may be written in either of the following ways :

$$\begin{array}{r} 9 - 6 = 3 \text{ Remainder} \\ \phantom{9 - 6 = 3 \text{ Remainder}} 9 \\ \phantom{9 - 6 = 3 \text{ Remainder}} - \underline{6} \\ \phantom{9 - 6 = 3 \text{ Remainder}} 3 \text{ Remainder} \end{array}$$

3-2. Add-or-Subtract Method of Simplifying Numbers

You can simplify subtraction by changing the number to be subtracted into a number ending in zero. This can be done with an easy addition or subtraction.

Example: $123 - 96$

Solution by steps:

(a) Add 4 to 96 to make it an easy 100

(b) $123 - 100 = 23$; this is 4 less than the real difference

(c) Add the 4 back: $23 + 4 = 27$, answer

Or you can add the same amount to both numbers, like this:

(a) $123 + 4 = 127$

(b) $96 + 4 = 100$

(c) $127 - 100 = 27$

Short way: Add 4 to both numbers: $127 - 100 = 27$

Why it works: In subtraction, if you add the same amount to both numbers, you do not change the difference between the numbers. Try it with any numbers, such as $5 - 1 = 4$.

Now add 2 to both numbers: $(5 + 2) - (1 + 2) = 7 - 3 = 4$.

Or add 3 to both numbers: $(5 + 3) - (1 + 3) = 8 - 4 = 4$.

Instead of adding, you may find it easier to subtract the same amount from both numbers. Again, this will not change the difference between the numbers.

Example: $224 - 173$

Solution: Subtract 3 from both numbers; this makes 173 end in zero; then: $221 - 170 = 51$, answer

3-3. Subtraction without Borrowing

You can sometimes avoid borrowing if you subtract by pairs of digits instead of by individual digits.

Example (1):

$$\begin{array}{r} 5,381 \\ - 4,863 \\ \hline 518 \end{array}$$

Solution: Instead of borrowing 1 from 8 and then subtracting 3 from 11, you can do this subtraction in two easy steps:

(a) $81 - 63 = 18$

(b) $53 - 48 = 5$; answer 518

Example (2):

$$\begin{array}{r} 47,635 \\ - 38,519 \\ \hline 9,116 \end{array}$$

Solution:

(a) $35 - 19 = 16$

(b) $6 - 5 = 1$

(c) $47 - 38 = 9$; answer 9,116

3-4. Splitting Numbers into Easy Parts

You can split numbers into smaller parts for easier subtraction.

Example: $3,862 - 2,428$

Solution by steps:

(a) Split 3,862 into $3,800 + 62$

(b) Split 2,428 into 2,400 + 28

(c) Subtract: $1,400 + 34 = 1,434$, answer

Or you can do it this way:

$$3,800 - 2,400 = 1,400; 62 - 28 = 34; 1,400 + 34 = 1,434$$

3-5. Counting Change after a Purchase

If you buy some groceries for \$3.69 and pay for them with a \$5 bill, how much change should you get?

Perhaps you can subtract \$3.69 from \$5.00 mentally in one step and see the

correct remainder 'as \$1.31. If so, fine! If not, you may find it easier to calculate the cents and the dollars separately. Most of the numbers you use will end in zero or zeros so that they are easy to handle.

Solution by steps:

(a) Add 1¢ to the 69¢; = 70¢. Add 30¢ to reach the next dollar (\$4.00). So your change in coins is 31¢.

(b) Subtract \$4.00 from \$5.00; = \$1.00, your change in dollars

(c) Total change: 31¢ + \$1.00 = \$1.31

Or you can check your change *after* you get it by using the money as “counters.” Start with the purchase amount and add the coins up to the next dollar; then add the dollars in change until you reach the denomination of the bill you gave in payment.

3-6. Checking Your Answer in Subtraction

The fastest, easiest way to prove your subtraction is to add the remainder to the number that was subtracted. If the sum of these two numbers equals the number from which the subtraction was made, your answer may be considered correct.

Or, to put it another way, you add the remainder to the number above it to see if their sum equals the number on top.

Example:

$$\begin{array}{r} 634,715 \\ - 491,284 \\ \hline 143,431 \text{ Remainder} \end{array}$$

Check: Adding 143,431 to 491,284 gives 634,715

Since you don't have to write any additional numbers to make this check. the

Since you don't have to write any additional numbers to make this check, the method is fast and easy, and should be used to prove every subtraction. Many errors in calculations are the result of errors in subtraction so that taking the time to check remainders will not only promote greater accuracy but will save time and effort in the long run.

Of course, if you want to be even more certain of your answer, you can also subtract the remainder from the number on top to see if you get the number that was subtracted. Again, this can be done without writing any additional numbers. In the example given, you would subtract 143,431 from 634,715 to get 491,284, indicating that the answer is correct.

You can also check subtraction by casting out 9's but the method is not included here because the check by adding the remainder to the number subtracted is faster, easier, and surer.

IV. MULTIPLICATION

4-1. Basic Principles

Multiplication is already a fast process. It is a rapid method for adding the same number over and over.

Example:

$$\begin{array}{r} 513 \\ 513 \\ 513 \\ \underline{513} \\ 2,052 \end{array} \qquad \begin{array}{r} 513 \\ \times 4 \\ \hline 2,052 \end{array}$$

Multiplication may be written in one of the following ways:

$$\begin{array}{ll} 3 \times 8 = 24 & \text{Product} \qquad \begin{array}{l} 3 \text{ Factor (multiplicand)} \\ \times 8 \text{ Factor (multiplier)} \\ \hline 24 \text{ Product} \end{array} \end{array}$$

The numbers 3 and 8 are called *factors* of 24 because, when multiplied together, they give 24. Other factors of 24 are 2, 4, 6, and 12 since $2 \times 12 = 24$, $4 \times 6 = 24$, $2 \times 2 \times 2 \times 3 = 24$, etc.

Besides being a *product* of 3×8 , 4×6 , etc., 24 is also said to be a *multiple* of 2, 3, 4, 6, 8, and 12 because each of these factors can be multiplied by another factor or factors to produce 24.

Numbers may be multiplied in any order. Thus, $3 \times 8 = 24$; $8 \times 3 = 24$; $3 \times 8 = 8 \times 3$. You can take advantage of this fact by multiplying in whatever order is

easiest for you.

Example: $25 \times 42 \times 4$

Solution: $25 \times 4 = 100$; $\times 42 = 4,200$, answer

Another basic way to simplify multiplication is to split a number into smaller parts by units, tens, hundreds, etc.

Example: 47×6

Solution by steps:

(a) Think of 47 as $40 + 7$

(b) Multiply by parts: $40 \times 6 = 240$; $7 \times 6 = 42$

(c) Add the partial products: $240 + 42 = 282$, answer

Or you can think of it this way:

$$\begin{array}{r} 40 \times 6 = 240 \\ \underline{7 \times 6 = 42} \\ 47 \times 6 = 282 \end{array}$$

Or do it this way: $(40 \times 6) + (7 \times 6) = 240 + 42 = 282$

To simplify multiplication, see if you can change a multiplier to one of the easiest numbers: 0, 1, 10 (also 100, 1,000, etc.), or 2. The products of these multipliers are as follows:

Any number multiplied by zero gives zero.

Examples: $1 \times 0 = 0$; $2 \times 0 = 0$; $3 \times 0 = 0$

Any number multiplied by 1 gives the same number.

Examples: $2 \times 1 = 2$; $3 \times 1 = 3$; $4 \times 1 = 4$

Any number multiplied by 10 gives the same number with a zero added or the decimal point moved one place to the right.

Examples: $36 \times 10 = 360$; $4.3 \times 10 = 43$; $62 \times 10 = 6.2$

Any number multiplied by 2 equals double the number.

Examples: $24 \times 2 = 48$; $45 \times 2 = 90$

Numbers ending in 5 (15, 25, 35, etc.) and numbers near a 10-level (9, 11, 19, 21, etc.) can be made into easy multipliers by transforming them into numbers ending in zero.

Example (1): 34×15

Solution by steps:

(a) Double 15 = 30

(b) Halve 34 = 17

(c) $17 \times 30 = 510$, answer

Example (2): 27×9

(a) Change 9 to a multiplier made up of two easy numbers: 10- 1

(b) $27 \times 10 = 270$; - $(27 \times 1) = 243$, answer

Applications and further details of these short-cut principles will be found in the methods that follow.

4-2. Multiplication When One Number Has All Digits Alike

If the digits of a number are all the same, make this number the multiplier.

Example: 88×76

Solution: Make 88 the multiplier. This will require multiplying only by 8 instead of by 7 and 6.

$$\begin{array}{r} 76 \\ \times \underline{88} \\ 608 \\ 608 \\ \hline 6688 \end{array}$$

With larger numbers, the method becomes even faster and easier.

Example:

$$\begin{array}{r} : \quad 888 \times 764 \\ 764 \\ \times \underline{888} \\ 6112 \\ 6112 \\ 6112 \\ \hline 678432 \end{array}$$

$$\begin{array}{r} 888 \\ \times \underline{764} \\ 3552 \\ 5328 \\ 6216 \\ \hline 678432 \end{array}$$

4-3. To Multiply by a Number Made Up of Multiples

Example (1) : 753×842

In the multiplier, 4 is a multiple of 2 since $2 \times 2 = 4$; 8 is a multiple of 4 since $4 \times 2 = 8$.

Solution:

$$\begin{array}{r}
 753 \\
 \times 842 \\
 \hline
 1506 \\
 3012 \\
 6024 \\
 \hline
 634026
 \end{array}$$

Multiply this by 2 instead of 753 by 4

Multiply this by 2 instead of 753 by 8

Example (2): 243×186

In this multiplier, 18 is a multiple of 6 since $6 \times 3 = 18$.

Solution:

$$\begin{array}{r}
 243 \\
 \times 186 \\
 \hline
 1458 \\
 4374 \\
 \hline
 45198
 \end{array}$$

Obtained by multiplying 1458 by 3 since 18 is 3 times 6. This is instead of multiplying 243 by 8 and then by 1.

Proving the answer:

$$\begin{array}{r}
 243 \\
 \times 186 \\
 \hline
 1458 \\
 1944 \\
 243 \\
 \hline
 45198
 \end{array}$$

4-4. No-Carry Multiplication

If carrying is something you would like to avoid, you can get around it this

way:

$$\begin{array}{r} 634 \\ \times 9 \\ \hline 36 \\ 27 \\ \hline 54 \\ \hline 5706 \end{array}$$

You can also do no-carry multiplication by splitting a number into units, tens, hundreds, etc., and multiplying from left to right. The zeros in the partial products help to make the addition easier.

$$\begin{array}{r} 634 \\ \times 9 \\ \hline 5400 \\ 270 \\ \hline 36 \\ \hline 5706 \end{array}$$

$$\begin{array}{l} \text{Also: } 600 \times 9 = 5,400 \\ 30 \times 9 = 270 \\ 4 \times 9 = 36 \\ \hline 634 \times 9 = 5,706 \end{array}$$

Or you can do it this way:

$$(600 \times 9) + (30 \times 9) + (4 \times 9) = 5,400 + 270 + 36 = 5,706$$

4-5. To Multiply a Number by 10, 100, 1,000, etc.

Multiplying by 10, 100, etc., is so simple that it hardly need be considered as a separate step. However, the process is reviewed here because the use of these multipliers is very important to methods that come later.

A. If the number has a decimal point:

Move the point one place to the right for each zero in the multiplier.

To multiply by 10, move the decimal point one place to the right; to multiply by 100, move it two places to the right; to multiply by 1,000, move the decimal point three places to the right, etc.

Examples:

$$\begin{aligned}6.27 \times 10 &= 62.7; .183 \times 10 = 1.83 \\6.27 \times 100 &= 627; .183 \times 100 = 18.3 \\6.27 \times 1,000 &= 6,270; .183 \times 1,000 = 183\end{aligned}$$

B. If the number has no decimal point:

Add a zero after the number for each zero in the multiplier.

To multiply by 10, add one zero after the number; to multiply by 100, add two zeros; to multiply by 1,000, add three zeros, etc.

Examples:

$$\begin{aligned}73 \times 10 &= 730; 192 \times 10 = 1,920 \\73 \times 100 &= 7,300; 192 \times 100 = 19,200 \\73 \times 1,000 &= 73,000; 192 \times 1,000 = 192,000\end{aligned}$$

In the following methods, where a number is to be multiplied by 10, 100, or 1,000, simply move the decimal point to the right as required, or add the proper number of zeros.

Multiplying by Aliquot Parts of 1, 10, 100, and 1,000

An aliquot part is a number that is contained in another number an exact number of times. For example, 5 is an aliquot part of 10 because 5 can be divided into 10 without a remainder.

Aliquot parts of 1, 10, 100, and 1,000 (such as .5, 5, 2.5, 25, 50, 125, etc.) can be simplified as multipliers by changing them into easy fractions, as shown in the following methods.

4-6. General Method for Multiplying by Aliquot

Parts

To multiply by an aliquot part, change it to its fractional equivalent and multiply. The numerator (top) of the fraction will be 1, 10, 100, or 1,000 while the denominator of the fraction will be 2, 4, or 8. By this method, .5 is changed to the fraction $\frac{1}{2}$, 5 is changed to $\frac{10}{2}$, 25 to $\frac{100}{4}$, 125 to $\frac{1000}{8}$, etc.

Example (1): 16×25

Solution by steps:

(a) Change 25 to fractional equivalent: $\frac{100}{4}$

(b) $16 \times \frac{100}{4} = \frac{1600}{4} = 400$, answer

Short way: $16 \div 4 = 4$; add two zeros = 400

Proving the answer:

$$\begin{array}{r} 16 \\ \times 25 \\ \hline 80 \\ 32 \\ \hline 400 \end{array}$$

Example (2): 16×2.5

Solution by steps:

(a) Fractional equivalent of $2.5 = \frac{10}{4}$

(b) $16 \times \frac{10}{4} = \frac{160}{4} = 40$, answer

Short way: $16 \div 4 = 4$; add a zero = 40

Why it works: The multipliers (2.5, 25, etc.) are not altered in value but are simply changed to an easier form. This does not change the result or product.

Although 1 is included among the multipliers (along with 10, 100, and 1,000) in the methods that follow, actually multiplying by 1 should not be done since it will not change the number multiplied. Also keep in mind that multiplying by

$\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$ is the same as dividing by 2, 4, or 8. (For details of multiplying and dividing by fractions, see Chapter VI.)

4-7. To Multiply by .5, 5, 50, or 500

The general method for this group of multipliers is to divide by 2 and then multiply by 10, 100, or 1,000. (No need to multiply by 1 as it will not change the number multiplied.)

A. To multiply a number by .5, divide the number by 2. Dividing by 2 is the same as multiplying by $\frac{1}{2}$ which is equivalent to $5 (\frac{5}{10})$.

Example: 362×5

Solution: $362 \div 2 = 181$; or, $362 \times \frac{1}{2} = 181$, answer

Proving the answer:

$$\begin{array}{r} 362 \\ \times \underline{.5} \\ 181.0 \end{array}$$

B. To multiply a number by 5, divide the number by 2 and multiply by 10. This is the same as multiplying by $\frac{10}{2}$ which is equivalent to 5.

Example: 482×5

Solution: $482 \div 2 = 241$; $\times 10 = 2,410$, answer

Proving the answer:

$$\begin{array}{r} 482 \\ \times \underline{5} \\ 2,410 \end{array}$$

C. To multiply a number by 50, divide the number by 2 and multiply by 100. This is the same as multiplying by $\frac{100}{2}$ which is equivalent to 50.

Example: 138×50

Solution: $138 \div 2 = 69$; $\times 100 = 6,900$, answer

Proving the answer:

$$\begin{array}{r} 138 \\ \times \underline{50} \\ 6,900 \end{array}$$

D. To multiply a number by 500, divide the number by 2 and multiply by 1,000. This is the same as multiplying by $\frac{1000}{2}$ which is equivalent to 500.

Example: 45.62×500

Solution: $45.62 \div 2 = 22.81$; $\times 1,000 = 22,810$, answer

Proving the answer:

$$\begin{array}{r} 45.62 \\ \times \underline{500} \\ 22810.00 \end{array} \quad \begin{array}{l} \text{Short way: } 4562 \\ \times \underline{5} \\ 22810 \end{array}$$

E. If the number to be multiplied is odd, there will be a remainder of $\frac{1}{2}$ after dividing by 2 but the same steps apply.

Examples:

$$\begin{array}{l} 47 \times 5: 47 \div 2 = 23\frac{1}{2} = 23.5; \times 10 = 235 \\ 23 \times 50: 23 \div 2 = 11\frac{1}{2} = 11.5; \times 100 = 1,150 \\ 85 \times 500: 85 \div 2 = 42\frac{1}{2} = 42.5; \times 1,000 = 42,500 \end{array}$$

However, you can multiply odd numbers by these multipliers in a still shorter way: simply ignore the fraction ($\frac{1}{2}$) after dividing by 2 and attach the multiplier (.5, 5, 50, or 500) to the result.

Examples:

$$23 \times .5 = 11.5 \text{ (halve 22 and attach .5)}$$

$$69 \times 5 = 345 \text{ (halve 68 and attach 5)}$$

$$57 \times 50 = 2,850 \text{ (halve 56 and attach 50)}$$

$$73 \times 500 = 36,500 \text{ (halve 72 and attach 500)}$$

4-8. To Multiply by .25, 2.5, 25, or 250

The general method for this group of multipliers is to divide by 4 and then multiply by 10, 100, or 1,000. (No need to multiply by 1 as it will not change the number multiplied.)

A. To multiply a number by .25, divide the number by 4. Dividing by 4 is the same as multiplying by $\frac{1}{4}$ which is equivalent to $.25 (\frac{25}{100})$.

Example: $72 \times .25$

Solution: $72 \div 4 = 18$; or, $72 \times \frac{1}{4} = 18$, answer

Proving the answer:

$$\begin{array}{r} 72 \\ \times .25 \\ \hline 360 \\ 144 \\ \hline 18.00 \end{array}$$

B. To multiply a number by 2.5, divide the number by 4 and multiply by 10. This is the same as multiplying by $\frac{10}{4}$ which is equivalent to $2.5 (2\frac{1}{2})$.

Example: 96×2.5

Solution: $96 \div 4 = 24$; $\times 10 = 240$, answer

Proving the answer:

$$\begin{array}{r} 96 \\ \times 2.5 \\ \hline 480 \\ 192 \\ \hline 240.0 \end{array}$$

C. To multiply a number by 25, divide the number by 4 and multiply by 100. This is the same as multiplying by $\frac{100}{4}$ which is equivalent to 25.

Example: 428×25

Solution: $428 \div 4 = 107$; $\times 100 = 10,700$, answer

Proving the answer:

$$\begin{array}{r} 428 \\ \times 25 \\ \hline 2140 \\ 856 \\ \hline 10700 \end{array}$$

D. To multiply a number by 250, divide the number by 4 and multiply by 1,000. This is the same as multiplying by $\frac{1000}{4}$ which is equivalent to 250.

Example: 848×250

Solution: $848 \div 4 = 212$; $\times 1,000 = 212,000$, answer

Proving the answer:

$$\begin{array}{r} 848 \\ \times 250 \\ \hline 42400 \\ 1696 \\ \hline 212000 \end{array}$$

E. If the number to be multiplied is not evenly divisible by 4, there will be a remainder of $\frac{1}{4}$ (.25), $\frac{1}{2}$ (.5) or $\frac{3}{4}$ (.75) but the same steps apply.

Examples:

$$\begin{array}{l} 17 \times .25: 17 \div 4 = 4\frac{1}{4} \text{ or } 4.25 \\ 18 \times 2.5: 18 \div 4 = 4\frac{1}{2} = 4.5; \times 10 = 45 \\ 19 \times 25: 19 \div 4 = 4\frac{3}{4} = 4.75; \times 100 = 475 \\ 21 \times 250: 21 \div 4 = 5\frac{1}{4} = 5.25; \times 1,000 = 5,250 \end{array}$$

4-9. To Multiply by .125, 1.25, 12.5, or 125

The general method for this group of multipliers is to divide by 8 and then multiply by 10, 100, or 1,000. (No need to multiply by 1 as it will not change the number multiplied.)

A. To multiply a number by .125, divide the number by 8. Dividing by 8 is the same as multiplying by $\frac{1}{8}$ which is equivalent to .125 $125 (\frac{125}{1000})$.

Example: $168 \times .125$

Solution: $168 \div 8 = 21$; or, $168 \times \frac{1}{8} = 21$, answer

Proving the answer:

$$\begin{array}{r}
 168 \\
 \times .125 \\
 \hline
 840 \\
 336 \\
 168 \\
 \hline
 21.000
 \end{array}$$

B. To multiply a number by 1.25, divide the number by 8 and multiply by 10. This is the same as multiplying by which is equivalent to $1.25 (1\frac{1}{4})$

Example: 328×1.25

Solution: $328 \div 8 = 41$; $\times 10 = 410$, answer

Proving the answer:

$$\begin{array}{r}
 328 \\
 \times 1.25 \\
 \hline
 1640 \\
 656 \\
 328 \\
 \hline
 410.00
 \end{array}$$

C. To multiply a number by 12.5, divide the number by 8 and multiply by 100. This is the same as multiplying by $\frac{100}{8}$ which is equivalent to $12.5 (12\frac{1}{2})$

Example: 48×12.5

Solution: $48 \div 8 = 6$; $\times 100 = 600$, answer

Proving the answer:

$$\begin{array}{r}
 48 \\
 \times 12.5 \\
 \hline
 240 \\
 96 \\
 48 \\
 \hline
 600.0
 \end{array}$$

D. To multiply a number by 125, divide the number by 8 and multiply by 1,000. This is the same as multiplying by $\frac{1000}{8}$ which is equivalent to 125.

