Department of Physics
IIT Kanpur

PHY 112- Mid Sem Exam
Date: 22 Feb 2024
Time: 1 PM – 3 PM

Question 1: A particle is subjected to a central force $f(r)\,\hat{r}$. [10+10+10+5]

(a) Define $u = \frac{1}{r}$, and show that the trajectory of the particle $u\,(\theta)$ is $\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}\,f(u)$. Here m is the mass of particle and L is the angular momentum.

(b) Prove that under the inverse square attractive force, $\vec{F}(r) = -\frac{k}{r^2}\,\hat{r}$, the trajectory of the particle is given by $r = \frac{L^2}{mk}\frac{1}{(1-e\ Cos\theta)}$. Here, e is eccentricity.

(c) If we perturb the inverse square attractive force with $-\frac{\delta}{r^3}\,\hat{r}$, (assume very small perturbation), How the eccentricity of the orbit changes and thereby the trajectory of the particle?

(d) Sketch the trajectory of the particle under the condition (b) and (c).

Question 2. An overdamped harmonic oscillator satisfies the equation: [15]

$$\ddot{x} + 10\dot{x} + 16x = 0.$$

At time t=0 the particle is projected from the point $x = 1$ towards the origin with speed u. Find x in the subsequent motion. Show that the particle will reach the origin at some later time t if

$$e^{6t} = \frac{u-2}{u-8}.$$

How large must u be so that the particle will pass through the origin?

Question 3. A spacecraft is going in a circular orbit of radius $2R$ around the earth. Here R is the radius of the earth. It is to be taken in a circular orbit of radius $4R$. [7.5+7.5]

(a) What is the minimum energy to be spent in the operation?

(b) It is done in two stages. In the first stage spacecraft is fixed at a point A, taking it to an elliptical orbit with the nearest point at a distance $2R$ and the farthest point B, at a distance $4R$ as shown in Figure below. Find the change in the speed at a point A when the rocket is fired.

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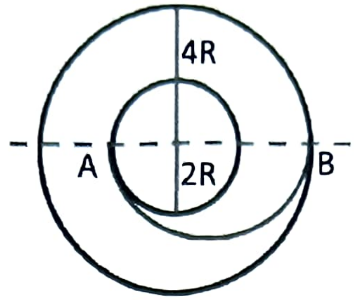
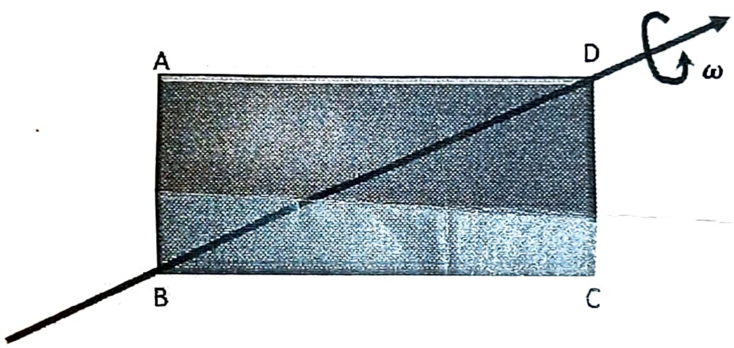


Question 4. A solid rectangular plate of length l and width b is rotated about the one of its diagonals with a constant angular velocity $\vec{\omega}$. Find the torque needed to maintain the constant velocity. [15]



Question 5: The oscillations of the tip of the galvanometer satisfy the equation [15+5]

$$\frac{d^2x}{dt^2} + 2K\left(\frac{dx}{dt}\right) + \gamma^2 x = 0.$$

The galvanometer is released from rest with $x = a$, and we wish to bring the reading permanently within the interval $-\epsilon a \le x \le \epsilon a$ as quickly as possible. Here ϵ is a small positive constant.

(a) Prove that this can be achieved by setting the value of $K = \gamma\left[1 + \left(\frac{\pi}{\ln\left(\frac{1}{\epsilon}\right)}\right)^2\right]^{-\frac{1}{2}}$.

(b) Sketch the graph of $x(t)$. [Hint: Assume the sub critical value of K such that the first minimum point of $x(t)$ occurs when $x = -\epsilon a$.]

