$$* \quad \frac{\partial L}{\partial x} = \frac{m}{2t^2} \cdot 2(x - \dot{x}t) = \frac{mx}{t^2} - \frac{m\dot{x}}{t}$$

$\therefore$ EL eqn :→ $m\ddot{x} = 0 \Rightarrow \ddot{x} = 0 \Rightarrow$ free particle.

PHY112 Endsem

Tentative question paper

Part I: Short answer type questions $10 \times 2 = 20$

1. (a) Determine the nature of the phase space trajectories corresponding to the coefficient matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the neighbourhood of the fixed point.

If the co-efficient matrix is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

the equation will be $\begin{matrix} \dot{x} = y \\ \dot{y} = x \end{matrix} \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow y^2 - x^2 =$ constant.

$\Rightarrow$ hyperbolae.

(*) If done using

$\mathrm{Tr}\, A = 0$, $\det A = -1$

$\Rightarrow \lambda_1, \lambda_2 = 1, -1 \Rightarrow$ Saddle point (give only 1)

(b) Without solving the Euler-Lagrange equations, recognize the system described by the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + 2\beta\dot{x}$, where β is a positive constant. Justify your answer.

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + 2\beta\dot{x} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + \underbrace{\frac{d}{dt}(2\beta x)}_{= \frac{dF(x)}{dt}}$$

So, the L will be the Lagrangian equivalent to $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ which describes a simple harmonic oscillator.

(c) Using any correct method, identify the motion corresponding to the Lagrangian $L = \frac{1}{2}\frac{m}{t^2}(x - \dot{x}t)^2$.

$$L = \frac{m}{2t^2}(x - \dot{x}t)^2 = \frac{m}{2t^2}\left[x^2 - 2x\dot{x}t + \dot{x}^2t^2\right]$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\frac{x^2}{t^2} - \frac{m}{t}x\dot{x}$$

$$= \underbrace{\frac{1}{2}m\dot{x}^2 - \left[\frac{d}{dt}\left(\frac{mx^2}{2t}\right)\right]}$$

equivalent to $L = \frac{1}{2}m\dot{x}^2$ free particle

OR:

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{m}{t^2}(x - \dot{x}t)(-t) = \frac{m}{t}(\dot{x}t - x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -\frac{m}{t^2}(\dot{x}t - x) + \frac{m}{t}(\ddot{x}t + \dot{x} - \dot{x})$$

$$= m\ddot{x} - \frac{m\dot{x}}{t} + \frac{mx}{t^2} \quad *$$

** $\dot{x}\,dx + \dot{y}\,dy = d(xy) + d(x^2 y) - d\left(\frac{y^3}{3}\right)$

1.5 $= d\left(xy + x^2 y - \frac{y^3}{3}\right) = -dV$

0.5 The system does not permit closed orbits and hence No Limit cycle.

(d) Calculate the energy function of the Lagrangian $L = \frac{1}{2}\dot{x}\sin^2 x$.

$$L = \frac{1}{2}\dot{x}\sin^2 x \Rightarrow \left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{1}{2}\sin^2 x$$

$$\therefore \mathcal{E} = \dot{x}\left(\frac{\partial L}{\partial \dot{x}}\right) - L = \frac{1}{2}\dot{x}\sin^2 x - \frac{1}{2}\dot{x}\sin^2 x = 0$$

(e) Find the law of central force for the trajectory $r^2\theta = 1$.

One possible method: $r^2\theta = 1 \Rightarrow u^2 = \theta \Rightarrow 2u\frac{du}{d\theta} = 1$

$$\xrightarrow{\frac{d}{d\theta}}$$

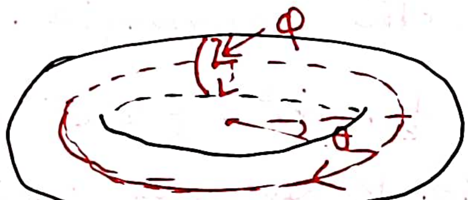
$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{1}{4u^2} \text{ and } \left(\frac{du}{d\theta}\right)^2 + u\frac{d^2u}{d\theta^2} = 0$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = -\frac{1}{4u^3} \qquad \therefore -\frac{f\left(\frac{1}{u}\right)}{mh^2u^2} = -\frac{1}{4u^3} + u$$

$$\Rightarrow f(r) = Ar + Br^{-3} = Ar + B/r^3$$

(f) A particle is moving on the surface of a solid torus. What is the number of effective degrees of freedom? Can you suggest a set of generalized coordinates for the motion? If it helps, you may explain your answer with a neat diagram.