Example: 728×125

Solution: $728 \div 8 = 91$; $\times 1,000 = 91,000$, answer

Proving the answer:

$$\begin{array}{r}
 728 \\
 \times 125 \\
 \hline
 3640 \\
 1456 \\
 728 \\
 \hline
 91000
 \end{array}$$

E. *Alternate methods* of multiplying by .125, 1.25, 12.5, and 125.

1. To multiply a number by .125, instead of dividing by 8, divide by 2 and then 4, or divide by 2, then 2, and by 2 again.

Example:

$$432 \times .125: 432 \div 2 = 216; \div 2 = 108; \div 2 = 54, \text{ answer}$$

2. To multiply a number by 1.25, add $\frac{1}{4}$ of the number to the number.

Example: $168 \times 1.25 = 168 + 42 = 210$, answer

3. To multiply a number by 12.5, multiply by 10 and add $\frac{1}{4}$ of the product to the product.

Example: $48 \times 12.5 = 480 + 120 = 600$, answer

4. To multiply a number by 125, multiply by 100 and add $\frac{1}{4}$ of the product to the product.

Example: $728 \times 125 = 72,800 + 18,200 = 91,000$, answer

4-10. To Multiply by .75, 7.5, 75, or 750

While .75, 7.5, 75, and 750 are not evenly divisible into 1, 10, 100, or 1,000 and are therefore not aliquot parts of these numbers, they are included here because they are closely related to the aliquot parts just discussed and can be converted into easy multipliers by similar steps.

The general method for this group of multipliers is to multiply by $\frac{3}{4}$ and then by 10, 100, or 1,000. (No need to multiply by 1 as it will not change the number multiplied.)

A. To multiply a number by .75, multiply the number by $\frac{3}{4}$ which is equivalent to $.75 (\frac{75}{100})$.

Example: $64 \times .75$

Solution: $64 \times \frac{3}{4} = 48$; or, $64 \div 4 = 16$; $\times 3 = 48$, answer

Proving the answer:

$$\begin{array}{r} 64 \\ \times .75 \\ \hline 320 \\ 448 \\ \hline 48.00 \end{array}$$

B. To multiply a number by 7.5, divide the number by 4 and multiply by 30

(or by 3 and then by 10). This is the same as multiplying by $\frac{30}{4}$ which is equivalent to 7.5 ($7\frac{1}{2}$).

Example: 92×7.5

Solution: $92 \div 4 = 23$; $\times 3 = 69$; $\times 10 = 690$, answer

Proving the answer:

$$\begin{array}{r} 92 \\ \times 7.5 \\ \hline 460 \\ 644 \\ \hline 690.0 \end{array}$$

C. To multiply a number by 75, divide the number by 4 and multiply by 300 (or by 3 and then by 100). This is the same as multiplying by $\frac{300}{4}$ which is equivalent to 75.

Example: 48×75

Solution: $48 \div 4 = 12$; $\times 3 = 36$; $\times 100 = 3,600$, answer

Proving the answer:

$$\begin{array}{r} 48 \\ \times 75 \\ \hline 240 \\ 336 \\ \hline 3600 \end{array}$$

D. To multiply a number by 750, divide the number by 4 and multiply by 3,000 (or by 3 and then by 1,000). This is the same as multiplying by $\frac{3000}{4}$ which is equivalent to 750.

Example: 648×750

Solution:

$$648 \div 4 = 162; \times 3 = 486; \times 1,000 = 486,000, \text{ answer}$$

Proving the answer:

$$\begin{array}{r} 648 \\ \times 750 \\ \hline 32400 \\ 4536 \\ \hline 486000 \end{array}$$

E. If the number to be multiplied is not evenly divisible by 4, there will be a remainder of $\frac{1}{4}$ (.25), $\frac{1}{2}$ (.5), or $\frac{3}{4}$ (.75) but the same steps apply.

Examples:

$$\begin{array}{l} 21 \times .75: 21 \div 4 = 5\frac{1}{4}; \times 3 = 15\frac{3}{4} \text{ or } 15.75 \\ 22 \times 7.5: 22 \div 4 = 5\frac{1}{2}; \times 3 = 16\frac{1}{2} = 16.5; \times 10 = 165 \\ 23 \times 75: 23 \div 4 = 5\frac{3}{4}; \times 3 = 17\frac{1}{4} = 17.25; \times 100 = 1,725 \\ 26 \times 750: 26 \div 4 = 6\frac{1}{2}; \times 3 = 19\frac{1}{2} = 19.5; \times 1,000 = 19,500 \end{array}$$

F. *Alternate method* of multiplying by .75, 7.5, 75, and 750.

1. To multiply a number by .75, halve the number and to this add $\frac{1}{2}$ of the half.

Example: $64 \times .75 = 32 + 16 = 48, \text{ answer}$

2. To multiply a number by 7.5, halve the number, to this add $\frac{1}{2}$ of the half, and multiply by 10.

Example: $92 \times 7.5: 46 + 23 = 69; \times 10 = 690, \text{ answer}$

3. To multiply a number by 75, halve the number, to this add $\frac{1}{2}$ of the half, and multiply by 100.

Example:

$$48 \times 75: 24 + 12 = 36; \times 100 = 3,600, \text{ answer}$$

4. To multiply a number by 750, halve the number, to this add $\frac{1}{2}$ of the half, and multiply by 1,000.

Example:

$$648 \times 750: 324 + 162 = 486; \times 1,000 = 486,000, \text{ answer}$$

G. A *third method* of multiplying by .75, 7.5, 75, and 750.

1. To multiply a number by .75, subtract $\frac{1}{4}$ of the number.

Example: $64 \times .75 = 64 - 16 = 48, \text{ answer}$

2. To multiply a number by 7.5, subtract $\frac{1}{4}$ of the number and multiply by 10.

Example: $92 \times 7.5: 92 - 23 = 69; \times 10 = 690, \text{ answer}$

3. To multiply a number by 75, subtract $\frac{1}{4}$ of the number and multiply by 100.

Example: $48 \times 75: 48 - 12 = 36; \times 100 = 3,600, \text{ answer}$

4. To multiply a number by 750, subtract $\frac{1}{4}$ of the number and multiply by 1,000.

Example: $648 \times 750: 648 - 162 = 486; \times 1,000$

$= 486,000, \text{ answer}$

Summary of Multiplication by Aliquot Parts and 75-Numbers

N stands for the number to be multiplied

To Multiply by Method

.5	$N/2 \quad (N \div 2)$
5	$N/2 \times 10$
50	$N/2 \times 100$
500	$N/2 \times 1,000$
.25	$N/4$
2.5	$N/4 \times 10$
25	$N/4 \times 100$
250	$N/4 \times 1,000$
.125	$N/8$
1.25	$N/8 \times 10$
12.5	$N/8 \times 100$
125	$N/8 \times 1,000$
.75	$N/4 \times 3$
7.5	$N/4 \times 3 \times 10$
75	$N/4 \times 3 \times 100$
750	$N/4 \times 3 \times 1,000$

Other aliquot parts: Multiplication by other aliquot parts in the form of mixed numbers (whole numbers with fractions, such as $12\frac{1}{2}$, $16\frac{2}{3}$, and $33\frac{1}{3}$) is discussed in Chapter VI.

To Obtain the Squares of Certain Numbers

As you probably know, the “square” of a number is simply the product obtained by multiplying the number by itself. Thus, 16 is the square of 4 since $4 \times 4 = 16$.

4-11. To Square a Number Ending in 1

Example: 41×41

Solution by steps:

- (a) Square the next lower number (always ends in zero): $40 \times 40 = 1,600$
- (b) Obtain the sum of the next lower number and the number being squared:
 $40 + 41 = 81$
- (c) Add (a) and (b): $1,600 + 81 = 1,681$, answer

Short way: $40 \times 40 = 1,600$; $40 + 41 = 81$; answer 1,681

Proving the answer:

$$\begin{array}{r} 41 \\ \times 41 \\ \hline 41 \\ 164 \\ \hline 1681 \end{array}$$

Other examples:

$$\begin{array}{l} 21 \times 21 = 441 \\ 31 \times 31 = 961 \\ 61 \times 61 = 3,721 \end{array}$$

4-12. To Square a Number Ending in 4

Example: 34×34

Solution by steps:

- (a) Square the next higher number: $35 \times 35 = 1,225$ (see Method 4-13)
- (b) Obtain the sum of the number being squared and the next higher number:
 $34 + 35 = 69$
- (c) Subtract (b) from (a): $1,225 - 69 = 1,156$, answer (simplify your subtraction by adding 1 to both numbers: $1,226 - 70 = 1,156$)

Short way: $35 \times 35 = 1,225$; $-(34 + 35) = 1,156$

Proving the answer:

$$\begin{array}{r} 34 \\ \times 34 \\ \hline 136 \\ 102 \\ \hline 1156 \end{array}$$

Why it works: How the square of a number equals the square of the next higher number minus the sum of the higher number and the number being

squared is shown in the following:

$$5 \times 5 : (6 \times 6) - (6 + 5) = 36 - 11 = 25$$

$$4 \times 4: (5 \times 5) - (5 + 4) = 25 - 9 = 16$$

$$3 \times 3: (4 \times 4) - (4 + 3) = 16 - 7 = 9$$

$$2 \times 2: (3 \times 3) - (3 + 2) = 9 - 5 = 4$$

$$1 \times 1: (2 \times 2) - (2 + 1) = 4 - 3 = 1$$

4-13. To Square a Number Ending in 5

Example : 35×35

Solution by steps:

(a) Multiply the first digit by the digit plus 1: $3 \times 4 = 12$

(b) Attach 25: 1,225, answer

Short way: $3 \times 4 = 12$; answer 1,225

Proving the answer:

$$\begin{array}{r} 35 \\ \times 35 \\ \hline 175 \\ 105 \\ \hline 1225 \end{array}$$

Other examples:

$$4.5 \times 4.5 = 20.25$$

$$65 \times 65 = 4,225$$

$$.75 \times .75 = .5625$$

$$115 \times 115 = 13,225 \text{ (treat 11 like a "first digit")}$$

4-14. To Square a Number Ending in 6

This works by the same steps as for numbers ending in 1 (Method 4-11).

Example: 36×36

Solution by steps:

(a) Square the next lower number: $35 \times 35 = 1,225$ (by Method 4-13)

(b) Obtain the sum of the next lower number and the number being squared:

$$35 + 36 = 71$$

(c) Add (a) and (b): $1,225 + 71 = 1,296$, answer

Short way: $35 \times 35 = 1,225$; $+ (35 + 36) = 1,296$

Proving the answer:

$$\begin{array}{r} 36 \\ \times 36 \\ \hline 216 \\ 108 \\ \hline 1296 \end{array}$$

Why it works: How the square of a number equals the square of the next lower number plus the sum of the lower number and the number being squared is shown in the following:

$$2 \times 2: (1 \times 1) + (1 + 2) = 1 + 3 = 4$$

$$3 \times 3: (2 \times 2) + (2 + 3) = 4 + 5 = 9$$

$$4 \times 4: (3 \times 3) + (3 + 4) = 9 + 7 = 16$$

$$5 \times 5: (4 \times 4) + (4 + 5) = 16 + 9 = 25$$

$$6 \times 6: (5 \times 5) + (5 + 6) = 25 + 11 = 36$$

4-15. To Square a Number Ending in 9

This works by the same steps as for numbers ending in 4 (Method 4-12).

Example: 39×39

Solution by steps:

(a) Square the next higher number: $40 \times 40 = 1,600$

(b) Obtain the sum of the number being squared and the next higher number:

$$39 + 40 = 79$$

(c) Subtract (b) from (a): $1,600 - 79 = 1,521$, answer

Short way: $40 \times 40 = 1,600$; $-(39 + 40) = 1,521$

Proving the answer:

$$\begin{array}{r} 39 \\ \times 39 \\ \hline 351 \\ 117 \\ \hline 1521 \end{array}$$

Other examples:

$$\begin{array}{l} 19 \times 19 = 361 \\ 49 \times 49 = 2,401 \\ 79 \times 79 = 6,241 \end{array}$$

Summary of Squaring Certain Numbers

N stands for the Number to be squared

Number ends in 1 or 6: $(N - 1) \times (N - 1) + (N + N - 1)$

Number ends in 4 or 9: $(N + 1) \times (N + 1) - (N + N + 1)$

Number ends in 5 (two-digit N): 1st digit \times (1st digit + 1); attach 25

Number ends in 5 (three-digit N): 1st two digits \times (1st two digits + 1); attach 25

Multiplying Numbers That Have Small Differences

4-16. To Multiply Two Numbers with a Difference of 1

Square either number, whichever is easier. If you square the larger number, subtract it from its square; if you square the smaller number, add it to its square.

Example (1): 24×23

Solution by steps:

(a) Square the larger number; this is easy to do by Method 4-12: $24 \times 24 = 576$

(b) Subtract the larger number: $576 - 24 = 552$, answer *Short way:* $24 \times 24 = 576$; $- 24 = 552$

Proving the answer:

$$\begin{array}{r} 24 \\ \times 23 \\ \hline 72 \\ 48 \\ \hline 552 \end{array}$$

Why it works: Squaring a number and subtracting the number from its square is equivalent to multiplying the number by the number minus 1.

Example (2): 35×36

Solution by steps:

(a) Square the smaller number; this is easy to do by Method 4-13: $35 \times 35 = 1,225$

(b) Add the smaller number: $1,225 + 35 = 1,260$, answer

Short way: $35 \times 35 = 1,225$; $+ 35 = 1,260$

Proving the answer:

$$\begin{array}{r} 35 \\ \times \underline{36} \\ 210 \\ \underline{105} \\ 1260 \end{array}$$

Why it works: Squaring a number and adding the number to its square is equivalent to multiplying the number by the number plus 1.

Other examples:

Squaring the higher number: $40 \times 39 = 1,600 - 40 = 1,560$

Squaring the lower number: $40 \times 41 = 1,600 + 40 = 1,640$

4-17. To Multiply Two Numbers with a Difference of 2

Square the average of the two numbers and subtract 1.

Example: 19×21

Solution by steps:

(a) The average is the middle or “in-between” number: 20

(b) Square the average: $20 \times 20 = 400$

(c) Subtract 1: $400 - 1 = 399$, answer

Short way: $20 \times 20 = 400$; $-1 = 399$

Proving the answer:

$$\begin{array}{r}
 19 \\
 \times 21 \\
 \hline
 19 \\
 38 \\
 \hline
 399
 \end{array}$$

Why it works: If you multiply two small numbers with a difference of 2, you can see how the method works. For example, 8×6 . The average of these numbers is 7 and the problem could be written as $(7 + 1) \times (7 - 1)$. By cross multiplying, here is how it works out:

$$\begin{array}{r}
 7 + 1 \\
 \times 7 - 1 \\
 \hline
 49 + 7 \\
 - 7 - 1 \\
 \hline
 49 \quad - 1
 \end{array}$$

The +7 and —7 cancel each other, leaving $49 - 1$ which equals 48, the product of 8×6 . Thus, the square of the average (7×7) minus 1 equals 8×6 .

4-18. To Multiply Two Numbers with a Difference of 3

Example: 27×24

Solution by steps:

- (a) Add 1 to the smaller number and square the sum: $24 + 1 = 25$; $\times 25 = 625$
(by Method 4-13)
- (b) Take 1 from the smaller number and add to (a): $24 - 1 = 23$; $+ 625 = 648$,
answer

Short way: $25 \times 25 = 625$; $+ 23 = 648$

Proving the answer:

$$\begin{array}{r} 27 \\ \times 24 \\ \hline 108 \\ 54 \\ \hline 648 \end{array}$$

Why it works: If you multiply two small numbers with a difference of 3, you can see how the method works. For example, 8×11 . $(8 + 1) = 9$; $9 \times 9 = 81$; $(8 - 1) = 7$; $+ 81 = 88$

Other examples:

$$\begin{aligned} 92 \times 89 &= 8,100 + 88 = 8,188 \\ 57 \times 54 &= 3,025 + 53 = 3,078 \end{aligned}$$

4-19. To Multiply Two Numbers with a Difference of 4

Square the average of the two numbers and subtract 4.

Example: 67×63

Solution by steps:

- (a) Average: 65 ($67 - 2$ or $63 + 2$)
- (b) $65 \times 65 = 4,225$ (by Method 4-13)
- (c) $4,225 - 4 = 4,221$, answer

Short way: $65 \times 65 = 4,225$; $-4 = 4,221$

Proving the answer:

$$\begin{array}{r}
 67 \\
 \times 63 \\
 \hline
 201 \\
 402 \\
 \hline
 4221
 \end{array}$$

Why it works: If you multiply two small numbers with a difference of 4, you can see how the method works.

Example:

$$\begin{aligned}
 &5 \times 9; \text{ average, } 7; \times 7 = 49; -4 = 45 \\
 &\text{Or, consider } 5 \text{ as } (7 - 2) \text{ and } 9 \text{ as } (7 + 2); \\
 &(7 - 2) \times (7 + 2) = 49 - 4 = 45
 \end{aligned}$$

Other examples:

$$\begin{aligned}
 &78 \times 82 = 6,400 - 4 = 6,396 \\
 &63 \times 59 = 3,721 - 4 = 3,717 \\
 &(61 \times 61 = 3,721 \text{ by Method 4-11})
 \end{aligned}$$

4-20. To Multiply Two Numbers with a Difference of 6

Square the average of the two numbers and subtract 9.

Example: 34×28

Solution by steps:

(a) Average: 31 ($34 - 3$ or $28 + 3$)

(b) $31 \times 31 = 961$ (by Method 4-11)

(c) $961 - 9 = 952$, answer

Short way: $31 \times 31 = 961$; $- 9 = 952$

Why it works: If you multiply two small numbers with a difference of 6, you can see how the method works.

Example:

2×8 ; average, 5; $\times 5 = 25$; $- 9 = 16$

Or, consider 2 as $(5 - 3)$ and 8 as $(5 + 3)$;

$(5 - 3) \times (5 + 3) = 25 - 9 = 16$

Other examples:

$83 \times 77 = 6,400 - 9 = 6,391$

$48 \times 42 = 2,025 - 9 = 2,016$

Summary of Multiplying Numbers Having Small Differences

The following table is based on 10 and numbers near 10

Difference of 1: $10 \times 9 = 90 = (10 \times 10) - 10 = 90$

Difference of 1: $10 \times 9 = 90 = (9 \times 9) + 9 = 90$

Difference of 2: $11 \times 9 = 99 = (10 \times 10) - 1 = 99$

Difference of 3: $11 \times 8 = 88 = (8 + 1) \times (8 + 1)$

$+ (8 - 1) = (9 \times 9) + 7 = 88$

Difference of 4: $12 \times 8 = 96 = (10 \times 10) - 4 = 96$

Difference of 6: $13 \times 7 = 91 = (10 \times 10) - 9 = 91$

Multiplying by Numbers Ending in 1

4-21. General Method for Multiplying by Numbers Ending in 1

Since $11 = 10 + 1$, $21 = 20 + 1$, etc., you can multiply by 11, 21, etc., more easily if you multiply by 10, 20, etc., and then add the number multiplied. The number you add is the “+1” of $10 + 1$, $20 + 1$, and so on.

Example:

$$67 \times 11$$

Solution:

$$67 \times (10 + 1) = 670 + 67 = 737, \text{ answer}$$

Other examples:

$$36 \times 21 = 720 + 36 = 756$$

$$27 \times 31 = 810 + 27 = 837$$

$$49 \times 101 = 4,900 + 49 = 4,949$$

Further details and alternate short-cuts for multiplying by 1-numbers will be found in the following methods.

4-22. To Multiply a Number by 11

Method A: This is the method explained in 4-21. Multiply the number by 10 and add the number.

Method B: Write the number to be multiplied and add 10 times the number.

Example: 369×11

Solution:

$$\begin{array}{r} 369 \\ 3690 \\ \hline 4059 \end{array}$$

Also:
$$\begin{array}{r} 369 \\ 369 \\ \hline 4059 \end{array}$$

Method C: To multiply a two-digit number by 11, add the number's digits and insert their sum between the digits.

Example: 27×11

Solution by steps:

(a) Sum of the number's digits: $2 + 7 = 9$

(b) Insert the 9 between the 2 and 7: 297, answer

Short way: $2 + 7 = 9$; answer 297

Other examples:

$$43 \times 11 = 473$$

$$72 \times 11 = 792$$

$$85 \times 11 = 935 \text{ (} 8 + 5 = 13; \text{ place 3 in the middle; carry 1 to 8; becomes 9)}$$

4-23. To Multiply a Number by 21

Double the number, multiply this by 10, and add the number.

