INDIAN INSTITUTE OF TECHNOLOGY KANPUR

PHY112 (Classical dynamics) 2023-24 Odd semester

Mid-semester examination

Date: 18. 09. 2023 **Time: 15h30-17h30** **Total points: 40**

Note: Please do all the objectives at one place. For the long answer type questions, please do all the all sub-parts of a question at one place.

Part I: Short answer type questions (2 points each) **6x2=12**

1. (a) Find the scalar potential $V(r,\theta)$ for the conservative force field

$$\mathbf{F} = r\sin 2\theta\ \hat{r} + r\cos 2\theta\ \hat{\theta}.$$

(b) If the trajectory of a particle of mass m is given by $r = a(1-\cos\theta)$ and $\dot{\theta} = 3$ units, show that the cross-radial (tangential) component of acceleration equals to $18a\sin\theta$.

(c) A particle of mass m moves in one dimension under a potential $V(x)$. For such a particle, if we define a function $L = L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - V$, then calculate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}.$$

(d) In Kepler's problem, calculate the angle between the angular momentum $\mathbf{L}$ and the Laplace Runge-Lenz vector $\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\ \hat{r}$, where the symbols have their usual meaning. What can you conclude from this?

(e) Find the radius a of a circular orbit under the potential $U = kr^4$, where $k > 0$. The final answer should be expressed in terms of $\mathbf{L}, k$, and the particle mass m.

(f) The equation of motion of a particle is given by $\ddot{x} + \alpha x + \beta x^3 = 0$, where α and β are non-zero real constants. Find an integral of motion for the particle. What will be the integral of motion if the right hand side of the equation is replaced by a positive constant q?

Part II: Long answer type questions (7 points each) **4x7=28**

2. The position vector of a particle of mass m is given by

$$\mathbf{r} = a\cos\omega t\ \hat{\mathbf{i}} + b\sin\omega t\ \hat{\mathbf{j}}\ ,$$

where a, b and ω are positive constants.

(a) Show that the force acting on the particle is always directed towards the origin.

(b) Calculate the curl of the force field and hence conclude if the force field is conservative? Also calculate the work done in moving the particle from $r = a$ to $r = b$.

(c) Find the total mechanical energy of the particle and show that it is constant.

1+(2+2)+2 =7

3. (a) The orbit of a particle of mass m, moving in a central force $\mathbf{F} = f(r)\hat{r}$, is given by $\theta = \theta(r)$. Using the polar expression of acceleration, show that the law of force is given by

$$f(r) = -\frac{mh^2[2\theta' + r\theta'' + r^2\theta'^3]}{r^5\theta'^3},$$

where $\theta' = \frac{d\theta}{dr}$ and $\theta'' = \frac{d^2\theta}{dr^2}$and h is the angular momentum per unit mass. (Hint: use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$)

(b) Using the derived formula, find the force law for the orbit $\theta = 1/r$.

(c) Write the expression for the effective potential of the particle and plot that in the energy diagram and briefly discuss the motion.

3+1+(1+1+1)=7

4. A particle of mass m moves with angular momentum $\mathbf{L}$ in a central force field

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3},$$

where k and λ are positive constants.

(a) Prove that orbit has the form

$$r(\theta) = \frac{\alpha}{1 + \epsilon\cos(\beta\theta)},$$

where α, ϵ, and β are positive constants. Express α, ϵ, and β in terms of the given parameters.

(b) Describe the nature of the orbit for the case when $0 < \epsilon < 1$. Find out if the orbit will be closed for any value(s) of β? What happens to your result as $\lambda \to 0$?

4+(1+1+1)=7

5. Consider a spring-mass system with the mass m attached to the end of a massless spring (with spring constant k). The mass can move only in one dimension, with the direction aligned with the length of the spring (say $\hat{x}$ direction). The mass m at the end of the spring is subject to two forces: i) a drag force proportional to its speed v, $F_{\text{drag}} = -bv$ (with $b > 0$ being a constant); ii) a driving force given by $F_{\text{drive}}(t) = F\sin(\gamma t)$. Here, t is the time, and $\gamma > 0$ denotes the driving frequency. The mass is initially at rest at $x = 0$.

(a) Calculate its position $x(t)$.

(b) Consider the real amplitude A of the oscillating part of the particular integral. For what value of the γ (which can be varied in experiments) will A be maximum?

(c) If we express the time-independent part of the particular integral in the form $Ae^{i\phi}$, what does the phase ϕ tell us about the motion?

4+2+1=7