It will have 2 degrees of freedom. 1 pt.



1 pt.

(g) Show that the following system

$$\dot{x} = y(1+2x); \quad \dot{y} = x(1+x) - y^2$$

is a gradient system. What can you conclude from this?

$$\dot{x} = y(1+2x), \quad \dot{y} = x(1+x) - y^2$$

for gradient system, $\frac{\partial V}{\partial x} = -y(1+2x)$ and $\frac{\partial V}{\partial y} = -x(1+x) + y^2$ needed.

then, $\dot{x}\,dx + \dot{y}\,dy = y\,dx + 2xy\,dx + x\,dy + x^2 dy - y^2 dy$

$$= (y\,dx + x\,dy) + (2xy\,dx + x^2 dy) - y^2\,dy$$ **

(h) The motion of a damped harmonic oscillator is described by

$$m\ddot{x} + b\dot{x} + kx = 0\,,$$

Re-write the above equation in the form of $\dot{X} = AX$ in the phase space and hence find the fixed point(s).

$\dot{x} = v$ and $m\ddot{x} + b\dot{x} + kx = 0$

$\Rightarrow m\dot{v} + bv + kx = 0 \Rightarrow \dot{v} = -\frac{b}{m}v - \frac{k}{m}x$

1 pt. So, $\dot{x} = v$

$\dot{v} = -\frac{k}{m}x - \frac{b}{m}v \Rightarrow \begin{pmatrix}\dot{x}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{pmatrix}\begin{pmatrix}x\\ v\end{pmatrix}$

At fixed pt. $\dot{x} = 0$, $\dot{v} = 0 \Rightarrow$ $\underline{(0,0)}$ is the fixed point.

(i) State Bertrand's theorem of central force.

$\longrightarrow$ 1 pt.

There are only two central force laws ($\frac{1}{r^2}$ and r) or potentials ($\frac{1}{r}$ and r^2), 1pt. where every bounded motion is associated with a closed orbit.

1 pt.

(j) If the force acting on a particle is parallel to its angular momentum vector, then show that the kinetic energy of the particle is conserved in time.

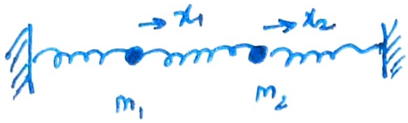
$$m\ddot{\vec{r}} = \alpha\vec{L} = \alpha(\vec{r}\times m\dot{\vec{r}})$$

[α = constant of proportion-ality].

$$\therefore\ m\dot{\vec{v}} = \alpha(\vec{r}\times m\vec{v})$$

$$\Rightarrow m\vec{v}\cdot\dot{\vec{v}} = \alpha(\vec{r}\times m\vec{v})\cdot\vec{v} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = 0 \Rightarrow \frac{1}{2}mv^2 = \text{Constant}.$$



Equation of motion: $\ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 = 0$

$$2\ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 = 0$$

let $x_1 = A e^{i\alpha t}$, $x_2 = A_2 A e^{i\alpha t}$ $\Rightarrow$

$$0 = \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -2\alpha^2 + 2\omega^2 \end{vmatrix}$$

Solving for α^2 $\Rightarrow$

$$\alpha = \pm\omega\sqrt{\frac{3+\sqrt{3}}{2}} \quad \& \quad \alpha = \pm\omega\sqrt{\frac{3-\sqrt{3}}{2}}.$$

Normal modes can be solved using

$$\begin{pmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -2\alpha^2 - 2\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

& substituting the above α's in it

Solving for normal modes $\Rightarrow$

for $\alpha = +\omega\sqrt{\frac{3+\sqrt{3}}{2}}$ $\Rightarrow$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha t + \phi_1)$

for $\alpha = +\omega\sqrt{\frac{3-\sqrt{3}}{2}}$ $\Rightarrow$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha t + \phi)$

Solution to Question 3 (End-Sem)

(3.)(a) Two masses are of masses m_1 and m_2 and of velocities $\vec{v_1}$ and $\vec{v_2}$ respectively.

