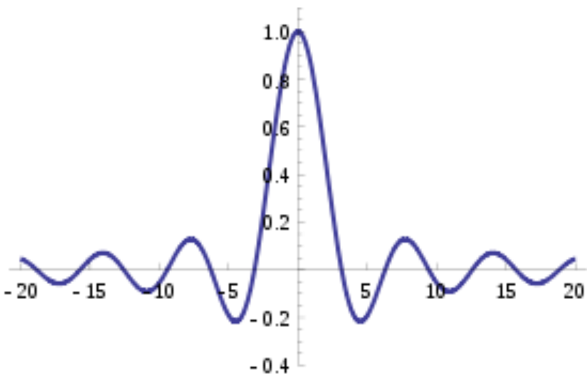
WIKIPEDIA

Characteristic function (probability theory)



The characteristic function of a uniform U(–1,1) random variable. This function is real-valued because it corresponds to a random variable that is symmetric around the origin; however characteristic functions may generally be complex-valued.

In probability theory and statistics, the **characteristic function** of any real-valued random variable completely defines its probability distribution. If a random variable admits a probability density function, then the characteristic function is the Fourier transform of the probability density function. Thus it provides the basis of an alternative route to analytical results compared with working directly with probability density functions or cumulative distribution functions. There are particularly simple results for the characteristic functions of distributions defined by the weighted sums of random variables.

In addition to univariate distributions, characteristic functions can be defined for vector or matrix-valued random variables, and can also be extended to more generic cases.

The characteristic function always exists when treated as a function of a real-valued argument, unlike the moment-generating function. There are relations between the behavior of the characteristic function of a distribution and properties of the distribution, such as the existence of moments and the existence of a density function.

Introduction

The characteristic function provides an alternative way for describing a random variable. Similar to the cumulative distribution function,

$$F_X(x) = \mathbf{E}\left[\mathbf{1}_{\{X \leq x\}}\right]$$

(where $\mathbf{1}_{\{X \leq x\}}$ is the indicator function — it is equal to 1 when $X \leq x$, and zero otherwise), which completely determines the behavior and properties of the probability distribution of the random variable X, the **characteristic function**,

$$\varphi_X(t) = \mathbf{E}\left[e^{itX}\right],$$

also completely determines the behavior and properties of the probability distribution of the random variable X. The two approaches are equivalent in the sense that knowing one of the functions it is always possible to find the other, yet they provide different insights for understanding the features of the random variable. However, in particular cases, there can be differences in whether these functions can be represented as expressions involving simple standard functions.

If a random variable admits a density function, then the characteristic function is its dual, in the sense that each of them is a Fourier transform of the other. If a random variable has a moment-generating function $M_X(t)$, then the domain of the characteristic function can be extended to the complex plane, and

$$\varphi_X(-it) = M_X(t).$$[1]

Note however that the characteristic function of a distribution always exists, even when the probability density function or moment-generating function do not.

The characteristic function approach is particularly useful in analysis of linear combinations of independent random variables: a classical proof of the Central Limit Theorem uses characteristic functions and Lévy's continuity theorem. Another important application is to the theory of the decomposability of random variables.

Definition

For a scalar random variable X the

characteristic function is defined as the expected value of e^{itX}, where i is the imaginary unit, and $t \in \mathbf{R}$ is the argument of the characteristic function:

$$\begin{cases} \varphi_X : \mathbb{R} \to \mathbb{C} \\ \varphi_X(t) = \operatorname{E}\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx}\, dF_X(x) = \int_{\mathbb{R}} e^{itx} f_X(x)\, dx = \int_0^1 e^{itQ_X(p)}\, dp \end{cases}$$

Here F_X is the cumulative distribution function of X, and the integral is of the Riemann–Stieltjes kind. If a random variable X has a probability density function f_X, then the characteristic function is its Fourier transform with sign reversal in the complex exponential,[2][3] and the last formula in parentheses is valid. $Q_X(p)$ is the inverse cumulative distribution function of X also called the quantile function of X.[4]

It should be noted though, that this convention for the constants appearing in the definition of the characteristic function differs from the usual convention for the Fourier transform.[5] For example, some authors[6] define $\phi_X(t) = \mathrm{E}e^{-2\pi itX}$, which is essentially a change of parameter. Other notation may be encountered in the literature: $\hat{p}$ as the characteristic function for a probability measure *p*, or $\hat{f}$ as the characteristic function corresponding to a density *f*.