1) (a)

we know,

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and, & $L = mr^2\dot{\theta}$

$$\dot{\theta} = \frac{L}{mr^2}$$

Given $r = \frac{1}{u}$

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta}$$

$$\dot{r} = -\frac{1}{u^2}\left(\frac{du}{d\theta}\right)\left(\frac{L}{mr^2}\right)$$

$$\dot{r} = -\frac{L}{m}\frac{du}{d\theta}$$

$$\ddot{r} = -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

Now,

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_r = \left(-\frac{L^2u^2}{m^2}\right)\frac{d^2u}{d\theta^2} - r\left(\frac{L^2}{m^2r^4}\right)$$

$$f(r) \longrightarrow f(u)$$

$$r = \frac{1}{u}$$

$$\frac{f(u)}{m} = a$$

$$\frac{f(u)}{m} = \frac{d^2u}{d\theta^2}\left(-\frac{L^2u^2}{m^2}\right) - \frac{1}{u}\left(\frac{L^2u^4}{m^2}\right)$$

After rearranging,

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{-m}{L^2u^2} f(u)}$$

b) $f(r)\,\hat{r} \longrightarrow \frac{-k}{r^2}\,\hat{r}$

$$F(r) = \frac{-k}{r^2}\,\hat{r}$$

$$f(u) = -ku^2$$

Substituting this in eq^n obtained in (a) part

$$\frac{d^2u}{d\theta^2} + u = \frac{-m}{L^2u^2}(-ku^2)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2}$$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{mk}{L^2}\right) = 0$$

(Let) $u' = u - \dfrac{mk}{L^2}$

$$\frac{du'}{d\theta} = \frac{du}{d\theta}$$

so,

$$\frac{d^2u'}{d\theta^2} + u' = 0$$

This is well known eq^n of SHM

$$u' = A\cos(\theta - \theta_0)$$

$$u' = u - \frac{mk}{L^2}$$

$$u' = -A\cos\theta$$

$$u = \frac{mk}{L^2} - A\cos\theta$$

$$\frac{1}{r} = \frac{mk}{L^2} - A\cos\theta$$

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 - \frac{AL^2}{mk}\cos\theta\right)$$

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 - e\cos\theta\right)$$

(say) $\cancel{e = \frac{mk}{AL^2}}$ $\quad e = \dfrac{AL^2}{mk}$

$$\frac{1}{r} = \frac{mk}{L^2}(1 - e\cos\theta)$$

$$\boxed{r = \frac{L^2}{mk}\,\frac{1}{(1 - e\cos\theta)}}$$

c) Now if we add a very small perturbation

$$F(r) = f(r)\,\hat{r}$$

$$= -\frac{k}{r^2}\,\hat{r} - \frac{\delta}{r^3}$$

$$f(r) = \frac{-k}{r^2} - \frac{\delta}{r^3}$$

Again. $$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}\,f(u)$$

$$f(r) \longrightarrow f(u)$$

$$f(r) = -\frac{k}{r^2} - \frac{\delta}{r^3}$$

$$= -ku^2 - \delta u^3$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}\left(-ku^2 - \delta u^3\right)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2} + \frac{m\delta u}{L^2}$$

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{m\delta}{L^2}\right)u - \frac{mk}{L^2} = 0$$

(say) $$1 - \frac{m\delta}{L^2} = \omega^2$$

$$\frac{d^2u}{d\theta^2} + \omega^2 u - \frac{mk}{L^2} = 0$$

$$\frac{d^2u}{d\theta^2} + \omega^2\left(u - \frac{mk}{L^2\omega^2}\right) = 0$$

$$u' = u - \frac{mk}{\omega^2 L^2}$$

$$\frac{du'}{d\theta} = \frac{du}{d\theta}$$

$$\frac{d^2u'}{d\theta^2} + \omega^2 u' = 0 \quad \Rightarrow \text{SHM}$$

Solⁿ,

$$u' = -A\cos\omega\theta$$

$$u' = u - \frac{mk}{\omega^2 L^2}$$

$$\Rightarrow \quad -A\cos\omega\theta = u - \frac{mk}{\omega^2 L^2}$$

$$\Rightarrow \quad u = \frac{mk}{L^2\omega^2}\left(1 - \frac{AL^2\omega^2}{mk}\cos\omega\theta\right)$$

$$\Rightarrow \quad u = \frac{mk}{L^2\omega^2}\left(1 - e\cos\omega\theta\right)$$