Hence, $\vec{V} = \dfrac{m_1\vec{v_1} + m_2\vec{v_2}}{m_1+m_2}$ (velocity of the C.M.)

$$= \frac{m_1\vec{v_1} + m_2\vec{v_2}}{M} \qquad (M = m_1+m_2)$$

and $\vec{v} = \vec{v_1} - \vec{v_2}$ ($\vec{v_2} - \vec{v_1}$ is also accepted)

$\Rightarrow \vec{v_1} = \vec{v} + \vec{v_2}$

$$\therefore \vec{V} = \frac{m_1(\vec{v}+\vec{v_2}) + m_2\vec{v_2}}{M} = \frac{m_1\vec{v} + (m_1+m_2)\vec{v_2}}{M}$$

$$\Rightarrow \vec{v_2} = \vec{V} - \frac{m_1\vec{v}}{M} \quad \text{and} \quad \vec{v_1} = \vec{v} + \vec{V} - \frac{m_1\vec{v}}{M}$$

$$= \vec{V} + \frac{m_2\vec{v}}{M}$$

and the total kinetic energy

$$= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$= \frac{1}{2}m_1\left(\vec{V} + \frac{m_2\vec{v}}{M}\right)^2 + \frac{1}{2}m_2\left(\vec{V} - \frac{m_1\vec{v}}{M}\right)^2$$

$$= \frac{1}{2}(m_1+m_2)V^2 + \cancel{\frac{m_1m_2}{M}\vec{V}\cdot\vec{v}} - \cancel{\frac{m_2m_1}{M}\vec{V}\cdot\vec{v}}$$

(2)

$$+ \frac{1}{2} \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2} v^2$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)} v^2$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$$

$$\left(\text{reduced mass} = \frac{m_1 m_2}{m_1+m_2}\right)$$

* Any other correct method was accepted.

(3.)(b.) Euler equations of rigid body :→

$$I_{xx} \dot{\omega}_x + \omega_y \omega_z (I_{zz} - I_{yy}) = M_x$$

$$I_{yy} \dot{\omega}_y + \omega_z \omega_x (I_{xx} - I_{zz}) = M_y$$

$$I_{zz} \dot{\omega}_z + \omega_x \omega_y (I_{yy} - I_{xx}) = M_z$$

where, M_x, M_y, M_z are the components of torque in the three principal directions ~~as~~ as measured by a space fixed frame.

I_{xx}, I_{yy} and I_{zz} are the three principal moments of Inertia and ω_x, ω_y and ω_z are the components of the total angular velocity in the three principal directions.

* Small errors are partially penalized But fundamental errors were not considered for any credit.

* Any correct method has been accepted!

(3.)(c.) For a system (rigid body), where the net torque ~~on~~ vanishes in the lab frame, the Euler's equations are given by

$$I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz}-I_{yy}) = 0$$

$$I_{yy}\dot{\omega}_y + \omega_z\omega_x(I_{xx}-I_{zz}) = 0$$

$$I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy}-I_{xx}) = 0$$

Now, multiplying the above three equations respectively with the terms $I_{xx}\omega_x$, $I_{yy}\omega_y$ and $I_{zz}\omega_z$, we obtain

$$I_{xx}^2\frac{d}{dt}\left(\frac{1}{2}\omega_x^2\right) + I_{xx}(I_{zz}-I_{yy})\omega_x\omega_y\omega_z = 0$$

$$I_{yy}^2\frac{d}{dt}\left(\frac{1}{2}\omega_y^2\right) + I_{yy}(I_{xx}-I_{zz})\omega_x\omega_y\omega_z = 0$$

$$I_{zz}^2\frac{d}{dt}\left(\frac{1}{2}\omega_z^2\right) + I_{zz}(I_{yy}-I_{xx})\omega_x\omega_y\omega_z = 0$$