Example: 67×21

Solution:

$$2 \times 67 = 134; \times 10 = 1,340; + 67 = 1,407, \text{ answer}$$

Proving the answer:

$$\begin{array}{r} 67 \\ \times 21 \\ \hline 67 \\ 134 \\ \hline 1407 \end{array}$$

4-24. To Multiply a Number by 31, 41, 51, etc.

Multiply the number by the first digit of the multiplier, multiply the result by 10, and add the number.

Example: 24×31

Solution by steps:

(a) Multiply 24 by the first digit of 31: $24 \times 3 = 72$

(b) $72 \times 10 = 720; + 24 = 744, \text{ answer}$

Short way: $24 \times 3 = 72$; $\times 10 = 720$; $+ 24 = 744$

Proving the answer:

$$\begin{array}{r} 24 \\ \times 31 \\ \hline 24 \\ 72 \\ \hline 744 \end{array}$$

Multiplying by Numbers Ending in 5

4-25. To Multiply by 15, 25, 35, 45, etc. Halve-and-Double Method

Halve the number to be multiplied and double the multiplier. This changes the multiplier ending in 5 to a multiplier ending in zero.

Example: 28×15

Solution:

Half of 28 = 14; double 15 = 30; $\times 14 = 420$, answer

Proving the answer:

$$\begin{array}{r}
 28 \\
 \times 15 \\
 \hline
 140 \\
 28 \\
 \hline
 420
 \end{array}$$

Why it works: In multiplication, halving one number (dividing it by 2) and doubling the other number (multiplying it by 2) does not change the product. This is easy to see in an example using small numbers, such as $4 \times 6 = 24$. Double 4 = 8; halve 6 = 3; $8 \times 3 = 24$. Or, halve 4 = 2; double 6 = 12; $2 \times 12 = 24$, again.

Other examples:

$$\begin{aligned}
 64 \times 35 &= 32 \times 70 = 2,240 \\
 22 \times 45 &= 11 \times 90 = 990 \\
 48 \times 3.5 &= 24 \times 7 = 168
 \end{aligned}$$

If the number to be multiplied is odd, the method still applies.

Example: 17×45

Solution:

$$\text{Half of } 17 = 8\frac{1}{2}; \text{ double } 45 = 90; \times 8\frac{1}{2} = 765, \text{ answer}$$

Sometimes you can halve-and-double twice to simplify the multiplication further.

Example: 175×24

Solution: $350 \times 12 = 700 \times 6 = 4,200$, answer

4-26. Alternate Method for Multiplying a Number by 15

Add half the number to itself and multiply by 10.

Example (1): 64×15

Solution: $64 + 32 = 96$; $\times 10 = 960$, answer

Proving the answer:

$$\begin{array}{r} 64 \\ \times 15 \\ \hline 320 \\ 64 \\ \hline 960 \end{array}$$

Example (2): 65×15

Solution: $65 + 32\frac{1}{2} = 97\frac{1}{2} = 97.5$; $\times 10 = 975$, answer

Why it works: Adding half the number to itself is the same as multiplying the number by $1\frac{1}{2}$; then multiplying this by 10 is the same as multiplying the original number by 15.

4-27. Alternate Method for Multiplying a Number by 45

Multiply the number by 50 and subtract $\frac{1}{10}$ of the product.

Example: 36×45

Solution by steps:

(a) $36 \times 50 = 1,800$ ($36 \div 2 = 18$; $\times 100 = 1,800$ by Method 4-7C)

(b) $\frac{1}{10} \times 1,800 = 180$

(c) $1,800 - 180 = 1,620$, answer

Short way: $36 \times 50 = 1,800$; $- 180 = 1,620$

Why it works: Multiplying by 50 and subtracting $\frac{1}{10}$ of the product is the same as multiplying by $(50 - 5)$ which is equivalent to 45.

4-28. Alternate Method for Multiplying a Number by 55

Multiply the number by 50 and add $\frac{1}{10}$ of the product.

Example: 48×55

Solution by steps:

(a) $48 \times 50 = 2,400$ ($48 \div 2 = 24$; $\times 100 = 2,400$)

(b) $\frac{1}{10} \times 2,400 = 240$

(c) $2,400 + 240 = 2,640$, answer

Short way: $48 \times 50 = 2,400$; $+ 240 = 2,640$

Why it works: Multiplying by 50 and adding $\frac{1}{10}$ of the product is the same as multiplying by $(50 + 5)$ which is equivalent to 55.

4-29. To Multiply One Number by Another When They Both End in 5 and the Sum of Their Other Digits is Even

Example: 35×55

Solution by steps:

(a) Multiply the first digits: $3 \times 5 = 15$

(b) Obtain half the sum of the first digits: $3 + 5 = 8; \times \frac{1}{2} = 4$

(c) Add (a) and (b): $15 + 4 = 19$

(d) Attach 25 to (c): 1,925, answer

Short way: $3 \times 5 = 15; + \frac{8}{2} = 19$; answer 1,925

Proving the answer:

$$\begin{array}{r} 35 \\ \times 55 \\ \hline 175 \\ 175 \\ \hline 1925 \end{array}$$

Why it works:

(a) Multiplying 3 by 5 is really $30 \times 50 = 1,500$

(b) Adding 4 to 15 is really adding 400 to $1,500 = 1,900$

(c) The 25 attached to 19 is the product of 5×5 : answer 1,925

Other examples:

$$\begin{array}{l} 85 \times 45 = 3,825 \\ 4.5 \times 6.5 = 29.25 \\ 75 \times .35 = 26.25 \\ 125 \times 85 = 10,625 \end{array}$$

4-30. To Multiply One Number by Another When They Both End in 5 and the Sum of Their Other Digits Is Odd

This follows the same steps as Method 4-29. However, there is a fraction, which is dropped, and the answer ends in 75 instead of 25.

Example: 65×35

Solution by steps:

(a) Multiply the first digits: $6 \times 3 = 18$

(b) Obtain half the sum of the first digits: $6 + 3 = 9$; $\times \frac{1}{2} = 4\frac{1}{2}$; drop the $\frac{1}{2}$ to get 4

(c) Add (a) and (b): $18 + 4 = 22$

(d) Attach 75 to (c): 2,275, answer

Proving the answer:

$$\begin{array}{r} 65 \\ \times 35 \\ \hline 325 \\ 195 \\ \hline 2275 \end{array}$$

Why it works:

(a) Multiplying 6 by 3 is really $60 \times 30 = 1,800$

(b) The $4\frac{1}{2}$ is really 450; dropping the $\frac{1}{2}$ is dropping 50

(c) Adding 4 to 18 is really adding 400 to 1,800: $= 2,200$; this is 50 too little because of the 50 dropped in (b)

(d) The 75 attached to 22 (really added to 2,200) is the product of 5×5 plus the 50 dropped in (b): answer 2,275

Other examples:

$$\begin{array}{l} 75 \times 45 = 3,375 \\ 3.5 \times 6.5 = 22.75 \\ .95 \times 85 = 80.75 \\ 115 \times 125 = 14,375 \end{array}$$

Multiplying by 9, by Numbers Ending in 9, and by Multiples of 9

The general method for multiplying by these numbers is to increase them to the next number ending in zero; then, after multiplying, to subtract the excess amount from the product, as shown in the following methods.

4-31. To Multiply a Number by 9

Multiply the number by 10 and subtract the number.

Example: 37×9

Solution: $37 \times 10 = 370$; $— 37 = 333$, answer

Proving the answer:

$$\begin{array}{r} 37 \\ \times \underline{9} \\ 333 \end{array}$$

Why it works: Multiplying a number by 10 and subtracting the number is the same as multiplying by $(10 - 1)$ which is equivalent to 9. You get the same result if you subtract $\frac{1}{10}$ of the product after multiplying the number by 10.

4-32. To Multiply a Number by 19, 29, 39, etc.

Add 1 to the multiplier (19, 29, etc.) to make it end in zero; then multiply the given number, and subtract the number from the result.

Example: 24×19

Solution: $19 + 1 = 20$; $\times 24 = 480$; $— 24 = 456$, answer

Proving the answer:

$$\begin{array}{r} 24 \\ \times \underline{19} \\ 216 \\ \underline{24} \\ 456 \end{array}$$

Why it works: Multiplying a number by 20 and subtracting the number is the same as multiplying by $(20 - 1)$ which is equivalent to 19. Other numbers ending in 9 work the same way: $29 = (30 - 1)$, $39 = (40 - 1)$, etc.

Other examples:

$$\begin{aligned}36 \times 29 &= 1,044 \\48 \times 59 &= 2,832 \\2.1 \times 69 &= 144.9\end{aligned}$$

4-33. To Multiply by a Multiple of 9 from 18 to 81

This refers to multiplying by 18, 27, 36, 45, 54, 63, 72, and 81 which are multiples of 9 produced by 9×2 , 9×3 , 9×4 , etc.

Example: 35×27

Solution by steps:

- (a) Increase the multiple of 9 to the next number ending in zero. Thus, increase 27 to 30
- (b) Multiply: $35 \times 30 = 1,050$
- (c) Subtract $\frac{1}{10}$ of the product: $1,050 - 105 = 945$, answer

Short way: $35 \times 30 = 1,050$; $- 105 = 945$

Proving the answer:

$$\begin{array}{r}35 \\ \times 27 \\ \hline 245 \\ 70 \\ \hline 945\end{array}$$

Why it works: Multiplying a number by 30 and subtracting $\frac{1}{10}$ of the product is the same as multiplying the number by $(30 - 3)$ which is equivalent to 27. The other multiples of 9 work the same way: $18 = 20 - 2$; $36 = 40 - 4$; $45 = 50 - 5$; $54 = 60 - 6$, etc.

Other examples:

$$\begin{aligned} 65 \times 18 &= 1,170 \\ 92 \times 36 &= 3,312 \\ 2.4 \times 6.3 &= 15.12 \end{aligned}$$

4-34. To Multiply a Number by 99

Multiply the number by 100 and subtract the number.

Example: 83×99

Solution: $83 \times 100 = 8,300$; $- 83 = 8,217$

Why it works: Multiplying a number by 100 and subtracting the number is the same as multiplying the number by $(100 - 1)$ which is equivalent to 99.

Other examples:

$$\begin{aligned} 46 \times 99 &= 4,554 \\ 3.7 \times 99 &= 366.3 \\ 248 \times 99 &= 24,552 \end{aligned}$$

4-35. To Multiply by a Multiple of 99

This refers to multiplying by 198, 297, 396, 495, 594, 693, 792, 891, and 990 which are multiples of 99 produced by 99×2 , 99×3 , 99×4 , etc.

Example: 13×297

Solution by steps:

(a) Increase the multiple of 99 to the next hundred. Thus, increase 297 to 300

(b) Multiply: $13 \times 300 = 3,900$

(c) Subtract $\frac{1}{100}$ of the product: $3,900 - 39 = 3,861$, answer

Proving the answer:

$$\begin{array}{r}
 297 \\
 \times \underline{13} \\
 891 \\
 \underline{297} \\
 3861
 \end{array}$$

Why it works: Multiplying a number by 300 and subtracting $\frac{1}{100}$ of the product is the same as multiplying the number by $(300 - 3)$ which is equivalent to 297. The other multiples of 99 work the same way: $198 = 200 - 2$; $396 = 400 - 4$; $495 = 500 - 5$, etc.

Other examples:

$$\begin{array}{l}
 24 \times 198 = 4,752 \\
 62 \times 396 = 24,552 \\
 .71 \times 4.95 = 3.5145
 \end{array}$$

Multiplying by Numbers Near 100

The general method for simplifying multiplication by numbers near 100 is to change the number into an easy multiplier consisting of 100 and a small number, as follows: change 98 to $100 - 2$; 99 to $100 - 1$; 101 to $100 + 1$; 102 to $100 + 2$, etc.

4-36. To Multiply a Number by 98

Multiply the number by 100 and subtract twice the number.

Example: 64×98

Solution:

$$64 \times 100 = 6,400; 64 \times 2 = 128; 6,400 - 128 = 6,272, \text{ answer}$$

Proving the answer:

$$\begin{array}{r} 64 \\ \times 98 \\ \hline 512 \\ 576 \\ \hline 6272 \end{array}$$

Why it works: Multiplying a number by 100 and subtracting twice the number is the same as multiplying the number by $(100 - 2)$ which is equivalent to 98. Other numbers near 100 work the same way.

To Multiply a Number by 99:

Multiply the number by 100 and subtract the number, as previously described in Method 4-34.

4-37. To Multiply a Number by 101

Multiply the number by 100 and add the number.

Example: 64×101

Solution: $64 \times 100 = 6,400$; $+ 64 = 6,464$, answer

To multiply any number from 10 to 99 by 101, simply write the number twice.

Examples:

$$\begin{array}{l} 24 \times 101 = 2,424 \\ 76 \times 101 = 7,676 \\ 93 \times 101 = 9,393 \end{array}$$

4-38. To Multiply a Number by 102

Multiply the number by 100 and add twice the number.

Example: 46×102

Solution:

$$46 \times 100 = 4,600; 46 \times 2 = 92; + 4,600 = 4,692, \text{ answer}$$

Proving the answer:

$$\begin{array}{r} 102 \\ \times \quad 46 \\ \hline 612 \\ 408 \\ \hline 4692 \end{array}$$

Other Short-Cuts in Multiplication

4-39. To Multiply One “Teen” Number by Another

Actually, this method can be used with 11 and 12 as well as the “teen” numbers from 13 to 19.

Example: 16×13

Solution by steps:

(a) Add either number to the units digit of the other: $16 + 3$ (or $13 + 6$) = 19

(b) Multiply by 10: $19 \times 10 = 190$

(c) Multiply the units digits: $6 \times 3 = 18$

(d) Add (b) and (c): $190 + 18 = 208$, answer

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Short way: $16 + 3 = 19$; $\times 10 = 190$; $+ (6 \times 3) = 208$

Proving the answer:

$$\begin{array}{r} 16 \\ \times 13 \\ \hline 48 \\ 160 \\ \hline 208 \end{array}$$

Other examples:

$$\begin{array}{l} 14 \times 19 = 266 \\ 18 \times 17 = 306 \\ 15 \times 13 = 195 \end{array}$$

If the last two digits add to 10, simply obtain the product of the last two digits and place a 2 in front of it.

Example: 17×13

Solution: $7 \times 3 = 21$; answer 221

Other examples:

$$\begin{array}{l} 19 \times 11 = 209 \text{ (zero must be inserted)} \\ 18 \times 12 = 216 \\ 16 \times 14 = 224 \\ 15 \times 15 = 225 \text{ (also by Method 4-13)} \end{array}$$

The next method extends this idea to use with other numbers and shows that the 2 in the above examples is derived by multiplying $1 \times (1+1)$, or 1×2 .

4-40. To Multiply Two Numbers When Their End Digits Add to 10 and Their Other Digits Are the Same

Example: 74×76

Solution by steps:

(a) Multiply the first digit by the first digit + 1: $7 \times 8 = 56$

(b) Multiply the end digits: $4 \times 6 = 24$

(c) Combine (a) and (b): 5,624, answer

Short way: $7 \times 8 = 56$; $4 \times 6 = 24$; answer 5,624

Proving the answer:

$$\begin{array}{r} 74 \\ \times 76 \\ \hline 444 \\ 518 \\ \hline 5624 \end{array}$$

Why it works: Multiplying 7 by 8 is really $70 \times 80 = 5,600$. Why multiply by 80? Because there are seventy 70's plus ten 70's from multiplying by 6 and by 4. The 24 in the answer is, of course, the product of 4×6 .

You can get a better look at how this works by using no-carry multiplication (Method 4-4), as follows:

$$\begin{array}{r} 74 \\ \times 76 \\ \hline 4900 \text{ (} 70 \times 70 \text{)} \\ 280 \text{ (} 70 \times 4 \text{)} \\ 420 \text{ (} 6 \times 70 \text{)} \\ 24 \text{ (} 6 \times 4 \text{)} \\ \hline 5624 \end{array} \quad \left. \begin{array}{l} 280 \\ 420 \end{array} \right\} = 10 \times 70 = 700$$

This can be shortened to:

$$\begin{array}{rcl}
 70 \times 70 = 4900 & \} & \\
 10 \times 70 = 700 & \} & = 80 \times 70 = 5,600 \\
 6 \times 4 = 24 & & \\
 \hline
 & & 5624
 \end{array}$$

And shortened again to:

$$(80 \times 70) + (6 \times 4) = 5,600 + 24 = 5,624$$

And finally to: $8 \times 7 = 56$; attach 24; = 5,624

Each set of numbers in the teens, 20's, 30's, 40's, etc., contains five pairs of numbers that can be multiplied by this method. For example, in the 60's:

$$69 \times 61 = 4,209 \text{ (zero must be inserted)}$$

$$68 \times 62 = 4,216$$

$$67 \times 63 = 4,221$$

$$66 \times 64 = 4,224$$

$$65 \times 65 = 4,225 \text{ (same as by Method 4-13)}$$

These pairs, of course, can also be reversed since $69 \times 61 = 61 \times 69$, $68 \times 62 = 62 \times 68$, etc.

The idea can be extended to larger numbers too, with the first two digits on the left treated as a "first digit."

Example: 113×117

Solution:

$$11 \times (11 + 1) = 11 \times 12 = 132; 3 \times 7 = 21; \text{ answer } 13,221$$

Other examples:

$$254 \times 256 = 65,024$$

$$418 \times 412 = 172,216$$

4-41. To Multiply Two Two-Digit Numbers When Their First Digits Add to 10 and Their End Digits Are the Same

Example: 36×76

Solution by steps:

(a) Multiply the first digits and add an end digit: $3 \times 7 = 21; + 6 = 27$

(b) Multiply the end digits: $6 \times 6 = 36$

(c) Combine (a) and (b): 2,736, answer

Short way: $(3 \times 7) + 6 = 27; 6 \times 6 = 36; \text{ answer } 2,736$

Proving the answer:

$$\begin{array}{r} 36 \\ \times 76 \\ \hline 216 \\ 252 \\ \hline 2736 \end{array}$$

Further examples are given in the following complete set of numbers in the 20's paired with numbers in the 80's having the same end digits. Notice that a zero must be inserted in the tens place of the first four products.

$$20 \times 80 = 1,600$$

$$21 \times 81 = 1,701$$

$$22 \times 82 = 1,804$$

$$23 \times 83 = 1,909$$

$$24 \times 84 = 2,016$$

$$25 \times 85 = 2,125$$

$$26 \times 86 = 2,236$$

$$27 \times 87 = 2,349$$

$$28 \times 88 = 2,464$$

$$29 \times 89 = 2,581$$

Starting with 10×90 , 11×91 , 12×92 , etc., you can multiply 50 pairs of two-digit numbers by this method. These pairs can also be reversed since $10 \times 90 = 90 \times 10$, etc.

Checking Results in Multiplication

4-42. Estimate the Answer Before You Multiply

Estimating your answer before you multiply will help to avoid big mistakes, such as getting an answer in the hundreds when it should be in the thousands, or the other way around, as may happen if you misplace a decimal point or put the right digit in the wrong place. If your answer comes out very different in magnitude from your estimate, check to see what is wrong.

One way to estimate the answer is to round off the numbers for an easy multiplication. Change the numbers in opposite directions, that is, increase one number and decrease the other. This will produce a smaller change in the product than if you increase or decrease both numbers.

For example, you might estimate the product of 486×112 like this:

Rounded off to the nearest hundred: $500 \times 100 = 50,000$

Rounded off to the nearest ten: $490 \times 110 = 53,900$

You can also estimate the answer by a short-cut method, such as by multiplying 486 by 100 to get 48,600 and adding $\frac{1}{8}$ of 48,600, since 112 is approximately $\frac{1}{8}$ larger than 100. This would give you an estimate of 54,675.

Any of these estimates will serve to show you in advance that your answer should be somewhere around 50,000. After you do the multiplication, check your estimate against your answer.

$$\begin{array}{r}
 486 \\
 \times 112 \\
 \hline
 972 \\
 486 \\
 \hline
 54432
 \end{array}$$

Now you can use one of the following methods to see if your multiplication is correct.

4-43. Checking Multiplication by Multiplying in Reverse

To make this check, reverse the numbers and multiply again.

Example:

$$\begin{array}{r}
 49 \\
 \times 36 \\
 \hline
 294 \\
 147 \\
 \hline
 1764
 \end{array}$$

Check:

$$\begin{array}{r}
 36 \\
 \times 49 \\
 \hline
 324 \\
 144 \\
 \hline
 1764
 \end{array}$$

4-44. Checking Multiplication by Using a Short-Cut

See if you can use a short-cut for a fast, easy check on your answer. For example, you could check the product (1,764) obtained in Method 4-43 as follows:

$$49 \times 36 = (50 \times 36) - 36 = 1,800 - 36 = 1,764 \text{ (by Method 4-32)}$$

If you use a short-cut in the first place, you can check your answer by another short-cut or by conventional multiplication.