Generalizations

The notion of characteristic functions generalizes to multivariate random variables and more complicated random elements. The argument of the characteristic function will always belong to the continuous dual of the space where the random variable *X* takes its values. For common cases such definitions are listed below:

- If *X* is a *k*-dimensional random vector, then for $t \in \mathbf{R}^k$
$$\varphi_X(t) = \operatorname{E}\left[\exp(it^T X)\right],$$
where t^T is the transpose of the matrix t,
- If *X* is a *k* × *p*-dimensional

random matrix, then for $t \in \mathbf{R}^{k \times p}$

$$\varphi_X(t) = \operatorname{E}\left[\exp\left(i \operatorname{tr}(t^T X)\right)\right],$$

where $\operatorname{tr}(\cdot)$ is the trace operator,

- If X is a complex random variable, then for $t \in \mathbf{C}$ [7]

$$\varphi_X(t) = \operatorname{E}\left[\exp\left(i \operatorname{Re}\left(\overline{t} X\right)\right)\right],$$

where $\overline{t}$ is the complex conjugate of t and $\operatorname{Re}(z)$ is the real part of the complex number z,

- If X is a k-dimensional complex random vector, then for $t \in \mathbf{C}^k$ [8]

$$\varphi_X(t) = \operatorname{E}[\exp(i \operatorname{Re}(t^* X))],$$

where t^* is the conjugate transpose of the matrix t,

- If $X(s)$ is a stochastic process, then for all functions $t(s)$ such that the integral $\int_{\mathbb{R}} t(s) X(s) \, \mathrm{d}s$ converges for almost all realizations of X [9]

$$\varphi_X(t) = \operatorname{E}\left[\exp\left(i \int_{\mathbf{R}} t(s) X(s) \, ds\right)\right].$$

Examples

Distribution	Characteristic function $\varphi(t)$
Degenerate δ_a	e^{ita}
Bernoulli Bern(p)	$1-p+pe^{it}$
Binomial B(n, p)	$(1-p+pe^{it})^n$
Negative binomial NB(r, p)	$\left(\frac{1-p}{1-pe^{i\,t}}\right)^r$
Poisson Pois(λ)	$e^{\lambda(e^{it}-1)}$
Uniform (continuous) U(a, b)	$\frac{e^{itb}-e^{ita}}{it(b-a)}$
Uniform (discrete) DU(a, b)	$\frac{e^{ait}-e^{(b+1)it}}{(b-a+1)(1-e^{it})}$
Laplace L(μ, b)	$\frac{e^{it\mu}}{1+b^2t^2}$
Normal $N(\mu, \sigma^2)$	$e^{it\mu-\frac{1}{2}\sigma^2t^2}$
Chi-squared χ^2_k	$(1-2it)^{-k/2}$
Cauchy C(μ, θ)	$e^{it\mu-\theta\|t\|}$
Gamma Γ(k, θ)	$(1-it\theta)^{-k}$
Exponential Exp(λ)	$(1-it\lambda^{-1})^{-1}$
Geometric Gf(p) (number of failures)	$\frac{p}{1-e^{it}(1-p)}$
Geometric Gt(p) (number of trials)	$\frac{p}{e^{-it}-(1-p)}$
Multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	$e^{\mathbf{t}^{\mathrm{T}}\left(i\boldsymbol{\mu}-\frac{1}{2}\boldsymbol{\Sigma}\mathbf{t}\right)}$
Multivariate Cauchy *MultiCauchy*($\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$)[10]	$e^{it^{\mathrm{T}}\boldsymbol{\mu}-\sqrt{\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}}}$

Oberhettinger (1973) provides

extensive tables of characteristic functions.