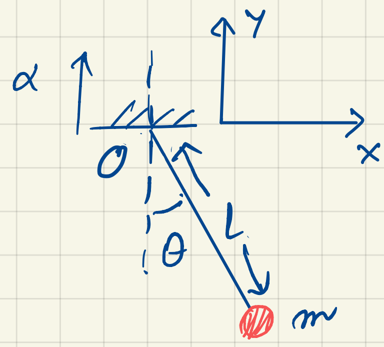
Summing all the three, one can show that,

$$\frac{d}{dt}\left(\frac{1}{2}I_{xx}^2\omega_x^2 + \frac{1}{2}I_{yy}^2\omega_y^2 + \frac{1}{2}I_{zz}^2\omega_z^2\right) = 0$$

$$\Rightarrow I_{xx}^2\omega_x^2 + I_{yy}^2\omega_y^2 + I_{zz}^2\omega_z^2 = \text{Constant.}$$

* Magnitude of Angular momentum conserved in Body frame.

Q.4(a)



(i)

$$y(t) = \frac{1}{2}\alpha t^2 - l\cos\theta(t)$$

$$x(t) = l\sin\theta(t)$$

$$\dot{x} \equiv \frac{dx}{dt} = l\cos\theta\,\dot{\theta} \qquad \text{where } \dot{\theta} \equiv \frac{d\theta}{dt}$$

$$\dot{y} \equiv \frac{dy}{dt} = \alpha t + l\sin\theta\,\dot{\theta}$$

$$T \equiv K.E. = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\times m\dot{y}^2 = \frac{1}{2}m\left[l^2\cos^2\theta\,\dot{\theta}^2 + (\alpha t + l\sin\theta\,\dot{\theta})^2\right]$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2 + \alpha^2 t^2 + 2\alpha l\sin\theta\,\dot{\theta}\,t\right]$$

$$U = P.E. = mgy = mg\left[\frac{1}{2}\alpha t^2 - l\cos\theta\right]$$

$$L = T - U$$

$$= \frac{1}{2}m\left[l^2\dot{\theta}^2 + \alpha^2 t^2 + 2\alpha l\sin\theta\,\dot{\theta}\,t\right] + mg\left[l\cos\theta - \frac{1}{2}\alpha t^2\right]$$

(ii)

$$\frac{\partial L}{\partial \theta} = m\alpha l t\cos\theta\,\dot{\theta} - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + m\alpha l t\sin\theta$$

Lagrange's equation of motion for θ:

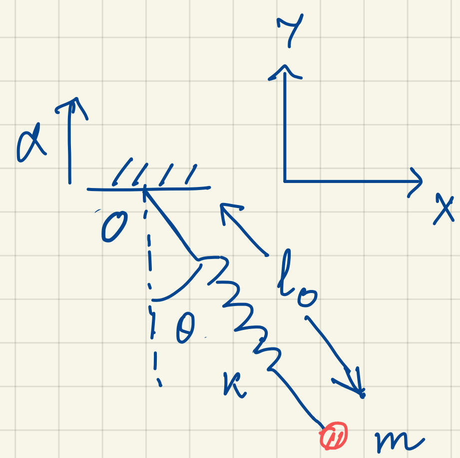
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left[ml^2\dot{\theta} + m\alpha l t\sin\theta\right] - \left[m\alpha l t\cos\theta\,\dot{\theta} - mgl\sin\theta\right] = 0$$

$$\Rightarrow \quad l^2\ddot{\theta} + \alpha l \sin\theta + \cancel{\alpha l t \cos\theta\,\dot{\theta}} = \cancel{\alpha l t \cos\theta\,\dot{\theta}} - g l \sin\theta$$

$$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{(\alpha+g)}{l}\sin\theta = 0}$$

Q.4.(b)



(i) Since the motion of the string pendulum is constrained in a plane, the dynamics of the system is defined by $\theta(t)$ = angular seperation of the bob, at time t, from the equilibrium position.