(say) $\quad e = \dfrac{AL^2\omega^2}{mk}$

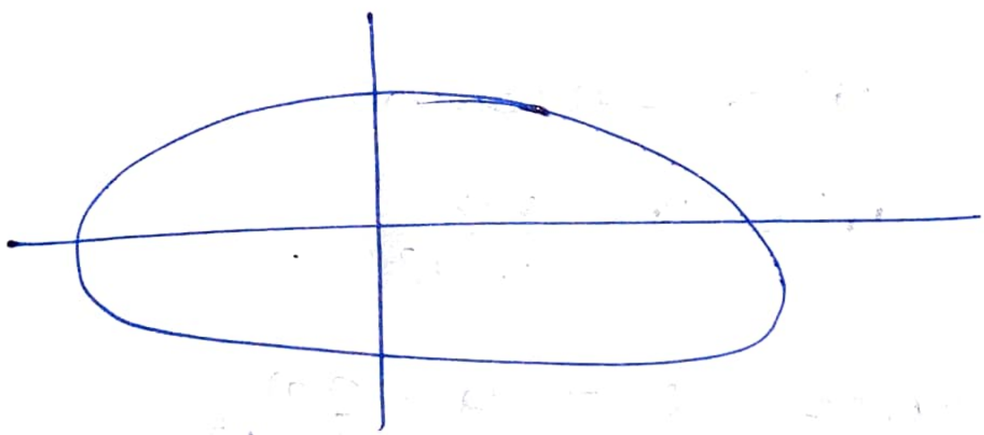
$$\frac{1}{r} = \frac{mk}{L^2\omega^2}(1 - e\cos\omega\theta)$$

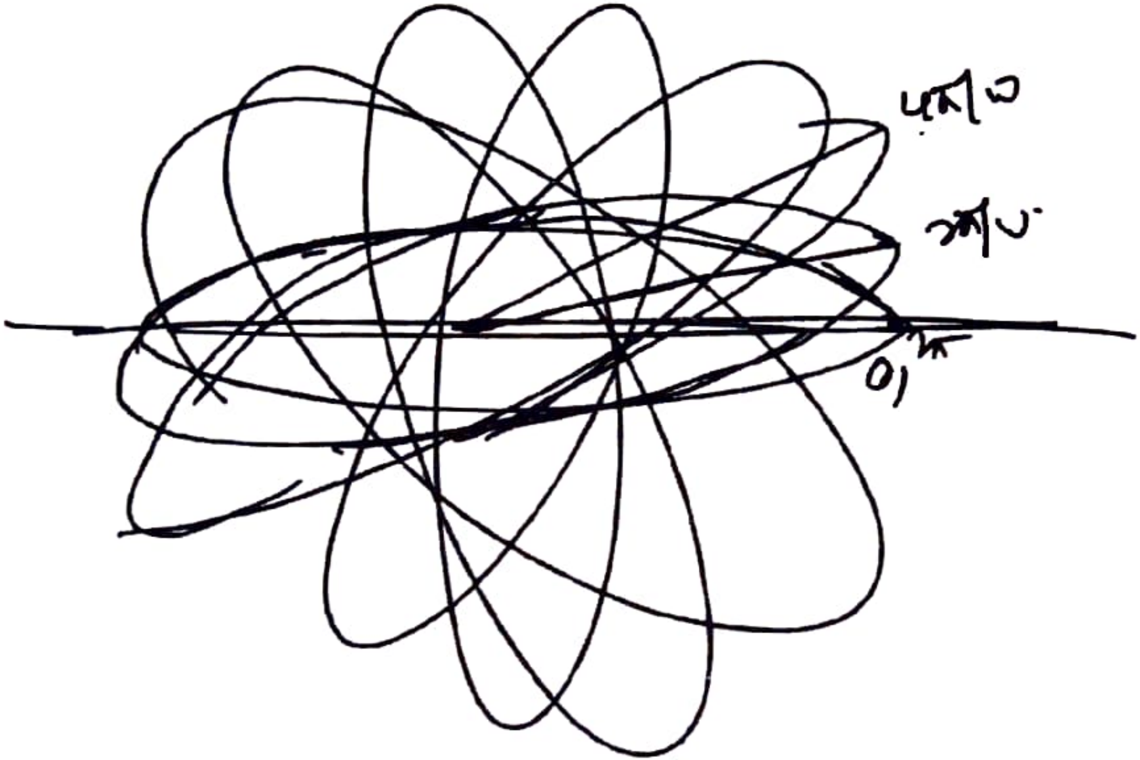
$$r = \frac{L^2\omega^2}{mk}\frac{1}{(1 - e\cos\omega\theta)}$$

$$r_0 = \frac{L^2\omega^2}{mk} \quad (\text{say})$$

$$\boxed{r = \frac{r_0}{1 - e\cos\omega\theta}}$$

d) (i)




4π/ω
2π/ω
0, π

Q2: An overdamped harmonic oscillator satisfies the equation $\ddot{x} + 10\dot{x} + 16x = 0$

at time $t = 0$, the particle is projected from the point $x = 1$ towards the origin with speed u, find x in the subsequent motion. Show that the particle will reach the origin at some later time t if

$$\frac{u-2}{u-8} = e^{6t}$$

How large must u be so that the particle will pass through the origin?