Properties

- The characteristic function of a real-valued random variable always exists, since it is an integral of a bounded continuous function over a space whose measure is finite.
- A characteristic function is uniformly continuous on the entire space
- It is non-vanishing in a region

around zero: $\phi(0) = 1$.

- It is bounded: $|\phi(t)| \le 1$.
- It is Hermitian: $\phi(-t) = \overline{\phi(t)}$. In particular, the characteristic function of a symmetric (around the origin) random variable is real-valued and even.
- There is a bijection between probability distributions and characteristic functions. That is, for any two random variables X_1, X_2, both have the same probability distribution if and

only if $\varphi_{X_1} = \varphi_{X_2}$.

- If a random variable *X* has moments up to *k*-th order, then the characteristic function φ_X is *k* times continuously differentiable on the entire real line. In this case
$$\mathbf{E}[X^k] = i^{-k} \varphi_X^{(k)}(0).$$
- If a characteristic function φ_X has a *k*-th derivative at zero, then the random variable *X* has all moments up to *k* if *k* is even, but only up to *k* – 1 if *k* is odd.[11]

$$\varphi_X^{(k)}(0) = i^k \, \mathbf{E}[X^k]$$

- If $X_1, \ldots, X_n$ are independent random variables, and $a_1, \ldots, a_n$ are some constants, then the characteristic function of the linear combination of the X_i 's is

  $$\varphi_{a_1X_1+\cdots+a_nX_n}(t) = \varphi_{X_1}(a_1t)\cdots\varphi_{X_n}(a_nt).$$

  One specific case is the sum of two independent random variables X_1 and X_2 in which case one has

  $$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t)\cdot\varphi_{X_2}(t).$$
- The tail behavior of the characteristic function determines the smoothness of

the corresponding density function.

- Let the random variable $Y = aX + b$ be the linear transformation of a random variable X. The characteristic function of Y is $\varphi_Y(t) = e^{itb}\varphi_X(at)$. For random vectors X and $Y = AX + B$, we have $\varphi_Y(t) = e^{it^\top B}\varphi_X(A^\top t)$.[12]

Continuity

The bijection stated above between probability distributions and characteristic functions is *sequentially continuous*. That is, whenever a sequence of distribution functions $F_j(x)$ converges (weakly) to some distribution $F(x)$, the corresponding sequence of characteristic functions $\phi_j(t)$ will also converge, and the limit $\phi(t)$ will correspond to the characteristic function of law F. More formally, this is stated as

> **Lévy's continuity theorem:** A sequence X_j of n-variate

random variables converges in distribution to random variable X if and only if the sequence ϕ_{X_j} converges pointwise to a function ϕ which is continuous at the origin. Then ϕ is the characteristic function of X.[13]

This theorem is frequently used to prove the law of large numbers, and the central limit theorem.

Inversion formulae

There is a one-to-one correspondence between cumulative distribution functions and characteristic functions, so it is possible to find one of these functions if we know the other. The formula in the definition of characteristic function allows us to compute ϕ when we know the distribution function F (or density f). If, on the other hand, we know the characteristic function ϕ and want to find the corresponding distribution function, then one of the following **inversion theorems** can be used.

Theorem. If characteristic function ϕ_X is integrable, then F_X is absolutely continuous, and therefore X has a probability density function. In the univariate case (i.e. when X is scalar-valued) the density function is given by

$$f_X(x) = F_X'(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-itx} \varphi_X(t)\, dt.$$

In the multivariate case it is

$$f_X(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-i(t\cdot x)} \varphi_X(t) \lambda(dt)$$

where $t \cdot x$ is the dot-product.