$\gamma(t)$ = lateral displacement of the spring, at time t, from its equilibrium length l_0.

$$x(t) = \gamma \sin\theta \Rightarrow \dot{x} = \dot{\gamma}\sin\theta + \gamma\cos\theta\,\dot{\theta}$$

$$y(t) = \frac{1}{2}\alpha t^2 - \gamma\cos\theta \Rightarrow \dot{y} = \alpha t - \dot{\gamma}\cos\theta + \gamma\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}m\left[\dot{\gamma}^2\sin^2\theta + 2\gamma\dot{\gamma}\dot{\theta}\sin\theta\cos\theta + \gamma^2\dot{\theta}^2\cos^2\theta + \alpha^2t^2 + \dot{\gamma}^2\cos^2\theta + \gamma^2\dot{\theta}^2\sin^2\theta + 2\alpha t\gamma\dot{\theta}\sin\theta - 2\alpha t\dot{\gamma}\cos\theta - 2\gamma\dot{\gamma}\dot{\theta}\sin\theta\cos\theta\right]$$

$$U = mgy + \frac{1}{2}\kappa\left(l_0 - r\right)^2$$

$$= mg\left[\frac{1}{2}\alpha t^2 - r\cos\theta\right] + \frac{1}{2}\kappa\left(l_0 - r\right)^2$$

$$L = T - U$$

$$= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \alpha^2 t^2 + 2\alpha t\left(r\dot{\theta}\sin\theta - \dot{r}\cos\theta\right)\right]$$

$$+ mg\left[r\cos\theta - \frac{1}{2}\alpha t^2\right] - \frac{1}{2}\kappa\left(l_0 - r\right)^2$$

$$\frac{\partial L}{\partial \theta} = m\alpha t\left[r\dot{\theta}\cos\theta + \dot{r}\sin\theta\right] - mg\, r\sin\theta$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + m\alpha t\dot{\theta}\sin\theta + mg\cos\theta + \kappa\left(l_0 - r\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + m\alpha t\, r\sin\theta$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} - m\alpha t\cos\theta$$

Lagrange's equation of motion for r

θ :
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left[m r^2 \dot{\theta} + m\alpha t r \sin\theta \right] - m\alpha t \left[r \dot{\theta} \cos\theta + \dot{r} \sin\theta \right] + m g r \sin\theta = 0$$

$$\Rightarrow 2 m r \dot{r} \dot{\theta} + m r^2 \ddot{\theta} + m\alpha r \sin\theta + m\alpha t \cancel{\dot{r} \sin\theta} + m\alpha t \cancel{r \cos\theta \dot{\theta}} - m\alpha t \cancel{r \dot{\theta}} \cos\theta - m\alpha t \cancel{\dot{r}} \sin\theta + m g r \sin\theta = 0$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} + \frac{m(\alpha + g)}{r} \sin\theta = 0}$$

r :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}\left[m\dot{r} - m\alpha t \cos\theta \right] - m r \dot{\theta}^2 - m\alpha t \dot{\theta} \sin\theta - m g \cos\theta - k(l_0 - r) = 0$$

$$\Rightarrow m\ddot{r} - m\alpha \cos\theta + m\alpha t \cancel{\sin\theta \dot{\theta}} - m r \dot{\theta}^2 - m\alpha t \cancel{\dot{\theta} \sin\theta} - m g \cos\theta - k(l_0 - r) = 0$$

$$\Rightarrow \boxed{\ddot{r} - r\dot{\theta}^2 - (g + \alpha)\cos\theta - \frac{k}{m}(l_0 - r) = 0}$$

(ii) Conjugate momenta: $p_\theta = \dfrac{\partial L}{\partial \dot\theta}$

$$\boxed{p_\theta = m\gamma^2\dot\theta + m\alpha t\gamma\sin\theta}$$

$$\Rightarrow \boxed{\dot\theta = \frac{p_\theta}{m\gamma^2} - \frac{\alpha t}{\gamma}\sin\theta} \quad \text{— I}$$

$$p_\gamma = \frac{\partial L}{\partial \dot\gamma}$$

$$\boxed{p_\gamma = m\dot\gamma - m\alpha t\cos\theta}$$

$$\Rightarrow \boxed{\dot\gamma = \frac{p_\gamma}{m} + \alpha t\cos\theta} \quad \text{— II}$$