4-45. Checking Multiplication by Dividing the Product by One of the Factors Used

Dividing the product by one of the factors should give the other factor as the quotient. This check takes longer but you might want to make it if the result is of great importance. Divide by the multiplier (rather than by the number multiplied) since this will change the order in which the digits are multiplied.

Example:

$$\begin{array}{r} 374 \text{ Factor} \\ \times \underline{85 \text{ Factor}} \\ 1870 \\ 2992 \\ \hline 31790 \end{array}$$

Check:

$$\begin{array}{r} 374 \text{ Quotient} \\ 85 \overline{) 31790} \\ \underline{255} \\ 629 \\ \underline{595} \\ 340 \\ \underline{340} \\ 0 \end{array}$$

4-46. Checking Multiplication by Casting Out 9's

Various ways to cast out 9's are shown in 1-6, Chapter I. The application of this method to checking multiplication is discussed here. Use of the method is recommended in particular when large numbers are involved. One of its

recommended in particular when large numbers are involved. One of its advantages is that you do not have to write the numbers again.

The remainders obtained after casting out 9's are used as check numbers for testing the answer in multiplication, as follows:

Example (1):

$$\begin{array}{r}
 752 \text{ Digit sum: } 14; \text{ Remainder: } 5 \\
 \times \underline{436} \text{ Digit sum: } 13; \text{ Remainder: } 4 \\
 4512 \quad 5 \times 4 = 20; \text{ Remainder: } 2 \\
 2256 \\
 3008 \\
 \hline
 327872 \text{ Digit sum: } 29; \text{ Remainder: } 2
 \end{array}$$

Multiplying the remainders of the factors gives a remainder of

2. The remainder of the answer is also 2. Therefore, the answer checks and may be considered correct.

In the following example, no digit sums are shown; the 9's are cast out by omitting them.

Example (2):

$$\begin{array}{r}
 639 \text{ Remainder: } 0 \\
 \times \underline{453} \text{ Remainder: } 3 \\
 1917 \quad 0 \times 3 = 0 \text{ Remainder} \\
 3195 \\
 2556 \\
 \hline
 289467 \text{ Remainder: } 0
 \end{array}$$

Since the remainders are the same (0), the answer checks as correct.

If the remainder of either factor is zero, the product of the remainders of the factors will also be zero since zero multiplied by any number is always zero.

Remember that casting out 9's does not give positive proof of the answer. The method will not catch errors of the following types:

method will not catch errors of the following types.

1. Where digits have been transposed, such as in 3,451 and 3,541. Since the same digits appear in both numbers, their digit sums and remainders will naturally be the same.

2. Where the decimal point has been misplaced. Again, it is obvious that 3.451, 34.51, 345.1, and 3,451 will all give the same digit sums and remainders. (You can guard against errors of this type by estimating your answer in advance.)

However, if casting out 9's does *not* confirm your answer, you can be sure that there is something wrong in the multiplication, in the check, or in both.

V. DIVISION

5-1. Basic Principles

Division is the process of finding how many times one number is contained in another. It is a rapid way of subtracting. Thus, $8 \div 2 = 4$ means that 2 is contained in 8 four times, or that 2 can be subtracted from 8 four times.

Division may also be shown in the form of a fraction:

$$8 \div 2 = \frac{8}{2} = 4.$$

Division is the inverse of multiplication. If a product is divided by one of two factors, the result (quotient) will be the other factor.

Multiplication

$$\begin{array}{r} 154 \text{ Factor} \\ \times \underline{6} \text{ Factor} \\ 924 \text{ Product} \end{array}$$

Division

$$\begin{array}{r} \text{Divisor } 6 \overline{) 924} \text{ Dividend} \\ \underline{154} \text{ Quotient} \end{array}$$

$$\begin{array}{r} \text{Divisor } 154 \overline{) 924} \text{ Dividend} \\ \underline{924} \text{ Quotient} \end{array}$$

One basic way to simplify division is to break up the divisor into easy factors. You can then divide by these smaller numbers in whatever order is easiest for you. To divide 378 by 18, for example, you can factor 18 into 6×3 and divide in either order: $378 \div 6 = 63$; $\div 3 = 21$; or $378 \div 3 = 126$; $\div 6 = 21$. Or you can use the factors 2×9 : $378 \div 2 = 189$; $\div 9 = 21$.

Another basic way of simplifying division depends on the fact that the divisor and the dividend can be multiplied by the same number, or be divided by the

same number, without changing the answer (quotient). For example, to divide 385 by 35, you can double both numbers to get $770 \div 70$ which, of course, is an easy 11.

These and other ways of making division faster and easier are covered in detail under the methods that follow.

5-2. Doing Short Division

In “short” division, the multiplying and dividing steps are not written out. This method is generally employed for dividing by numbers up to 12 although you can use it with larger divisors, also. Two simple procedures can help to make your short division faster, easier, and surer.

A. Write each digit of your answer directly under the proper digit in the dividend.

B. Write each partial remainder as a small number in front of and above (or below) the next digit in the dividend. This will save you from having to remember each remainder and will make it easier to check your division in case of a mistake.

Example:

$$\begin{array}{r} \text{Divisor } 6 \overline{) 174,972} \text{ Dividend} \\ \underline{29,162} \text{ Quotient} \end{array}$$

5-3. Doing Long Division

In “long” division, the multiplying and subtracting steps are written out so that the remainders are shown for each step. The following procedures can help you get the right answer faster and easier.

A. Write each digit of your answer directly above the proper digit in the dividend.

B. When a digit repeats in the answer, use the product previously obtained with this digit instead of multiplying again.

Example:

$$\begin{array}{r} 717 \\ 432 \overline{) 309744} \\ \underline{3024} \\ 734 \\ \underline{432} \\ 3024 \\ \underline{3024} \\ 0 \end{array}$$

The first product of 7×432 , 3024, should simply be written for the second 7 instead of multiplying by 7 again.

C. When a digit in the answer is a multiple of a preceding digit in the answer, you can simplify your multiplication as shown below.

Example:

$$\begin{array}{r} 36 \\ 48 \overline{) 1728} \\ \underline{144} \\ 288 \\ \underline{288} \\ 0 \end{array}$$

Instead of multiplying 48 by 6, multiply 144 by 2 since 6 is twice 3.
Thus, $6 \times 48 = 288$; $2 \times 144 = 288$

D. If the divisor ends in one or more zeros, cancel these before you divide; move the decimal point in the dividend one place to the left for each zero cancelled.

Example: $173,216 \div 3,200$

$$\begin{array}{r}
 54.13 \\
 3200 \overline{) 1732.16} \\
 \underline{160} \\
 132 \\
 \underline{128} \\
 41 \\
 \underline{32} \\
 96 \\
 \underline{96}
 \end{array}$$

Why it works: Cancelling the two zeros and moving the decimal point two places to the left is the same as dividing both numbers by 100. In division, the answer is not changed if the divisor and the dividend are both divided by the same number (or multiplied by the same number).

If you do the division **WITHOUT** cancelling the zeros, you have to handle much larger numbers.

$$\begin{array}{r}
 54.13 \\
 3200 \overline{) 173216.00} \\
 \underline{16000} \\
 13216 \\
 \underline{12800} \\
 4160 \\
 \underline{3200} \\
 9600 \\
 \underline{9600}
 \end{array}$$

5-4. To Divide by 10, 100, 1,000, etc.

Dividing by 10, 100, etc., is so simple that it hardly need be considered as a separate step. However, the process is reviewed here because the use of these divisors is very important to methods that come later.

A. To divide a number by 10, move the decimal point one place to the left.

Examples: $682 \div 10 = 68.2$ $72.31 \div 10 = 7.231$

B. To divide a number by 100, move the decimal point two places to the left.

Examples: $361 \div 100 = 3.61$ $4.26 \div 100 = .0426$

C. To divide a number by 1,000, move the decimal point three places to the left.

Examples: $945 \div 1,000 = .945$ $80 \div 1,000 = .08$

In the following methods, where a number is to be divided by 10, 100, or 1,000, simply move the decimal point the required number of places to the left.

Dividing by Aliquot Parts of 1, 10, 100, and 1,000

As mentioned in Chapter IV on multiplying, an aliquot part is a number that goes into another number an exact number of times. For example, 5 is an aliquot part of 10 because 5 can be divided into 10 “evenly”, that is, without a remainder.

Aliquot parts of 1, 10, 100, and 1,000 (such as .5, 5, .25, 2.5, 12.5, and 125) can be simplified as divisors by changing them into easy fractions, as shown in the following methods.

5-5. General Method for Dividing by Aliquot Parts

Change the aliquot part to its fractional equivalent, invert the fraction, and multiply.

Example: $641 \div 25$

Solution by steps:

(a) 25 is an aliquot part of 100. Change it to its fractional equivalent: $\frac{100}{25}$

(b) Invert the fraction: $\frac{25}{100}$ (equivalent to $\frac{1}{4}$)

(c) Multiply: $= 641 \times \frac{4}{100} = \frac{2564}{100} = 25.64$, answer

Short way: $641 \times 4 = 2,564$; answer 25.64

Why it works: Dividing by 25 is the same as multiplying by $\frac{1}{25}$ which is equivalent to $\frac{4}{100}$. Multiplying by 4 and pointing off two decimal places to the left is the same as multiplying by $\frac{4}{100}$.

Dividing by other aliquot parts works on the same principle. The divisors discussed in the next methods, with their fractional equivalents, are:

$$\begin{aligned} .5 &= \frac{1}{2}; & 5 &= \frac{10}{2}; & 50 &= \frac{100}{2}; & 500 &= \frac{1000}{2} \\ .25 &= \frac{1}{4}; & 2.5 &= \frac{10}{4}; & 25 &= \frac{100}{4}; & 250 &= \frac{1000}{4} \\ .125 &= \frac{1}{8}; & 1.25 &= \frac{10}{8}; & 12.5 &= \frac{100}{8}; & 125 &= \frac{1000}{8} \end{aligned}$$

For multiplying and dividing by fractions, see Chapter VI.

5-6. To Divide by .5, 5, 50, or 500

A. To divide a number by .5, multiply the number by 2. Multiplying by 2 is the same as dividing by $\frac{1}{2}$ which is equivalent to $5 (\frac{5}{10})$.

Example: $85 \div .5$

Solution: $85 \times 2 = 170$, answer

Proving the answer:

$$\begin{array}{r} .5 \overline{) 85} = 5 \overline{) 850} \\ \underline{170} \end{array}$$

B. To divide a number by 5, multiply the number by 2 and divide by 10. This

is the same as multiplying by $\frac{2}{10}$ which is equivalent to dividing by $\frac{10}{2}$ which equals 5.

Example: $243 \div 5$

Solution: $243 \times 2 = 486; \div 10 = 48.6$, answer

Proving the answer:

$$\begin{array}{r} 5 \overline{) 243.0} \\ \underline{48.6} \end{array}$$

C. To divide a number by 50, multiply the number by 2 and divide by 100. This is the same as multiplying by $\frac{2}{100}$ which is equivalent to dividing by $\frac{100}{2}$ which equals 50.

Example: $215 \div 50$

Solution: $215 \times 2 = 430; \div 100 = 4.3$, answer

Proving the answer:

$$\begin{array}{r} 4.3 \\ 50 \overline{) 215.0} \\ \underline{200} \\ 150 \\ \underline{150} \end{array} \quad \text{Short way: } \begin{array}{r} 5 \overline{) 21.5} \\ \underline{4.3} \end{array}$$

D. To divide a number by 500, multiply the number by 2 and divide by 1,000. This is the same as multiplying by $\frac{2}{1000}$ which is equivalent to dividing by $\frac{1000}{2}$ which equals 500.

Example: $164 \div 500$

Solution: $164 \times 2 = 328; \div 1,000 = .328$, answer

Proving the answer:

$$\begin{array}{r}
 .328 \\
 500 \overline{) 164.000} \\
 \underline{1500} \\
 1400 \\
 \underline{1000} \\
 4000 \\
 \underline{4000} \\
 0
 \end{array}
 \quad \text{Short way: }
 \begin{array}{r}
 5 \overline{) 1.640} \\
 \underline{.328} \\
 0
 \end{array}$$

5-7. To Divide by .25, 2.5, 25, or 250

A. To divide a number by .25, multiply the number by 4. Multiplying by 4 is the same as dividing by $\frac{1}{4}$ which is equivalent to $.25 (\frac{25}{100})$.

Example: $71 \div .25$

Solution: $71 \times 4 = 284$, answer

Proving the answer:

$$\begin{array}{r}
 .25 \overline{) 71} = 284 \\
 \underline{50} \\
 210 \\
 \underline{200} \\
 100 \\
 \underline{100} \\
 0
 \end{array}$$

B. To divide a number by 2.5, multiply the number by 4 and divide by 10. This is the same as multiplying by $\frac{4}{10}$ which is equivalent to dividing by $\frac{10}{4}$ which equals $2.5 (2\frac{1}{2})$.

Example: $52 \div 2.5$

Solution: $52 \times 4 = 208; \div 10 = 20.8$, answer

Proving the answer:

$$\begin{array}{r} 20.8 \\ 2.5 \overline{) 52} = 25 \overline{) 520.0} \\ \underline{50} \\ 200 \\ \underline{200} \end{array}$$

C. To divide a number by 25, multiply the number by 4 and divide by 100.
This is the same as multiplying by $\frac{4}{100}$ which is equivalent to dividing by $\frac{100}{4}$ which equals 25.

Example: $.161 \div 25$

Solution: $161 \times 4 = 644; \div 100 = 6.44$, answer

Proving the answer:

$$\begin{array}{r} 6.44 \\ 25 \overline{) 161.00} \\ \underline{150} \\ 110 \\ \underline{100} \\ 100 \\ \underline{100} \end{array}$$

D. To divide a number by 250, multiply the number by 4 and divide by 1,000.
This is the same as multiplying by $\frac{4}{1000}$ which is equivalent to dividing by $\frac{1000}{4}$ which equals 250.

Example: $832 \div 250$

Solution: $832 \times 4 = 3,328$; $\div 1,000 = 3.328$, answer

Proving the answer:

$$\begin{array}{r} 3.328 \\ 250 \overline{) 832.000} \\ \underline{750} \\ 820 \\ \underline{750} \\ 700 \\ \underline{500} \\ 2000 \\ \underline{2000} \end{array} \quad \text{or:} \quad \begin{array}{r} 3.328 \\ 25 \overline{) 83.200} \\ \underline{75} \\ 82 \\ \underline{75} \\ 70 \\ \underline{50} \\ 200 \\ \underline{200} \end{array}$$

5-8. To Divide by .125, 1.25, 12.5, or 125

A. To divide a number by .125, multiply the number by 8. This is the same as dividing by $\frac{1}{8}$ which is equivalent to .125 ($\frac{125}{1000}$).

Example: $32 \div .125$

Solution: $32 \times 8 = 256$, answer

Proving the answer:

$$\begin{array}{r} 256 \\ .125 \overline{) 32} = 125 \overline{) 32000} \\ \underline{250} \\ 700 \\ \underline{625} \\ 750 \\ \underline{750} \end{array}$$

B. To divide a number by 1.25, multiply the number by 8 and divide by 10. This is the same as multiplying by $\frac{8}{10}$ which is equivalent to dividing by $\frac{10}{8}$ which equals 1.25 ($1\frac{1}{4}$ or $\frac{5}{4}$) or.

Example: $53 \div 1.25$

Solution: $53 \times 8 = 424$; $\div 10 = 42.4$, answer

Proving the answer:

$$\begin{array}{r} 1.25 \overline{)53} = 125 \overline{)5300.0} \\ \underline{500} \\ 300 \\ \underline{250} \\ 500 \\ \underline{500} \\ 0 \end{array}$$

C. To divide a number by 12.5, multiply the number by 8 and divide by 100. This is the same as multiplying by $\frac{8}{100}$ which is equivalent to dividing by $\frac{100}{8}$ which equals 12.5 ($12\frac{1}{2}$ or $\frac{25}{2}$) or .

Example: $315 \div 12.5$

Solution: $315 \times 8 = 2,520$; $\div 100 = 25.2$, answer

Proving the answer:

$$\begin{array}{r} 12.5 \overline{)315} = 125 \overline{)3150.0} \\ \underline{250} \\ 650 \\ \underline{625} \\ 250 \\ \underline{250} \\ 0 \end{array}$$

D. To divide a number by 125, multiply the number by 8 and divide by 1,000. This is the same as multiplying by $\frac{8}{1000}$ which is equivalent to dividing by $\frac{1000}{8}$ which equals 125.

Example: $441 \div 125$

Solution: $441 \times 8 = 3,528$; $\div 1,000 = 3.528$, answer

Proving the answer:

$$\begin{array}{r}
 3.528 \\
 125 \overline{) 441.000} \\
 \underline{375} \\
 660 \\
 \underline{625} \\
 350 \\
 \underline{250} \\
 1000 \\
 \underline{1000} \\
 0
 \end{array}$$

5-9. To Divide by .75, 7.5, 75, or 750

While .75, 7.5, 75, and 750 are not evenly divisible into 1, 10, 100, or 1,000 and are therefore not aliquot parts of these numbers, they are included here because they are closely related to the, aliquot parts just discussed and can be converted into easy divisors by similar steps,

The fractional equivalents of these divisors are: $.75 = \frac{3}{4}$; $7.5 = \frac{30}{4}$; $75 = \frac{300}{4}$; and $750 = \frac{3000}{4}$. When these fractions are inverted and used as multipliers, this involves multiplying by 4, dividing by 3, and then dividing by 10, 100, or 1,000, as the case may be. (No need to multiply by 1 as it will not change the number multiplied.) Since multiplying by 4 and dividing by 3 is the same as multiplying by $\frac{4}{3}$, you may find it easier to simply increase the multiplied number by $\frac{1}{3}$ of the number. This is the same as multiplying by $1\frac{1}{3}$ which equals $\frac{4}{3}$.

A. To divide a number by .75, increase the number by $\frac{1}{3}$ of the number..

Example: $36 \div .75$

Solution:

$$36 \times \frac{1}{3} = 12; \quad + 36 = 48, \text{ answer}$$

also:

$$36 \times \frac{4}{3} = 48$$

Proving the answer:

$$\begin{array}{r} .75 \overline{)36} = 75 \overline{)3600} \\ \underline{300} \\ 600 \\ \underline{600} \\ 0 \end{array}$$

B. To divide a number by 7.5, increase the number by $\frac{1}{3}$ of the number and divide by 10.

Example: $144 \div 7.5$

Solution:

$$\begin{array}{l} 144 \times \frac{1}{3} = 48; \quad + 144 = 192; \quad \div 10 = 19.2, \\ \text{answer} \end{array}$$

also:

$$144 \times \frac{4}{3} = 192; \quad \div 10 = 19.2$$

Proving the answer:

$$7.5 \overline{)144} = 75 \overline{)1440.0} \begin{array}{r} 19.2 \\ 75 \\ \hline 690 \\ 675 \\ \hline 150 \\ 150 \\ \hline \end{array}$$

C. To divide a number by 75, increase the number by $\frac{1}{3}$ of the number and divide by 100.

Example: $825 \div 75$

Solution:

$$825 \times \frac{1}{3} = 275; \quad + 825 = 1,100; \quad \div 100 = 11, \\ \text{answer}$$

also:

$$825 \times \frac{4}{3} = 1,100; \quad \div 100 = 11$$

Proving the answer:

$$75 \overline{)825} \begin{array}{r} 11 \\ 75 \\ \hline 75 \\ 75 \\ \hline \end{array}$$

D. To divide a number by 750, increase the number by $\frac{1}{3}$ of the number and divide by 1,000.

Example: $126 \div 750$

Solution:

$$126 \times \frac{1}{3} = 42; \quad + 126 = 168; \quad \div 1,000 = .168, \\ \text{answer}$$

also:

$$126 \times \frac{4}{3} = 168; \quad \div 1,000 = .168$$

Proving the answer:

$$\begin{array}{r} .168 \\ 750 \overline{) 126.000} \text{ or: } 75 \overline{) 12.600} \\ \underline{750} \qquad \qquad \underline{75} \\ 5100 \qquad \qquad 510 \\ \underline{4500} \qquad \qquad \underline{450} \\ 6000 \qquad \qquad 600 \\ \underline{6000} \qquad \qquad \underline{600} \end{array}$$

Summary of Division by Aliquot Parts and 75-Numbers

N stands for the Number to be divided

<i>To Divide by</i>	<i>Method</i>
.5	$2N \quad (2 \times N)$
5	$2N \div 10$
50	$2N \div 100$
500	$2N \div 1,000$
.25	$4N$
2.5	$4N \div 10$
25	$4N \div 100$
250	$4N \div 1,000$
.125	$8N$
1.25	$8N \div 10$
12.5	$8N \div 100$
125	$8N \div 1,000$
.75	$4N/3 \text{ (or } N + N/3)$
7.5	$4N/3 \div 10$
75	$4N/3 \div 100$
750	$4N/3 \div 1,000$

Other aliquot parts: Division by other aliquot parts in the form of mixed numbers ($2\frac{1}{2}$, $12\frac{1}{2}$, $16\frac{2}{3}$, etc.) is discussed in Chapter VI.