<u>Solution:</u> In a similar fasion (as in eqn 1)

Let the general solution of eqn. is $x = e^{\lambda t}$

Now λ must satisfy the eqn.

$$\lambda^2 + 10\lambda + 16 = 0$$

solving for the root of the eqn.

$$\lambda = -2, -8$$

we have now pair of soln:

$$x = \begin{cases} e^{-2t} \\ e^{-8t} \end{cases}$$

Now the general solutn.

$$x = Ae^{-2t} + Be^{-8t}$$; where A and B are arbitary cost.

Initial condition.

$$x = 1 , \dot{x} = -4 \text{ when } t = 0$$

$$\boxed{1 = A + B} \quad \text{—— ①}$$

$$\boxed{-4 = \dot{x}}$$

$$\dot{x} = -2Ae^{-2t} - 8Be^{-8t}$$

at $t = 0$

$$-4 = -2A - 8B \Rightarrow \boxed{4 = 2A + 8B} \quad \text{—— ②}$$

Solving eqn ① and ②

$$A = -\frac{1}{6}(4-8) \; ; \; B = \frac{1}{6}(4-2)$$

Now we can re-write $x(t)$

$$x = Ae^{-2t} + Be^{-8t}$$

$$= -\frac{1}{6}(4-8)e^{-2t} + \frac{1}{6}(4-2)e^{-8t}$$

$$\boxed{x = \frac{1}{6}(4-2)e^{-8t} - \frac{1}{6}(4-8)e^{-2t}}$$

∵ If the particle is at origin $x = 0$

$$0 = \frac{1}{6}(u-2)e^{-8t} - \frac{1}{6}(u-8)e^{-2t}$$

$$\frac{(u-2)}{(u-8)} = \frac{e^{-2t}}{e^{-8t}} = e^{6t}$$

$$\boxed{\frac{u-2}{u-8} = e^{6t}}$$

such a ~~value~~ of e^{6t} is posible if

$$\frac{(u-2)}{(u-8)} > 1$$

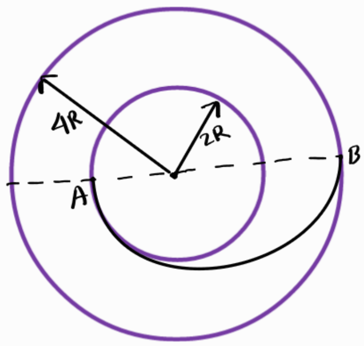
Therefore: $F(u) = \frac{(u-2)}{(u-8)} > 1$

This is only posible if $u > 8$

Hence the particle will pass through the origin if $u > 8$

Q3.

(a) Energy in the orbit of radius 2R:



Since, $\vec{F}_{cent} = \vec{F}_{grav}$

$$\Rightarrow \frac{mv_{2R}^2}{2R} = \frac{GMm}{(2R)^2}$$

m: mass of the spacecraft
M: mass of the earth

$$\left.\begin{aligned} K.E. &= \frac{1}{2}mv_{2R}^2 = \frac{GMm}{4R} \\ P.E. &= -\frac{GMm}{2R} \end{aligned}\right\} \Rightarrow E_{2R} = K.E + P.E = -\frac{GMm}{4R}$$

Similarly, energy in the orbit of radius 4R:

$$\frac{mv_{4R}^2}{4R} = \frac{GMm}{(4R)^2}$$

$$\left.\begin{aligned} K.E. &= \frac{1}{2}mv_{4R}^2 = \frac{GMm}{8R} \\ P.E. &= -\frac{GMm}{4R} \end{aligned}\right\} \Rightarrow E_{4R} = K.E + P.E = -\frac{GMm}{8R}$$