Energy function: $\mathcal{H} = \sum_i p_i \dot q_i - L$

$$= p_\theta\dot\theta + p_\gamma\dot\gamma - L$$

$$\Rightarrow p_\theta\left(\frac{p_\theta}{m\gamma^2} - \frac{\alpha t}{\gamma}\sin\theta\right) + p_\gamma\left(\frac{p_\gamma}{m} + \alpha t\cos\theta\right)$$

$$-\frac{1}{2}m\left[\left(\frac{p_\gamma}{m} + \alpha t\cos\theta\right)^2 + \gamma^2\left(\frac{p_\theta}{m\gamma^2} - \frac{\alpha t}{\gamma}\sin\theta\right)^2\right.$$

$$+ \alpha^2 t^2 + 2\alpha t\left[\gamma\left(\frac{p_\theta}{m\gamma^2} - \frac{\alpha t}{\gamma}\sin\theta\right)\sin\theta\right.$$

$$\left.\left. - \left(\frac{p_\gamma}{m} + \alpha t\cos\theta\right)\cos\theta\right]\right] - mg\left[\gamma\cos\theta - \frac{1}{2}\alpha t^2\right]$$

$$+ \frac{1}{2} k (l_0 - r)^2$$

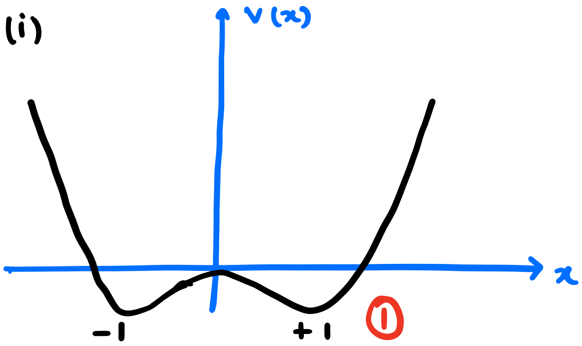
upon simplification the energy function is [using I & II]

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\alpha t}{r} p_\theta \sin\theta + \alpha t p_r \cos\theta + \frac{1}{2} k (l_0 - r)^2 + \frac{1}{2} m \alpha g t^2 - m g r \cos\theta}$$

END SEM

5a) $V(x) = -x^2/2 + x^4/4$.

(i)



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(ii) • The force is $-\frac{dV}{dx} = x - x^3$.

$\therefore$ Equation of motion $\ddot{x} = x - x^3$

This can be rewritten as $\dot{x} = v$

$\dot{v} = x - x^3$.

Fixed points occur for $(\dot{x}, \dot{v}) = (0,0)$;

$v = 0$ &

$x - x^3 = 0$

$\Rightarrow x[1-x^2] = 0$

$\Rightarrow x = 0, \pm 1$

$\therefore$ Fixed points $(0,0), (\pm 1, 0)$

• In the neighborhood of $(0,0)$; $x = \epsilon \to 0$

$v = \dot{x} = \dot{\epsilon}$

$\ddot{x} = \epsilon - \cancel{\epsilon^3} \to 0$

$\Rightarrow \dot{v} = \epsilon = x$

$\therefore$ We have $\left.\begin{array}{l}\dot{x} = v \\ \dot{v} = x\end{array}\right\}$

$\Rightarrow \frac{dv}{dx} = \frac{x}{v}$

$\Rightarrow v^2/2 - x^2/2 = \text{const}$ $\Rightarrow$ Hyperbola. 1 Saddle point.

• $(+1, 0)$, we have $x = 1 + \epsilon$

$\dot{x} = v \Rightarrow \dot{\epsilon} = v$

$\dot{v} = x - x^3 = (1+\epsilon) - (1+\epsilon)^3 = (1+\epsilon) - [1 + 3\epsilon + 3\epsilon^2 + \epsilon^3]$

$= -2\epsilon$

$\therefore$ $\dot{\epsilon} = v$

$\dot{v} = -2\epsilon$

$\Rightarrow \dot{\epsilon}/\dot{v} = -\frac{v}{2\epsilon}$

$$\Rightarrow 2\epsilon\, d\epsilon = -v\, dv$$

$$\Rightarrow \epsilon^2 + \frac{v^2}{2} = \text{const} \Rightarrow \text{Ellipse}$$ 1 <u>Centres</u>