Testing the Divisibility of a Number

It is often useful to be able to tell quickly if one number can be divided evenly by another. The following tests make this easy to do.

5-10. Test for Divisibility by Odd and Even Numbers

A. *Odd numbers:* Numbers ending in 1, 3, 5, 7, or 9 can only be divided evenly by another odd number. Therefore, if a number's last digit is odd, you know at once that the number is not evenly divisible by any even number.

B. *Even numbers:* Numbers ending in 2, 4, 6, 8, or 0 may be divisible evenly by either odd or even numbers. For example, 12 is evenly divisible by the even numbers 2, 4, and 6, as well as by the odd number 3.

5-11. Test for Divisibility by 2

If a number ends in 0, 2, 4, 6, or 8, it is, of course, evenly divisible by 2.

5-12. Test for Divisibility by 3

If the sum of a number's digits is evenly divisible by 3, the number can be evenly divided by 3.

Example: 573

Test: Digit sum is $5 + 7 + 3 = 15$. Since 15 is evenly divisible by 3, so is the number 573.

Check:

$$\begin{array}{r} 3 \overline{) 573} \\ \underline{191} \end{array}$$

Since changing the order of the digits does not change the digit sum, it follows that 375, 357, 537, 735, and 753 are also evenly divisible by 3.

Instead of getting the sum of all the digits, you can make the test faster and easier by omitting 3's, 6's, and 9's, and combinations that add up to 3, 6, or 9 (2 and 1, 2 and 4, 4 and 5, etc.), and testing the sum of the other digits.

Example: 1,945,369

Test: By omitting the two 9's, the 4 and 5, and the 3 and 6, only the 1 remains. This not only shows that the number is not evenly divisible by 3 but also that there will be a remainder of 1 if the number is divided by 3.

Check:

$$\begin{array}{r} 3 \overline{) 1,945,369} \\ \underline{648,456\frac{1}{3}} \end{array} \text{ (1 remainder)}$$

5-13. Test for Divisibility by 4

If a number's last two digits are zeros or are evenly divisible by 4, the number can be evenly divided by 4.

Example: 9,537,724

Test: Since the last two digits (24) are evenly divisible by 4, the number itself

is evenly divisible by 4.

You can simplify this test further by dividing the last two digits by 2. If the result is an even number, it shows that the last two digits are evenly divisible by 4 and that, therefore, the entire number is, also.

Check:

$$\begin{array}{r} 4 \overline{) 9,537,724} \\ \underline{2,384,431} \end{array} \quad \begin{array}{r} 2 \overline{) 9,537,724} \\ \underline{2 \overline{) 4,768,862}} \\ \underline{2,384,431} \end{array}$$

5-14. Test for Divisibility by 5

If a number ends in 5 or 0, it is evenly divisible by 5.

5-15. Test for Divisibility by 6

If a number is even and its digit sum is evenly divisible by 3, the number can be evenly divided by 6.

Example: 23,754

Test: The number is even and its digit sum, 21, is evenly divisible by 3. Hence, the number can be evenly divided by 6.

Check:

$$\begin{array}{r} 6 \overline{) 23,754} \\ \underline{3,959} \end{array}$$

5-16. Test for Divisibility by 7

There is no simple test for divisibility by 7. The best way is to try dividing the number by 7, especially if the number has fewer than six digits. However, you may find the following method interesting and, at times, useful.

Example (1): 1,224,251

Steps in test:

- (a) Starting from the right, split the number into groups of three digits: 1 224 251. Numbers of 1,000 or more are generally marked off into groups of three by commas, so you don't have to do this separately.
- (b) Add the odd groups (first, third, etc.), add the even groups (second, fourth, etc.; 224 stands alone here, so there is nothing to add), and subtract one from the other: $1 + 251 = 252$; $- 224 = 28$.
- (c) The original number is evenly divisible by 7 if the remainder in (b) is zero or can be evenly divided by 7. Since the remainder is 28, the number can be evenly divided by 7.

Check:

$$\begin{array}{r} 7 \overline{) 1,224,251} \\ \underline{174,893} \end{array}$$

You can use this same test for checking divisibility by 11 and 13, also. Since the remainder (28) cannot be evenly divided by either 11 or 13, neither can the original number.

Check:

$$\begin{array}{r} 11 \overline{) 1,224,251} \\ \underline{111,295} \quad \underline{6} \\ \quad \quad \quad \underline{11} \end{array} \qquad \begin{array}{r} 94173 \\ 13 \overline{) 1224251} \\ \underline{117} \\ 54 \\ \underline{52} \\ 22 \\ \underline{13} \\ 95 \\ \underline{91} \\ 41 \\ \underline{39} \\ 2 \text{ Remainder} \end{array}$$

Example (2): Test 220,324 for divisibility by 7, 11, or 13.

Example (2): Test 220,324 for divisibility by 7, 11, or 13.

Test: $324 - 220 = 104$ which is evenly divisible by 13 but not by 7 or 11.
Therefore, the original number can be evenly divided by 13 but not by 7 or 11.

Check:

$$\begin{array}{r}
 16948 \\
 13 \overline{) 220324} \\
 \underline{13} \\
 90 \\
 \underline{78} \\
 123 \\
 \underline{117} \\
 62 \\
 \underline{52} \\
 104 \\
 \underline{104} \\
 0
 \end{array}
 \qquad
 \begin{array}{r}
 7 \overline{) 220,324} \\
 \underline{31,474} \overset{6}{7} \\
 0
 \end{array}
 \qquad
 \begin{array}{r}
 11 \overline{) 220,324} \\
 \underline{20,029} \overset{5}{11} \\
 0
 \end{array}$$

Example (3): Test 676,522 for divisibility by 7, 11, or 13.

Test: $676 - 522 = 154$ which is evenly divisible by both 7 and 11 but not by 13. Therefore, the same applies to the original number.

Check:

$$\begin{array}{r}
 7 \overline{) 676,522} \\
 \underline{96,646} \\
 0
 \end{array}
 \qquad
 \begin{array}{r}
 52040 \\
 13 \overline{) 676522} \\
 \underline{65} \\
 26 \\
 \underline{26} \\
 52 \\
 \underline{52} \\
 2 \text{ Remainder}
 \end{array}$$

A separate test for divisibility by 11 is given in Method 5-20.

5-17. Test for Divisibility by 8

If the last three digits of a number are zeros or are evenly divisible by 8, the number can be evenly divided by 8.

Example: 727,648

Test: The last three digits are evenly divisible by 8 ($648 \div 8 = 81$); therefore, the number can be evenly divided by 8.

Check:

$$\begin{array}{r} 8 \overline{) 727,648} \\ \underline{90,956} \end{array}$$

You can simplify this test further by dividing the last three digits by 4. If the result is an even number, it shows that the last three digits are evenly divisible by 8 and that the original number is, also. Or, instead of dividing by 4, you can divide twice by 2 to see if you get an even number.

Of course, if the last two digits are not evenly divisible by 4, the number is not evenly divisible by either 4 or 8. And keep in mind that, if the number's last digit is odd, you need go no further since the number cannot be evenly divided by *any* even number.

Here is another quick test. A number is evenly divisible by 8 if (a) the hundreds digit is even and the last two digits are evenly divisible by 8, such as in 464; or (b) the hundreds digit is odd and the last two digits are evenly divisible by 4 but not by 8, such as in 512.

5-18. Test for Divisibility by 9

If a number's digit sum is evenly divisible by 9, the number can be evenly divided by 9.

Examples:

18: digit sum: 9
36: digit sum: 9
666: digit sum: 18
111,111,111: digit sum: 9

Test: The digit sums of all four numbers are evenly divisible by 9; therefore, the numbers themselves are evenly divisible by 9.

Check:

$$\begin{array}{r} 9 \overline{) 666} \\ 74 \end{array}$$

$$\begin{array}{r} 9 \overline{) 111,111,111} \\ 12,345,679 \end{array}$$

Instead of getting the digit sum, it is faster and easier to cast out 9's. If there is no remainder after casting out 9's, the number can be evenly divided by 9.

Example: 993,645

Test: Casting out 9, 9, 3 + 6 (= 9), and 4 + 5 (= 9), there is no remainder. Therefore, the number is evenly divisible by 9.

Check:

$$\begin{array}{r} 9 \overline{) 993,645} \\ 110,405 \end{array}$$

However, if there is a remainder after casting out 9's, there will be the same remainder when the number is divided by 9.

Example: 2,081

Test: Casting out 8 and 1 (= 9) leaves a remainder of 2. Therefore, there will also be a remainder of 2 when the number is divided by 9.

Check:

$$\begin{array}{r} 9 \overline{) 2,081} \\ 231 \frac{2}{9} \end{array}$$

5-19. Test for Divisibility by 10

If a number ends in zero, it is, of course, evenly divisible by 10.

Summary of Tests for Divisibility: 2 to 10

*Number is Divisible
by*

If Number's

2

Last digit is even.

3	Digit sum is evenly divisible by 3.
4	Last two digits are 00 or are evenly divisible by 4.
5	Last digit is 5 or 0.
6	Last digit is even and digit sum is evenly divisible by 3.
7	(See Method 5-16)
8	Last three digits are 000 or are evenly divisible by 8.
9	Digit sum is evenly divisible by 9.
10	Last digit is 0.

With regard to these tests, the following should also be noted: (a) The tests for divisibility by 2, 5, and 10 involve checking only the last digit.

(b) The test for divisibility by 4 depends on the last two digits.

(c) The test for divisibility by 8 depends on the last three digits.

(d) The tests for divisibility by 3, 6, and 9 are based on digit sums; in the case of 6, the number must also be even.

You can extend these tests to certain larger divisors by applying two of the tests to a number. For example, if the number is evenly divisible by both 3 and 4, it is also evenly divisible by 12 (product of 3×4); if evenly divisible by 3 and 5, the number is also evenly divisible by 15.

However, this will not work if the factors that make up the larger divisor can be evenly divided into each other. For instance, you cannot test for divisibility by 16 by testing for 2 and 8 because 2 goes into 8 evenly and any number evenly divisible by 8 must also be evenly divisible by 2. Therefore, this does not provide two distinct tests. Of course, if you divide a number by 8 and get an even number as a result, this does tell you that the number is evenly divisible by 16.

Testing for Divisibility by Larger Divisors

5-20. Test for Divisibility by 11

Starting from the left or right, add the digits in the “odd” places and the digits in the “even” places. If the difference between these sums is 0 or is evenly divisible by 11, the number is evenly divisible by 11.

Example (1): 3,861

Steps in test:

(a) Sum of first and third digits from the right: $1 + 8 = 9$

(b) Sum of second and fourth digits from the right: $6 + 3 = 9$

(c) $9 - 9 = 0$; the number is evenly divisible by 11

Short way: $(1 + 8) - (6 + 3) = 0$

Check:

$$\begin{array}{r} 11 \overline{) 3,861} \\ \underline{351} \end{array}$$

Example (2): 937,651

Test: $1 + 6 + 3 = 10$; $9 + 7 + 5 = 21$; $21 - 10 = 11$; the number is evenly divisible by 11.

Check:

$$\begin{array}{r} 11 \overline{) 937,651} \\ \underline{85,241} \end{array}$$

Checking three-digit numbers for divisibility by 11 is especially easy by this method. If the sum of the two outside digits equals the middle digit, or differs by 11 from the middle digit, the number is evenly divisible by 11. (Note that the two outside digits are in “odd” places, the middle digit in an “even” place.)

Example (1): 792 and 297

Test: $7 + 2 = 9$, the middle digit; both 792 and 297 are evenly divisible by 11

Test: $7 + 2 = 9$, the middle digit, both 792 and 297 are evenly divisible by 11.

Check:

$$\begin{array}{r} 11 \overline{) 792} \\ 72 \end{array}$$

$$\begin{array}{r} 11 \overline{) 297} \\ 27 \end{array}$$

Example (2): 968 and 869

Test: $9 + 8 = 17$; $6 = 11$; both 968 and 869 are evenly divisible by 11.

Check:

$$\begin{array}{r} 11 \overline{) 968} \\ 88 \end{array}$$

$$\begin{array}{r} 11 \overline{) 869} \\ 79 \end{array}$$

If a number is made up of a repeating digit, it is evenly divisible by 11 if it has an even number of digits.

Examples: 66, 5,555, and 777,777 are all evenly divisible by 11 but 666, 66,666, and 7,777,777 are not.

Check:

$$\begin{array}{r} 11 \overline{) 5,555} \\ 505 \end{array}$$

$$\begin{array}{r} 11 \overline{) 55,555} \\ 5,050 \underline{5} \\ 11 \end{array}$$

Another test for divisibility by 11 is included under Method 5-16.

5-21. Test for Divisibility by 12

If a number is evenly divisible by both 3 and 4, it can be evenly divided by 12.

Example: 3,625,020

Steps in test:

- (a) Digit sum is 18; number is evenly divisible by 3 (Method 5-12)
- (b) Last two digits (20) are evenly divisible by 4; number is evenly divisible by 4 (Method 5-13)
- (c) Since the number is evenly divisible by both 3 and 4, it is evenly divisible by 12

Check:

$$\begin{array}{r} 12 \overline{) 3,625,020} \\ \underline{302,085} \end{array}$$

5-22. Test for Divisibility by 15

If a number is evenly divisible by both 3 and 5, it can be evenly divided by 15.

Example: 7,335

Steps in test:

- (a) Digit sum is 18; number is evenly divisible by 3
- (b) Number ends in 5; is evenly divisible by 5
- (c) Since the number meets both tests, it is evenly divisible by 15

Check:

$$\begin{array}{r} 489 \\ 15 \overline{) 7335} \\ \underline{60} \\ 133 \\ \underline{120} \\ 135 \\ \underline{135} \end{array}$$

5-23. Test for Divisibility by 18

If a number is evenly divisible by both 2 and 9, it can be evenly divided by 18.

Example: 2,952

Steps in test:

- (a) Number ends in 2; is evenly divisible by 2
- (b) Digit sum is 18; number is evenly divisible by 9
- (c) Since the number meets both tests, it is evenly divisible by 18

Check:

$$\begin{array}{r}
 164 \\
 18 \overline{) 2952} \\
 \underline{18} \\
 115 \\
 \underline{108} \\
 72 \\
 \underline{72} \\
 0
 \end{array}
 \quad \text{or:} \quad
 \begin{array}{r}
 2) 2,952 \\
 \underline{9) 1,476} \\
 164
 \end{array}$$

5-24. Extending the Tests to Divisibility by Larger Divisors

Tests for divisibility can be extended to larger divisors, as shown by the following examples.

<i>Number is Divisible by</i>	<i>If the Number</i>
20	Ends in 0 and the preceding digit is even.
22	Is even and evenly divisible by 11.
24	Is evenly divisible by both 3 and 8.
25	Ends in 00, 25, 50, or 75.
30	Ends in 0 and is evenly divisible by 3.
33	Is evenly divisible by both 3 and 11.
36	Is evenly divisible by both 4 and 9.

Simplifying Dividends and Divisors

5-25. Simplifying the Dividend by Adding or Subtracting

Example (1): $3,196 \div 4$

Solution by steps:

- (a) Simplify 3,196 to an easy zero-number by adding 4: $3,196 + 4 = 3,200$
- (b) Divide the simplified dividend: $3,200 \div 4 = 800$; which means that there are 800 4's in 3,200, but since one 4 was added, this is 1 too many
- (c) Subtract the 1: $800 - 1 = 799$, answer

Short way:

$$3,196 \div 4 = \frac{3,200 - 4}{4} = 800 - 1 = 799$$

Proving the answer:

$$\begin{array}{r} 4 \overline{) 3,196} \\ 799 \end{array}$$

Example (2): $711 \div 9$

Solution by steps:

- (a) Digit sum of 711 shows that the number is evenly divisible by 9. Simplify to a zero-number by adding 9: $711 + 9 = 720$
- (b) $720 \div 9 = 80$; since a 9 was added, this is one 9 too many, so 1 must be subtracted.
- (c) $80 - 1 = 79$, answer

Short way:

$$711 \div 9 = \frac{720 - 9}{9} = 80 - 1 = 79$$

Proving the answer:

$$\begin{array}{r} 9 \overline{) 711} \\ 79 \end{array}$$

Example (3): $558 \div 9$

Solution by steps:

(a) Digit sum of 558 shows that the number is evenly divisible by 9.

Subtracting 18 will give a zero-number that is also divisible by 9: $558 - 18 = 540$

(b) $540 \div 9 = 60$; this represents two 9's fewer than were in the original number; therefore 2 must be added.

(c) $60 + 2 = 62$, answer

Short way:

$$558 \div 9 = \frac{540 + 18}{9} = 60 + 2 = 62$$

Proving the answer:

$$\begin{array}{r} 9 \overline{) 558} \\ \underline{62} \end{array}$$

You can use this method even if there is going to be a remainder.

Example: $3,197 \div 4$

Solution by steps:

(a) Test for divisibility by 4 (Method 5-13) shows that there will be a remainder of 1 ($97 \div 4 = 24\frac{1}{4}$)

(b) Adding 3 to the dividend will make it a zero-number evenly divisible by 4: $3,197 + 3 = 3,200$

(c) $3,200 \div 4 = 800$; this is $\frac{3}{4}$ of a 4 too much because of the 3 that was added; therefore, $\frac{3}{4}$ must be subtracted.

(d) $800 - \frac{3}{4} = 799\frac{1}{4}$, answer

Short way:

$$3,197 \div 4 = \frac{3,200 - 3}{4} = 800 - \frac{3}{4} = 799\frac{1}{4}$$

Proving the answer:

$$\begin{array}{r} 4 \overline{) 3,197} \\ \underline{799} 1 \\ 4 \end{array}$$

If the dividend is odd and the divisor even, you may find it helpful to make the dividend even, also.

Example (1): $111 \div 2$

Solution by steps:

(a) $111 - 1 = 110$; the 1 subtracted must be added as a remainder of $\frac{1}{2}$ in the answer.

(b) $110 \div 2 = 55$; $+ \frac{1}{2} = 55\frac{1}{2}$, answer

Short way:

$$111 \div 2 = \frac{110 + 1}{2} = 55\frac{1}{2}$$

Example (2): $69 \div 4$

Solution:

$$69 - 1 = 68; \div 4 = 17; + \frac{1}{4} = 17\frac{1}{4}$$

5-26. Simplifying the Dividend by Breaking It into Parts

Example: $168 \div 3$ (test by Method 5-12 shows 168 is evenly divisible by 3)

Solution by steps:

(a) Break the dividend into easier numbers that are also evenly divisible by 3:

$$168 = 150 + 18$$

(b) $\frac{150}{3} + \frac{18}{3} = 50 + 6 = 56$, answer

Breaking the dividend into parts by units, tens, hundreds, etc., is another way

to simplify it for easier division.

Example: $4,264 \div 8$ (test by Method 5-17 shows number is evenly divisible by 8)

Solution:

$$4,264 = 4,000 + 200 + 64; \div 8 = 500 + 25 \\ + 8 = 533, \text{ answer}$$

Proving the answer:

$$\begin{array}{r} 8 \overline{) 4,264} \\ \underline{533} \end{array}$$

5-27. Simplifying the Divisor by Factoring It

Factoring the divisor simply means breaking it into factors — into two or more smaller numbers which, when multiplied together, give the divisor as the product, such as breaking 6 into 2×3 . You may find it easier to divide by such factors than by the larger divisor.

In dividing by factors, it is usually better to start with the smallest one (or smaller one) since this will reduce the dividend for easier division by the larger factors. However, you may sometimes find it easier to start with the larger factor, such as in dividing 81 by 18. After factoring 18 into 2×9 , it is obviously better to divide first by 9 rather than by 2, thus: $81 \div 9 = 9; \div 2 = 4\frac{1}{2}$.

In considering which factor to start with, test the dividend for divisibility and try to divide first by a factor that will not leave a remainder, if you have such a choice.

Example (1): $216 \div 27$

Solution by steps:

(a) Factor 27 into 3×9 (test by Method 5-18 shows that 216 is evenly

divisible by 9 and, therefore, also by 3)
 (b) $216 \div 3 = 72$; $\div 9 = 8$, answer

Proving the answer:

$$\begin{array}{r} 8 \\ 27 \overline{) 216} \\ \underline{216} \end{array} \quad \text{also:} \quad \begin{array}{r} 3 \overline{) 216} \\ 9 \overline{) 72} \\ \underline{72} \\ 8 \end{array}$$

Why it works: Dividing by the factors of a number is the same as dividing by the number which is a product of those factors.

Thus,

$$\frac{24}{12} = 2; \frac{24}{3 \times 4} = \frac{8}{4} = 2, \text{ or } \frac{6}{3} = 2;$$

also,

$$\frac{24}{2 \times 6} = \frac{12}{6} = 2, \text{ or } \frac{4}{2} = 2$$

Example (2): $2,184 \div 24$

Solution by steps:

(a) 24 can be split into three pairs of factors: 2×12 , 3×8 , and 4×6 . Use any pair you please. Tests for divisibility show that 2,184 can be evenly divided by all of these factors, and also by 24.