The minimum energy to be spent in the operation is

$$\boxed{\Delta E_{min} = E_{4R} - E_{2R} = \frac{GMm}{8R}}$$

(b) We know the energy of aircraft in the circular orbit of radius 2R at point A,

$$E_A = -\frac{GM}{4R}$$

Since, $E_A = K.E. + P.E.$

$$\Rightarrow \quad -\frac{GMm}{4R} = \frac{1}{2} m v_A^2 - \frac{GMm}{2R}$$

v_A: velocity of the spacecraft in the circular orbit at A.

$$\Rightarrow \quad v_A = \sqrt{\frac{GM}{2R}}$$

Energy when spacecraft goes to the elliptical orbit,

$$E = -\frac{GMm}{2R+4R} = -\frac{GMm}{6R}$$

and $E = K.E + P.E.$

v_A': velocity of the spacecraft in the elliptical orbit at A.

$$\Rightarrow \quad -\frac{GMm}{6R} = \frac{1}{2} m v_A'^2 - \frac{GMm}{2R}$$

$$\Rightarrow \quad v_A' = \sqrt{\frac{2GM}{3R}}$$

Therefore, the change in the speed at A:

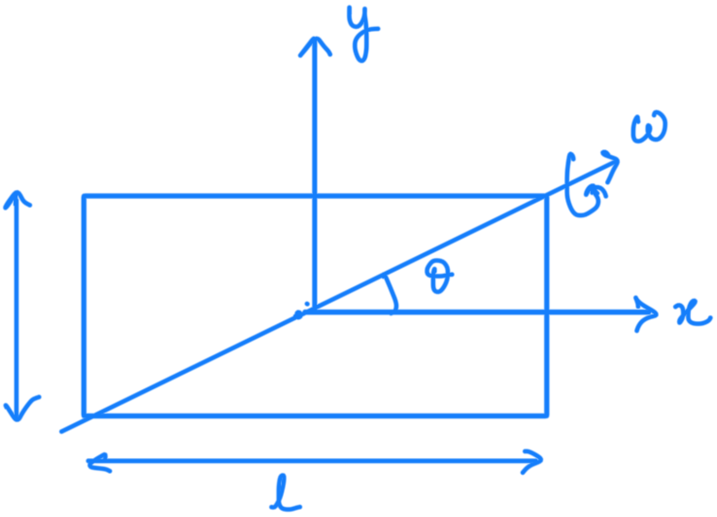
$$\boxed{\Delta v_A = v_A' - v_A = \left(\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{2}}\right)\sqrt{\frac{GM}{R}}}$$

Answer → 4

$$I_x = \frac{Mb^2}{12}$$

$$I_y = \frac{Ml^2}{12}$$

$$I_z = \frac{M(l^2+b^2)}{12}$$



ω along x : $\omega_x = \omega\cos\theta$

ω along y : $\omega_y = \omega\sin\theta$

ω along z : $\omega_z = 0$

Using Euler's eq. :→

$$\tau_x = I_x\dot{\omega}_x + \omega_y \xcancel{\omega_z}^{\,0} (I_z - I_y)$$

$$\tau_y = I_y\dot{\omega}_y + \omega_x \xcancel{\omega_z}^{\,0} (I_x - I_z)$$

$$\tau_z = I_z\dot{\omega}_z + \omega_x\omega_y (I_y - I_x)$$

$$\tau_x = 0 \quad , \quad \tau_y = 0$$

$$\tau_z = 0$$

$\Downarrow$

$$\tau_z = \omega_x \omega_y \, (I_y - I_x)$$

$$\because \quad \cos\theta = \frac{l}{\sqrt{l^2+b^2}} \quad \& \quad \sin\theta = \frac{b}{\sqrt{l^2+b^2}}$$

$$\therefore \quad \omega\cos\theta = \frac{\omega l}{\sqrt{l^2+b^2}}$$

$$\& \quad \omega\sin\theta = \frac{\omega b}{\sqrt{l^2+b^2}}$$

$$\Rightarrow \quad \tau_z = \frac{\omega^2 l b}{\left(\sqrt{l^2+b^2}\right)^2} \frac{M(l^2-b^2)}{12}$$

$$= \frac{M\omega^2}{12} \frac{(l^2-b^2)}{(l^2+b^2)}$$

Answer 5:

$$\ddot{x} + 2\kappa\dot{x} + \Omega^2 x = 0 \longrightarrow (1)$$

General solution of equation (1) is : $[\kappa < \gamma]$

$$x = e^{-\kappa t}(A\cos\Omega_D t + B\sin\Omega_D t)$$

here, $\Omega_D = (\gamma^2 - \kappa^2)^{1/2}$ $\longrightarrow$ (5)

$$\dot{x} = e^{-\kappa t}(-A\sin\Omega_D t + B\cos\Omega_D t)\Omega_D + (A\cos\Omega_D t + B\sin\Omega_D t)e^{-\kappa t}(-\kappa)$$

$$\dot{x} = e^{-\kappa t}\left[(\Omega_D B - \kappa A)\cos\Omega_D t - (\Omega_D A + \kappa B)\sin\Omega_D t\right].$$