- $(-1,0)$; $x = -1+\epsilon$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$\dot{x} = v = \dot{\epsilon}$

$\dot{v} = x - x^3 = (-1+\epsilon) - (-1+\epsilon)^3$

$= (-1+\epsilon) - [-1 + \epsilon^3 - 3\epsilon^2 + 3\epsilon]$

$= \epsilon - \epsilon^3 + 3\epsilon^2 - 3\epsilon \simeq -2\epsilon$

$\therefore \quad \dot{x} = \dot{\epsilon} = v$

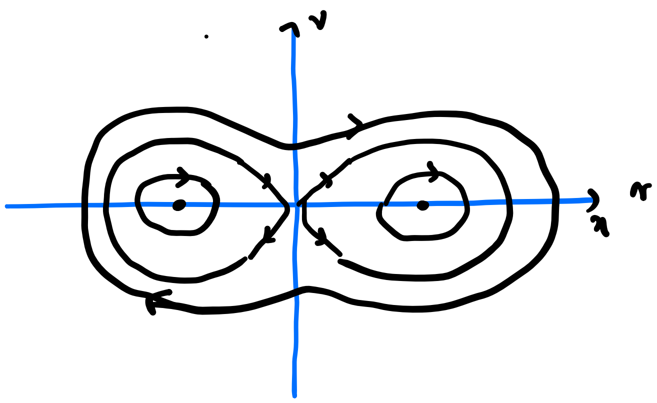
$\dot{v} = -2\epsilon$

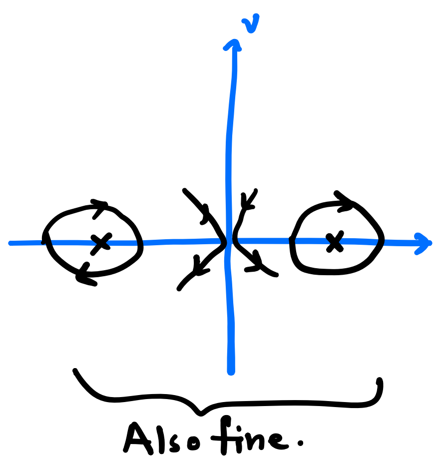
$\Rightarrow \dot{\epsilon}/\dot{v} = -\frac{v}{2\epsilon}$

$\Rightarrow 2\epsilon\, d\epsilon = -v\, dv$

$\Rightarrow \epsilon^2 + v^2/2 = \text{const} \Rightarrow$ Ellipse. 1 <u>Centres</u>

(iii)





1

5b) $\frac{d^2u}{d\theta^2} + u = \alpha + \epsilon u^2$.

$u = 1/r$, $\alpha > 0$ & ϵ is very small.

$$J = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix}.$$

τ = sum of diag.

(i) Rewrite using u & $v = \frac{du}{d\theta}$

$$\frac{du}{d\theta} = v$$

$$\frac{dv}{d\theta} = \alpha + \epsilon u^2 - u$$ 1

(ii) $(\dot{u}^*, \dot{v}^*) = (0,0)$

$$\epsilon u^2 - u + \alpha = 0$$

$$\Rightarrow u = \frac{1 \pm \sqrt{1-4\alpha\epsilon}}{2\epsilon}$$

So, $(u_1^*, v_1^*) = \left(\frac{1+\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right)$ &

$$(u_2^*, v_2^*) = \left(\frac{1-\sqrt{1-4\alpha\epsilon}}{2\epsilon}, 0\right).$$

2

Now, Jacobian

$$\begin{pmatrix} 0 & 1 \\ 2\epsilon u - 1 & 0 \end{pmatrix} \quad \therefore J_1 = \begin{pmatrix} 0 & 1 \\ \sqrt{1-4\alpha\epsilon} & 0 \end{pmatrix}$$

$\Delta_1 = -\sqrt{1-4\alpha\epsilon}$; $\tau = 0 \Rightarrow$ Saddle point.

$$J_2 = \begin{pmatrix} 0 & 1 \\ -\sqrt{1-4\alpha\epsilon} & 0 \end{pmatrix}$$; $\Delta_2 = \sqrt{1-4\alpha\epsilon}$; $\tau = 0 \Rightarrow$ linear Centre. 1