(b)

$$\begin{aligned} 2,184 \div 2 &= 1,092; \div 12 = 91, \text{ answer} \\ 2,184 \div 3 &= 728; \div 8 = 91 \\ 2,184 \div 4 &= 546; \div 6 = 91 \end{aligned}$$

Proving the answer:

$$\begin{array}{r} 91 \\ 24 \overline{) 2184} \\ \underline{216} \\ 24 \\ \underline{24} \end{array}$$

Dividing by factors may enable you to simplify a problem from “long” division to easy “short” divisions.

Example: $2,679,568 \div 16$

Solution by short divisions:

(a) Factor 16 into 2×8 or 4×4 and divide by either pair of factors.

(b)

$$\begin{array}{r} 2 \overline{) 2,679,568} \\ 8 \overline{) 1,339,784} \\ \hline 167,473 \end{array}$$

$$\begin{array}{r} 4 \overline{) 2,679,568} \\ 4 \overline{) 669,892} \\ \hline 167,473 \end{array}$$

Solution by long division:

$$\begin{array}{r} \therefore 167473 \\ 16 \overline{) 2679568} \\ \underline{16} \\ 107 \\ \underline{96} \\ 119 \\ \underline{112} \\ 75 \\ \underline{64} \\ 116 \\ \underline{112} \\ 48 \\ \underline{48} \end{array}$$

Other examples:

$$384 \div 24: 384 \div 2 = 192; \div 2 = 96; \div 6 = 16$$

$$644 \div 16: 644 \div 2 = 322; \div 2 = 161; \div 4 = 40\frac{1}{4}$$

If a factor is not evenly divisible into a number, and you do not need the exact answer, you can ignore the remainder after each division. This will give you an answer within 1 less than the exact answer, no matter how many factors you divide by.

Example (1): $913 \div 27$

Solution by steps:

- (a) Factor 27 into 9×3 (test for divisibility shows that 913 is not evenly divisible by either factor)
- (b) $913 \div 9 = 101\frac{4}{9}$; drop the $\frac{4}{9}$
- (c) $101 \div 3 = 33\frac{2}{3}$, approximate answer

Exact answer:

$$\begin{array}{r} 33 \\ 27 \overline{) 913} \\ \underline{81} \\ 103 \\ \underline{81} \\ 22 \end{array} \quad \text{answer: } 33\frac{2}{27}$$

The approximate answer is only $\frac{4}{27}$ less than the exact answer:
 $33\frac{2}{27} - 33\frac{2}{3} (33\frac{18}{27}) = \frac{4}{27}$

Example (2): $22,841 \div 132$

Solution by steps:

- (a) 132 may be factored into $2 \times 2 \times 3 \times 11$ (tests for divisibility show that 22,841 is not evenly divisible by any of these factors)
- (b) In the following divisions, the remainders are ignored and not included in the next division: $22,841 \div 2 = 11,420\frac{1}{2}$;
 $\div 2 = 5,710$; $\div 3 = 1,903\frac{1}{3}$; $\div 11 = 173$, approximate answer

Exact answer:

$$\begin{array}{r}
 173 \\
 132 \overline{) 22841} \\
 \underline{132} \\
 964 \\
 \underline{924} \\
 401 \\
 \underline{396} \\
 5
 \end{array}
 \quad \text{answer: } 173\frac{5}{132}$$

The approximate answer is only $\frac{5}{132}$ less than the exact answer.

5-28. Simplifying the Dividend and Divisor by Multiplying or Dividing Both by the Same Number

If the divisor ends in 5, doubling it and the dividend can simplify the division by changing the divisor into a number ending in zero.

Example (1): $138 \div 15$

Solution:

Double both numbers and divide: $276 \div 30 = 27.6 \div 3 = 9.2$, answer

Proving the answer:

$$\begin{array}{r}
 9.2 \\
 15 \overline{) 138.0} \\
 \underline{135} \\
 30 \\
 \underline{30} \\
 0
 \end{array}$$

Example (2): $560 \div 35$

Solution: $1,120 \div 70 = 112 \div 7 = 16$, answer

Alternate solution: Simplify both numbers first by dividing by 7:

$$\frac{560}{7} \div \frac{35}{7} = 80 \div 5 = 16$$

Proving the answer:

$$\begin{array}{r} 16 \\ 35 \overline{) 560} \\ \underline{35} \\ 210 \\ \underline{210} \\ 0 \end{array}$$

Why it works: The dividend and divisor can be multiplied by the same number, or be divided by the same number, without changing the answer. You can easily see how this works by using small numbers, such as $12 \div 6 = 2$. Halve both numbers: $6 \div 3 = 2$; double both numbers: $24 \div 12 = 2$; triple both numbers: $36 \div 18 = 2$, etc.

5-29. Problems Involving Both Multiplication and Division

If a problem involves both multiplication and division, look it over to see which operation will be easier to do first. Also, whether the multiplier and divisor can be used to cancel or reduce each other.

Example (1): Multiply 484 by 16 and divide by 32

Solution:

$$16 \div 32 = \frac{1}{2}; \times 484 = 242, \text{ answer}$$

also:

$$\frac{484 \times 16}{32} = 242$$

Proving the answer:

$$\begin{array}{r} 484 \\ \times 16 \\ \hline 2904 \\ 484 \\ \hline 7744 \div 32 \end{array}$$

$$\begin{array}{r} 242 \\ 32 \overline{) 7744} \\ \underline{64} \\ 134 \\ \underline{128} \\ 64 \\ \underline{64} \\ 0 \end{array}$$

Example (2): Divide 336 by 21 and multiply by 45

Solution:

$$\frac{336 \div 21}{3} = 720, \text{ answer } (48 \times 15 = 24 \times 30 = 720)$$

also:

$$\begin{array}{r} 16 \\ 48 \\ \hline 336 \times 45 = 720 \quad (16 \times 45 = 8 \times 90 = 720) \\ 21 \\ 3 \end{array}$$

Why it works: When multiplication and division are to be done together, it does not matter in what order you perform the operations. You can easily see this by using small numbers, such as in multiplying 3 by 4 and dividing by 6.

$$\begin{aligned} 3 \times 4 &= 12; \div 6 = 2 \\ 3 \div 6 &= \frac{1}{2}; \times 4 = 2 \\ 4 \div 6 &= \frac{2}{3}; \times 3 = 2 \end{aligned}$$

Checking Results in Division

5-30. Estimate the Answer Before You Divide

Making an estimate before you divide can help to avoid big mistakes, such as getting an answer in the thousands when it should be in the hundreds, or vice versa. If you get an answer that is very different in magnitude from your estimate, check to see what is wrong.

One way to make an estimate is to round off the dividend and the divisor for easy division. In rounding off, change both numbers in the same direction; that is, either increase both or decrease both. (In rounding off for multiplication, you would increase one number and decrease the other, as mentioned in Method 4-42.)

Example: $45,4967 \div 517$

Estimate: $45,000 \div 500 = 450 \div 5 = 90$, approximate answer

Exact answer:

$$\begin{array}{r}
 88 \\
 517 \overline{) 45496} \\
 \underline{4136} \\
 4136 \\
 \underline{4136}
 \end{array}$$

In some problems, an estimate may be all you need; this can save you unnecessary figuring. It all depends upon the degree of accuracy you require for your purpose.

5-31. Check the Subtractions in Long Division As You Go

As you complete each subtraction in long division, add the remainder back to the number above it as a check. This will help to catch an error before you go on and will save time in the long run.

Example:

$$\begin{array}{r}
 341 \\
 371 \overline{) 126511} \\
 \underline{1113} \\
 1521 \text{ As you go, add 152 back to 1113 to get 1265;} \\
 \underline{1484} \text{ add 37 back to 1484 to get 1521.} \\
 371 \\
 \underline{371}
 \end{array}$$

After you obtain your answer, use one of the following methods to check it.

5-32. Checking Division by Multiplying Your Answer by the Divisor

Multiplying your answer (quotient) by the divisor should give the dividend as the product.

Example:

<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;"> <div style="text-align: right;">34 Quotient</div> <div style="text-align: right;">Divisor 48</div> </div> <div style="margin-right: 10px;"> $\begin{array}{r} 48 \overline{) 1632} \\ \underline{144} \\ 192 \\ \underline{192} \\ 0 \end{array}$ </div> <div> <div style="text-align: left;">Dividend</div> </div> </div>	<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;"> <div>Check:</div> </div> <div> $\begin{array}{r} 34 \\ \times 48 \\ \hline 272 \\ 136 \\ \hline 1632 \end{array}$ </div> </div>
---	--

Multiplying the divisor by the quotient (48×34) will also give the dividend as the product but this check is not as good since the same multiplications will be done as in the division and a mistake in multiplying may be repeated.

$$\begin{array}{r} 48 \\ \times 34 \\ \hline 192 \\ 144 \\ \hline 1632 \end{array}$$

When the check is made this way, the multiplications, 4×48 and 3×48 , are the same as in the division.

If there is a remainder in the division, add this to the product in checking your answer.

Example:

<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;"> <div style="text-align: right;">46</div> <div style="text-align: right;">35</div> </div> <div style="margin-right: 10px;"> $\begin{array}{r} 46 \\ 35 \overline{) 1626} \\ \underline{140} \\ 226 \\ \underline{210} \\ 16 \end{array}$ </div> <div> <div>Remainder</div> </div> </div>	<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;"> <div>Check:</div> </div> <div> $\begin{array}{r} 46 \\ \times 35 \\ \hline 230 \\ 138 \\ \hline 1610 \\ + 16 \\ \hline 1626 \end{array}$ </div> </div>
--	---

5-33. Checking Division by Dividing Your Answer into the Dividend

Dividing your answer (quotient) into the dividend should give the divisor as the new quotient.

Example:

$$\begin{array}{r} 74 \\ 38 \overline{) 2812} \\ \underline{266} \\ 152 \\ \underline{152} \end{array} \quad \text{Check:} \quad \begin{array}{r} 38 \\ 74 \overline{) 2812} \\ \underline{222} \\ 592 \\ \underline{592} \end{array}$$

5-34. Checking Division by Using a Short-Cut

Any short-cut that could be used to simplify the division may be used to check the answer.

Example (1):

$$\begin{array}{r} 3456 \\ 18 \overline{) 62208} \\ \underline{54} \\ 82 \\ \underline{72} \\ 100 \\ \underline{90} \\ 108 \\ \underline{108} \end{array} \quad \text{Check:} \quad \begin{array}{r} 2) 62208 \\ 9 \overline{) 31104} \\ \underline{3456} \end{array}$$

Example (2):

$$\begin{array}{r}
 539.448 \\
 125 \overline{) 67431.000} \\
 \underline{625} \\
 493 \\
 \underline{375} \\
 1181 \\
 \underline{1125} \\
 560 \\
 \underline{500} \\
 600 \\
 \underline{500} \\
 1000 \\
 \underline{1000} \\
 0
 \end{array}$$

Check: $67,431 \times 8 = 539,448;$
 $\div 1,000 = 539.448$
 (by Method 5-8D)

If you use a short-cut in doing the division the first time, you can check your answer by another short-cut or by conventional division.

5-35. Checking Division by Casting Out 9's

Casting out 9's is probably the fastest and easiest way to check division if the numbers are large. You can cast out 9's by any of the methods described in 1-6, Chapter I.

Small numbers are used in the next example to make it easier to show how the remainders from casting out 9's are applied to checking division.

Example (1):

$$\begin{array}{r}
 311 \\
 24 \overline{) 7468} \\
 \underline{72} \\
 26 \\
 \underline{24} \\
 28 \\
 \underline{24} \\
 4
 \end{array}$$

Check by steps:

(a) Cast out 9's from the divisor (24):

(b) Remainder

- 6 Remainder
- (b) Cast out 9's from the quotient (311):
- 5 Remainder
- (c) Multiply the remainders of (a) and (b): $6 \times 5 = 30$; cast out 9's:
- 3 Remainder
- (d) Cast out 9's from the remainder of the division (4) and add to the remainder of (c): $4 + 3 = 7$:
- 7 Remainder
- (e) Cast out 9's from the dividend (7468):
- 7 Remainder

The remainder (7) is the same in (d) and (e); the answer may therefore be assumed to be correct.

Example (2):

$$\begin{array}{r}
 1897 \\
 4364 \overline{) 8279461} \\
 \underline{4364} \\
 39154 \\
 \underline{34912} \\
 42426 \\
 \underline{39276} \\
 31501 \\
 \underline{30548} \\
 953
 \end{array}$$

Check by casting out 9's

- (a) Remainder of divisor: 8
- (b) Remainder of quotient: 7
- (c) $8 \times 7 = 56$; remainder: 2
- (d) Remainder of division's remainder (953): 8; add to (c): $8 + 2 = 10$; remainder: 1
- (e) Remainder of dividend: 1

Since the remainder (1) is the same in (d) and (e), the answer may be considered correct.

Keep in mind that a check by casting out 9's will not catch an error caused by digits being transposed or a decimal point misplaced since such errors will not affect the remainders. However, if the check does *not* prove your answer, there is definitely something wrong in the division, in the check, or in both.

VI. FRACTIONS, MIXED NUMBERS, AND PERCENTAGES

6-1. Definitions and Basic Principles Common fractions:

A common fraction is written with a whole number above a line and a whole number below the line, thus: $\frac{3}{4}$ ^{Numerator} _{Denominator}. Common fractions are usually referred to simply as “fractions.”

Fractions are a form of division. The line of a fraction means to divide the upper number (numerator) by the lower number (denominator); it has the same meaning as a division sign: $\frac{3}{4}$ or $3/4 = 3 \div 4 = 4)3$. When “of” is used with a fraction, it means to multiply: $\frac{3}{4}$ of $\bar{8} = \frac{3}{4} \times 8 = 6$.

Multiplying the numerator and denominator by the same number, or dividing both by the same number, does not change the value of a fraction, as shown here:

$$\underline{2} \times \underline{4} = \underline{8} \div \underline{2} = \underline{4} \div \underline{2} = \underline{2}$$

$$5 \times 4 = 20 \div 2 = 10 \div 2 = 5$$

Multiplying by $\frac{4}{4}$ is the same as multiplying by 1; dividing by $\frac{2}{2}$ is the same as dividing by 1. Multiplying or dividing by 1 leaves the value of the fraction unchanged.

Decimal fractions:

A decimal fraction is a common fraction that has 10, 100, 1,000, etc., as the denominator.

denominator.

Examples:

$$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$$

Such fractions may be expressed in decimal form by using a decimal point. Thus, $\frac{1}{10} = .1$, $\frac{1}{100} = .01$, $\frac{1}{1000} = .001$, etc. Fractions written this way are usually referred to as “decimals.”

As you know, the value of a digit is decreased by one-tenth for each place to the right of the decimal point and is increased ten times for each place to the left of the decimal point.

In general, you can use the same short-cuts with decimals as with whole numbers, but be careful to keep decimal points in their proper places. When multiplying with decimals, there must be as many decimal places in the product as the total of decimal places in the factors.

Mixed numbers:

A mixed number is a whole number with a fraction, such as $1\frac{1}{2}$, $2\frac{1}{4}$, $3\frac{3}{4}$. Such numbers may be reduced to fractional form, thus: $1\frac{1}{2} = \frac{3}{2}$; $2\frac{1}{4} = \frac{9}{4}$; $3\frac{3}{4} = \frac{15}{4}$.

Adding and Subtracting Fractions

6-2. Finding a Common Denominator

Before fractions can be added or subtracted, they must have the same denominator — a common denominator that is evenly divisible by the denominator of each fraction.

Example of addition: $\frac{1}{4} + \frac{2}{3}$

Solution by steps:

- (a) Since “fourths” and “thirds” cannot be added directly, obtain a common denominator by multiplying the denominators: $4 \times 3 = 12$

(b) Change the fractions to the common denominator: $\frac{1}{4} = \frac{3}{12}$; $\frac{2}{3} = \frac{8}{12}$.

Now you can add “twelfths”.

(c) Add the numerators and place their sum over the common denominator: $3 + 8 = 11$ twelfths = $\frac{11}{12}$, answer

Other examples:

$$\frac{1}{3} + \frac{1}{5} + \frac{3}{8}: 3 \times 5 \times 8 = 120, \text{ common denominator};$$

$$\frac{40}{120} + \frac{24}{120} + \frac{45}{120} = \frac{109}{120}, \text{ answer}$$

$$\frac{1}{9} + \frac{3}{4} + \frac{1}{7}: 9 \times 4 \times 7 = 252, \text{ common denominator};$$

$$\frac{28}{252} + \frac{189}{252} + \frac{36}{252} = \frac{253}{252}, \text{ or } 1\frac{1}{252}, \text{ answer}$$

Example of subtraction: $\frac{2}{5} - \frac{1}{4}$

Solution by steps:

(a) Common denominator: $5 \times 4 = 20$

(b) Change fractions to common denominator:

$$\frac{2}{5} = \frac{8}{20}; \quad \frac{1}{4} = \frac{5}{20}$$

(c) Subtract: $\frac{8}{20} - \frac{5}{20} = \frac{3}{20}$, answer also:

$$\frac{8-5}{20} = \frac{3}{20}$$

Other examples:

$$\frac{1}{6} - \frac{1}{9} = \frac{9}{54} - \frac{6}{54} = \frac{3}{54} = \frac{1}{18}, \text{ answer}$$

$$\frac{3}{5} - \frac{2}{7} = \frac{21}{35} - \frac{10}{35} = \frac{11}{35}, \text{ answer}$$

6-3. Converting Fractions to Decimals for Easier Adding or Subtracting

If fractions have denominators of 2, 4, 5, 8, or 10, you can change them easily into decimals to simplify adding or subtracting. This avoids the need to find a common denominator.

Example:

$$\frac{1}{5} + \frac{3}{4}$$

Solution:

$$\frac{1}{5} = .2; \quad \frac{3}{4} = .75; \quad + .2 = .95, \text{ or } \frac{95}{100},$$

or $\frac{19}{20}$, answer

also:

$$\frac{1}{5} + \frac{3}{4} = \frac{4}{20} + \frac{15}{20} = \frac{19}{20}$$

Other examples:

$$\frac{1}{8} + \frac{2}{5} = .125 + .4 = .525 \left(\frac{525}{1000} \text{ or } \frac{21}{40} \right)$$

$$\frac{1}{2} - \frac{1}{5} = .5 - .2 = .3 \left(\frac{3}{10} \right)$$

$$\frac{3}{10} - \frac{1}{4} = .3 - .25 = .05 \left(\frac{5}{100} \text{ or } \frac{1}{20} \right)$$

Fractions with denominators of 3, 6, 7, or 9 cannot be converted into decimals that come out “even.” For example, $\frac{1}{3} = .3333 \dots \frac{1}{3}$; $\frac{1}{6} = .1666 \dots \frac{2}{3}$, etc. It is therefore easier to add and subtract such fractions in their fractional form.

6-4. Fastest Way to Add Any Pair of Fractions

Example: $\frac{1}{4} + \frac{2}{3}$

Solution by steps:

(a) Multiply the denominators: $4 \times 3 = 12$, new denominator

(b) Cross multiply the numerators and denominators, and add: $1 \times 3 = 3$; 4×2

= 8; + 3 = 11, new numerator
(c) Answer: $\frac{11}{12}$

Short way:

$$\frac{1}{4} + \frac{2}{3} = \frac{3+8}{4 \times 3} = \frac{11}{12}$$

Other examples:

$$\frac{1}{6} + \frac{2}{5} = \frac{17}{30}$$
$$\frac{4}{9} + \frac{3}{8} = \frac{59}{72}$$

6-5. Fastest Way to Subtract Any Pair of Fractions

Example: $\frac{2}{5} - \frac{1}{4}$

Solution by steps:

(a) Multiply the denominators (same step as in adding fractions): $5 \times 4 = 20$, new denominator

(b) Cross multiply the numerators and denominators (as in adding), and subtract: $5 \times 1 = 5$; $4 \times 2 = 8$; $- 5 = 3$, new numerator

(c) Answer: $\frac{3}{20}$

Short way:

$$\frac{2}{5} - \frac{1}{4} = \frac{8-5}{5 \times 4} = \frac{3}{20}$$

Other examples:

$$\frac{5}{7} - \frac{3}{11} = \frac{34}{77}$$
$$\frac{11}{12} - \frac{5}{8} = \frac{28}{96} = \frac{7}{24}$$

6-6. To Add Two Fractions with Numerators of 1

The sum of the fractions = $\frac{\text{sum of the denominators}}{\text{product of the denominators}}$

Example:

$$\frac{1}{5} + \frac{1}{6}$$

Solution:

$$\frac{5+6}{5 \times 6} = \frac{11}{30}, \text{ answer}$$

Proving the answer:

$$\frac{1}{5} = \frac{6}{30}; \quad \frac{1}{6} = \frac{5}{30}; \quad \frac{6}{30} + \frac{5}{30} = \frac{11}{30}$$

Other examples:

$$\begin{aligned} \frac{1}{8} + \frac{1}{7} &= \frac{15}{56} \\ \frac{1}{3} + \frac{1}{11} &= \frac{14}{33} \end{aligned}$$

6-7. To Subtract Two Fractions with Numerators of 1

The difference of the fractions = $\frac{\text{difference of the denominators}}{\text{product of the denominators}}$

Example:

$$\frac{1}{5} - \frac{1}{6}$$

Solution:

$$\frac{6-5}{6 \times 5} = \frac{1}{30}, \text{ answer}$$

Proving the answer:

$$\frac{1}{5} = \frac{6}{30}; \quad \frac{1}{6} = \frac{5}{30}; \quad \frac{6}{30} - \frac{5}{30} = \frac{1}{30}$$

Other examples:

$$\begin{aligned} \frac{1}{7} - \frac{1}{9} &= \frac{2}{63} \\ \frac{1}{3} - \frac{1}{11} &= \frac{8}{33} \end{aligned}$$

Multiplying and Dividing by Fractions

6-8. To Multiply by a Fraction

Multiplying is probably the easiest of the operations with fractions.