Initial Conditions : $x = a$ and $\dot{x} = 0$ at $t = 0$.

$$a = A$$

& $0 = [(\Omega_D B - \kappa A)] \Rightarrow \Omega_D B = \kappa A$

$$B = \frac{\kappa a}{\Omega_D}.$$

$$\Rightarrow \quad x = a e^{-\kappa t}\left[\cos\Omega_D t + \frac{\kappa}{\Omega_D}\sin\Omega_D t\right] \longrightarrow (3)$$

and $\dot{x} = -a e^{-\kappa t}\left(\Omega_D + \frac{\kappa^2}{\Omega_D}\right)\sin\Omega_D t$

$$\dot{x} = -a\left(\frac{\gamma^2}{\Omega_D}\right)e^{-\kappa t}\sin\Omega_D t.$$

Stationary Points is given by $\dot{x} = 0$

$$\Rightarrow \sin\Omega_D t = 0 = \sin n\pi$$

$$t = \frac{n\pi}{\Omega_D}$$

minimum $t = \dfrac{\pi}{\Omega_D}$. $\longrightarrow$ (3)

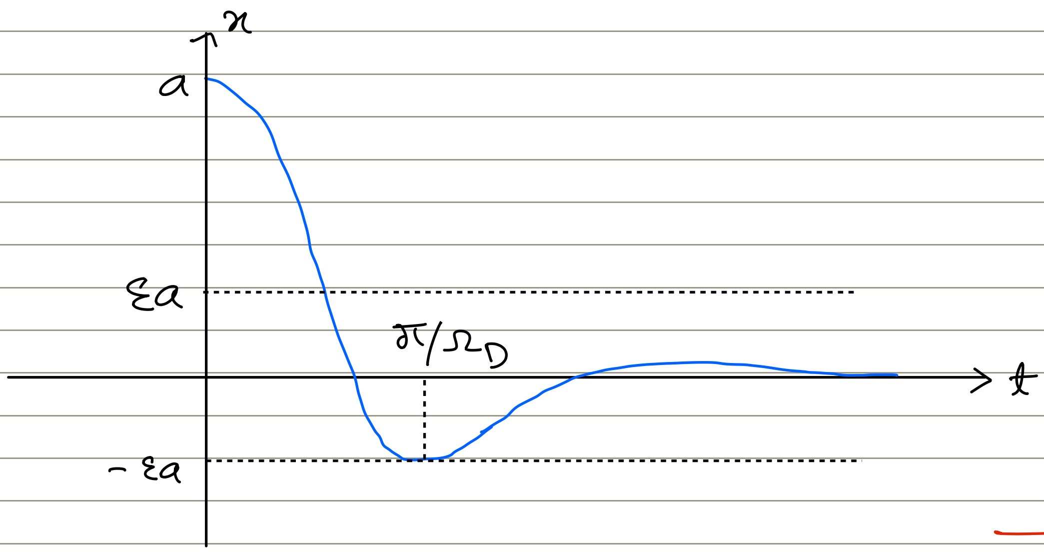
$\Rightarrow$ At this instant t, galvanometer should be at $x = -\epsilon a$

$$\Rightarrow \quad x\left(t = \frac{\pi}{\Omega_D}\right) = -\varepsilon a$$

$$-\varepsilon a = a e^{-\kappa\pi/\Omega_D}\left[\cos\left(\Omega_D \frac{\pi}{\Omega_D}\right) + \sin\left(\Omega_D \frac{\pi}{\Omega_D}\right)\frac{\kappa}{\Omega_D}\right]$$

$$-\varepsilon \cancel{a} = -\cancel{a}\, e^{-\kappa\pi/\Omega_D}$$

$$\Rightarrow \quad \kappa = \gamma\left(1 + \frac{\pi^2}{\left(\ln\left(1/\varepsilon\right)\right)^2}\right)^{-1/2}. \longrightarrow (4)$$



$\longrightarrow$ (5)