The pdf is the Radon–Nikodym derivative of the distribution μ_X with respect to the Lebesgue measure λ:

$$f_X(x) = \frac{d\mu_X}{d\lambda}(x).$$

Theorem (Lévy).[note 1] If ϕ_X is characteristic function of distribution function F_X, two points $a < b$ are such that $\{x \mid a < x < b\}$ is a continuity set of μ_X (in the univariate case this condition is equivalent to continuity of F_X at points a and b), then

- If X is scalar:

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{+T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_X(t)\, dt.$$

  This formula can be re-stated in a form more convenient for numerical computation as [14]

$$\frac{F(x+h) - F(x-h)}{2h} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin ht}{ht} e^{-itx} \varphi_X(t)\, dt.$$

  For a random variable bounded from below one can obtain $F(b)$ by taking a such that $F(a) = 0.$ Otherwise, if a random variable is not bounded from below, the limit for $a \to -\infty$ gives $F(b)$, but is numerically impractical.[14]

- If *X* is a vector random variable:

$$\mu_X\big(\{a < x < b\}\big) = \frac{1}{(2\pi)^n} \lim_{T_1\to\infty} \cdots \lim_{T_n\to\infty} \int\limits_{-T_1\le t_1\le T_1} \cdots \int\limits_{-T_n\le t_n\le T_n} \prod_{k=1}^{n} \left(\frac{e^{-it_k a_k} - e^{-it_k b_k}}{it_k}\right) \varphi_X(t)\lambda(dt_1 \times \cdots \times dt_n)$$

Theorem. If *a* is (possibly) an atom of *X* (in the univariate case this means a point of discontinuity of F_X) then

- If *X* is scalar:

$$F_X(a) - F_X(a-0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} e^{-ita} \varphi_X(t)\, dt$$

- If *X* is a vector random

variable[15]:

$$\mu_X(\{a\}) = \lim_{T_1\to\infty} \cdots \lim_{T_n\to\infty} \left(\prod_{k=1}^{n} \frac{1}{2T_k}\right) \int_{[-T_1,T_1]\times\cdots\times[-T_n,T_n]} e^{-i(t\cdot a)} \varphi_X(t) \lambda(dt)$$

Theorem (Gil-Pelaez).[16] For a univariate random variable *X*, if *x* is a continuity point of F_X then

$$F_X(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}[e^{-itx} \varphi_X(t)]}{t} \, dt.$$

where the imaginary part of a complex number z is given by $\operatorname{Im}(z) = (z - z^*)/2i$. The integral may be not Lebesgue-integrable; for example, when *X* is the discrete random variable that is always 0, it becomes the Dirichlet integral.

Inversion formulas for multivariate distributions are available.[17]

Criteria for characteristic functions

The set of all characteristic functions is closed under certain operations:

- A convex linear combination $\sum_n a_n \varphi_n(t)$ (with

$a_n \geq 0,\ \sum_n a_n = 1$) of a finite or a countable number of characteristic functions is also a characteristic function.

- The product of a finite number of characteristic functions is also a characteristic function. The same holds for an infinite product provided that it converges to a function continuous at the origin.
- If ϕ is a characteristic function and α is a real number, then $\bar{\varphi}$, Re(ϕ), $|\phi|^2$, and $\phi(\alpha t)$ are also characteristic functions.

It is well known that any non-decreasing càdlàg function F with limits $F(-\infty) = 0, F(+\infty) = 1$ corresponds to a cumulative distribution function of some random variable. There is also interest in finding similar simple criteria for when a given function ϕ could be the characteristic function of some random variable. The central result here is Bochner's theorem, although its usefulness is limited because the main condition of the theorem, non-negative definiteness, is very hard to verify. Other theorems also exist, such as Khinchine's, Mathias's, or Cramér's, although their application is just as difficult. Pólya's theorem, on the other hand, provides a very simple convexity condition which is sufficient but not necessary. Characteristic functions which satisfy this condition are called Pólya-type.[18]

Bochner's theorem. An arbitrary function $\phi : \mathbf{R}^n \to \mathbf{C}$ is the characteristic function of some random variable if and only if ϕ is positive definite, continuous at the origin, and if $\phi(0) = 1$.