A. To multiply a whole number by a fraction, divide the number by the denominator and multiply by the numerator; or multiply first by the numerator and then divide by the denominator, whichever way is easier.

Examples:

$$\begin{aligned} 15 \times \frac{2}{3}: \quad 15 \div 3 &= 5; \times 2 = 10 \\ \text{or: } 15 \times 2 &= 30; \div 3 = 10 \\ 16 \times \frac{3}{4}: \quad 16 \div 4 &= 4; \times 3 = 12 \\ 23 \times \frac{3}{10}: \quad 23 \times 3 &= 69; \div 10 = 6.9 \end{aligned}$$

B. To multiply fractions together, multiply numerators by numerators and denominators by denominators but first simplify by reducing and cancelling, where possible.

Example:

$$\frac{3}{5} \times \frac{1}{6} \times \frac{10}{11}$$

Solution:

$$\frac{3}{5} \times \frac{1}{6} \times \frac{10}{11} = \frac{1}{11}, \text{ answer}$$

Proving the answer:

$$\frac{3 \times 1 \times 10}{5 \times 6 \times 11} = \frac{30}{330} = \frac{1}{11}$$

Other examples:

$$\frac{3}{7} \times \frac{8}{5} \times \frac{4}{13} = \frac{3 \times 8 \times 4}{7 \times 5 \times 13} = \frac{96}{455}, \text{ answer}$$

$$\frac{5}{18} \times \frac{9}{10} \times \frac{3}{4} = \frac{5 \times 9 \times 3}{18 \times 10 \times 4} = \frac{3}{16}, \text{ answer}$$

To save time in simplifying a fraction, apply the tests for divisibility (Methods 5-10 to 5-24) to see if the numerator and denominator can be divided evenly by the same number (a common factor). Keep dividing by common factors until the fraction has been reduced to its simplest form.

6-9. To Divide by a Fraction

Invert the fraction and proceed as in multiplication.

Example: $8 \div \frac{1}{2}$ (when $\frac{1}{2}$ is inverted it becomes $\frac{2}{1}$ or simply 2)

Solution:

$8 \div \frac{1}{2} = 8 \times 2 = 16$, answer; which means that $\frac{1}{2}$ goes into 8 16 times, or that there are $16 \frac{1}{2}$'s in 8,

Other examples:

$$9 \div \frac{3}{4} = 9 \times \frac{4}{3} = 12$$

$$16 \div \frac{8}{9} = 16 \times \frac{9}{8} = 18$$

If fractions have a common denominator, you can ignore the denominators and divide the numerators like whole numbers.

Examples:

$$\frac{4}{7} \div \frac{2}{7} = 4 \div 2 = 2, \text{ answer; or } \frac{4}{\cancel{7}} \times \frac{\cancel{7}}{2} = 2$$

$$\frac{2}{7} \div \frac{4}{7} = 2 \div 4 = \frac{1}{2}, \text{ answer; or } \frac{2}{\cancel{7}} \times \frac{\cancel{7}}{4} = \frac{1}{2}$$

$$\frac{5}{9} \div \frac{2}{9} = 5 \div 2 = 2\frac{1}{2}, \text{ answer}$$

6-10. To Multiply a Number by $\frac{3}{4}$

Method A: Take $\frac{1}{4}$ of the number and subtract it from the number.

Example: $1,848 \times \frac{3}{4}$

Solution: $1,848 \times \frac{1}{4} = 462$; $1,848 - 462 = 1,386$, answer

Proving the answer:

$$\begin{array}{r} 1,848 \\ \times 3 \\ \hline 4 \overline{) 5,544} \\ 1,386 \end{array}$$

Why it works: Subtracting $\frac{1}{4}$ of a number from the number leaves $\frac{3}{4}$ of the number. This is the same as multiplying the number by $\frac{3}{4}$

Method B: Halve the number and, to this, add $\frac{1}{4}$ of the half.

Example: $128 \times \frac{3}{4}$

Solution: $128 \times \frac{1}{2} = 64$; $\times \frac{1}{2} = 32$; $+ 64 = 96$, answer

Proving the answer:

$$\begin{array}{r} 128 \\ \times 3 \\ \hline 4 \overline{) 384} \\ 96 \end{array}$$

Why it works:

$$\frac{1}{2} + (\frac{1}{2} \text{ of } \frac{1}{2}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Multiplying and Dividing with Mixed Numbers

6-11. Simplifying a Mixed Number Before Multiplying

Example:

$$19\frac{7}{8} \times 16$$

Solution by steps:

(a) Simplify $19\frac{7}{8}$ to a whole number ending in zero by adding $\frac{1}{8} = 20$

(b) Multiply: $20 \times 16 = 320$; this is $\frac{1}{8}$ of 16 too much

(c) Subtract $\frac{1}{8}$ of 16: $320 - 2 = 318$, answer

Proving the answer:

$$\begin{array}{r} 19\frac{7}{8} \\ \times 16 \\ \hline 14 \quad (16 \times \frac{7}{8} = 14) \\ 114 \\ \hline 19 \\ \hline 318 \end{array}$$

Other examples:

$$\begin{array}{l} 15\frac{3}{4} \times 25: 16 \times 25 = 400; - (\frac{1}{4} \times 25) = 393\frac{3}{4} \\ 18\frac{2}{3} \times 39: 20 \times 39 = 780; - (1\frac{1}{3} \times 39) = 728 \end{array}$$

You can also simplify a mixed number by splitting it into parts — into the whole number and the fraction — and multiplying these separately.

Example: $19\frac{7}{8} \times 16$

Solution by steps:

(a) $19 \times 16 = 304$ (by Method 4-32 or 4-39)

(b) $\frac{7}{8} \times 16 = 14$

(c) Add (a) and (b): $304 + 14 = 318$, answer

Short way: $19 \times 16 = 304; + (\frac{7}{8} \times 16) = 318$

6-12. To Square a Mixed Number Ending in $\frac{1}{2}$

Example: $4\frac{1}{2} \times 4\frac{1}{2}$ (same as 4.5 x 4.5 in Method 4-13)

Solution by steps:

(a) Multiply the whole number by the number + 1: $4 \times 5 = 20$

(b) Place $\frac{1}{4}$ after the product: $20\frac{1}{4}$, answer

Short way: $4 \times 5 + \frac{1}{4} = 20\frac{1}{4}$

Proving the answer: $4\frac{1}{2} \times 4\frac{1}{2} = \frac{9}{2} \times \frac{9}{2} = \frac{81}{4} = 20\frac{1}{4}$

Other examples:

$$\begin{aligned} 6\frac{1}{2} \times 6\frac{1}{2} &= 42\frac{1}{4} \\ 7\frac{1}{2} \times 7\frac{1}{2} &= 56\frac{1}{4} \\ 9\frac{1}{2} \times 9\frac{1}{2} &= 90\frac{1}{4} \end{aligned}$$

6-13. To Multiply a Number by a Mixed Number Ending in $\frac{1}{2}$: Halve-and-Double Method

Halve the number to be multiplied, double the mixed number, and multiply. This gets rid of the fraction for easier multiplication.

Example:

$$48 \times 3\frac{1}{2}$$

Solution: $\frac{1}{2} \times 48 = 24$; $2 \times 3\frac{1}{2} = 7$; $24 \times 7 = 168$, answer

Proving the answer:

$$48 \times 3\frac{1}{2} = \overset{24}{\cancel{48}} \times \frac{7}{2} = 168$$

Why it works: In multiplication, halving one number and doubling the other does not change the product of the two numbers. (Method 4-25 for multiplying by 15, 1.5, 35, 3.5, etc., works the same way.)

Other examples:

$$64 \times 3\frac{1}{2} = 32 \times 7 = 224$$

$$22 \times 4\frac{1}{2} = 11 \times 9 = 99$$

$$4.8 \times 3\frac{1}{2} = 2.4 \times 7 = 16.8$$

6-14. To Divide a Number by a Mixed Number Ending in $\frac{1}{2}$

Double both numbers and divide. This gets rid of the fraction for easier division.

Example:

$$42 \div 3\frac{1}{2}$$

Solution:

$$2 \times 42 = 84; 2 \times 3\frac{1}{2} = 7; 84 \div 7 = 12, \text{ answer}$$

Proving the answer:

$$42 \div 3\frac{1}{2} = 42 \div \frac{7}{2} = 42 \times \frac{2}{7} = 12$$

Why it works: Multiplying the dividend and the divisor by the same number does not change the answer in division. (Method 5-28 for dividing by whole numbers ending in 5 works the same way.)

Other examples:

$$37 \div 4\frac{1}{2} = 74 \div 9 = 8\frac{2}{9}$$

$$51 \div 5\frac{1}{2} = 102 \div 11 = 9\frac{3}{11}$$

6-15. To Multiply Two Mixed Numbers When They Both End in $\frac{1}{2}$ and the Sum of Their Whole Numbers Is Even

Example: $4\frac{1}{2} \times 6\frac{1}{2}$ (same as 4.5 x 6.5 in Method 4-29)

Solution by steps:

(a) Multiply the whole numbers: $4 \times 6 = 24$

(b) Obtain $\frac{1}{2}$ the sum of the whole numbers: $\frac{4+6}{2} = 5$

(c) Add (a) and (b) and attach $\frac{1}{4}$: $24 + 5 = 29$; $29\frac{1}{4}$, answer

Short way: $(4 \times 6) + \frac{1}{2}(4 + 6) = 29$, answer $29\frac{1}{4}$

Proving the answer: $4\frac{1}{2} \times 6\frac{1}{2} = \frac{9}{2} \times \frac{13}{2} = \frac{117}{4} = 29\frac{1}{4}$

Other examples:

$$8\frac{1}{2} \times 4\frac{1}{2} = 38\frac{1}{4}$$

$$7\frac{1}{2} \times 3\frac{1}{2} = 26\frac{1}{4}$$

$$12\frac{1}{2} \times 8\frac{1}{2} = 106\frac{1}{4}$$

6-16. To Multiply Two Mixed Numbers When They Both End in $\frac{1}{2}$ and the Sum of Their Whole Numbers Is Odd

Example: $7\frac{1}{2} \times 2\frac{1}{2}$ (same as 7.5 x 2.5 in Method 4-30)

Solution by steps: same as Method 6-15 but there is a fraction to be dropped and the answer ends in $\frac{3}{4}$ instead of $\frac{1}{4}$.

(a) Multiply the whole numbers: $7 \times 2 = 14$

(b) Obtain $\frac{1}{2}$ the sum of the whole number: $\frac{7+2}{2} = 4\frac{1}{2}$ drop the $\frac{1}{2}$, = 4

(c) Add (a) and (b) and attach $\frac{3}{4}$: $14 + 4 = 18$; $18\frac{3}{4}$ answer

Short way: $(7 \times 2) + \frac{1}{2}(7 + 2) - \frac{1}{2} = 18$ answer $18\frac{3}{4}$

Other examples:

$$5\frac{1}{2} \times 4\frac{1}{2} = 24\frac{3}{4}$$

$$6\frac{1}{2} \times 9\frac{1}{2} = 61\frac{3}{4}$$

$$11\frac{1}{2} \times 12\frac{1}{2} = 143\frac{3}{4}$$

6-17. To Multiply Two Mixed Numbers Having the Same Whole Numbers, and Fractions That Add to 1

Example: $6\frac{3}{4} \times 6\frac{1}{4}$

Solution by steps:

(a) Multiply the whole number by the number + 1: $6 \times 7 = 42$

(b) Find the product of the fractions: $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$

(c) Add (a) and (b): $42 + \frac{3}{16} = 42\frac{3}{16}$, answer

Short way: $(5 \times 7) + (\frac{3}{4} \times \frac{1}{4}) = 42\frac{3}{16}$

Proving the answer:

$$6\frac{3}{4} \times 6\frac{1}{4} = \frac{27}{4} \times \frac{25}{4} = \frac{675}{16} = 42\frac{3}{16}$$

Other examples: $5\frac{1}{8} \times 5\frac{7}{8} = 30\frac{7}{64}$

$$5\frac{1}{8} \times 5\frac{7}{8} = 30\frac{7}{64}$$

$$7\frac{1}{9} \times 7\frac{8}{9} = 56\frac{8}{81}$$

$$16\frac{1}{4} \times 16\frac{3}{4} = 272\frac{3}{16}$$

6-18. To Multiply or Divide by $2\frac{1}{2}$

A. To multiply a number by $2\frac{1}{2}$, divide the number by 4 and multiply by 10 (same as Method 4-8B for multiplying by 2.5).

Example: $144 \times 2\frac{1}{2}$

$$144 \times 2\frac{1}{2}$$

Solution:

$$144 \div 4 = 36; \times 10 = 360, \text{ answer}$$

Proving the answer:

$$144 \times 2\frac{1}{2} = \overset{72}{144} \times \frac{5}{2} = 360$$

Why it works: Dividing by 4 and multiplying by 10 is the same as multiplying by $\frac{10}{4}$ which equals $2\frac{1}{2}$.

B. To divide a number by $2\frac{1}{2}$, multiply the number by 4 and divide by 10 (same as Method 5-7B for dividing by 2.5).

Example:

$$72 \div 2\frac{1}{2}$$

$$\text{Solution: } 72 \times 4 = 288; \div 10 = 28.8, \text{ answer}$$

Proving the answer:

$$72 \div 2\frac{1}{2} = 72 \div \frac{5}{2} = 72 \times \frac{2}{5} \\ = \frac{144}{5} = 28.8$$

Why it works: Multiplying by 4 and dividing by 10 is the same as multiplying by $\frac{4}{10}$, which is equivalent to dividing by $\frac{10}{4}$ which equals $2\frac{1}{2}$.

Other examples:

$$635 \div 2\frac{1}{2} = 254 \\ 24 \div 2\frac{1}{2} = 9.6$$

6-19. To Multiply or Divide by $12\frac{1}{2}$

A. To multiply a number by $12\frac{1}{2}$, divide the number by 8 and multiply by 100 (same as Method 4-9C for multiplying by 12.5).

Example:

$$168 \times 12\frac{1}{2}$$

Solution: $168 \div 8 = 21$; $\times 100 = 2,100$, answer

Proving the answer:

$$168 \times 12\frac{1}{2} = \overset{84}{168} \times \frac{25}{2} = 2,100$$

Why it works: Dividing by 8 and multiplying by 100 is the same as multiplying by $\frac{100}{8}$ which equals $12\frac{1}{2}$.

Other examples: $38 \times 12\frac{1}{2}$: $38 \div 8 = 4\frac{3}{4} = 4.75$; $\times 100 = 475$
 $81 \times 12\frac{1}{2}$: $81 \div 8 = 10\frac{1}{8} = 10.125$; $\times 100 = 1,012.5$

B. To divide a number by $12\frac{1}{2}$, multiply the number by 8 and divide by 100

(same as Method 5-8C for dividing by 12.5).

Example :

$$52 \div 12\frac{1}{2}$$

Solution: $52 \times 8 = 416; \div 100 = 4.16$, answer

Proving the answer:

$$52 \div 12\frac{1}{2} = 52 \div \frac{25}{2} = 52 \times \frac{2}{25} = \frac{104}{25} = 4\frac{4}{25} = 4.16$$

Why it works: Multiplying by 8 and dividing by 100 is the same as multiplying by $\frac{8}{100}$, which is equivalent to dividing by $\frac{100}{8}$ which equals $12\frac{1}{2}$.

Other examples:

$$\begin{aligned} 83 \div 12\frac{1}{2}: 83 \times 8 &= 664; \div 100 = 6.64 \\ 315 \div 12\frac{1}{2}: 315 \times 8 &= 2,520; \div 100 = 25.2 \end{aligned}$$

6-20. To Multiply or Divide by $16\frac{2}{3}$

A. To multiply a number by $16\frac{2}{3}$, divide the number by 6 and multiply by 100.

Example:

$$72 \times 16\frac{2}{3}$$

Solution: $72 \div 6 = 12; \times 100 = 1,200$, answer

Proving the answer:

$$72 \times 16\frac{2}{3} = \overset{24}{\cancel{72}} \times \frac{50}{3} = 1,200$$

Why it works: Dividing by 6 and multiplying by 100 is the same as multiplying by $\frac{100}{6}$ which equals $16\frac{2}{3}$.

Other examples:

$$39 \times 16\frac{2}{3}: 39 \div 6 = 6\frac{1}{2} = 6.5; \times 100 = 650$$

$$44 \times 16\frac{2}{3}: 44 \div 6 = 7\frac{1}{3}; \times 100 = 733\frac{1}{3}$$

B. To divide a number by $16\frac{2}{3}$, multiply the number by 6 and divide by 100.

Example :

$$91 \div 16\frac{2}{3}$$

Solution :

$$91 \times 6 = 546 ; \div 100 = 5.46, \text{ answer}$$

Proving the answer:

$$91 \div 16\frac{2}{3} = 91 \div \frac{50}{3} = 91 \times \frac{3}{50} = \frac{273}{50} = 5.46$$

Why it works: Multiplying by 6 and dividing by 100 is the same as multiplying by $\frac{6}{100}$, which is equivalent to dividing by $\frac{100}{6}$ which equals $16\frac{2}{3}$.

Other examples:

$$39 \div 16\frac{2}{3}: 39 \times 6 = 234; \div 100 = 2.34$$

$$141 \div 16\frac{2}{3}: 141 \times 6 = 846; \div 100 = 8.46$$

6-21. To Multiply or Divide by $33\frac{1}{3}$

A. To multiply a number by $33\frac{1}{3}$, divide the number by 3 and multiply by 100.

Example:

$$93 \times 33\frac{1}{3}$$

Solution: $93 \div 3 = 31$; $\times 100 = 3,100$, answer

Proving the answer:

$$93 \times 33\frac{1}{3} = \overset{31}{\cancel{93}} \times \frac{100}{\cancel{3}} = 3,100$$

Why it works: Dividing by 3 and multiplying by 100 is the same as multiplying by $\frac{100}{3}$ which equals $33\frac{1}{3}$.

Other examples:

$$\begin{aligned} 28 \times 33\frac{1}{3} &: 28 \div 3 = 9\frac{1}{3}; \times 100 = 933\frac{1}{3} \\ 47 \times 33\frac{1}{3} &: 47 \div 3 = 15\frac{2}{3}; \times 100 = 1,566\frac{2}{3} \end{aligned}$$

B. To divide a number by $33\frac{1}{3}$, multiply the number by 3 and divide by 100.

Example:

$$53 \div 33\frac{1}{3}$$

Solution: $53 \times 3 = 159$; $\div 100 = 1.59$, answer

Proving the answer:

$$53 \div 33\frac{1}{3} = 53 \div \frac{100}{3} = 53 \times \frac{3}{100} = \frac{159}{100} = 1.59$$

Why it works: Multiplying by 3 and dividing by 100 is the same as

multiplying by $\frac{3}{100}$, which is equivalent to dividing by $\frac{100}{3}$ which equals $33\frac{1}{3}$.

Other examples:

$$\begin{aligned} 67 \div 33\frac{1}{3}: 67 \times 3 = 201; \div 100 = 2.01 \\ 156 \div 33\frac{1}{3}: 156 \times 3 = 468; \div 100 = 4.68 \end{aligned}$$

6-22. To Multiply or Divide by $66\frac{2}{3}$

A. To multiply a number by $66\frac{2}{3}$, divide the number by 3 and multiply by 200 (or by 2, then by 100).

Example:

$$48 \times 66\frac{2}{3}$$

Solution: $48 \div 3 = 16; \times 200 = 3,200$, answer

Proving the answer:

$$48 \times 66\frac{2}{3} = \overset{16}{48} \times \frac{200}{3} = 3,200$$

Why it works: Dividing by 3 and multiplying by 200 is the same as multiplying by $\frac{200}{3}$ which equals $66\frac{2}{3}$.

Other examples:

$$\begin{aligned} 37 \times 66\frac{2}{3}: 37 \div 3 = 12\frac{1}{3}; \times 2 = 24\frac{2}{3}; \times 100 = 2,466\frac{2}{3} \\ 18.63 \times 66\frac{2}{3}: 18.63 \div 3 = 6.21; \times 2 = 12.42; \times 100 \\ = 1,242 \end{aligned}$$

Alternate method of multiplying by $66\frac{2}{3}$:

Multiply the number by 100 and subtract $\frac{1}{3}$ of the product.