Khinchine's criterion. A complex-valued, absolutely continuous function ϕ, with $\phi(0) = 1$, is a characteristic function if and only if it admits the representation

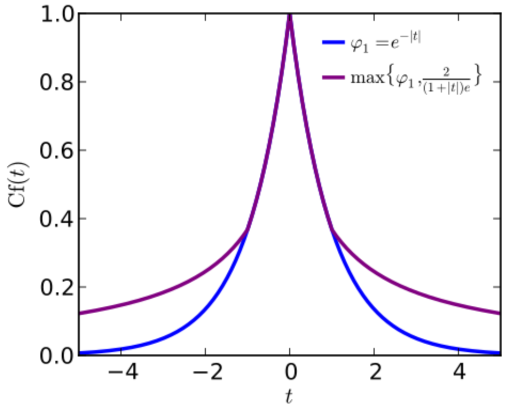
$$\varphi(t)=\int_{\mathbf{R}} g(t+\theta)\overline{g(\theta)}\,d\theta.$$

Mathias' theorem. A real-valued, even, continuous, absolutely integrable function ϕ, with $\phi(0) =$

1, is a characteristic function if and only if

$$(-1)^n \left(\int_{\mathbf{R}} \varphi(pt) e^{-t^2/2} H_{2n}(t)\, dt \right) \geq 0$$

for $n = 0,1,2,\ldots$, and all $p > 0$. Here H_{2n} denotes the Hermite polynomial of degree $2n$.



Pólya's theorem can be used to construct an example of two random variables whose characteristic functions coincide over a finite interval but are different elsewhere.

Pólya's theorem. If φ is a real-valued, even, continuous function which satisfies the conditions

- $\varphi(0) = 1$,

- φ is convex for $t > 0$,
- $\varphi(\infty) = 0$,

then $\phi(t)$ is the characteristic function of an absolutely continuous distribution symmetric about 0.

Uses

Because of the continuity theorem, characteristic functions are used in the most frequently seen proof of the central limit theorem. The main technique involved in making calculations with a characteristic function is recognizing the function as the characteristic function of a particular distribution.

Basic manipulations of distributions

Characteristic functions are particularly useful for dealing with linear functions of independent random variables. For example, if $X_1, X_2, \ldots, X_n$ is a sequence of independent (and not necessarily identically distributed) random variables, and

$$S_n = \sum_{i=1}^{n} a_i X_i \, ,$$

where the a_i are constants, then the characteristic function for S_n is given by

$$\varphi_{S_n}(t) = \varphi_{X_1}(a_1 t)\varphi_{X_2}(a_2 t)\cdots\varphi_{X_n}(a_n t)$$

In particular, $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$. To see this, write out the definition of characteristic function:

$$\varphi_{X+Y}(t) = \mathrm{E}\left[e^{it(X+Y)}\right] = \mathrm{E}\left[e^{itX}e^{itY}\right] = \mathrm{E}\left[e^{itX}\right]E\left[e^{itY}\right] = \varphi_X(t)\varphi_Y(t)$$

The independence of X and Y is

required to establish the equality of the third and fourth expressions.

Another special case of interest for identically distributed random variables is when $a_i = 1/n$ and then S_n is the sample mean. In this case, writing $\overline{X}$ for the mean,

$$\varphi_{\overline{X}}(t) = \varphi_X\!\left(\tfrac{t}{n}\right)^n$$

Moments

Characteristic functions can also be used to find moments of a random variable. Provided that the n^{th} moment exists, the characteristic function can be differentiated n times and

$$\operatorname{E}[X^n] = i^{-n}\, \varphi_X^{(n)}(0) = i^{-n} \left[\frac{d^n}{dt^n}\varphi_X(t)\right]_{t=0}$$

For example, suppose X has a standard Cauchy distribution. Then $\phi_X(t) = e^{-|t|}$. This is not differentiable at $t = 0$, showing that the Cauchy distribution has no expectation. Also, the characteristic function of the sample mean $\overline{X}$ of n independent observations has characteristic function $\phi_{\overline{X}}(t) = (e^{-|t|/n})^n = e^{-|t|}$, using the result from the previous section. This is the characteristic function of the standard Cauchy distribution: thus, the sample mean has the same distribution as the population itself.

The logarithm of a characteristic function is a cumulant generating function, which is useful for finding cumulants; some instead define the cumulant generating function as the logarithm of the moment-generating function, and call the logarithm of the characteristic function the *second* cumulant generating function.

Data analysis

Characteristic functions can be used as part of procedures for fitting probability distributions to samples of data. Cases where this provides a practicable option compared to other possibilities include fitting the stable distribution since closed form expressions for the density are not available which makes implementation of maximum likelihood estimation difficult. Estimation procedures are available which match the theoretical characteristic function to the empirical characteristic function, calculated from the data. Paulson et al. (1975) and Heathcote (1977) provide some theoretical background for such an estimation procedure. In addition, Yu (2004) describes applications of empirical characteristic functions to fit time series models where likelihood procedures are impractical.

Example

The gamma distribution with scale parameter θ and a shape parameter *k* has the characteristic function

$$(1 - \theta\, i\, t)^{-k}.$$

Now suppose that we have

$$X \ \sim \Gamma(k_1, \theta) \text{ and } Y \sim \Gamma(k_2, \theta)$$

with *X* and *Y* independent from

each other, and we wish to know what the distribution of $X + Y$ is. The characteristic functions are

$$\varphi_X(t) = (1 - \theta\, i\, t)^{-k_1}, \qquad \varphi_Y(t) = (1 - \theta\, i\, t)^{-k_2}$$

which by independence and the basic properties of characteristic function leads to

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = (1 - \theta\, i\, t)^{-k_1}(1 - \theta\, i\, t)^{-k_2} = (1 - \theta\, i\, t)^{-(k_1+k_2)}.$$

This is the characteristic function of the gamma distribution scale parameter θ and shape parameter $k_1 + k_2$, and we therefore conclude

$$X + Y \sim \Gamma(k_1 + k_2, \theta)$$

The result can be expanded to *n* independent gamma distributed random variables with the same scale parameter and we get

$$\forall i \in \{1, \dots, n\} : X_i \sim \Gamma(k_i, \theta) \qquad \Rightarrow \qquad \sum_{i=1}^{n} X_i \sim \Gamma\left(\sum_{i=1}^{n} k_i, \theta\right).$$

Entire characteristic functions

This section needs expansion.

Learn more

As defined above, the argument of the characteristic function is treated as a real number: however, certain aspects of the theory of characteristic functions are advanced by extending the definition into the complex plane by analytical continuation, in cases where this is possible.[19]

Related concepts

Related concepts include the moment-generating function and the probability-generating function. The characteristic function exists for all probability distributions. This is not the case for the moment-generating function.

The characteristic function is closely related to the Fourier transform: the characteristic function of a probability density function $p(x)$ is the complex conjugate of the continuous Fourier transform of $p(x)$ (according to the usual convention; see continuous Fourier transform – other conventions).

$$\varphi_X(t) = \langle e^{itX} \rangle = \int_{\mathbf{R}} e^{itx} p(x)\, dx = \overline{\left(\int_{\mathbf{R}} e^{-itx} p(x)\, dx \right)} = \overline{P(t)},$$

where $P(t)$ denotes the continuous Fourier transform of the probability density function $p(x)$. Likewise, $p(x)$ may be recovered from $\phi_X(t)$ through the inverse Fourier transform:

$$p(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{itx} P(t)\, dt = \frac{1}{2\pi} \int_{\mathbf{R}} e^{itx} \overline{\varphi_X(t)}\, dt.$$

Indeed, even when the random variable does not have a density, the characteristic function may be seen as the Fourier transform of the measure corresponding to the random variable.

Another related concept is the representation of probability distributions as elements of a reproducing kernel Hilbert space via the kernel embedding of distributions. This framework may be viewed as a generalization of the characteristic function under specific choices of the kernel function.

See also

- Subindependence, a weaker condition than independence, that is defined in terms of characteristic functions.
- Entropic value at risk
- Cumulant, a term of the *cumulant generating functions*, which are logs of the characteristic functions.