Example:

$$48 \times 66\frac{2}{3}$$

Solution: $48 \times 100 = 4,800$; $- 1,600 = 3,200$, answer

Why it works: $66\frac{2}{3}$ is $\frac{1}{3}$ less than 100 so that multiplying by 100 and subtracting $\frac{1}{3}$ of the product is the same as multiplying by $66\frac{2}{3}$.

B. To divide a number by $66\frac{2}{3}$, multiply the number by 3 and divide by 200 (or by 2, then by 100).

Example:

$$46 \div 66\frac{2}{3}$$

Solution: $46 \times 3 = 138$; $\div 2 = 69$; $\div 100 = .69$, answer *Proving the answer:*

$$46 \div 66\frac{2}{3} = 46 \div \frac{200}{3} = 46 \times \frac{3}{200} = \frac{138}{200} = \frac{69}{100} = .69$$

Why it works: Multiplying by 3 and dividing by 200 is the same as multiplying by $\frac{3}{200}$, which is equivalent to dividing by $\frac{200}{3}$ which equals $66\frac{2}{3}$.

Other examples:

$$\begin{array}{l} 154 \div 66\frac{2}{3}: 154 \times 3 = 462; \div 2 = 231; \div 100 = 2.31 \\ 47 \div 66\frac{2}{3}: 47 \times 3 = 141; \div 2 = 70.5; \div 100 = .705 \end{array}$$

Alternate method of dividing by $66\frac{2}{3}$:

Divide the number by 100 and increase the result by half.

Example:

$$284 \div 66\frac{2}{3}$$

Solution:

$$284 \div 100 = 2.84; \times \frac{1}{2} = 1.42; + 2.84 = 4.26, \text{ answer}$$

Proving the answer:

$$\text{Proving the answer:}$$

$$284 \div 66\frac{2}{3} = 284 \div \frac{200}{3} = 284 \times \frac{3}{200} = \frac{426}{100} = 4.26$$

Why it works: 100 is $\frac{1}{2}$ greater than $66\frac{2}{3}$ ($66\frac{2}{3} + \frac{1}{2}$ of $66\frac{2}{3} = 66\frac{2}{3} + 33\frac{1}{3} = 100$) so that dividing a number by 100 and adding $\frac{1}{2}$ of the result gives the same answer as dividing the number by $66\frac{2}{3}$.

Table of Multiplying by Certain Mixed Numbers

N stands for the Number to be multiplied

To Multiply by Method

$$2\frac{1}{2} \quad N/4 \times 10$$

$$12\frac{1}{2} \quad N/8 \times 100$$

$$16\frac{2}{3} \quad N/6 \times 100$$

$$33\frac{1}{3} \quad N/3 \times 100$$

$$37\frac{1}{2} \quad N/8 \times 3 \times 100$$

$$62\frac{1}{2} \quad N/8 \times 5 \times 100$$

$$66\frac{2}{3} \quad N/3 \times 2 \times 100$$

$$66\frac{2}{3} \quad 100N - 100N/3$$

$$87\frac{1}{2} \quad 100N - 100N/8$$

Table of Dividing by Certain Mixed Numbers

N stands for the Number to be divided

<i>To Divide by</i>	<i>Method</i>
$2\frac{1}{2}$	$4N/10$
$12\frac{1}{2}$	$8N/100$
$16\frac{2}{3}$	$6N/100$
$33\frac{1}{3}$	$3N/100$
$37\frac{1}{2}$	$8N/3 \div 100$
$62\frac{1}{2}$	$8N/5 \div 100$
$66\frac{2}{3}$	$3N/2 \div 100$
$66\frac{2}{3}$	$N/100 + ()$
$87\frac{1}{2}$	$(N + N/7) \div 100$

Working with Percentages

“Per cent” simply means “per hundred” or “per 100”. Thus, 6% means 6 per 100, or 6 in every 100, or 6 out of every 100. This may also be written as a decimal, .06, or as a common fraction, $\frac{6}{100}$

Accordingly, 1 % of a number is $\frac{1}{100}$ of the number, 10% of a number is $\frac{10}{100}$ or $\frac{1}{10}$ of the number, 50% of a number is $\frac{50}{100}$ or $\frac{1}{2}$ of the number, etc.

To answer the question as to what per cent one number is of another, simply set up the numbers as a fraction and change this to per cent, as follows:

1. Make the number that follows the words. “what per cent is.” the numerator

... make the number that follows the words, "what per cent is," the numerator of the fraction.

2. Make the other number the denominator of the fraction.

3. Reduce the fraction, if possible, and then change it to per cent by multiplying by 100.

Example: What per cent is 16 of 64?

Solution by steps:

(a) Make 16 the numerator and 64 the denominator: $\frac{16}{64}$; this can be reduced to $\frac{1}{4}$

(b) Multiply the fraction by 100 to change it to per cent: $\frac{1}{4} \times 100 = 25\%$; which means, of course, that 16 is 25% of 64

Short way: $\frac{16}{64} = \frac{1}{4}; \times 100 = 25\%$

Other examples:

What per cent is 64 of 16? $\frac{64}{16} = 4; \times 100 = 400\%$

What per cent is 5 of 10? $\frac{5}{10} = \frac{1}{2}; \times 100 = 50\%$

What per cent is 4 of 36? $\frac{4}{36} = \frac{1}{9}; \times 100 = 11\frac{1}{9}\%$

6-23. To Find $16\frac{2}{3}\%$ of a Number

Divide the number by 6.

Example: Find

$16\frac{2}{3}\%$ of 90.

Solution:

$$90 \div 6 = 15; \text{ or } 90 \times \frac{1}{6} = 15, \text{ answer}$$

Proving the answer: What per cent is 15 of 90?

$$\frac{15}{90} = \frac{1}{6}; \times 100 = 16\frac{2}{3}\%$$

Other examples:

$$16\frac{2}{3}\% \text{ of } 54 = 9$$

$$16\frac{2}{3}\% \text{ of } \$7.50 = \$1.25$$

6-24. To Find 20 % of a Number

Divide the number by 5.

Example: Find 20% of 95.

Solution: $95 \div 5 = 19$; or $95 \times \frac{1}{5} = 19$, answer

Proving the answer: What per cent is 19 of 95?

$$\frac{19}{95} = \frac{1}{5}; \times 100 = 20\%$$

Other examples:

$$20\% \text{ of } 65 = 13 \quad 20\% \text{ of } \$4.50 = \$.90$$

6-25. To Find $33\frac{1}{3}\%$ of a Number

Divide the number by 3.

Example: Find $33\frac{1}{3}\%$ of 87.

Solution: $87 \div 3 = 29$; or $87 \times \frac{1}{3} = 29$, answer

Proving the answer:

What per cent is 29 of 87?

$$\frac{29}{87} = \frac{1}{3}; \times 100 = 33\frac{1}{3}\%$$

Other examples:

$$33\frac{1}{3}\% \text{ of } 72 = 24$$

$$33\frac{1}{3}\% \text{ of } \$10.50 = \$3.50$$

6-26. To Find 50% of a Number

Divide the number by 2.

Example: Find 50% of 96.

Solution:

$$96 \div 2 = 48; \text{ or } 96 \times \frac{1}{2} = 48, \text{ answer}$$

Proving the answer: What per cent is 48 of 96%?

$$\frac{48}{96} = \frac{1}{2}; \times 100 = 50\%$$

Other examples:

$$50\% \text{ of } 185 = 92\frac{1}{2}$$

$$50\% \text{ of } \$3.50 = \$1.75$$

6-27. To Find $66\frac{2}{3}\%$ of a Number

Divide the number by 3 and multiply by 2.

Example: Find

$$66\frac{2}{3}\% \text{ of } 108$$

Solution: $108 \div 3 = 36$; $\times 2 = 72$, answer or:

$$108 \times \frac{2}{3} = 72$$

Proving the answer: What per cent is 72 of 108?

$$\frac{72}{108} = \frac{2}{3}; \times 100 = 66\frac{2}{3}\%$$

Other examples:

$$66\frac{2}{3}\% \text{ of } 45 = 30$$

$$66\frac{2}{3}\% \text{ of } \$4.86 = \$3.24$$

Table for Finding Certain Per Cents

To Find This Per Cent of a Number (N) Method

$$2\frac{1}{2}\% \qquad N/40$$

$$12\frac{1}{2}\% \qquad N/8$$

$$16\frac{2}{3}\% \qquad N/6$$

$$20\% \qquad N/5$$

$$25\% \qquad N/4$$

$$33\frac{1}{3}\% \qquad N/3$$

$$37\frac{1}{2}\% \qquad N/8 \times 3$$

$$\dots\dots\dots$$

50%

$N/2$

$62\frac{1}{2}\%$

$N/8 \times 5$

$66\frac{2}{3}\%$

$N/3 \times 2$

$66\frac{2}{3}\%$

$N - N/3$

75 %

$N \times 3/4$

75 %

$N - N/4$

$87\frac{1}{2}\%$

$N - N/8$

INDEX

FUNDAMENTALS OF SHORT-CUT METHODS

- 1-1. Basic Ways of Simplifying Calculations
- 1-2. Results are What You Want
- 1-3. Combining Mental and Written Math for Best Results
- 1-4. Approximating and Rounding Off
- 1-5. Achieving Accuracy with Speed Checking
- 1-6. Testing Answers by Casting Out's

ADDITION

- 2-1. Basic Principles
- 2-2. Adding by 10-Groups
- 2-3. Reaching 10-Levels
- 2-4. Adding by Other Combinations of Numbers
- 2-5. Adding by Multiplying
- 2-6. Adding Two or More Columns
- 2-7. Adding Long Columns
- 2-8. No-Carry Addition
- 2-9. Rounding Off Numbers for Addition
- 2-10. Rounding Off Dollars and Cents for Addition
- 2-11. Adding a Regular Series of Numbers
- 2-12. Add-and-Subtract Method of Simplifying Numbers
- 2-13. Splitting Numbers into Easy Parts
- 2-14. Thinking of Numbers as "Dollars" and "Cents" Checking Your Answer in Addition
- 2-15. Checking Your Answer by Adding Again in Reverse
- 2-16. Checking Addition by the No-Carry Method

2-16. Checking Addition by the NO-Carry Method

2-17. Checking Addition by Casting Out 9's

SUBTRACTION

3-1. Introduction

3-2. Add-or-Subtract Method of Simplifying Numbers

3-3. Subtraction without Borrowing

3-4. Splitting Numbers into Easy Parts

3-5. Counting Change after a Purchase

3-6. Checking Your Answer in Subtraction

MULTIPLICATION

4-1. Basic Principles

4-2. Multiplication When One Number Has All Digits Alike

4-3. To Multiply by a Number Made Up of Multiples

4-4. No-Carry Multiplication

4-5. To Multiply a Number by, etc.

A. If the number has a decimal point

B. If the number has no decimal point

Multiplying by Aliquot Parts of 1, 10, 100, and 1,000

4-6. General Method for Multiplying by Aliquot Parts

4-7. To Multiply by .5, 5, 50, or 500

4-8. To Multiply by .25, 2.5, 25, or 250

4-9. To Multiply by .125, 1.25, 12.5, or

4-10. To Multiply by .75, 7.5, 75, or 750

Summary of Multiplication by Aliquot Parts and 75-Numbers

To Obtain the Squares of Certain Numbers

4-11. To Square a Number Ending in 1

4-12. To Square a Number Ending in 4

4-13. To Square a Number Ending in 5

4-14. To Square a Number Ending in 6

4-15. To Square a Number Ending in 9

Summary of Squaring Certain Numbers

Multiplying Numbers That Have Small Differences

4-16. To Multiply Two Numbers with a Difference of 1

4-17. To Multiply Two Numbers with a Difference of 2

4-18. To Multiply Two Numbers with a Difference of 3

4-19. To Multiply Two Numbers with a Difference of 4

4-20. To Multiply Two Numbers with a Difference of

Summary of Multiplying Numbers Having Small Differences

Multiplying by Numbers Ending in 1

4-21. General Method for Multiplying by Numbers Ending in 1

4-22. To Multiply a Number by 11

4-23. To Multiply a Number by 21

4-24. To Multiply a Number by, etc.

Multiplying by Numbers Ending in 5

4-25. To Multiply by, etc.

4-26. Alternate Method for Multiplying a Number by 15

4-27. Alternate Method for Multiplying a Number by

4-28. Alternate Method for Multiplying a Number by

4-29. To Multiply One Number by Another When They Both End in 5 and the Sum of Their Other Digits is Even

4-30. To Multiply One Number by Another When They Both End in 5 and the Sum of Their Other Digits is Odd

Multiplying by 9, by Numbers Ending in 9, and by Multiples of 9

4-31. To Multiply a Number by 9

4-32. To Multiply a Number by 19, 29, 39, etc.

4-33. To Multiply by a Multiple of 9 from 18 to 81

4-34. To Multiply a Number by 99

4-35. To Multiply by a Multiple of 99

Multiplying by Numbers Near 100

4-36. To Multiply a Number by 98

To Multiply a Number by 99

4-37. To Multiply a Number by 101

4-38. To Multiply a Number by 102

Other Short-Cuts in Multiplication

4-39. To Multiply One “Teen” Number by Another

4-40. To Multiply Two Numbers When Their End Digits Add to 10 and Their Other Digits are the Same

4-41. To Multiply Two Two-Digit Numbers When Their First Digits Add to 10 and Their End Digits Are the Same

Checking Results in Multiplication

4-42. Estimate the Answer Before You Multiply

4-43. Checking Multiplication by Multiplying in Reverse

4-44. Checking Multiplication by Using a Short-Cut

4-45. Checking Multiplication by Dividing the Product by One of the Factors Used

4-46. Checking Multiplication by Casting Out 9’s

DIVISION

5-1. Basic Principles

5-2. Doing Short Division

5-3. Doing Long Division

5-4. To Divide by 10, 100, 1,000, etc.

Dividing by Aliquot Parts of 1, 10, 100, and 1,000

5-5. General Method for Dividing by Aliquot Parts

5-6. To Divide by .5, 5, 50, or 500

5-7. To Divide by .25, 2.5, 25, or 250

5-8. To Divide by .125, 1.25, 12.5 or 125

5-9. To Divide by .75, 7.5, 75, or 750

Summary of Division by Aliquot Parts and-Numbers

Testing the Divisibility of a Number

5-10. Test for Divisibility by Odd and Even Numbers

5-11. Test for Divisibility by 2

5-12. Test for Divisibility by 3

5-13. Test for Divisibility by 4

5-14. Test for Divisibility by 5

5-15. Test for Divisibility by 6

5-16. Test for Divisibility by 7

5-17. Test for Divisibility by 8

5-18. Test for Divisibility by 9

5-19. Test for Divisibility by 10

Summary of Tests for Divisibility: 2 to 10

Testing for Divisibility by Larger Divisors

5-20. Test for Divisibility by 11

5-21. Test for Divisibility by 12

5-22. Test for Divisibility by 15

5-23. Test for Divisibility by 18

5-24. Extending the Tests to Divisibility by Larger Divisors

Simplifying Dividends and Divisors

5-25. Simplifying the Dividend by Adding or Subtracting

5-26. Simplifying the Dividend by Breaking It into Parts

5-27. Simplifying the Divisor by Factoring It

5-28. Simplifying the Dividend and Divisor by Multiplying or Dividing Both by the Same Number

5-29. Problems Involving Both Multiplication and Division

Checking Results in Division

5-30. Estimate the Answer Before You Divide

5-31. Check the Subtractions in Long Division As You Go

5-32. Checking Division by Multiplying Your Answer by the Divisor

- 5-32. Checking Division by Multiplying Your Answer by the Divisor
- 5-33. Checking Division by Dividing Your Answer into the Dividend.
- 5-34. Checking Division by Using a Short-Cut
- 5-35. Checking Division by Casting Out's

FRACTIONS, MIXED NUMBERS, AND PERCENTAGES

6-1. Definitions and Basic Principles

- Common fractions
- Decimal fractions
- Mixed Numbers

Adding and Subtracting Fractions.

6-2. Finding a Common Denominator

6-3. Converting Fractions to Decimals for Easier Adding or Subtracting

6-4. Fastest Way to Add Any Pair of Fractions

6-5. Fastest Way to Subtract Any Pair of Fractions

6-6. To Add Two Fractions with Numerators of 1

6-7. To Subtract Two Fractions with Numerators of 1

Multiplying and Dividing by Fractions

6-8. To Multiply by a Fraction

6-9. To Divide by a Fraction

6-10. To Multiply a Number by $\frac{3}{4}$

Multiplying and Dividing with Mixed Numbers

6-11. Simplifying a Mixed Number Before Multiplying

6-12. To Square a Mixed Number Ending in

6-13. To Multiply a Number by a Mixed Number Ending in $\frac{1}{2}$: Halve-and-Double Method

6-14. To Divide a Number by a Mixed Number Ending in $\frac{1}{2}$

6-15. To Multiply Two Mixed Numbers When They Both End in $\frac{1}{2}$ and the Sum of Their Whole Numbers Is Even

6-16. To Multiply Two Mixed Numbers When They Both End in $\frac{1}{2}$ and the Sum of Their Whole Numbers Is Odd.

6-17. To Multiply Two Mixed Numbers Having the Same Whole Numbers, and Fractions That Add to 1

6-18. To Multiply or Divide by $2\frac{1}{2}$

6-19. To Multiply or Divide by $12\frac{1}{2}$

6-20. To Multiply or Divide by $16\frac{2}{3}$

6-21. To Multiply or Divide by $3\frac{1}{3}$

6-22. To Multiply or Divide by $6\frac{2}{3}$

Alternate Method of Multiplying by $6\frac{2}{3}$

Alternate Method of Dividing by $6\frac{2}{3}$

Table of Multiplying by Certain Mixed Numbers

Table of Dividing by Certain Mixed Numbers

Working With Percentages

6-23. To Find $16\frac{2}{3}$ % of a Number

6-24. To Find 20% of a Number

6-25. To Find $33\frac{1}{3}$ % of a Number

6-26. To Find 50% of a Number

6-27. To Find $66\frac{2}{3}$ % of a Number

Table for Finding Certain Per Cents

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DOVER BOOKS
IN ALL FIELDS OF INTEREST**



A CATALOG OF SELECTED DOVER BOOKS IN ALL FIELDS OF INTEREST

100 BEST-LOVED POEMS, Edited by Philip Smith. "The Passionate Shepherd to His Love," "Shall I compare thee to a summer's day?" "Death, be not proud," "The Raven," "The Road Not Taken," plus works by Blake, Wordsworth, Byron, Shelley, Keats, many others. 96pp. $5\frac{3}{16}$ x $8\frac{1}{4}$.

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0-486-40158-8

101 GREAT SAMURAI PRINTS, Utagawa Kuniyoshi. Kuniyoshi was a master of the warrior woodblock print — and these 18th-century illustrations represent the pinnacle of his craft. Full-color portraits of renowned Japanese samurais pulse with movement, passion, and remarkably fine detail. 112pp. $8\frac{1}{2}$ x 11.

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ABC OF BALLET, Janet Grosser. Clearly worded, abundantly illustrated little guide defines basic ballet-related terms: arabesque, battement, pas de chat, relevé, sissonne, many others. Pronunciation guide included. Excellent primer. 48pp. $4\frac{3}{16}$ x $5\frac{3}{4}$.

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ACCESSORIES OF DRESS: An Illustrated Encyclopedia, Katherine Lester and Bess Viola Oerke. Illustrations of hats, veils, wigs, cravats, shawls, shoes, gloves, and other accessories enhance an engaging commentary that reveals the humor and charm of the many-sided story of accessorized apparel. 644 figures and 59 plates. 608pp. $6\frac{1}{8}$ x $9\frac{1}{4}$.

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ADVENTURES OF HUCKLEBERRY FINN, Mark Twain. Join Huck and Jim as their boyhood adventures along the Mississippi River lead them into a world of excitement, danger, and self-discovery. Humorous narrative, lyrical descriptions of the Mississippi valley, and memorable characters. 224pp. $5\frac{5}{16}$ x $8\frac{1}{4}$.

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ALICE STARMORE'S BOOK OF FAIR ISLE KNITTING. Alice Starmore. A

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ALICE'S ADVENTURES IN WONDERLAND, Lewis Carroll. Beloved classic about a little girl lost in a topsy-turvy land and her encounters with the White Rabbit, March Hare, Mad Hatter, Cheshire Cat, and other delightfully improbable characters. 42 illustrations by Sir John Tenniel. 96pp. 5³/₁₆ x 8¹/₄.

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AMERICA'S LIGHTHOUSES: An Illustrated History, Francis Ross Holland. Profusely illustrated fact-filled survey of American lighthouses since 1716. Over 200 stations — East, Gulf, and West coasts, Great Lakes, Hawaii, Alaska, Puerto Rico, the Virgin Islands, and the Mississippi and St. Lawrence Rivers. 240pp. 8 x 10³/₄.

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AN ENCYCLOPEDIA OF THE VIOLIN, Alberto Bachmann. Translated by Frederick H. Martens. Introduction by Eugene Ysaye. First published in 1925, this renowned reference remains unsurpassed as a source of essential information, from construction and evolution to repertoire and technique. Includes a glossary and 73 illustrations. 496pp. 6¹/₈ x 9¹/₄.

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