Notes

1. *named after the French mathematician Paul Lévy*

References

Citations

1. *Lukacs (1970) p. 196*
2. *Statistical and Adaptive Signal Processing (2005)*
3. *Billingsley (1995)*
4. *Shaw, W. T.; McCabe, J. (2009). "Monte Carlo sampling given a Characteristic Function: Quantile Mechanics in Momentum Space". arXiv:0903.1592 [q-fin.CP].*
5. *Pinsky (2002)*

6. *Bochner (1955)*
7. *Andersen et al. (1995, Definition 1.10)*
8. *Andersen et al. (1995, Definition 1.20)*
9. *Sobczyk (2001, p. 20)*
10. *Kotz et al. p. 37 using 1 as the number of degree of freedom to recover the Cauchy distribution*
11. *Lukacs (1970), Corollary 1 to Theorem 2.3.1*
12. *"Joint characteristic function"* .

www.statlect.com. Retrieved 7 April 2018.

13. *Cuppens (1975, Theorem 2.6.9)*
14. *Shepard, N.G. (1991a)*
15. *Cuppens (1975, Theorem 2.3.2)*
16. *Wendel, J.G. (1961)*
17. *Shephard (1991a,b)*
18. *Lukacs (1970), p. 84*
19. *Lukacs (1970, Chapter 7)*

Sources

- Andersen, H.H., M. Højbjerre, D. Sørensen, P.S. Eriksen (1995).

Linear and graphical models for the multivariate complex normal distribution. Lecture Notes in Statistics 101. New York: Springer-Verlag. ISBN 978-0-387-94521-7.

- Billingsley, Patrick (1995). *Probability and measure* (3rd ed.). John Wiley & Sons. ISBN 978-0-471-00710-4.
- Bisgaard, T. M.; Sasvári, Z. (2000). *Characteristic functions and moment sequences*. Nova Science.
- Bochner, Salomon (1955). *Harmonic analysis and the theory of probability*. University of California Press.

- Cuppens, R. (1975). *Decomposition of multivariate probabilities*. Academic Press.
- Heathcote, C.R. (1977). "The integrated squared error estimation of parameters". *Biometrika*. **64** (2): 255–264. doi:10.1093/biomet/64.2.255.
- Lukacs, E. (1970). *Characteristic functions*. London: Griffin.
- Kotz, Samuel; Nadarajah, Saralees (2004). *Multivariate T Distributions and Their Applications*. Cambridge University Press.
- Oberhettinger, Fritz (1973).

"Fourier Transforms of Distributions and their Inverses: A Collection of Tables". Academic Press.

- Paulson, A.S.; Holcomb, E.W.; Leitch, R.A. (1975). "The estimation of the parameters of the stable laws". *Biometrika*. **62** (1): 163–170. doi:10.1093/biomet/62.1.163.
- Pinsky, Mark (2002). *Introduction to Fourier analysis and wavelets*. Brooks/Cole. ISBN 978-0-534-37660-4.
- Sobczyk, Kazimierz (2001). *Stochastic differential equations*. Kluwer Academic Publishers. ISBN 978-1-4020-0345-5.

- Wendel, J.G. (1961). "The non-absolute convergence of Gil-Pelaez' inversion integral". *The Annals of Mathematical Statistics*. **32** (1): 338–339. doi:10.1214/aoms/1177705164.
- Yu, J. (2004). "Empirical characteristic function estimation and its applications". *Econometrics Reviews*. **23** (2): 93–1223. doi:10.1081/ETC-120039605.
- Shephard, N. G. (1991a). "From characteristic function to distribution function: A simple framework for the theory". *Econometric Theory*. **7** (4): 519–529.

doi:10.1017/s0266466600004746.

- Shephard, N. G. (1991b). "Numerical integration rules for multivariate inversions". *J. Statist. Comput. Simul.* **39** (1–2): 37–46. doi:10.1080/00949659108811337.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Characteristic function", *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers,

ISBN 978-1-55608-010-4

Retrieved from "https://en.wikipedia.org/w/index.php?title=Characteristic_function_(probability_theory)&oldid=933704349"

Last edited 14 days ago by an anonymous user